Algebraic Aspects of Differential Equations

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I would like to dedicate this thesis to my loving parents, and my nephew and niece—Neelesh and Karunya.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. Chapter 1 and 4 contains necessary introduction and background material, for which no originality is claimed unless stated otherwise. Chapter 2 and 3 includes original results which are published in the following papers [41] and [42] respectively. Chapter 5 includes new results which has not been submitted to any journal so far.

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Abstract

This thesis considers algebraic properties of differential equations, and can be divided into two parts. The major distinction among them is that the first part deals with the theory of linear ordinary differential equations, while the second part deals with the nonlinear partial differential equations.

In the first part, we present a method to transform the Green's operator into the Green's function. This transformation is already known in the classical case of well-posed two-point boundary value problems, here we extend it to the whole class of Stieltjes boundary problems. In comparison, Stieltjes boundary problems have more freedom from which stems more difficulties. In view of the specification of the boundary conditions: (1) they allow more than two evaluation points. (2) they allow derivatives of arbitrary order; (3) global terms in the form of definite integrals are also allowed. Our results show that the resulting Green's function is not only a piecewise function but also a distribution. Using suitable differential and Rota-Baxter structures, we aim to provide the algebraic underpinning for symbolic computation systems handling such objects. In particular, we show that the Green's function of regular boundary problems (for linear ordinary differential equations) can be expressed naturally in the new setting and that it is characterized by the corresponding distributional differential equation known from analysis.

In the second part we concern ourselves with integrable systems. A system of partial differential equations is called *integrable* if it exhibits infinitely many symmetries. Master symmetries provide a tool which guarantees the existence of infinitely many symmetries and thus help in determining proof of integrability. Using the $\mathscr{O}$-scheme developed by Wang (2015), we compute master symmetries for three new two-component third order Burgers' type systems with non-diagonal constant matrix of leading order terms. These systems can be found in the work of Talati and Turhan (2016). Two more systems with the same dimension are also presented from the ongoing work of Wang et al. In the end, we compute a master symmetry for a Davey–Stewartson type system which is a $(2+1)$-dimensional system.

List of Symbols

Here we present a list of symbols that we have used in this thesis.

Through out the first part of the thesis (Ch. 2 and 3), $\eth_{\mathscr{M}}$ and $\fint_{\mathscr{M}}$ denotes an extension of the differential operator ∂ and the integral operator $\int$ respectively, from the base algebra.

Part I: Chapter 2 and Chapter 3

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
K	Commutative ring/field
∂	Derivation
$\int$	Integration
$\mathscr{F}$	Ground algebra
$(\mathscr{F}, \partial)$	Differential algebra
$(\mathscr{F}, \int)$	Rota-Baxter algebra
$(\mathscr{F}/M, \int, \partial)$	Integro-differential algebra/module
$(\mathscr{F}/M, \partial, \int)$	Differential Rota-Baxter algebra/module
$(\mathscr{F}/M, \partial, \int, S)$	Shifted Differential Rota-Baxter algebra/module

$\mathscr{C}$	Ring of constant functions in $\mathscr{F}$ used to denote $\mathrm{Ker}(\partial)$
$\mathscr{I}$	Ideal of initialized functions in $\mathscr{F}$ used to denote $\mathrm{Im}(\int)$
$\mathtt{j}$	Initialisation of $\mathscr{F}$
$\mathtt{E}$	Evaluation of $\mathscr{F}$
f	function in $\mathscr{F}$
$f^{(i)}$	i-th derivative of f
C^∞	Algebra of smooth functions
$\varphi \in \Phi$	Algebra homomorphisms from $\mathscr{F} \to K$ (characters)
$\{1\} \cup \mathscr{F}^\#$	K-basis of $\mathscr{F}$
$\mathscr{F}^\bullet$	Non-zero characters on $\mathscr{F}$
$\mathscr{F}_\Phi[\partial, \int]$	Algebra of integro-differential operators
$\mathscr{F}[\partial, \int_\Phi]$	Equitable ring of integro-differential operators
$(T, \mathscr{B})$	Boundary problem
T	Differential operator
$\beta \in \mathscr{B}$	Boundary conditions in the space of boundary conditions
$\mathscr{B}^\perp$	Orthogonal space of $\mathscr{B}$
G or $G_{x\xi}$	Green's operator
W	Wronskian matrix
$\mathscr{F}[\partial]$	Ring of differential operators
$\mathscr{F}[\int]$	Ring of integral operators
$T^\diamond$	Fundamental right inverse
$g(x, \xi)$	Green's function
$\tilde{G}_{x\xi}$	Functional part of $G_{x\xi}$

$\hat{G}_{x\xi}$	Distributional part of $G_{x\xi}$
$J \subset \mathbb{R}$	Interval containing evaluation points
$C(J^2)$	Continuous functions on rectangular region $J^2 \in \mathbb{R}^2$
δ	Dirac distribution
$\delta^{(i)}$	i-th derivative of Dirac distribution
$\mathtt{E}$	Distinguished evaluation $\mathtt{E}\int = 0$ (Ch. 3 only)
$\eth$	Derivation on Module M
$\oint$	Integration on Module M
$\hat{\mathtt{E}}$	Induced pseudo-evaluation of $(\mathscr{P}\mathscr{F}, \partial, \int)$
$\acute{\mathtt{E}}$	Induced evaluation of an integro-differential module $(\mathscr{D}\mathscr{F}, \oint, \eth)$
$[P]$	If condition P is true then $[P] = 1$, else 0
H_a	Heaviside function used to denote $H(x-a) = 1 - \bar{H}_a$
$H_a^{(i)}$	i-th derivative of Heaviside function
sgn	Sign function
$a \sqcup b$	$\max(a,b)$
$a \sqcap b$	$\min(a,b)$
$(K, <)$	Ordered ring/field
$K_\sqcup$	Monoid K with the binary operation $\sqcup$
$K_\sqcap$	Monoid K with the binary operation $\sqcap$
$\mathscr{P}\mathscr{F}$	Piecewise extension of ground algebra $\mathscr{F}$
a^+	Positive part of a
a^-	Negative part of a
$S_a f$	Shift map taking $f(x)$ to $f(x+a)$

Ś	Shift map on module
$\mathscr{D}\mathscr{F}$	Distribution module over ground algebra $\mathscr{F}$
$f^{(m)}$	m-th derivative of f
ψ	Element in module M
Ψ	Integro-differential morphism
Ś	Shift map on module
$\hat{\mathscr{D}}\mathscr{F}$	Slim distribution module
$\hat{\mathscr{P}}\mathscr{F}$	Slim piecewise extension
S_a^x	Shift map on algebra
$\acute{\mathrm{S}}_a^x$	Shift map on bivariate module
$\acute{\mathrm{e}}\mathrm{v}_a^x$	Evaluation on bivariate module
$\mathscr{D}_2\mathscr{F}$	Bivariate distribution module
$\mathscr{D}_{x-\xi}\mathscr{F}$	Diagonal distribution module
$\mathscr{D}_x\xi\mathscr{F}$	Tensorial distribution module
$\mathscr{M}$	Dirac module

Part II: Chapter 4 and Chapter 5

$\mathtt{ad}_K Q$	Adjoint action of K on Q, also denoted by $[K,Q]$
$\mathscr{O}$	Bernstein-Gelfand-Gelfand (BGG) category of $\mathfrak{sl}(2,\mathbb{C})$-modules
$\mathscr{A}$	Space of all differential functions
$\mathbf{v}$	Vector field
$\mathtt{pr}\,\mathbf{v}$	Prolongation of a vector field $\mathbf{v}$
$\mathfrak{sl}(2,\mathbb{C})$	Special linear Lie algebra of 2×2 matrices with basis e, f and h
$\triangle_\nu\,[u]$	System of a differential equations

τ	Master Symmetry
$B_3[u]$	Burger equation of order 3, $u_3 + 3uu_2 + 3u^2u_x + 3u_x^2$
D_t	Differentiation with respect to t
D_x	Differentiation with respect to x
$L(\lambda)$	Finite dimensional simple module in $\mathscr{O}$
$M(\lambda)$	Verma module in $\mathscr{O}$
$M^\vee(\lambda)$	Dual Verma module in $\mathscr{O}$
$P(\lambda)$	Projective module in $\mathscr{O}$
Q	Characteristic of a vector field
$u(x,t)$	Differential function of two independent variables
u_i, v_i	i-th derivative with respect to x

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Chapter 1

Introduction

It is indisputable that differential equations are absolutely fundamental to modern science and engineering. They provide mathematical methods that can be used in medicine (cancer growth), engineering (movement of electricity), chemistry (chemical reactions), economics (optimum investment strategies) and of course, physics (motion of waves, pendulums or chaotic systems). However, these mathematical models can fail to provide the whole picture. Therefore, to comprehend differential equations, it is important to investigate them not only analytically, but also algebraically as they can provide exact solutions. In this thesis, we concern ourselves with two different topics on differential equations focusing on their algebraic aspects.

A boundary value problem (bvp) is a differential equation together with additional constraints, called the boundary conditions. A solution to a bvp is a solution to the differential equation subjected to the corresponding boundary conditions. For instance, a classical well-posed bvp can take the below form

$$\boxed{\begin{array}{l} u'' + u = 0, \\ u(0) = 0, u(\pi/2) = 2. \end{array}} \tag{1.1}$$

A general solution to the differential equation is

$$u(x) = A\sin(x) + B\cos(x),$$

where A and B are constants. On satisfying the boundary conditions one gets $u(x) = 2\sin(x)$ as a unique solution to the bvp (1.1). Here, the term *well-posed* means that there exists a unique solution to a given bvp.

Different techniques can be employed to solve a boundary problem, we are interested in a method that uses the so-called *Green's function*. If T is a linear differential operator, then the Green's function is the solution for u of the equation $Tu = \delta$, where δ is a Dirac delta function defined by

$$\int_{-\infty}^{+\infty} f(t)\delta(t-a)dt = f(a).$$

On the operator level, we call the Green's function the *Green's operator*. Rosenkranz and Regensburger presented a method in [39, 40] for solving a linear bvp using the Green's operator.

In the first part of this thesis, we present a method to transform the Green's operator into the Green's function for linear order differential equations (LODEs). This transformation is already known in the classical case of well-posed two-point boundary value problems, here we extend it to a more general class of linear bvps. In comparison, we allow boundary conditions with more than two evaluation points, derivatives of arbitrary order or global terms in the form of definite integrals (see Eq. (1.6)). Our results show that the resulting Green's function is not only a piecewise function but also a distribution (Chapter 2). Using suitable differential and Rota-Baxter structures, we aim to provide the algebraic underpinning needed for symbolic computation systems handling such objects. In particular, we show that the Green's function of regular boundary problems (for linear ordinary differential equations) can be expressed naturally in the new setting and that it is characterized by the corresponding distributional differential equation known from analysis (Chapter 3).

In the second part we concern ourselves with integrable systems. A system of partial differential equation (PDE) is called *integrable* if it exhibits infinitely many symmetries; loosely speaking this means that there exists infinitely many functions that maps one solution to another solution of the system. Master symmetries provide is a tool which guarantees the existence of infinitely many symmetries and thus help in determining proof of integrability. Using the $\mathscr{O}$-scheme developed by Wang (2015), we compute master symmetries for three new two-component third order Burgers' type systems with non-diagonal constant matrix of leading order terms. These systems can be found in the work of Talati and Turhan (2016). Two more systems with the same dimension are also presented from the ongoing work of Wang et al. In the end, we compute a master symmetry for a Davey–Stewartson type system which is a $(2+1)$-dimensional system.

Now let us discuss the different parts of the thesis one by one. In the next section, we give a brief introduction to the problems that we will investigate in Chapters 2 and 3.

1.1 An algebraic setting for Green's functions and Dirac distributions

As differential equations arise naturally in many disciplines, it is important to find an algorithmic approach to solve them using computer algebra systems. So far, the treatment of boundary problems has been done using analysis. For that reason, we present a method to solve boundary problems in a symbolic way; and the first step is to transform our knowledge from analysis to an algebraic setting. This systematic treatment was carried out by Rosenkranz and Regensburger in their papers [39, 40], where they presented a method for solving regular boundary value problems for linear ordinary differential equations in terms of the *Green's operator*[1]. Their inspiration comes from the paper [17] which describes the use of noncommutative Gröbner bases for simplifying huge terms arising in operator control theory. This new approach has some advantages as it has a greater potential for generalisation. For example, the theory of Green's functions presupposes the linear structure on differential operators and thus, it is far less apparent to the nonlinear partial differential operators. Here, we restrict ourselves to linear ordinary differential equations; for partial differential equations, see [36].

Rosenkranz and Regensburger worked on the level of operators and thus did not provide a way to determine the *Green's function* for a given boundary value problem. We tackle this problem in the Chapter 2, where we present a method to transform the problem to a functional setting and provide an algorithm to extract the Green's function of a boundary value problem from its Green operator. This transformation is already known in the classical case of well-posed two point boundary value problems which we describe below with a simple example.

Given a function $f \in C^\infty[0,1]$, we want to find a function $u \in C^\infty[0,1]$ such that

$$\boxed{\begin{aligned} &u'' = f, \\ &u(0) = u(1) = 0, \end{aligned}} \tag{1.2}$$

where $'$ is the usual notation representing derivative of u with respect to the variable $x \in [0,1]$. Then the Green's operator $G\colon C^\infty[0,1] \to C^\infty[0,1]$ of this problem is the right inverse of the differential operator $\frac{d^2u}{dx^2}$ such that $Gf = u$ and (u, f) satisfy (1.2). Using the standard

[1] The operator is called the Green's operator, because it is the integral operator induced by the Green's function. This name was introduced by Neumann [31] and Riemann [18, Sect. 23] in honor of the mathematician George Green (1793–1841), who invented the concept in [15, p. 12].

reduction system[2] of [40], we obtain the Green's operator as

$$G = x\int_0^x - \int_0^x x + x\mathrm{E}_1\int_0^x x - x\mathrm{E}_1\int_0^x. \tag{1.3}$$

where E_α denotes the evaluation functional $f \to f(\alpha)$ for any real number $\alpha \in \mathbb{R}$. However it is easy to extract the Green's function if we use the form after substituting $\mathrm{E}_\alpha \int_0^x = \int_0^x - \int_\alpha^x$ in (1.3), which in turn yields

$$G = x\int_0^x x - \int_0^x x - x\int_1^x x + x\int_1^x.$$

If we rewrite the above expression in the form

$$Gf(x) = \int_0^1 g(x,\xi)\, f(\xi)\, d\xi, \tag{1.4}$$

then $g(x,\xi)$ is found to be

$$g(x,\xi) = \begin{cases} (x-1)\xi & \text{for } 0 \le \xi \le x \le 1, \\ x(\xi-1) & \text{for } 0 \le x \le \xi \le 1, \end{cases} \tag{1.5}$$

which is nothing but its Green's function.

Now, the question arises as to how do we extend this transformation to *ill-posed boundary problems*. Loosely speaking, this means that we allow boundary conditions with integrals and higher order derivatives compared to the operator of the problem. For example, consider the following boundary problem

$$\boxed{\begin{array}{l} u'' - u = f, \\ u'''(-1) - \int_0^1 u(\xi)\,\xi\, d\xi = 0, \\ u'(-1) - u''(1) + \int_{-1}^1 u(\xi)\, d\xi = 0. \end{array}} \tag{1.6}$$

Here, we have boundary conditions of order 3 for a boundary problem of order 2, along with nonlocal terms (integrals).

[2]This algorithm is mentioned in the Sec. 2.3

Let us summarise this—for a boundary problem with differential operator T of order n and boundary conditions $\beta_1, \cdots, \beta_n$.

$$\boxed{\begin{array}{l} Tu = f, \\ \beta_1 u = \cdots = \beta_n u = 0, \end{array}} \tag{1.7}$$

first we find an "inverse" operator of T called the *Green's operator* G such that $Gf = u$, and then extract the Green's function form its Green's operator. We present this result in the *Structure Theorem* 2.3.

Our result proves that the Green's function is not only a piecewise function (continuous/ smooth), but moreover, a proper distribution in the case of *ill-posed* boundary problems (see Example 2.4.4). It encouraged us to further investigate an algebraic setting for treating distributions which we describe in Chapter 3.

Interestingly, the *theory of distributions* has received little attention in symbolic analysis, even though in modern analysis it provides a solid background to support the theory of linear (ordinary and partial) differential equations. One reason for this might be the widespread limitation of differential algebra to structures having only derivations, as in differential rings/fields/algebras/modules. In such a setting one can only treat the Dirac distribution as a differential indeterminate, and cannot say anything more than that a distribution δ_a has arbitrary formal derivatives $\delta_a', \delta_a'', \cdots$. Here, we developed algebraic structures involving not only derivatives but also integrals, which allows us to explore the algebraic properties of distributions. The Heaviside function $H(x-a) := H_a$ is the integral of the Dirac delta distribution, so let us recall its definition before further discussion. When $a = 0$, it is defined as $H(x)\colon K \to K$ by

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

If $a > 0$ then the graph of H_a can be obtained by shifting the graph of $H(x)$ by a units along the positive x-axis. Similarly, if $a < 0$ then we shift the graph a units along the negative x-axis. We revisit the definition in detail in Chapter 3 (Def. 3.7). We also consider piecewise functions as they can be built using the Heaviside function. This may tempt one to build up distributions via this route— adding a derivation that maps H_a to δ_a—but it fails to capture the essence of distributions (see Remark 3.3). Instead, we introduce *distributions* as a differential Rota-Baxter module, following an independent route, but such that the piecewise functions reappear as a Rota-Baxter subalgebra (Thm. 3.1).

In short, we show that the Green's function of a regular boundary problem (for a linear ordinary differential equation) can be expressed naturally in the new setting, and that it is characterized by the corresponding distributional differential equation known from analysis.

1.2 Symmetry structures for nonlinear partial differential equations

The second part of the thesis is devoted to *integrable systems* or *exactly solvable equations*. These systems are essential to theoretical and mathematical physics. In fact, one may argue that they constitute the “mathematical nucleus" of theoretical physics with an aim to describe real classical or quantum systems. Yet, it is hard to define the notion of integrability. This has led to a discussion among scientists which resulted into the book, “What Is integrability” [6].

Several methods can be employed to define *integrability* of a system, for example, reductions to known integrable systems, bilinear (Hirota) representation, the Painléve test and the symmetry approach. Throughout this thesis, we follow the symmetry approach where a system of partial differential equations is called integrable if it exhibits infinitely many higher order[3] symmetries. Before we begin, we first give a sketch of the development of this field based on [6, 44, 51] and the references therein. The purpose is purely motivational, we do not intend to give a complete historical survey of the field. For references to some of the original sources, see Table 1.1.

In 1834, John Scott Russell, a Scottish naval engineer was observing the passage of a boat along a canal and noticed a very strange type of wave travelling along the canal. His famous—often repeated—summary of the event states [43]-

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after

[3]Order of a differential equation is the highest order of derivative in the equation.

> a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

The mathematical theory explaining the existence of these solitary heaps of water was first developed by Boussinesq in 1871 and, independently, by Rayleigh in 1876. Later in 1895, Korteweg and de Vries derived a model equation for water waves in shallow channels which first appeared in the work of Boussinesq as a footnote. The equation now is named after them and known as the KdV equation, given by

$$u_t = u_3 + uu_1, \tag{1.8}$$

where $u_i = \frac{\partial^i u}{\partial x^i}$. It is probably one of the most celebrated evolution equations. They showed that periodic solutions could be found in closed form without any approximations. This is arguably the first stage of discovery of solitary waves but it was not until 1955 that this subject formally began. In 1955, Fermi, Pasta and Ulam (FPU) undertook a numerical study of the one-dimensional harmonic chain model that provided the required thrust to investigate such systems—all thanks to the Maniac I computer. In their study, they expected the nonlinear interactions to result in thermal equilibrium but the system returned to its originally excited state and a few nearby modes. This strange behaviour attracted Kruskal and Zabusky. In 1965, they tackled the FPU problem from the continuous viewpoint and amazingly rediscovered the KdV equation. They found stable pulse-like waves with computer simulation and named them *solitons*. They had a remarkable property— after collision, they preserved their shapes and speeds, and simply spread apart again.

It was around this time that many physicists and mathematicians took interest in the subject and soon, the KdV equation started to appear everywhere—in fluid dynamical applications, plasma physics and the study of dispersive waves in elastic rods. The situation changed dramatically in 1967, when Gardner, Greene, Kruskal and Miura introduced a new and very powerful method—the *inverse scattering transform* (IST). They used this method to solve the KdV equation and showed that any finite number of solitons can be expressed in closed form. The following year, Lax provided a clear interpretation of their result in terms of the notion of a *Lax pair* (or L-A pair). It played an important role in extending the applicability of the method. Later in 1971, Zakharov and Shabat made an influential contribution and showed that IST is indeed a method and not a trick suitable only for a single equation.

An explanation was needed to explain the existence of solitons exhibiting this remarkable stability and it came in the form of *conservation laws* and consequently the notion of *symmetry* was developed. Originally, the symmetry approach began from the work of Sophus Lie. He studied the symmetry groups of differential equations and developed a more universal method for solving differential equations rather than the familiar *cookbook* methods. Roughly speaking, a symmetry group of a system consists of transformations on the space of independent and dependent variables which leave the system invariant; such transformations lead to so-called *geometric symmetries*.

Generalised symmetries first appeared in the fundamental work of Noether in 1918. The difference with the classical Lie symmetries (geometric symmetries) is their dependence on the highest derivatives of u and thus they lack proper geometrical meaning. She proved the remarkable theorem, giving a one-to-one correspondence between symmetry groups and conservation laws, now known as the Noether's theorem. Unfortunately, her work was neglected for many years and rediscovered several times since.

Magri [28] studied the connections between conservation laws and symmetries from the geometric point of view in terms of Hamiltonian and symplectic operators. He found that some systems admit two distinct but compatible Hamiltonian pairs (they are bi-Hamiltonian), and showed that KdV is in fact a bi-Hamiltonian equation. It can be written as-

$$u_t = D_x\left(u_2 + \frac{1}{2}u^2\right) = \left(D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_1\right)u.$$

However, these operators had already been observed by Lenard [35], who used them to redrive the KdV hierarchy. This construction is now called the *Lenard scheme*, formulated in [13]. In 1977, Olver presented the theory of *recursion operators* which can generate a hierarchy of generalised symmetries, originally due to Lenard. A recursion operator for the KdV equation is

$$D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}, \tag{1.9}$$

with D_x^{-1} being the left inverse of D_x.

Later on, at the first Kiev conference in September 1979, Mikhailov, Shabat and Sokolov altered the symmetry approach based on explicit integrability conditions [23]. In 1980, Fokas [10] also discussed the symmetry approach, where, he found all equations of the form

$$u_t = u_3 + f(u, u_1)$$

having one nonclassical symmetry of a fixed order. However, the first publication [23] on the formal symmetries of evolution equations with explicit integrability conditions is due to Ibragimov and Shabat, 1980. After this time, there has been a significant development of the

1834	John Scott Russell discovered a physical soliton [43].
1871,76	Boussinesq and Rayleigh [3, 49].
1895	Korteweg and de Vries derived the famous KdV equation $u_t = u_3 + uu_1$ [25].
1918	Emmy Noether showed the correspondence between symmetry groups and conservation laws [32].
1955	Fermi, Pasta and Ulam (FPU problem) [9].
1965	Kruskal and Zabusky tackled the FPU problem from the continuum viewpoint [53].
1967	Gardner, Greene, Kruskal and Miura developed the Inverse Scattering Transform (IST) method [12].
1968	Lax introduced the so called L-A pair [27].
1971	Zakharov and Faddeev introduced bi-Hamiltonian system [55].
1977	Olver introduced the recursion operator [33].
1983	Fuchssteiner introduced master symmetry [11].

Table 1.1 Timeline of the development of the field of integrable systems.

theory. Of course, the development of this field has seen other milestones, but we mention only few of them in the timeline below as appropriate to this thesis, see Table 1.1.

The notion of the master symmetry first appeared in the work of Fuchssteiner [11] where he treated the Benjamin-Ono and the KP equation. In his own words—

> With the new discovery of so many completely integrable evolution equations, there is a growing demand for simple, transparent and direct methods to obtain these quantities in an explicit form. In order to contribute to a partial satisfaction of this demand, we introduce in this paper the notion of *master symmetry*.

The notion of integrability can be classified into “C-integrability" and “S-integrability", which was introduced by Calogero in his 1987 paper [5]. S-integrable equations are those solvable via the IST method whereas C-integrable equations are those solvable via an appropriate change of variables; they are therefore generally easier to investigate. The first three examples considered in this thesis (5.4.2, 5.4.3, 5.4.4) are in fact C-integrable.

Following the symmetry approach, we construct master symmetries for nonlinear partial differential equations.

1.3 Overview and a suggestion for the reader

The two main topics—LODEs and PDEs—discussed in this thesis are somewhat independent of each other and thus can be read separately. The first topic comprises of Chapter 2 and 3, which are recommended to be read in sequence. Chapter 4 and 5 make up the second half of this thesis focusing on integrability. Readers familiar with vector fields and the concept of symmetry in differential equations can easily skip the introductory Chapter 4 and move directly to Chapter 5.

Let us end our introduction with an overview of all chapters.

Chapter 2

We begin this chapter by setting up an algebraic structure for the treatment of boundary value problems (bvp) (Sec. 2.1). In the following (Sec. 2.2), we construct the ring of *integro-differential operators* in which the Green's operator resides. The Green's operators are introduced explicitly in the next section (Sec. 2.3), together with the algorithm for their construction. We end this chapter with the *Structure Theorem* (Thm. 2.3) which fulfils our goal of extracting the Green's function from its corresponding Green's operator, alongside some examples.
Goal: To extract the Green's function from the Green's operator of a boundary value problem.
Concepts: Differential algebra $(\mathscr{F}, \partial)$ (Def. 2.1).
Integro-differential algebra $(\mathscr{F}, \int, \partial)$ (Def. 2.2).
Rota-Baxter (RB) algebra $(\mathscr{F}, \int)$ (Def. 2.6).
Differential RB algebra:= Differential algebra with RB algebra $(\mathscr{F}, \partial, \int)$ (Def. 2.7).
Integro-differential operators $\mathscr{F}_\Phi[\partial, \int]$ (Def. 2.8).
Equitable integro-differential operators $\mathscr{F}[\partial, \int_\Phi]$ (Def. 2.9).
Green's operator $G = (1-P)T^\lozenge$ (Def. 2.15).
Results: Extraction of the Green's function (Thm. 2.3).
New examples.
Observation: The Green's function of a boundary problem is a piecewise function or even a

distribution in the case of ill-posed boundary problems. This observation becomes the theme for the next chapter.

Chapter 3

We briefly review the theory of the *Rota-Baxter algebra and modules* (Sec. 3.1) which forms the basic algebraic framework for the rest of chapter. In the next section (Sec. 3.2), we treat piecewise functions and show that they are Rota-Baxter extension of the ground algebra which generalise the familiar setting of smooth functions. Distributions are introduced in the following section as a differential Rota-Baxter module (Def. 3.11), and we also show that the piecewise functions reappear as a Rota-Baxter subalgebra. We see that the distributions even form an integro-differential module (Prop. 3.3) and that they can be characterised by a universal property (Prop 3.4). Afterwards, we introduce the *bivariate distributions* which contain the bivariate piecewise functions as a subalgebra relative to both Rota-Baxter structures (Def. 3.31). Using these tools, we provide applications in the theory of LODE boundary problems (Sec. 3.4.1).

Goal: Develop a new algebraic model to accommodate piecewise and distribution nature of the Green's function corresponding to a boundary problem.

Concepts: Differential Rota Baxter module (Def. 3.4).

Evaluations.

Construct Piecewise extension $\mathscr{PF}$ (Def. 3.8) for treating piecewise nature of the Greens' function using Heaviside functions.

- Introduce derivation ∂ and integral $\int$ on $\mathscr{PF}$ (Eq. (3.10))

$$\partial H_a = 0, \qquad \textstyle\int f H_a = (\int_a f)\, H_a + \bar{H}(a) \int_0^a f.$$

- $(\mathscr{PF}, \partial, \int)$ is a differential RB algebra (Prop. 3.1).

Construct Distribution module $\mathscr{DF}$ for treating Dirac distributions appearing in the Green's functions using the sifting property $f\,\delta_a = f(a)\,\delta_a$ (Def. 3.11).

- Introduce derivation $\eth$ and integral $\oint$ on $\mathscr{DF}$ (Eq. (3.22))

$$\eth H_a = \delta_a \qquad \oint \delta_a = H_a - \bar{H}(a).$$

- $(\mathscr{DF}, \oint, \eth)$ is an integro-differential module over (Prop. 3.3)

Construct Bivariate distribution module $\mathscr{D}_2\mathscr{F}$ for treating elements like $\delta(x-\xi)$ and $H(x-\xi)$ (Eq. (3.31))

$$\mathscr{D}_2\mathscr{F} := \mathscr{D}_{x\xi}\mathscr{F} \oplus \mathscr{D}_{x-\xi}\mathscr{F}.$$

- Introduce derivations $\eth_x, \eth_\xi$ and integrals $\oint^x, \oint^\xi$ on $\mathscr{D}_2\mathscr{F}$ (Eq. (3.32))
- $(\mathscr{D}_2\mathscr{F}, \eth_x, \eth_\xi, \oint^x, \oint^\xi)$ is a duplex differential RB module.

Results: Express the Green's functions in this algebraic language (Thm. 3.3).

Characterisation of the Green's function (Thm. 3.4).
Allowing a piecewise function to be a forcing function for a boundary problem.

Chapter 4

This chapter covers essential terminology required to understand the concept of integrability. We begin by defining a vector field and its prolongation (Sec. 4.2), and then use the concept of prolongation to define a symmetry (Sec. 4.3) of a system of differential equations. We introduce a Lie bracket on the differential algebra of vector fields using the Fréchet derivative, which in turn makes it a Lie algebra (Sec. 4.4). The relation between prolongation and the Fréchet derivative (Prop.4.2) helps us to rewrite the symmetry condition in terms of the Lie bracket (Def. 4.8). We end this chapter by giving an example of the second order Burgers' equation.
Goal: To introduce notations, terms and concepts in order to understand partial differential equations via *master symmetry*.
Concepts: One parameter group of transformations (Eq. 4.1).
Infinitesimal form (Eq. (4.6)).
Symmetry (Def. 4.1).
Generalised vector field (Def. 4.2).
Prolongation (Eq. (4.17), (4.19)).
Evolutionary vector field and its characteristic (Def. 4.3).
Generalised symmetry (Eq. (4.27)).
Fréchet derivative (Eq. (4.41)).
Lie bracket (Def. 4.7).
Conclusions: Symmetry method is a powerful technique to solve ordinary differential equations. Generalised symmetries can be used to explicitly determine special type of solutions of PDEs (see Example 4.3.1). However, this method has the drawback of not generating infinitely many symmetries at once for integrable systems.

Chapter 5

In this chapter, we shall see how we can use the algebra $\mathfrak{sl}(2,\mathbb{C})$ to compute a master symmetry and time dependent symmetries for a given evolution equation. We begin this chapter by giving some motivation (Sec. 5.1) for the appearance of $\mathfrak{sl}(2,\mathbb{C})$ in integrable systems. We consider $\mathfrak{sl}(2,\mathbb{C})$ modules which live in the Bernstein-Gelfand-Gelfand (BGG) category, which is briefly explained in Section 5.2. In the next section, we describe the main theorem, which provides an algorithmic approach to compute master symmetry (Thm. 5.1). In the following section on applications, we present new results—master symmetries for three new two-component Burgers' type $(1+1)$-dimensional systems that appeared in [50] (Examples 5.4.2, 5.4.3, 5.4.4) and for two new systems from the ongoing work of Wang et al. [29] (Examples 5.4.5, 5.4.6). We end this chapter by introducing quasilocal polynomials which provides an algebraic framework to deal with $(2+1)$-dimensional systems together with an example of a Davey-Stewartson type system (Example 5.5.1).

Goal: To compute master symmetries of nonlinear partial differential equations using $\mathfrak{sl}(2,\mathbb{C})$ algebra.

Concepts: Lie algebra (Def. 5.1).

BGG category $\mathscr{O}$ of $\mathfrak{sl}(2,\mathbb{C})$ modules (Sec. 5.2.1).

Homogeneous evolution equation (Def. 5.3).

Master symmetry (Def. 5.4).

Construction of master symmetries (Thm. 5.1).

$\mathscr{O}$-scheme (Dia. 5.1).

Construction of time-dependent symmetries (Thm. 5.2).

Results: Master symmetries for five new two-component $(1+1)$-dimensional systems (Examples 5.4.2 - 5.4.6).

Master symmetry for a two-component $(2+1)$-dimensional Davey-Stewartson type system (Example 5.5.1).

Chapter 2

Symbolic Solution of Boundary Problems

Boundary problems for linear ordinary differential equations (LODEs) or partial differential equations (LPDEs) are among the most important model types in the engineering sciences. However, their systematic treatment in symbolic computation is rather recent; carried out by Rosenkranz and Regensbuger [39, 40]. They present a new approach for expressing and solving a linear boundary problem in the language of differential algebras. Taking an algebra $\mathscr{F}$ together with a derivation and integration operator, they construct an algebra of linear integro-differential operators. In such a setting, one can discuss regular boundary problems along with their solution operator in an algebraic manner.

An abstract boundary problem can be defined as a pair $(T, \mathscr{B})$ consisting of a surjective linear map T (differential operator) and an orthogonally closed subspace (boundary conditions) of the dual space of $\mathscr{F}$. The Green's operator G is the right inverse of T mapping forcing function f to solution u of a given bvp, that is, $Gf = u$. It is called the Green's operator since it is the integral operator induced by the Green's function. In the case of LODEs, where the "industrial standard" for solving boundary problems is via the Green's function. The algorithm of [39, 40] computes the solution of a boundary problem in the form of its Green's operator. Nonetheless, most engineers prefers the language of the Green's function.

In the classical case of *well-posed two-point boundary value problems*, it is known how to transform the Green's operator into the corresponding Green's function (Chapter 1, [48]). In this chapter, we extend this transformation to the whole class of *Stieltjes boundary problems*. In view of the specification of the boundary conditions they have more freedom in the following sense: (1) they allow more than two evaluation points. (2) they allow derivatives of arbitrary order; (3) global terms in the form of definite integrals are also allowed. The *structure theorem* depicting this result is published in [41] and is explained in the Section 2.4.

A detailed outline of this chapter is as follows. The goal here is to extract the Green's function from the Green's operator of a boundary value problem. We introduce the notions of a differential algebra $(\mathscr{F}, \partial)$ (Def. 2.1), an integro-differential algebra $(\mathscr{F}, \int, \partial)$ (Def. 2.2) and a Rota-Baxter (RB) algebra $(\mathscr{F}, \int)$ (Def. 2.6), followed by that of a differential RB algebra, which is a differential algebra together with an RB algebra $(\mathscr{F}, \partial, \int)$ (Def. 2.7). Next, we define integro-differential operators $\mathscr{F}_\Phi[\partial, \int]$ (Def. 2.8) along with the equitable integro-differential operators $\mathscr{F}[\partial, \int_\Phi]$ (Def. 2.9) which contains the Green's operator $G = (1-P)T^\Diamond$ (Def. 2.15). In Section 2.4, we present our main result, the *structure theorem* (Thm. 2.3) which depicts an algorithm for extracting the Green's function from its corresponding Green's operator. At the end, we demonstrate this algorithm with some new examples of boundary problems.

2.1 Algebraic setting

In what follows we take $\mathscr{F}$ to be an algebra over a fixed commutative ring K.
We begin this section by defining a differential algebra $(\mathscr{F}, \partial)$ which is well explained in the literature [38]. It is well known that they provide rich algebraic properties capturing essential properties of a derivation which helps in understanding differential equations. But this is not the case for integrals. Therefore our main focus here is to lay foundation for the treatment of integrals. We will see that one can define a similar object $(\mathscr{F}, \int)$ which we call a Rota-Baxter algebra.

Since a boundary problem involves both the operations, a derivation and an integral—we need to understand how one can combine these two structures systematically. Here, we will describe two different algebraic ways to construct such objects; an integro-differential algebra and a differential Rota-Baxter algebra denoted by $(\mathscr{F}, \int, \partial)$ and $(\mathscr{F}, \partial, \int)$ respectively.

Definition 2.1. A differential ring $(\mathscr{F}, \partial)$ is a ring $\mathscr{F}$ together with a derivation $\partial : \mathscr{F} \to \mathscr{F}$ meaning, ∂ is a linear map and satisfies the Leibniz rule, $\partial(fg) = \partial(f)g + f\partial(g)$. One calls $(\mathscr{F}, \partial)$ a differential algebra if $\mathscr{F}$ is an algebra over K and ∂ is a K-derivation.

In the following definition we incorporate the integral operation in a differential algebra.

Definition 2.2. Let $\mathscr{F}$ be a commutative K-algebra with K-linear operations ∂ and $\int$. Then the triple $(\mathscr{F}, \int, \partial)$ is an *integro-differential algebra* over K if the operations satisfy the axioms below:

- Section axiom

$$(\textstyle\int f)' = f, \tag{2.1}$$

- Leibniz axiom

$$(fg)' = f'g + fg', \tag{2.2}$$

- Strong Rota-Baxter axiom

$$f \textstyle\int g = \int fg + \int (f' \int g), \tag{2.3}$$

with the usual notation $\partial = '$.

An integro-differential algebra is called *ordinary* if $\dim_K \operatorname{Ker}(\partial) = 1$.

We refer to ∂ and $\int$ respectively as the derivation and integral of $\mathscr{F}$. Axiom (2.1) is called section axiom since $\partial \circ \int = 1$,i.e, $\int$ is a *right inverse* of ∂. The product rule of differentiation which is commonly called the Leibniz axiom is Axiom (2.2) and Axiom (2.3) captures integration by parts which was introduced by Rosenkranz in [39]. The usual formulation $\int fG' = fG - \int f'G$ is only satisfied "up to a constant", or if one restricts G to $\operatorname{Im}(\int)$. On substituting $G = \int g$ one can obtain the strong Rota-Baxter axiom. The notion of *strong* will become clearer later. For the future we denote $\int f \int g := \int (f \int g)$ implying that multiplication has precedence over integration.

Definition 2.3. We call a right inverse operator $\int$ of ∂ a *section* of ∂ and the pair $(\int, \partial)$ is called a section pair.

Example 2.1.1. Let $\mathscr{F}$ be the algebra of *holomorphic functions* on a simply connected domain S, where S is a subset of the complex plane $\mathbb{C}$. For a fixed point $z_0 \in S$, define $\int f = \int_{z_0}^{z} f(\zeta)\, d\zeta$ in the sense of a complex integral along any path within S that connects z_0 and z together with the derivation $\partial f = \frac{df}{dz}$. Then $(\mathscr{F}, \int, \partial)$ is an integro-differential algebra.

For any section pair between modules, we can construct two complementary associated projectors [2, p.209]. As in our setting, the operations ∂ and $\int$ always form a section pair, so we shall see what are the complementary projectors in this case.

Definition 2.4. Let $(\mathscr{F}, \partial)$ be a differential algebra and $\int$ a section of ∂. Then

$$\mathtt{j} = \textstyle\int \circ \partial \qquad \text{and} \qquad \mathtt{E} = 1 - \mathtt{j} \tag{2.4}$$

are respectively called the *initialization* and the *evaluation* of $\mathscr{F}$.

Notice that initialisation is not an identity map since for $\partial = \frac{d}{dx}$ and $\int = \int_a^x$, we obtain $\mathtt{j}f(x) = \int_a^x f'(x) = f(x) - f(a)$.

It is easy to check that they are indeed projectors since $\mathtt{j} \circ \mathtt{j} = \int \circ (\partial \circ \int) \circ \partial = \mathtt{j}$ by (2.1), which implies $\mathtt{E} \circ \mathtt{E} = 1 - \mathtt{j} - \mathtt{j} + \mathtt{j} \circ \mathtt{j} = \mathtt{E}$. It is well known that every projector induces a unique direct decomposition of the module into two submodules characterised by its kernel and image.

Definition 2.5. Let $(\mathscr{F}, \int, \partial)$ be an integro-differential algebra. Then the modules

$$\mathscr{C} = \mathrm{Ker}(\partial) = \mathrm{Ker}(\mathtt{j}) = \mathrm{Im}(\mathtt{E}) \quad \text{and} \quad \mathscr{I} = \mathrm{Im}(\textstyle\int) = \mathrm{Im}(\mathtt{j}) = \mathrm{Ker}(\mathtt{E})$$

are respectively called the *constant functions* and the *initialized functions* with the direct decomposition

$$\mathscr{F} = \mathscr{C} \oplus \mathscr{I}.$$

If we consider smooth functions $\mathscr{F} = C^\infty[a,b]$ with $\partial = \frac{d}{dx}$ and $\int = \int_a^x$, then $(\mathscr{F}, \int, \partial)$ is a *standard* model of an integro-differential algebra. In this case, $\mathscr{C}$ consists of constant functions $f(x) = c$ with $c \in \mathbb{C}$, while $\mathscr{I}$ consists of those $f \in C^\infty[a,b]$ that satisfy the homogeneous initial condition $f(a) = 0$. Now the terminology for the projectors also makes sense, since $\mathtt{E}f = f(a)$ evaluates f at the initialization point a, and $\mathtt{j}f = f - f(a)$ enforces the initial condition.

Lemma 2.1. *For an integro-differential algebra $(\mathscr{F}, \int, \partial)$ the following results always hold:*

1. *Module $\mathscr{C}$ is a subalgebra of $\mathscr{F}$ and module $\mathscr{I}$ is an ideal in $\mathscr{F}$.*

2. *The projector $\mathtt{E}$ is multiplicative.*

Proof. Since the submodule $\mathscr{C}$ is the image of the algebra endomorphism, that is $\mathrm{Im}(\mathtt{E}) = \mathscr{C}$, therefore it is straightforward that $\mathscr{C}$ is a subalgebra, whereas the claim that $\mathscr{I}$ is an ideal follows directly from the strong Rota-Baxter axiom (2.3).

For the Item 2, we write functions $f, g \in \mathscr{F}$ as $f = f_\mathscr{C} + f_\mathscr{I}$ and $g = g_\mathscr{C} + g_\mathscr{I}$. Then

$$\begin{aligned}
\mathtt{E}(fg) &= \mathtt{E}((f_\mathscr{C} + f_\mathscr{I})(g_\mathscr{C} + g_\mathscr{I})) \\
&= \mathtt{E}(f_\mathscr{C} g_\mathscr{C} + f_\mathscr{C} g_\mathscr{I} + f_\mathscr{I} g_\mathscr{C} + f_\mathscr{I} g_\mathscr{I}) \\
&= \mathtt{E}(f_\mathscr{C} g_\mathscr{C}),
\end{aligned}$$

because the last three summand must vanish since $\mathscr{I}$ is an ideal. Moreover, $\mathscr{C}$ is a subalgebra and $\mathtt{E}$ is a projector with image $\mathscr{C}$. Hence, $\mathtt{E}(f_\mathscr{C} g_\mathscr{C}) = \mathtt{E}(f_\mathscr{C})\mathtt{E}(g_\mathscr{C})$. $\square$

Corollary 2.1. Let $(\mathscr{F}, \partial)$ be a differential algebra with a section $\int$ of ∂. Then $(\mathscr{F}, \int, \partial)$ is an integro-differential algebra iff its evaluation $\mathtt{E}$ is multiplicative iff $\mathscr{I} = \mathrm{Im}(\int)$ is an ideal.

In general the ideal $\mathscr{I}$ corresponding to an integral is *not a differential ideal* of $\mathscr{F}$. It is easy to see in the standard example $(\mathscr{F}, \int_0^x, \frac{d}{dx})$, where $\mathscr{I}$ consists of all $f \in C^\infty[0,1]$ with $f(0) = 0$. Here $x \in \mathscr{I}$ but $x' = 1 \notin \mathscr{I}$ implying $\mathscr{I}$ is not a differential ideal.

Corollary 2.2. An integro-differential algebra is never a field.

Proof. It is immediate from Corollary 2.1. Since then the only possibilities for $\mathscr{I}$ would be 0 and $\mathscr{F}$. In the first case when $\mathscr{I} = 0$ meaning $\mathrm{Im}(\int) = 0$ we have $\mathrm{Ker}(\partial) = \mathscr{F}$, which contradicts the surjectivity of ∂. Whereas in the second case of $\mathscr{I} = \mathscr{F}$ implies $\mathrm{Ker}(\partial) = 0$, which is not possible because $\partial 1 = 0$. □

In some sense, this observation ensures that all integro-differential algebras are *fairly complicated*. The next result points in the same direction.

Proposition 2.1. *The iterated integrals* $1, \int 1, \int\int 1, \ldots$ *are all linearly independent. Hence every integro-differential algebra is infinite-dimensional.*

Proof. Denote $1, \int 1, \int\int 1, \ldots$ by $u_0, u_1, u_2 \cdots$,i.e, u_n is the sequence of iterated integrals. Use induction on n. The base case when $n = 0$ is trivial. For the induction step from n to $n+1$, assume $c_0u_0 + \cdots + c_{n+1}u_{n+1} = 0$. Applying ∂^{n+1} yields $c_{n+1} = 0$. But by the induction hypothesis, we already have $c_0 = \cdots = c_n = 0$ therefore $u_0, \ldots, u_{n+1}$ are linearly independent. □

In view of Corollary 2.1, if we extract the differential part from an integro-differential algebra $(\mathscr{F}, \partial, \int)$, then we can retain the structure of *differential algebra* $(\mathscr{F}, \partial)$, meaning a K-algebra $\mathscr{F}$ with a K-linear operation ∂ that satisfies the Leibniz axiom (2.2). But in general one cannot expand a given differential algebra to an integro-differential algebra. For example, the differential algebra $(K[x^2], x\partial)$ cannot be a integro-differential algebra since the derivation $x\partial$ is not surjective ($1 \notin \mathrm{Im}(x\partial)$) and thus there is no section satisfying Axiom (2.1).

However, it is not straightforward to extract the *integro* part. To isolate the integro part from an integro-differential algebra we need to introduce the *Rota-Baxter* algebra which captures the property of integrals.

Definition 2.6. Let $\mathscr{F}$ be a K-algebra and $\int$ a K-linear operation satisfying the *weak* Rota-Baxter axiom

$$(\textstyle\int f)(\int g) = \int f \int g + \int g \int f. \tag{2.5}$$

Then $(\mathscr{F}, \int)$ is called a *Rota-Baxter algebra*.

The terminology for *strong/weak* originates from the fact that (2.5) is a consequence of the strong Rota-Baxter axiom if one replaces f by $\int f$ in (2.3).

It seems natural to think that an integro-differential algebra $(\mathscr{F}, \int, \partial)$ is a differential algebra $(\mathscr{F}, \partial)$ combined with a Rota-Baxter algebra $(\mathscr{F}, \int)$ such that the section axiom (2.1) is satisfied, but this is not the case. In fact, we define this new structure below where $(\mathscr{F}, \partial)$ and $(\mathscr{F}, \int)$ are coupled only by the section axiom $\partial \circ \int = \mathrm{id}_{\mathscr{F}}$.

Definition 2.7. The triple $(\mathscr{F}, \partial, \int)$ is called a *differential Rota-Baxter algebra* if $(\mathscr{F}, \partial)$ is a differential algebra together with the Rota-Baxter algebra $(\mathscr{F}, \int)$ such that $\partial \circ \int = \mathrm{id}_{\mathscr{F}}$.

Because of the Equation (2.3), the coupling is a little stronger in an integro-differential algebra $(\mathscr{F}, \int, \partial)$. Therefore, there are Rota-Baxter algebra which are not integro-differential algebra. The first such example was found by G. Regensburger in [40, Ex. 3] which is presented here as Example 2.1.2.

Recall that both ∂ and $\int$ were introduced as K-linear operations on $\mathscr{F}$. Using the Leibniz axiom (2.2), one sees immediately that ∂ is $\mathscr{C}$-linear. It is natural to expect the same from $\int$, but this is exactly the difference between $(\mathscr{F}, \partial, \int)$ and $(\mathscr{F}, \int, \partial)$. Moreover, the $\mathscr{I}$ is not an ideal in $(\mathscr{F}, \partial, \int)$ as a direct consequence of the Lemma 2.1.

Proposition 2.2. *Let $(\mathscr{F}, \partial)$ be a differential algebra and $(\mathscr{F}, \int)$ a Rota-Baxter algebra then $(\mathscr{F}, \int, \partial)$ is an integro-differential algebra iff $(\mathscr{F}, \partial, \int)$ is a differential Rota-Baxter algebra and $\int$ is $\mathscr{C}$-linear*

Proof. $\Longrightarrow$ We only need to verify the second part of the statement—$\int$ is $\mathscr{C}$-linear. Since it is already given that $(\mathscr{F}, \partial)$ is a differential algebra and $(\mathscr{F}, \int)$ is a Rota-Baxter algebra, therefore their coupling $(\mathscr{F}, \partial, \int)$ is a differential Rota-Baxter algebra by definition. For $c \in \mathscr{C}$ and any function $f \in \mathscr{F}$, strong Rota-Baxter axiom (2.3) implies

$$\int cf = c \int f - \int c' \int f = c \int f.$$

$\Longleftarrow$ Conversely, it suffices to prove the strong Rota-Baxter axiom for $f, g \in \mathscr{F}$. Since $\mathscr{F} = \mathscr{C} \oplus \mathscr{I}$, we may first consider the case $f \in \mathscr{C}$ and then the case $f \in \mathscr{I}$. But the first case follows from $\mathscr{C}$-linearity; the second case means $f = \int \tilde{f}$ for $\tilde{f} \in \mathscr{F}$. Then the weak Rota-Baxter axiom (2.5) becomes the strong Rota-Baxter (2.3) for $\tilde{f}$ and g. $\square$

Now for the promised *counterexample* to the claim that there are differential Rota-Baxter algebra which are not integro-differential algebra.

Example 2.1.2. Consider the ring $R = K[y]/y^4$ with K being a field of characteristic zero. Set $\mathscr{F} = R[x]$ and define the usual derivation $\partial = \frac{\partial}{\partial x}$, then $(\mathscr{F}, \partial)$ is a differential algebra. Define a K-linear map $\int$ on $\mathscr{F}$ by

$$\textstyle\int f = \int^* f + f(0,0)\, y^2. \tag{2.6}$$

with $\int^*$ being the usual integration meaning $x^k \mapsto x^{k+1}/(k+1)$, Since the second term vanishes under ∂, we see immediately that $\int$ is a section (right inverse) of ∂. For verifying weak Baxter axiom (2.5), we compute

$$\begin{aligned}
&(\textstyle\int f)(\int g) = (\int^* f)(\int^* g) + y^2 \int^* \big(g(0,0)\, f + f(0,0)\, g\big) + f(0,0)\, g(0,0)\, y^4,\\
&\textstyle\int f \int g = \int f \big(\int^* g + g(0,0)\, y^2\big) = \int^* f \int^* g + g(0,0)\, y^2 \int^* f.
\end{aligned}$$

Since $y^4 \equiv 0$, and the ordinary integral $\int^*$ already fulfills the weak Baxter axiom, this implies immediately that the $\int$ does also. However, it does not fulfill the strong Rota-Baxter Baxter axiom (2.3) because it is not $\mathscr{C}$-linear: Observe that $\mathscr{C}$ is here $\operatorname{Ker}(\partial) = R$, so in particular we should have $\int (y \cdot 1) = y \int 1$. But one checks immediately that the left-hand side yields xy, while the right-hand side yields $xy + y^3$.

2.2 Integro-differential operators

Our goal of this section is to build an algebra of operators which encodes boundary conditions of a boundary problem and provides a way to represent its Green's operator. The algebra of *integro-differential operators* offers a unified language which provides a solid algebraic structure for constructing the Green's operator but before this we recall the familiar algebra of *differential operators*.

For a given differential algebra $(\mathscr{F}, \partial)$ over a ground field K, a differential operator T of degree $n \in \mathbb{N}$ can be written as

$$T = \sum_{i=0}^{n} c_i \partial^i,$$

with coefficients $c_0, \ldots, c_n \in \mathscr{F}$. The collection of these operators becomes a K-algebra $\mathscr{F}[\partial]$ by defining addition and scalar multiplication in the obvious way along with the commutator relation $\partial c = c\partial + \partial(c)$. For the obvious reason we call this relation the Leibniz rule.

The algebra $\mathscr{F}[\partial]$ contains all arithmetic terms in ∂ like $\partial^2 c_0(\partial + 2\partial c_1)$, and together with the Leibniz rule it provides *canonical forms* for them. Furthermore, the Leibniz rule extracts all the essential algebraic properties that we know from analysis.

We want to do the same for integro-differential algebras $(\mathscr{F}, \int, \partial)$, so our notion of *integro-differential operator* should capture arithmetic terms in ∂ and $\int$. In the standard model where $\mathscr{F} = C^\infty[a,b]$, one may encounter boundary conditions like $u'(0) - 3u(1) = 0$ or even $\int_0^{1/2} u(\xi)\, d\xi = 0$ as a composition of $\int = \int_0^x$ and evaluation at $1/2$. To formulate such boundary conditions, our natural choice is to consider all characters on $\mathscr{F}$ since they are algebraic counterparts of point evaluation. By a *character* of $\mathscr{F}$ we mean an algebra homomorphism $\mathscr{F} \to K$. Now we construct this algebra of integro-differential operators which we will denote by $\mathscr{F}_\Phi[\partial, \int]$.

As for the differential operators in $\mathscr{F}[\partial]$ we had only one relation, namely the Leibniz rule, but of course in the case of $\mathscr{F}_\Phi[\partial, \int]$ we have more relations. There are four types of basic operators: derivation ∂, integral $\int$, multiplication operators f, and characters φ. We need appropriate relations so that we can write operators in a canonical form which captures essential algebraic properties from analysis.

Let us describe how one can derive these rules for few of the entries in the table. For example, the relation

$$\int f \int \to (\int f) \int + \int (\int f)$$

comes from the weak Rota-Baxter axiom if one substitutes $g = 1$ in (2.5). Similarly, the reduction $\int f \varphi \to (\int f) \varphi$ stems from the fact that φ maps any function to a constant. Since $\int$ is linear over constants so φ can be pulled out from the integral. The relation $\varphi\psi \to \psi$ is the easiest one to understand. We find that $\varphi\psi f = \varphi(\psi(f)) = \psi(f)$, since $\psi(f)$ is a constant and a character maps a constant to itself.

There are nine such relations which are required for the construction of integro-differential operators (for detail please see [40, §3]). We list them in the table below along with the formal definition of $\mathscr{F}_\Phi[\partial, \int]$.

For K-algebra $\mathscr{F}$, we fix a set $\mathscr{F}^\#$ of $\mathscr{F}$ such that $\{1\} \cup \mathscr{F}^\#$ is a K-basis of $\mathscr{F}$, that is a basis of $\mathscr{F}$ over the field K. Denote the space of all non-zero characters on $\mathscr{F}$ by $\mathscr{F}^\bullet$. We shall make a distinction here, the projector $\mathtt{E} = 1 - \int \circ \partial$ will be a called a distinguished character for a section pair ∂ and $\int$.

Definition 2.8. Let$(\mathscr{F}, \int, \partial)$ be an ordinary integro-differential algebra over a field K and $\Phi \subseteq \mathscr{F}^\bullet$. The *integro-differential operators* $\mathscr{F}_\Phi[\partial, \int]$ are defined as the K-algebra generated by the symbols ∂ and $\int$, the "functions" $f \in \mathscr{F}^\#$ and the "characters" $\varphi \in \Phi \cup \{\mathtt{E}\}$, modulo the rewrite rules given in Table 2.1.

$fg \quad \to \quad fg$	$\partial f \quad \to \quad \partial(f) + f\partial$	$\int f \int \quad \to \quad (\int f) \int - \int (\int f)$
$\varphi\psi \quad \to \quad \psi$	$\partial \varphi \quad \to \quad 0$	$\int f \partial \quad \to \quad f - \int (\partial(f)) - \mathrm{E}(f)\mathrm{E}$
$\varphi f \quad \to \quad \varphi(f)\varphi$	$\partial \int \quad \to \quad 1$	$\int f \varphi \quad \to \quad (\int f)\varphi$

Table 2.1 Rewrite Rules for Integro-Differential Operators

Remark Notice that, in the above table we distinguish ∂f from $\partial(f)$. The expression $\partial(f)$ denotes derivation of f, whereas ∂f represents action on the operator level which simplifies to $\partial(f) + f\partial$. The first entry in the table simply means that the product of two functions remains the same, that is, $fg \to fg$. We follow this notation through out this chapter.

The algebra $\mathscr{F}_\Phi[\partial, \int]$ has a standard decomposition

$$\mathscr{F}_\Phi[\partial, \textstyle\int] = \mathscr{F}[\partial] \oplus \mathscr{F}[\textstyle\int] \oplus (\Phi). \tag{2.7}$$

The above ring structure seems promising to give the symbolic representation for the Green's operators. However, to understand the relationship between the Green's operators and Green's functions, we view the latter as a certain canonical form. Therefore, we equip the ring of integro-differential operators with a slightly different set of reduction rules leading to the ring of *equitable integro-differential operators* $\mathscr{F}[\partial, \int_\Phi]$. The notation is used to emphasize that the integral operators are parametrized by Φ in contrast to $\mathscr{F}_\Phi[\partial, \int]$ where its dependence is based on the chosen set of characters Φ.

The *fundamental theorem of calculus* provides a passage to move from the integro-differential operator ring $\mathscr{F}_\Phi[\partial, \int]$ to its equitable clone $\mathscr{F}[\partial, \int_\Phi]$. The theorem states that for any function $f \in C^\infty(\mathbb{R})$ and initialization point $\varphi \in \mathbb{R}$,

$$\int_\varphi^x f'(\xi)\, d\xi = f(x) - f(\varphi).$$

Therefore, for an integro-differential algebra $(\mathscr{F}, \int, \partial)$ we can rewrite this in the operator form

$$\textstyle\int_\varphi \partial = 1 - \varphi.$$

which gives

$$\textstyle\int_\varphi := (1 - \varphi)\int.$$

We shall understand the integral $\int_\varphi = \int_\varphi^x$ as it provides a way to distinguish in the case of a bivariate function where one can also introduce an operator like $\int_{\varphi}^{y}$. We shall see this in the next chapter where we introduce *bivariate Green's function* (3.4). With this understanding it is easy to see that for any other character ψ, one can obtain the relation

$$\psi\int_{\varphi} = \int_{\psi} - \int_{\varphi} = \int_{\varphi}^{\psi}. \tag{2.8}$$

It is immediate that $(\mathscr{F}, \partial, \int_{\varphi})$ is also an ordinary integro-differential algebra. To obtain the algebra of equitable integro-differential operators $\mathscr{F}[\partial, \int_{\Phi}]$ we can adjoin *all* $\int_{\varphi}$ to $\mathscr{F}[\partial]$. As in the case of $\mathscr{F}_{\Phi}[\partial, \int]$, here we also need to introduce essential relations which we list in the table below. For the precise formulation of $\mathscr{F}[\partial, \int_{\Phi}]$ as a quotient ring please see [37, §5.1].

Definition 2.9. Let $(\mathscr{F}, \int, \partial)$ be an ordinary integro-differential algebra. The *equitable integro-differential operators* $\mathscr{F}[\partial, \int_{\Phi}]$ are defined as the K-algebra generated by ∂ and $\int_{\varphi}^{x}$ and f with φ ranging over Φ and f over $\mathscr{F}^{\#}$, modulo the rewrite rule given in Table 2.2.

$fg \quad \to \quad f{\cdot}g$	$\partial f \quad \to \quad \partial(f) + f\partial$	$\partial \int_{\varphi}^{x} \quad \to \quad 1$
$\int_{\varphi}^{x} f \int_{\psi}^{x} \quad \to \quad (\int_{\varphi}^{x} f)\int_{\psi}^{x} - \int_{\varphi}^{x}(\int_{\varphi}^{x} f)$		
$\int_{\varphi}^{x} f\partial \quad \to \quad f - \int_{\varphi}^{x}(\partial(f)) - \varphi(f) + \varphi(f)\int_{\varphi}^{x}\partial$		

Table 2.2 Equitable Integro-Differential Relations

Since the algorithm in Section 2.3 computes the Green's operator in the algebra $\mathscr{F}_{\Phi}[\partial, \int]$, we introduce the so-called *translation isomorphism* to rewrite the operator in $\mathscr{F}[\partial, \int_{\Phi}]$, which makes it easy to extract the corresponding Green's function.

Definition 2.10. The *translation isomorphism* $\iota \colon \mathscr{F}_{\Phi}[\partial, \int] \to \mathscr{F}[\partial, \int_{\Phi}]$ fixes $f \in \mathscr{F}$ and ∂ while using the above fundamental relation in the form

$$\iota(\varphi) = 1 - \int_{\varphi}\partial \quad \text{and} \quad \iota^{-1}(\int_{\varphi}) = (1-\varphi)\int. \tag{2.9}$$

Note that this includes also the character $\mathtt{E} := 1 - \int\partial$ associated to the distinguished integral $\int = \int_{\mathtt{E}}$ underlying $\mathscr{F}_{\Phi}[\partial, \int]$.

Now we are ready to describe an algorithm to compute the Green's operator, but before that let us define a boundary problem formally.

Definition 2.11. Let $(\mathscr{F}, \int, \partial)$ be an ordinary integro-differential algebra. A n-th order *boundary problem* is given by a pair $(T, \mathscr{B})$, where T is a monic differential operator $T \in \mathscr{F}[\partial]$ of order n and the boundary space $\mathscr{B} \in \mathscr{F}^\bullet$ is the linear span $\mathscr{B} = [\beta_1, \ldots, \beta_n]$ of n linearly independent boundary conditions. We represent such a boundary problem by

$$\boxed{\begin{aligned} &Tu = f, \\ &\beta_1 u = \cdots = \beta_n u = 0. \end{aligned}} \tag{2.10}$$

We say that u is a solution of $(T, \mathscr{B})$ for a given $f \in \mathscr{F}$ if

$$Tu = f \quad \text{and} \quad u \in \mathscr{B}^\perp, \tag{2.11}$$

where $\mathscr{B}^\perp$ is an *orthogonal* complement of $\mathscr{B}$ containing all functions $u \in \mathscr{F}$ which satisfy the given boundary conditions, i.e, $\mathscr{B}^\perp = \{u \in \mathscr{F} \mid \beta(u) = 0 \text{ for all } \beta \in \mathscr{B}\}$. A boundary problem $(T, \mathscr{B})$ is *regular* if

$$\operatorname{Ker}(T) \oplus \mathscr{B}^\perp = \mathscr{F}. \tag{2.12}$$

Regularity guarantees that the boundary problem (2.10) has a unique solution $u \in \mathscr{F}$ for all given $f \in \mathscr{F}$. To check if the boundary problem is regular we make use of the following *evaluation matrix*

$$\beta(u) = \begin{pmatrix} \beta_1(u_1) & \cdots & \beta_1(u_n) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \cdots & \beta_n(u_n) \end{pmatrix} \in K^{n \times n} \tag{2.13}$$

where $u_1, \ldots, u_n \in \mathscr{F}$ forms the *fundamental system* for T, that is, it is a basis for $\operatorname{Ker}(T)$. If the evaluation matrix is regular (non singular) then so is (2.10) and vice-versa.

Construct another space of boundary conditions $\tilde{\mathscr{B}}$ as a linear span $\tilde{\mathscr{B}} = [\tilde{\beta}_1, \ldots, \tilde{\beta}_n]$ where

$$(\tilde{\beta}_i, \ldots, \tilde{\beta}_n)^t = \beta(u)^{-1}(\beta_1, \ldots, \beta_n)^t \tag{2.14}$$

and t stands for transpose of a matrix. It is always possible to find such boundary conditions for a regular boundary problem because $\beta(u)$ is invertible. The advantage of this is $\operatorname{Ker}(T) = [u_1, \cdots, u_n]$ and $\tilde{\mathscr{B}} = [\tilde{\beta}_1, \ldots, \tilde{\beta}_n]$ form a *biorthogonal* system meaning

$$\tilde{\beta}_i(u_i) = \delta_{ij}. \tag{2.15}$$

We will find its use later in the Theorem 2.1.

In the introduction we have already mentioned the nature of *Stieltjes boundary problems*. To define these problems formally first we need to give an algebraic meaning to the corresponding boundary conditions.

Definition 2.12. The *right* ideal $|\Phi) = \Phi \cdot \mathscr{F}_\Phi[\partial, \textstyle\int]$ is defined as the ideal of *Stieltjes conditions* and a boundary problem (2.10) is called a *Stieltjes boundary problem* if the boundary space $\mathscr{B}$ is the linear span of n linearly independent Stieltjes conditions.

From the viewpoint of applications, Stieltes conditions $\beta \in (\Phi)$ are easier to comprehend in terms of their $\mathscr{F}_\Phi[\partial, \textstyle\int]$-normal form: They can be described uniquely as sums

$$\beta = \sum_{\varphi\in\Phi}\sum_{i\geq 0} a_{\varphi,i}\varphi\partial^i + \sum_{\varphi\in\Phi}\varphi \textstyle\int f_\varphi \tag{2.16}$$

with only finitely many $a_{\varphi,i} \in K$ and $f_\varphi \in \mathscr{F}$ nonzero. The double sum involving derivation in (2.16) is called the *local part* of β and the subsequent sum its *global/nonlocal part*. In the important $C^\infty(\mathbb{R})$ case with distinguished integral $\textstyle\int = \int_0^x$, this yields

$$\beta(u) = \sum_{\varphi,i} a_{\varphi,i}u^{(i)}(\varphi) + \sum_{\varphi} \textstyle\int_0^\varphi f_\varphi(\xi)\, u(\xi)\, d\xi,$$

for certain $a_{\varphi,i} \in \mathbb{R}$ and $f_\varphi \in C^\infty(\mathbb{R})$.

Definition 2.13. A Stieltjes boundary problem $(T, \mathscr{B})$ of order n with $\mathscr{B} = [\beta_1, \ldots, \beta_n]$ is called *well-posed* if it is regular and the β_i can be chosen with all derivatives having order below n; otherwise it is called *ill-posed*.

Well-posed: In the standard model $(C^\infty(\mathbb{R}), \int_0^x, \frac{d}{dx} = D)$, consider a boundary problem

$$\boxed{\begin{array}{l} u'' = f(x), \\ u'(0) - u(1) = 0, u'(1) = 0, \end{array}}$$

which can be denoted by $(D^2, [\mathrm{E}_0 D - \mathrm{E}_1, \mathrm{E}_1 D])$. The notation is clear here, E_a is the evaluation at point a. The fundamental system of this problem is $\{1, x\}$ leading to the evaluation matrix

$$\beta(u) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which is invertible. Hence, it is a well-posed two-point boundary value problem.

Ill-posed: Our next example is totally unclassical (despite being a regular boundary problem)—ill-posed—it involves more than two evaluation points with a third order boundary condition for a second order differential operator, and have non-local conditions. Later in the applications section we compute its Green's operator along with its Green's function,

$$\boxed{\begin{array}{l} u'' - u = f, \\ u'''(-1) - \int_0^1 u(\xi)\,\xi\,d\xi = 0, \\ u'(-1) - u''(1) + \int_{-1}^1 u(\xi)\,d\xi = 0, \end{array}} \tag{2.17}$$

2.3 Green's operator

Definition 2.14. Let $(\mathscr{F}, \int, \partial)$ be an ordinary integro-differential algebra. Then the operator G is called the *Green's operator* of a regular boundary problem $(T, \mathscr{B})$ if $TG = 1$ and $\mathrm{Im}(G) = \mathscr{B}^{\perp}$.

To construct this operator we borrow a result from abstract algebra. For proof, see [37, Prop. 4.1]

Proposition 2.3. *Let T be a surjective linear map between modules M and N*

$$T : M \to N$$

so that $M = \mathrm{Ker}(T) + \mathscr{I}$ with $\mathscr{I}$ being a complement of $\mathrm{Ker}(T)$ in M. Then there exists a unique section G of T with $\mathrm{Im}(G) = \mathscr{I}$. Moreover, G is the unique solution of the equation

$$GT = 1 - P \tag{2.18}$$

for projector P with $\mathrm{Im}(P) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(P) = \mathscr{I}$.

Notice that if $\tilde{G}$ is any section meaning $T\tilde{G} = 1$ then $GT\tilde{G} = G = (1-P)\tilde{G}$. We formulate this as a corollary.

Corollary 2.3. Given any section $\tilde{G}$ of T, the section G corresponding to a complement $\mathscr{I}$ of $\mathrm{Ker}(T)$ is given by

$$G = (1-P)\tilde{G},$$

where P is the projector with $\mathrm{Im}(P) = \mathrm{Ker}(T)$ and $\mathrm{Ker}(P) = \mathscr{I}$.

In view of the above corollary—for a regular boundary problem $(T, \mathscr{B})$ if we can find a section $\tilde{G}$ and projector P such that

$$T\tilde{G} = 1 \quad \text{with} \quad \mathrm{Im}(P) = \mathrm{Ker}(T) = \mathscr{C}, \mathrm{Ker}(P) = \mathscr{B}^{\perp} = \mathscr{I},$$

then we can easily construct its Green's operator.

Theorem 2.1. *Let* $(T,\mathscr{B})$ *be a regular boundary problem with* $u_1,\dots,u_n$ *and* $\beta_1,\dots,\beta_n$ *be respectively a basis for* $\mathrm{Ker}(T)$ *and* $\mathscr{B}$. *Then* $P:\mathscr{F}\to\mathscr{F}$ *is the* projector *defined by*

$$Pu=\sum_{i=1}^{n}\tilde{\beta}_i(u)u_i$$

where $(\tilde{\beta}_i,\dots,\tilde{\beta}_n)^t=\beta(u)^{-1}(\beta_1,\dots,\beta_n)^t$ *with* $\mathrm{Im}(P)=\mathrm{Ker}(T)$ *and* $\mathrm{Ker}(P)=\mathscr{B}^{\perp}$.

Proof. Since the boundary conditions are linear functionals, we can write

$$P(Pu)=\sum_{i=1}^{n}\tilde{\beta}_i(u)P(u_i).$$

By biorthogonality of $\{u_i\}$ and $\{\tilde{\beta}_j\}$ (2.15), we get $P(u_i)=u_i$ and thus $P^2=P$. From the construction of the projector P, it is clear that $\mathrm{Im}(P)=\mathrm{Ker}(T)$ and $\mathrm{Ker}(P)=\mathscr{B}^{\perp}$. $\square$

Example 2.3.1. Lets compute this projector for a well-posed two point boundary value problem which was mentioned before

$$\boxed{\begin{array}{l} u''=f(x),\\ u'(0)-u(1)=0,u'(1)=0.\end{array}}$$

Here boundary conditions are $\beta_1=\mathtt{E}_0D-\mathtt{E}_1,\beta_2=\mathtt{E}_1D$ with $D=\frac{d}{dx}$, and the fundamental system is $\{x,1\}$ meaning $\mathrm{Ker}(T)$ is the linear span $\mathrm{Ker}(T)=[u_1,u_2]=[1,x]$. To find the required boundary conditions $\tilde{\beta}_i$ as in the theorem, we first find its evaluation matrix

$$\beta(u)=\begin{pmatrix}\beta_1(u_1) & \beta_1(u_2)\\ \beta_1(u_2) & \beta_2(u_2)\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

which yields

$$\begin{pmatrix}\tilde{\beta}_1\\ \tilde{\beta}_2\end{pmatrix}=\beta(u)^{-1}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}=\begin{pmatrix}\beta_2\\ -\beta_1\end{pmatrix}.$$

From Theorem 2.1, we know that the projector is

$$\begin{aligned}P&=x\tilde{\beta}_1+\tilde{\beta}_2\\ &=x\mathtt{E}_1D-\mathtt{E}_0D+\mathtt{E}_1.\end{aligned}$$

Now we have the projector which satisfies the necessary conditions of the Corollary 2.3. Similarly, can we find a section $\tilde{G}$? As it turns out we can always construct a section for a differential operator T. The following theorem describes such a construction which is well known as “variation of constants". For the proof, see Theorem 3.30 in [37].

Theorem 2.2. *Let $(\mathcal{F}, \int, \partial)$ be an ordinary integro-differential algebra and let be $T \in \mathcal{F}[\partial]$ monic with regular fundamental system $u_1, \dots, u_n$. Then its fundamental right inverse is given by*

$$T^{\Diamond} = \sum_{i=1}^{n} u_i \textstyle\int d^{-1} d_i \in \mathcal{F}[\partial, \textstyle\int]$$

where d denotes the determinant of the Wronskian matrix W associated with $u_1, \dots, u_n$

$$W(u) = \begin{pmatrix} u_1 & \cdots & u \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)} & \cdots & u_n^{(n-1)} \end{pmatrix}$$

and d_i the determinant of the matrix W_i obtained from W by replacing the i-th column by the n-th unit vector.

Furthermore, for an initial value problem

$$\boxed{\begin{aligned} &Tu = f, \\ &u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0, \end{aligned}}$$

the operator $T^{\Diamond}$ is in fact its Green’s operator.

Bringing all pieces together we can define the Green’s operator for a regular boundary problem in a constructive way.

Definition 2.15. The *Green’s operator* $G \in \mathcal{F}[\partial, \int]$ for any regular boundary problem $(T, \mathcal{B})$ is given by

$$G = (1 - P)T^{\Diamond}, \tag{2.19}$$

where the operators P and $T^{\Diamond}$ are as in the Theorem 2.1 and 2.2 respectively.

Example 2.3.2. Let us compute the Green’s operator for the same problem 2.3.1

$$\boxed{\begin{aligned} &u'' = f(x), \\ &u'(0) - u(1) = 0, u'(1) = 0. \end{aligned}}$$

We have already computed the projector for this problem. To find d_i as in the Theorem 2.2, write the Wronskian matrix for the corresponding fundamental system $\{x, 1\}$ which is

$$W(u) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}.$$

This provides $d = -1, d_1 = -1$ and $d_2 = x$ with the fundamental right inverse

$$T^{\diamondsuit} = \sum_{i=1}^{2} u_i \int d^{-1} d_i.$$

After substitution we find that $T^{\diamondsuit} = x.A - A.x$ where A is the usual integral operator $\int_0^x$. Using the result from the previous example we find that the Green's operator is

$$\begin{aligned} G =&(1-P)T^{\diamondsuit} \\ =&xA - Ax + (-1-x)\mathrm{E}_1 A + \mathrm{E}_1 Ax \end{aligned}$$

2.4 Extraction of Green's function

We now turn to the central task of this chapter, the extraction of the Green's function $g(x,\xi)$ corresponding to the Green's operator $G_{x\xi}$. Notice that here we used the notation $G_{x\xi}$ for the Green's operator in the equitable ring $\mathscr{F}[\partial, \int_\Phi]$ as compared to the Green's operator G in the ring $\mathscr{F}_\Phi[\partial, \int]$.

In the case of ill-posed boundary problems where we also allow a nonlocal part in boundary conditions, the corresponding Green's function in general contain Dirac distributions and their derivatives [48, §2]. Since all boundary problems considered in this chapter have only finitely many evaluation points $\alpha \in \Phi \subset \mathbb{R}$, one may choose an interval $J \subset \mathbb{R}$ containing all the α and denote continuous functions on a rectangular region $J^2 \in \mathbb{R}^2$ by $C(J^2)$. Hence the $C(J^2)$-module $\mathscr{G} \subset \mathscr{D}'(J^2)$ generated by the *Dirac distributions* δ_α and their derivatives will be sufficient to capture all Green's functions. We will follow the common practice of denoting this "function" as

$$\delta_\alpha = \delta(\xi - \alpha) \ \text{ if } \ x \neq \alpha, \tag{2.20}$$

with the defining property

$$\int_J \delta(\xi - \alpha) f(\xi)\, d\xi = f(\alpha). \tag{2.21}$$

The transformation from the Green's operators to Green's functions

$$\mathscr{F}[\partial, {\textstyle\int_\Phi}] \to \mathscr{G},\; G \mapsto G_{x\xi}$$

is clearly an $\mathbb{R}$-linear map, hence it will be sufficient to define it on the *canonical* $\mathbb{R}$*-basis* of $\mathscr{F}_\Phi[\partial, \textstyle\int] \cong \mathscr{F}[\partial, \textstyle\int_\Phi]$. Following the strategy of the example in the Introduction, the easiest part is $\mathscr{F}[\textstyle\int] \subseteq \mathscr{F}_\Phi[\partial, \textstyle\int]$, which is handled by setting

$$(f \textstyle\int g)_{x\xi} = f(x)\, g(\xi)\, [0 \le \xi \le x] - f(x)\, g(\xi)\, [x \le \xi \le 0],$$

where we use the Iverson bracket notation $[P]$ signifying 1 if the property P is true and zero otherwise. Note that at most one of the two summands above is nonzero for fixed (x, ξ). Since $(\Phi) \subset \mathscr{F}_\Phi[\partial, \textstyle\int]$ is a left $\mathscr{F}$-module over $|\Phi)$, we settle this part via

$$\begin{aligned} (f\,\mathrm{E}_\alpha \partial^i)_{x\xi} &= (-1)^i f(x)\, \delta^{(i)}(\xi - \alpha), \\ (f\,\mathrm{E}_\alpha \textstyle\int g)_{x\xi} &= \operatorname{sgn}(\alpha)\, f(x)\, g(\xi)\, [0 \le \xi \le \alpha]. \end{aligned}$$

Finally, on $\mathscr{F}[\partial]$ we define

$$(f \partial^i)_{x\xi} = f(x)\, \delta^{(i)}(x - \xi),$$

and the definition is complete.

In order to accommodate these Dirac distributions we introduce two lemmas below which will help us to prove the main Theorem 2.3.

Lemma 2.2. *Let $T \in \mathscr{F}[\partial]$ be any monic differential operator of order n, and choose a fundamental system $u_1, \ldots, u_n$ for T with Wronskian matrix W. Then we have*

$$\begin{aligned} \partial^k T^\diamond &= \sum_{j=1}^{n} u_j^{(k)} \textstyle\int \dfrac{d_j}{d} + \displaystyle\sum_{j=1}^{k} \partial^{k-j} \rho_j \quad \textit{with} \\ \rho_k &:= \frac{1}{d} \sum_{j=1}^{n} u_j^{(k-1)} d_j \in \mathscr{F}. \end{aligned} \tag{2.22}$$

Here $d = \det(W)$ and $d_j = \det(W_j)$, where W_i denotes the matrix resulting from W when replacing the j-th column by the n-th unit vector of K^n.

Proof. Before we begin the proof of this theorem, notice that $\rho_1 = \cdots = \rho_{n-1} = 0$ and $\rho_n = 1$, by the definition of d_j and d respectively. So equivalently, we can say that the second sum in (2.22) survives only for $k > n$.

Let us prove our claim using induction on k. The base case $k = 0$ follows from the Theorem 2.2. By the induction hypothesis we obtain

$$\partial^k T^\diamond = \sum_{j=1}^{n} u_j^{(k)} \smallint \frac{d_j}{d} + \sum_{j=1}^{k} \partial^{k-j} \rho_j,$$

which in turn after taking derivation gives

$$\partial^{k+1} T^\diamond = \sum_{j=1}^{n} u_j^{(k+1)} \smallint \frac{d_j}{d} + \frac{1}{d} \sum_{j=1}^{n} u_j^{(k)} d_j + \sum_{j=1}^{k} \partial^{k-j+1} \rho_j.$$

This is just (2.22) for $k+1$ since the middle sum is ρ_{k+1} and can be absorbed into the third. □

Lemma 2.3. *The Green's operator of any regular Stieltjes boundary problem is contained in $\mathscr{F}[\smallint_\Phi] + \mathscr{L}$, where $\mathscr{L}$ denotes the left $\mathscr{F}$-module generated by the local Stieltjes conditions. Therefore, the Green's operator can be written in the form $G = \tilde{G} + \hat{G}$ with $\tilde{G} \in \mathscr{F}[\smallint_\Phi]$ and $\hat{G} \in \mathscr{L}$.*

Proof. For a regular Stieltjes boundary problem $(T, \mathscr{B})$ of order n, we showed in Section 2.3 that its Green's operator $G = (1-P)T^\diamond$ with operators

$$T^\diamond = \sum_{i=1}^{n} u_i \smallint d^{-1} d_i \quad \text{and} \quad P = \sum_{i=1}^{n} u_i \tilde{\beta}_i,$$

where

$$\tilde{\beta}(u) = \sum_{\varphi,i} a_{\varphi,i} u^{(i)}(\varphi) + \sum_{\varphi} \smallint_0^\varphi f_\varphi(\xi)\, u(\xi)\, d\xi.$$

Since $T^\diamond$ clearly resides in $\mathscr{F}[\smallint_\Phi]$ so it suffices to show $PT^\diamond \in \mathscr{F}[\smallint_\Phi] + \mathscr{L}$. Notice that every summand of P is either of the form $fE_\alpha \partial^k$ or $fE_\alpha \smallint$ where the latter is an expression in $\mathscr{F}[\smallint_\Phi]$. It remains to prove $fE_\alpha \partial^k T^\diamond \in \mathscr{F}[\smallint_\Phi] + \mathscr{L}$. From (2.22) we see that

$$fE_\alpha \partial^k T^\diamond = \sum_{j=1}^{n} f u_j^{(k)}(\alpha) \smallint^\alpha \frac{d_j}{d} + \sum_{j=1}^{k} fE_\alpha \partial^{k-j} \rho_j.$$

The first sum is clearly contained in $\mathscr{F}[\smallint_\Phi]$, while the second is in $\mathscr{L}$ because $\partial^{k-j}\rho_j \in \mathscr{F}[\partial]$ and thus the result follows. □

We are now ready to state the main *structure theorem for Green's functions* of regular Stieltjes boundary problems.

Theorem 2.3. *The Green's function of any regular Stieltjes boundary problem with m evaluations $\alpha_1, \dots, \alpha_m$ has the form $g(x,\xi) = \tilde{g}(x,\xi) + \hat{g}(x,\xi)$, where the functional part $\tilde{g} \in C(J^2)$ is defined by the $2(m-1)$ case branches*

$$\xi \in [\alpha_i, \alpha_{i+1}] \ (0 < i < m), x \leq \xi;$$
$$\xi \in [\alpha_i, \alpha_{i+1}] \ (0 < i < m), \xi \leq x,$$

while the distributional part $\hat{g}(x,\xi)$ is an $\mathscr{F}$-linear combination of the $\delta(\xi - \alpha_i)$ and their derivatives.

Proof. Recalling the notation for the Green's operator in $\mathscr{F}[\partial, \textstyle\int_\Phi]$ and using Lemma 2.3 we can write $G_{x\xi} = \tilde{G}_{x\xi} + \hat{G}_{x\xi}$ with $\tilde{G}_{x\xi} \in \mathscr{F}[\textstyle\int_\Phi]$ and $\hat{G}_{x\xi} \in \mathscr{L}$. To extract the corresponding parts of the Green's function we first begin with the $\hat{g}(x,\xi)$. We may write

$$\tilde{G} = \sum_{i=1}^{r} f_i \textstyle\int_{\alpha_i} g_i,$$

where $\alpha_i = \alpha_j$ is possible for $i \neq j$. Using the transformation $\mathscr{F}[\partial, \textstyle\int_\Phi] \to \mathscr{G}$, we obtain $\tilde{g}(x,\xi)$ as

$$\begin{aligned}
&\sum_{i=1}^{r} \Big(f_i(x)\, g_i(\xi)\, [\alpha_i \leq \xi][\xi \leq x] - f_i(x)\, g_i(\xi)\, [\xi \leq \alpha_i][x \leq \xi] \Big) \\
&\quad = \sum_{i=1}^{r} \Big(\sum_{\alpha_j \leq \alpha_i} f_j(x)\, g_j(\xi) \Big) [\alpha_{j-1} \leq \xi \leq \alpha_j][\xi \leq x] \\
&\qquad - \sum_{i=1}^{r} \Big(\sum_{\alpha_j \geq \alpha_i} f_j(x)\, g_j(\xi) \Big) [\alpha_j \leq \xi \leq \alpha_{j+1}][x \leq \xi],
\end{aligned}$$

where the two inner sums are restricted by $j > 0$ and $j < n$. Collecting terms, this is a sum of $2(m-1)$ characteristic functions over disjoint domains in $\mathbb{R}^2$, hence one may also write $\tilde{g}(x,\xi)$ in terms of a corresponding case distinction with $2(m-1)$ branches.

The distributional part $\hat{g}(x,\xi)$ is even easier. Writing $\hat{G}$ as an $\mathscr{F}$-linear combination of local conditions we obtain $\hat{g}(x,\xi)$ via

$$\hat{G}_{x\xi} = \Big(\sum_{\alpha,i} f_{i,\alpha} \alpha \partial^i \Big)_{x\xi} = \sum_{\alpha,i} (-1)^i f_{i,\alpha}(x)\, \delta^{(i)}(\xi - \alpha),$$

which is clearly of the stated form. $\square$

Remark 2.1. In the case of the standard model the distinguished character $\mathtt{E} = 1 - \textstyle\int \circ \partial$ is simply an evaluation at 0. If it is not used in the boundary conditions, a straightforward translation of the Green's operator G may introduce two spurious extra case branches in the Green's function $G_{x\xi}$ since $\int_0^x$ occurs in the formula for G. For avoiding this, one has to use a different version of $T^\diamondsuit$ that replaces $\mathtt{E}_0$ by any one of the characters $\mathtt{E}_\alpha$ used in the boundary conditions.

2.4.1 Examples for Stieltjes boundary problems

From the view point of applications here we only consider the standard model

$$\left(C^\infty(\mathbb{R}), \textstyle\int_0^x, \frac{d}{dx} = D\right).$$

For easy comprehension we recall all necessary identities and terminologies which we will use extensively in the rest of this section.

Let $(T, \mathscr{B})$ be a regular boundary problem of order n with the regular fundamental system $u_1, \ldots, u_n$ and $\mathscr{B}$ as the linear span of n linearly independent boundary conditions $\beta_1, \ldots, \beta_n$. Then the boundary conditions $\tilde{\beta}_i$ can be found by (2.14)

$$(\tilde{\beta}_1, \ldots, \tilde{\beta}_n)^t = \beta(u)^{-1}(\beta_1, \ldots, \beta_n)^t,$$

such that $\{u_i\}$ and $\{\tilde{\beta}_j\}$ form a biorthogonal system. In order to compute the corresponding Green's operator and Green's function for $(T, \mathscr{B})$, we summarise the whole procedure in five steps:

1. Construct the fundamental right inverse $T^\diamondsuit = \sum_{i=1}^n u_i \int d^{-1} d_i \in \mathscr{F}[\partial, \int]$ as in the Theorem 2.2.

2. Determine the projector $P = \sum_{i=1}^n u_i \tilde{\beta}_i \in \mathscr{F}[\partial, \int]$ by Theorem 2.1.

3. Compute the Green's operator $G = (1 - P)T^\diamondsuit$.

4. Find $G_{x\xi}$ as an equitable integro-differential operator equivalent of G using the translation isomorphism (Definition 2.10).

$$\mathtt{E}_\alpha \textstyle\int \to \int_0 - \int_\alpha. \tag{2.23}$$

5. Finally, extract the Green's function from respective functional and distributive part of $G_{x\xi}$ by the following maps (Theorem 2.3).

$$\tilde{g} \to f \textstyle\int_\alpha g = f(x)\, g(\xi)\, [\alpha \leq \xi][\xi \leq x] \tag{2.24}$$
$$\hat{g} \to f\, \mathrm{E}_\alpha \partial^i = (-1)^i f(x)\, \delta^{(i)}(\xi - \alpha). \tag{2.25}$$

Having this algorithm to hand we are ready to move towards examples.

Example 2.4.1. Consider, the boundary problem $(T, \mathscr{B})$ with $T = D^3 - 6D^2 - 6$ and $\mathscr{B} = [\beta_1, \beta_2, \beta_3]$ for $\beta_1 = \mathrm{E}_0, \beta_2 = \mathrm{E}_0 D + \mathrm{E}_1 D^2$ and $\beta_3 = \mathrm{E}_1 D^2$:

$$\boxed{\begin{array}{l} u''' - 6u'' + 11u' - 6u = f(x), \\ u(0) = 0, u'(0) + u''(1) = 0, u''(1) = 0. \end{array}}$$

Observe that for the corresponding homogeneous differential equation the characteristic polynomial has no repeated roots, giving $\{e^x, e^{2x}, e^{3x}\}$ as a fundamental system for the operator T. Following the algorithmic steps, first we compute the fundamental right inverse

$$T^\Diamond = \sum_{i=1}^{3} u_i \int d^{-1} d_i,$$

where d_i are as in Theorem 2.2. The Wronskian matrix for the fundamental system $\{e^x, e^{2x}, e^{3x}\}$ is

$$W = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}$$

with determinants $d = 2e^{6x}, d_1 = e^{5x}, d_2 = -2e^{4x}$ and $d_3 = e^{3x}$. Consequently, we obtain the fundamental right inverse as

$$T^\Diamond = \frac{1}{2} e^x A e^{-x} - e^{2x} A e^{-2x} + \frac{1}{2} e^{3x} A e^{-3x},$$

where A is the usual integral operator $\int_0^x$.

For the second part of the algorithm above, first we compute the evaluation matrix

$$\beta(u) = \begin{pmatrix} 1 & 1 & 1 \\ 3 + 9e^3 & 1 + e & 2 + 4e^2 \\ 9e^3 & e & 4e^2 \end{pmatrix}$$

which has inverse

$$\beta(u)^{-1} = \frac{-1}{\rho}\begin{pmatrix} 6(2e-3e^2) & 9e^2-4e & 4e-e^{-1}-9e^2 \\ 3(3e^2-1) & 1-9e^2 & 2e^{-1}+9e^2-1 \\ -2(2e-1) & 4e-1 & 1-e^{-1}-4e \end{pmatrix}$$

with $\rho = 9e^2-8e+1$. From Theorem 2.1, we know that $P = u_1\tilde{\beta}_1 + u_2\tilde{\beta}_2 + u_3\tilde{\beta}_3$. Using the relation $(\tilde{\beta}_1,\dots,\tilde{\beta}_n)^t = \beta(u)^{-1}(\beta_1,\dots,\beta_n)$ we find that the projector

$$P = \frac{1}{\rho}\left(p_1\mathrm{E}_0 + p_2\mathrm{E}_0 D + p_3\mathrm{E}_1 D^2\right)$$

where

$$\begin{aligned} p_1 &= -9e^{2x+2} + 4e^{3x+1} + 18e^{x+2} - 12e^{x+1} + 3e^{2x} - 2e^{3x}, \\ p_2 &= -9e^{2x+2} + 4e^{3x+1} + 9e^{x+2} - 4e^{x+1} + e^{2x} - e^{3x}, \\ p_3 &= -2e^{2x-1} + e^{x-1} + e^{3x-1}. \end{aligned}$$

To compute the *Green's operator* we use the identity $G = (1-P)T^{\Diamond}$, which in our case after reduction becomes

$$\begin{aligned} G = \frac{1}{2}e^{3x}Ae^{-3x} + \frac{1}{2}e^{x}Ae^{-x} - e^{2x}Ae^{-2x} \\ + g_1\mathrm{E}_1 Ae^{-3x} + g_2\mathrm{E}_1 Ae^{-x} + g_3\mathrm{E}_1 Ae^{-2x}, \end{aligned}$$

$$\begin{aligned} \text{where} \quad & g_1(x) = \tfrac{-9}{2\rho}e^{x+2}(e^{2x}-2e^x+1), \quad g_2(x) = \tfrac{-1}{2\rho}e^{x}(e^{2x}-2e^x+1) \\ \text{and} \quad & g_3(x) = \tfrac{4}{\rho}e^{x+1}(e^{2x}-2e^x+1). \end{aligned}$$

To rewrite G as an *equitable* integro-differential operator, we replace each evaluation $\mathtt{E}$ by $1 - A_{\mathtt{E}}D$ which comes from the step 4. In our case, we just need to replace each $\mathtt{E}_1$ by $1 - A_1 D$. After this translation we find that

$$\begin{aligned} G_{x\xi} = \frac{1}{2}e^{3x}Ae^{-3x} + \frac{1}{2}e^{x}Ae^{-x} - e^{2x}A_0e^{-2x} + g_1Ae^{-3x} + g_2Ae^{-x} \\ + g_3Ae^{-2x} - g_1A_1e^{-3x} - g_2A_1e^{x} - g_3A_1e^{-2x}. \end{aligned}$$

Finally, to extract the Green's function we use the step 5 which yields

$$g(x,\xi) = \begin{cases} \frac{1}{2}e^{3x-3\xi} + \frac{1}{2}e^{x-\xi} - e^{2x-2\xi} + g_1 e^{-3\xi} + g_2 e^{-\xi} + g_3 e^{-2\xi} & 0 \leq \xi \leq x \leq 1, \\ g_1 e^{-3\xi} - g_2 e^{\xi} - g_3 e^{-2\xi} & 0 \leq x \leq \xi \leq 1. \end{cases}$$

Notice that since the problem is well-posed therefore the distributional part $\hat{g}$ of the Green's function is zero.

Example 2.4.2. Consider another well-posed two point boundary problem $(T, \mathscr{B})$ with $T = D^3 - 12D^2 + 45D - 50$ and $\mathscr{B} = [\beta_1, \beta_2, \beta_3]$ for $\beta_1 = \mathrm{E}_0, \beta_2 = \mathrm{E}_0 D + \mathrm{E}_1 D^2$ and $\beta_3 = \mathrm{E}_0 D^2$:

$$\boxed{\begin{array}{l} u''' - 12u'' + 45u' - 50u = f(x), \\ u(0) = 0, u'(0) - u''(1) = 0, u''(0) = 0. \end{array}}$$

The fundamental system for this problem is $\{e^{2x}, e^{5x}, e^{5x}x\}$ and we find that the corresponding fundamental right inverse

$$T^{\Diamond} = \frac{1}{9}e^{2x}Ae^{2x} - \frac{1}{3}e^{5x}Ae^{-5x}x + \frac{1}{9}e^{5x}(3x-1)Ae^{-5x}.$$

Following the same procedure as in Example 1, we compute the evaluation matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 2-4e^2 & 5-25e^5 & 1-35e^5 \\ 4 & 25 & 10 \end{pmatrix}$$

and its inverse

$$\frac{1}{\rho}\begin{pmatrix} 25(1+25e^5) & -10 & 1-35e^5 \\ -4(4-10e^2+35e^5) & 10 & -1+35e^5 \\ 10(3-10e^2+10e^5) & -21 & 3-25e^5+4e^2 \end{pmatrix}$$

with $\rho = 9 + 485e^5 + 40e^2$. Using the relation $P = \sum_{i=1}^{3} u_i \tilde{\beta}_i$, we find the projector

$$P = \frac{1}{\rho}\left(p_1 \mathrm{E}_0 + p_2 \mathrm{E}_0 D + p_3 \mathrm{E}_0 D^2 + p_4 \mathrm{E}_1 D^2\right),$$

where

$$\begin{aligned}
p_1 =& 100e^{5x+5}x - 100e^{5x+2}x + 30e^{5x}x - 140e^{5x+5} \\
& + 625e^{2x+5} + 40e^{5x+2} + 25e^{2x} - 16e^{5x}, \\
p_2 =& -21e^{5x}x + 10e^{2x} - 10e^{5x}, \\
p_3 =& -25e^{5x+5}x - 4e^{5x+2}x - 3e^{5x}x - 35e^{5x+5} \\
& + 35e^{2x+5} - e^{2x} + e^{5x}, \\
p_4 =& 21e^{5x}x + 10e^{2x} - 10e^{5x}.
\end{aligned}$$

Using the identity $G = (1-P)T^{\diamondsuit}$, we find that the Green's operator

$$G = \frac{1}{9}e^{2x}Ae^{-2x} - \frac{1}{3}e^{5x}Ae^{-5x}x + \frac{1}{9}e^{5x}(3x-1)Ae^{-5x} + g_1e^2RAe^{-2x} \\ + g_2e^5RAe^{-5x}x + g_3e^5RAe^{-5x}$$

with

$$\begin{aligned}
\text{with} \quad & g_1 = \tfrac{-4}{9\rho}(21e^{5x}x - 10e^{5x} + 10e^{2x}), g_2 = \tfrac{25}{3\rho}(21e^{5x}x - 10e^{5x} + 10e^{2x}), \\
\text{and} \quad & g_3 = \tfrac{-80}{9\rho}(21e^{5x}x - 10e^{5x} + 10e^{2x}).
\end{aligned}$$

In this case, we see that the Green's operator G in equitable integro-differential operator is

$$G_{x\xi} = \frac{1}{9}e^{2x}A_0e^{-2x} - \frac{1}{3}e^{5x}A_0e^{-5x} + \frac{1}{9}e^{5x}(3x-1)A_0e^{-5x} + g_1e^2A_0e^{-2x} + g_2e^5A_0e^{-5x}x + \\ g_3e^5A_0e^{-5x} - g_1e^2A_1e^{-2x} - g_2e^5A_1e^{-5x}x - g_3e^5A_1e^{-5x}.$$

with the Green's function

$$g = \begin{cases} \tfrac{1}{9}e^{2x-2\xi} - \tfrac{1}{3}e^{5x-5\xi} + \tfrac{1}{9}e^{5x-5\xi}(3x-1) + g_1e^{2-2\xi} & \\ \qquad + g_2e^{5-5\xi}\xi + g_3e^{5-5\xi} & ; 0 \le \xi \le x \le 1, \\ g_1e^{2-2\xi} + g_2e^{5-5\xi}\xi + g_3e^{5-5\xi} & ; 0 \le x \le \xi \le 1. \end{cases}$$

Above examples are relatively easy where all the computations can be carried out by hand. But for all the following examples we use the `Maple` package `IntDiffOp` [24] for computing the Green's operator. On top of this, we write an extra code to extract the corresponding Green's function which is described in the Appendix A.

Example 2.4.3. Unlike our examples so far, this example is special in the sense that it is a four point boundary value problem:

$$\boxed{\begin{array}{l} -u'' = f, \\ u(0) + u\left(\frac{1}{3}\right) = u(1) + u\left(\frac{2}{3}\right) = 0. \end{array}}$$

This is a boundary problem $(T, \mathscr{B})$ with $T = -D^2$ and $\mathscr{B} = [\beta_1, \beta_2]$ for $\beta_1 = \mathtt{E}_0 + \mathtt{E}_1/3$ and $\beta_2 = \mathtt{E}_2/3 + \mathtt{E}_1$. Since the fundamental right inverse does not depends up on boundary conditions, we obtain $T^\lozenge = xA_0 - A_0x$ as in the Example 2.3.2. In this case, the evaluation matrix is

$$\beta(u) = \begin{pmatrix} \frac{1}{3} & 2 \\ \frac{5}{3} & 2 \end{pmatrix}.$$

We find that the projector

$$P = \left(\frac{5}{8} - \frac{3}{4x}\right)\mathtt{E}_0 + \left(\frac{5}{8} - \frac{3}{4x}\right)\mathtt{E}_{\left(\frac{1}{3}\right)} + \left(\frac{-1}{8} + \frac{3}{4x}\right)\mathtt{E}_1 + \left(\frac{-1}{8} + \frac{3}{4x}\right)\mathtt{E}_{\left(\frac{2}{3}\right)},$$

and the Green's operator

$$\begin{aligned} G = x{\textstyle\int} - {\textstyle\int} x + \left(\frac{-5}{24} + \frac{x}{4}\right)\mathtt{E}_{\left(\frac{1}{3}\right)}{\textstyle\int} + \left(\frac{5}{8} - \frac{3x}{4}\right)\mathtt{E}_{\left(\frac{1}{3}\right)}{\textstyle\int} x \\ + \left(\frac{1}{8} - \frac{3x}{4}\right)\mathtt{E}_1{\textstyle\int} x + \left(\frac{1}{12} - \frac{x}{2}\right)\mathtt{E}_{\left(\frac{2}{3}\right)}{\textstyle\int} + \left(\frac{-1}{8} + \frac{3x}{4}\right)\mathtt{E}_{\left(\frac{2}{3}\right)}{\textstyle\int} x. \end{aligned}$$

Here, it is not straightforward to extract the Green's function as in Example 2.4.1. Our `Maple` code which is based on the Step 5 of the algorithm gives $g(x,\xi) = \tilde{g}(x,\xi)$ with 6 cases

Case	Term
$0 \le \xi \le 1/3, \xi \le x$	$(-1/8 - x/4) + (5/8 - 3x/4)\xi$
$0 \le \xi \le 1/3, x \le \xi$	$-1/8 - 5x/4 + 13\xi/8 - 3x\xi/4$
$1/3 \le \xi \le 2/3, \xi \le x$	$1/12 - x/2$
$1/3 \le \xi \le 2/3, x \le \xi$	$-3x/2 + 1/12 + \xi$
$2/3 \le \xi \le 1, \xi \le x$	$(1/8 - 3x/4)\xi$
$2/3 \le \xi \le 1, x \le \xi$	$-x - (3x/4 - 9/8)\xi$

Example 2.4.4. As promised before, we look at the ill-posed boundary problem

$$\boxed{\begin{aligned} &u'' - u = f,\\ &u'''(-1) - \textstyle\int_0^1 u(\xi)\,\xi\,d\xi = 0,\\ &u'(-1) - u''(1) + \textstyle\int_{-1}^1 u(\xi)\,d\xi = 0, \end{aligned}} \tag{2.26}$$

for functions $u, f \in C^\infty[-1,1]$. Here, the Green's operator is

$$\begin{aligned} \sigma G = \sigma/2\,(e^x{\textstyle\int} e^{-x} - e^{-x}{\textstyle\int} e^x) &\\ + 2(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})(\mathtt{E}_{-1}\partial + \mathtt{E}_1{\textstyle\int} x) &\\ + (e-1)(-e^{x+2} - 2e^{x+1} + e^{-x+1})(\mathtt{E}_{-1}{\textstyle\int} + \mathtt{E}_1{\textstyle\int}) &\\ + (3e^{x+2} - e^{x+1} - 3e^{-x+1} + 3e^{-x})\mathtt{E}_1{\textstyle\int} e^x &\\ + (2e^{x+2} - 3e^{x+1})(e^{-1}\mathtt{E}_{-1}{\textstyle\int} e^{-x} + e\mathtt{E}_{-1}{\textstyle\int} e^x) &\\ + (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1})\mathtt{E}_1 & \end{aligned}$$

using the abbreviation $\sigma := 2(2e-3)(e-1)$ while collecting and factoring some terms for enhanced readability.

Case	Term
$\left.\begin{array}{l} -1 \le \xi \le 0 \\ \xi \le x \end{array}\right\}$	$3e^{x+2+\xi} + 3e^{x-\xi} - 2e^{x+1-\xi} - 2e^{3+x+\xi}$ $+e^{3+x} + e^{-x+1} + e^{x+2} - e^{-x+2} - 2e^{x+1}$
$\left.\begin{array}{l} -1 \le \xi \le 0 \\ x \le \xi \end{array}\right\}$	$-2e^{x+1} + 2e^{-x+2+\xi} - 5e^{-x+1+\xi} - 2e^{x+2-\xi}$ $-2e^{3+x+\xi} + 3e^{-x+\xi} + e^{-x+1} + e^{x+2}$ $+e^{3+x} + 3e^{x+1-\xi} + 3e^{x+2+\xi} - e^{-x+2}$
$\left.\begin{array}{l} 0 \le \xi \le 1 \\ \xi \le x \end{array}\right\}$	$-2e^{3+x}\xi - 2e^{-x+1}\xi + 2e^{x+2}\xi + 2e^{-x+2}\xi$ $-2e^{x+1}\xi + 3e^{x+2+\xi} + 3e^{x-\xi} - 5e^{x+1-\xi}$ $+2e^{-x+1+\xi} - e^{x+1+\xi} - 2e^{-x+2+\xi} + 2e^{x+2-\xi}$ $-e^{3+x} - e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1}$
$\left.\begin{array}{l} 0 \le \xi \le 1 \\ x \le \xi \end{array}\right\}$	$-2e^{3+x}\xi - 2e^{-x+1}\xi + 2e^{x+2}\xi + 2e^{-x+2}\xi$ $-2e^{x+1}\xi + 3e^{-x+\xi} + 3e^{x+2+\xi} - e^{3+x}$ $-e^{-x+1} - e^{x+2} + e^{-x+2} + 2e^{x+1}$ $-3e^{-x+1+\xi} - e^{x+1+\xi}$

After transforming this to equitable form (which is again straightforward), we can then apply Theorem 2.3 to extract the Green's function $g(x,\xi) = \tilde{g}(x,\xi) + \hat{g}(x,\xi)$ with the distributional part

$$\begin{aligned}\sigma\,\hat{g}(x,\xi) = (-e^{x+3} - e^{x+2} + 2e^{x+1} + e^{-x+2} - e^{-x+1})\,\delta(\xi-1)\\ +2\,(-e^{x+3} + e^{x+2} - e^{x+1} + e^{-x+2} - e^{-x+1})\,\delta'(\xi-1)\end{aligned}$$

coming from the $(\dots)\,\mathtt{E}_1$ and $(\dots)\,\mathtt{E}_{-1}\partial$ terms, and with the functional part defined by the case distinction for $\sigma\,\tilde{g}(x,\xi)$ as given in the table above. This example shows also that the *representation of Green's operators* in terms of the Green's functions is not always the most useful and economical way of representing the Green's operator. For many purposes it is better to take the Green's operator just as an element of the operator ring $\mathscr{F}_\Phi[\partial,\textstyle\int]$ or $\mathscr{F}[\partial,\textstyle\int_\Phi]$.

2.5 Conclusions

It was already known, how one can extract the Green's function from the corresponding Green's operator for a well-posed two-point boundary problem. In this chapter, we extended this transformation to a more general class of linear bvps, namely to the Stieltjes boundary problems (Def. 2.12). In particular, we allowed boundary problems with boundary conditions having more than two evaluation points, derivatives of arbitrary order and global terms in the form of definite integrals (See e.g. (1.6)). The extraction algorithm is mentioned at the beginning of Section 2.4.1, followed by some examples demonstrating this procedure. The main result of this chapter is the Structure theorem 2.3.

Our results showed that the resulting Green's function is not only a piecewise function but also a distribution. This observation encouraged us to look for a suitable differential and Rota-Baxter structures handling such Green's functions. In the next chapter, we aim to provide the algebraic underpinning needed for symbolic computation systems handling such objects.

Chapter 3

Algebraic Diracs

In modern analysis, the theory of distributions provides a solid footing in order to understand the theory of linear differential equations. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense and any locally integrable function has a distributional derivative. The most basic application is the use of the fundamental solution (Green's function for bvps) to solve inhomogeneous linear problems.

If convolution $*$ among two functions f, g is defined by

$$f * g = \int_{-\infty}^{\infty} f(s-a)g(s)\,ds,$$

and if δ is the Dirac delta function then, $\delta * f = f$ (sifting property (3.17)) for all functions f continuous at $s = a$. On the other hand, if L is a linear differential operator (ordinary or partial) then $Lu * f = L(u * f)$. This means that if you could find solution to $Lu = \delta$ then on convoluting both sides with f gives $L(u * f) = f$. So $v = u * f$ is solution to $Lv = f$ whenever u is a solution to $Lu = \delta$. This u is called the fundamental solution for the operator L. This example shows the importance of distributions.

It is evident that differential algebra has made outstanding contributions to the theory of differential equations—yet distributions have received little attention from algebraic vantage point, possibly because integration has not found its place in algebraic structures. However, derivations are widespread in differential algebras, differential rings and differential modules. In such structures one cannot say more than that a distribution δ has arbitrary formal derivatives $\delta', \delta'', \cdots$. Here we investigate an algebraic setting including integrals to accommodate distributions.

We developed a method in Theorem 2.3 which provides an algorithmic approach to extract the Green's function for a regular Stieltjes boundary problem from its Green operator.

As seen in the Example 2.4.4 before, the Green's function is naturally a piecewise function or even a proper distribution in the case of ill-posed boundary problems. Here, we provide an algebraic model of the *bivariate Green's function* corresponding to a given boundary problem. In view of this, we develop a new algebraic setting for treating piecewise functions and distributions together with suitable differential and Rota-Baxter structures. Our aim is not only to show that the Green's function of a regular boundary problem (for a linear ordinary differential equation) can be expressed naturally in this new setting, but also to provide the algebraic underpinning needed for symbolic computation systems handling such objects. These results are published in [42].

A detailed outline of this chapter is as follows. The goal is to develop a new algebraic model to accommodate piecewise and distribution nature of the Green's function corresponding to a boundary problem. To achieve this, we define notions of a differential Rota Baxter module (Def. 3.4) and evaluations, followed by that we construct a Piecewise extension $\mathscr{P}\mathscr{F}$ for treating piecewise nature of Greens' function using Heaviside functions (Def. 3.7). To introduce distributions in our setting, we construct a distribution module $\mathscr{D}\mathscr{F}$ (Def. 3.11) for treating Dirac distributions appearing in the Green's functions using the sifting property $f\delta_a = f(a)\,\delta_a$. Finally, we introduce a Bivariate distribution module $\mathscr{D}_2\mathscr{F}$ for treating elements like $\delta(x-\xi)$ and $H(x-\xi)$ (Eq. (3.31)). At the end, we present some applications of our new algebraic setting to express the Green's functions in this algebraic language (Thm. 3.3, Thm. 3.4).

Since our algebraic setting is heavily based upon the differential and Rota-Baxter structures, we begin this chapter with a short review of them.

3.1 Differential Rota-Baxter algebras and modules

Let us begin with reviewing the definition of a Rota-Baxter algebra and a differential Rota-Baxter algebra from the last chapter.

Definition 3.1. Let $\mathscr{F}$ be a K-algebra and $\int$ a K-linear operation satisfying the *weak* Rota-Baxter axiom

$$(\textstyle\int f)(\int g) = \int f \int g + \int g \int f. \tag{3.1}$$

Then $(\mathscr{F}, \int)$ is called a *Rota-Baxter algebra*.

Now we combine this Rota-Baxter algebra with a differential algebra to give rise to a differential Rota-Baxter algebra.

Definition 3.2. The triple $(\mathscr{F},\partial,\int)$ is called a *differential Rota-Baxter algebra* if $(\mathscr{F},\partial)$ is a differential algebra together with the Rota-Baxter algebra $(\mathscr{F},\int)$ such that $\partial\circ\int=\mathrm{id}_{\mathscr{F}}$.

We discussed the difference between a differential Rota-Baxter algebra and an integro-differential algebra [Cor. 2.1, Prop.2.2]. To recall: in an integro-differential algebra $(\mathscr{F},\int,\partial)$, the image of the integral operator $\mathrm{Im}\int\subset\mathscr{F}$ is an ideal rather than a subalgebra. Moreover, the operator $\int$ is linear not only over the ground ring K but also over $\mathrm{Ker}(\partial)$. This means that an integro-differential algebra has more structure compared to a differential Rota-Baxter algebra, implying that all integro-differential algebras are differential Rota-Baxter algebra but not vice-versa, see Example 2.1.2.

If the evaluation $\mathtt{E}:=1_{\mathscr{F}}-\int\partial$ is multiplicative in a differential Rota-Baxter algebra $(\mathscr{F},\partial,\int)$ then it becomes an integro-differential algebra $(\mathscr{F},\int,\partial)$. For more information review Definition 2.7 and the following text. The notation is motivated by the fact that in an **integro**-differential algebra, the integral $\int$ appears first and thus we write $(\mathscr{F},\int,\partial)$, where as for a **differential** Rota-Baxter algebra we write $(\mathscr{F},\partial,\int)$.

Since our focus here is to understand Dirac distributions algebraically, we begin with a key observation from analysis.

Evaluations: The characteristic feature of the Dirac distribution δ_a is that it effects an evaluation at a when it appear under the integral

$$\int_{\infty}^{-\infty} f(s)\delta(s-a)\,ds=f(a).$$

To make this idea precise one must provide an algebraic treatment of evaluations. In the more general setting of a plain Rota-Baxter algebra $(\mathscr{F},\int)$, an evaluation is defined as any character $\mathtt{E}\colon\mathscr{F}\to K$ such that $\mathtt{E}\int=0$. We can think of a Rota-Baxter operator $\int$ as integration $\int_\alpha^x$ with initialization point α, therefore we can introduce as many Rota-Baxter operators as there are choices for an initialization point. We think of evaluation as evaluating at a certain point α, namely the initialization point of the Rota-Baxter operator $\int=\int_\alpha^x$. Therefore, the evaluation for the operator $\int_\alpha^x$ will be $\mathtt{E}_\alpha$ since $\mathtt{E}_\alpha\int_\alpha^x=\int_\alpha^\alpha=0$. In this context we will often use $f(\alpha)$ as a suggestive notation for E_α.

To reiterate, we begin with defining an evaluation $\mathtt{E}$ on a structure with integration $\int$ only.

Definition 3.3. In a Rota-Baxter algebra setting $(\mathscr{F},\int)$, an evaluation is defined as any character $\mathtt{E}\colon\mathscr{F}\to K$ such that $\mathtt{E}\int=0$.

Example 3.1.1. The standard example from analysis is when we take the algebra of smooth functions $\mathscr{F} = C^\infty(\mathbb{R})$ with the derivation $\partial f(x) = df/dx$ and the Rota-Baxter operator $\int f(x) = \int_0^x f(s)\,ds$. Here, the initialized functions $f(x)$ are those with $f(0) = 0$, corresponding to the distinguished evaluation $\mathtt{E}(f) = f(0)$. Any other evaluation $\mathtt{E}_c(f) := f(c)$ may be used for generating additional Rota-Baxter operators $\int_c f = \int_c^x f(x)\,dx$.

All the ring-theoretic structures that we have discussed in the previous chapter—differential algebras $(\mathscr{F}, \partial)$, Rota-Baxter algebras $(\mathscr{F}, \int)$, differential Rota-Baxter algebras $(\mathscr{F}, \partial, \int)$, integro-differential algebras $(\mathscr{F}, \int, \partial)$—have natural module-theoretic analogues.

Definition 3.4. The triple $(M, \eth, \oint)$ is a *differential Rota-Baxter module* over a differential Rota-Baxter algebra $(\mathscr{F}, \partial, \int)$ if the derivation $\eth \colon M \to M$ satisfies

$$\eth f \psi = (\partial f)\, \psi + f \eth \psi, \tag{3.2}$$

and a Rota-Baxter operator $\oint \colon M \to M$ is characterized by the (weak) Rota-Baxter axiom (see (2.5))

$$\int f \cdot \oint \psi = \int f \oint \psi + \oint (\int f) \psi, \tag{3.3}$$

for $f \in \mathscr{F}$ and $\psi \in M$.
We call $(M, \eth, \oint)$ *ordinary* if and only if $\mathrm{Ker}(\eth) = K$.

We will see that the *distribution module* (3.11) is in fact a differential Rota-Baxter module. Similarly one can define module-theoretic analogues for remaining structures. However, the notion of *integro-differential module* (Def. 3.12) is slightly more subtle since we must now distinguish the strong Rota-Baxter axiom (see (2.3)) for coefficients and the one for module elements. We shall postpone this discussion to later when it is needed (Lemma 3.2).

Now when both the operations, derivation ∂ and integral $\int$ are present in an algebraic structure. We see that they induce an evaluation and we call it an *induced evaluation* denoted by $\hat{\mathtt{E}}$ (algebra setting) or $\acute{\mathtt{E}}$ (module setting).

Definition 3.5. Let $(\mathscr{F}, \partial, \int)$ be a differential Rota-Baxter module then

$$\hat{\mathtt{E}} := 1_{\mathscr{F}} - \int \partial \tag{3.4}$$

is called an *induced evaluation* of $\mathscr{F}$.

Remark 3.1. If an (induced) evaluation is not multiplicative then we call it a *pseudo*-(induced) evaluation.

Note, however, that we have already defined induced evaluation in Definition 2.4 of the previous chapter and simply called it an evaluation, but this distinction between an *induced evaluation* and an *evaluation* is needed here.

3.2 Piecewise extension

Our goal in this section is to describe the passage from smooth functions $\mathscr{F} = \mathbb{C}^\infty(\mathbb{R})$ to piecewise smooth functions $\mathscr{P}\mathscr{F}$ in an abstract algebraic manner. One should be careful with the notations here; we will use the notation $\mathscr{P}\mathscr{F}$ to denote the algebraic construction of the piecewise extension of $\mathscr{F}$, whereas *PC* for the standard notion of piecewise functions in real analysis. See Example 3.2.2.

To accommodate piecewise functions we extend our ground algebra by adjoining the characteristic functions for all intervals $[a,b] \subset \mathbb{R}$, and these can in turn be generated by the well-known *Heaviside function* $H(x)$ in the sense that

$$1_{[a,b]}(x) = H(x-a)H(b-x)$$

Before we begin the formal treatment, let us introduce some notations and terminologies.

The Heaviside function requires an ordered field, so we begin by taking the ground algebra $\mathscr{F}$ over a fixed ordered field $(K,<)$[1]. The minimum and maximum of two elements $a,b \in K$ are denoted by $a \sqcap b$ and $a \sqcup b$ respectively, and we agree that the operators $\sqcap, \sqcup$ have precedence over $+,-$. Notice that these binary operations are associative which reminds us of the following structure.

Definition 3.6. A *semigroup* is a set S together with a binary operation $\cdot : S \to S$ that satisfies the associative property

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

for all $a,b,c \in S$.

Therefore, $(K,\sqcup)$ and $(K,\sqcap)$ are semigroups. Moreover, every element a in K can be divided into its positive and negative part given by $a^+ := a \sqcup 0$ and $a^- := a \sqcap 0$, so that $a = a^+ + a^-$. In the following we denote semigroups $(K,\sqcup)$ and $(K,\sqcap)$ by $K_\sqcup$ and $K_\sqcap$ respectively.

[1]Distributions are usually defined as generalizations of functions of a *real* variable, meaning either $\mathbb{R}^n \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{C}$. The case of a complex variable $\mathbb{C}^n \to \mathbb{C}$ is effectively treated as $\mathbb{R}^{2n} \to \mathbb{C}$, ignoring the field structure of $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$. Starting from an ordered field thus seems plausible.

Definition 3.7. We define the *Heaviside function* $H(x)\colon K \to K$ by

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ \eta & \text{if } x = 0, \\ 1 & \text{if } x > 0, \end{cases}$$

along with the *dual Heaviside function* $\bar{H}(x) := 1 - H(x)$. Shifted function $H(x-a)$ will be denoted by H_a and its dual with $\bar{H}_a$.

The choice of $\eta \in K$ is somewhat subtle; in fact we can have three distinguished cases (the terminology is again motivated by the case $K = \mathbb{R}$):

- The *left continuous* convention uses $\eta = 0$.
- Similarly for the *right continuous* choice, we put $\eta = 1$.
- Whereas $\eta = 1/2$ yields a *symmetric* setting which is neither left nor right continuous.

Here we use the left continuous case $\eta = 0$, unless stated otherwise.

To extend the ground algebra to its piecewise extension, we make use of so called *semigroup algebra*.
Let A be a commutative ring. Let G be a semigroup written multiplicatively.
Let $A[G]$ be the set of all maps $\gamma : G \to G$ such that $\gamma(x) = 0$ for almost all $x \in G$, that is $\gamma(x) \neq 0$ for finitely many $x \in G$. Define addition in $A[G]$ to be the ordinary addition of mappings. If $\alpha, \beta \in A[G]$, we define their product $\alpha\beta$ by the rule

$$\alpha\beta(z) = \sum_{xy=z} \alpha(x)\beta(y).$$

This sum is finite since there are finitely many pairs $(x,y) \in G \times G$ such that $\alpha(x)\beta(y) \neq 0$. One calls $A[G]$ the *semigroup algebra* of G over A. For more information please see [26, p. 104].

In the definition below, we define the piecewise extension in terms of the *semigroup algebra* for the semigroup $K_\sqcup$. We write $\mathscr{F}[K_\sqcup]$ for the semigroup algebra of $K_\sqcup$ over $\mathscr{F}$.

Definition 3.8. Let $\mathscr{F}$ be an algebra over an ordered ring $(K,<)$. Then we define its *piecewise extension* as $\mathscr{P}\mathscr{F} := \mathscr{F}[K_\sqcup]$.

We denote the identity element of $\mathscr{P}\mathscr{F}$ by 1 and the other generators by $H_a (a \in K)$. Then, one can view $\mathscr{P}\mathscr{F}$ as the quotient ring of the polynomial ring $\mathscr{F}[H_a]$,

$$\mathscr{P}\mathscr{F} = \mathscr{F}[H_a] \big/ < H_a H_b - H_{a\sqcup b} >, \tag{3.5}$$

where $< H_aH_b - H_{a\sqcup b} >$ denotes the ideal generated by the relations $H_aH_b - H_{a\sqcup b}$ for all $a,b \in K$. In relation to the semigroup $K_\sqcap$, we see that the piecewise extension $\mathscr{P}\mathscr{F} = \mathscr{F}[K_\sqcup]$ is in fact isomorphic to its dual $\mathscr{F}[K_\sqcap]$ under the map $H_a \mapsto 1 - \bar{H}_a$, $\bar{H}_a$ denote the generator of the dual. This can also be seen from the *exchange law* which stems from the linearity of the order on K:

$$H_{a\sqcup b} + H_{a\sqcap b} = H_a + H_b. \tag{3.6}$$

We will restrict ourselves to the setting $\mathscr{P}\mathscr{F} = \mathscr{F}[K_\sqcup]$. In the sequel, we will use the notation $H(x-a) := H_a$ and $H(a-x) := \bar{H}_a$ interchangeably. With this, the relation $H_aH_b - H_{a\sqcup b}$ can be written as

$$H(x-a)H(x-b) = H(x-a\sqcup b).$$

Similarly the other relations becomes

$$H(a-x)H(b-x) = H(x-a\sqcap b), \tag{3.7}$$

$$H(a-x)H(x-b) = 0 \quad \text{if } a < b. \tag{3.8}$$

We can generalise this setting and provide an algebraic characterisation.

Definition 3.9. We call an algebra *order-related* if it encodes the order of the ground ring K within its multiplicative structure, that is, if there exists a semigroup embedding

$$H\colon (K,\sqcup) \hookrightarrow (\mathscr{F},\cdot) \quad \text{so that} \quad H_aH_b = H_{a\sqcup b}\,(a,b\in K).$$

An order-related morphism between order-related rings is an algebra homomorphism $\rho\colon \mathscr{F} \to \tilde{\mathscr{F}}$ such that $\rho(H_a) = \tilde{H}_a\,(a\in K)$. Then the piecewise extension $\mathscr{P}\mathscr{F}$ can be characterized as the *universal order-related extension algebra* of $\mathscr{F}$, meaning every embedding $\mathscr{F} \hookrightarrow A$ into an order-related algebra A factors through the algebra embedding $\mathscr{F} \hookrightarrow \mathscr{P}\mathscr{F}$ via a unique order-related morphism $\mathscr{P}\mathscr{F} \to A$.

To define a Rota-Baxter operator (integration) on $\mathscr{P}\mathscr{F}$, we need an algebraic domain with multiple evaluation points. Intuitively, this is because integrating against a step function based at $a \in K = \mathbb{R}$ amounts to starting off the integral at a, with integration constant induced by evaluation at a, see (3.10). To make it precise we introduce a *shift map* S_c for shifting evaluations with $E_c := \mathtt{E} \circ S_c$ and define a *shifted Rota-Baxter algebra* as below.

Definition 3.10. By a *shift map* on an algebra $\mathscr{F}$ we mean a group homomorphism

$$S\colon (K,+) \to \big(Aut_K(\mathscr{F}),\circ\big) \;\text{ such that }\; S_a f = f(x+a) \;\text{ for } a\in K,\, f\in\mathscr{F}.$$

If $\mathscr{F}$ is equipped with ∂ and $\int$ then we require the following compatibility conditions. We use the usual notation for the commutator of operators $[P,Q] := PQ - QP$.

1. We call $(\mathscr{F}, \int, S)$ a *shifted Rota-Baxter algebra* if S is a shift map on a Rota-Baxter algebra $(\mathscr{F}, \int)$ with evaluation $\mathtt{E}$ such that $[S_c, \int] = \mathtt{E}_c \int$ for all $c \in K$, where $\mathtt{E}_c := \mathtt{E} \circ S_c$ is called the evaluation at c.

2. We call $(\mathscr{F}, \partial, S)$ a *shifted differential algebra* if S is a shift map on a differential algebra $(\mathscr{F}, \partial)$ such that $[S_c, \partial] = 0$ for all $c \in K$.

3. We call $(\mathscr{F}, \partial, \int, S)$ a *shifted differential Rota-Baxter algebra* if $(\mathscr{F}, \partial, \int)$ is a differential Rota-Baxter algebra such that both $(\mathscr{F}, \int, S)$ and $(\mathscr{F}, \partial, S)$ are shifted.

In the sequel, we suppress the shift map S when referring to structures such as $(\mathscr{F}, \partial, \int, S)$.

Example 3.2.1. The two most important *examples* are the Rota-Baxter algebra $\big(C(\mathbb{R}), \int_0^x\big)$ and the integro-differential algebra $\big(C^\infty(\mathbb{R}), \int_0^x, \frac{d}{dx}\big)$. Both structures satisfies the above conditions with a shift map $f(x) \mapsto f(x+a)$ and an evaluation $\mathtt{E}_c f(x) = f(c)$.

Using evaluations, we can introduce *shifted Rota-Baxter operators* $\int_c \colon \mathscr{F} \to \mathscr{F}$ and the *definite integrals* $\int_c^d \colon \mathscr{F} \to K$ by

$$\int_c := (1 - \mathtt{E}_c) \int \qquad \text{and} \qquad \int_c^d := \mathtt{E}_d \int_c.$$

One checks immediately that $\int_c = S_{-c} \int S_c$ and $\int_c^d = \int_c - \int_d$ are equivalent definitions. Now of course, each $(\mathscr{F}, \int_c)$ is a shifted Rota-Baxter algebra with evaluation $\mathtt{E}_c$ for $c \in K$.

Let us bring our attention to the main task of this section—defining the Rota-Baxter operator on $\mathscr{P}\mathscr{F}$. First, we observe that every element $v \in \mathscr{P}\mathscr{F}$ can be written uniquely as

$$v = f + \sum_{a \in K} f_a H_a \tag{3.9}$$

with finitely many $f_a \neq 0$. Since by assumption $(\mathscr{F}, \int)$ is a shifted Rota-Baxter algebra, it suffices to define $\int \colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ as the unique extension of $\int \colon \mathscr{F} \to \mathscr{F}$ such that

$$\int f H_a = (\int_{a^+} f) H_a - (\int_{a^-}^0 f) \bar{H}_a = (\int_a f) H_a + \bar{H}(a) \int_0^a f \tag{3.10}$$

for all $f \in \mathscr{F}$ and $a \in K$. Equivalently, we can define

$$\int f \bar{H}_a = (\int_{a^-} f) \bar{H}_a - (\int_{a^+}^0 f) H_a = (\int_a f) \bar{H}_a + H(a) \int_0^a f. \tag{3.11}$$

The motivation for this definition comes from the standard *Riemann integral* $\int = \int_0^x$. For instance, if we take $f(x) = \cosh x$ then $\int f H_a$ can be visualised with four different cases depending upon the sign of x and a. This is illustrated in Figure 3.1.

In the standard example, the shift map $S_a(f) = f(x+a)$ shifts the graph of f by a units to the left. Similarly, in the case of the piecewise functions, we can define $S_a\colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ by $S_a(H_b) := H_{b-a}$.

Finally, we can extend the character $\mathtt{E}$ uniquely from the ground algebra by defining

$$\mathtt{E}\colon \mathscr{P}\mathscr{F} \to K \quad \text{such that} \quad \mathtt{E}(\bar{H}_a) = H(a).$$

This gives $\mathtt{E}_c(H_a) = \bar{H}(a-c)$ and $\mathtt{E}_c(\bar{H}_a) = H(a-c)$. Since the relation $H_a H_b = H_{a \sqcup b}$ implies that $\bar{H}(a)\bar{H}(b) = \bar{H}(a \sqcup b)$ which in turn yields $\bar{H}(0)^2 = \bar{H}(0)$. Therefore $\bar{H}(0) \in \{0,1\}$, and thus this set up works for both the cases $\eta = 0 \,\text{or}\, 1$, but it rules out the symmetric setting $\eta = 1/2$.

Proposition 3.1. *Let $(\mathscr{F}, \int)$ be an ordinary shifted Rota-Baxter algebra over an ordered field K. Then $(\mathscr{P}\mathscr{F}, \int)$ is a shifted Rota-Baxter algebra extending $(\mathscr{F}, \int)$.*

Proof. To prove it is a Rota-Baxter algebra, we need to show

$$(\textstyle\int f H_a)(\int g H_b) = \int f H_a \int g H_b + \int g H_b \int f H_a \tag{3.12}$$

holds (Def. 3.1), for all $f, g \in \mathscr{F}$. Since $\mathscr{P}\mathscr{F}$ is commutative and the right-hand of the above expression suggests that order of the functions does not matter, so it suffices to prove

$$(\textstyle\int f H_a)^2 = 2 \int f H_a \int f H_a \tag{3.13}$$

for $f \in \mathscr{F}$ and $a \in K$. Since H_a is idempotent, the definition of $\int\colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ and the Rota-Baxter axioms of $\int_{a^+}$ and $\int_{a^-}$ give $2\,(\int_{a^+} f \int_{a^+} f)\, H_a + 2\,(\int_{a^-}^0 f \int_{a^-} f)\, \bar{H}_a$ for the left-hand side of (3.13). Likewise, we get $2\,(\int_{a^+} f \int_{a^+} f)\, H_a - 2\,(\int_{a^-}^0 f \int_{a^+} f)\, \bar{H}_a$ on the right-hand side of (3.13), using twice the definition of $\int\colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$. It remains to check that the second terms are equal on both sides. For $a \geq 0$ both terms vanish while for $a < 0$ the problem reduces to checking $\int_a^0 f \int_a f = \int_0^a f \int f$. Splitting the inner integral $\int_a f = \int_a^0 f + \int f$ on the left-hand side yields $(\int_0^a f)^2 - \int_0^a f \int f$ since $\int_a^0$ is K-linear and $\int_a^0 f \in K$ by $(\mathscr{F}, \int)$ being ordinary. Then the result follows from the Rota-Baxter axiom of $(\mathscr{F}, \int)$. From the construction, it is clear that $(\mathscr{P}\mathscr{F}, \int)$ is a Rota-Baxter extension of $(\mathscr{F}, \int)$.

To show that it is indeed a shifted Rota-Baxter extension, it is enough to prove the compatibility relation $[S_c, \int] = \mathtt{E}_c \int$ (Def. 3.10) for the induced evaluations $\mathtt{E}_c = \mathtt{E} \circ S_c$.

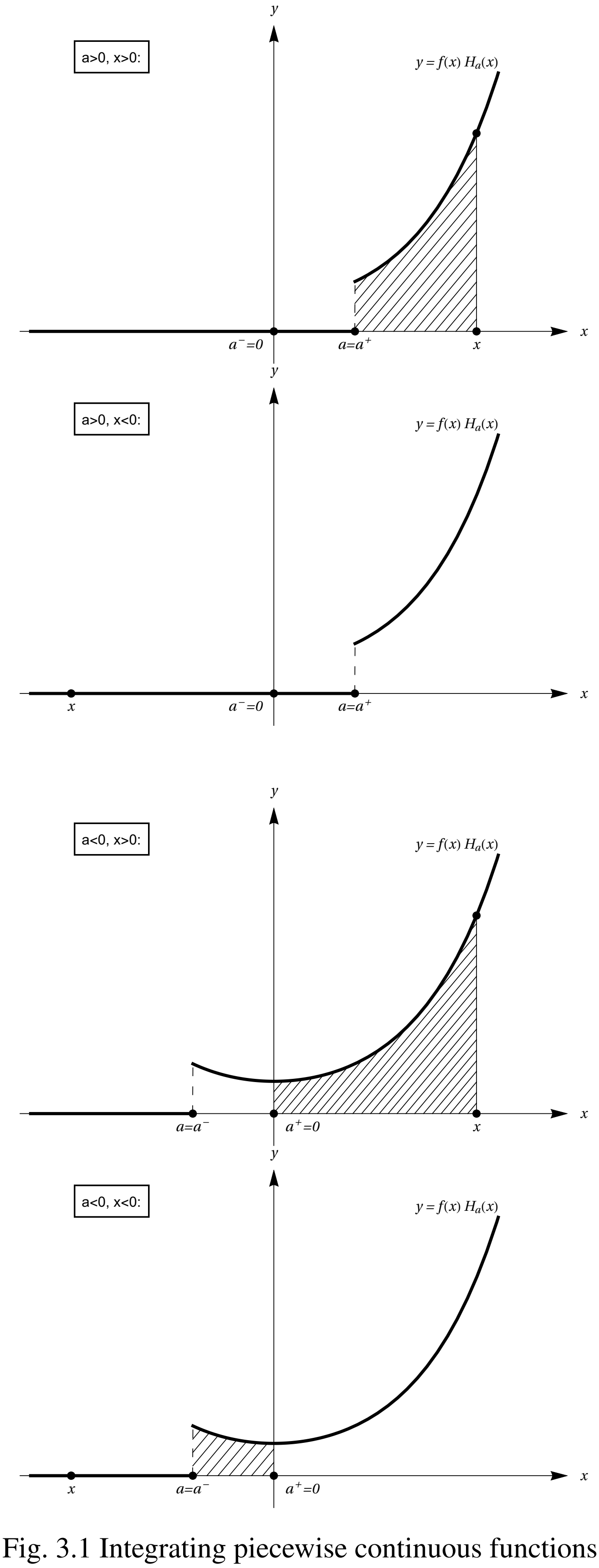

Fig. 3.1 Integrating piecewise continuous functions.

By (3.9), it is enough to verify this relation on the elements of the form fH_a, since it is already satisfied for $f \in \mathscr{F}$ due to the shift relation on $\mathscr{F}$. From the Figure 3.1, one can derive the generic identities which are valid for Rota-Baxter algebras over ordered fields,

$$\textstyle\int_{s^+} = \int + H(s) \int_s^0, \quad \int_{s^-} = \int + \bar{H}(s) \int_s^0 \qquad \text{and} \qquad \int_{s^+}^0 = H(s) \int_s^0, \quad \int_{s^-}^0 = \bar{H}(s) \int_s^0, \tag{3.14}$$

together with the simple consequence $\int_0^{a-c} S_c f = \int_c^a f$ of the shift relation on $\mathscr{F}$. Doing so yields $\int_0^c f + H(a) \int_a^0 f + H(a-c) \int_c^a f$ for both sides of $[S_c, \int] fH_a = \mathrm{E}_c \int fH_a$. $\square$

The above proposition gives an algebraic description of integration on piecewise functions. Now, our next aim here is to add a derivation on this structure, and then, as discussed before, the pair $(\int, \partial)$ induces an evaluation

$$\hat{\mathrm{E}} = 1 - \textstyle\int \circ \partial. \tag{3.15}$$

In the next proposition we prove that the induced evaluation $\hat{\mathrm{E}}$ is not multiplicative and hence we will call it *induced pseudo-evaluation*.

We can extend the derivation from the ground algebra $\mathscr{F}$ to $\mathscr{P}\mathscr{F}$ in two different ways: either we set $\partial(H_a) = 0$ or we define $\partial(H_a) = \delta_a$. It is tempting to use the second option and build distributions via this route as a differential ring extension of $\mathscr{P}\mathscr{F}$, but this gives rise to some complications, see Remark 3.3 below.
Therefore, here we define $\partial \colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ by setting $\partial H_a = 0$ for all $a \in K$. Observe that the ring of constants is enlarged to $\mathrm{Ker}(\partial) = K[H_a \mid a \in K]$. In the proposition below, we extend the ordinary shifted differential Rota-Baxter algebraic structure of $(\mathscr{F}, \partial, \int)$ to $(\mathscr{P}\mathscr{F}, \partial, \int)$ using the derivation defined above.

Proposition 3.2. *Let $(\mathscr{F}, \partial, \int)$ be an ordinary shifted differential Rota-Baxter algebra over the ordered field $(K, <)$. Then $(\mathscr{P}\mathscr{F}, \partial, \int)$ is a shifted differential Rota-Baxter extension algebra whose induced pseudo-evaluation*

$$\hat{\mathrm{E}}(fH_a) = \mathrm{E}_a(f) H_a + \mathrm{E}_0(f) - \mathrm{E}_{a^-}(f) = \begin{cases} \mathrm{E}_0(f) - \mathrm{E}_a(f) \bar{H}_a & \text{if } a \leq 0, \\ \mathrm{E}_a(f) H_a & \text{if } a \geq 0, \end{cases} \tag{3.16}$$

is not multiplicative. Hence $(\mathscr{P}\mathscr{F}, \int, \partial)$ is not an integro-differential algebra.

Proof. From the definition it is clear that $\int \colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ is a section of $\partial \colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$, so $(\mathscr{P}\mathscr{F}, \partial, \int)$ is a differential Rota-Baxter algebra by Proposition 3.1. For showing that it is shifted, it remains to prove the compatibility relation $[S_c, \partial] = 0$. Since it is true by hypothesis on $\mathscr{F}$, we need only check that $S_c \partial f H_a = f'(x+c) H_{a-c} = \partial S_c f H_a$.

One checks immediately that the pseudo-evaluation of $\mathscr{P}\mathscr{F}$ is given by (3.16), using the relation $\mathtt{E}_a = \mathtt{E}_{a^+} - \mathtt{E}_0 + \mathtt{E}_{a^-}$. As for every integro-differential algebra, we have $(K[x], \textstyle\int) \subseteq (\mathscr{F}, \textstyle\int)$. Since $K \supseteq \mathbb{Q}$ is an ordered field, we have $0 < 1 < 2$ so that

$$\hat{\mathtt{E}}(xH_1 \cdot xH_2) = \hat{\mathtt{E}}(x^2 H_2) = 4H_2 \neq 2H_2 = H_1 \cdot 2H_2 = \hat{\mathtt{E}}(xH_1) \cdot \hat{\mathtt{E}}(xH_2),$$

which shows that $\hat{\mathtt{E}}$ fails to be multiplicative. □

From (3.16), it is clear that $\hat{\mathtt{E}}(H_a) = H_a$, which is in agreement with the fact that $H_a \in \operatorname{Ker}(\partial)$. We should not confuse this evaluation with the distinguished evaluation $\mathtt{E}(H_a) = \bar{H}(a)$, which has image K rather than $K[H_a \mid a \in K]$.

Example 3.2.2. Let us describe the piecewise extension of continuous functions $\mathscr{F} = C(\mathbb{R})$ and smooth functions $\mathscr{F} = C^\infty(\mathbb{R})$ simultaneously.

Let $f\colon D \to \mathbb{R}$ be continuous/smooth function on an open set $D \subseteq \mathbb{R}$. Then f is called a *piecewise continuous/smooth* if D has finite complement in $\mathbb{R}$ and f has one-sided limits at each $x \in \mathbb{R} \setminus D$. We call $x \in \mathbb{R} \setminus D$ *regular* if f can be extended continuously/smoothly to the domain $D \cup \{x\}$. In this case, we define $f(x) = \lim_{\xi \to x} f(\xi)$ and write $\tilde{f}$ for its maximal extension. With this understanding, we define[2] $PC(\mathbb{R})$ and $PC^\infty(\mathbb{R})$ as the set of piecewise functions $f\colon D \to \mathbb{R}$ with $\tilde{f} = f$. They both become rings under the operations

$$f_1 + f_2 := \widetilde{f_1 \oplus f_2}, \qquad f_1 \cdot f_2 = \widetilde{f_1 \odot f_2},$$

where $f_1 \oplus f_2$ and $f_1 \odot f_2$ denote the pointwise sum and product of functions $f_i\colon D_i \to \mathbb{R}$ after restricting each to their common domain $D_1 \cap D_2$. Naturally, the usual Rota-Baxter operator $\int = \int_0^x$ yields Rota-Baxter algebras $(PC^\infty(\mathbb{R}), \int) \subset (PC(\mathbb{R}), \int)$. Moreover, the derivation $\partial = \frac{d}{dx}$ on the piecewise smooth function gives rise to a differential Rota-Baxter algebra $(PC^\infty(\mathbb{R}), \partial, \int)$.

There is an algebra homomorphism $\pi\colon \mathscr{P}C(\mathbb{R}) \to PC(\mathbb{R})$ that fixes $C(\mathbb{R})$ and that sends each H_a $(a \in \mathbb{R})$ to $H(x-a) \in PC(\mathbb{R})$. Clearly, we have also $\mathscr{P}C^\infty(\mathbb{R}) \to PC^\infty(\mathbb{R})$ by restriction. We show that both homomorphisms π are surjective: Each $f \in PC(\mathbb{R})$ or

[2]Note the difference between P and $\mathscr{P}$ in this example; the latter stands for the algebraic construction described above while the former denotes the standard notion of piecewise functions in real analysis.

$f \in PC^\infty(\mathbb{R})$ with regular part $f\colon D \to \mathbb{R}$ can be written as

$$f(x) = \sum_{i=0}^{n} f_i(x) H(x - x_i) H(x_{i+1} - x)$$

where $\mathbb{R} \setminus D = \{x_1 < \cdots < x_n\}$ and $f_i\colon \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous/smooth extension of the *function pieces* $f|_{(x_i,x_{i+1})}$. Here we set $x_0 = -\infty$ and $x_{n+1} = +\infty$ with the understanding that $H(x+\infty) = H(\infty - x) = 1$. With this choice of pieces $f_0, \ldots, f_n$ we have $f = \pi\big(\sum_i f_i H_{x_i} \bar{H}_{x_{i+1}}\big)$, so π is indeed surjective. However, π is not injective: it has respective kernels

$$\mathscr{R} := \mathrm{Ker}(\pi\colon \mathscr{P}C(\mathbb{R}) \to PC(\mathbb{R})) \quad \text{and} \quad \mathscr{R}_\infty := \mathrm{Ker}(\pi\colon \mathscr{P}C^\infty(\mathbb{R}) \to PC^\infty(\mathbb{R})).$$

These ideals encode the algebraic relations between continuous/smooth functions and Heavisides, for instance $b(x)H(x-2) = 0$ where $b(x)$ is any bump function supported in $[-1,1]$. Only when we quotient these kernels out, we get isomorphisms $PC(\mathbb{R}) \cong \mathscr{P}C(\mathbb{R})/\mathscr{R}$ and $PC^\infty(\mathbb{R}) \cong \mathscr{P}C^\infty(\mathbb{R})/\mathscr{R}_\infty$.

3.3 The Distribution module

Our construction of the distribution module is based on a free differential module over $\mathscr{F}$, rather than a module over $\mathscr{P}\mathscr{F}$ (see Remark 3.3). Here we introduce distributions using the derivation $\partial(H_a) = \delta_a := \delta(x-a)$. The aim of this section is to start from an ordinary shifted integro-differential algebra $(\mathscr{F}, \int, \partial)$, and obtain the *distribution module* $(\mathscr{D}\mathscr{F}, \oint, \eth)$, which is an ordinary shifted integro-differential module over $\mathscr{F}$. This section is rather technical. Therefore, for an easy comprehension we give an outline of our approach.

1. Construct the differential module $\mathscr{D}\mathscr{F}$ over $\mathscr{F}$ as a quotient module by extending the derivation on the ground algebra $(\mathscr{F}, \partial)$ (Def. 3.11).
2. Find a "normal form" for elements in the distributional module (Lemma 3.1).
3. Introduce a Rota-Baxter operator ("integral") on the distributional module (Eq. (3.22)).
4. Show that if $(\mathscr{F}, \int, \partial)$ is an ordinary shifted integro-differential algebra then
   (a) $(\mathscr{D}\mathscr{F}, \eth, \oint)$ is a differential Rota-Baxter module over $(\mathscr{F}, \int, \partial)$ (Thm. 3.1) .
   (b) $(\mathscr{D}\mathscr{F}, \oint, \eth)$ is an ordinary integro-differential module over $(\mathscr{F}, \int, \partial)$ (Prop. 3.3).

(c) $(\mathscr{D}\mathscr{F}, \oint, \eth)$ is an ordinary *shifted* integro-differential module over $(\mathscr{F}, \int, \partial)$ (Thm. 3.2).

In the theory of distributions, δ_a is considered not a function in itself but only in relation to how it affects other functions when "integrated" against them. Its basic property is that $f\delta_a$ vanishes identically when $f(a) = 0$. That is, $f\delta_a$ only depends on $f(a)$ and not on all of f. One may define it using its *sifting property*

$$\int_{-\infty}^{\infty} f(s)\delta(s-a)\,ds = f(a), \tag{3.17}$$

which reflect that it helps in "extracting" the source value. We use $f\delta_a = f(a)\,\delta_a$ as the basis for our algebraic construction.

If $(\mathscr{F}, \partial)$ is a differential algebra, we denote as usual $f' := \partial(f)$ and $f^{(i)} := \partial^i(f)$ for the derivatives of an element $f \in \mathscr{F}$. We also employ the abbreviation $\delta_a := \partial(H_a) = H_a'$, and for their higher order derivatives we write $\delta^{(i)}$ and $H_a^{(i)}$. For a set of differential indeterminates X, the algebra of differential polynomials $\mathscr{F}\{X\}$ is the free object in the category of differential $\mathscr{F}$ algebras. Similarly the $\mathscr{F}$-submodule $\mathscr{F}\{X\}_1$ consisting of *affine differential polynomials*, that is, those having total degree atmost 1, is the free object in category of differential $\mathscr{F}$-modules. Affine differential polynomials are required so that we can avoid terms consisting of $(\delta_a')^2$, $f(\delta_a'')^3$ and so on.

Definition 3.11. Let $(\mathscr{F}, \partial)$ be a differential algebra over a ring K. We define the *distribution module* $(\mathscr{D}\mathscr{F}, \eth)$ as the differential $\mathscr{F}$-module $\mathscr{F}\{H_a \mid a \in K\}_1/Z$, where Z denotes the differential $\mathscr{F}$-submodule generated by $\{f\,\delta_a - \mathtt{E}_a(f)\,\delta_a \mid f \in \mathscr{F},\ a \in K\}$.

Using the order on K, we can induce the ranking $\prec$ on $\mathscr{F}\{H_a \mid a \in K\}$. We say

$$H_a^{(m)} \prec H_b^{(n)} \quad \text{iff} \ \ a < b \quad \text{or} \quad a = b \ \ \text{and} \ \ m < n. \tag{3.18}$$

Moreover, the direct sum decomposition

$$\mathscr{F}\{H_a \mid a \in K\}_1 = \bigoplus_{a \in K} \mathscr{F}\{H_a\}_1$$

of differential $\mathscr{F}$-modules induces the direct decomposition $Z = \bigoplus Z_a$, and we write

$$\zeta = \sum_{a \in K} \zeta_a \quad (\zeta_a \in Z_a)$$

for the corresponding sum representation of an arbitrary $\zeta \in Z$. We can use the ranking $\prec$ to get a kind of Gröbner basis for Z which is mentioned in the lemma below. This lemma provides a way to write elements in the distribution module $\mathscr{D}\mathscr{F}$.

Lemma 3.1. *The differential $\mathscr{F}$-module Z in Definition 3.11 is generated as an $\mathscr{F}$-module by*

$$\Big\{ f\,\delta_a^{(k)} - \sum_{i=0}^{k} \binom{k}{i} (-1)^i E_a(f^{(i)})\,\delta_a^{(k-i)} \,\Big|\, a \in K, f \in \mathscr{F}, k \geq 0 \Big\}, \tag{3.19}$$

which forms a Gröbner basis of Z. For every element $\zeta \in Z$, the leading coefficient f_a of each ζ_a has the property $\mathrm{E}_a(f_a) = 0$. Relative to this Gröbner basis, the elements $\psi + Z \in \mathscr{D}\mathscr{F}$ of the quotient have the canonical representatives

$$\psi = f + \sum_{a\in K} f_a H_a + \sum_{a\in K}\sum_{k\geq 0} \lambda_{a,k}\,\delta_a^{(k)} \quad (f, f_a \in \mathscr{F};\, \lambda_{a,k} \in K) \tag{3.20}$$

with only finitely many f_a and $\lambda_{a,k}$ nonzero.

Proof. We split the proof into several steps.

1. Let us first show that Z contains the $\mathscr{F}$-module generated by (3.19). Since the components Z_a are independent, we fix an $a \in K$ and write the corresponding elements of (3.19) by $\zeta_{f,k}$. We prove by induction on k that all $\zeta_{f,k}$ are contained in Z. For $k = 0$ this is clear since $\zeta_{f,0}$ is a (differential) generator of Z. Assume that all $\zeta_{f,j}$ with $j < k$ and arbitrary $f \in \mathscr{F}$ are contained in Z; we show that $\zeta_{f,k} \in Z$ for a fixed $f \in \mathscr{F}$. Differentiating an arbitrary generator $f\,\delta_a - \mathrm{E}_a(f)\,\delta_a$ of Z, we obtain

$$\eth^k \zeta_{f,0} = f\,\delta_a^{(k)} + \sum_{i=0}^{k-1} \binom{k}{i} (\partial^{k-i} f)\,\delta_a^{(i)} - \mathrm{E}_a(f)\,\delta_a^{(k)} \in Z.$$

Eliminating the terms $f^{(i)}\,\delta_a^{(k-i)}$ yields

$$\eth^k \zeta_{f,0} - \sum_{i=0}^{k-1} \binom{k}{i} \zeta_{f^{(k-i)},i} = f\,\delta_a^{(k)} + \sum_{j=0}^{k-1}\sum_{i=j}^{k-1} \binom{k}{i}\binom{i}{j} (-1)^{i+j} \mathrm{E}_a(f^{(k-j)})\,\delta_a^{(j)} - \mathrm{E}_a(f)\,\delta_a^{(k)}$$

after an index transformation. The double sum simplifies to

$$\begin{aligned}\sum_{j=0}^{k-1}\sum_{i=j}^{k-1}\cdots &= \sum_{j=0}^{k-1}(-1)^j \mathrm{E}_a(f^{(k-j)})\,\delta_a^{(j)}\sum_{i=j}^{k-1}\binom{k}{i}\binom{i}{j}(-1)^i\\ &= (-1)^{k+1}\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\mathrm{E}_a(\partial^{k-j}f)\,\delta_a^{(j)}\\ &= -\sum_{j=1}^{k}\binom{k}{j}(-1)^j\mathrm{E}_a(f^{(j)})\,\delta_a^{(k-j)},\end{aligned}$$

using the fact that the inner sum above evaluates to $(-1)^{k+1}\binom{k}{j}$. Extending the range of the last sum to include $j = 0$ incorporates the remaining term so that

$$\eth^k\zeta_{f,0} - \sum_{i=0}^{k-1}\binom{k}{i}\zeta_{f^{(k-i)},i} = f\,\delta_a^{(k)} - \sum_{j=0}^{k}\binom{k}{j}(-1)^j\mathrm{E}_a(f^{(j)})\,\delta_a^{(k-j)} = \zeta_{f,k}, \tag{3.21}$$

which shows that $\zeta_{f,k} \in Z$ since all $\zeta_{\partial^{k-i}f,i} \in Z$ by the induction hypothesis.

2. For the converse that Z is contained in the $\mathscr{F}$-module generated by (3.19), it suffices to show that all the derivatives $\eth^k\zeta_{f,0}$ are $\mathscr{F}$-linear combinations of the $\zeta_{f,j}$. But this is clear from (3.21).

3. We proceed now to the statement about the leading coefficients. To this end, we rewrite the module generators as

$$\zeta_{f,k} = (f - \mathrm{E}_a f)\,\delta_a^{(k)} - \sum_{i=1}^{k}\binom{k}{i}(-1)^i\mathrm{E}_a(f^{(j)})\,\delta_a^{(k-i)},$$

from which the claim is evident.

4. Next we must show that (3.19) forms a Gröbner basis for the $\mathscr{F}$-module Z. This involves a slight variation of the usual setting of Gröbner bases for commutative polynomials [4] since we have infinitely many indeterminates and the coefficient ring $\mathscr{F}$ may have zero divisors (it is certainly not a field). Since we need only the linear fragment of the polynomial ring, we may use the approach of [1, §9.5a], which also allows for infinitely many generators. In the notation of [1, §9.5a], we set $k = K$ and $R = \mathscr{F}$ with trivial presentation (every element of $\mathscr{F}$ is a generator, and there are no relations) and the module $M = Z$ with generators $\delta_a^{(k)}$ and relations (3.19). The only S-polynomials σ arise from the self-overlaps of (3.19), namely $f\bar{f}\,\delta_a^{(k)}$, and this

yields

$$\begin{aligned}
\sigma &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \Big(\mathtt{E}_a(f^{(i)})\, \bar f - \mathtt{E}_a(\bar f^{(i)})\, f \Big)\, \delta_a^{(k-i)} \\
&\to \sum_{i=0}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^{i+j} \mathtt{E}_a(f^{(i)}\, \bar f^{(j)} - \bar f^{(i)}\, f^{(j)})\, \delta_a^{(k-i-j)} \\
&= \sum_{i+j \leq k} e_{ij}\, \eta_{ij},
\end{aligned}$$

which vanishes since the summation is over a triangle $i+j \leq k$, symmetric with respect to $i \leftrightarrow j$, while the evaluation term $e_{ij} = \mathtt{E}_a(\ldots)$ is antisymmetric and the trinomial term $\eta_{ij} = k!/i!j!(k-i-j)!\,(-1)^{i+j}\,\delta_a^{(k-i-j)}$ symmetric.

5. The analog of the Diamond Lemma in [1, §9.5a] ensures that the normal forms of (3.19) are canonical representatives of the congruence classes $\psi + Z \in \mathscr{D}\mathscr{F}$. Hence it suffices to characterize the normal forms of an arbitrary (noncanonical) representative ψ. Clearly, every such ψ is reducible as long as it contains any $\delta_a^{(k)}$ with a coefficient in $\mathscr{F} \setminus K$; hence we can achieve (3.20), which is clearly irreducible with respect to (3.19).

This completes the proof of the Presentation Lemma. $\square$

Now we can endow $(\mathscr{D}\mathscr{F}, \eth)$ with a Rota-Baxter operator. We define $\oint \colon \mathscr{D}\mathscr{F} \to \mathscr{D}\mathscr{F}$ as an extension of $\int \colon \mathscr{P}\mathscr{F} \to \mathscr{P}\mathscr{F}$ via the recursion

$$\oint f\,\delta_a^{(k)} = \begin{cases} \mathtt{E}_a(f) \oint \delta_a & \text{for } k = 0, \\ f\,\delta_a^{(k-1)} - \oint f'\,\delta_a^{(k-1)} & \text{for } k > 0, \end{cases} \tag{3.22}$$

where $\mathtt{E}_a$ denotes the evaluation in $\mathscr{F}$ and

$$\oint \delta_a = H_a - \bar H(a) = H(a) - \bar H_a, \tag{3.23}$$

which may also be written symmetrically as $\oint \delta_a = H(a)\,H_a - \bar H(a)\,\bar H_a$. If we set $f = 1$ in (3.22) then we obtain $\oint \delta_a^{(k)} = \delta_a^{(k-1)}$ for $k > 0$. Hence, the *induced evaluation* $\acute{\mathtt{E}} = 1_{\mathscr{D}\mathscr{F}} - \oint \eth$ gives

$$\acute{\mathtt{E}}(H_a) = H_a - \oint \delta_a = \bar H(a) \quad \text{and} \quad \acute{\mathtt{E}}(\delta_a^{(k)}) = 0 \quad (k \geq 0). \tag{3.24}$$

Remark 3.2. The definition of the Rota-Baxter operator $\oint\colon \mathscr{D}\mathscr{F} \to \mathscr{D}\mathscr{F}$ in (3.10) and (3.22)–(3.23) can be combined together to form the single formula

$$\oint f H_a^{(k+1)} = f H_a^{(k)} - \oint f' H_a^{(k)} \quad (k \in \mathbb{N}). \tag{3.25}$$

It is obvious for $k > 0$. It can be proved for $k = 0$ by using the relation $\mathtt{E}_a = \mathtt{E}_{a^+} - \mathtt{E}_0 + \mathtt{E}_{a^-}$ and the fact that $f(a^+) = f(a)H(a) + f(0)\bar{H}(a)$. However, we prefer to use the split definition (3.22)–(3.23) as it is more intuitive.

Since the derivation of H_a is different in both the modules $\mathscr{P}\mathscr{F}$ and $\mathscr{D}\mathscr{F}$, we can view $\mathscr{P}\mathscr{F} \subset \mathscr{D}\mathscr{F}$ as plain $\mathscr{F}$-modules but not as differential $\mathscr{F}$-modules. Our next result states that the distribution module $\mathscr{D}\mathscr{F}$ is an extension of the ground algebra $\mathscr{F}$ that contains the piecewise extension $\mathscr{P}\mathscr{F}$ as a Rota-Baxter module. See the figure below, where ι is the embedding of Rota-Baxter $\mathscr{F}$-modules while $u_{\mathscr{P}}$ and $u_{\mathscr{D}}$ are the structure maps of the $\mathscr{F}$-modules $\mathscr{P}\mathscr{F}$ and $\mathscr{D}\mathscr{F}$, respectively.

$$\begin{tikzcd}
\mathscr{P}\mathscr{F} \arrow[r, hook, "\iota"] & \mathscr{D}\mathscr{F} \\
\mathscr{F} \arrow[u, "u_{\mathscr{P}}"] \arrow[ur, "u_{\mathscr{D}}"'] &
\end{tikzcd}$$

Fig. 3.2 Embedding of Rota-Baxter $\mathscr{F}$-modules.

Theorem 3.1. *Let $(\mathscr{F}, \int, \partial)$ be an ordinary shifted integro-differential algebra. Then the distribution module $(\mathscr{D}\mathscr{F}, \eth, \oint)$ is a differential Rota-Baxter module over $\mathscr{F}$ that extends $(\mathscr{P}\mathscr{F}, \int)$ as a Rota-Baxter module.*

Proof. It suffices to prove the following statements:

1. *The map $\oint\colon \mathscr{D}\mathscr{F} \to \mathscr{D}\mathscr{F}$ is well-defined.* For this we have to show that $\oint Z \subseteq Z$, which we do with the help of Lemma 3.1. For fixed $a \in K$, we prove $\oint \zeta_{f,k} \in Z$ for all $f \in \mathscr{F}$ and $k \geq 0$. Using induction on k, the base case $k = 0$ follows immediately from (3.22). For the induction step it is enough to prove that $\oint \zeta_{f,k+1} = \zeta_{f,k} - \oint \zeta_{f',k}$ for all $f \in \mathscr{F}$.

Using the generators (3.19) we have

$$\oint \zeta_{f,k+1} = \oint f\, \delta_a^{(k+1)} - \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^i \mathrm{E}_a(f^{(i)}) \oint \delta_a^{(k-i+1)},$$

which simplifies by (3.22) and the binomial recursion $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$ to

$$\begin{aligned} f\, \delta_a^{(k)} - \sum_{i=0}^{k} \binom{k}{i} (-1)^i \mathrm{E}_a(f^{(i)})\, \delta_a^{(k-i)} - \Big(\oint f'\, \delta_a^{(k)} - \sum_{i=0}^{k} \binom{k}{i} (-1)^i \mathrm{E}_a(f'^{(i)}) \oint \delta_a^{(k-i)} \Big) \\ = \zeta_{f,k} - \oint \zeta_{f',k} \end{aligned}$$

and thus completes the induction.

2. *The map $\oint \colon \mathscr{D}\mathscr{F} \to \mathscr{D}\mathscr{F}$ is a Rota-Baxter operator.* Hence we must prove the weak RB axiom, for any $f, g \in \mathscr{F}$ and $a \in K$ and $k \geq 0$,

$$\textstyle\int f \cdot \oint g\, \delta_a^{(k)} = \oint f \oint g\, \delta_a^{(k)} + \oint (\int f) g\, \delta_a^{(k)}. \tag{3.26}$$

We fix $a \in K$ and use induction on k to prove (3.26) for all $f, g \in \mathscr{F}$. In the base case, exploring definition (3.22) reveals that $\mathrm{E}_a(g)$ factors on both sides of (3.26); hence it suffices to take $g = 1$. The left-hand side is then $\int f \cdot \oint \delta_a$ while we obtain

$$\textstyle\big(H(a) \int f H_a - \bar{H}(a) \int f \bar{H}_a\big) + \int_0^a f \cdot \oint \delta_a$$

for the right-hand side. Using the definition (3.10), (3.11) of the Rota-Baxter operator on the piecewise extension $\mathscr{P}\mathscr{F} \subset \mathscr{D}\mathscr{F}$ and properties of the Heaviside function, the first summand becomes $\int_a f \cdot \oint \delta_a$ and then combines with the remaining term to $\int f \cdot \oint \delta_a$; this completes the base case of the induction. Assume now that (3.26) holds for k; we show that it holds for $k+1$. Using the definition (3.22) once, the left-hand side is $\int f \cdot (g \delta_a^{(k)} - \oint g'\, \delta_a^{(k)})$. On the right-hand side we use (3.22) on each term to get

$$\begin{aligned} \textstyle\oint f g\, \delta_a^{(k)} - \oint f \oint g'\, \delta_a^{(k)} + (\int f) g\, \delta_a^{(k)} - \oint f g\, \delta_a^{(k)} - \oint (\int f) g'\, \delta_a^{(k)} \\ \textstyle= (\int f) g\, \delta_a^{(k)} - \oint f \oint g'\, \delta_a^{(k)} - \oint (\int f) g'\, \delta_a^{(k)}. \end{aligned}$$

Canceling the first terms on both sides, we end up with (3.26) where g is replaced by g', and this holds by the induction hypothesis.

3. *The map* $\eth\colon \mathscr{DF} \to \mathscr{DF}$ *is a well-defined derivation.* In fact, it suffices to prove well-definedness since the derivation property then follows immediately from the definition of $\mathscr{DF}$ as a quotient of a differential module. Hence we must prove $\partial Z \subset Z$, but this follows directly from $\eth\zeta_{f,k} = \zeta_{f,k+1} + \zeta_{f',k}$, obtained by differentiating the identity of Item (1).

4. *The Rota-Baxter operator* $\oint$ *is a section of the derivation* $\eth$. We start by showing that $\eth\oint fH_a = fH_a$ holds for all $f \in \mathscr{F}$. Using definition (3.10) for the Rota-Baxter operator on $\mathscr{PF}$ and the Leibniz rule together with the basic relation $f\delta_a = \mathtt{E}_a(f)\,\delta_a$ of Z yields

$$\eth\oint fH_a = fH_a + \left(\int_{a^+}^{a} f\right)\delta_a + \left(\int_{a^-}^{0} f\right)\delta_a \tag{3.27}$$

whose last two terms combine to $0+0$ in the case $a \geq 0$ and again to $\int_0^a f + \int_a^0 f = 0$ in the case $a \leq 0$. Hence the right-hand side of (3.27) is indeed fH_a. Now for elements of the form $f\delta_a^{(k)}$ we use induction on k. In the base case we have

$$\eth\oint f\delta_a = \mathtt{E}_a(f)\,\eth\big(H_a - H(a)\big) = \mathtt{E}_a(f)\,\delta_a = f\delta_a,$$

where the last step uses again the basic relation of Z. Now assume $\eth\oint f\delta_a^{(k)} = f\delta_a^{(k)}$ for a fixed k. Then we have

$$\eth\oint f\delta_a^{(k+1)} = \eth(f\delta_a^{(k)}) - \eth\oint f'\delta_a^{(k)} = f\delta_a^{(k+1)},$$

where the last step uses the Leibniz rule for $\eth$ and the induction hypothesis. This completes the proof of the section axiom for $\oint$.

□

Now we address the remark promised earlier on the "construction design" of $\mathscr{DF}$. We mention a complication that can arise if one constructs $\mathscr{DF}$ by simply defining derivation $H_a' = \delta_a$ in $\mathscr{PF}$.

Remark 3.3. Let us pose our problem as a question: Why can we not introduce distributions as a *differential ring extension* of $\mathscr{PF}$? The problem stems from the fact that $H_a^2 = H_a$ in $\mathscr{PF}$. If we employ the derivation $H_a' = \delta_a$ in $\mathscr{PF}$ instead of $H_a' = 0$, then after differentiating $H_a^2 = H_a$, we obtain $2H_a\,\delta_a = \delta_a$. This in turn implies that

$$\delta_a = 2H_a\,\delta_a = 2H_a\,(2H_a\,\delta_a) = (4H_a^2)\,\delta_a = 4H_a\,\delta_a = 2\delta_a,$$

meaning $\delta_a = 0$ which gives us an absurd result $\mathscr{DF} = \mathscr{PF}$.

It is now clear why our construction of $\mathscr{DF}$ was based on a free differential module over $\mathscr{F}$ rather than some module over $\mathscr{PF}$. It should be clear that $\mathscr{PF} \subset \mathscr{DF}$ together with the relation $H_a^2 = H_a \in \mathscr{DF}$ but we are barred from differentiating this relation since $\mathscr{DF}$ is a differential module over $\mathscr{F}$ and not over $\mathscr{PF}$. This is in accordance with the famous result due to Schwartz that distributions cannot be multiplied [47]. $\square$

The distribution module do, in fact, have a richer structure, more than was stated in Theorem 3.1. To capture this we need to introduce the module-theoretic analogue of the *integro-differential algebra*.

Definition 3.12. We call a module $(M, \oint, \eth)$ an *integro-differential module* over an integro-differential algebra $(\mathscr{F}, \int, \partial)$ if the induced evaluation $\acute{\mathtt{E}} := 1_M - \oint \eth$ is multiplicative in the following sense:

$$\acute{\mathtt{E}}(f\psi) = \mathtt{E}(f)\,\acute{\mathtt{E}}(\psi), \tag{3.28}$$

where $\mathtt{E} := 1_{\mathscr{F}} - \int \partial$ is the evaluation on the ground algebra $\mathscr{F}$.

In the following lemma we characterise this module.

Lemma 3.2. *Let $(M, \eth, \oint)$ be a differential Rota-Baxter module over the integro-differential algebra $(\mathscr{F}, \int, \partial)$. Then we have the following equivalences (where $f, c \in \mathscr{F}$ and $\psi, \gamma \in M$):*

1. $\oint c\psi = c\,(\oint \psi)$ *(for all $c \in \operatorname{Ker}(\partial)$)* $\quad \Leftrightarrow \quad$ $\oint f\psi = f \oint \psi - \oint f' \oint \psi$
2. $\oint f\gamma = (\int f)\,\gamma$ *(for all $\gamma \in \operatorname{Ker}(\eth)$)* $\quad \Leftrightarrow \quad$ $\oint f\psi = (\int f)\,\psi - \oint (\int f)\,\psi'$
3. $\acute{\mathtt{E}}(f\psi) = E(f)\,\acute{\mathtt{E}}(\psi)$ $\quad \Leftrightarrow \quad$ *(1a) & (2a)* $\quad \Leftrightarrow \quad$ *(1b) & (2b)*

If M is ordinary, then properties (1a) and (1b) are immediate; if $\mathscr{F}$ is ordinary, the same holds for properties (2a) and (2b).

Proof. The implications are similar to the corresponding ones given in [16, Thm. 2.5] for noncommutative rings, provided one splits the properties of the ring into its left-hand and right-hand versions.

Let us start with (1). The implication from right to left is obvious, so assume the homogeneity condition (1a) for $c \in \operatorname{Ker}(\partial)$. Then we have

$$f \oint \psi = (f - \textstyle\int f') \oint \psi + (\int f')(\oint \psi) = \oint f\psi - \oint (\int f')\psi + (\int f')(\oint \psi),$$

where we have used the homogeneity condition for $c = f - \int f' \in \operatorname{Ker}(\partial)$. By the (plain) Rota-Baxter axiom the last term above is $(\int f')(\oint \psi) = \oint (\int f')\psi + \oint f' \oint \psi$, hence one immediately obtains (1b). The proof of the equivalence (2a) $\Leftrightarrow$ (2b) is completely analogous.

Turning to (3), let us first assume the multiplicativity condition $\acute{\mathtt{E}}(f\psi) = \mathtt{E}(f)\,\acute{\mathtt{E}}(\psi)$. Specializing to $f = c \in \operatorname{Ker}(\partial)$ yields $\oint c\psi' = c\oint \psi'$, which is (1a) since $\eth$ is surjective; likewise specializing to $\psi = \gamma \in \operatorname{Ker}(\eth)$ gives $\oint f'\gamma = (\int f')\,\gamma$, which is (2a) since ∂ is surjective as well. For the converse statement, we may assume (1b) and (2b) to prove the multiplicativity condition for the evaluations. From the plain Rota-Baxter axiom we have

$$(\textstyle\int f')(\oint \psi') = \oint (\int f')\,\psi' + \oint f' \oint \psi' = \Big((\int f')\,\psi - \oint f'\,\psi \Big) + \Big(f \oint \psi' - \oint f\,\psi' \Big),$$

where the first and the second parenthesized terms come from applying (2b) and (1b), respectively. Subtracting $f\psi$ from both sides of the above identity and rearranging, one obtains exactly $\acute{\mathtt{E}}(f\psi) = \mathtt{E}(f)\,\acute{\mathtt{E}}(\psi)$. □

Now, we can prove our claim that $\mathscr{D}\mathscr{F}$ of Theorem 3.1 is indeed an ordinary integro-differential module.

Proposition 3.3. *If $(\mathscr{F}, \int, \partial)$ is an ordinary shifted integro-differential algebra, $(\mathscr{D}\mathscr{F}, \oint, \eth)$ is an ordinary integro-differential module over $\mathscr{F}$.*

Proof. Let us first prove that $\mathscr{D}\mathscr{F}$ is ordinary, meaning $\operatorname{Ker}(\eth) = K$. Hence assume $\eth\psi = 0$ for an arbitrary element $\psi \in \mathscr{D}\mathscr{F}$. By Lemma 3.1 we may assume

$$\psi = f + \sum_{a\in K} f_a H_a + \sum_{a\in K}\sum_{k\geq 0} \lambda_{a,k}\, \delta_a^{(k)}$$

for some $f, f_a \in \mathscr{F}$ and $\lambda_{a,k} \in K$ so that

$$f' + \sum_{a\in K} (f_a' H_a + f_a \delta_a) + \sum_{a\in K}\sum_{k\geq 0} \lambda_{a,k}\, \delta_a^{(k+1)} = 0.$$

In view of Lemma 3.1, we obtain $f' = f_a' = f_a = \lambda_{a,k} = 0$. But then we have $\psi = f \in \operatorname{Ker}(\partial) = K$, so the differential module $(\mathscr{D}\mathscr{F}, \eth)$ is ordinary. From Lemma 3.2 it follows immediately that $(\mathscr{D}\mathscr{F}, \oint, \eth)$ is also an integro-differential module. □

To show that the module $\mathscr{D}\mathscr{F}$ *inherits* all the properties from the ground algebra we introduce shifted evaluations on it by extending the shift map of the ground algebra. If we define $\acute{\mathrm{S}} \colon K \to \operatorname{Aut}_K(\mathscr{D}\mathscr{F})$ such that

$$\acute{\mathrm{S}}_c H_a = H_{a-c} \quad \text{and} \quad \acute{\mathrm{S}}_c\, \delta_a = \delta_{a-c}\ (a, c \in K),$$

and extend it through linearity and multiplicativity, then $\acute{\mathrm{S}}$ is a shift map on $\mathscr{D}\mathscr{F}$. Similarly one can obtain the shifted evaluations on $\mathscr{D}$ by setting $\acute{\mathtt{E}}_c := \acute{\mathtt{E}} \circ \acute{\mathrm{S}}_c$. Clearly, this yields $\acute{\mathtt{E}}_c H_a =$

$\bar{H}(a-c)$ and $\acute{\mathtt{E}}_c\delta_a^{(k)} = 0$ on the generators as per (3.24). As usual we write $\oint_b$ $(b \in K)$ for the resulting shifted Rota-Baxter operators.

Theorem 3.2. *If $(\mathscr{F}, \int, \partial)$ is an ordinary shifted integro-differential algebra, $(\mathscr{D}\mathscr{F}, \oint, \eth)$ is an ordinary shifted integro-differential module over $\mathscr{F}$. Its shifted Rota-Baxter operators are given by the recursion* (3.22)*, with $\oint$ replaced by $\oint_b$, and by the base case* (3.23)*, with $\bar{H}(a)$ replaced by $\bar{H}(a-b)$ or $H(a)$ replaced by $H(a-b)$.*

Proof. The recursive description of the shifted Rota-Baxter operators follows immediately from the definition $\oint_b := (1-\acute{\mathtt{E}}_b)\int$. In view of Proposition 3.3, it then remains to prove the compatibility relations $[\acute{\mathrm{S}}_c, \oint] = \acute{\mathtt{E}}_c \oint$ and $[\acute{\mathrm{S}}_c, \eth] = 0$. Let us start with the former.

Since $\acute{\mathrm{S}}_c$ and $\oint$ as well as $\acute{\mathtt{E}}_c$ agree on $\mathscr{P}\mathscr{F} \subset \mathscr{D}\mathscr{F}$ by definition, it suffices to consider elements of the form $f\delta_a^{(k)}$ $(k \geq 0)$. We apply induction on k. For the base case $k = 0$, we obtain $\mathtt{E}_a(f)\,\big(\bar{H}(a-c) - \bar{H}(a)\big)$ for both the left-hand and the right-hand side of the relation $[\acute{\mathrm{S}}_c, \oint] = \acute{\mathtt{E}}_c \oint$ applied to $f\delta_a$. Now assume the relation for all $f\delta_a^{(k)}$ with fixed $k \geq 0$; we must show it for $f\delta_a^{(k+1)}$. A straightforward computation, using the induction hypothesis on $\oint f'\delta_a^{(k)}$, yields $-\acute{\mathtt{E}}_c \oint f'\delta_a^{(k)}$ for both sides of $[\acute{\mathrm{S}}_c, \oint] = \acute{\mathtt{E}}_c \oint$ as applied to $f\delta_a^{(k+1)}$.

Let us now turn to the commutation identity $\acute{\mathrm{S}}_c\eth = \eth\acute{\mathrm{S}}_c$. Since $\mathscr{F}$ is a shifted integro-differential algebra by hypothesis, we need only consider elements of the form $fH_a^{(k)}$ $(k \geq 0)$. For those one obtains indeed $\acute{\mathrm{S}}_c\eth\, fH_a^{(k)} = \eth\acute{\mathrm{S}}_c\, fH_a^{(k)} = S_c(f')\,\delta_{a-c}^{(k)} + S_c(f)\,H_{a-c}^{(k+1)}$, making use of the commutation identity on $\mathscr{F}$. □

The distribution module $\mathscr{D}\mathscr{F}$ is a universal object, that is, we can characterize it in terms of a *universal mapping property*. Let us briefly review this universal property in terms of initial and terminal object.

An *initial object* A in a category $\mathscr{C}$ is an object such that for any object X of $\mathscr{C}$, there is a unique morphism $f : A \to X$. An initial object, if it exists is unique up to isomorphism. Initial objects are the dual concept to *terminal objects*, meaning an object B is called terminal if for every object $X \in \mathscr{C}$ there is a unique morphism $g : X \to B$.

The existence of a unique morphism is the same as saying that the corresponding Hom set consists of a single element. Therefore, in both the cases here, the sets $\mathrm{Hom}_{\mathscr{C}}(A,X)$ and $\mathrm{Hom}_{\mathscr{C}}(X,B)$ are singletons. We already know of some universal objects; let us consider a few examples below.

In the category of sets, the single element set is a terminal object since there is only one set-function to a single element set. However, there are infinitely many singleton sets but they all are isomorphic. On the other hand, the empty set is an initial object, since the "empty function" is the only set-mapping from the empty set to any set. Notice that the initial object here is truly unique and not just up to isomorphism.

In the category of groups, the identity group or trivial group $\{1\}$ is both an initial and a terminal object, since group homomorphism must preserve the identity element. Here, this trivial group is not unique but unique up to isomorphism.

For a slightly more interesting example, consider the category of rings. If R is any ring with identity element 1_R then the map $f : \mathbb{Z} \to R$ defined by $f(n) = n.1_R$ is a unique morphism, hence this makes the ring of integers $\mathbb{Z}$ an initial object in the category of rings.

Now we come back to the task at hand. To characterize universality of the Dirac module first we need to fulfil the following requirements:

- first, we adjoin a family of distributions δ_a $(a \in K)$ to the given integro-differential algebra $(\mathscr{F}, \partial, \int)$ and then,
- algebraically characterize them by the sifting property (3.17) and the integro-differential relation
$$\delta^{(k)} \underset{\oint}{\overset{\eth}{\rightleftarrows}} \delta^{(k+1)} \quad \text{for } k \geq 0,$$
together with the conditions $H_a' = \delta_a$ and $\oint = H_a - \bar{H}(a)$.
- finally, the multiplication of distributions is not allowed.

Definition 3.13. Let $(\mathscr{F}, \int, \partial)$ be an integro-differential algebra. An integro-differential module $(\mathscr{M}, \oint_{\mathscr{M}}, \eth)$ over $\mathscr{F}$ is called a *Dirac module* if $\mathscr{P}\mathscr{F} \hookrightarrow \mathscr{M}$ as Rota-Baxter modules such that (3.17) holds and $\delta_a := \eth_{\mathscr{M}} H_a$ satisfies $\oint_{\mathscr{M}} \delta_a = H_a - \bar{H}(a)$ as well as $\oint_{\mathscr{M}} \delta_a^{(k+1)} = \delta_a^{(k)}$, for all $a \in K$ and $k \geq 0$.

With the above definition, we are equipped to show that the distribution module $\mathscr{D}\mathscr{F}$ is an initial object in the category of Dirac modules.

Proposition 3.4. *The differential Rota-Baxter module $(\mathscr{D}\mathscr{F}, \eth, \oint)$ is the universal Dirac module over $(\mathscr{F}, \partial, \int)$ that extends $(\mathscr{P}\mathscr{F}, \int)$ as a Rota-Baxter module. In other words, for every Dirac module $\mathscr{M}$ there is a unique integro-differential morphism $\Psi\colon \mathscr{D}\mathscr{F} \to \mathscr{M}$ that respects the canonical embedding of $\mathscr{P}\mathscr{F}$.*

Proof. Let $\kappa\colon \mathscr{P}\mathscr{F} \hookrightarrow \mathscr{M}$ be the embedding of Rota-Baxter modules from Definition 3.13, and let $u_{\mathscr{P}}$, $u_{\mathscr{D}}$, ι be as in the diagram below.

$$
\begin{tikzcd}
\mathscr{F} \arrow[d, "u_{\mathscr{P}}"'] \arrow[dr, "u_{\mathscr{D}}"] \arrow[ddr, "u_{\mathscr{M}}"] & \\
\mathscr{P}\mathscr{F} \arrow[r, hook, "\iota"] \arrow[dr, hook, "\kappa"'] & \mathscr{D}\mathscr{F} \arrow[d, dashed, "\Psi"] \\
 & \mathscr{M}
\end{tikzcd}
$$

Fig. 3.3 Universal mapping property of distribution module.

Furthermore, we will write $u_{\mathscr{M}}$ for the structure map of the $\mathscr{F}$-module $\mathscr{M}$. We construct a morphism of integro-differential modules $\Psi\colon \mathscr{D}\mathscr{F} \to \mathscr{M}$ that makes the right-hand diagram commute. It suffices to show $\Psi\iota = \kappa$ since then $\Psi u_{\mathscr{D}} = u_{\mathscr{M}}$ follows from the module structures $\iota u_{\mathscr{P}} = u_{\mathscr{D}}$ and $\kappa u_{\mathscr{P}} = u_{\mathscr{M}}$.

If the required map Ψ exists, it must be $\mathscr{F}$-linear and send $(\iota H_a)^{(k)}$ to $(\kappa H_a)^{(k)}$. But this defines Ψ uniquely because $\mathscr{D}\mathscr{F}$ is generated by $(\iota H_a)^{(k)}$ as an $\mathscr{F}$-module. Defining first $\tilde{\Psi}\colon \mathscr{F}\{H_a \mid K\}_1 \to \mathscr{M}$ by these requirements, it follows at once that $\tilde{\Psi}$ is in fact a morphism of differential $\mathscr{F}$-modules. To see that it lifts to a map $\Psi\colon \mathscr{D}\mathscr{F} \to \mathscr{M}$, we must show $\tilde{\Psi}(Z) = 0$. Since Ψ respects the derivation, it suffices to prove that Ψ annihilates the differential generators $f\,\delta_a - \mathtt{E}_a(f)\,\delta_a$ or, more precisely, the corresponding elements $u_{\mathscr{D}}(f)\,\iota(H_a)' - \mathtt{E}_a(f)\,\iota(H_a)'$. But this follows immediately from the sifting property (3.17) of the Dirac module $\mathscr{M}$.

We have now a differential morphism $\Psi\colon \mathscr{D}\mathscr{F} \to \mathscr{M}$ that clearly satisfies the required commutation property $\Psi\iota = \kappa$. Moreover, it is clear from the construction that Ψ is unique. Hence it only remains to prove that Ψ is also a morphism of Rota-Baxter algebras over $\mathscr{F}$. To this end, we show first that

$$\oint_{\mathscr{M}} \Psi(f\,\iota H_a) = \Psi \oint (f\,\iota H_a). \tag{3.29}$$

Note that the left-hand side may be written as $\oint_{\mathscr{M}} \kappa(f\,\iota H_a)$ since $\Psi\iota = \kappa$. Since by hypothesis we have $\mathscr{P}\mathscr{F} \hookrightarrow \mathscr{M}$ as Rota-Baxter $\mathscr{F}$-modules, we may now apply $\oint_{\mathscr{M}} \kappa = \kappa \int$ and then expand the integral $\int$ of $\mathscr{P}\mathscr{F}$ to obtain

$$\kappa\Big(\big(\textstyle\int_{a^+} f\big) H_a - \big(\int_{a^-}^{0} f\big)\bar{H}_a\Big) = \Psi\Big(\big(\textstyle\int_{a^+} f\big)\,\iota H_a - \big(\int_{a^-}^{0} f\big)\,\iota\bar{H}_a\Big)$$

for the left-hand side of (3.29), using again $\Psi\iota = \kappa$ for the last step. Recalling that $\oint$ on $\mathscr{D}\mathscr{F}$ was defined as an extension of $\int$ on $\mathscr{D}\mathscr{F}$, this yields the right-hand side of (3.29). It remains to prove

$$\oint_{\mathscr{M}} \Psi(f\,\delta_a^{(k)}) = \Psi \oint (f\,\delta_a^{(k)}) \tag{3.30}$$

for all $k \geq 0$. By the sifting property (3.17), valid in $\mathscr{D}\mathscr{F}$ as well as $\mathscr{M}$, we may replace f by $\mathtt{E}_a(f)$ on both sides of (3.30). Hence we may set $f = 1$ for the proof of (3.30). For $k = 0$, we use the antiderivative relation of the Dirac module $\mathscr{M}$ in the precise form $\oint(\kappa H_a)' = \kappa H_a - \bar{H}(a)$ to obtain

$$\oint_{\mathscr{M}} \Psi\delta_a = \oint(\kappa H_a)' = \kappa\big(H_a - \bar{H}(a)\big) = \Psi(\iota H_a - \bar{H}(a)) = \Psi(\oint \delta_a)$$

as required. For $k > 0$, Equation (3.30) follows immediately from $\oint_{\mathscr{M}} (\kappa H_a)^{(k)} = (\kappa H_a)^{(k-1)}$, which holds since $\mathscr{M}$ is a Dirac module. $\square$

3.4 Bivariate distributional module and applications

From the point of view of applications, we need to find an algebraic setting where the functions $\delta(x-a)$, $\delta(\xi - b)$ and $\delta(x-\xi)$ "live". The construction of the distributional module was the first step in this direction and now we expand it via the tensor product to achieve our goal. This construction was carried out by Rosenkranz. Here we rewrite the description of this construction without including rigorous proofs. For a detailed description, please read [42].

To accommodate the distributions $\delta(\xi - b)$, it is enough to introduce the tensor product and build functions in the variable ξ. To do this, we define $\mathscr{F}_2 := \mathscr{F} \otimes_K \mathscr{F}$ and then introduce the corresponding differential operators ∂_x, ∂_ξ, together with the Rota-Baxter operators $\int^x, \int^\xi$. This is the first essential tensor product that is required on the ground algebra and then slowly we expand it to the piecewise extension and the distributional module.

However, it is tricky to find a setting for "diagonal distributions" $\delta(x-\xi)$ and "diagonal Heavisides" $H(x-\xi)$. To construct such elements, first we need to introduce a *slim distributional module* where we get rid of the whole gamut of H_a and retain a single Heaviside $\hat{H}$. Taking motivation from analysis then we introduce an algebraic setting which contains

$\delta(x-\xi)$ and $H(x-\xi)$. Finally, we bring together all these structures to build the bivariate distributional module. Let us begin this construction step by step.

1. **Slim distribution module**

   For our purpose we need only a few Heavisides rather than the whole range of $H_a (a \in K)$. To get rid of all such Heavisides and pick their one representative, we construct a quotient module.

   Let $N_{\mathscr{D}}$ be the differential Rota-Baxter submodule generated by the set $\{H_a \mid a \in K^\times\}$. Then the module
   $$\hat{\mathscr{D}}\mathscr{F} = \mathscr{D}\mathscr{F}/N_{\mathscr{D}}$$
   is called the *slim distributional module*. The Heaviside is denoted by $\hat{H} := H_0 + N_{\mathscr{D}}$ and consequently we write $\hat{\delta} := \hat{H}'$. Similarly one can obtain the *slim piecewise extension*
   $$\hat{\mathscr{P}}\mathscr{F} = \mathscr{P}\mathscr{F}/N_{\mathscr{P}},$$
   where $N_{\mathscr{P}}$ is the *ideal* generated by $\{H_a \mid a \in K^\times\}$. We may view $\hat{\mathscr{D}}\mathscr{F}$ as a module over $\hat{\mathscr{P}}\mathscr{F}$. In fact, we shall only need the K-subspace generated by $\hat{H}$ and its derivatives; let us denote this space by $\hat{\mathscr{D}}K \subset \hat{\mathscr{D}}\mathscr{F}$. Likewise, we shall write $\hat{\mathscr{P}}K \subset \hat{\mathscr{P}}\mathscr{F}$ for the K-subalgebra generated by $\hat{H}$ alone.

2. **$\mathscr{F}$-bimodule with differential and Rota-Baxter operators**

   To introduce the counterparts of univariate $H(x-a)$ and its derivatives, we take the tensor product on the ground algebra $\mathscr{F}_2 := \mathscr{F} \otimes_K \mathscr{F}$ with the following derivations and RB operators
   $$\begin{aligned} \partial_x(f_1 \otimes f_2) &= (\partial f_1) \otimes f_2, & \partial_\xi(f_1 \otimes f_2) &= f_1 \otimes (\partial f_2), \\ {\textstyle\int^x}(f_1 \otimes f_2) &= ({\textstyle\int} f_1) \otimes f_2, & {\textstyle\int^\xi}(f_1 \otimes f_2) &= f_1 \otimes ({\textstyle\int} f_2). \end{aligned}$$
   We have two embeddings $\iota_x, \iota_\xi : \mathscr{F} \to \mathscr{F}_2$ with $\iota_x(f) = f \otimes 1$ and $\iota_\xi(f) = 1 \otimes f$; we denote their images by $\mathscr{F}_x$ and $\mathscr{F}_\xi$, respectively. For a ground element $f \in \mathscr{F}$, their embeddings are also written as $f(x) := \iota_x(f) \in \mathscr{F}_x$ and $f(\xi) := \iota_\xi(f) \in \mathscr{F}_\xi$. It is clear that both the structures $(\mathscr{F}_2, \partial_x, \int^x)$ and $(\mathscr{F}_2, \partial_\xi, \int^\xi)$ are *integro-differential algebras* over K. Although, they are not the ordinary ones since $\operatorname{Ker}(\partial_x) = \mathscr{F}_\xi$ and $\operatorname{Ker}(\partial_\xi) = \mathscr{F}_x$. One can also extend the shift operators as $S_a^x(f_1 \otimes f_2) := (S_a f_1) \otimes f_2$ and $S_a^\xi(f_1 \otimes f_2) := f_1 \otimes (S_a f_2)$. If $\tau \colon \mathscr{F}_2 \to \mathscr{F}_2$ is the usual exchange automorphism $\tau(f_1 \otimes f_2) = f_2 \otimes f_1$ then the derivations, Rota-Baxter and shift operators are conjugate under τ,

meaning

$$\partial_\xi = \tau\partial_x\tau,\ {\textstyle\int}^\xi = \tau{\textstyle\int}^x\tau, S^\xi_a = \tau S^x_a\tau.$$

3. **Pure distributional module and bivariate piecewise extension**
The natural thing is to extend the structure $(\mathscr{F}_2, \partial_x, \int^x)$ and $(\mathscr{F}_2, \partial_\xi, \int^\xi)$ such that the resulting setting also contains Heavisides and Diracs.

Definition 3.14. The *pure distribution modules* are introduced by $\mathscr{D}_x\mathscr{F} := \mathscr{D}(\mathscr{F}_2, \partial_x, \int^x)$ and $\mathscr{D}_\xi\mathscr{F} := \mathscr{D}(\mathscr{F}_2, \partial_\xi, \int^\xi)$. We write $H(x-a) \in \mathscr{D}_x\mathscr{F}$ and $H(\xi - a) \in \mathscr{D}_\xi\mathscr{F}$ for the corresponding differential generators ($a \in K$).

It should be clear that we view $H(x-a) \in \mathscr{P}_x\mathscr{F}$ and $H(\xi - a) \in \mathscr{P}_\xi\mathscr{F}$. One can induce a *duplex* structure by defining the action of $\eth_x, \oint^x, \acute{S}^x_a$ on $\mathscr{D}_\xi\mathscr{F}$ by regarding "foreign" factors as constants.

$$\eth_x f H^{(k)}_a := (\partial_x f)\, H^{(k)}_a, \qquad \oint^x f H^{(k)}_a := ({\textstyle\int}^x f)\, H^{(k)}_a, \quad \acute{S}^x_a f H^{(k)}_a := (S^x_a f)\, H^{(k)}_a.$$

Analogously, one can define $\eth_\xi, \oint^\xi, \acute{S}^\xi_a$ on $\mathscr{D}_x\mathscr{F}$. Altogether, this gives *duplex shifted differential Rota-Baxter modules*, $(\mathscr{D}_x\mathscr{F}, \eth_x, \eth_\xi, \oint^x, \oint^\xi)$ and $(\mathscr{D}_\xi\mathscr{F}, \eth_x, \eth_\xi, \oint^x, \oint^\xi)$. Their induced evaluations are written as $\acute{\mathtt{E}}_x := 1 - \oint^x \eth_x$ and $\acute{\mathtt{E}}_\xi := 1 - \oint^\xi \eth_\xi$, along with the shifted versions $\acute{\mathrm{e}}\mathrm{v}^x_a := \acute{\mathtt{E}}_x \acute{S}^x_a$ and $\acute{\mathrm{e}}\mathrm{v}^\xi_a := \acute{\mathtt{E}}_\xi \acute{S}^\xi_a$.

One can identify the corresponding piecewise extension as $\mathscr{P}_x\mathscr{F} \subset \mathscr{D}_x\mathscr{F}$ and $\mathscr{P}_\xi\mathscr{F} \subset \mathscr{D}_\xi\mathscr{F}$. With this understanding, we can introduce the *bivariate piecewise extension*

$$\mathscr{P}_{x\xi}\mathscr{F} := \mathscr{P}_x\mathscr{F} \otimes_{\mathscr{F}} \mathscr{P}_\xi\mathscr{F}.$$

This setting is helpful to represent a rectangular region $(x,\xi) \in [a,b] \times [c,d]$ in the $\mathbb{R}^2$ plane. With the same analogy as in $\mathscr{F}_2$, we shall drop the $\otimes$ symbol and write $H(x-a)H(\xi - b)$ for $H_a \otimes H_b$. Now we can represent the characteristic function[3] in our language

$$[a \le x \le b] \otimes [c \le \xi \le d] := H(x-a)\,H(b-x)H(\xi - c)\,H(d-\xi) \in \mathscr{P}_\xi\mathscr{F}.$$

4. **Diagonal distribution module**
The algebraic description of the diagonal Heavisides $H(x-\xi)$ and diagonal Diracs

[3] As in [41] we use the *Iverson bracket notation* [14, §2.2]

$\delta(x-\xi)$ is somewhat more complicated, so we turn to analysis for better understanding. In analysis ($K = \mathbb{R}$), with fixed $a \in \mathbb{R}$ and variables x, ξ ranging over $\mathbb{R}$. We have the relation

$$(x \geq a) \wedge x(\geq \xi) \Longleftrightarrow (x \geq a \wedge a \geq \xi) \vee (x \geq \xi \wedge a \leq \xi),$$

since we may split the case $a < \xi$ and $a > \xi$, while the case $a = \xi$ holds in both the above cases. In our algebraic language this translates to

$$H(x-a)\,H(x-\xi) = H(x-a)\,H(a-\xi) + H(x-\xi)\,H(\xi-a)$$

or $H_a(x)\,\hat{H} = H_a(x)\,\bar{H}_a(\xi) + H_a(\xi)\,\hat{H}$ where $H_a(x) := H_a \otimes 1 \in \mathscr{P}_{x\xi}\mathscr{F}$ and $H_a(\xi) := 1 \otimes H_a \in \mathscr{P}_{x\xi}\mathscr{F}$. With this understanding, we can formulate the diagonal distributions. Let $\hat{Z}$ be the $\mathscr{P}_{x\xi}\mathscr{F}$-submodule of $\mathscr{P}_{x\xi}\mathscr{F} \otimes_K \hat{\mathscr{D}}K$ that is generated by the set $\{\big(H_a(x) - H_a(\xi)\big)\hat{H} - H_a(x)\,\bar{H}_a(\xi) \mid a \in K\}$. Then the $\mathscr{P}_{x\xi}\mathscr{F}$-module

$$\mathscr{D}_{x-\xi}\mathscr{F} := \frac{\mathscr{P}_{x\xi}\mathscr{F} \otimes_K \hat{\mathscr{D}}K}{\hat{Z}}$$

is called the *diagonal distribution module*. We shall denote the (congruence class of) its slim generator $\hat{H} \in \hat{\mathscr{D}}K$ by $H(x-\xi)$, and its derivative $\hat{\delta} \in \hat{\mathscr{D}}K$ by $\delta(x-\xi)$. Analogously to the univariate case, we set also $H(\xi - x) := 1 - \hat{H}$.

We emphasize again that the submodule $\hat{Z}$ is *not differentially generated*. In other words, one is not supposed to differentiate the relation (4) as this would once again lead to inconsistencies (Remark 3.3).

5. **Bivariate distributional module**
To obtain the bivariate distributional module, first we need to introduce the *tensorial distribution module*. It can be done by combining the univariate distribution module $\mathscr{D}_x\mathscr{F}$ and $\mathscr{D}_\xi\mathscr{F}$ along with the bivariate piecewise extension $\mathscr{P}_{x\xi}\mathscr{F}$ into a single module given by

$$\mathscr{D}_{x\xi}\mathscr{F} := \big(\mathscr{D}_x\mathscr{F} \otimes_{\mathscr{F}} \mathscr{P}_\xi\mathscr{F}\big) \oplus \big(\mathscr{P}_x\mathscr{F} \otimes_{\mathscr{F}} \mathscr{D}_\xi\mathscr{F}\big).$$

Following the same routine (comments after Def. 3.14), we can combine all structures into a duplex shifted differential Rota-Baxter module $(\mathscr{D}_{x\xi}\mathscr{F}, \eth_x, \eth_\xi, \oint^x, \oint^\xi)$

over $\mathscr{F}_2$, which is also a module over $\mathscr{P}_{x\xi}\mathscr{F}$. So far, the situation is parallel to that of Theorem 3.1.

At this point we have two $\mathscr{P}_{x\xi}\mathscr{F}$-modules, $\mathscr{D}_{x\xi}\mathscr{F}$ and $\mathscr{D}_{x-\xi}\mathscr{F}$. Since $\mathscr{F}_2 \subset \mathscr{P}_{x\xi}$, we may also view them as $\mathscr{F}_2$-modules. They are both *free modules* just as $\mathscr{P}_{x\xi}$ itself is free as an $\mathscr{F}_2$-module. Indeed, the bivariate piecewise extension $\mathscr{P}_{x\xi}$ has the $\mathscr{F}_2$-basis $\mathscr{B} := \{1, H_a(x), H_a(\xi), H_a(x)\,H_b(\xi) \mid a,b \in K\}$, while the tensorial distribution module $\mathscr{D}_{x\xi}\mathscr{F}$ has $\mathscr{B}_{x\xi} := \mathscr{B} \cup \{H_a(x)\,\delta^{(n)}(b-\xi), H_a(\xi)\,\delta^{(n)}(b-x) \mid a,b \in K; n \in \mathbb{N}\}$ as an $\mathscr{F}_2$-basis. Finally, using the relation (4), the diagonal distribution module $\mathscr{D}_{x-\xi}\mathscr{F}$ can be equipped with the "left-focused" $\mathscr{F}_2$-basis $\mathscr{B}_x := \mathscr{B} \cup \{H^{(n)}(x-\xi), H_a(x)\,H^{(n)}(x-\xi) \mid a \in K, n \geq 0\}$ or with its "right-focused" companion $\mathscr{B}_\xi := \mathscr{B} \cup \{H^{(n)}(x-\xi), H_a(\xi)\,H^{(n)}(x-\xi) \mid a \in K, n \geq 0\}$. Now we can put together the tensorial and the diagonal distribution module to obtain the full *bivariate distribution module*

$$\mathscr{D}_2\mathscr{F} := \mathscr{D}_{x\xi}\mathscr{F} \oplus \mathscr{D}_{x-\xi}\mathscr{F} \tag{3.31}$$

as a direct sum of $\mathscr{P}_{x\xi}\mathscr{F}$-modules. We use the basis mentioned above to define operators on $\mathscr{D}_2\mathscr{F}$.

$\mathscr{D}_{x\xi}\mathscr{F}$ is already equipped with a duplex differential Rota-Baxter structure. Therefore, to extend derivation to $\mathscr{D}_2\mathscr{F}$, first we need to define derivation on $\mathscr{D}_{x-\xi}\mathscr{F}$. To do this, we use the $\mathscr{F}_2$-basis $\mathscr{B}_\xi := \mathscr{B} \cup \{H^{(n)}(x-\xi), H_a(\xi)\,H^{(n)}(x-\xi) \mid a \in K, n \geq 0\}$ of $\mathscr{D}_{x-\xi}\mathscr{F}$, by setting

$$\begin{aligned}
\eth_x H^{(n)}(x-\xi) &:= H^{(n+1)}(x-\xi),\\
\eth_x H_a(\xi) H^{(n)}(x-\xi) &:= H_a(\xi) H^{(n+1)}(x-\xi).
\end{aligned}$$

Then, the derivative map $\eth_x\colon \mathscr{D}_{x-\xi}\mathscr{F} \to \mathscr{D}_2\mathscr{F}$ is uniquely determined as an extension of $\eth_x\colon \mathscr{P}_{x\xi} \to \mathscr{D}_{x\xi} \subset \mathscr{D}_2\mathscr{F}$. Analogously, the derivative $\eth_\xi\colon \mathscr{D}_{x-\xi}\mathscr{F} \to \mathscr{D}_2\mathscr{F}$ is introduced as an extension of the derivative $\partial_\xi\colon \mathscr{P}_{x\xi} \to \mathscr{D}_{x\xi} \subset \mathscr{D}_2\mathscr{F}$ with $\eth_\xi H^{(n)}(x-\xi) := -H^{(n+1)}(x-\xi)$, via the $\mathscr{F}_2$-basis $\mathscr{B}_x$. Now, the resulting maps can be combined together with the existing derivation on $\mathscr{D}_{x\xi}\mathscr{F}$ which will provide a canonical derivation on the direct sum $\mathscr{D}_2\mathscr{F} := \mathscr{D}_{x\xi}\mathscr{F} \oplus \mathscr{D}_{x-\xi}\mathscr{F}$.

To introduce RB operaotors on the diagonal Heavisides and its derivatives, we define $\oint^x\colon \mathscr{D}_{x-\xi}\mathscr{F} \to \mathscr{D}_2\mathscr{F}$ using the $\mathscr{F}_2$-basis $\mathscr{B}_\xi$. If we view "foreign" factors as constants then we can settle the base case similar to (3.10)–(3.11):

$$\oint^x f(x)\,H(x-\xi) := \Big(\int_\xi^x f(x)\Big)\,H(x-\xi) + \Big(\int^\xi f(x)\Big)\bar{H}_0(\xi), \tag{3.32}$$

$$\text{likewise,}\quad \oint^\xi g(\xi)\,H(x-\xi) := \Big(\int_x^\xi g(\xi)\Big)\,H(x-\xi) + \Big(\int^x g(\xi)\Big)H_0(x), \tag{3.33}$$

with abbreviation $\int^\xi f(x) := \tau\big(\int^x f(x)\big) \in \mathscr{F}_\xi$ and $\int^x g(\xi) := \tau\big(\int^\xi g(\xi)\big) \in \mathscr{F}_x$. We have then $\int_\xi^x f(x) = (1-\tau)\int^x f(x)$ and $\int_x^\xi g(\xi) = (1-\tau)\int^\xi g(\xi)$. Here it is important to distinguish carefully $H(x-\xi) = \hat{H} \in \mathscr{D}_{x\xi}\mathscr{F}$ and $H(\xi - x) = 1 - \hat{H} \in \mathscr{D}_{x\xi}\mathscr{F}$ from $H_0(\xi) = 1 \otimes H_0 \in \mathscr{P}_{x\xi}\mathscr{F} \subset \mathscr{D}_{x\xi}\mathscr{F}$ and $H_0(x) = H_0 \otimes 1 \in \mathscr{P}_{x\xi}\mathscr{F} \subset \mathscr{D}_{x\xi}\mathscr{F}$. Furthermore, it should be noted that while the x-integral (3.32) corresponds to (3.10), the ξ-integral (3.33) corresponds to (3.11) since $H(x-\xi)$ behaves like $\bar{H}_x(\xi)$ from the ξ perspective; this is the reason for having $H_0(x)$ in (3.33) as opposed to $\bar{H}_0(\xi)$ in (3.32).

Before we complete our definition of RB operators on $\mathscr{D}_{x-\xi}\mathscr{F}$, we present the *diagonal piecewise extension* as the $\mathscr{P}_{x\xi}\mathscr{F}$-submodule

$$\mathscr{P}_{x-\xi}\mathscr{F} := \frac{\mathscr{P}_{x\xi}\mathscr{F} \otimes_K K\hat{H}}{\hat{Z}} \subset \mathscr{D}_{x-\xi}\mathscr{F}, \tag{3.34}$$

where $K\hat{H} \subset \hat{\mathscr{D}}K$ is the K-subspace generated by $\hat{H} = H(x-\xi) \in \hat{\mathscr{D}}K$. It is clear that $\mathscr{P}_{x-\xi}\mathscr{F}$ is free over $\mathscr{F}_2$ with basis $\mathscr{B}_x^0 := \{H(x-\xi), H_a(x)H(x-\xi) \mid a \in K\} \subset \mathscr{B}_x$ or again alternatively $\mathscr{B}_\xi^0 := \{H(x-\xi), H_a(\xi)H(x-\xi) \mid a \in K\} \subset \mathscr{B}_\xi$. In analogy to Definition 3.31, the *bivariate piecewise extension* $\mathscr{P}_2\mathscr{F} := \mathscr{P}_{x\xi}\mathscr{F} \oplus \mathscr{P}_{x-\xi}\mathscr{F}$ is a $\mathscr{P}_{x\xi}\mathscr{F}$-module consisting of tensorial and diagonal components. If $(\mathscr{F}, \int)$ is an ordinary shifted Rota-Baxter algebra over an ordered field K then $(\mathscr{P}_2\mathscr{F}, \oint^x, \oint^\xi)$ is a duplex Rota-Baxter algebra that extends $(\mathscr{F}_2, \int^x, \int^\xi)$.

We return now to the definition of the Rota-Baxter operators $\oint^x$ and $\oint^\xi$ on the bivariate distribution module $\mathscr{D}_{x\xi}\mathscr{F}$, which is in fact dictated by the Rota-Baxter axiom for modules. Having settled the base case in (3.32)–(3.33), we apply the reasoning of Remark 3.2 to continue the definition by setting

$$\oint^x f(x)\,H^{(n+1)}(x-\xi) := f(x)\,H^{(n)}(x-\xi) - \oint^x f'(x)\,H^{(n)}(x-\xi), \tag{3.35}$$

$$-\oint^\xi g(\xi)\,H^{(n+1)}(x-\xi) := g(\xi)\,H^{(n)}(x-\xi) - \oint^\xi g'(\xi)\,H^{(n)}(x-\xi) \tag{3.36}$$

for all $f(x) \in \mathscr{F}_x, g(\xi) \in \mathscr{F}_\xi$ and $n \in \mathbb{N}$. Note the distinct sign in (3.36), due to the fact that $\eth_\xi = -\eth_x$ on the diagonal distribution module $\mathscr{D}_{x-\xi}\mathscr{F}$.

One can also define evaluation operators $\acute{E}_a^x\colon \mathscr{D}_2\mathscr{F} \to \mathscr{D}_\xi\mathscr{F}$ and $\acute{E}_a^\xi\colon \mathscr{D}_2\mathscr{F} \to \mathscr{D}_x\mathscr{F}$ by

$$\acute{E}_a^x H(x-\xi) := \bar{H}(\xi - a), \qquad \acute{E}_a^\xi H(x-\xi) := H(x-a). \tag{3.37}$$

For evaluating diagonal Diracs, we use again the analogy to our earlier definition $\acute{E}_a\, \delta_\xi^{(k)} := 0$ set up earlier (see the paragraph before Theorem 3.2). Thus setting

$$\acute{E}_a^x\, \delta^{(k)}(x-\xi) = 0 \qquad \acute{E}_a^\xi\, \delta^{(k)}(x-\xi) = 0, \tag{3.38}$$

completes the definition of $\acute{E}_a^x$ and $\acute{E}_a^\xi$ on the diagonal distribution module $\mathscr{D}_{x-\xi}\mathscr{F} \subset \mathscr{D}_2\mathscr{F}$. If $(\mathscr{F}, \int, \partial)$ be an ordinary shifted integro-differential algebra. Then the bivariate distribution module $(\mathscr{D}_2\mathscr{F}, \eth_x, \eth_\xi, \oint^x, \oint^\xi)$ is a duplex differential Rota-Baxter module containing two isomorphic copies $(\mathscr{D}_x\mathscr{F}, \eth_x, \oint^x)$ and $(\mathscr{D}_\xi\mathscr{F}, \eth_\xi, \oint^\xi)$ of the given $(\mathscr{F}, \partial, \int)$. As a duplex Rota-Baxter module, $\mathscr{D}_2\mathscr{F}$ extends $\mathscr{P}_2\mathscr{F}$.

3.4.1 Applications to boundary problems

We have developed a rather modest algebraic structure to handle Heavisides and Diracs. This construction stems from the fact that the Green's function can also be a distribution. Since our theory concerns the treatment of boundary value problems, we can think of three essential applications of our new theory:

- Express the Green's functions in the algebraic language of Heavisides and Diracs.
- Characterisation of the Green's function by the corresponding distributional differential equation known from analysis.
- Allowing a piecewise function to be a forcing function for a boundary problem.

The first item is the generalisation of the structure theorem mentioned in the previous chapter. We shall provide a purely algebraic framework for accommodating the Green's function. In particular, we present that the Green's function of a regular Stieltjes boundary problem that lies in the bivariate distributional module $\mathscr{D}_2\mathscr{F}$, while for well-posed problems in the bivariate piecewise extension $\mathscr{P}_2\mathscr{F}$.

The procedure is similar to that in the structure theorem (Thm. 2.3) of the previous chapter. Here, we need to interpret Heavisides and Diracs in the sense of $\mathscr{D}_2\mathscr{F}$. When $a < b$ and $a < x$, we write $[a \le \xi \le b] := H(\xi - a)\,\bar{H}(\xi - b)$ for the characteristic function of the interval $[a,b]$ and similarly for $[a,x]$ we write $[a \le \xi \le x] := H(\xi - a)\,H(x-\xi)$. To take the relative order in account, we can define $[a \le \xi \le b]_\pm := [a \le \xi \le b] - [b \le \xi \le x]$ and $[a \le$

$\xi \leq x]_{\pm} := [a \leq \xi \leq x] - [x \leq \xi \leq a]$ so that we do not need to worry about by the ordering of the field elements. With this notation, one can immediately check that

$$\begin{aligned}
[a \leq \xi \leq b]_{\pm} &:= [a \leq \xi \leq b] - [b \leq \xi \leq x] \\
&= H(\xi - a)(1 - H(\xi - b)) - H(\xi - b)(1 - H(\xi - a)) \\
&= H(\xi - a)H(\xi - b)
\end{aligned}$$

Similarly, $[a \leq \xi \leq x]_{\pm} = H(x - \xi) + H(\xi - a) - 1$. With this we are equipped with Heavisides to replace characteristic functions. In the case of Diracs, the rule is

$$\oint_{\alpha}^{\beta} \delta^{(i)}(x - \xi)\, f(\xi) = f^{(i)}(x),$$

which is written as

$$\int_{-\infty}^{\infty} \delta^{(i)}(\xi - x)\, f(\xi)\, d\xi = (-1)^i f^{(i)}(x)$$

in analysis.

Remark 3.4. In our original formulation [41], there was an extra *alternating sign* present erroneously which we have corrected here.

$G \in \mathscr{F}_{\Phi}[\partial, \int]$	$G_{x\xi} \in \mathscr{D}_2\mathscr{F}$
$u\,\partial^i$	$u(x)\,\delta^{(i)}(x - \xi)$
$u \int v$	$u(x)\,v(\xi)\,[o \leq \xi \leq x]_{\pm}$
$u\,\mathrm{E}_a \partial^i$	$(-1)^i u(x)\,\delta^{(i)}(\xi - a)$
$u\,\mathrm{E}_a \int v$	$u(x)\,v(\xi)\,[o \leq \xi \leq a]_{\pm}$

Table 3.1 Extraction Map $\eta : \mathscr{F}_{\Phi}[\partial, \int] \to \mathscr{D}_2\mathscr{F}$

To show that $g(x,\xi) \in \mathscr{D}_2\mathscr{F}$, one needs to go through the elements of the Table 3.1 row by row. The action of the element on the left-hand side must coincides with the action of the corresponding right-hand side element. For example, the left-hand side of the first row yields $Gf(x) = u(x)\,f^{(i)}(x)$, whereas the right-hand side provides $u(x) \oint_{\alpha}^{\beta} f(\xi)\,\delta^{(i)}(x - \xi)$. Therefore, one needs to show that

$$f^{(i)}(x) = \oint_{\alpha}^{\beta} f(\xi)\,\delta^{(i)}(x - \xi) \;\text{ on } [\alpha, \beta]. \tag{3.39}$$

Theorem 3.3. *Let $\mathscr{F}$ be an ordinary shifted integro-differential algebra over any ordered field K, and let $\eta : \mathscr{F}_{\Phi}[\partial, \int] \to \mathscr{D}_2\mathscr{F}$ be as in Table 3.1. Choose $\alpha, \beta \in K$ with $\alpha \leq a_1 <$*

$\cdots < a_k \leq \beta$ then we have

$$Gf(x) = \oint_{\alpha}^{\beta} g(x,\xi)\, f(\xi) \quad \in \quad \mathscr{F}_x \tag{3.40}$$

on $[\alpha,\beta]$, for all $f \in \mathscr{F}$ and $G \in \mathscr{F}_{\Phi}[\partial,\textstyle\int]$ with extraction $g(x,\xi) := G_{x\xi}$. If G is the Green's operator of a regular Stieltjes boundary problem, $g(x,\xi)$ is thus its Green's function.

Now for the second part of the result—$g(x,\xi) \in \mathscr{P}_2\mathscr{F}$ for well-posed boundary problems—we use the previous result from the structure theorem that the Green's function splits as

$$g(x,\xi) = \tilde{g}(x,\xi) + \hat{g}(x,\xi)$$

with a functional part $\tilde{g}(x,\xi) \in \mathscr{P}_2\mathscr{F}$ and a distributional part $\hat{g}(x,\xi) \in \mathscr{D}_2\mathscr{F} \setminus \mathscr{P}_2\mathscr{F}$. Since $\hat{g}(x,\xi) = 0$ or $\hat{G} = 0$ for well-posed problems, we obtain $g(x,\xi) \in \mathscr{P}_2\mathscr{F}$.

Proposition 3.5. *Let $\mathscr{F}$ and $\eta\colon \mathscr{F}_{\Phi}[\partial,\textstyle\int] \to \mathscr{D}_2\mathscr{F}$ be as in Theorem 3.3. If the regular boundary problem $(T,\mathscr{B})$ is well-posed, then we have $g(x,\xi) \in \mathscr{P}_2\mathscr{F}$ for the Green's function $g(x,\xi) = G_{x\xi}$ extracted from its Green's operator $G = (T,\mathscr{B})^{-1}$.*

The second item of interest in this section is to characterize the Green's function and Dirac by the corresponding differential equation known from analysis. To do so, first we show that the function $\delta(x-\xi)$ has essential properties required from analysis and then in the similar fashion we extend our result for the Green's function. We call a differential algebra $(\mathscr{F},\partial)$ *strongly ordinary* if $\dim \operatorname{Ker}(T)$ is finite for any differential operator $T \in \mathscr{F}[\partial]$. We shall point out that all the usual examples of ordinary differential algebras in analysis are strongly ordinary.

To recover Dirac distributions uniquely, we need to define *degenerate* functions; this definition plays the role of the *fundamental lemma of the variational calculus*.

Definition 3.15. We call a function $\o \in \mathscr{F}$ *degenerate on* $[\alpha,\beta]$ if

$$\textstyle\int_{\alpha}^{\beta} \o(\xi)\, f(\xi) = 0$$

for all $f \in \mathscr{F}$.

In the same vein, we call $k(x,\xi) \in \mathscr{D}_2\mathscr{F}$ *nondegenerate* if there is no degenerate function $\o(\xi) \in \mathscr{F}_{\xi}$. If $(\mathscr{F},\textstyle\int,\partial)$ is a strongly ordinary shifted integro-differential algebra and choose any bivariate distribution $k(x,\xi) \in \mathscr{D}_2\mathscr{F}$ that is nondegenerate on $[\alpha,\beta]$. Given $\oint_{\alpha}^{\beta} k(x,\xi)\, f(\xi) = f(x)$ on $[\alpha,\beta]$ for all $f \in \mathscr{F}$, then necessarily $k(x,\xi) = \delta(x-\xi)$. Now, we can say the following and have achieved our second goal.

Theorem 3.4. *Let $(\mathscr{F}, \int, \partial)$ be a strongly ordinary shifted integro-differential algebra and $(T, \mathscr{B})$ a regular Stieltjes boundary problem over $(\mathscr{F}, \partial, \int)$. Then there exists a bivariate distribution $g(x,\xi) \in \mathscr{D}_2\mathscr{F}$ such that*

$$\boxed{\begin{aligned} &T_x\, g(x,\xi) = \delta(x-\xi), \\ &\beta_x\, g(x,\xi) = 0 \quad (\beta \in \mathscr{B}). \end{aligned}} \tag{3.41}$$

Moreover, this $g(x,\xi)$ coincides with the Green's function of Theorem 3.3.

Our last goal—allowing a *piecewise forcing function* for boundary problems—is immediate but with a reservation. Since the multiplication of Dirac and Heaviside is undefined, we need to restrict ourselves to well-posed boundary problems so that $g(x,\xi) \in \mathscr{P}_2\mathscr{F}$ (see Prop.3.5). Now we can easily apply the same method for handling piecewise functions. We end this chapter by iterating this result below.

If $(\mathscr{F}, \int, \partial)$ is a strongly ordinary shifted integro-differential algebra and let $(T, \mathscr{B})$ be a well-posed Stieltjes boundary problem, then (1.7) admits exactly one solution $u \in \mathscr{P}\mathscr{F}$ for any given forcing function $f \in \mathscr{P}\mathscr{F}$. If $G \in \mathscr{F}_\Phi[\partial, \int]$ is the corresponding Green's operator with the Green's function $g(x,\xi)$, we can compute the solution either via $u = Gf$ or via $u(x) = \oint_\alpha^\beta g(x,\xi)\, f(\xi)$.

3.5 Conclusions

The treatment of boundary problems (linear ordinary differential equations) is a major application area for our algebraic approach to piecewise smooth functions and Dirac distributions. This algebraic treatment is only a starting point for algebraic characterization of distributions and Heavisides. We established their relationship with the only link among them $H_a' = \delta_a$. The multiplication of Heavisides (Eq. (3.7)) reflects an order structure in the ground field whereas multiplication of distributions is not allowed (Rmk. 3.3).

We generalised the algorithm (Thm. 2.3) of extracting the Green's function from the Green's operators to bivariate distributions over ordinary shifted integro-differential algebras (Thm. 3.3). We showed that the Green's function satisfies an algebraic version of well-known distributional differential equations with $\delta(x-\xi)$ on the right-hand side (Thm. 3.4). Finally, we confirmed that the corresponding Green's operator of an arbitrary well-posed boundary problem may actually be applied to piecewise functions.

In future, one might investigate allowing *piecewise continuous coefficients* in the differential operator T. However, it would be difficult to adapt such a setting in our present approach. This will require the ground algebra $\mathscr{F}$ to be a differential Rota-Baxter algebra rather than an integro-differential algebra, and it will result in losing the strong Rota-Baxter axiom. As a result, the Green's operator/functions cannot be computed in the usual way—it might need a different justification.

Chapter 4

Symmetries and Generalised Symmetries

Integrable nonlinear equations play an important role in understanding coherent structures and patterns arising in many phenomena in nature. Moreover, they are interesting by themselves as they possess rich algebraic and analytic structures. One of the criteria used to establish integrability is through the existence of infinitely many commuting symmetries. *Master symmetry* provides a tool which guarantees this existence for these equations and thus helps in constructing hierarchies of integrable evolution equations.

This chapter is arranged as follows. The goal is to introduce notations, terms and concepts in order to understand partial differential equations via the master symmetry approach that is carried out in the next chapter. We begin by defining the one parameter group of transformations (Eq. (4.1)) and the infinitesimal form of a group (Eq. (4.6)). In the next section, we define a generalised vector field (Def. 4.2) in n variables and some results around vector fields—prolongation (Eq. (4.17)) and characteristic (Def. 4.3). Then, we specialise our theory to differential functions $u = u(x,t)$ of two independent variables. Later, in the following Section 4.3, we compute *generalised symmetries* for Burgers' equation. Finally, we end the chapter by defining a Lie bracket (Def. 4.7) on differential functions so that we can construct Lie algebras, which will be exploited in the next chapter.

4.1 A brief introduction

There are many ingenious techniques/tricks available for solving differential equations, but they usually works for a limited class of problems. Fortunately, it was discovered that these techniques essentially exploit *symmetries* of differential equations. The concept of one-parameter transformation groups which leave the differential equation invariant provides the unified understanding of special solution techniques. This subject was initiated by Sophus Lie over a century ago, who put forward many of the fundamental ideas behind symmetry methods. Let us begin by defining a one-parameter group of transformation since symmetries[1] of a differential equation form such a group.

In the case of (x, y) plane, we say that the transformation

$$x_1 = f(x, y, \varepsilon), \quad y_1 = g(x, y, \varepsilon) \tag{4.1}$$

is a one-parameter group of transformation if it satisfies the following conditions:

- *Identity*: $\varepsilon = 0$ characterizes the identity transformation,

$$x = f(x, y, 0), \quad y = g(x, y, 0).$$

- *Inverse*: $-\varepsilon$ characterizes the inverse transformation

$$x = f(x_1, y_1, -\varepsilon), \quad y = g(x_1, y_1, -\varepsilon).$$

- *Closure*: Product of two transformation is characterised by the sum of parameters, meaning, if

$$x_2 = f(x_1, y_1, \delta), \quad y_2 = g(x_1, y_1, \delta),$$

  then it can be rewritten as

$$x_2 = f(x, y, \varepsilon + \delta), \quad y_2 = g(x, y, \varepsilon + \delta).$$

We remark that the associativity law follows from the closure property.

Example 4.1.1. The following transformation forms a one-parameter group.

- Translation group

$$x_1 = x, \quad y_1 = y + \varepsilon. \tag{4.2}$$

- Stretching group

$$x_1 = e^{\varepsilon} x, \quad y_1 = e^{\varepsilon} y. \tag{4.3}$$

[1] This is not always the case. For example, one can consider discrete symmetries which fails to form a one-parameter group of transformation.

- Rotation group

$$x_1 = x\cos\varepsilon - y\sin\varepsilon, \quad y_1 = x\sin\varepsilon + y\cos\varepsilon. \tag{4.4}$$

- Projective group

$$x_1 = \frac{x}{1-\varepsilon x}, \quad y_1 = \frac{y}{1-\varepsilon x}. \tag{4.5}$$

The functions $f(x,y,\varepsilon)$ and $g(x,y,\varepsilon)$ in (4.1) are called the *global form* of the group. We obtain the so-called *infinitesimal form* of a group by Taylor expansion of (4.1). Since $\varepsilon = 0$ gives the identity element, the expansion yields

$$x_1 = x + \varepsilon \frac{dx_1}{d\varepsilon}\Big|_{\varepsilon=0} + \mathbf{O}(\varepsilon^2), \quad y_1 = y + \varepsilon \frac{dy_1}{d\varepsilon}\Big|_{\varepsilon=0} + \mathbf{O}(\varepsilon^2). \tag{4.6}$$

We introduce functions

$$\frac{dx_1}{d\varepsilon}\Big|_{\varepsilon=0} = \xi(x,y) \quad \text{and} \quad \frac{dy_1}{d\varepsilon}\Big|_{\varepsilon=0} = \eta(x,y)$$

then (4.6) can be rewritten as

$$x_1 = x + \varepsilon\xi(x,y) + \mathbf{O}(\varepsilon^2), \quad y_1 = y + \varepsilon\eta(x,y) + \mathbf{O}(\varepsilon^2), \tag{4.7}$$

which is the *infinitesimal form* of the group. The vector (ξ,η) is the *tangent vector* at the point (x,y), to the curve described by the transformed points (x_1,y_1). The crucial property of a one-parameter transformation group is that given the infinitesimal form, one can deduce the global form by solving the following system

$$\frac{dx_1}{d\varepsilon} = \xi(x,y), \quad \frac{dy_1}{d\varepsilon} = \eta(x,y), \tag{4.8}$$

with initial conditions

$$x_1(0) = x, \quad y_1(0) = y. \tag{4.9}$$

The converse is also true since there is a unique local solution of the first order equation (4.8) subject to the initial conditions (4.9).

Example 4.1.2. For the rotation group (4.4), we have the global form

$$x_1 = x\cos\varepsilon - y\sin\varepsilon, \quad y_1 = x\sin\varepsilon + y\cos\varepsilon.$$

To obtain the infinitesimal form, we compute

$$\frac{dx_1}{d\varepsilon} = -x\sin\varepsilon - y\cos\varepsilon \quad \text{and} \quad \frac{dy_1}{d\varepsilon} = x\cos\varepsilon - y\sin\varepsilon$$

subjected to $\varepsilon = 0$, which yields

$$\xi(x,y) = -y, \quad \eta(x,y) = x,$$

and therefore, the infinitesimal form is

$$x_1 = x - \varepsilon y + \mathbf{O}(\varepsilon^2), \quad y_1 = y + \varepsilon x + \mathbf{O}(\varepsilon^2). \tag{4.10}$$

To retrieve back the global form we solve the initial value problem

$$\begin{aligned} &\frac{dx_1}{d\varepsilon} = -y_1, \quad \frac{dy_1}{d\varepsilon} = x_1 \quad \text{with} \\ &x_1(0) = x, \quad y_1(0) = y. \end{aligned}$$

For easy computation, we introduce the complex variable $z = x + iy$ and $z = x_1 + iy_1$, then

$$\begin{aligned} \frac{dz_1}{d\varepsilon} &= \frac{dx_1}{d\varepsilon} + i\frac{dy_1}{d\varepsilon}, \\ &= -y_1 + ix_1 = iz, \end{aligned}$$

which upon solving gives

$$z_1 = e^{i\varepsilon}z.$$

On comparing the corresponding parts—real and imaginary—after applying the formula $e^{i\varepsilon} = \cos\varepsilon + i\sin\varepsilon$ gives back the original global form.

To find symmetries of a differential equation, one needs to satisfy the corresponding symmetry condition. Let us make this clear for a general first order differential equation.

Definition 4.1. A general first order equation

$$\frac{dy}{dx} = f(x,y) \tag{4.11}$$

is said to be *invariant* under the transformation (x_1, y_1), if it satisfies the following the *symmetry condition*

$$\frac{dy_1}{dx_1} = f(x_1, y_1). \tag{4.12}$$

Collection of such transformations form the corresponding *symmetry group* or simply one can say, a symmetry group of the system of differential equations is a one parameter transformation group which maps any solution to another solution of the equation.

In the next example we show that how one can use the knowledge of symmetry to solve a first order differential equation.

Example 4.1.3. Consider the first order differential equation

$$\frac{dy}{dx} = \frac{y^2}{x^3} + \frac{2y}{x} + x. \tag{4.13}$$

If we apply the transformation

$$x_1 = xe^{\alpha}, \quad y_1 = ye^{\beta} \quad \text{(for real parameters } \alpha \text{ and } \beta\text{)},$$

to Equation (4.13) then it gives

$$\begin{aligned} e^{\alpha-\beta}\frac{dy_1}{dx_1} &= e^{3\alpha-2\beta}\frac{y_1^2}{x_1^3} + 2e^{\alpha-\beta}\frac{y_1}{x_1} + e^{-\alpha}x_1, \\ \implies \frac{dy_1}{dx_1} &= e^{2\alpha-\beta}\frac{y_1^2}{x_1^3} + 2\frac{y_1}{x_1} + e^{\beta-2\alpha}x_1. \end{aligned}$$

Therefore, Equation (4.13) is *invariant* under this transformation iff $2\alpha = \beta$, that is,

$$x_1 = xe^{\alpha}, \quad y_1 = xe^{2\alpha}.$$

We observe that

$$\frac{y_1}{x_1^2} = \frac{ye^{2\alpha}}{x^2e^{2\alpha}} = \frac{y}{x^2}.$$

Hence, the function $\nu(x) = \frac{y}{x^2}$ is an invariant of the Equation (4.13). Substituting $y(x) = x^2\nu(x)$ in (4.13) yields

$$\begin{aligned} 2xv + x^2\frac{dv}{dx} &= x\left(v^2 + 2v + x\right), \\ \implies \frac{dv}{dx} &= \frac{1+v^2}{x}, \\ \implies v(x) &= \tan\left(\ln x + c\right), \end{aligned}$$

where c is the constant of integration and thus the general solution is

$$y(x) = x^2 \tan(\ln x + c).$$

Remark 4.1. In general, a symmetry group of an ordinary differential equation leads to a simplification of the equation. If the equation is of first order then it becomes separable and so is solvable by quadrature. For higher order, use of an invariant typically allows a reduction in the order of the equation by one.

More details on this subject can be found in [22, 18], where authors have well explained the group method, especially for ODEs. Since our main focus is to find symmetries for partial differential equations, so in the next section we begin by describing vector fields and their prolongation. They serve as essential tools in the classical Lie group method.

4.2 Vector fields and prolongation

A general system of partial differential equations involves p independent variables on the Euclidean space $X \cong \mathbb{R}^p$ and q dependent variables on $U \cong \mathbb{R}^q$. The total space is the Euclidean space $E = X \times U \cong \mathbb{R}^{p+q}$ coordinated by the independent and dependent variables. A *differntial function* is defined on this total space. In particular, a function $P(x, u^{(n)})$ is called a differential function if it is a smooth function of x, u and derivatives of u up to order n. Through out this chapter, we only consider the scalar case with $p = 2$ and $q = 1$, i.e, $u = u(x,t)$. We will denote such a function by $P[u]$ where the square brackets will serve us to remind that P depends on x, u and derivatives of u. We let $\mathscr{A}$ denote the space of all such differentials functions. Note that $\mathscr{A}$ is an algebra over $\mathbb{R}$, moreover, we can turn this in to a differential algebra with the derivation defined as below.
Given $P[u]$, the i-th total derivative of P with respect to x and t is defined as

$$D_x P = \frac{\partial P}{\partial x} + u_x \frac{\partial P}{\partial u} + u_{xx} \frac{\partial P}{\partial u_x} + u_{xt} \frac{\partial P}{\partial u_t} + \cdots, \tag{4.14}$$

$$D_t P = \frac{\partial P}{\partial t} + u_t \frac{\partial P}{\partial u} + u_{tt} \frac{\partial P}{\partial u_t} + u_{xt} \frac{\partial P}{\partial u_x} + \cdots, \tag{4.15}$$

respectively. For instance, if $P = x u_x u_{xt}$ then

$$D_x P = u_x u_{xt} + x u_{xt} u_{xx} + x u_x u_{xxt} \quad \text{and} \quad D_t P = x u_{xt}^2 + x u_x u_{xtt}.$$

Now, we are ready to define one of the main tool that will be required for our investigation of integrability.

Definition 4.2. A *generalised vector field* is defined to be a formal expression of the form

$$\mathbf{v} = \xi^1[u]\frac{\partial}{\partial x} + \xi^2[u]\frac{\partial}{\partial t} + \varphi[u]\frac{\partial}{\partial u}, \tag{4.16}$$

where $\xi^i[u]$ and $\varphi[u]$ are smooth differential functions. If all $\xi^i = \xi^i(x,t,u)$ and $\varphi = \varphi(x,t,u)$, that is they depends only on x,t and u then it is called a *geometric vector field*.

For every generalised vector field $\mathbf{v}$ there is an induced function $\mathtt{pr}^{(n)}\mathbf{v}$ called the n-th prolongation which is given by the *prolongation formula* (Theorem 2.36, [34])

$$\mathtt{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_J \varphi^J[u]\frac{\partial}{\partial u_J}, \tag{4.17}$$

where, for $J = (j_i, j_2)$; $1 \leq j_k \leq 2$, $1 \leq k \leq n$ and n is the highest order appearing in (4.17). The coefficient functions φ^J of $\mathtt{pr}^{(n)}\mathbf{v}$ are given by the following formula:

$$\varphi^J[u] = D_J\left(\varphi - \xi^1\frac{\partial u}{\partial x} - \xi^2\frac{\partial u}{\partial t}\right) + \sum_{i=1}^{2}\xi^i\frac{\partial u_J}{\partial x^i}. \tag{4.18}$$

For more details, please see the Theorem 2.36 in [34].

In the above formula, if we do not restrict the value of n and let $n \geq 0$, then the corresponding prolongation is called *infinite prolongation* (or *prolongation* for short). This formal infinite sum is written as

$$\mathtt{pr}\,\mathbf{v} = \xi^1[u]\frac{\partial}{\partial x} + \xi^2[u]\frac{\partial}{\partial t} + \sum_J \varphi^J[u]\frac{\partial}{\partial u_J}. \tag{4.19}$$

Example 4.2.1. Let us develop our understanding on the prolongation formula when $p = q = 1$, i.e, $u = u(x)$. Let

$$\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \tag{4.20}$$

be a vector field then by (4.17)

$$\mathtt{pr}^{(2)}\mathbf{v} = \mathbf{v} + \varphi^x\frac{\partial}{\partial u_x} + \varphi^{xx}\frac{\partial}{\partial u_{xx}},$$

with $\varphi^x = D_x(x - (-uu_x)) - uu_{xx} = 1 + u_x^2$ and $\varphi^{xx} = 3u_x u_{xx}$. Upon substitution, we obtain

$$\mathrm{pr}^{(2)}\mathbf{v} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}}.$$

Example 4.2.2. We repeat a similar example as before but this time, let us consider a more general geometric vector field

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u}. \tag{4.21}$$

Then its prolongation is

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + \varphi^x\frac{\partial}{\partial u_x} + \varphi^t\frac{\partial}{\partial u_t}$$

with $\varphi^x = D_x(\varphi - (\xi u_x + \tau u_t)) + \xi u_{xx} + \tau u_{xt}$ and $\varphi^t = D_x(\varphi - (\xi u_x + \tau u_t)) + \xi u_{xt} + \tau u_{tt}$. After expanding, we obtain

$$\begin{aligned}\varphi^x =& \varphi_x + \varphi_u u_x - \xi_x u_x - \xi_u u_x^2 - \xi u_{xx} - \tau_x u_t - \tau_u u_t u_x - \tau u_{xt} + \xi u_{xx} + \tau u_{xt}\\ =& \varphi_x + (\varphi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t\end{aligned}$$

and $\varphi^t = \varphi_t - \xi_t u_x + (\varphi_u - \tau_t)u_t - \xi_u u_x u_t - \tau_u u_t^2$.

Similarly, we can compute the second prolongation,

$$\mathrm{pr}^{(2)}\mathbf{v} = \mathbf{v} + \varphi^{xx}\frac{\partial}{\partial u_x x} + \varphi^{xt}\frac{\partial}{\partial u_{xt}} + \varphi^{tt}\frac{\partial}{\partial u_{tt}}$$

with

$$\varphi^{xt} = D_x D_t(\varphi - (\xi u_x + \tau u_t)) + \xi u_{xxt} + \tau u_{xtt}. \tag{4.22}$$

In practice, it is easy to work with so called *evolutionary vector fields*, specifically with their corresponding *characteristic*. We introduce them in the definition below.

Definition 4.3. For a generalised vector field $\mathbf{v}$ given by (4.16), we define the corresponding *evolutionary vector field*

$$\mathbf{v}_Q = Q[u]\frac{\partial}{\partial u}, \tag{4.23}$$

where $Q[u] \in \mathscr{A}$ is called its *characteristic* with

$$Q = \varphi - \xi^1\frac{\partial u}{\partial x} - \xi^2\frac{\partial u}{\partial t}, \tag{4.24}$$

Now, we notice that the prolongation takes a simpler form for $\mathbf{v}_Q$ compared to (4.17),

$$\mathrm{pr}\,\mathbf{v}_Q = \sum_J D_J Q[u] \frac{\partial}{\partial u_J}. \tag{4.25}$$

Example 4.2.3. Let $\mathbf{v} = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2+2t)\frac{\partial}{\partial u}$ be a generalised vector field then we compute the characteristic using (4.24). Since $q = 1$, we simply obtain $Q = -(x^2+2t) - 4txu_x - 4t^2u_t$, which further implies that

$$\mathbf{v}_Q = (-x^2 - 2t - 4txu_x - 4t^2u_t)\frac{\partial}{\partial u} \tag{4.26}$$

is the associative evolutionary representative of $\mathbf{v}$.

Similarly, we find the corresponding evolutionary vector field of the vector field (4.21), $\mathbf{v}_Q = (\varphi - (\xi u_x + \tau u_t))\frac{\partial}{\partial u}$ with characteristic $Q = \varphi - (\xi u_x + \tau u_t)$.

Remark 4.2. It is always possible to go from $\mathbf{v} \to \mathbf{v_Q}$ but the reverse direction is possible only for the geometric vector fields ((4.21)). Also, notice that characteristic functions of evolutionary vector fields forms a Lie algebra, see Definition 4.7

It is now clear that we can always write a vector field in the evolutionary form.

4.3 Generalised symmetries

Definition 4.4. Given a system of differential equations

$$\triangle_\nu\,[u] = \triangle_\nu\,(x, u^{(n)}) = 0,$$

a generalised vector field $\mathbf{v}$ is a *generalised symmetry* if and only if

$$\mathrm{pr}\,\mathbf{v}[\triangle_\nu] = 0, \tag{4.27}$$

for every smooth solution of the equation, where $\nu = 1, \cdots, l$. If there exists infinitely many symmetries then we call that system to be *integrable*.

We already mentioned in the previous section that it is easy to work with evolutionary vector fields rather than generalised vector fields, and the following proposition illustrates how evolutionary vector fields can be exploited.

Proposition 4.1. *A generalised vector field* $\mathbf{v}$ *is a symmetry of a system of differential equations if and only if its evolutionary representative* $\mathbf{v}_Q$ *is a symmetry.*

Proof. If we substitute (4.18) in (4.19) and expand, then we obtain

$$\operatorname{pr}\mathbf{v} = \sum_{i=1}^{p}\xi^i[u]\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_J\left(D_J\left(\varphi_\alpha - \sum_{i=1}^{p}\xi^i\frac{\partial u^\alpha}{\partial x^i}\right) + \sum_{i=1}^{q}\xi^i\frac{\partial u_J^\alpha}{\partial x^i}\right)\frac{\partial}{\partial u_J^\alpha} \tag{4.28}$$

$$= \sum_{i=1}^{p}\xi^i[u]\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_J\left(D_J\left(\varphi_\alpha - \sum_{i=1}^{p}\xi^i\frac{\partial u^\alpha}{\partial x^i}\right)\right)\frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^{q}\xi^i\frac{\partial u_J^\alpha}{\partial x^i}\frac{\partial}{\partial u_J^\alpha} \tag{4.29}$$

$$= \sum_{\alpha,J} D_j Q_\alpha \frac{\partial}{\partial u_J^\alpha} + \sum_{i=1}^{p}\xi^i\left(\frac{\partial}{\partial x^i} + \sum_{\alpha,J}\frac{\partial u_J^\alpha}{\partial x^i}\frac{\partial}{\partial u_J^\alpha}\right), \tag{4.30}$$

where the last equality is obtained by rearranging the terms and using 4.24 and Definition 4.2. Now, if we compare the above result with (4.25) then we deduce that

$$\operatorname{pr}\mathbf{v} = \operatorname{pr}\mathbf{v_Q} + \sum_{i=1}^{p}\xi^i D_i. \tag{4.31}$$

Since $D_i \triangle_\nu = 0$ on all solutions of $\triangle_\nu$, we obtain $\operatorname{pr}\mathbf{v}[\triangle_\nu] = 0$ if and only if $\operatorname{pr}\mathbf{v_Q}[\triangle_\nu] = 0$ and hence, follows the result. □

Now, we demonstrate briefly a general method to compute generalised symmetries. For more information, please see the Section 5.1 of [34]. It proceeds in the same way as one computes geometrical symmetries which is well explained in the Section 2.4 of [34]. Here, we start with the evolutionary representative $\mathbf{v}_Q$ of the generalised symmetry $\mathbf{v}$. This gives us the advantage of reducing unknown function from $p+q$ to q, simultaneously giving a simpler computation for the prolongation of $\mathbf{v}_Q$. Before we start the computation, we fix the order of derivatives of the function u in Q. Consider an example of scalar case with $p = 2$ and $q = 1$. To compute a second order symmetry we assume characteristic $Q = Q(x,u,u_x,u_t,u_{xx},u_{xt},u_{tt})$. Before we move to examples, we shall define the type of equations we want to investigate. From now on, we will denote $\dfrac{\partial^n u}{\partial x^n}$ by u_n.

Definition 4.5 ([52, Section 3])**.** An equation of the form

$$u_t = K[u]$$

is called an *evolution equation*, where $K[u]$ is a differential function in $\mathscr{A}$. We call an equation $(n+1)-$*dimensional*, if it involves n spatial variables and a time variable t.

The below example is $(1+1)$-dimensional, that is $u = u(x,t)$ and we will also consider an example of $(2+1)$-dimensional system in the next chapter where $u = u(x,y,t)$.

The goal of this section is to demonstrate the complexity of computing symmetries for a given problem without getting in to much details.

Example 4.3.1. Consider the following non-linear wave equation

$$u_t = uu_1.$$

Notice that any t-derivatives of u can be replaced by their corresponding $x-$ derivatives of u. For example, by differentiating the given equation with respect to x, one can replace u_{xt} by $uu_2 + u_1^2$. To compute the second order generalised symmetry v_Q in the evolutionary form, we suppose $v_Q = Q[u]\frac{\partial}{\partial u}$ with $Q = Q(x,t,u,u_1,u_2)$. We find the prolongation of v_Q

$$\mathtt{pr}^{(1)}\mathbf{v}_Q = Q\frac{\partial}{\partial u} + D_x Q\frac{\partial}{\partial u_1} + D_t Q\frac{\partial}{\partial u_t},$$

then by Symmetry condition (4.27), we get,

$$D_t Q = uD_x Q + u_1 Q \tag{4.32}$$

where all the total derivatives can be computed using (4.14), for example,

$$D_t Q = \frac{\partial Q}{\partial t} + uu_1\frac{\partial Q}{\partial u} + (u_1^2 + uu_2)\frac{\partial Q}{\partial u_1} + (3u_1u_2 + uu_3)\frac{\partial Q}{\partial u_2}.$$

Notice that all the occurrence of t-derivatives of u have been replaced by their corresponding x-derivatives of u. Upon substitution in (4.32) and after simplification we obtain

$$\frac{\partial Q}{\partial t} - u\frac{\partial Q}{\partial x} + u_1^2\frac{\partial Q}{\partial u_1} + 3u_1u_2\frac{\partial Q}{\partial u_2} = u_1 Q.$$

Corresponding *characteristic system* of this first order partial differential equation is,

$$\frac{\partial t}{\partial s} = 1, \frac{\partial x}{\partial s} = -u, \frac{\partial u}{\partial s} = 0, \frac{\partial u_1}{\partial s} = u_1^2, \frac{\partial u_2}{\partial s} = 3u_1u_2.$$

This gives that, Q is of the form

$$Q = u_1 R\left(x + tu, u, t + \frac{1}{u_1}, \frac{u_2}{u_1^3}\right).$$

If we focus only on the *projectable symmetries*, in which the action on the independent variables does not depend on the dependent variables, meaning the characteristic takes the form $Q = \varphi(x,t,u) - \xi^1(x,t)u_1 - \xi^2(x,t)uu_1$ (see (4.24)), then one can find 8 such functions which spans this subgroup of symmetries. Here it looks like

$$Q = u_1\left(t\varphi(x+tu,u) + \psi(x+tu,u)\right) + \varphi(x+tu,u)$$

for some ψ such that $\psi + t\varphi = \xi^1(x,t)$. Imposing Symmetry condition (4.32) yields

$$\varphi_t = u\varphi_x, \qquad \xi^1_t - u\xi^1_x = \varphi.$$

Assuming $\varphi = 0$ allows ξ^1 to be any constant, so Q can be u_1. If $\varphi = x+tu$ then Q becomes $-(xt+t^2u)$. Similarly one can find all eight generators for projectable symmetry group

$$\begin{aligned}
&u_1, && uu_1,\\
&xu_1 + tuu_1, && xu_1 - u,\\
&tu_1 + 1, && xuu_1 - u^2,\\
&x + tu + xtu_1 + t^2uu_1, && x^2u_1 + xtuu_1 - (x+tu)u.
\end{aligned}$$

In the next example, complexity of the problem becomes more apparent.

Example 4.3.2. Consider the Burgers' equation in *potential form*

$$u_t = u_2 + u_1^2. \tag{4.33}$$

Let $v_Q = Q\frac{\partial}{\partial u}$ be a generalised symmetry of (4.33) in the evolutionary form. Proceeding similarly to our previous example, here we compute third order symmetries. Since we can replace all the t- derivatives of u using (4.33), we can assume $Q = Q(x,t,u,u_1,u_2,u_3)$. In this case, Symmetry condition (4.27) is

$$D_tQ = D_x^2Q + 2u_1D_xQ. \tag{4.34}$$

Consider the higher order terms occurring in (4.34)

$$\begin{aligned}
D_tQ &= Q_{u_3}u_{3xt} + \cdots,\\
D_xQ &= Q_{u_3}u_4 + \cdots,\\
D_x^2Q &= Q_{u_3}u_5 + Q_{u_3u_3}u_4^2 + \cdots,
\end{aligned}$$

where $u_{3xt} = u_5 + 6u_2u_3 + 2u_1u_4$. Upon substitution in (4.34), we notice that the coefficients of u_5 cancel and since, u_4^2 does not appear on the right hand side of (4.34), we obtain

$Q_{u_{3x}u_{3x}} = 0$ which implies that

$$Q = F(x,t,u,u_1,u_2)u_3 + F'(x,t,u,u_1,u_2)$$

Now working with the above form of Q and repeating all the steps again, we find that

$$Q = \alpha(t)u_3 + G(x,t,u,u_1,u_2). \tag{4.35}$$

Proceeding with this equation, we find that the only term involving u_3^2 must have coefficient equal to zero i.e $G_{u_2u_2} = 0$. Therefore, we can assume G to be a function of G^1 and G^2 such that

$$G = G^1(x,t,u,u_1)u_2 + G^2(x,t,u,u_1). \tag{4.36}$$

Simultaneously comparing the coefficients of u_3 appearing in (4.34), we get

$$6\alpha u_2 + \alpha_t = 2G_{u_1u_2}u_2 + 2G_{uu_2}u_1 + 2G_{xu_2},$$

which gives

$$6\alpha = 2G_{u_1u_2}, \;\; 2G_{uu_2} = 0 \text{ and } \; 2G_{xu_2} = \alpha_t \tag{4.37}$$

Now from Equation (4.36) together with (4.37), we can find that

$$G = 3\alpha u_1u_2 + (\frac{1}{2}\alpha_t x + \beta(t))u_2 + G^2(x,t,u,u_1)$$

Now the coefficient of u_2^2 in (4.34) reads as

$$6\alpha u_1 + \alpha_t x + 2\beta = G^2_{u_1u_1}.$$

On integrating the above equation and putting together in (4.35) gives

$$Q = \alpha(u_3 + 3u_xu_2 + u_1^3) + (\frac{1}{2}\alpha_t x + \beta(t))(u_2 + u_1^2) + A(x,t,u)u_x + B(x,t,u). \tag{4.38}$$

Applying the same method continuously we can find different possible values for Q. For example, it can be

$$u_1,\, u_2 + u_1^2,\, t^2(u_2 + u_1^2) + txu_1 + \frac{1}{2}t + \frac{1}{4}x^2, \cdots \tag{4.39}$$

One can continue in the same fashion to obtain higher order symmetries but the computations will grow more and more complex. Again, this is to remind that our goal here is not to compute all symmetries but rather to demonstrate the process and complexity involved.

In the next chapter we will provide a more systematic approach to deal with this problem where we can find higher order symmetries by simply taking Lie brackets with a special vector field. So far our theory is worked out in the language of prolongation, but fortunately it can be simplified in terms of Lie brackets. This is done in the below section.

4.4 Lie Brackets

We equip our algebra $\mathscr{A}$ of differential functions with a Lie bracket. To do this, we need to define the concept of *Fréchet derivative*.

Definition 4.6. Let $P[u] \in \mathscr{A}$ be a differential function then the *Fréchet derivative* P_* is the differential operator defined as

$$P_*(Q) = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} P[u+\varepsilon Q[u]]. \tag{4.40}$$

For example, if $P[u] = uu_x + u_{xx}$ then

$$P_*(Q) = \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} (u+\varepsilon Q)(u_x+\varepsilon D_x Q) + (u_{xx}+\varepsilon D_x^2 Q) = u_x Q + u D_x Q + D_x^2 Q,$$

therefore as an operator, $P_* = u_x + uD_x + D_x^2$. This definition can be easily generalised, such that for any differential function $P[u]$

$$P_* = \sum_j \frac{\partial P}{\partial u_{jx}} D_x^j. \tag{4.41}$$

Now, we are ready to define a Lie bracket on the algebra $\mathscr{A}$.

Definition 4.7. For any two evolutionary vector fields with characteristics P and Q, we define a bracket as

$$[P,Q] = Q_*(P) - P_*(Q). \tag{4.42}$$

This is a Lie bracket as it satisfies the following properties for any differential functions $P,Q,R \in \mathscr{A}$-

(i) Bilinear

$$[cP + c'Q, R] = c[P,R] + c'[Q,R], \qquad c,c' \in \mathbb{R}$$

(ii) Skew-Symmetry

$$[P,Q] = -[Q,P]$$

(iii) Jacobi Identity

$$[P,[Q,R]]+[R,[P,Q]]+[Q,[R,P]]=0.$$

The proposition given below is the key observation which will help us to work with the language of Lie brackets instead of prolongation.

Proposition 4.2. *Let $P,Q \in \mathscr{A}$, then*

$$P_*(Q) = \mathtt{pr}\, v_Q(P).$$

Proof. We already know from the Equation (4.41) and (4.25) that

$$P_* = \sum_j \frac{\partial P}{\partial u_j} D_x^j \quad \text{and} \quad \mathtt{pr}\, \mathbf{v}_Q = \sum_j D_x^j Q \frac{\partial}{\partial u_j}, \tag{4.43}$$

which simply gives $P_*(Q) = \mathtt{pr}\, v_Q(P)$. □

Now, we can rewrite the symmetry condition in terms of the Lie bracket. If $u_t = K[u]$, then Symmetry condition (4.27) will become $\mathtt{pr}\, v_Q(u_t - K) = (u_t - K)_* Q = D_t Q - K_* Q = 0$. Since,

$$D_t Q = Q_t + \sum_{i \geq 0} \frac{\partial Q}{\partial u_i} D_x^i u_t = Q_t + \sum_{i \geq 0} \frac{\partial Q}{\partial u_i} D_x^i K = Q_t + Q_*(K),$$

we obtain that $Q_t + Q_*(K) - K_*(Q) = Q_t + [K,Q] = 0$. From now on, we will use this as a symmetry condition. Let us define this formally.

Definition 4.8. An evolution vector field with a characteristic Q is a symmetry of an evolution equation $u_t = K[u]$ if and only if

$$\frac{\partial Q}{\partial t} + [K,Q] = 0. \tag{4.44}$$

We also define the *adjoint action* of K on Q by $\mathtt{ad}_K Q = [K,Q]$.

Notice that if Q does not depend on t explicitly then the symmetry condition reduces to $[K,Q] = 0$.

Now we have completely defined symmetry in terms of the Lie bracket. We give an example below to illustrate this. We provide symmetries for the Burgers' equation without explicitly showing the process which is detailed in the next chapter.

Example 4.4.1. Consider the second order Burgers' equation

$$u_t = u_2 + u u_1. \tag{4.45}$$

Its *master symmetry* (see Section 5.3) is

$$\begin{aligned}\tau =& x(uu_1+u_2)+\frac{1}{2}u^2+\frac{3}{2}u_1\\ =& xu_t+\frac{1}{2}u^2+\frac{3}{2}u_1\end{aligned}$$

which helps us to to generate symmetries by its adjoint action on u_t. We denote these symmetries by $S_n=\mathtt{ad}_\tau^n K$. For example, we compute a third and a fourth order symmetry as

$$\begin{aligned}S_1 =& 2u_3+3uu_2+\frac{3}{2}u^2u_1+3u_1^2,\\ S_2 =& 6u_4+12uu_3+3u^3u_1+9u^2u_2+18uu_1^2+30u_1u_2,\end{aligned}$$

and the direct computation of Lie bracket gives $[u_t,S_1]=[u_t,S_2]=0$ which agrees with our Definition 4.8.

This equation also have infinitely many t-dependent symmetries Q_n. To systematically compute such symmetries we state a theorem in the next chapter, one can also look in to [45]. We list few of them here to check validity of the condition (4.44). Each Q_n is a polynomial in t of degree n.

$$\begin{aligned}Q_1 &= tu_1+1,\\ Q_2 &= t^2(u_2+uu_1)+t(xu_1+u)+x,\\ Q_3 &= t^3\left(u_3+\frac{3}{2}uu_2+\frac{3}{2}u_1^2+\frac{3}{4}u^2u_1\right)+\frac{3}{2}t^2\left(xu_2+xuu_1+2u_1+\frac{1}{2}u^2\right)\\ &\quad +\frac{3}{4}t(x^2u_1+2xu+2)+\frac{3}{4}x^2.\end{aligned}$$

We find that

$$\frac{\partial Q_2}{\partial t}=2tu_t+xu_1+u \quad \text{and} \quad [u_t,Q_2]=-(2tu_t+xu_1+u)$$

which implies that Q_2 satisfies Symmetry condition (4.44). Similarly, one can check for Q_1 and Q_3.

4.5 Conclusions

Symmetry method is a powerful technique to solve ordinary differential equations. Other well-known techniques are special cases of a few symmetry methods. Unfortunately, for systems of partial differential equations, the symmetry group is usually of no help in determining the general solution. However, generalised symmetries can be used to explicitly determine special type of solutions which are invariant under some subgroup of the full symmetry group of the system (see Example 4.3.1). However, our approach in this example has the drawback of not generating infinitely many symmetries at once. We tackle this problem in the next chapter.

In the last Example 4.4.1, we computed three symmetries Q_1, Q_2, Q_3 for the Burgers' equation $u_t = u_2 + uu_1$. We had a hidden motive to list these symmetries here. The coefficients of the linear terms in Q_1, Q_2 and Q_3 with constant factors form an $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra, meaning, if

$$e = u_1, \quad h = 2(xu_1 + u), \quad f = -(x^2u_1 + 2xu + 2),$$

then the Lie bracket

$$\begin{aligned}
[e,h] &= [u_x, 2(xu_x + \alpha u)] \\
&= 2(xD + \alpha)(u_x) - D(2(xu_x + \alpha u)) \\
&= 2xu_{xx} + 2\alpha u_x - 2(u_x + xu_{xx} + \alpha u_x) = -2e.
\end{aligned}$$

Similarly, it is easy to check that $[e,f] = h$ and $[h,f] = -2f$. Hence e, f, h generates $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra (see (5.4)). This property is exploited in the next chapter; moreover, we will find that we can always construct $\mathfrak{sl}(2,\mathbb{C})$-modules of an integrable system.

Chapter 5

$\mathfrak{sl}(2,\mathbb{C})$ Representation and Master Symmetry

In the previous chapter, we described a method in Section 4.3 which gives a systematic approach to compute the generalised symmetries of a system of differential equations but it requires the order of the symmetry to be defined in advance. Thus, it has the drawback of *not* generating infinitely many symmetries simultaneously. As promised earlier, here we will formally define a *master symmetry* which will help us to generate infinitely many symmetries at once. We shall see how we can incorporate our understanding of $\mathfrak{sl}(2,\mathbb{C})$ Lie algebra in constructing master symmetry and time dependent symmetries. More details can be found in [45, 52] and references therein.

A detailed outline of this chapter is as follows. Our goal is to compute master symmetries of nonlinear partial differential equations using $\mathfrak{sl}(2,\mathbb{C})$ algebra. We introduce the notions of a Lie algebra (Def. 5.1) and the Bernstein-Gelfand-Gelfand (BGG) category $\mathscr{O}$ of $\mathfrak{sl}(2,\mathbb{C})$ modules (Sec. 5.2.1), followed by the definition of a homogeneous evolution equation (Def. 5.3) and a master symmetry (Def. 5.4). Next, we present a theorem for constructing master symmetries (Thm. 5.1) along with the so-called $\mathscr{O}$-scheme which illustrates the role of $\mathfrak{sl}(2,\mathbb{C})$ modules, see Dia. 5.1. A method to construct time-dependent symmetries is depicted in Thm. 5.2. Using this powerful approach we construct master symmetries for five new two-component $(1+1)$-dimensional systems (Examples 5.4.2 - 5.4.6) and a master symmetry for a two-component $(2+1)$-dimensional Davey-Stewartson type system (Example 5.5.1).

5.1 Motivation

We recall Symmetry condition (4.44) which tells us that $Q \in \mathscr{A}$ is a symmetry for an evolution equation $u_t = K[u]$ if

$$\frac{\partial Q}{\partial t} + \mathtt{ad}_K Q = 0. \tag{5.1}$$

Then, it is natural to look for a function Q such that $\frac{\partial Q}{\partial t} = -\mathtt{ad}_K Q$. Keeping this in mind, we consider the function

$$Q = \exp(-t\,\mathtt{ad}_K)Q_0 \tag{5.2}$$

with $\mathtt{ad}_K^{m+1} Q_0 = 0$ for some $m \in \mathbb{N}$, so that the above sum is finite and topology becomes irrelevant. Then Q is a symmetry of K and moreover, $\frac{\partial^r Q}{\partial t^r}$ is also a symmetry, since

$$\frac{\partial^{r+1} Q}{\partial t^{r+1}} + \mathtt{ad}_K \frac{\partial^r Q}{\partial t^r} = \frac{\partial^r}{\partial t^r}\left(\frac{\partial Q}{\partial t} + \mathtt{ad}_K Q\right) = 0. \tag{5.3}$$

Notice that, this argument works only when the differential functions K and Q are independent of the variable t. In particular, if Q is a polynomial of degree p in t, then $\frac{\partial^p Q}{\partial t^p}$ is a time-independent symmetry of K.

If m is minimum possible such that $\mathtt{ad}_K^{m+1} Q_0 = 0$ then we call Q_0 to be a K- generator of degree m. This condition reminds us of the representation theory of $\mathfrak{sl}(2,\mathbb{C})$. In particular, we will like to scrutinize the vector space spanned by the set $\{\mathtt{ad}_K^j Q \mid j = 0, \cdots, m\}$. In the previous chapter we mentioned that the coefficients of the linear terms appearing in the t-dependent symmetries S_n of the Burgers' equation form an $\mathfrak{sl}(2,\mathbb{C})$ algebra (see Example 4.4.1). In fact, the vector space spanned by all the linear terms of S_n is an infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module in the Bernstein-Gelfand-Gelfand (BGG) category $\mathscr{O}$.

This gives us enough reason to investigate the role of $\mathfrak{sl}(2,\mathbb{C})$ algebra in order to understand nonlinear integrable system.

5.2 The BGG category $\mathscr{O}$ of $\mathfrak{sl}(2,\mathbb{C})$-modules

We begin this section by a short review of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and its representations.

Definition 5.1. A Lie algebra is *simple* if it has no non-trivial ideals and is not abelian. It is called *semisimple* if it is isomorphic to a direct sum of simple Lie algebras.

The special linear Lie algebra $\mathfrak{sl}(n,\mathbb{C})$ is the semisimple Lie algebra of $n \times n$ matrices with zero trace and the Lie bracket is given by $[X,Y] = XY - YX$ for all matrices X, Y in

$\mathfrak{sl}(n,\mathbb{C})$. As our subject of interest is $\mathfrak{sl}(2,\mathbb{C})$ we begin with its standard basis which consists of

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\, f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\, h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Then, we have

$$[h,e]=2e, \quad [h,f]=-2f, \quad [e,f]=h. \tag{5.4}$$

Definition 5.2. A vector space V together with an operation $\cdot : L\times V\to V$ is called an L-module (or equivalently a representation of L) if the following conditions are satisfied:

- $(ax+by)\cdot v=a(x\cdot v)+b(y\cdot v)$,
- $x\cdot(av+bw)=a(x\cdot v)+b(y\cdot w)$,
- $[x,y]\cdot v=x\cdot(y\cdot v)-y\cdot(x\cdot v)$,

for $x,y\in L$; $v,w\in V$ and $a,b\in\mathbb{C}$.

A nonzero L-module V is called *irreducible* if it has precisely two L-submodules, i.e., 0 and L.

5.2.1 Nonisomorphic indecomposable modules in $\mathscr{O}$

Before we start discussing the modules in the category $\mathscr{O}$, we reiterate our motivation. In Example 4.4.1, the vector space spanned by all the linear terms of the symmetries of the Burgers' equation is an infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module in the Bernstein-Gelfand-Gelfand (BGG) category $\mathscr{O}$. Therefore, there knowledge may shed more information in understanding the nonlinear integrable equations. So here we will describe the indecomposable modules in the BGG category $\mathscr{O}$.

Let V be an arbitrary $\mathfrak{sl}(2,\mathbb{C})$-module over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed and h is semisimple, we can infer that h acts diagonally on V (Section 6.4, [21]). This yields an eigenspace decomposition of V i.e. $V=\bigoplus V_\lambda$ with finitely many $V_\lambda=\{v\in V\mid h\cdot v=\lambda v\}$, $\lambda\in\mathbb{C}$. If λ is not an eigenvalue of the representation $h\cdot$, then $V_\lambda=\{0\}$. Whenever, $V_\lambda\neq 0$, we call λ a *weight* of h in V and V_λ its associated *weight space*.

Lemma 5.1. *If $v\in V_\lambda$ then $e\cdot v\in V_{\lambda+2}$ and $f\cdot v\in V_{\lambda-2}$.*

Proof. By the Jacobi identity,

$$h\cdot(f\cdot v)=[h,f]\cdot v+f\cdot h\cdot v=-2f\cdot v+\lambda f\cdot v=(\lambda-2)f\cdot v.$$

Similarly, one can show that $h\cdot(e\cdot v)=(\lambda+2)e\cdot v$. □

A *highest weight vector* of weight λ is an element $v \in V$ such that $e \cdot v = 0$ and $h \cdot v = \lambda v$, and the corresponding weight space is called *highest weight module*.
The objects of the BGG category $\mathscr{O}$ of $\mathfrak{sl}(2,\mathbb{C})$-Modules are M such that

- M is finitely generated
- M is a weight module, i.e., M is the direct sum of weight spaces, and
- all weight spaces of M are finite-dimensional.

Notice that all finite dimensional modules are in this category $\mathscr{O}$ and every module can be written as the direct sum of indecomposable modules. Some basic results for these indecomposable modules are listed below. To read more on this please refer to [20].

The finite dimensional simple module $L(\lambda)$
For $\lambda \in \mathbb{N}$, $L(\lambda)$ is the finite dimensional simple module of dimension $\lambda+1$ with basis vectors $\{v_i \mid 0 \le i \le \lambda\}$ and action of $\mathfrak{sl}(2,\mathbb{C})$ as-

$$f.v_i = v_{i+1}, \quad h.v_i = (\lambda - 2i)v_i, \quad e.v_i = i(\lambda - i + 1)v_{i-1} \text{ with } \quad e.v_0 = f.v_\lambda = 0.$$

Its one dimensional weight space has weights $\lambda, \lambda-2, \cdots, -(\lambda-2), -\lambda$.

The Verma module $M(\lambda)$
The Verma module is a *maximal* highest-weight module with a highest weight vector v_0. That is, every other highest-weight module with highest weight λ is a quotient of the Verma module. Its weights are $\lambda, \lambda-2, \lambda-4\cdots$ each with multiplicity one. This module $M(\lambda)$ is an infinite dimensional module with basis vectors $\{v_i \mid i \ge 0\}$ with

$$f.v_i = v_{i+1}, \quad h.v_i = (\lambda - 2i)v_i, \quad e.v_i = i(\lambda - i + 1)v_{i-1}.$$

If $\lambda \in \mathbb{N}$ then for $i = \lambda+1$, $e.v_{\lambda+1} = 0$ and the maximal submodule of $M(\lambda)$ is $M(-\lambda-2)$. Moreover, $L(\lambda) \cong M(\lambda)/M(-\lambda-2)$.

Remark 5.1. When $\lambda \in \mathbb{N}$ then $M(-\lambda-2)$ is the only non-trivial subspace of $M(\lambda)$, else $M(\lambda)$ is a simple module.

The dual Verma module $M^\vee(\lambda)$
This is also an infinite dimensional weight module with basis vectors $\{v_i(i \ge 0)\}$ such that

$$e \cdot v_i = v_{i-1}, \quad h \cdot v_i = (\lambda - 2i)v_i, \quad f \cdot v_i = (i+1)(\lambda - i)v_{i+1}, \quad e \cdot v_0 = 0.$$

Observe that the highest weight vector v_0 can be reached from any other vector by the action of e whereas in the case of Verma module it can be achieved by the action of f which indicates the duality property.

When $\lambda \in \mathbb{N}$, then for $i = \lambda, f \cdot v_\lambda = 0$ which implies that $L(\lambda)$ is the maximal submodule of $M^\vee$ and consequently, the Verma module $M(-\lambda-2) \cong M^\vee(\lambda)/L(\lambda)$.

The projective module $P(\lambda-2)$
When $\lambda \in \mathbb{N}$ then there exists a nontrivial projective module $P(\lambda-2)$. Its weights with multiplicity one are $\lambda, \lambda-2, \cdots, -\lambda$ and the weights $-\lambda-2, -\lambda-4, \cdots$ are with multiplicity two. It is generated by the basis vectors $\{v_i \mid i \geq 0\}$ and $\{w_j \mid j \geq 0\}$ such that

$$f.v_i = v_{i+1}, \quad h.v_i = (\lambda-2i)v_i, \quad e.v_i = i(\lambda-i+1)v_{i-1} \text{ and,}$$

$$f.w_j = w_{j+1}, h.w_i = (-\lambda-2-2i)w_i, e.w_i = -i(\lambda+i+1)w_{i-1}, e \cdot w_0 = v_\lambda.$$

From the actions defined above, it is clear that the projective module $P(\lambda-2)$ has the Verma submodules $M(\lambda)$ and $M(-\lambda-2)$. Moreover, we have

$$\begin{aligned} P(\lambda-2)/M(\lambda) &\cong M(-\lambda-2), \\ P(\lambda-2)/M(-\lambda-2) &\cong M^\vee(\lambda). \end{aligned}$$

In the next section, we give examples of such modules.

5.3 Master symmetries

Here, we will consider only *homogeneous* evolution equations $u_t = K[u]$ (defined below) with restriction on the differential function $K[u]$ such that it depends only on u and the x-derivatives of u up to a finite order. This is needed to make sure that u_x is a symmetry for the given evolution equation, so that we can always construct an $\mathfrak{sl}(2,\mathbb{C})$ algebra. It is explained in the lemma below.

Definition 5.3. Equation $u_t = K[u]$ is said to be *homogeneous* (or α- homogeneous) if $[xu_x + \alpha u, K] = \kappa K$ for some constants α and κ.

Lemma 5.2. *For an homogeneous equation with a scaling $h = 2(xu_x + \alpha u)$, where $\alpha =$ constant, the elements $e = u_x, f = -(x^2 u_x + 2\alpha x u)$ and h form an $\mathfrak{sl}(2,\mathbb{C})$ algebra.*

Proof. $[e,h] = [u_x, 2(xu_x+\alpha u)] = 2(xD+\alpha)(u_x) - D(2(xu_x+\alpha u)) = 2xu_{xx}+2\alpha u_x - 2(u_x + xu_{xx}+\alpha u_x) = -2e$. Similarly, it is easy to check that $[e,f] = h$ and $[h,f] = -2f$. □

Observe that the element f in the above lemma is not unique for given elements h and e. In fact any element $f' = f + f_0$ will suffice given $[e, f_0] = 0$ and $[h, f_0] = -2f_0$ so that condition 5.4 is satisfied.

It is now clear that we can always construct an $\mathfrak{sl}(2,\mathbb{C})$ algebra for a given homogeneous evolution equation regardless of its integrability.

Now we will construct an infinite dimensional $\mathfrak{sl}(2,\mathbb{C})$-module in the BGG category $\mathscr{O}$. We begin by noticing that for any element $w \in \mathscr{A}$

$$\mathtt{ad}_e w = [u_x, w] = \sum_{j\geq 0} \frac{\partial w}{\partial u_j} D^j(u_x) - \frac{\partial w}{\partial x} - \sum_{j\geq 0} \frac{\partial w}{\partial u_j} D^j(u_x) = -\frac{\partial w}{\partial x}. \tag{5.5}$$

Therefore, for any given element $g \in \mathscr{A}$, we can find $w \in \mathscr{A}$ such that $\mathtt{ad}_e w = g$, i.e, we can compute the inverse action of e.

Lemma 5.3. *Let $w = (x^3 u_x + 3\alpha x^2 u)/3$. Then, $ad_e w = f$ and*

$$\mathtt{ad}_f^n w = \frac{n!}{3}(x^{n+3}u_x + (n+3)\alpha x^{n+2} u), \quad n = 0,1,2,\cdots \tag{5.6}$$

with $\mathtt{ad}_h \mathtt{ad}_f^n w = -2(n+2)\,\mathtt{ad}_f^n w$.

Proof. We showed above $\mathtt{ad}_e w = -\frac{\partial w}{\partial x}$, therefore

$$\begin{aligned} \mathtt{ad}_e w &= -\frac{1}{3}\frac{\partial x^3 u_x + 3\alpha x^2 u}{\partial x} \\ &= -(x^2 u_x + 2\alpha x u) = f. \end{aligned}$$

Other results can also be proved by an easy induction on n and using the definition of Lie bracket. For the complete proof please see [52]. □

Similar to the element f in Lemma 5.2, the element w also have freedom. For example, any element $w + w_0$ satisfying $\mathtt{ad}_e w = f$ and $\mathtt{ad}_h w = -4w$ can be chosen such that $\mathtt{ad}_e w_0 = 0$ and $\mathtt{ad}_h w_0 = -4w_0$. Lemma 5.3 tells us $\mathtt{ad}_f w = \int w dx$ and $\mathtt{ad}_e w = -\frac{\partial w}{\partial x}$, therefore, we can say that $\mathtt{ad}_e \mathtt{ad}_f w = -w$.

Since $\mathtt{ad}_f^n w$ represents a polynomial in x of order n, therefore it is clear that $\{\mathtt{ad}_f^n w, n = 0,1,2,\cdots\}$ generate an infinite-dimensional space over $\mathbb{C}$ with the highest weight vector w.

The immediate consequence of Lemma 5.2 is we can construct the below $\mathfrak{sl}(2)$-module

$$\underset{e}{\overset{f}{\leftrightarrows}} f \underset{e}{\leftarrow} w \underset{f}{\overset{e}{\leftrightarrows}} \mathtt{ad}_f w \underset{f}{\overset{e}{\leftrightarrows}} \mathtt{ad}_f^2 w \underset{f}{\overset{e}{\leftrightarrows}} \cdots \tag{5.7}$$

where the $q \overset{p}{\to} r$ means $\mathtt{ad}_p q = \gamma r$ with $\gamma \neq 0$ This module and its freedom over choosing f and w plays an important role in constructing master symmetries and time-dependent symmetries for homogeneous evolution equations.

Definition 5.4. Master symmetry is an evolution vector field τ whose adjoint action $\mathtt{ad}_\tau = [\tau, \cdot]$ maps a symmetry to a new symmetry.

The following theorem provides a constructive approach to find a master symmetry. Its proof can be found in Theorem 2 [52].

Theorem 5.1. *For a homogeneous evolution equation $u_t = K[u]$ satisfying*

$$[u_x, K] = 0, \quad [xu_x + \alpha u, K] = \kappa K, \kappa > 1,$$

with a certain constant α, let

$$\tau = \frac{1}{2\kappa}[x^2 u_x + 2\alpha x u, K] \tag{5.8}$$

and $a_{n+1} = [\tau, a_n]$ with $a_0 = e = u_x$. If

- $[[\tau, K], K] = 0$ *and*
- *there is a Lie subalgebra $\mathfrak{h}$ such that $a_n \in \mathfrak{h}$ for all $n = 0, 1, 2, \cdots$ and, moreover, for any $P, Q \in \mathfrak{h}$ satisfying $[P, e] = [Q, e] = [P, K] = [Q, K] = 0$, it follows that $[P, Q] = 0$,*

then τ is a master symmetry of the equation $u_t = K[u]$, i.e., all a_n mutually commute.

The factor $\frac{1}{2\kappa}$ in τ ((5.8)) does not plays any important role. It is needed to avoid unwanted constant factor appearing in the master symmetry.

Since in our case the second condition in the theorem is always satisfied so we can define master symmetry alternatively.

Definition 5.5. A t-independent evolution vector field τ is a master symmetry of $u_t = K$ if and only if

$$[[\tau, K], K] = 0 \text{ provided } [\tau, K] \neq 0. \tag{5.9}$$

Lemma 5.3 and Theorem 5.1 helps us to develop the $\mathscr{O}$-scheme (see Diagram 5.1 below). It also includes the τ-scheme formulated by Dorfman [8] which can be seen in the elements in the first vertical line.

Some notes on the $\mathscr{O}$-scheme

- Each a_n is a highest weight vector therefore we can construct an infinite dimensional Verma module $< \mathtt{ad}^i_f a_n \mid i \geq 0 >$.
- If there exists a k such that $\mathtt{ad}^{k+1}_f a_n = 0$ then $< \mathtt{ad}^i_f a_n \mid 0 \leq i \leq k >$ is a finite dimensional simple module isomorphic to $L((\lambda-2)n+2)$. In this case, we can find a vector w such that $\mathtt{ad}_e = w$. Then $< \mathtt{ad}^i_f a_n, \mathtt{ad}^j_f w \mid 0 \leq i \leq k, j = 0,1,2\cdots >$ is a $\mathfrak{sl}(2)$-module isomorphic to $P(-(\lambda-2)n-4)/M(-(\lambda-2)n-4)$.
- Elements on the edge of Diagram 5.1 generate t-dependent symmetries.

$$
\begin{tikzcd}
u_x \arrow[d, "\tau"] \arrow[dr, "f"] & & & & \\
K \arrow[d, "\tau"] \arrow[dr, "f"] & h \arrow[dr, "f"] & & & \\
a_2 \arrow[d, "\tau"] \arrow[dr, "f"] & \tau \arrow[dr, "f"] & f & & \\
a_3 \arrow[d, "\tau"] \arrow[dr, "f"] & \mathtt{ad}_f a_2 \arrow[dr, "f"] & \mathtt{ad}^2_f a_1 \arrow[dr, "f"] & w \arrow[ul, "e"'] \arrow[dr, "f"] & \\
{} & {} & {} & {} & {}
\end{tikzcd}
$$

Diagram 5.1 $\mathscr{O}$-scheme for constructing a master symmetry for $u_t = K[u]$

Remark 5.2. Two symmetries are said to be *dependent* if their difference can be expressed as sum of lower order symmetries. For example, the following two symmetries are dependent

$$\frac{\partial}{\partial t}\left(\exp(-t\mathtt{ad}_K)\mathtt{ad}^2_f a_1\right), \quad \exp(-t\mathtt{ad}_K)\mathtt{ad}_f a_2. \tag{5.10}$$

Therefore, for a given degree in t and order of the polynomial, there is a unique independent symmetry which can be generated by the elements on the edge in the $\mathscr{O}$-scheme.

To find t-dependent symmetries, the element w (5.3) needs to satisfy some necessary conditions which are mentioned in the theorem below. Its proof can be found in Theorem 3 of [52].

Theorem 5.2. *Let the homogeneous equation $u_t = K[u]$ satisfy the conditions in Theorem 5.1. Let the evolution vector field w satisfy*

$$ad_e w = f, \qquad ad_K w = \frac{1}{2-4\kappa} ad_f^2 K$$

Then

$$\frac{\partial^j}{\partial t^j} \exp(-t\, ad_K)(ad_f^n w), n = 0, 1, 2, \cdots \quad j = 0, 1, 2, \cdots, n+2$$

are t-dependent symmetries of the equation.

Most of the theory developed so far boils downs to computing Lie brackets for constructing master symmetries. To do this, we turn towards computer algebra system to help us in this endeavour. For the case of $(1+1)$-dimensional case, we successfully implemented the algorithm of Theorem 5.1 in Maple. But, the $(2+1)$-dimensional case is problematic as it involves local terms. Therefore, we describe a natural grading on differential functions in $\mathscr{A}$ so that it can help us to simplify cumbersome calculations.

Consider the system of 2 components u and v with weights λ_1 and λ_2 respectively. Such a system of order n can be written as

$$u_t \frac{\partial}{\partial u} + v_t \frac{\partial}{\partial v} = \sum_{i,j} K^{(i,j)}_{n-i\lambda_1 - j\lambda_2}, \quad i, j \geq -1, \tag{5.11}$$

where $K^{(i,j)}$ indicates degree i in u and j in v with m number of x-derivatives altogether. The term $K^{(i,j)}$ makes sense only when $n - i\lambda_1 - j\lambda_2 \in \mathbb{N}$ and can appear in the sum. We read $u_t = K[u]\frac{\partial}{\partial u}$ and $v_t = Q[u]\frac{\partial}{\partial v}$ therefore, degree can be -1, eg: $K_0^{(-1,0)} = \frac{\partial}{\partial u}$. For a non-negative degree example,

$$K^{(2,0)} = \begin{pmatrix} u_1\theta^{-1}(u\theta^{-1}u) \\ 3v\theta^{-1}(u_1\theta^{-1}u) \end{pmatrix}.$$

Notice that as a result on taking the Lie brackets of two differential functions, corresponding degrees get added, i.e, $[K^{(m_1,n_1)}, Q^{(m_2,n_2)}] \sim R^{(m_1+m_2, n_1+n_2)}$.

We will use this grading for calculating Lie brackets in Section 5.5.2. More details on this can be found in [46].

5.4 Applications

We have gathered enough information to proceed with our goal to construct master symmetries for new systems. First we compute master symmetries for three new two-component Burgers' type $(1+1)$-dimensional systems that appeared in [50] and then two new systems from the ongoing work of Wang (et al.) [29].

Talati and Turhan [50] classified $(1,1)$-homogeneous third order Burgers' type systems with weight 3 having nondiagonal constant matrix of leading order terms

$$J^{(a,\varepsilon)} = \begin{pmatrix} a & \varepsilon \\ 0 & a \end{pmatrix}, \quad \varepsilon \neq 0. \tag{5.12}$$

The class of such systems with undetermined constant coefficients have the form

$$\left.\begin{aligned} u_t &= l_1^1 u_{xxx} + l_2^1 v_{xxx} + l_3^1 u u_{xx} + l_4^1 u v_{xx} + l_5^1 v u_{xx} + l_6^1 v v_{xx} + l_7^1 u_x^2 + l_8^1 u_x v_x \\ &\quad + l_9^1 v_x^2 + l_{10}^1 u^2 u_x + l_{11}^1 u^2 v_x + l_{12}^1 v u u_x + l_{13}^1 u v v_x + l_{14}^1 v^2 u_x \\ &\quad + l_{15}^1 v^2 v_x + l_{16}^1 u_4 + l_{17}^1 u^3 v + l_{18}^1 u^2 v^2 + l_{19}^1 u v^3 + l_{20}^1 v^4, \\ v_t &= l_1^2 u_{xxx} + l_2^2 v_{xxx} + l_3^2 u u_{xx} + l_4^2 u v_{xx} + l_5^2 v u_{xx} + l_6^2 v v_{xx} + l_7^2 u_x^2 + l_8^2 u_x v_x \\ &\quad + l_9^2 v_x^2 + l_{10}^2 u^2 u_x + l_{11}^2 u^2 v_x + l_{12}^2 v u u_x + l_{13}^2 u v v_x + l_{14}^2 v^2 u_x \\ &\quad + l_{15}^2 v^2 v_x + l_{16}^2 u_4 + l_{17}^2 u^3 v + l_{18}^2 u^2 v^2 + l_{19}^2 u v^3 + l_{20}^2 v^4. \end{aligned}\right\} = K \tag{5.13}$$

They set $l_1^1 = l_2^2 = 1, l_1^2 = 0, l_2^1 = \varepsilon \neq 0$, i.e., $(J^{(1,\varepsilon)})$ in (5.13) and looked for systems possessing a symmetry from the class of $(1,1)$-homogeneous systems of weight 5. Then they found that the class of systems (5.13) is equivalent to only eight systems up to a linear change of variables u and v, and rescaling of x and t. For the list please see [50, Proposition 2]. All the systems obtained having a higher symmetry turned out to be systems admitting the Hopf-Cole transformation $u = ln(w)_x$ after which the coefficient matrix of leading order terms becomes the identity matrix. Since they are C-integrable, therefore they could be useful nonlinear forms of some linear systems.

Here we will compute master symmetries for three systems in Examples 5.4.2, 5.4.3 and 5.4.4. These systems appears to be new among the list of eight systems found by Talati and Turhan. They also obtained the master symmetries of these systems by investigating their t-dependent symmetries but here we use the $\mathscr{O}$-scheme. All the calculations in this section are carried out in the Maple software and the required code can be found in Appendix B.

Before we proceed with two-component systems, we give an easy example in the scalar case.

Example 5.4.1. Consider the Burgers' equation of order 3,

$$u_t = B_3[u] = u_3 + 3uu_2 + 3u^2 u_x + 3u_x^2. \tag{5.14}$$

It is an homogeneous equation for $\alpha = 1$ (5.3), as for $h = xu_x + u$ we obtain,

$$[h, u_t] = 9u^2 u_x + 9uu_2 + 9u_x^2 + 3u_3 = 3K.$$

Therefore, according to Theorem 5.1

$$\begin{aligned}\tau &= \frac{1}{6}[x^2 u_x + 2xu, K] \\ &= 3xu^2 u_x + 3xuu_2 + 3xu_x^2 + u^3 + xu_3 + 5uu_x + 2u_2\end{aligned}$$

is a master symmetry if $[[\tau, K], K] = 0$, which is indeed the case as

$$\begin{aligned}[\tau, K] = &-15u^4 u_x - 30u^3 u_2 - 90u^2 u_x^2 - 30u^2 u_3 - 150uu_x u_2 \\ &- 45u_x^3 - 15uu_4 - 45u_x u_3 - 30u_2^2 - 3u_5,\end{aligned}$$

and the further computations shows $[[\tau, K], K] = 0$.

Now, we look at the two component systems of non-linear partial differential equations. For such systems, we can extend the definition of Lie brackets (4.7).

Definition 5.6. For any two evolutionary vector fields with characteristics $\begin{pmatrix} P \\ Q \end{pmatrix}$ and $\begin{pmatrix} J \\ K \end{pmatrix}$, the Lie bracket takes the form

$$\left[\begin{pmatrix} P \\ Q \end{pmatrix} \quad \begin{pmatrix} J \\ K \end{pmatrix} \right] = \begin{pmatrix} J_u^*(P) + J_v^*(Q) - P_u^*(J) - P_v^*(K) \\ K_u^*(P) + K_v^*(Q) - Q_u^*(J) - Q_v^*(K) \end{pmatrix}.$$

where $A_u^*(B)$ means the Fréchet derivative of A with respect to u acting on B.

Example 5.4.2. This examples comes from a new system which appears as System (12) in [50],

$$\left.\begin{aligned} u_t &= B_3[u] + 2\varepsilon(v_2 + 3uv_x + 2vu_1 + 2u^2 v)_x \\ v_t &= v_3 + 3uv_2 + 6u_x v_x + 3u^2 v_x - \varepsilon(4vv_2 - v_x^2 + 8u_x v^2 + 8uvv_x + 4u^2 v^2) \end{aligned}\right\} = K. \tag{5.15}$$

For easy comprehension, we divide this example into subsections.

Construction of master symmetry

It is $(1,1)$ homogeneous, i.e, the following Lie brackets gives,

$$\left[\begin{pmatrix} xu_x+u \\ xv_x+v \end{pmatrix} \quad \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = 3\begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

and the underlying Lie algebra (5.2) is generated by the elements

$$e = \begin{pmatrix} u_x \\ v_x \end{pmatrix}, h = 2\begin{pmatrix} xu_x+u \\ xv_x+v \end{pmatrix}, f = -\begin{pmatrix} x^2u_x+2xu-1 \\ x^2v_x+2xv+\gamma \end{pmatrix}.$$

Notice that $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ has a free parameter γ.

Now, we compute a master symmetry $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ using Theorem 5.2. Please be aware that the theorem make use of $-f$ to avoid the minus sign. However, we simply use the original f as in the Lie algebra above, then

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \frac{1}{6}\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right],$$

which gives a one parameter family of master symmetries with

$$\begin{aligned}
\tau_1 = {} & -xu_t - \tfrac{2}{3}\gamma\varepsilon u_2 - \tfrac{3}{2}u_2 - 3\varepsilon v_2 - \tfrac{4}{3}\gamma\varepsilon uu_x - 6\varepsilon vu_x \\
& - 4uu_x - 8\varepsilon uv_x - 4\varepsilon u^2v - u^3, \\
\tau_2 = {} & -xv_t + \tfrac{2}{3}\gamma\varepsilon v_2 - \tfrac{3}{2}v_2 + \tfrac{8}{3}\gamma\varepsilon vu_x - 2vu_x + \tfrac{4}{3}\gamma\varepsilon uv_x \\
& + 2\varepsilon vv_x - 4uv_x + \tfrac{4}{3}\gamma\varepsilon u^2v + 4\varepsilon uv^2 - u^2v.
\end{aligned}$$

Moreover, if we put $\gamma = \frac{3}{4}$ then we obtain the same master symmetry $\mathbf{M} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ (up to sign) as in [50] where,

$$\begin{aligned}
M_1 = {} & xu_3 + 2\varepsilon xv_3 + 4\varepsilon xvu_2 + 3xuu_2 + 2u_2 + 6\varepsilon xuv_2 + 3\varepsilon v_2 \\
& + 3xu_x^2 + 10\varepsilon xu_xv_x + 8\varepsilon xuvu_x + 3xu^2u_x + 6\varepsilon vu_x \\
& + 5uu_x + 4\varepsilon xu^2v_x + 8\varepsilon uv_x + 4\varepsilon u^2v + u^3, \\
M_2 = {} & xv_3 - 4\varepsilon xvv_2 + 3xuv_2 + v_2 + 6xu_xv_x - 8\varepsilon xv^2u_x \\
& + \varepsilon xv_x^2 - 8\varepsilon xuvv_x + 3xu^2v_x - 2\varepsilon vv_x + 3uv_x \\
& - 4\varepsilon xu^2v^2 - 4\varepsilon uv^2.
\end{aligned}$$

Since, $\mathbf{M}$ and τ are dependent master symmetries (see Remark 5.2), we can construct a lower order master symmetry $\mathscr{M} = \mathbf{M} + \tau$ with

$$\begin{aligned}
\mathscr{M}_1 = {} & -\tfrac{2}{3}\gamma\varepsilon u_2 + \tfrac{1}{2}u_2 - \tfrac{4}{3}\gamma\varepsilon uu_x + uu_x \\
\mathscr{M}_2 = {} & -\tfrac{1}{2}v_2 + \tfrac{2}{3}\gamma\varepsilon v_2 - 2vu_x + \tfrac{8}{3}\gamma\varepsilon vu_x - uv_x + \tfrac{4}{3}\gamma\varepsilon uv_x \\
& - u^2v + \tfrac{4}{3}\gamma\varepsilon u^2v.
\end{aligned}$$

Observe that, $\mathscr{M}_1$ depends only on the variable u and $\mathscr{M}_2$ is linear in v, i.e, we have found a triangular master symmetry. Moreover, it generates triangular symmetries by a suitable linear combination of symmetries obtained from τ and $\mathscr{M}$. For example, the following linear combination

$$S = \left(\frac{8}{3}\gamma\varepsilon^2 - 12\varepsilon\right)[\mathscr{M}, K] - \left(\frac{8}{3}\gamma\varepsilon^2 - 2\varepsilon\right)[\tau, K], \tag{5.16}$$

generates a triangular symmetry $S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}$ with

$$
\begin{aligned}
S_1 = {} & 8\varepsilon^2 u_5 - 6\varepsilon u_5 + 40\gamma\varepsilon^2 u u_4 - 30\varepsilon u u_4 + 120\gamma\varepsilon^2 u_3 u_1 - 90\varepsilon u_3 u_1 \\
& + 80\gamma\varepsilon^2 u^2 u_3 - 60\varepsilon u^2 u_3 + 80\gamma\varepsilon^2 u_2^2 - 60\varepsilon u_2^2 + 400\gamma\varepsilon^2 u u_2 u_1 \\
& - 300\varepsilon u u_2 u_1 + 80\gamma\varepsilon^2 u^3 u_2 - 60\varepsilon u^3 u_2 + 120\gamma\varepsilon^2 u_1^3 - 90\varepsilon u_1^3 \\
& + 240\gamma\varepsilon^2 u^2 u_1^2 - 180\varepsilon u^2 u_1^2 + 40\gamma\varepsilon^2 u^4 u_1 - 30\varepsilon u^4 u_1,
\end{aligned}
$$

$$
\begin{aligned}
S_2 = {} & 8\gamma\varepsilon^2 v_5 - 6\varepsilon v_5 + 40\gamma\varepsilon^2 u v_4 - 30\varepsilon u v_4 + 80\gamma\varepsilon^2 u_3 v_1 \\
& - 60\varepsilon u_3 v_1 + 120\gamma\varepsilon^2 v_3 u_1 - 90\varepsilon v_3 u_1 + 80\gamma\varepsilon^2 u^2 v_3 \\
& - 60\varepsilon u^2 v_3 + 120\gamma\varepsilon^2 u_2 v_2 - 90\varepsilon u_2 v_2 + 240\gamma\varepsilon^2 u u_2 v_1 \\
& - 180\varepsilon u u_2 v_1 + 320\gamma\varepsilon^2 u v_2 u_1 - 240\varepsilon u v_2 u_1 + 80\gamma\varepsilon^2 u^3 v_2 \\
& - 60\varepsilon u^3 v_2 + 280\gamma\varepsilon^2 u_1^2 v_1 - 210\varepsilon u_1^2 v_1 + 320\gamma\varepsilon^2 u^2 u_1 v_1 \\
& - 240\varepsilon u^2 u_1 v_1 + 40\gamma\varepsilon^2 u^4 v_1 - 30\varepsilon u^4 v_1.
\end{aligned}
$$

This means that we can find a suitable transformation to linearise this system.

Hierarchy of symmetries

By the property of Master symmetry, the adjoint action of τ on K, i.e, $\mathtt{ad}_\tau^n K$ generates infinitely many symmetries, each of order $3+2n$.

Theorem 5.3. *$\mathtt{ad}_\tau^n K$ generates symmetry of order $3+2n$ for a positive integer n.*

Proof. We need to find the order of symmetries, therefore we focus only on the highest order terms appearing in the equation. For example, we write

$$
K = \begin{pmatrix} u_3 + 2\varepsilon v_3 \\ v_3 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} -x u_3 - 2\varepsilon x v_3 \\ -x v_3 \end{pmatrix}, \tag{5.17}
$$

and forget about the lower order terms, then

$$
\mathtt{ad}_\tau K = \begin{pmatrix} -3u_5 - 12\varepsilon v_5 \\ -3v_5 \end{pmatrix}, \tag{5.18}
$$

which is a symmetry of order 5.

We prove the theorem by induction on n. The above equation shows that our hypothesis is true for $n=1$. Assume, the result holds for arbitrary n, then for $n+1$ we have

$$\mathtt{ad}_\tau^{n+1}K = [\tau, \mathtt{ad}_\tau^n K].$$

By induction hypothesis, $\mathtt{ad}_\tau^n K = S$ is a symmetry of order $3+2n$, therefore we can write,

$$\begin{aligned}
\mathtt{ad}_\tau^{n+1}K =& [\tau, S] \\
=& \left[\begin{pmatrix} -xu_3 - 2\varepsilon xv_3 \\ -xv_3 \end{pmatrix} \quad \begin{pmatrix} au_{3+2n} \\ bv_{3+2n} \end{pmatrix} \right] \\
=& \begin{pmatrix} -a(3+2n)u_{3+2(n+1)} + 2a\varepsilon(3+2n)v_{(3+2(n+1))} \\ -b(3+2n)v_{(3+2(n+1))} \end{pmatrix}
\end{aligned}$$

where a and b are constants. It is now clear that the highest order is $(3+2(n+1))$ and thus, the result follows. $\square$

Time dependent symmetries

For time-dependent symmetries we make use of Theorem 5.2. It requires us to find an element w such that

$$\mathtt{ad}_e w = f, \qquad \mathtt{ad}_K w = \frac{1}{2-4\kappa}\mathtt{ad}_f^2 K.$$

Keeping in mind Lemma 5.3, we find this element $w = -\int f\, dx$ which provides

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x^3u_x + x^2u - x \\ \frac{1}{3}x^3v_x + x^2v + \gamma x \end{pmatrix},$$

with $\mathtt{ad}_e w = f$. To satisfy the second condition we need to find a value of γ such that the following identity holds, we already know that $\frac{1}{6}[f,K] = \tau$, therefore condition 5.4.2 becomes

$$\left[\begin{pmatrix} u_t \\ v_t \end{pmatrix} \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] - \frac{-3}{5} \left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \right] = 0.$$

On inspection we find that for $\gamma = 0$ and $\gamma = 3$, i.e,

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x^3u_x + x^2u - x \\ \frac{1}{3}x^3v_x + x^2v \end{pmatrix}, \text{and} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x^3u_x + x^2u - x \\ \frac{1}{3}x^3v_x + x^2v + 3x \end{pmatrix},$$

satisfies all the necessary conditions of Theorem 5.2. Therefore,

$$\frac{\partial^j}{\partial t^j}\exp(-t\,\mathtt{ad}_K)(\mathtt{ad}_F^n W)$$

are time-dependent symmetries for $n = 0,1,\cdots$ and $j = 0,1,\cdots n+2$.

Above example illustrates in detail how we can compute master and t-dependent symmetries. In the following examples we continue in the same fashion without providing much details.

Example 5.4.3. Consider the System

$$\left.\begin{aligned} u_t &= B_3[u] + \varepsilon(v_2 + 2uv_x + vu_x + u^2 v)_x, \\ v_t &= v_3 + 6u_x v_x - \varepsilon(vv_2 - v_x^2 + v^2 u_x). \end{aligned}\right\} = K \tag{5.19}$$

It is also $(1,1)$ homogeneous with

$$\left[\begin{pmatrix} xu_x + u \\ xv_x + v \end{pmatrix} \quad \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = 3\begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

Similar technique as in the previous example suggests $f = \int h dx$ which gives,

$$f = -\begin{pmatrix} x^2 u_x + 2xu - 2 \\ x^2 v_x + 2xv + 6 \end{pmatrix}.$$

This produces a master symmetry τ,

$$\frac{1}{6}\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

provided $[[\tau,K],K] = 0$, i.e, we need to show $[\tau,K]$ is a symmetry of the system. We find that

$$\begin{aligned} \tau_1 = &-xu_3 - \varepsilon xv_3 - \varepsilon xvu_2 - 3xuu_2 - 2u_2 - 2\varepsilon xuv_2 - \tfrac{4}{3}\varepsilon v_2 \\ &- 3xu_x^2 - 3\varepsilon xu_x v_x - 2\varepsilon xuvu_x - 3xu^2 u_x - \tfrac{4}{3}\varepsilon vu_x \\ &- 5uu_x - \varepsilon xu^2 v_x - \tfrac{7}{3}\varepsilon uv_x - \varepsilon u^2 v - u^3, \\ \tau_2 = &-xv_3 + \varepsilon xvv_2 - v_2 - 6xu_x v_x + \varepsilon xv^2 u_x - \varepsilon xv_x^2 + \tfrac{1}{3}\varepsilon vv_x \\ &- 2uv_x + \tfrac{1}{3}\varepsilon uv^2, \end{aligned}$$

and the maple computation (see code in B) indeed shows $[[\tau, K], K] = 0$.

Master symmetry $\mathbf{M}$ computed by Talati and Turhan [50] is

$$\begin{aligned} M_1 = xu_3 &+ \varepsilon xv_3 + \varepsilon xvu_2 + 3xuu_2 + 2u_2 + 2\varepsilon xuv_2 + 3\varepsilon v_2 \\ &+ 3xu_x^2 + 3\varepsilon xu_xv_x + 2\varepsilon xuvu_x + 3xu^2u_x + 6\varepsilon vu_x \\ &+ 5uu_x + \varepsilon xu^2v_x + 8\varepsilon uv_x + 4\varepsilon u^2v + u^3, \\ M_2 = xv_3 &- \varepsilon xvv_2 + v_2 + 6xu_xv_x - \varepsilon xv^2u_x \\ &+ \varepsilon xv_x^2 - 2\varepsilon vv_x + 3uv_x - 4\varepsilon uv^2. \end{aligned}$$

Similar to the previous example, here we can also find a lower order master symmetry $\mathscr{M} = \mathbf{M} + \tau$ with order 2

$$\begin{aligned} \mathscr{M}_1 =& \tfrac{5}{3}\varepsilon v_2 + \tfrac{14}{3}\varepsilon vu_x + \tfrac{17}{3}\varepsilon uv_x + 3\varepsilon u^2v, \\ \mathscr{M}_2 =& uv_x - \tfrac{5}{3}\varepsilon vv_x - \tfrac{11}{3}\varepsilon uv^2. \end{aligned}$$

In this case, $\mathtt{ad}^n_{\mathscr{M}}$ generates symmetry of order $3+n$ whereas $\mathtt{ad}^n_{\tau}$ generates symmetry of order $3+2n$. Proofs are similar to Theorem 5.3 in Example 5.4.2.

Example 5.4.4. The following system

$$\left.\begin{aligned} u_t &= B_3[u] + 3\varepsilon(v_2 - vu_x)_x + 3\varepsilon^2(4vv_2 + 3v_x^2 - v^2u_x) + 12\varepsilon^3v^2v_x, \\ v_t &= v_3 - 3(uv_x - u^2v)_x + 3\varepsilon(3vv_2 + 3v_x^2 - 2(v^2u)_x + 21\varepsilon^2v^2v_x. \end{aligned}\right\} = K \tag{5.20}$$

is $(1,1)$ homogeneous similar to our previous examples with the same κ factor,

$$\left[\begin{pmatrix} xu_x + u \\ xv_x + v \end{pmatrix} \quad \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = 3\begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

With $f = -\begin{pmatrix} x^2u_x + 2xu + 1 \\ x^2v_x + 2xv \end{pmatrix}$, the following Lie bracket

$$\frac{1}{6}\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix},$$

provides a master symmetry τ with

$$\begin{aligned}
\tau_1 = {} & -xu_3 - 3\varepsilon xv_3 + 3\varepsilon xvu_2 - 3xuu_2 - \tfrac{5}{2}u_2 - 12\varepsilon^2 xvv_2 - 6\varepsilon v_2 \\
& - 3xu_x^2 + 3\varepsilon xu_x v_x + 3\varepsilon^2 xv^2 u_1 - 3xu^2 u_x + 4\varepsilon vu_x - 6uu_x - 9\varepsilon^2 xv_x^2 \\
& - 12\varepsilon^3 xv^2 v_x - 18\varepsilon^2 vv_x + \varepsilon uv_x - 4\varepsilon^3 v^3 + \varepsilon^2 uv^2 - u^3, \\
\tau_2 = {} & -xv_3 - 9\varepsilon xvv_2 + 3xuv_2 - \tfrac{3}{2}v + 3xu_x v_x + 6\varepsilon xv^2 u_x - 6xuvu_x \\
& - 9\varepsilon xv_x^2 - 21\varepsilon^2 xv^2 v_x + 12\varepsilon xuvv_x - 3xu^2 v_x - 13\varepsilon vv_x + 3uv_x \\
& - 7\varepsilon^2 v^3 + 6\varepsilon uv^2 - 3u^2 v.
\end{aligned}$$

After comparing the master symmetry $\mathbf{M}$ as in [50],

$$\begin{aligned}
M_1 = {} & xu_3 + 3\varepsilon xv_3 - 3\varepsilon xvu_2 + 3xuu_2 + 2u_2 \\
& + 12\varepsilon^2 xvv_2 + 3\varepsilon v_2 + 3xu_x^2 - 3\varepsilon xu_x v_x \\
& - 3\varepsilon^2 xv^2 u_x + 3xu^2 u_x + 6\varepsilon vu_x + 5uu_x \\
& + 9\varepsilon^2 xv_x^2 + 12\varepsilon^3 xv^2 v_x + 8\varepsilon uv_x + 4\varepsilon u^2 v + u^3 \\
M_2 = {} & xv_3 + 9\varepsilon xvv_2 - 3xuv_2 + v_2 - 3xu_x v_x \\
& - 6\varepsilon xv^2 u_x + 6xuvu_x + 9\varepsilon xv_x^2 \\
& + 21\varepsilon^2 xv^2 v_x - 12\varepsilon xuvv_x \\
& + 3xu^2 v_x - 2\varepsilon vv_x + 3uv_x - 4\varepsilon uv^2,
\end{aligned}$$

we find a lower order master symmetry $\mathscr{M} = \mathbf{M} + \tau$ of order 2,

$$\begin{aligned}
\mathscr{M}_1 = {} & -\tfrac{1}{2}u_2 - 3\varepsilon v_2 - uu_x + 10\varepsilon vu_x + 9\varepsilon uv_x \\
& - 18\varepsilon^2 vv_x + 4\varepsilon u^2 v - 4\varepsilon^3 v^3 + \varepsilon^2 uv^2 \\
\mathscr{M}_2 = {} & -\tfrac{1}{2}v_2 + 6uv_x - 15\varepsilon vv_x - 3u^2 v \\
& + 2\varepsilon uv^2 - 7\varepsilon^2 v^3.
\end{aligned}$$

Here again we find that $\mathtt{ad}^n_{\mathscr{M}}$ generates symmetry of order $3+n$ whereas $\mathtt{ad}^n_\tau$ generates symmetry of order $3+2n$.

For time-dependent symmetries we find the element $w = -\int f\,dx$ which provides

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}x^3 u_x + x^2 u + x \\ \frac{1}{3}x^3 v_x + x^2 v \end{pmatrix}.$$

Notice that the required constant of integrations are 0 so that the following conditions are satisfied

$$\mathtt{ad}_e w = f, \quad \left[\begin{pmatrix} u_t \\ v_t \end{pmatrix} \;\; \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] - \tfrac{-3}{5}\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \;\; \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}\right] = 0.$$

Now,

$$\frac{\partial^j}{\partial t^j} \exp(-t\,\mathtt{ad}_K)(\mathtt{ad}_f^n w)$$

are time-dependent symmetries for $n = 0, 1, \cdots$ and $j = 0, 1, \cdots n+2$.

So far all the two-component systems that we have considered have appeared in [50]. The following two examples are from current work (in progress) of Wang et al. [29].

Example 5.4.5. Consider the following system,

$$\left.\begin{aligned} u_t =& u_3 + (\lambda - 1)\left[2(u+v)v_2 + 6u^2 v_1 - 12uvv_1 - 10v^2 v_1 + (u-v)^2(u+3v)(v+3u)\right] \\ &+ 6uu_2 + 9u_1^2 - 6(vu_1)_x + (\lambda+2)\left[-v_1^2 + 4(u^2+v^2)u_1\right] + 8(\lambda-4)uvu_1, \\ v_t =& \lambda v_3 + (\lambda - 1)\left[2(u+v)u_2 + 10u^2 u_1 + 12uvu_1 - 6v^2 u_1 + (u-v)^2(u+3v)(v+3u)\right] \\ &+ (2\lambda+1)\left[u_1^2 + 4(u^2+v^2)v_1\right] + 3\lambda\left[2(uv_1)_x - 2vv_2 - 3v_1^2\right] - 8(4\lambda-1)uvv_1, \\ &\lambda \in \mathbb{C}\setminus\{0,1\}, \end{aligned}\right\} = K \tag{5.21}$$

which is $(1,1)$ homogeneous with weight 3. We find a master symmetry τ by

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = -\frac{1}{6}\left[\begin{pmatrix} x^2 u_1 + 2xu - \frac{1}{4} \\ x^2 v_1 + 2xv + \frac{1}{4} \end{pmatrix} \;\; \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right]$$

with,

$$\begin{aligned} \tau_1 = & -xu_t - \tfrac{3}{2}u_2 + 6vu_1 - 10uu_1 + 3uv_1 + 3vv_1 - \lambda uv_1 - \lambda vv_1 + 11u^2 v \\ & - 5uv^2 - \lambda u^3 + 3\lambda v^3 - 3u^3 - 3v^3 - 3\lambda u^2 v + \lambda uv^2, \\ \tau_2 = & -xv_t - \tfrac{3}{2}\lambda v_2 + vu_1 + uu_1 - 3\lambda uu_1 - 3\lambda vu_1 - 6\lambda uv_1 + 10\lambda vv_1 \\ & + u^2 v - 3uv^2 - 3\lambda u^3 - 3\lambda v^3 + 3u^3 - v^3 - 5\lambda u^2 v \\ & + 11\lambda uv^2 + 4xu^3 v - 14xu^2 v^2 + 4xuv^3. \end{aligned}$$

Since the order of the master symmetry is 3, one can prove that $\mathtt{ad}_\tau^n$ generates symmetry of order $3+2n$.

Example 5.4.6. The following system

$$\left.\begin{aligned}u_t =&u_3+3((u^2+v^2)u_1)_x-2(\lambda-1)uvv_2+(\lambda+2)uv_1^2+3(u^4+v^4)u_1\\&+3uu_1^2-2(\lambda-1)(u^2+3v^2)uvv_1-2(2\lambda-5)u^2v^2u_1-(\lambda-1)uv^2(u^2+v^2)^2,\\v_t =&\lambda(v_3)+2(\lambda-1)uvu_2+6\lambda uu_1v_1+9\lambda vv_1^2+3\lambda(u^2+v^2)v_2\\&+(2\lambda+1)vu_1^2+2(\lambda-1)uv(v^2+3u^2)u_1+3\lambda(u^4+v^4)v_1+2(5\lambda-2)u^2v^2v_1\\&+(\lambda-1)u^2v(u^2+v^2)^2.\end{aligned}\right\} = K \tag{5.22}$$

is the only system so far which is $(\frac{1}{2},\frac{1}{2})$ homogeneous with weight 3. Here, we find a master symmetry as

$$-\frac{1}{6}\left[\begin{pmatrix}x^2u_1+xu\\x^2v_1+xv\end{pmatrix}\quad\begin{pmatrix}u_t\\v_t\end{pmatrix}\right]=\begin{pmatrix}\tau_1\\\tau_2\end{pmatrix},$$

with

$$\begin{aligned}\tau_1 &= -xu_t-\tfrac{3}{2}u_2-5u^2u_1-3v^2u_1-2u^3v^2-\tfrac{3}{2}uv^4-\tfrac{1}{2}u^5\\&\qquad+\lambda uvv_1+\lambda u^3v^2+\lambda uv^4-3uvv_1,\\\tau_2 &= -xv_t-\tfrac{3}{2}\lambda v_2-\tfrac{3}{2}\lambda u^4v-2\lambda u^2v^3-3\lambda u^2v_1-5\lambda v^2v_1+uvu_1\\&\qquad-\tfrac{1}{2}\lambda v^5+u^4v+u^2v^3-3\lambda uvu_1.\end{aligned}$$

which generates symmetries $S_n=\mathtt{ad}_\tau^n$ of order $3+2n$.

5.5 Two-component (2+1)-dimensional partial differential equations

The $\mathscr{O}$-scheme (Diagram 5.1) is a powerful tool for testing the integrability of $(1+1)$-dimensional nonlinear partial differential equations. We wish to extend this method for the $(2+1)$-dimensional case but we encounter few obstacles, chiefly non-locality, i.e, the appearance of the formal integral D_x^{-1} or D_y^{-1}. The higher order symmetries and the equation themselves are non-local in their evolutionary form for integrable equations. This was noted by Mikhailov and Yamilov in [30], where they introduced a concept of *quasi-local polynomials* to characterize nonlocalities. The appearance of such operators forces us to extend the differential algebra $\mathscr{A}$ which is defined in the Section 4.2. The naive approach is to adjoin all possible integrals, i.e, to construct the differential algebra $\mathscr{A}(D_x^{-1},D_y^{-1})$. But now, any $f \in \mathscr{A}(D_x^{-1}, D_y^{-1})$ is a total derivative $f \in D\mathscr{A}(D_x^{-1}, D_y^{-1})$ and consequently,

$$\mathscr{A}(D_x^{-1}, D_y^{-1})/D\mathscr{A}(D_x^{-1}, D_y^{-1}) = \mathbb{C}.$$

This implies that all conservation laws are trivial. Therefore, such a construction seems not to be very fruitful.

Mikhailov and Yamilov also made another very important observation that the operators D_x^{-1} and D_y^{-1} never appear alone, but always in pairs like $D_x^{-1}D_y$ and $D_y^{-1}D_x$ for all known integrable equations and their hierarchies of symmetries. Based on this observation they introduced operators

$$\theta = D_x^{-1}D_y \text{ and } \theta^{-1} = D_y^{-1}D_x, \tag{5.23}$$

then many classes of equations and their symmetry hierarchies can be written without D_x^{-1} and D_y^{-1}. In the following section we construct not only such an extension but also the base ring $\mathscr{A}$ formally.

5.5.1 Quasi-local polynomials

Quasi-local polynomials are elements in the non-local extension of a base differential ring. We discuss the base ring first and then show how to construct such an extension.

Differential polynomials over an algebra K forms a differential ring

$$\mathscr{A} = \bigoplus_{k \geq 1} \mathscr{A}^k,$$

where $\mathscr{A}^k$ denotes the set of differential polynomials of degree k. Notice that since $k \geq 1$, $\mathscr{A}$ is a non-unital ring. We consider two derivations on this ring, total x-derivation and y-derivation given by

$$D_x = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{i+1,j} \frac{\partial}{\partial u_{ij}} \text{ and } D_y = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} u_{i,j+1} \frac{\partial}{\partial u_{ij}}. \tag{5.24}$$

This base ring is $\mathscr{A}$ (non-) commutative if the given algebra K is (non-) commutative.

To extend this ring non-locally, we introduce new operators

$$\theta = D_x^{-1}D_y \text{ and } \theta^{-1} = D_y^{-1}D_x \tag{5.25}$$

with

$$\theta\mathscr{A} = \{\theta f : f \in \mathscr{A}\} \text{ and } \theta^{-1}\mathscr{A} = \{\theta^{-1} f : f \in \mathscr{A}\}.$$

With this notation in hand we can construct the ring of quasi-local polynomials in a recursive manner. Define $\mathscr{A}_0(\theta) = \mathscr{A}$ and $\mathscr{A}_k(\theta)$ as the ring closure of the union given below

$$\mathscr{A}_{k-1}(\theta)\bigcup\theta\mathscr{A}_{k-1}(\theta)\bigcup\theta^{-1}\mathscr{A}_{k-1}(\theta). \tag{5.26}$$

It is clear that $\mathscr{A}_{k-1} \subset \mathscr{A}_k$. Now we can define the non-local extension of $\mathscr{A}$ as $\mathscr{A}(\theta) = \lim_{k\to\infty}\mathscr{A}_k$ such that for every $f \in \mathscr{A}$ there exists a k such that $f \in \mathscr{A}_k$. There is a natural gradation on this extension given by number of u and its derivatives

$$\mathscr{A}(\theta) = \bigoplus_{l\geq 1}\mathscr{A}^l(\theta), \quad \mathscr{A}^p(\theta).\mathscr{A}^q(\theta) \in \mathscr{A}^{p+q}(\theta). \tag{5.27}$$

We should note that $\mathscr{A}_k(\theta)$ is not invariant under change of variables. For instance, if

$$x' \to x+y \quad \text{and} \quad y' \to y$$

then

$$\theta = D_{x'}^{-1}D_{y'} \to (D_x^{-1}+D_y^{-1})D_{y'} = 1+\theta$$

Thus, $\theta^{-1} \to (1+\theta^{-1})^{-1}$ which is not in $\mathscr{A}_k(\theta)$.

5.5.2 Davey-Stewartson type system

Classical Davey–Stewartson system first appeared in [7], it provides a canonical description of the evolution of surface waves with slowly varying amplitude is written in the form

$$i\psi_t = \psi_{xx}+\psi_{yy}+2\psi\varphi, \quad \varphi_{xy} = |\psi|^2_{xx}+|\psi|^2_{yy}. \tag{5.28}$$

Here, we will consider a Davey-Stewartson type system from the work of Huard and Novikov [19]. They classified integrable dispersive $(2+1)$- dimensional equations of second order Davey-Stewartson type, which are of the form

$$u_t = F(u,v,w,Du,Dv,Dw), \quad w_y = u_x \tag{5.29}$$

$$v_t = G(u,v,w,Du,Dv,Dw). \tag{5.30}$$

Here $u(x,y,t), v(x,y,t), w(x,y,t)$ are scalar variables with Du, Dv, Dw as the collection of partial derivatives of u, v, w with respect to x, y up to the second order, and F, G are polynomials in derivatives with coefficients depending only on u, v, w.
System (5.28) can be transformed into a system of the form (5.29) by a change of variables

u and v, and rescaling of x and t. This connection is shown by Zakharov in [54]. We will compute a master symmetry for such a system in Example 5.5.1 but before that we introduce some results that are helpful in computing Lie brackets.

For a differential function P which is independent of x and t, i.e, if $P = P(u, u_1, ..u_n)$ then one may find the below equality useful,

$$\begin{aligned}[xP, P] =& P_*(xP) - (xP)_*(P), \\ =& \sum_{j\geq 0}^{n} \frac{\partial P}{\partial u_j} D^j(xP) - \sum_{i\geq 0}^{n} \frac{\partial xP}{\partial u_i} D^i(P), \\ =& \sum_{j\geq 1}^{n} j \frac{\partial P}{\partial u_j} D^{j-1}(P),\end{aligned}$$

where we used the fact that $(xP)_* = xP_*$ and $D^m(xP) = mD^{m-1}(P) + xD^m(P)$ for $m \geq 0$. This result can be generalised easily in the $(2+1)$- dimensional case.

Similarly, one may come across terms like $\theta^{-1}(u_1\theta^{-1}u)$, $\theta^{-1}(xu_1\theta^{-1}u)$ and $\theta^{-1}(x^k u_m)$ which requires simplified form for the calculations. Let us prove them one by one in the following proposition.

Proposition 5.1. *Following results holds,*

1. $\theta^{-1}(x^k u_m) = kx^{k-1}\theta^{-1}u_{m-1} + x^k\theta^{-1}u_m$ *for integers* $k, m \geq 1$.

2. $\theta^{-1}(u_1\theta^{-1}u) = \theta^{-1}(u_1)\theta^{-1}(u)$.

3. $\theta^{-1}(xu_1\theta^{-1}u) = \frac{1}{2}(\theta^{-1}u)^2 + x\theta^{-1}(u_1)\theta^{-1}(u)$.

Proof. We list the proof corresponding to the items.

1. This item is straightforward, first we replace the θ^{-1} by $D_xD_y^{-1}$ and pull out x^k since it acts as a constant for the operator D_y^{-1}. Finally expand the resulting expression using the product rule of D_x.

$$\begin{aligned}\theta^{-1}(x^k u_m) &= D_xD_y^{-1}(x^k u_m), \\ &= D_x\left(x^k D_y^{-1} u_m\right), \\ &= kx^{k-1}\theta^{-1}u_{m-1} + x^k\theta^{-1}u_m.\end{aligned}$$

2. For Item 2 we use the integration by parts in (5.31), then on replacing $D_y^{-1}u_1 = D_y^{-1}D_x u = \theta u$ and cancelling D_y^{-1} with D_y we obtain (5.32). Now we can expand

using the linearity and product rule of the operator D_x and up on bringing the second term on the left hand side we obtain the desired result

$$\begin{aligned}
\theta^{-1}(u_1\theta^{-1}u) &= D_xD_y^{-1}(u_1\theta^{-1}u), \\
&= D_x\left((\theta^{-1}u)D_y^{-1}u_1 - D_y^{-1}\left(D_y(\theta^{-1}u)(D_y^{-1}u_1)\right)\right), \qquad (5.31)\\
&= D_x\left((\theta^{-1}u)^2 - D_y^{-1}(u_1\theta^{-1}u)\right), \qquad (5.32)\\
&= 2(\theta^{-1}u)(\theta^{-1}u_1) - \theta^{-1}(u_1\theta^{-1}u).
\end{aligned}$$

3. Proof of the item (3) is also very similar.

$$\theta^{-1}(xu_1\theta^{-1}u) = D_xD_y^{-1}(xu_1\theta^{-1}u) \tag{5.33}$$

$$= D_x\left(\frac{1}{2}x(\theta^{-1}u)^2\right) \tag{5.34}$$

$$= \frac{1}{2}(\theta^{-1}u)^2 + x\theta^{-1}(u_1)\theta^{-1}(u), \tag{5.35}$$

where we used the fact that $D_y^{-1}(u_1\theta^{-1}u) = \frac{1}{2}(\theta^{-1}u)^2$.

□

Now we are ready to extend our theory for the $(2+1)$-dimensional case. The below example can be found in the Huard and Novikov [19].

Example 5.5.1. We follow the same routine procedure as before based on Theorem 5.1. The following system,

$$\left.\begin{aligned} u_t &= u_x\theta^{-1}u + \varepsilon u_2 + (uv)_x, \\ v_t &= (v\theta^{-1}u)_x - \varepsilon v_2 + vv_x. \end{aligned}\right\} = K \tag{5.36}$$

is homogeneous with

$$\left[\begin{pmatrix} xu_x \\ xv_x + v \end{pmatrix} \ \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = 2\begin{pmatrix} u_t \\ v_t \end{pmatrix}. \tag{5.37}$$

We obtain a master symmetry by the following action

$$\begin{aligned}
\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} &= \left[\begin{pmatrix} x^2u_x \\ x^2v_x + 2xv \end{pmatrix} \ \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] \\
&= 4\begin{pmatrix} xu_t + \frac{1}{2}\varepsilon u_x + \frac{1}{2}uv \\ xv_t + v\theta^{-1}u - \frac{3}{2}\varepsilon v_x + \frac{1}{2}v^2 \end{pmatrix}.
\end{aligned}$$

Once we have a master symmetry τ, we can proceed to compute a symmetry by computing the action of τ on the K, which yields

$$S = \begin{pmatrix} 2\varepsilon^2 u_3 + 3\varepsilon(u_1 v_1 + u_2 v + u_2\theta^{-1}u + u_1\theta^{-1}u_1) + 3(uv)_x\theta^{-1}u \\ +\frac{3}{2}uv\theta^{-1}u_1 + \frac{3}{2}u_1\theta^{-1}(uv) + 3uvv_1 + \frac{3}{2}u_1v^2 + \frac{3}{2}u_1(\theta^{-1}u)^2 \\ 2\varepsilon^2 v_3 - 3\varepsilon(u_2 v + v_1^2 + v_2\theta^{-1}u + v_1\theta^{-1}u_1) \\ 3vv_1\theta^{-1}u + \frac{3}{2}v^2\theta^{-1}u_1 + \frac{3}{2}v\theta^{-1}(uv)_x + \frac{3}{2}v_1\theta^{-1}(uv) \\ +\frac{3}{2}v_1v^2 + 3v\theta^{-1}u\theta^{-1}u_1 + \frac{3}{2}v_1(\theta^{-1}u)^2 \end{pmatrix} \tag{5.38}$$

It is indeed a symmetry since one can compute that $[\tau,[\tau,K]=0$.

In the following an attempt is made to give an idea how such calculations were carried out. If one wishes to find a second order symmetry S for 5.36, then we use the grading on $\mathscr{A}$ mentioned in the beginning of the application section. It tells us that S can be written in the form

$$S = S^{(0,0)} + S^{(0,1)} + S^{(1,0)} + S^{(1,1)} + S^{(0,2)} + S^{(2,0)}. \tag{5.39}$$

To compute S we need to compute the each Lie bracket relative to their grading. For example, the zero degree term corresponds to $S^{(0,0)} = [\tau^{(0,0)}, K^{(0,0)}]$. For our purpose, we only present the computation for $S^{(2,0)}$. Since $K^{(2,0)} = \tau^{(2,0)} = 0$, what remains is $S^{(2,0)} = [\tau^{(1,0)}, K^{(1,0)}]$ with

$$\tau^{(1,0)} = \begin{pmatrix} xu_1\theta^{-1}u \\ (xv\theta^{-1}u)_x \end{pmatrix} \text{and } K^{(1,0)} = \begin{pmatrix} u_1\theta^{-1}u \\ (v\theta^{-1}u)_x \end{pmatrix}. \tag{5.40}$$

It makes it easy to compute the Lie brackets and one finds that

$$S^{(2,0)} = \begin{pmatrix} \frac{3}{2}u_1\theta^{-1}(u\theta^{-1}u) \\ 3v\theta^{-1}(u_1\theta^{-1}u) + \frac{3}{2}v_1\theta^{-1}(u\theta^{-1}u) \end{pmatrix}. \tag{5.41}$$

5.6 Conclusions

We showed that regardless of integrability, there exists a natural $\mathfrak{sl}(2,\mathbb{C})$ algebra for a homogeneous equation and moreover, we can construct an infinite dimensional module in the BGG category. Based on these observations, we presented a new structure, called the $\mathscr{O}$ scheme. This method can be used to construct master symmetries and consequently symmetries for homogeneous equations under some technical conditions. We present new results—master symmetries for three new two-component Burgers' type $(1+1)$-dimensional systems that appeared in [50] (Examples 5.4.2, 5.4.3, 5.4.4) and for two new systems from the ongoing work of Wang et al. [29] (Examples 5.4.5, 5.4.6). Towards the end of the chapter we introduce quasilocal polynomials which provides an algebraic framework to deal with $(2+1)$-dimensional systems together with an example of a Davey-Stewartson type system (Example 5.5.1). In our results, we observed that for an evolution equation $u_t = K[u]$, their master symmetry takes the form $\tau = xu_t + r$ with some "remaining" differential function $r \in \mathscr{A}$. So for a future project, it might be interesting to start with an integrable equation explicitly depending on the space variable x and study the structure of its master symmetry.

Bi-Hamiltonian systems cover a high percentage of the known integrable systems but there are integrable equations like Burgers' equation and Ibragimov-Shabat equation which fall outside of this category. Our approach can be used to construct time-dependent symmetries for these equations. These symmetries can be seen as a part of $\mathfrak{sl}(2,\mathbb{C})$-module, however our understanding of their appearance in the construction of symmetries is still limited. Constructing a scheme based on algebra of higher rank is a promising direction of research which would allow us to study a wider class of PDEs. In future, extending this approach to integrable differential-difference and discrete systems could also be very important.

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Appendix A

Maple Code for Green's Function

This chapter provides the details on the extra code needed to extract Green's function in Examples 2.4.1 – 2.4.4.

```
[breaklines=true]
 create_cond1 := proc(P)
  local rng, fnc, cond;
  rng := P[1];
  fnc := P[2];
  cond:= x <= xi and (xi<=lhs(rhs(rng)));
  [cond, -fnc]
end proc:

create_cond2 := proc(P)
  local rng, fnc, cond;
  rng := P[1];
  fnc := P[2];
  cond:= (lhs(rhs(rng)) <= xi) and xi <=x;
  [cond, fnc]
end proc:

create_piecewise1 := proc(L)
  local L0;
  L0 := map(create_cond1, map(Reverse, L));
  piecewise(op(Flatten(L0)));
end proc:
```

```
create_piecewise2 := proc(L)
  local L1;
  L1 := map(create_cond2, map(Reverse, L));
  piecewise(op(Flatten(L1)));
end proc:

Greenop_eq:=proc(Gop)
  local G1, G2, G3;
  G1:=expand(subs(int=Int,Gop &* f(x)));
  G2:=eval(G1, x=y);
  G3:=value(evalindets(evalindets(G2,specfunc(Int),
  x->IntegrationTools:-Split(x, y)),And(specfunc(Int),
  satisfies(x->type(op([2,2,1],x),identical(y)))),IntegrationTools:-Flip));
  expand(eval(G3, y=x));
end proc:

extract_gf:=proc(Gfeq, evpts)
  local Gf1, Gf2, Gf3, L1, L2, L3, i, cn1, cn2, S, con1,
   func1, pair1, con2, func2, pair2, S1, S2;
  Gf1:= eval(Gfeq, x=y);
  Gf2 := subs(x=xi, Gf1);
  Gf3 := eval(Gf2, y=x);
  L1:=convert(combine(Gf3),list);
  L2:=map(`[]`@op,L1);
  L3:=subs(f(xi)=1, L2);
  S1:=0;
  S2:=0;
  for i from 1 while i<= nops(L3)
   do
    cn1[i]:=xi-lhs(rhs(L3[i,2]));
    # To get conditions for Green's function
    cn2[i]:=lhs(rhs(L3[i,2]))-xi;
    S1:=(Heaviside(cn1[i])*L3[i,1])+S1;
        # To get respective terms in Green's function
```

```
    S2:=(Heaviside(cn2[i])*(-L3[i,1]))+S2;
   end do;

  for i from 1 while i<= nops(L3)-1
   do
    con1[i]:= (evpts[i]<=xi and xi<=evpts[i+1] and xi<=x);
    func1[i]:=(S1 assuming evpts[i]<xi and xi<evpts[i+1]);
    pair1[i]:=con1[i], simplify(func1[i]);
    con2[i]:= (evpts[i]<=xi and xi<=evpts[i+1] and x<=xi);
    func2[i]:=(S2 assuming evpts[i]<xi and xi<evpts[i+1]);
    pair2[i]:=con2[i], simplify(func2[i]);
   end do;
  piecewise(op(Flatten(convert(pair1, list))),
   op(Flatten(convert(pair2, list))));
end proc:

regular_part := proc(G)
  applyrule(t::EVDIFFOP=EVDIFFOP(), G)
end proc:

distributional_part := proc(G)
  G-regular_part(G)
end proc:

renaming_apply := proc(G,f)
  local xi;
  subs(xi=x,subs(x=xi, G) &* f)
end proc:

safe_apply := proc(G,f)
  local Greg,Gdist;
  Greg := regular_part(G);
  Gdist := distributional_part(G);
  renaming_apply(Gdist,f)+ApplyOperator(Greg,f)
end proc:
```

Following code is used to obtain the third order differential equation from a given set of fundamental solution. It is used to cross check the examples.

```
with(VectorCalculus):
ode2sol:= proc(v)
  local W, Wr, d, Wk, M, W1, W2, W3, q1, q2, q3;
  W:=Vector([v], readonly=true);
  Wr:=wronskian(W,x);
  d:=det(Wr);
  Wk:=Vector([y(x), v], readonly=true);
  M:=Matrix(wronskian(Wk,x));
  W1:=Adjoint(M)(1,3);
  W2:=Adjoint(M)(1,2);
  W3:=Adjoint(M)(1,1);
  q1:=-W1/d;
  q2:=-W2/d;
  q3:=-W3/d;
  diff(y(x), x, x, x)*1 + q1*(diff(y(x), x, x))+ q2*diff(y(x), x)+ q3*y(x)
end proc:
```

Appendix B

Maple Code for Computing Lie Brackets

This chapter provides details on the Maple code used to compute master symmetries and time-dependent symmetries in the examples of Chapter 4. The following code is used for scalar case in Example 5.4.1.

```
restart:
DD:=proc(f,n)#function to compute total derivative of f of order n
local w,ii,i,vv,ll;
w:=f;
for ii to n do
vv:=0;
ll:=sort(convert(map(op,indets(w)),list)):
 for i in ll do #this loop differentiate wrt u[i]
 vv:=vv+u[i+1]*diff(w,u[i]);
 od;
w:=diff(w,x)+vv; #this just do extra diff wrt x
od;
RETURN(sort(expand(w)));
end:
LieBracket:=proc(P,Q)
# computing Lie bracket of the given funxtions P and Q (By Myself)
local n1,n2,i, A, B;
n1:=max(convert(map(op,indets(P,name)),set) minus indets(P,name));
#highest order derivative appearing in P
n2:=max(convert(map(op,indets(Q,name)),set) minus indets(Q,name));
A:=0; B:=0;
```

```
for i from 0 to n2 do
A:=A+diff(Q,u[i])*DD(P,i);#Frechet derivative of acting on P ie P_*(Q)
od;
for i from 0 to n1 do
B:=B+diff(P,u[i])*DD(Q,i);
od;
return(expand(A-B));#[P,Q]=D[Q](P)-D[P](Q)
end:
LieBracketSys:=proc(P,Q,R,T)
# computing Lie bracket for a system (a11=P, a12=R, a21=Q, a22=T)
local n1,n2,i, A, B;
n1:=max(convert(map(op,indets(P,name)),set) minus indets(P,name));
#highest order derivative appearing in P
n2:=max(convert(map(op,indets(Q,name)),set) minus indets(Q,name));
A:=0; B:=0;
for i from 0 to n2 do
A:=A+diff(Q,u[i])*DD(P,i);#Frechet derivative of acting on P ie P_*(Q)
od;
for i from 0 to n1 do
B:=B+diff(P,u[i])*DD(Q,i);
od;
return(expand(A-B));#[P,Q]=D[Q](P)-D[P](Q)
end:
```

We use the below code for generalised two-component systems as in Examples 5.4.2 – 5.5.6.

```
restart:
DD:=proc(f,n)#function to compute total derivative of f of order n
local w,ii,i,vv,ll;
w:=f;
for ii to n do
vv:=0;
ll:=sort(convert(map(op,indets(w)),list)):
 for i in ll do #this loop differentiate wrt u[i]
 vv:=vv+u[i+1]*diff(w,u[i])+v[i+1]*diff(w,v[i]);
 od;
w:=diff(w,x)+vv; #this just do extra diff wrt x
```

```
od;
RETURN(sort(expand(w)));
end:
LieBracketFre:=proc(P,Q,R,T)# computing Lie bracket for
a system (a11=P, a12=R, a21=Q, a22=T)
local hdu,hdv,i, A,L,K, B,hdU,hdV,j, C, D,M,N;

L:=convert(map(op,subs(seq(u[i]=0, i = 0 .. 10),indets(P,name))),set)
minus subs(seq(u[i]=0, i = 0 .. 10),indets(P,name));
L:={op(L),0};
hdv:=max(L);#highest order derivative of v appearing in P
K:=convert(map(op,subs(seq(v[i]=0, i = 0 .. 10),indets(P,name))),set)
minus subs(seq(v[i]=0, i = 0 .. 10),indets(P,name));
K:={op(K),0};
hdu:=max(K);#highest order derivative of u appearing in P
A:=0; B:=0;
for i from 0 to hdu do
A:=A+diff(P,u[i])*DD(R,i);#Frechet derivative of P wrt u acting on R
od;
for i from 0 to hdv do
B:=B+diff(P,v[i])*DD(T,i);#Frechet derivative of P wrt v acting on T
od;
return(simplify(A+B));

#[P,Q]=D[Q](P)-D[P](Q)
end:
LB:=proc(P,Q,R,T)
return(<LieBracketFre(P,Q,R,T),LieBracketFre(Q,P,R,T) >);#<P,Q>_*
end:
LieBracketSys:=proc(P,Q,R,T)
return(-LB(P,Q,R,T)+LB(R,T,P,Q));
end:
collectsort:=proc(f)#sorting with order u[i+1]>u[i] and u[i]=v[i]
local g,In,h;
h:=expand(f);
```

```
In:=seq([u[i],v[i]][],i=20..1,-1);
g:=sort(h, order=plex(In));
return(g);
end:
adjointP:=proc(n,p,q,r,t)#to compute adnP i.e n times
 adjoint of P=<p,q> first two components
local M,L,i,j;
M:=LieBracketSys(p,q,r,t);
L:=convert(M,list);
for i from 1 to n-1 do
j:=nops(L);
L:=[op(L),op(convert(LieBracketSys(p,q,L[j-1],L[j]),list))];#recursion
od;
return(<collectsort(L[-2]),collectsort(L[-1])>);
end:
```