Applications for Smooth and Discrete Moving Frames

Ana Rojo-Echeburúa

A thesis submitted for the
Degree of Doctor of Philosophy
in the subject of
Applied Mathematics

School of Mathematics, Statistics and Actuarial Science
University of Kent
Canterbury

March 2019

Declaration

I affirm that the presented thesis has not been submitted before with the purpose of achieving any qualification at any University or Institution of any kind. I hereby declare that this project is based on genuine research in collaboration with my supervisor Professor Elizabeth Mansfield and other authors whose joint-work has been distinctly stated. Any additional sources of information have been duly cited and any other published and unpublished contributions have been fully quoted and referenced.

Contribution to the papers [74], [75], [76] and [77]

[74], Mansfield, E. L., Rojo-Echeburúa, A., Hydon, P. E., Peng, L., *Moving Frames and Noether's Finite Difference Conservation Laws I.*
My contribution to this paper was the development of the running example as well as the application to Euler's elastica. I also checked the theory and provided comments and observations.

[75], Mansfield, E. L., Rojo-Echeburúa, A., *Moving Frames and Noether's Finite Difference Conservation Laws II.*
In this paper, both Elizabeth Mansfield and I worked on the format and also on the abstract, introduction and conclusion. I calculated the adjoint action, discrete frame and invariants, Euler-Lagrange equations and conservation laws of the linear action of $SL(2)$ in the plane and the $SA(2)$ linear action. I checked the calculations of the general solution for all three examples and also for the adjoint action, discrete frame and invariants, Euler-Lagrange equations and conservation laws of the $SL(2)$ projective action.

[76], Mansfield, E. L., Rojo-Echeburúa, A., Wang, J. P., *Discrete Moving Frames, Evolution of Curvature Invariants and Integrability.*
The abstract and introduction were written by myself. I also computed the linear transformations running example and developed the invariant differentiation section. The examples were fully calculated by myself except for the integrability part in where I had some guidance from Jing Ping Wang. The rest of the paper is written by myself and the results are based on discussion with Elizabeth Mansfield.

[77], Mansfield, E. L., Rojo-Echeburúa, A., *Variational Systems with a Euclidean Symmetry using the Rotation Minimising Frame.*

Elizabeth Mansfield and I worked jointly on the introduction, section two, examples and application. The invariant calculus of variations was developed by myself.

Acknowledgements

Firstly, I would like to immensely thank my thesis supervisor, Elizabeth Mansfield, for pointing me in the right direction on many occasions and for her motivation and enthusiasm during all this time. Without her valuable input this project would not have been successfully conducted.

Secondly, I would also wish to express my sincere gratitude to Peter Hydon and Jing Ping Wang for their time, contribution and insightful comments.

I would like to offer my special thanks to Evelyne Hubert for pointing out the use of the Rotation Minimising frame in the Computer Aided Design literature, and the Maplesoft Customer Support for help with the plots of the sweep surfaces appearing in §7, which were performed using Maple 2018.

I would also like to thank my BSc and MSc final project supervisor Eduardo Martínez and my BSc and MSc lecturer Luis Randez for their support and encouragement since I started my mathematics degree, introducing me to the world of research and helping me during my application to this PhD position.

I am indebted to the internal and external examiners for accepting to evaluate and assess my work. I would also like to the SMSAS department at the University of Kent and the EPSRC for generously funding this research.

Finally, last but not least, I am beholden to my parents, the rest of my family and my friends for their unfailing supporting throughout these years of research and through the journey of writing this thesis.

This accomplishment would not have been achieved without you all.

Thank you so much.

Abstract

In this thesis, the calculation of Euler–Lagrange systems of ordinary difference equations is considered, including the difference Noether's Theorem. The discrete and difference moving frame is presented, and it is shown that for any Lagrangian that is invariant under a Lie group action on the space of dependent variables, the Euler–Lagrange equations can be calculated directly in terms of the invariants of the group action. Furthermore, Noether's conservation laws can be written in terms of a difference moving frame and the invariants. It is shown that this can significantly ease the problem of solving the Euler–Lagrange equations. We show the calculations for a discretisation of the Lagrangian for the Euler's elastica, and compare our discrete solution to that of its smooth continuum limit. We also study in depth some finite difference Lagrangians which are invariant under specific Lie group actions such as the special unitary action, the linear and projective actions of $SL(2)$, and the linear equi-affine action which preserves area in the plane. We first find the generating invariants, and then we write the Euler–Lagrange difference equations and Noether's difference conservation laws for any invariant Lagrangian, in terms of the invariants and a difference moving frame. We then give the details of the final integration step, assuming the Euler–Lagrange equations have been solved for the invariants. This last step relies on understanding the Adjoint action of the Lie group on its Lie algebra. Effectively, for all three actions, we show that solutions to the Euler–Lagrange equations, in terms of the original dependent variables, share a common structure for the whole set of Lagrangians invariant under each given group action, once the invariants are known as functions on the lattice. The projective special linear group action, and the special euclidean action in $\mathbb{R}^2$ are explored using multispace theory.

Moreover, we show how to compute the discrete correction matrices and prove that the curvature matrix can be computed simply by knowing only the correction matrix and the Lie algebra of the Lie group. We prove that the relationships between a discrete flow and its induced curvature flow is in terms of a syzygy operator and that it is a linear shift operator depending only on the curvature invariants. We also show how this is related to discrete integrable systems for some Lie group actions.

We also present the Rotation minimising frame and show how to use the known symbolic techniques despite the fact that it does not readily fit the known framework needed for these techniques. We derive the invariant differentiation formulae and the syzygy operator needed to obtain Noether's laws for variational problems with a Euclidean symmetry using the Rotation minimising frame and present some application in biological problems. We also develop the relationships between two frames differing by a gauge in the general case and prove that the curvature matrices of one frame can also be written in terms of the curvature matrices coming from the other frame and study some examples.

Contents

Introduction

The name of moving frames is associated with Élie Cartan, [14] who referred to them as *repère mobiles* and used them to solve equivalence problems in differential geometry. However, moving frames appear in earlier work by other authors such as Cotton, [17] and Darboux, [18] and continued to be studied as shown in Green and Griffiths, [35, 36]. Work by Fels and Olver, [24, 25] showed that, given a Lie group action, defining a frame as an equivariant map from the manifold to the group led to symbolic recurrence formulae for the differential invariants amongst many other insights.

The Fels and Olver approach is well suited to symbolic computation as presented in Olver, [87, 88], Hubert, [41, 43, 44], and Hubert and Kogan, [45, 46]. Thanks to the invariant calculus one can study differential systems which are either invariant or equivariant under the action of a Lie group, see Mansfield, [78], and it can be implemented using Mathematica or Maple. In Mansfield, [70], the author provides an introduction to the symbolic differential invariant calculus together with some applications.

The theory of Lie group based moving frames is now well established with significant applications. Related to this thesis are: construction of the invariant Euler-Lagrange equations from their invariant Lagrangian, (Kogan and Olver, [63]), computation of symmetry groups and classification of partial differential equations and integration of Lie group invariant differential equations, (Mansfield, [69], Morozov, [82]), the Noether correspondence between symmetries and invariant conservation laws, (Gonçalves and Mansfield, [32, 33]), integrable systems, (Beffa, [2, 3, 4], Mansfield and van der Kamp, [73], Mansfield and R–E, [76]), symmetry reduction of dynamical systems, (Hubert and Labahn, [47], Siminos and Cvitanovic, [101]), Lie pseudo-groups, (Olver and Pohjanpelto, [90]) and applications to computer aided design (Mansfield and R–E, [77]).

The theory of moving frames has been recently extended to the discrete case, leading to new applications such as integrable differential–difference systems, (Beffa, Mansfield and Wang, [6], Mansfield, R–E and Wang, [76]), invariant evolutions of projective polygons, (Beffa and Wang, [7]), computer vision, (Olver, [89]) and numerical schemes for systems with a Lie symmetry, (Kim, [56, 57, 58], Mansfield and Hydon, [72], Rebelo and Valiquette, [97]).

The first results for the computation of discrete invariants using group-based moving frames were given by Olver who called them joint invariants in [88]. But this approach was not computationally useful. However, in Beffa, Mansfield and Wang, [6], a notion of a discrete moving frame is introduced, which is essentially a sequence of frames, and which is adapted to discrete computation. In that paper, discrete recursion formulae were proven for small computable generating sets of invariants, called the discrete Maurer–Cartan invariants and their recursion relations called syzygies were studied. The theory of discrete moving frames is extended in Beffa and Mansfield, [5] by considering lattice based multispaces where the frame is simultaneously a smooth frame and a frame defined on local difference approximations. In Mansfield, R–E, Hydon and Peng, [74] and Mansfield and R–E, [75] a discrete analogue of the theorems appearing in Gonçalves and Mansfield, [32, 33, 34] is presented. In both smooth and discrete cases, it is shown how to calculate the invariant Euler–Lagrange system in terms of the standard Euler operator, a syzygy operator specific to the action, and the invariant Lie derivatives acting on the invariant volume form. It is also shown how to obtain the linear space of conservation laws in terms of vectors of invariants, and the Adjoint representation of a moving frame for the Lie group action. This new structure for the conservation laws allows the calculations for the extremals to be reduced and given in the original variables, once the Euler–Lagrange system is solved for the invariants.

This thesis will be divided in nine different chapters, this being the first one.

In §2, an introductory background is given in order to give context to the following chapters, where the notion of moving frame is presented.

In §3, we introduce the discrete and difference moving frames. Given an invariant discrete Lagrangian, a general formula for computing the discrete Euler–Lagrange equations in terms of the invariants of the symmetry group and a way of expressing Noether's conservation laws in terms of a difference moving frame and the invariants of the symmetry group is presented. The theory is illustrated with a running example.

In §4, we give some applications of the results presented in the previous chapter. Apart from the study of systems that are inherently discrete, one significant application is to obtain geometric (variational) integrators that have finite difference approximations of the continuous conservation laws embedded a priori. This is achieved by taking an invariant finite difference Lagrangian in which the discrete invariants have the correct continuum limit to their smooth counterparts. We show the calculations for a discretization of the Lagrangian for Euler's elastica, and compare our discrete solution to that of its smooth continuum limit. We also consider finite difference Lagrangians which are invariant under linear and projective actions of

$SL(2)$, and the linear equi-affine action which preserves the area in the plane. We first find the generating invariants, and then use the results appearing in §2 to write the Euler–Lagrange difference equations and Noether's difference conservation laws for any invariant Lagrangian, in terms of the invariants and a difference moving frame. We then give the details of the final integration step, assuming the Euler–Lagrange equations have been solved for the invariants. This last step relies on understanding the Adjoint action of the Lie group on its Lie algebra. For all three actions, we show that solutions to the Euler–Lagrange equations, in terms of the original dependent variables, share a common structure for the whole set of Lagrangians invariant under each given group action, once the invariants are known as functions on the lattice. The study of $SU(2)$ is also presented.

In §5, we explore when we have commuting flows on the invariants using discrete moving frames, given two commuting equivariant flows. We show that the relationships between a flow and its curvature flow is in terms of a syzygy operator. We prove that this is a linear shift operator depending only on the curvature invariants. We analyse the condition for discrete curve evolutions to commute in terms of a discrete moving frame. We exhibit two examples in order to illustrate the theory and relate them to discrete integrable systems.

In §6, we recall the basics of lattice based multispace theory and explore applications for two Lie groups studied in the previous chapters.

In §7, we show how to adapt the methods of Gonçalves and Mansfield, [32, 34] to study variational systems with an Euclidean symmetry, using the Rotation Minimising frame. We derive the recurrence formulae for the invariant differentiation expressions and the syzygy operator needed to obtain Noether's laws for variational problems with a Euclidean symmetry.

In §8, we develop the relationships between two frames differing by a gauge and explore a few examples in order to illustrate the theory.

In §9, we summarise what has been done in this thesis and present some questions that still need to be addressed.

Introductory Background

In this chapter we present the necessary background regarding groups, Lie groups, group actions, Lie algebras, infinitesimals, the Adjoint action, Calculus of Variations and moving frames in order to understand the next chapters. This section is based on some results appearing in Gonçalves and Mansfield, [32], Mansfield, [70], Mansfield and van der Kamp, [73], and Olver, [84]. Some of the examples, "easy to understand" explanations, as well as all the pictures have been developed by myself, most of them motivated by discussions with my supervisor and the lecture notes on *Lie groups and Lie algebras* given by her on my first year of my PhD. The examples and explanations that have not been developed by myself, have been referenced.

2.1 Groups, Lie groups and Lie algebras

The definition of a group is natural in the sense that there are lots of structures that consist of a set and a binary operation. For instance, the integers, the rational numbers, vectors, matrices, permutations, symmetries... and the list is almost endless. Therefore, it is logical to condense this feature of so many known objects in a definition.

Definition 2.1.1 (Group). *A group is a set G equipped with a binary operation*

$$\begin{aligned} G \times G &\rightarrow G, \\ (a, b) &\rightarrow a \cdot b \end{aligned}$$

satisfying the following properties

- *Closure:* $a \cdot b \in G$,
- *Associativity:* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
- *There exists* $e \in G$ *such that* $a \cdot e = e \cdot a = a$ *for all* $a \in G$. *We will call* e *the identity element.*

- *For all $a \in G$ there exists $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$. We will call a^{-1} the inverse element of a.*

Remark 2.1.2. *A group G is commutative or abelian if $a \cdot b = b \cdot a$.*

The groups we are going to be interested in are Lie groups. A Lie group is a group that is also a differentiable manifold - *which is just a (topological) space that locally looks like the Euclidean space near each point* - , so one can do calculus on it. Lie groups were named after Sophus Lie, a Norwegian mathematician, who introduced and developed them in order to integrate differential equations. Formally, we have the following definition:

Definition 2.1.3 (Lie group). *A Lie group is a finite dimensional smooth manifold G together with a group structure on G, such that the maps*

$$\begin{aligned} \mu\colon G \times G &\to G, \\ (a,b) &\to a \cdot b \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \nu\colon G &\to G, \\ a &\to a^{-1} \end{aligned}$$

are smooth.

Example 2.1.4. *The set of 2×2 rotation matrices form a group denoted by $\mathrm{SO}(2,\mathbb{R})$. It can be parametrised as follows*

$$\mathrm{SO}(2,\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \ \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

The multiplication, which corresponds to the addition on angles, and the inversion, which corresponds to the angle with opposite sign, are differentiable maps, where the binary operation is the matrix multiplication. Therefore $\mathrm{SO}(2,\mathbb{R})$ is a Lie group.

Example 2.1.5. *The Cantor set is created by iteratively removing the open middle third from a set of segments. First, the open middle third from the interval $[0,1]$ is deleted, leaving two line segments. Then, one removes the open middle third of each of these remaining segments, leaving four line segments. This process is continued infinitely, where the n-th set is given by $C_n = \dfrac{C_{n-1}}{3} \cup \left(\dfrac{2}{3} + \dfrac{C_{n-1}}{3}\right)$ for $n \geq 1$, and $C_0 = [0,1]$. The Cantor set can be seen as a (topological) group. This set cannot have the structure of a manifold as is totally disconnected and not discrete. Therefore, it is a group that is not a Lie group. This group is homeomorphic to the group of p-adic integers. One can create a continuous one to one mapping between the Cantor group and the dyadic integers as follows:*

$$\sum_{n=0}^{\infty} \frac{a_n}{3^{n+1}} \to \sum_{n=0}^{\infty} \frac{b_n}{2^{n+1}} \tag{2.1}$$

where $a_n \in \{0, 2\}$ *and* $b_n \in \{0, 1\}$*. In order to perform an addition of two sequences in the Cantor group, one maps these sequences to the dyadic group and performs the addition, which is coordinate-wise, with each coordinate addition in the integers mod* p^{i+1}*. If any of the sums is* p *or more, a carry of* 1 *needs to be taken to the next sum. Then, one take the result back to the Cantor group using the map* (2.1)*.*

From now on, the groups that we will consider will be Lie groups as they are the groups of interest in this thesis, as mentioned before. A very important Lie group is the general linear group $GL(n, \mathbb{F})$ - where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ - which is the group of square matrices with non zero determinant, together with the operation of matrix multiplication. It plays an important role in the theory of representations.

Definition 2.1.6 (Representation)**.** *A representation is a map*

$$\phi \colon G \to GL(n, \mathbb{F})$$

such that

$$\phi(g * h) = \phi(g) \cdot \phi(h)$$

where here $*$ *denotes the product in the Lie group* G *and* $\cdot$ *denotes the matrix product.*

Example 2.1.7. *The special euclidean group* $\mathrm{SE}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$ *is the Lie group of rotations and translations in* $\mathbb{R}^n$*. Let us denote* $R \in \mathrm{SO}(n)$ *the rotation part and* $v \in \mathbb{R}^n$ *the translation part. If we define*

$$\begin{aligned} \phi \colon \mathrm{SE}(n) &\to GL(n+1, \mathbb{R}) \\ (R, v) &\to \begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{2.2}$$

we obtain a matrix representation of $\mathrm{SE}(n)$*.*

2.1.1 Group actions

Let us consider a manifold M.

Definition 2.1.8 (Group action)**.** *A group* G *is said to act on a space* M *if there exists a map*

$$\alpha \colon G \times M \to M, \tag{2.3}$$

such that

$$\alpha(g_2, \alpha(g_1, z)) = \alpha(g_2 g_1, z), \tag{2.4a}$$

or

$$\alpha(g_2, \alpha(g_1, z)) = \alpha(g_1 g_2, z) \tag{2.4b}$$

are satisfied.

The actions that satisfy (2.4a) are called left actions, whereas the ones that satisfy (2.4b) are called right actions.

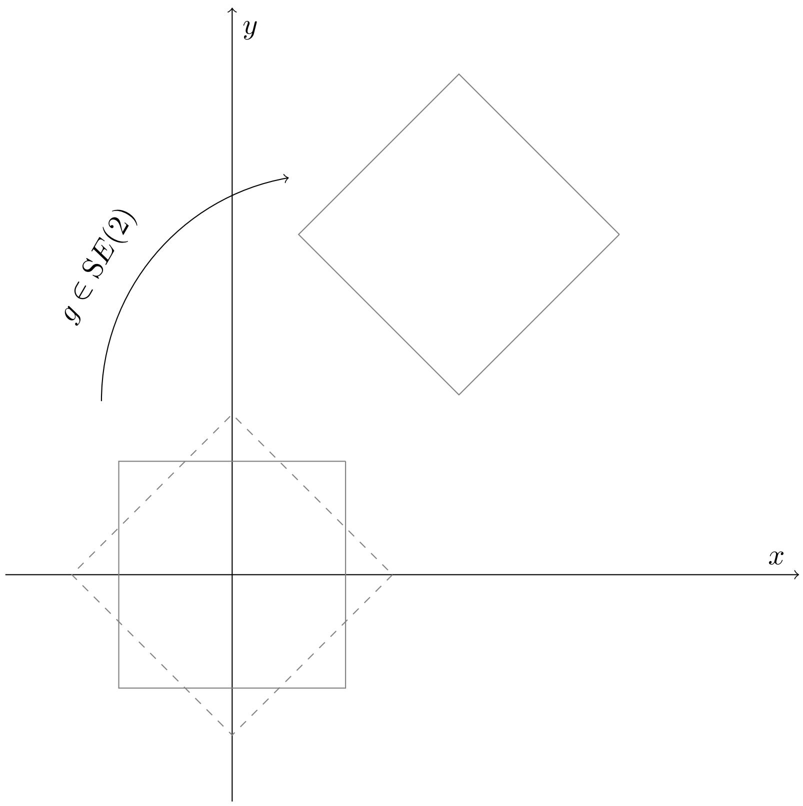
We will assume that the map (2.3) is smooth in both elements of the group G and elements of the space M.

Notation 2.1.9. *From now on we will denote a left action with* $*$ *and a right action with* $\bullet$. *When the parity of the action is implicit or is not specified we will denote the action by* $\cdot$.

Example 2.1.10. *Consider a square situated in the origin with base parallel to the* x*-axis. Suppose this square is rotated* 45 *degrees and translated* 2 *centimetres in the* x*-axis and* 3 *centimetres in the* y*-axis. What it is happening, mathematically speaking, is that the element*

$$g = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 3 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{S}E(2)$$

is acting on the points of the square situated in the origin and base parallel to the x*-axis of the form* $(x_0, y_0, 1)$*, as shown in the next picture.*



Example 2.1.11. *Consider the Lie group of* 2×2 *real matrices with determinant* 1 *denoted by* $\mathrm{SL}(2,\mathbb{R})$

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \quad a,b,c,d \in \mathbb{R} \quad \text{and} \quad ad - bc = 1 \right\} \tag{2.5}$$

and the manifold $M = \mathbb{R}^2$. *Let* $g \in \mathrm{SL}(2,\mathbb{R})$. *In this example, we will consider the projective action of* $\mathrm{SL}(2,\mathbb{R})$ *acting on curves* $(x, u(x))$ *in* $\mathbb{R}^2$ *given by*

$$(x, u(x)) \to g \cdot (x, u(x)) = \left(x, \frac{au(x) + b}{cu(x) + d} \right). \tag{2.6}$$

Note that this action is not well defined if $u(x) = -\dfrac{d}{c}$. *To "fix" this, we add a new point* ∞ *and we extend the map as*

$$g \cdot (x, \infty) = \left(x, \frac{a}{c} \right) \qquad \text{and} \qquad g \cdot \left(x, -\frac{d}{c} \right) = (x, \infty).$$

Let us consider $g_1, g_2 \in \mathrm{SL}(2,\mathbb{R})$ *such that*

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad \text{and} \qquad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

We have that

$$\begin{aligned} g_1 \cdot (g_2 \cdot u(x)) &= g_1 \cdot \left(\frac{a_2 u(x) + b_2}{c_2 u(x) + d_2} \right) \\ &= \frac{a_1 \left(\dfrac{a_2 u(x) + b_2}{c_2 u(x) + d_2} \right) + b_1}{c_1 \left(\dfrac{a_2 u(x) + b_2}{c_2 u(x) + d_2} \right) + d_1} \\ &= \frac{a_1(a_2 u(x) + b_2) + b_1(c_2 u(x) + d_2)}{c_1(a_2 u(x) + b_2) + d_1(c_2 u(x) + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2) u(x) + a_1 b_2 + b_1 d_2}{(c_1 a_2 + d_1 c_2) u(x) + c_1 b_2 + d_1 d_2} \\ &= (g_1 g_2) \cdot u(x) \end{aligned}$$

and therefore (2.6) *is a left action.*

Given a left action $(g, z) \mapsto g \cdot z$, we have that $(g, z) \mapsto g^{-1} \cdot z$ is a right action. In practice both right and left actions happen, and depending on the choice the difficulty of the calculations can differ considerably. In the theory, only one is needed, so from now on we will just consider left actions as the theory for right actions is parallel.

Notation 2.1.12. *The image of a variable under an action will be often denoted as*

$$g \cdot z = \widetilde{z}.$$

Properties of the actions

In this thesis, we will be interested in some specific type of actions; free and regular actions. In order to understand these actions we will first give a few definitions. Let G be a group acting on M and let $z \in M$.

Definition 2.1.13 (Orbit)**.** *The orbit of z is the set of points in M that are the image of z when acted upon by an element $g \in G$, i.e.*

$$\mathcal{O}(z) = \{g \cdot z \mid \forall g \in G\}.$$

Definition 2.1.14 (Stabilizer)**.** *For every $z \in M$ we define the stabilizer subgroup of G with respect to z as the set of all elements in G that fix z, i.e.*

$$G_z = \{g \in G \mid g \cdot z = z\}.$$

Definition 2.1.15 (Free action)**.** *A group action on M is said to be free, if for all points $z \in M$, their stabilizers are just composed of the identity element, i.e.*

$$G_z = \{g \in G | g \cdot z = z\} = \{e\},$$

for all $z \in M$.

Definition 2.1.16 (Regular action)**.** *A group action is regular if*

(i) all orbits have the same dimensions,

(ii) for each $z \in M$, there are arbitrary small neighbourhoods $\mathcal{U}(z)$ of z such that for all $z' \in \mathcal{U}(z)$, $\mathcal{U}(z) \cap \mathcal{O}(z')$ is connected - which is a set that cannot be partitioned into two non-empty subsets such that each subset has no points in common with the set closure of the other -.

Remark 2.1.17. *The majority of the actions are not free and regular. However, one can usually extend them in different ways in order to make them free and regular, as shown in the running example in §3.*

2.1.2 Induced actions

Even though there are many different types of induced actions, we will introduce here the ones that are relevant for this thesis.

Induced actions on functions

Let us denote by $C^\infty(M, \mathbb{R})$ the set of smooth functions mapping M to $\mathbb{R}^N$. The action induced by the left action $G \times M \to M$ on $C^\infty(M, \mathbb{R})$ in the following way

$$g \bullet (f_1(z), ..., f_n(z)) = (f_1(g * z), ..., f_n(g * z))$$

is a right action. Furthermore, due to the fact that a left action on M corresponds to a right action on the coordinates, the coordinates are functions from M to $\mathbb{R}$.

Definition 2.1.18 (Invariant of an action)**.** *Given an action $G \times M \to M$ we say that the function $f \colon M \to \mathbb{R}$ is an invariant of such action if it satisfies*

$$f(g \cdot z) = f(z)$$

for all $z \in M$.

If the property of a mathematical object does not change under a group action we say that the group action is a *symmetry* preserving such property.

Example 2.1.19. *Let us consider now another action of* (2.5) *given by*

$$(u(x), v(x)) \to g \cdot (u(x), v(x)) = (au(x) + bv(x), cu(x) + dv(x)) \tag{2.7}$$

where we have taken another parametrization of the curves in the plane. Note that this action is a linear action. Given two curves $(u_1(x), v_1(x))$ and $(u_2(x), v_2(x))$ we have that

$$\begin{aligned} g \cdot \begin{vmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{vmatrix} &= \begin{vmatrix} au_1(x) + bv_1(x) & cu_1(x) + dv_1(x) \\ au_2(x) + bv_2(x) & cu_2(x) + dv_2(x) \end{vmatrix} \\ &= adu_1(x)v_2(x) - adu_2(x)v_1(x) - bcu_1(x)v_2(x) + bcu_2v_1(x) \\ &= (u_1(x)v_2(x) - u_2(x)v_1(x))(ad - bc) \\ &= \begin{vmatrix} u_1(x) & v_1(x) \\ u_2(x) & v_2(x) \end{vmatrix}. \end{aligned}$$

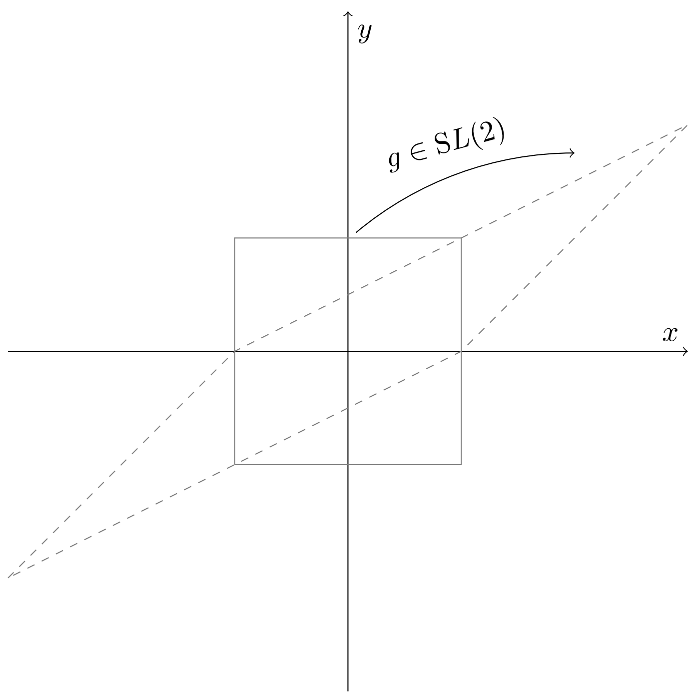
We can see that the area is preserved, and therefore (2.7) *is a symmetry preserving the area, which is an invariant of* (2.7).

For instance, let us suppose that the square centred at the origin with base parallel to the x*-axis with area* $1\ \mathrm{cm}^2$ *is transformed into a rhomboid of the same area.*

What it is happening, mathematically speaking, is that the element

$$g = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is acting on the points of the square preserving its area, as shown in the next picture.



Induced actions on derivatives

To understand the prolonged action, let us situate ourselves in the simplest scenario possible. If we have a group G acting on the curves $(x, u(x))$ where $\widetilde{x} = x$ then there is an induced action on its derivatives u_x, u_{xx},... etc. This action is known as the prolonged action and it is computed as follows:

$$
\begin{aligned}
g \cdot u_x &= g \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}(g\cdot u)}{\mathrm{d}(g\cdot x)} = \frac{\dfrac{\mathrm{d}(g\cdot u)}{\mathrm{d}x}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}},\\
g \cdot u_{xx} &= g \cdot \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = g \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}(g\cdot x)}\left(\frac{\dfrac{\mathrm{d}(g\cdot u)}{\mathrm{d}x}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\right) = \frac{1}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\dfrac{\mathrm{d}(g\cdot u)}{\mathrm{d}x}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\right),\\
g \cdot u_{xxx} &= g \cdot \frac{\mathrm{d}^3u}{\mathrm{d}x^3} = g \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}}{\mathrm{d}(g\cdot x)}\left(\frac{\mathrm{d}^2(g\cdot u)}{\mathrm{d}(g\cdot x)^2}\right)\\
&= \frac{\mathrm{d}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}(g\cdot x)}\left(\frac{\mathrm{d}(g\cdot u)}{\mathrm{d}(g\cdot x)}\right)\right) = \frac{\mathrm{d}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\dfrac{\mathrm{d}(g\cdot u)}{\mathrm{d}x}}{\dfrac{\mathrm{d}(g\cdot x)}{\mathrm{d}x}}\right)\right),\\
&\dots
\end{aligned}
$$

and so on.

Example 2.1.20. *For* (2.6) *note that*

$$\frac{\mathrm{d}(g\cdot x)}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}x} = 1$$

and therefore

$$
\begin{aligned}
g\cdot u_x &= \frac{\mathrm{d}(g\cdot u)}{\mathrm{d}x}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{au+b}{cu+d}\right)\\
&= \frac{au_x(cu+d) - cu_x(au+b)}{(cu+d)^2}\\
&= \frac{u_x(ad-bc)}{(cu+d)^2}\\
&= \frac{u_x}{(cu+d)^2},
\end{aligned}
$$

$$
\begin{aligned}
g\cdot u_{xx} &= \frac{\mathrm{d}(g\cdot u_x)}{\mathrm{d}x}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_x}{(cu+d)^2}\right)\\
&= \frac{u_{xx}}{(cu+d)^2} - 2\frac{cu_x^2}{(cu+d)^3}
\end{aligned}
$$

and

$$
\begin{aligned}
g \cdot u_{xxx} &= \frac{\mathrm{d}(g \cdot u_{xx})}{\mathrm{d}x} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u_{xx}}{(cu+d)^2} - 2\frac{cu_x^2}{(cu+d)^3}\right) \\
&= \frac{u_{xxx}}{(cu+d)^2} - \frac{6cu_xu_{xx}}{(cu+d)^3} + \frac{6c^2u_x^3}{(cu+d)^4}.
\end{aligned}
$$

Hence the prolonged action of (2.6) *on the space* $(x, u, u_x, u_{xx}, u_{xxx})$ *is*

$$
\begin{aligned}
\widetilde{x} &= x, \\
\widetilde{u} &= \frac{au+b}{cu+d}, \\
\widetilde{u_x} &= \frac{u_x}{(cu+d)^2}, \\
\widetilde{u_{xx}} &= \frac{u_{xx}}{(cu+d)^2} - 2\frac{cu_x^2}{(cu+d)^3}, \\
\widetilde{u_{xxx}} &= \frac{u_{xxx}}{(cu+d)^2} - \frac{6cu_xu_{xx}}{(cu+d)^3} + \frac{6c^2u_x^3}{(cu+d)^4}.
\end{aligned}
\tag{2.8}
$$

The prolonged action of (2.6) on the space (x, u, u_x, u_{xx}) was previously calculated in Gonçalves and Mansfield, [32].

Now we consider the general case.

Remark 2.1.21. *We will often use a multi-index notation to denote the derivatives.*

Example 2.1.22.

$$\frac{\partial^4 u^2}{\partial x_1 \partial x_2^2 \partial x_3}$$

will be denoted by

$$u^2_{1223}.$$

Let us consider p independent variables $\mathbf{x} = (x_1, ..., x_p)$ and q dependent variables $\mathbf{u} = (u^1, ..., u^q)$. The space containing $\mathbf{x}$ will be denoted by X and the space containing $\mathbf{u}$ will be denoted by U. The space containing finitely many derivatives of $\mathbf{u}$ will be denoted by $U^{(n)}$. An element of $U^{(n)}$ will be denoted by $\mathbf{u}^{(n)}$. The space containing $\mathbf{x}$ and $\mathbf{u}^{(n)}$ will be denoted by $M = J(X \times U^{(n)})$.

Example 2.1.23. *If* $p = 2$ *and* $q = 1$ *we have that*

$$(x, y, u, u_1, u_2, u_{11}, u_{22}, u_{12}) \in M = J(X \times U^{(2)})$$

in where we have assumed that the partial derivatives commute.

Definition 2.1.24 (Differentiation operator)**.** *The total differentiation operator is given by*

$$\mathrm{D}_i = \frac{\mathrm{D}}{\mathrm{D}x_i} = \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \sum_{K} u^{\alpha}_{Ki} \frac{\partial}{\partial u^{\alpha}_{K}}.$$

We will make the assumption of an R-dimensional group G acting on the left of the space $J(X \times U^{(n)})$. The prolonged action is obtained explicitly as follows:

$$g \cdot u^{\alpha}_{i..j} = \widetilde{\mathrm{D}_i} \cdots \widetilde{\mathrm{D}_j} \widetilde{u^{\alpha}}, \tag{2.9}$$

where

$$\widetilde{\mathrm{D}_i} = \sum_{k=1}^{p} \left((\mathrm{D}\widetilde{\mathbf{x}})^{-1} \right)_{ki} \mathrm{D}_k \tag{2.10}$$

and

$$\mathrm{D}\widetilde{\mathbf{x}} = \begin{pmatrix} \dfrac{\partial \widetilde{x_1}}{\partial x_1} & \cdots & \dfrac{\partial \widetilde{x_1}}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \widetilde{x_p}}{\partial x_1} & \cdots & \dfrac{\partial \widetilde{x_p}}{\partial x_p} \end{pmatrix}. \tag{2.11}$$

Definition 2.1.25 (Prolonged action invariant)**.** *An invariant under the induced prolonged action is called differential invariant.*

Example 2.1.26. *Now consider the group* $\mathrm{SL}(2, \mathbb{R})$ *acting on the variables* $(x, t, u(x,t))$ *as follows*

$$\widetilde{t} = t, \qquad \begin{pmatrix} \widetilde{x} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

so t is invariant. Using (2.11) *we obtain*

$$\mathrm{D}\widetilde{\mathbf{x}} = \begin{pmatrix} a + bu_x & bu_t \\ 0 & 1 \end{pmatrix}.$$

Therefore by (2.10)

$$\begin{pmatrix} \widetilde{\mathrm{D}}_x \\ \widetilde{\mathrm{D}}_t \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a + bu_x} & 0 \\ -\dfrac{bu_t}{a + bu_x} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{D}_x \\ \mathrm{D}_t \end{pmatrix}.$$

From (2.9) *we have that*

$$\widetilde{u_x} = \widetilde{\mathrm{D}}_x \widetilde{u} = \frac{ac + u_x(1 + bc)}{a(a + bu_x)}, \quad \widetilde{u_{xx}} = \widetilde{\mathrm{D}}_x^2 \widetilde{u} = \frac{u_{xx}}{(a + bu_x)^3}, \quad \widetilde{u_t} = \widetilde{\mathrm{D}}_t \widetilde{u} = \frac{u_t}{a + bu_x}.$$

This example appears in Mansfield, [70].

Induced actions on products

Let us denote the product manifold of N-copies of M by $\mathcal{M}$.

Definition 2.1.27. *The product action induced on* $\mathcal{M}$ *is given by*

$$g \cdot (z_1, ..., z_N) = (g \cdot z_1, ..., g \cdot z_N).$$

A N-point invariant of the action is an invariant of the product action on $\mathcal{M}$. These invariants, called joint-invariants, were introduced in Olver, [88], as mentioned in the introduction. However, the recursive expressions for these invariants does not seem to be computationally useful. In §3, we will present a tool previously introduced by Beffa, Mansfield and Wang, [6] that offers significant computational advantages.

2.1.3 Infinitesimals

Suppose that $a_1, a_2, ..., a_r$ are the parameters of groups elements near the identity of a Lie group G.

Definition 2.1.28 (Infinitesimals of a prolonged action)**.** *Given a group action of* G *on* $M = J(X \times U^{(n)})$, *the infinitesimals of the prolonged group action are defined to be the derivatives of the* $\widetilde{x}_i, \widetilde{u^\alpha_K}$ *with respect to the group parameters* a_j *at the identity, and are denoted as*

$$\left.\frac{\partial \widetilde{x}_i}{\partial a_j}\right|_{g=e} = \xi^i_j, \qquad \left.\frac{\partial \widetilde{u^\alpha}}{\partial a_j}\right|_{g=e} = \phi^\alpha_{,j}, \qquad \left.\frac{\partial \widetilde{u^\alpha_K}}{\partial a_j}\right|_{g=e} = \phi^\alpha_{K,j}.$$

A condensed form to write the infinitesimals is to write a table of infinitesimals of the form

	x_i	u^α	u^α_K
a_j	ξ^i_j	$\phi^\alpha_{,j}$	$\phi^\alpha_{K,j}$

.

The prolonged infinitesimals $\phi^\alpha_{K,j}$ can also be calculated using the formula

$$\phi^\alpha_{K,j}(x, u^{(n)}) = \mathrm{D}_K \left(\phi^\alpha_{,j} - \sum_i u^\alpha_i \xi^i_j \right) + \sum_i \xi^i_j u^\alpha_{Ki}$$

where D_K is a total derivative of order K. Setting

$$Q^\alpha_j(x, u) = \phi^\alpha_{,j} - \sum_i u^\alpha_i \xi^i_j$$

the tuple $Q^j(x, u) = (Q_1, ..., Q_q)$ will be called characteristic of the vector field $\mathbf{v}_j$, which is given by

$$\mathbf{v}_j = \sum_{i,\alpha,K} \xi^i_j \frac{\partial}{\partial x_i} + \phi^\alpha_{,j} \frac{\partial}{\partial u^\alpha} + \phi^\alpha_{K,j} \frac{\partial}{\partial u^\alpha_K}.$$

Definition 2.1.29. *Let $G \times U \to U$ be a smooth local Lie group action. If $\gamma(t)$ is a path in G with $\gamma(0) = e$, the identity element in G, then*

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma(t) \cdot \mathbf{u} \tag{2.12}$$

is called the infinitesimal generator of the group action at $\mathbf{u} \in U$, in the direction $\gamma'(0) \in T_eG$, where T_eG is the tangent space to G at e. In coordinates, the components of the infinitesimal generator are $\phi^\alpha = \mathbf{v}(u^\alpha)$, so

$$\mathbf{v} = \phi^\alpha \frac{\partial}{\partial u^\alpha}.$$

Example 2.1.30. *For* (2.8) *the table of infinitesimals is of the form*

$$\begin{array}{c|ccccc} & x & u & u_x & u_{xx} & u_{xxx} \\ \hline a & 0 & 2u & 2u_x & 2u_{xx} & 2u_{xxx} \\ b & 0 & 1 & 0 & 0 & 0 \\ c & 0 & -u^2 & -2uu_x & -2(u_x^2 + uu_{xx}) & -2(uu_{xxx} + 3u_xu_{xx}) \end{array} . \tag{2.13}$$

Therefore, the prolonged infinitesimal vector fields corresponding to the parameters a, b and c are

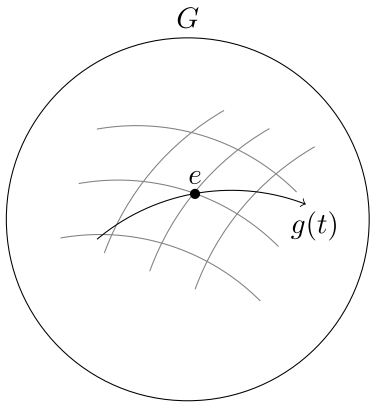
$$\begin{aligned} \mathbf{v}_a &= 2u\frac{\partial}{\partial u} + 2u_x\frac{\partial}{\partial u_x} + 2u_{xx}\frac{\partial}{\partial u_{xx}} + 2u_{xxx}\frac{\partial}{\partial u_{xxx}}, \\ \mathbf{v}_b &= \frac{\partial}{\partial u}, \\ \mathbf{v}_c &= -u^2\frac{\partial}{\partial u} - 2uu_x\frac{\partial}{\partial u_x} - 2(u_x^2 + uu_{xx})\frac{\partial}{\partial u_{xx}} - 2(uu_{xxx} + 3u_xu_{xx})\frac{\partial}{\partial u_{xxx}}. \end{aligned} \tag{2.14}$$

For the non prolonged action, we have that (see Gonçalves and Mansfield, [32])

$$\begin{aligned} \mathbf{v}_a &= 2u\frac{\partial}{\partial u}, \\ \mathbf{v}_b &= \frac{\partial}{\partial u}, \\ \mathbf{v}_c &= -u^2\frac{\partial}{\partial u}. \end{aligned} \tag{2.15}$$

2.1.4 From Lie group to Lie algebra

The key idea to go from a Lie group to a Lie algebra is to look at G near the identity element.



If we can represent our Lie group by matrices, we can consider a smooth path $t \to g(t)$ in G with $g(0) = e$ and we can differentiate it. We define the tangent space of G at e to be

$$T_eG = \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(t) \quad \Big| \quad g(0) = e \quad \text{and } g \text{ is smooth} \right\}.$$

It is easy to show that T_eG is a vector space.

Proposition 2.1.31. *The tangent space of G at e is a vector space.*

Proof. Let

$$v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(t) \qquad \text{and} \qquad w = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} h(t)$$

and assume that G is a matrix Lie group. Let us use the notation $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g(t) = g'(0)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (g(t)h(t)) = g'(0)h(0) + g(0)h'(0) = g'(0) + h'(0) = v + w$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (kg(t)) = kg'(0) = kv \quad \text{for all} \quad k \in \mathbb{R}.$$

□

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} h(t)^{-1} = -h(0)h'(0)h(0) = -h'(0).$$

Set now $X = g'(0) \in T_eG$ and consider the path $t \to h(t)Xh(t)^{-1}$ in T_eG where $h(t)$ is a smooth path in G and $h(0) = e$.

Consider now

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(h(t)Xh(t)^{-1}\right) = h'(0)X - Xh'(0) \in T_eG.$$

If $Y = h'(0) \in T_eG$ then $XY - YX \in T_eG$.

Definition 2.1.32. *We call $[X,Y] := XY - YX$ the matrix Lie bracket.*

Therefore we have that T_eG is a vector space with a product.

In fact, $T_eG := \mathfrak{g}$ is a Lie algebra.

Definition 2.1.33 (Lie algebra)**.** *A Lie algebra L is a vector space with a bracket*

$$[\ ,\ ] \colon L \times L \to L$$

such that $[\ ,\ ]$ is

- *Bilinear: $[aX_1 + bX_2, Y] = a[X_1,Y] + b[X_2,Y]$ and $[X, aY_1 + bY_2] = a[X,Y_1] + b[X,Y_2]$ where $a,b \in \mathbb{R}$ and X, X_1, X_2, Y, Y_1 and $Y_2 \in L$.*
- *Skew-symmetric: $[X,Y] = -[Y,X]$ where $X,Y \in L$.*
- *Satisfies the Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

where X, Y and $Z \in L$.

It is easy to check that T_eG satisfies the properties above.

Example 2.1.34. *The Lie algebra $T_e\mathrm{SL}(2,\mathbb{R}) = \mathfrak{sl}(2)$ is computed as follows:*

Set

$$g(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \in \mathrm{SL}(2).$$

In the identity we have that $a(0) = d(0) = 1$ and $b(0) = c(0) = 0$. From the condition

$$a(t)d(t) - b(t)c(t) = 1$$

differentiating with respect to t and evaluating in the identity we have

$$a'(0) + d'(0) = 0$$

and therefore

$$g'(0) = \begin{pmatrix} a'(0) & b'(0) \\ c'(0) & -a'(0) \end{pmatrix} \in \mathfrak{sl}(2).$$

So

$$T_eSL(2,\mathbb{R}) = \mathfrak{sl}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \Big| \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

A basis of $\mathfrak{sl}(2)$ *is*

$$\left\{ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}. \tag{2.16}$$

It is simple to check that the following Lie bracket table holds

[,]	h	e	f
h	0	$2e$	$-2f$
e	$-2e$	0	h
f	$2f$	$-h$	0

(2.17)

Now we give a list of some common Lie groups and their correspondent Lie algebras (see Kirillov, [59], Serre, [98] and Varadarajan, [107]).

Lie group	**Description**	**Lie algebra**	**Description**
$\mathbb{R}^n$	Euclidean space with addition	$\mathbb{R}^n$	Euclidean space with zero Lie bracket. For $n = 3$ one can identify the bracket with the cross product.
$\mathbb{C}^n$	Complex numbers with addition	$\mathbb{C}^n$	Complex numbers with zero Lie bracket
$\mathbb{R}^\times$	Real nonzero numbers with multiplication	$\mathbb{R}$	Real numbers with zero Lie bracket
$\mathbb{C}^\times$	Complex nonzero numbers with multiplication	$\mathbb{R}$	Complex numbers with zero Lie bracket
$\mathbb{R}^+$	Real positive numbers with multiplication	$\mathbb{R}$	Real numbers with zero Lie bracket
$S^1 = U$	Complex number of modulus 1 with multiplication. Also called the circle group, is isomorphic to SO(2), Spin(2) and $\mathbb{R}/\mathbb{Z}$	$\mathbb{R}$	Real numbers with zero Lie bracket
$S^3 = SP(1)$	Quaternions of modulus 1 with multiplication. Isomorphic to SU(2), Spin(3) and double cover of SO(3)	$\mathbb{H}$	Quaternions
$GL(n, \mathbb{R})$	General linear group $n \times n$ real matrices with non zero determinant	$\mathfrak{M}(n, \mathbb{R})$	$n \times n$ real matrices
$GL^+(n, \mathbb{C})$	General linear group of $n \times n$ complex matrices with non zero determinant. $GL(1, \mathbb{R})$ is isomorphic to $\mathbb{C}^\times$	$\mathfrak{M}(n, \mathbb{C})$	$n \times n$ complex matrices
$GL^+(n, \mathbb{R})$	General linear group of $n \times n$ real matrices with positive determinant. $GL^+(1, \mathbb{R})$ is isomorphic to $\mathbb{R}^+$	$\mathfrak{M}(n, \mathbb{R})$	$n \times n$ real matrices
$\mathrm{S}L(n, \mathbb{R})$	Special linear group of $n \times n$ real matrices with determinant 1. $\mathrm{S}L(2, \mathbb{R})$ is isomorphic to $SU(1, 1)$ and $Sp(2, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R})$	$n \times n$ real matrices
$\mathrm{S}L(n, \mathbb{C})$	Special linear group of $n \times n$ complex matrices with determinant 1. $\mathrm{S}L(2, \mathbb{C})$ is isomorphic to $Spin(3, \mathbb{C})$ and $Sp(2, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C})$	$n \times n$ complex matrices
$PSL(2, \mathbb{C})$	Projective special linear group of $n \times n$ complex matrices with determinant 1. Isomorphic to $SO(3, \mathbb{C})$ and the Möbius group	$\mathfrak{sl}(n, \mathbb{C})$	$n \times n$ real matrices with trace zero
$O(n, \mathbb{R})$	Orthogonal group of $n \times n$ real orthogonal matrices.	$\mathfrak{so}(n, \mathbb{R})$	skew–symmetric $n \times n$ real matrices
$O(n, \mathbb{C})$	Orthogonal group of $n \times n$ complex orthogonal matrices.	$\mathfrak{so}(n, \mathbb{C})$	skew–symmetric $n \times n$ complex matrices
$\mathrm{S}O(n, \mathbb{R})$	Orthogonal group of $n \times n$ real orthogonal matrices with determinant 1.	$\mathfrak{so}(n, \mathbb{R})$	skew–symmetric $n \times n$ real matrices
$\mathrm{S}O(n, \mathbb{C})$	Orthogonal group of $n \times n$ complex orthogonal matrices with determinant 1.	$\mathfrak{so}(n, \mathbb{C})$	skew–symmetric $n \times n$ complex matrices
$Spin(n)$	Spin group: double cover of SO(n). Spin(1) is isomorphic to $\mathbb{Z}_2$.	$\mathfrak{so}(n, \mathbb{R})$	skew–symmetric $n \times n$ real matrices
$Sp(2n, \mathbb{R})$	Symplectic group of real symplectic matrices.	$\mathfrak{sp}(2n, \mathbb{R})$	$n \times n$ quaternionic matrices satisfying $X = -X^*$
$Sp(2n, \mathbb{C})$	Symplectic group of complex symplectic matrices.	$\mathfrak{sp}(2n, \mathbb{C})$	$n \times n$ complex matrices satisfying $JX = -X^T J$ where J is the standard skew–symmetric matrix
$U(n)$	Unitary group of complex $n \times n$ unitary matrices.	$\mathfrak{u}(n)$	$n \times n$ complex matrices satisfying $X = -X^*$
$SU(n)$	Special nitary group of complex $n \times n$ unitary matrices with determinant 1	$\mathfrak{su}(n)$	$n \times n$ complex matrices with zero trace satisfying $X = -X^*$

Other famous Lie groups are the so–called exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 which are not easy to describe in terms of matrix groups.

Apart from representations of Lie groups one can also consider representations of Lie algebras. The most interesting one in this thesis is the adjoint representation.

Definition 2.1.35 (Adjoint representation)**.** *We define the adjoint representation as the map*

$$ad\colon L \to \mathfrak{gl}(L)$$

such that

$$\mathfrak{gl}(L)\colon L \to L$$

and

$$ad(x)(y) = [x, y]. \tag{2.18}$$

We will often write $ad(x) = ad_x$. Now we present an example in order to show how to compute the adjoint representation in practice.

Example 2.1.36. *Let us consider the Lie algebra* $\mathfrak{sl}(2)$ *of the special linear group* $\mathrm{SL}(2, \mathbb{R})$. *Recall the basis* (2.16). *Hence, an element of* $\mathfrak{sl}(2)$ *can be written as* $\alpha h + \beta e + \gamma f$. *Therefore, using* (2.17) *and* (2.18) *we have*

$$\begin{aligned}
ad_h(\alpha h + \beta e + \gamma f) &= \alpha[h, h] + \beta[h, e] + \gamma[h, f] = 2\beta e - 2\gamma f,\\
ad_e(\alpha h + \beta e + \gamma f) &= \alpha[e, h] + \beta[e, e] + \gamma[e, f] = -2\alpha e + \gamma h,\\
ad_f(\alpha h + \beta e + \gamma f) &= \alpha[f, h] + \beta[f, e] + \gamma[f, f] = 2\alpha f - \beta h
\end{aligned}$$

and hence

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \xrightarrow{ad_h} \begin{pmatrix}0\\ 2\beta\\ -2\gamma\end{pmatrix}, \quad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \xrightarrow{ad_e} \begin{pmatrix}\gamma\\ -2\alpha\\ 0\end{pmatrix}, \quad \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \xrightarrow{ad_f} \begin{pmatrix}-\beta\\ 0\\ 2\alpha\end{pmatrix}$$

so

$$ad_h = \begin{pmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -2\end{pmatrix}, \quad ad_e = \begin{pmatrix}0 & 0 & 1\\ -2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad ad_f = \begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 2 & 0 & 0\end{pmatrix}.$$

Definition 2.1.37 (Killing Form)**.** *The Killing form is the map* $\mathrm{B}\colon L \times L \to \mathbb{F}$, *such that*

$$\mathrm{B}(x, y) = trace(ad_x, ad_y). \tag{2.19}$$

It is simple to check that the Killing form is symmetric, bilinear and associative. We will also refer to the Killing form as the matrix associated to the map (2.19).

Example 2.1.38. *For* $\mathrm{S}L(2,\mathbb{R})$ *we have that*

$$\mathrm{B}(h,h)=8,\quad \mathrm{B}(h,e)=0\quad \mathrm{B}(h,f)=0,\quad \mathrm{B}(e,e)=0,\quad \mathrm{B}(e,f)=4,\quad \mathrm{B}(f,f)=0$$

and therefore (see Gonçalves and Mansfield, [32])

$$\mathrm{B}=\begin{array}{c} \\ h \\ e \\ f \end{array}\begin{array}{c} \begin{array}{ccc} h & e & f \end{array} \\ \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \end{array}. \tag{2.20}$$

Even though in the previous example B is non–degenerate this is not always the case.

Definition 2.1.39 (Cartan's Second Criterion)**.** *A Lie algebra L is semi-simple if and only if the Killing form* B *is non–degenerate.*

Without going into much detail, semi-simple Lie algebras over $\mathbb{C}$ are copies of $\mathrm{S}L(2,\mathbb{C})$ glued together in beautiful ways. In particular, $\mathrm{S}L(2,\mathbb{C})$ is a semi–simple Lie algebra.

2.2 Matrix of infinitesimals and the Adjoint action

We next define the matrix of infinitesimals and the Adjoint matrix which will play a very important role at the end of this chapter and §3. In this section, we make use of the theory developed in Gonçalves and Mansfield, [32], [33], [34] and Mansfield, [70]. In Mansfield, R–E, Hydon and Peng, [74] and Mansfield and R–E, [75] the Adjoint matrix is chosen to be the inverse transpose of the Adjoint matrix appearing in Gonçalves and Mansfield, [32], [33], [34] and Mansfield, [70]. In this thesis, we will be using the following convention: we will use the form of the Adjoint matrix appearing in Gonçalves and Mansfield, [32], [33], [34] and Mansfield, [70] in the smooth examples and the form of the Adjoint matrix appearing in Mansfield, R–E, Hydon and Peng, [74] and Mansfield and R–E, [75] in the discrete cases. The theory appearing in these last two papers concerning the Adjoint matrix will be presented §3.

2.2.1 Matrix of infinitesimals and the Adjoint action: form adopted for the smooth examples

Definition 2.2.1 (Matrix of infinitesimals)**.** *Let the group element near the identity be given as $g = g(a_1, \dots, a_R)$ so that the independent parameters of the group action are the a_i, and let $z = (z^1, z^2, \dots, z^p)$ be coordinates on M near $z \in M$. The matrix $\Omega(z)$ of infinitesimals is an $R \times p$ matrix, given by*

$$\Omega(z) = (\phi_{ij}), \qquad \phi_{ij} = \frac{\partial \tilde{z}^j}{\partial a_i}\Big|_{g=e}. \tag{2.21}$$

A vector field can be seen also as a map from the manifold to its tangent bundle,

$$\mathbf{v} : M \to TM, \qquad \mathbf{v}(z) \in T_z M, \quad \forall z \in M,$$

where the tangent bundle is essentially a manifold that assembles all the tangent vectors in M. We denote the set of all vector fields on M as $\mathcal{X}(M)$. In coordinates $z = (z^1, \dots, z^p)$ on M, vector fields can be rewritten of the form

$$\mathbf{v}(z) = \sum_{i=1}^{p} f_i(z) \frac{\partial}{\partial z^i} = \mathbf{f}^T \nabla$$

where

$$\nabla = \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^2}, \cdots, \frac{\partial}{\partial z^k} \right)^T.$$

For a smooth Lie group action G on a smooth manifold M, there is a corresponding Adjoint action on the set of all smooth vector fields $\mathcal{X}(M)$ of the manifold M.

Definition 2.2.2 (Adjoint action)**.** *The Adjoint action $\mathcal{A}d$ on vector fields is defined as*

$$\begin{aligned} \mathcal{A}d : G \times \mathcal{X}(M) &\to \mathcal{X}(M) \\ (g, \mathbf{v}) &\mapsto \mathcal{A}d_g(\mathbf{v}), \end{aligned} \tag{2.22}$$

such that $\mathcal{A}d_g(\mathbf{v})(z) = Tg^{-1}\mathbf{v}(g \cdot z)$. Here $Tg : TM \to TM$ is the tangent map with respect to the group action, $g\cdot : M \to M$.

Denoting $g \cdot z = \widetilde{z}$, we have

$$\begin{aligned} \mathcal{A}d_g(\mathbf{v}) &= f^i(\widetilde{z}) \frac{\partial}{\partial \widetilde{z}} \\ &= \frac{\partial z^i}{\partial \widetilde{z}^j} f^j(\widetilde{z}) \frac{\partial}{\partial z^i}. \end{aligned} \tag{2.23}$$

Writing $\mathbf{v} = \mathbf{f}^T(z)\nabla$, we can represent the Adjoint action as

$$\mathcal{A}d_g(\mathbf{v}) = \left(\left(\frac{\partial \widetilde{z}}{\partial z} \right)^{-1} \mathbf{f}(\widetilde{z}) \right)^T \nabla, \tag{2.24}$$

where $\left(\frac{\partial \widetilde{z}}{\partial z} \right)$ is the Jacobian of $\widetilde{z} = g \cdot z$ with respect to z.

Remark 2.2.3. *The adjoint action* (2.22) *is a right action, while the adjoint action appearing in §3 is a left action.*

It can be shown from the definition of the Adjoint action (2.22) that the map

$$\mathcal{A}d : \mathcal{X}(M) \to \mathcal{X}(M)$$
$$h \mapsto g^{-1}hg,$$

takes T_eG to itself.

The Adjoint action takes infinitesimal vector fields to infinitesimal vector fields, and one obtains a representation of G, called the Adjoint representation. In co-ordinates, this yields a representation of G in $GL(R)$, where $R = \dim(G)$.

The fact that $Ad_g(\mathbf{v}) \in \mathcal{X}_G(M)$ implies that for any basis $\mathbf{v}_i$ of $\mathcal{X}_G(M)$, for $i = 1, ..., R$, where $R = dim(G)$

$$Ad_g \left(\sum_i \alpha_i \mathbf{v}_i \right) = \sum_i \alpha_i Ad_g(\mathbf{v}_i) = \sum_j \sum_i \alpha_i (\mathcal{A}d(g))_{ij} \mathbf{v}_j. \tag{2.25}$$

Lemma 2.2.4. *Let the matrix of infinitesimals for the group action* $G \times \mathrm{M} \to \mathrm{M}$, $\widetilde{z} = g \cdot z$, *relative to given co-ordinates on* G *and* M, *be* $\Omega(z)$. *We denote the Jacobian matrix of the group action as* $\left(\frac{\partial \widetilde{z}}{\partial z} \right)$. *If the* $R \times R$ *matrix* $\mathcal{A}d(g)$ *denotes the Adjoint representation of* $g \in G$, *relative to same coordinates as for the infinitesimal matrix, then*

$$\mathcal{A}d(g)\Omega(z) = \Omega(\widetilde{z}) \left(\frac{\partial \widetilde{z}}{\partial z} \right)^{-T}. \tag{2.26}$$

Example 2.2.5. *Continuing with* (2.8), *restricting ourselves to the second prolongation, after calculating the table of infinitesimals it is easy to build the matrix of infinitesimals which has* *the form (see Mansfield, [70])*

$$\Omega(u,u_x,u_{xx}) = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} u & u_x & u_{xx} \end{array} \\ \begin{pmatrix} 2u & 2u_x & 2u_{xx} \\ 1 & 0 & 0 \\ -u^2 & -2uu_x & -2(u_x^2+uu_{xx}) \end{pmatrix} \end{array}. \tag{2.27}$$

In order to compute the Adjoint matrix associated to the Adjoint action, one can use the infinitesimal vector fields instead of the prolonged infinitesimal vector fields in order to ease the calculations.

Consider (2.15)

$$\mathbf{v}_a = 2u\frac{\partial}{\partial u}, \qquad \mathbf{v}_b = \frac{\partial}{\partial u}, \qquad \textit{and} \qquad \mathbf{v}_c = -u^2\frac{\partial}{\partial u}.$$

Recall from (2.6) *that*

$$\widetilde{u} = \frac{au+b}{cu+d}.$$

Also, using the chain rule

$$\frac{\partial}{\partial u} = \frac{\partial \widetilde{u}}{\partial u}\frac{\partial}{\partial \widetilde{u}} = \frac{1}{(cu+d)^2}\frac{\partial}{\partial \widetilde{u}}.$$

Therefore

$$\frac{\partial}{\partial \widetilde{u}} = (cu+d)^2\frac{\partial}{\partial u}.$$

Hence

$$\begin{aligned}
\widetilde{\mathbf{v}_a} &= 2\widetilde{u}\frac{\partial}{\partial \widetilde{u}} \\
&= 2\frac{au+b}{cu+d}(cu+d)^2\frac{\partial}{\partial u} \\
&= 2(au+b)(cu+d)\frac{\partial}{\partial u} \\
&= 2\left(acu^2 + (ad+bc)u + bd\right)\frac{\partial}{\partial u}, \\
\widetilde{\mathbf{v}_b} &= \frac{\partial}{\partial \widetilde{u}} \\
&= (cu+d)^2\frac{\partial}{\partial u} \\
&= \left(c^2u^2 + 2cdu + d^2\right)\frac{\partial}{\partial u},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathbf{v}_c} &= -\widetilde{u}^2 \frac{\partial}{\partial \widetilde{u}} \\
&= -\left(\frac{au+b}{cu+d}\right)^2 (cu+d)^2 \frac{\partial}{\partial u} \\
&= -(au+b)^2 \frac{\partial}{\partial u} \\
&= -\left(a^2u^2 + 2abu + b^2\right) \frac{\partial}{\partial u}.
\end{aligned}$$

In conclusion

$$\widetilde{\mathbf{v}_a} = (ad+bc)\mathbf{v}_a + 2bd\mathbf{v}_b - 2ac\mathbf{v}_c, \tag{2.28}$$

$$\widetilde{\mathbf{v}_b} = cd\mathbf{v}_a + d^2\mathbf{v}_b - c^2\mathbf{v}_c, \tag{2.29}$$

$$\widetilde{\mathbf{v}_c} = -ab\mathbf{v}_a - b^2\mathbf{v}_b + a^2\mathbf{v}_c. \tag{2.30}$$

$$\tag{2.31}$$

Therefore

$$\begin{pmatrix} \widetilde{\mathbf{v}_a} \\ \widetilde{\mathbf{v}_b} \\ \widetilde{\mathbf{v}_c} \end{pmatrix} = \mathcal{A}d(g) \begin{pmatrix} \mathbf{v}_a \\ \mathbf{v}_b \\ \mathbf{v}_c \end{pmatrix} \tag{2.32}$$

where

$$\mathcal{A}d(g) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} ad+bc & 2bd & -2ac \\ cd & d^2 & -c^2 \\ -ab & -b^2 & a^2 \end{pmatrix} \end{array}. \tag{2.33}$$

Lemma (2.2.4) *is straightforward to check taking into account that (see Mansfield, [70])*

$$\frac{\partial(\widetilde{u}, \widetilde{u_x}, \widetilde{u_{xx}})}{\partial(u, u_x, u_{xx})} = \begin{pmatrix} \dfrac{1}{(cu+d)^2} & 0 & 0 \\ -\dfrac{-2cu_x}{(cu+d)^3} & \dfrac{1}{(cu+d)^2} & 0 \\ \dfrac{-2c((cu+d)u_{xx} - 3u_x^2)}{(cu+d)^4} & -\dfrac{4cu_x}{(cu+d)^3} & \dfrac{1}{(cu+d)^2} \end{pmatrix}. \tag{2.34}$$

We will illustrate with an example how the Adjoint action is obtained in Mansfield, R–E, Hydon and Peng, [74] and Mansfield and R–E, [75] and we will present the theory in detail in §3.

Example 2.2.6. *From* (2.28) *we have that*

$$\begin{pmatrix} \widetilde{\mathbf{v}_a} & \widetilde{\mathbf{v}_b} & \widetilde{\mathbf{v}_c} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c \end{pmatrix} \begin{pmatrix} ad+bc & cd & -ab \\ 2bd & d^2 & -b^2 \\ -2ac & -c^2 & a^2 \end{pmatrix}.$$

Hence, we have that the induced action on these are

$$\begin{pmatrix} \widetilde{\mathbf{v}_a} & \widetilde{\mathbf{v}_b} & \widetilde{\mathbf{v}_c} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c \end{pmatrix} \mathcal{R}(g)^{-1}$$

where

$$\mathcal{R}(g) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix} \end{array}. \tag{2.35}$$

What we have is that $\mathcal{A}d(g)^T = \mathcal{R}(g)^{-1}$.

Finally, we give some interesting properties regarding the Adjoint matrix and the Killing form.

Lemma 2.2.7. *The Killing form is invariant under the Adjoint action,i.e,*

$$\mathrm{B}(\mathcal{A}d(g)x, \mathcal{A}d(g)y) = \mathrm{B}(x, y).$$

Corollary 2.2.8. *Let* B *the Killing form of the Lie algebra* L. *Since* B *is invariant under the Adjoint action and this action on the vector fields can be written as in* (2.25), *we have*

$$\mathrm{B} = \mathcal{A}d(g)\mathrm{B}\mathcal{A}d(g)^T. \tag{2.36}$$

2.3 Variational calculus and Noether's Theorem

The variational calculus generalises the problem of finding extrema of functions in several variables. It appears in so many disciplines, for instance in physics and engineering, and it allows one to transform the problem of optimisation of a functional into the problem of solving a differential equation. For systems of differential equations occurring in variational problems, each conservation law of these systems arises from a corresponding symmetry property. This was first proved by Emmy Noether in 1918 (see Noether, [83]). In order to use this Theorem to find these conservation laws we first need to introduce some background on Calculus of

Variations. Let us consider $\Omega \subset \mathbb{R}^n$ an open, connected subset with smooth boundary $\partial\Omega$.

Given a smooth function $u = f(x)$ there exists an induced function $u^{(n)} = \mathrm{pr}^{(n)} f(x)$ called the $n-$th prolongation of f and it is defined by the partial derivatives up to order n. We adopt Olver's notation in Olver, [84].

Example 2.3.1. *For $u = f(x)$ the second prolongation of this function is*

$$u^{(2)} = \mathrm{pr}^{(2)} f(x) = (u; u_x; u_{xx}).$$

This can be extended to many more variables. For example, for $u = f(x, y)$ the second prolongation is (see Olver, [84])

$$u^{(2)} = \mathrm{pr}^{(2)} f(x, y) = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}).$$

A variational problem consists of finding the extrema - which are the maxima or minima - of a functional

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})\ \mathrm{d}x$$

in some class of functions $u = f(x)$ defined over Ω. The element $L(x, u^{(n)})$ is called the Lagrangian of the variational problem $\mathcal{L}$ and it depends on x, u and derivatives of u.

Example 2.3.2. *One of the most famous variational problems is the minimisation of the Euclidean curvature squared of a curve $(x, u(x))$*

$$\mathcal{L}[u] = \int \kappa^2\ \mathrm{d}s = \int \frac{u_{xx}^2}{(1 + u_x^2)^{\frac{5}{2}}}\ \mathrm{d}x \tag{2.37}$$

where

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{\frac{3}{2}}} \qquad \textit{and} \qquad \mathrm{d}s = \sqrt{1 + u_x^2}\ \mathrm{d}x.$$

This problem was solved by Euler, [20] using elliptic functions. Solutions are known as Euler's elastica. A good historical report can be found in Levien, [66].

Remark 2.3.3. *The conditions of the class of functions over which $\mathcal{L}$ is extremised, will depend on the boundary conditions and also on differentiability conditions required of the extremals $u = f(x)$.*

We will assume that the extremals of the variational problem are smooth. To find the extrema of functionals $\mathcal{L}[u]$ we use the variational derivative of $\mathcal{L}$.

Definition 2.3.4 (Variational derivative)**.** *Let $\mathcal{L}[u]$ be a variational problem. The variational* *derivative of $\mathcal{L}$ is the unique q-tuple*

$$\delta\mathcal{L}[u] = (\delta_1\mathcal{L}, ..., \delta_q\mathcal{L}),$$

such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\mathcal{L}(u+\varepsilon v) = \int_\Omega \delta\mathcal{L}[f(x)]\cdot v(x)\ \mathrm{d}x \tag{2.38}$$

whenever $u = f(x)$ is a smooth function defined on Ω, and $v(s) = (v^1(x), ..., v^q(x))$ is a smooth function with compact support in Ω - so it is zero outside Ω-, so that $f+\varepsilon\eta$ still satisfies any boundary conditions that might be imposed on the space of functions over which we are extremising $\mathcal{L}$. The element

$$\delta_\alpha\mathcal{L} = \frac{\delta\mathcal{L}}{\delta u^\alpha}$$

is the variational derivative of $\mathcal{L}$ with respect to u^α.

Proposition 2.3.5. *If $u = f(x)$ is an extremal of $\mathcal{L}[u]$, then*

$$\delta\mathcal{L}[f(x)] = 0, \qquad x\in\Omega. \tag{2.39}$$

For the following, we need to introduce first the Divergence Theorem.

Theorem 2.3.6 (Divergence Theorem)**.** *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let $X = (X^1, ..., X^m)$ be a smooth vector field defined on $\Omega\cup\partial\Omega$ whose components have continuous first order derivatives. Then*

$$\int_\Omega\sum_{\mu=1}^{n}\partial_\mu X^\mu\ \mathrm{d}V = \int_{\partial\Omega} X\cdot\vec{n}\ \mathrm{d}S \tag{2.40}$$

where the integral on the left is a volume integral over the volume V and the integral on the right is a surface integral over the surface enclosing the volume. The surface has outward-pointing unit vector $\vec{n}$.

In order to find the general formula for the variational derivative, it is assumed that $L(x, \mathrm{pr}^{(n)}(f+\varepsilon\eta)(x))$ is continuous so the order of differentiation and integration can be interchanged.

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\mathcal{L}(u+\varepsilon v) &= \int_\Omega \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} L(x, \mathrm{pr}^{(n)}(f+\varepsilon v)(x)\mathrm{d}x\\ &= \int_\Omega\left\{\sum_{\alpha,J}\frac{\partial L}{\partial u^\alpha_J}(x, \mathrm{pr}^{(n)}f(x))\cdot\partial_J v^\alpha(x)\right\}\mathrm{d}x.\end{aligned}$$

Since v has compact support - it is zero outside of Ω - , integrating by parts the above and use (2.40) , the boundary terms can be eliminated as v vanishes in $\partial\Omega$. Hence

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathcal{L}(u+\varepsilon v) = \int_\Omega \left\{ \sum_{\alpha=1}^{q} \left[\sum_J (-\mathrm{D})_J \frac{\partial L}{\partial u_J^\alpha}(x, \mathrm{pr}^{(n)} f(x)) \right] v^\alpha(x) \right\} \mathrm{d}x. \tag{2.41}$$

The operator appearing in (2.41) is the well-known Euler–Lagrange operator.

Definition 2.3.7 (Euler–Lagrange operator)**.** *For $1 \le \alpha \le q$, the α-th Euler–Lagrange operator is given by*

$$\mathrm{E}^\alpha = \sum_J (-\mathrm{D})_J \frac{\partial}{\partial u_J^\alpha}$$

the sum extending over all multi-indices $J = (j_1, ..., j_k)$ with $1 \neq j_k \neq p, k \ge 0$.

Example 2.3.8. *In this example, $\alpha = 1$ and the Euler operator takes the form*

$$\mathrm{E} = \frac{\partial}{\partial u} - \mathrm{D}_x \frac{\partial}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial}{\partial u_{xx}} - ...$$

In conclusion, the variational derivative of $\mathcal{L}[u]$ gives us the same result as applying the Euler–Lagrange operator to the coefficient of the Lagrangian of $\mathcal{L}[u]$, i.e.

$$\delta\mathcal{L}[u] = (\delta_1\mathcal{L}[u], ..., \delta_q\mathcal{L}[u]) = (\mathrm{E}^1(L), ..., \mathrm{E}^q(L)) = \mathrm{E}(L).$$

Hence, equation (2.41) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathcal{L}(f+\varepsilon \eta) = \int_\Omega \left\{ \sum_{\alpha=1}^{q} \left[\sum_J \mathrm{E}^\alpha(L(x, \mathrm{pr}^{(n)} f(x))) \right] \eta^\alpha(x) \right\} \mathrm{d}x.$$

Theorem 2.3.9. *If $u = f(x)$ is an extremal of the variational problem $\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})\ \mathrm{d}x$, then $u = f(x)$ is a solution of the Euler–Lagrange equations*

$$\mathrm{E}^\alpha(L) = 0, \qquad \alpha = 1, ..., q.$$

This is possible thanks to the Fundamental Lemma of Calculus of Variations.

Theorem 2.3.10 (Fundamental Lemma of Calculus of Variations, Gelfand and Fomin, [31])**.** *If $g(x)$ is a locally integrable function on Ω and*

$$\int_\Omega g(x) \cdot h(x) \quad \mathrm{d}x = 0$$

where $h(x)$ has compact support, then $g(x) = 0$.

Example 2.3.11. *The Euler–Lagrange equation for the variational problem* (2.37) *is of the form*

$$0 = \mathrm{E}(L) = \frac{\partial L}{\partial u} - \mathrm{D}_x \frac{\partial L}{\partial u_x} + \mathrm{D}_x^2 \frac{\partial L}{\partial u_{xx}} - ...$$

Explicitly, we have that (see Euler, [20])

$$0 = \mathrm{E}(L) = \kappa_{ss} + \frac{1}{2}\kappa^3. \tag{2.42}$$

A symmetry group of a system of equations is a local group of transformations G that acts on an open subset $\Omega \subset X \times U$ such that it transforms solutions of the system into other of its solutions. In the case of the Euler–Lagrange equations not all symmetry groups of $\mathrm{E}(L) = 0$ are variational symmetry groups of the original variational problem. This motivates the following definition.

Definition 2.3.12 (Variational symmetry group, Olver, [84])**.** *A local group of transformations G acting on $M \subset \Omega_0 \times U$ is a variational symmetry group of the functional $\mathcal{L}[u] = \int_{\Omega_0} L(x, u^{(n)})\ \mathrm{d}x$ if whenever Ω is a subdomain with closure $\bar{\Omega} \subset \Omega_0, u = f(x)$ is a smooth function defined over Ω whose graph lies in M, and $g \in G$ is such that $\widetilde{u} = \widetilde{f}(\widetilde{x}) = g \cdot f(\widetilde{x})$ is a single-valued function defined over $\widetilde{\Omega} \subset \Omega_0$, then*

$$\int_{\widetilde{\Omega}} L(\widetilde{x}, \mathrm{pr}^{(n)}\widetilde{f}(\widetilde{x}))\mathrm{d}\widetilde{x} = \int_{\Omega} L(x, \mathrm{pr}^{(n)} f(x))\mathrm{d}x.$$

The following theorem tells us the necessary and sufficient condition for a connected group of transformations to be a variational symmetry group of a variational problem. But first we need to define the diveregence of smooth functions.

Definition 2.3.13 (Total divergence, Olver, [84])**.** *The total divergence of a p-tuple P of smooth functions of x, u and derivatives of u is the function*

$$\mathrm{Div}P = \mathrm{D}_1 P_1 + \mathrm{D}_2 P_2 + ... + \mathrm{D}_p P_p.$$

Example 2.3.14. *Suppose that $u = u(x, y)$. For $P = (u_x u_y, u^2)$ we have that*

$$\mathrm{Div}P = \mathrm{D}_x(u_x u_y) + \mathrm{D}_y u = u_{xx}u_y + u_x u_{xy} + 2uu_y.$$

Now we are ready to give the infinitesimal criterion of invariance Theorem.

Theorem 2.3.15 (Infinitesimal criterion of invariance, Olver, [84])**.** *A connected group of transformations G acting on $M \subset \Omega_0 \times U$ is a variational symmetry group of the functional*

$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})\ \mathrm{d}x$ *if and only if*

$$\mathrm{pr}^{(n)}\mathbf{v}(L) + L\ \mathrm{Div}\ \xi = 0$$

for all $(x, u^{(n)}) \in M$ *and every infinitesimal generator*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi^\alpha \frac{\partial}{\partial u^\alpha}$$

of G.

Example 2.3.16. *Consider the group action of rotations and translations of curves* $(x, u(x))$ *in the plane*

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.43}$$

The induced action on u_x *and* u_{xx} *are (see Mansfield, [70])*

$$\widetilde{u_x} = \frac{\mathrm{d}\widetilde{u}/\mathrm{d}x}{\mathrm{d}\widetilde{x}/\mathrm{d}x} = \frac{\sin\theta + \cos\theta u_x}{\cos\theta - \sin\theta u_x}, \qquad \widetilde{u_{xx}} = \frac{1}{\mathrm{d}\widetilde{x}/\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}\widetilde{u}/\mathrm{d}x}{\mathrm{d}\widetilde{x}/\mathrm{d}x} = \frac{u_{xx}}{(\cos\theta - \sin\theta u_x)^3}.$$

We therefore have that the table of infinitesimals is of the form

	x	u	u_x	u_{xx}
a	1	0	0	0
b	0	1	0	0
θ	$-u$	x	$1+u_x^2$	$3u_x u_{xx}$

. (2.44)

The prolonged infinitesimal vector fields are

$$\mathrm{pr}^{(2)}\mathbf{v}_a = \frac{\partial}{\partial x}, \quad \mathrm{pr}^{(2)}\mathbf{v}_b = \frac{\partial}{\partial u}, \quad \mathrm{pr}^{(2)}\mathbf{v}_\theta = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}}.$$

We have that

$$\begin{aligned} \mathrm{pr}^{(2)}\mathbf{v}_a(L) + L\mathrm{D}_x\xi = 0, \qquad \mathrm{pr}^{(2)}\mathbf{v}_b(L) + L\mathrm{D}_x\xi = 0,\\ \mathrm{pr}^{(2)}\mathbf{v}_\theta(L) + L\mathrm{D}_x\xi = (1+u_x^2)\frac{\partial}{\partial u_x}\left(\frac{u_{xx}^2}{(1+u_x^2)^{\frac{5}{2}}}\right) + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}}\left(\frac{u_{xx}^2}{(1+u_x^2)^{\frac{5}{2}}}\right)\\ -\left(\frac{u_{xx}^2}{(1+u_x^2)^{\frac{5}{2}}}\right)u_x = 0. \end{aligned}$$

Therefore by (2.3.15), *the group of transformations* (2.43) *is a variational symmetry group of* *the functional* (2.37).

Now we are finally ready to present Noether's First Theorem.

Theorem 2.3.17 (Noether's First Theorem, Noether, [83])**.** *Suppose G is a local one-parameter group of symmetries of the variational problem $\mathcal{L}[u] = \int L(x, u^{(n)})\mathrm{d}x$. Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi^\alpha \frac{\partial}{\partial u^\alpha}$$

be the infinitesimal generator of G, and

$$Q_\alpha(x, u) = \phi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha, \qquad \textit{with } \alpha = 1, ..., q$$

the corresponding components of the characteristics of $\mathbf{v}$*. Then $Q = (Q_1, ..., Q_q)$ is also a the characteristic of a conservation law of the Euler–Lagrange equations* $\mathrm{E}(L) = 0$*; in other words, there is a p-tuple $P(x, u^{(m)}) = (P_1, ..., P_p)$ such that*

$$\mathrm{Div} P = Q \cdot \mathrm{E}(L) = \sum_{\alpha=1}^{q} Q_\alpha \mathrm{E}^\alpha(L)$$

is a conservation law in characteristic form for the Euler–Lagrange equations $\mathrm{E}(L) = 0$.

For the special case of one-dimensional Lagrangians $L(x, u, u_x, u_{xx}, ..)\mathrm{d}x$ we have the following Theorem appearing in Mansfield, [70]:

Theorem 2.3.18. *Consider a one-dimensional Lagrangian $L(x, u, u_x, u_{xx}, ...)$ $\mathrm{d}x$ with arbitrary order, that is invariant under the one-parameter group action*

$$\varepsilon \cdot (x, u) = (\widetilde{x}, \widetilde{u}). \tag{2.45}$$

Let

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \widetilde{x} = \xi(x, u), \qquad \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \widetilde{u} = \phi(x, u),$$

be the infinitesimal generators of (2.45) *then*

$$(\phi - u_x\xi)\,\mathrm{E}(L) + \frac{\mathrm{d}}{\mathrm{d}x}\left(L\xi + \sum_{m=1} \sum_{k=0}^{m-1} (-1)^k \left(\frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{\partial L}{\partial u_m} \right) \left(\frac{\mathrm{d}^{m-1-k}\,(\phi - u_x\xi)}{\mathrm{d}x^{m-1-k}} \right) \right) = 0,$$

where we have denoted

$$u_m = \frac{\mathrm{d}^m u}{\mathrm{d}x^m}.$$

Therefore, the first integral for $\mathrm{E}(L)=0$ *is*

$$L\xi+\sum_{m=1}\sum_{k=0}^{m-1}(-1)^k\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\frac{\partial L}{\partial u_m}\right)\left(\frac{\mathrm{d}^{m-1-k}\,(\phi-u_x\xi)}{\mathrm{d}x^{m-1-k}}\right)=c,$$

where c is a constant. Moreover, if we consider the group action to be a translation in x*, with infinitesimals* $\xi=1$ *and* $\phi=0$*, then* $u_x\mathrm{E}(L)$ *is a total derivative when* L *does not depend explicitly on* x*, i.e,*

$$u_x\mathrm{E}(L)=\frac{\mathrm{d}}{\mathrm{d}x}\left(L-\sum_{m=1}\sum_{k=0}^{m-1}(-1)^k\left(\frac{\mathrm{d}^k}{\mathrm{d}x^k}\frac{\partial L}{\partial u_m}\right)u_{m-k}\right).$$

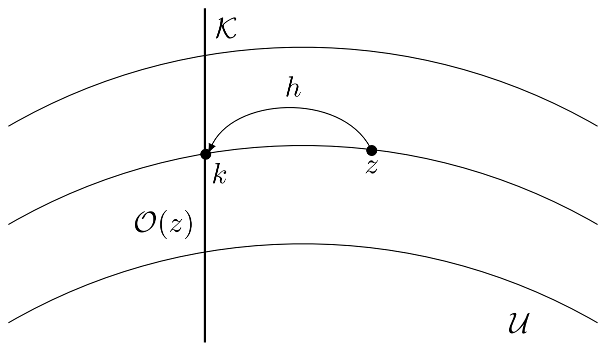
Example 2.3.19. *For our example, the three first integrals for the Euler–Lagrange equation* (2.42) *are (see Mansfield, [70])*

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}\dfrac{1}{\sqrt{1+u_x^2}} & -\dfrac{u_x}{\sqrt{1+u_x^2}} & 0\\ \dfrac{u_x}{\sqrt{1+u_x^2}} & \dfrac{1}{\sqrt{1+u_x^2}} & 0\\ \dfrac{xu_x-u}{\sqrt{1+u_x^2}} & \dfrac{uu_x+x}{\sqrt{1+u_x^2}} & 1\end{pmatrix}\begin{pmatrix}-\kappa^2\\-2\kappa_s\\2\kappa\end{pmatrix}$$

where the first component arises from the translation in x*, the second component arises from the translation in* u *and the third component arises from the rotation in the* (x,u) *plane about the origin.*

2.4 Moving frames

Consider a Lie group G whose action is free and regular in some domain M (see Definitions 2.1.15 and 2.1.16). Then the following holds (see picture below): for every $z\in M$ there exists a neighbourhood $\mathcal{U}$ of z such that the group orbits of $\mathcal{U}$ have the dimension of the Lie group G and they foliate $\mathcal{U}$. There exists a cross-section $\mathcal{K}\subset\mathcal{U}$ that intersects the group orbits of $\mathcal{U}$ transversally such that the intersection of a group orbit of $\mathcal{U}$ with the cross-section $\mathcal{K}$ is a single point. Finally, the element $h\in G$ taking $z\in\mathcal{U}$ to $\{k\}=\mathcal{O}(z)\bigcap\mathcal{K}$ is unique.



The cross-section is transverse to the orbits that foliate the space

A moving frame can be define by choosing a group action with features mentioned in the paragraph above.

Definition 2.4.1 (Moving Frame)**.** *Given a smooth Lie group action $G \times M \to M$, a moving frame is an equivariant map $\rho : \mathcal{U} \subset M \to G$ where $\mathcal{U}$ is the domain of the frame.*

A left equivariant map satisfies

$$\rho(g \cdot z) = g\rho(z) \tag{2.46}$$

and a right equivariant map satisfies

$$\rho(g \cdot z) = \rho(z)g^{-1}. \tag{2.47}$$

A frame satisfying (2.46) will be called left frame and a frame satisfying (2.47) will be called right frame.

The following table holds (see Mansfield, [70])

	left action	right action
right frame	$\rho(g * z) = \rho(z)g^{-1}$	$\rho(g \bullet z) = g^{-1}\rho(z)$
left frame	$\rho(g * z) = g\rho(z)$	$\rho(g \bullet z) = \rho(z)g$

.

In order to find the frame, we let the cross-section $\mathcal{K}$ be given by a system of equations $\psi_i(z) = 0$, for $i = 1, 2, \ldots, R$, where R is the dimension of the group G. We then solve the so-called normalization equations,

$$\psi_i(g \cdot z) = 0, \quad i = 1, \ldots, R, \tag{2.48}$$

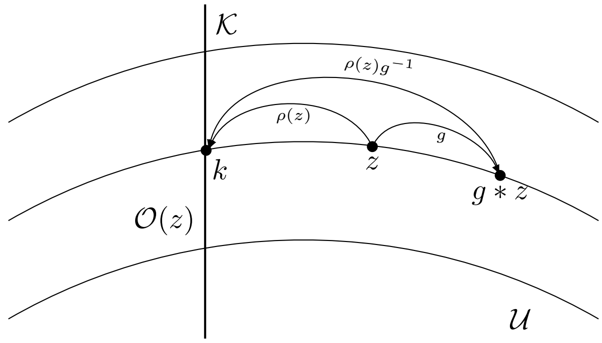
for g as a function of z. The solution is the group element $g \in G$ which maps z to k where $\{k\} = \mathcal{K} \cap \mathcal{O}(z)$, and is denoted by ρ. In other words, the frame ρ satisfies

$$\psi_i(\rho(z) \cdot z) = 0, \quad i = 1, \ldots, R.$$

The conditions on the action mentioned in the first paragraph of section (2.4) are those for the Implicit Function Theorem to hold (see Hirsch, [40]) so that the solution ρ is unique. A consequence of uniqueness is that

$$\rho(g \cdot z) = \rho(z)g^{-1}$$

that is, the frame is right equivariant, since both $\rho(g \cdot z)$ and $\rho(z)g^{-1}$ solve the equation $\psi_i\left(\rho(g \cdot z) \cdot (g \cdot z)\right) = 0$. A left frame is obtained by taking the inverse of a right frame. In practice, the ease of calculation can differ considerably depending on the choice of parity.



Using the cross-section we can construct a right moving frame

The cross-section $\mathcal{K}$ is selected by choice in order to simplify the calculations in the applications at hand. Also, the cross-section $\mathcal{K}$ is not unique.

Example 2.4.2. *Consider the action* (2.6) *and let us take the cross section* $\mathcal{K}$ *to be the coordinate plane*

$$u = 0, \qquad u_x = 1, \qquad u_{xx} = 0.$$

Therefore, the normalization equations are

$$\widetilde{u} = 0, \qquad \widetilde{u_x} = 1, \qquad \widetilde{u_{xx}} = 0. \tag{2.49}$$

Solving (2.49) *for the group parameters we obtain the frame*

$$a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2u_x^{3/2}}. \tag{2.50}$$

The frame (2.50) *can be represented as*

$$\rho = \begin{pmatrix} \dfrac{1}{\sqrt{u_x}} & -\dfrac{u}{\sqrt{u_x}} \\ \dfrac{u_{xx}}{2u_x^{3/2}} & \dfrac{2u_x^2 - uu_{xx}}{2u_x^{3/2}} \end{pmatrix} \tag{2.51}$$

as shown in Gonçalves and Mansfield, [32]. The square root restricts the domain of the frame. When making a choice of the root we make certain that the frame is the identity on the cross-section.

Note that

$$\rho(\widetilde{u},\widetilde{u_x},\widetilde{u_{xx}}) = \begin{pmatrix} \dfrac{1}{\sqrt{\widetilde{u_x}}} & -\dfrac{\widetilde{u}}{\sqrt{\widetilde{u_x}}} \\ \dfrac{\widetilde{u_{xx}}}{2\widetilde{u_x}^{3/2}} & \dfrac{2\widetilde{u_x}^2 - \widetilde{u}\widetilde{u_{xx}}}{2\widetilde{u_x}^{3/2}} \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{cu+d}{\sqrt{u_x}} & -\dfrac{au+b}{\sqrt{u_x}} \\ \dfrac{u_{xx}(cu+d)-2cu_x^2}{2u_x^{\frac{3}{2}}} & -\dfrac{u_{xx}(au+b)-2au_x^2}{2u_x^{\frac{3}{2}}} \end{pmatrix} = \rho(u,u_x,u_{xx})g^{-1}$$

so (2.51) *is equivariant.*

2.4.1 Invariants

Theorem 2.4.3. *Given a right frame, we have that $\iota(z) = \rho(z)\cdot z$ is an invariant.*

Definition 2.4.4 (invariantization operator)**.** *The map $z \mapsto \iota(z)$ will be called invariantizaton operator. This operator extends to functions as $f(z) \mapsto f(\iota(z))$, and we call $f(\iota(z))$ the invariantization of f.*

Definition 2.4.5 (normalized Invariants)**.** *Given a left or right action $G\times M \to M$ and a right frame ρ, the normalized invariants are the coordinates of $\iota(z) = \rho(z)\cdot z$.*

The components of $\iota(z)$ for any prolonged action in the $(x_i, u^\alpha, u^\alpha_K)$-space are represented as follows

$$J_i = I^{x_i} = \iota(x_i) = \widetilde{x}_i|_{g=\rho(z)}, \quad I^\alpha_K = \iota(u^\alpha_K) = \widetilde{u^\alpha_K}|_{g=\rho(z)}$$

where K is the multi-index of differentiation. For instance, $I^u_{111} = \iota(u_{xxx}) = \widetilde{u_{xxx}}|_{g=\rho(z)}$.

Theorem 2.4.6 (Replacement Rule)**.** *If $F(z)$ is an invariant of the Lie group action $G\times M \to M$, and $\iota(z)$ is the normalized invariant for a moving frame ρ on M, then $F(z) = F(\iota(z))$.*

It follows that the normalized invariants provide a set of generators for the algebra of invariants.

The Replacement rule allows us to express well-known invariants in terms of I^α_K even when we cannot solve for the frame. One can construct symbolic invariant calculus, formulated meticulously by Hubert [43], [44], [46], [45], from the normalization equations without solving the frame.

Example 2.4.7. *For our running example we have that (see Gonçalves and Mansfield, [32])*

$$I^u = \rho\cdot u = 0, \quad I^u_1 = \rho\cdot u_x = 1, \quad I^u_{11} = \rho\cdot u_{xx} = 0, \quad I^u_{111} = \rho\cdot u_{xxx} = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}.$$

This last invariant is commonly known as Schwarzian derivative of u, and it is usually denoted as $\{u;x\}$ (see Mansfield, [70]).

2.4.2 Invariant differentiation

Definition 2.4.8. *An invariant differential operator is defined by evaluating the transformed differential operator on the frame, i.e,*

$$\mathcal{D}_i = \frac{\mathrm{D}}{\mathrm{D}\widetilde{x}_i}\bigg|_{g=\rho(z)},$$

where $\dfrac{\mathrm{D}}{\mathrm{D}\widetilde{x}_i}$ *is defined as* (2.10).

Note that even though

$$\frac{\partial}{\partial x_i} u^\alpha_K = u^\alpha_{Ki},$$

the same does not hold for the invariant differential operators as

$$\mathcal{D}_i I^\alpha_K \neq I^\alpha_{Ki}.$$

Remark 2.4.9. *Note that if $\widetilde{x}_i = x_i$ then $\mathcal{D}_i = \mathrm{D}_i$.*

We define the correction terms N_{ij} and M^α_{Kj} as

$$\mathcal{D}_j J_i = \delta_{ij} + N_{ij} \qquad \text{and} \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{Kj} + M^\alpha_{Kj} \tag{2.52}$$

where δ^i_j is the Kronecker delta.

Example 2.4.10. *We introduce now an invariant variable t. For our running example, the following equations*

$$\begin{aligned} \mathcal{D}_x I^u_{12} &= I^u_{112}, \\ \mathcal{D}_x I^u_{112} &= I^u_{1112} - 2I^u_{12} I^u_{111}, \\ \mathcal{D}_x I^u_{111} &= I^u_{1111}, \\ \mathcal{D}_t I^u_{111} &= I^u_{1112} - I^u_{12} I^u_{111} \end{aligned} \tag{2.53}$$

are easy to obtain by using (2.52)*. Note that now our independent variables are x and t that correspond to the indices* 1 *and* 2 *respectively. For example $I^u_{112} = \iota(u_{xxt}) = \widetilde{u_{xxt}}|_{g=\rho(z)}$.*

The next theorem provides a formulae to compute correction terms.

Theorem 2.4.11. *There exists a $p \times R$ correction matrix $\mathbf{K}$ such that*

$$N_{kj} = \iota\left(\sum_{l=1}^{R} \mathbf{K}_{jl}\xi^k_l\right), \qquad M^\alpha_{Kj} = \iota\left(\sum_{l=1}^{R} \mathbf{K}_{jl}\phi^\alpha_{K,l}\right), \tag{2.54}$$

where l is the index for the group parameters and $R = dim(G)$.

The correction matrix $\mathbf{K}$ can be computed without explicit knowledge of the frame. In order to calculate it, we only need to know the normalization equations and the infinitesimals. Suppose that the variables appearing in the normalization equations are $\zeta_1, ..., \zeta_n$, p of which are independent, and the remaining $n-p$ are dependent variables and their derivatives. We define the matrix $\mathbf{T}$ to be the invariant $p \times n$ total derivative matrix

$$\mathbf{T}_{ij} = \iota\left(\frac{\mathrm{D}}{\mathrm{D}x_i}\zeta_j\right)$$

and let $\mathbf{\Phi}$ denote the $R \times n$ matrix of infinitesimals with invariantized arguments

$$\mathbf{\Phi}_{ij} = \iota\left(\left.\frac{\partial\widetilde{\zeta_j}}{\partial g_i}\right|_{g=e}\right).$$

Moreover, let $\mathbf{J}$ be the $n \times R$ transpose of the Jacobian matrix of the left-hand side of the normalization equations with invariantized arguments, i.e.

$$\mathbf{J}_{ij} = \frac{\partial(\iota(\psi_j))}{\partial(\iota(\zeta_i))}.$$

Using the above matrices we can obtain the correction matrix, as stated in the theorem below

Theorem 2.4.12. *The correction matrix* $\mathbf{K}$, *is given by*

$$\mathbf{K} = -\mathbf{T}\mathbf{J}(\mathbf{\Phi}\mathbf{J})^{-1}, \tag{2.55}$$

where $\mathbf{T}$, $\mathbf{J}$ *and* $\mathbf{\Phi}$ *are defined above.*

Example 2.4.13. *Consider* (2.6) *and let us induce a dummy variable t such that $\widetilde{t} = t$ and $u = u(x,t)$. Recall* (2.49). *We have that*

$$\zeta_1 = u, \quad \zeta_2 = u_x \quad \textit{and} \quad \zeta_3 = u_{xx}$$

and

$$\psi_1 = u, \quad \psi_2 = u_x - 1 \quad \textit{and} \quad \psi_3 = u_{xx}.$$

Taking into account the table of infinitesimals appearing in Example 2.1.30 it is easy to compute *the matrices* $\mathbf{\Phi}, \mathbf{J}$ *and* $\mathbf{T}$. *They are of the form*

$$\mathbf{\Phi} = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} u & u_x & u_{xx} \end{array} \\ \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array}, \quad \mathbf{J} = \begin{array}{c} \\ I^u \\ I^u_1 \\ I^u_{11} \end{array}\begin{array}{c} \begin{array}{ccc} \iota(\psi_1) & \iota(\psi_2) & \iota(\psi_3) \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}, \quad \mathbf{T} = \begin{array}{c} \\ x \\ t \end{array}\begin{array}{c} \begin{array}{ccc} u & u_x & u_{xx} \end{array} \\ \begin{pmatrix} 1 & 0 & I^u_{111} \\ I^u_2 & I^u_{12} & I^u_{112} \end{pmatrix} \end{array}$$

and therefore by (2.55), *(see Mansfield, [70]),*

$$\mathbf{K} = \begin{array}{c} \\ x \\ t \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & -1 & \frac{1}{2}I^u_{111} \\ -\frac{1}{2}I^u_{12} & -I^u_2 & \frac{1}{2}I^u_{112} \end{pmatrix} \end{array}. \tag{2.56}$$

One can check that

$$M_{11} = 0, \qquad M_{12} = -I^u_{12}, \qquad M_{21} = -I^u_{111}, \qquad M_{22} = -I^u_{112}$$

using (2.54) *obtaining the expected result. Note that* $N_{11} = N_{12} = N_{21} = N_{22} = 0$ *as* $\widetilde{x} = x$ *and* $\widetilde{t} = t$.

2.4.3 Syzygies and curvature matrices

We consider finite sets of generators of the differential algebra of invariants and the functional and differential relations they satisfy. These relations are called syzygies. Before obtaining the main differential syzygy we need to introduce the curvature matrices. Assume the Lie group G is given as a matrix group.

Definition 2.4.14 (Curvature matrices)**.** *The matrices*

$$Q^i = (\mathcal{D}_i\rho(z))\rho(z)^{-1} \tag{2.57}$$

are called curvature matrices where i *denotes an independent variable.*

The entries of the curvature matrices are called curvature invariants. It is possible to compute these matrices without explicit knowledge of the frame.

Theorem 2.4.15. *The curvature matrices can be computed using just the normalization* *equations and the infinitesimals. Indeed,*

$$Q^i = \sum_j \mathbf{K}_{ij}\mathfrak{a}_j$$

where $\{\mathfrak{a}_j\}$ *is a basis of the Lie algebra* $\mathfrak{g}$ *and* $\mathbf{K}$ *is the correction matrix given in* (2.55).

Definition 2.4.16 (Syzygy). *A syzygy on a set of invariants is a functional dependency relation between the invariants.*

Therefore, a syzygy on a set of invariants is a function of invariants, which is identically zero when the invariants are expressed in terms of the underlying variables.

Proposition 2.4.17. *The curvature matrices* (2.57) *satisfy the syzygy*

$$\mathcal{D}_j(Q^i) - \mathcal{D}_i(Q^j) = ([\mathcal{D}_j, \mathcal{D}_i]\rho)\rho^{-1} + [Q^j, Q^i]. \tag{2.58}$$

By equating components in (2.58), if the normalization equation do not involve time–derivatives, then one can express the evolution of the curvature invariants κ in terms of I_t as

$$\kappa_t = \mathcal{H} I_t \tag{2.59}$$

where $\mathcal{H}$ is an invariant differential operator matrix involving just curvature invariants. We will often call (2.59) the reduced form of (2.58).

Remark 2.4.18. *We denote* κ *the curvature invariants. Note that the* κ *appearing in* (2.37) *is not the same as the* κ *appearing in* (2.61)*. Both expressions are denoted by* κ *as they are curvature invariants for their examples respectively.*

Example 2.4.19. *Using* (2.16) *and* (2.56) *we obtain the curvature matrices*

$$Q^x = \begin{pmatrix} 0 & -1 \\ \frac{1}{2}I^u_{111} & 0 \end{pmatrix}, \qquad Q^t = \begin{pmatrix} -\frac{1}{2}I^u_{12} & -I^u_2 \\ \frac{1}{2}I^u_{112} & \frac{1}{2}I^u_{12} \end{pmatrix} \tag{2.60}$$

and also the commutator

$$[Q^x, Q^t] = \begin{pmatrix} \frac{1}{2}(I^u_{112} - I^u_2 I^u_{111}) & I^u_{12} \\ \frac{1}{2}I^u_{12}I^u_{111} & \frac{1}{2}(I^u_2 I^u_{111} - I^u_{112}) \end{pmatrix}.$$

Using (2.58) *and taking into account the relationships of the form* (2.53) *we obtain*

$$\kappa_t = (\mathcal{D}_x^3 + 2\kappa\mathcal{D}_x + \mathcal{D}_x\kappa)I^u_2$$

where we have set

$$\kappa = I^u_{111}. \tag{2.61}$$

The operator

$$\mathcal{H} = \mathcal{D}_x^3 + 2\kappa\mathcal{D}_x + \mathcal{D}_x\kappa \tag{2.62}$$

is a famous Hamiltonian operator of the KDV equation. These results were previously obtained by Gonçalves and Mansfield, [32] and Mansfield, [70].

2.4.4 Invariantized form of Calculus of Variations and Noether's Theorem

We can write the Euler–Lagrange equations of an invariant Lagrangian under a Lie group in terms of the invariants. Let us suppose that the Lagrangian depends on the independent variables $\mathbf{x} = (x_1, ..., x_p)$, the dependent variables $\mathbf{u} = (u^1, ..., u^q)$ and also finitely many derivatives of the dependent variables. We also assume that the action leaves $\mathbf{x}$ invariant. In order to obtain an invariantized analogue of (2.38), we introduce a dummy independent variable t that is invariant.

Note that the two variational problems

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathcal{L}[u^\alpha + \varepsilon v^\alpha] = \mathcal{D}_t\Big|_{u^\alpha_t = v^\alpha}\mathcal{L}[u^\alpha]$$

give the same symbolic result. Applying Calculus of Variations to the invariant variational problem

$$\int L[\boldsymbol{\kappa}]$$

where $\boldsymbol{\kappa}$ denotes the vector of curvature invariants, we obtain

$$\begin{aligned}
0 &= \mathcal{D}_t \int L[\boldsymbol{\kappa}]\mathrm{d}\mathbf{x} \\
&= \int \left[\sum_{j,K} \frac{\partial L}{\partial \mathcal{D}_K \kappa_j}\mathcal{D}_K\mathcal{D}_t\kappa_j\right]\mathrm{d}\mathbf{x} \\
&= \int \left[\sum_{j,K} (-1)^{|K|}\mathcal{D}_K \frac{\partial L}{\partial \mathcal{D}_K \kappa_j}\mathcal{D}_t\kappa_j\right]\mathrm{d}\mathbf{x} + \text{B.T.'s} \\
&= \int \left[\sum_{j,\alpha} \mathrm{E}^j(L)\mathcal{H}_{j,\alpha}I^\alpha_t\right]\mathrm{d}\mathbf{x} + \text{B.T.'s} \\
&= \int \left[\sum_{j,\alpha} \left(\mathcal{H}^*_{j,\alpha}\mathrm{E}^j(L)\right)I^\alpha_t] + \sum_i \frac{\mathrm{D}}{\mathrm{D}x_i}\left(\sum_{J,\alpha} I^\alpha_{t,J}C^\alpha_{i,J}\right)\right]\mathrm{d}\mathbf{x} + \text{B.T.'s}
\end{aligned}$$

where this defines the coefficients $C_{i,J}$ which are the coefficients of $I^{\alpha}_{t,J}$ coming from the integration by parts and where B.T.'s are the boundary terms, $\mathrm{E}^j(L)$ is the Euler operator corresponding to variations in the curvature invariants and $\mathcal{H}^*_{j,\alpha}$ is the adjoint of $\mathcal{H}_{j,\alpha}$. Note that I^{α}_t contains the factor u^{α}_t which is the independent variation in the dependent variable. Thus, from (2.3.10), the element I^{α}_t must be zero and therefore, the invariantized Euler–Lagrange equations are of the form

$$\mathrm{E}^{\alpha}(L) = \sum_j \mathcal{H}^*_{j,\alpha}\mathrm{E}^j(L). \tag{2.63}$$

In matrix form we can write (2.63) as

$$\mathrm{E}^{u}(L) = \mathcal{H}^*\mathrm{E}^{\boldsymbol{\kappa}}(L).$$

Example 2.4.20. *Let us consider the variational problem*

$$\int L(\kappa,\kappa_x)\mathrm{d}s$$

where κ was defined in (2.61)*. This variational problem is invariant under* (2.6)*. The operator* (2.62) *satisfies $\mathcal{H}^* = -\mathcal{H}$. Hence the invariantized Euler–Lagrange equation is*

$$\mathrm{E}^{u}(L) = (-\mathcal{D}_x^3 - 2\kappa\mathcal{D}_x - \kappa_x)\mathrm{E}^{\kappa}(L) = 0.$$

Remark 2.4.21. *This way of obtaining the invariantized Euler–Lagrange equations contrasts with the method proposed by Kogan and Olver, [63] as stated in Gonçalves and Mansfield, [32].*

The term $\sum_{J,\alpha} I^{\alpha}_{t,J}C^{\alpha}_{i,J}$ is a conservation law. Recall Noether's First Theorem stated in Theorem (2.3.17). We now give the invariant version of this Theorem appearing in Gonçalves and Mansfield, [32] which generalises the result obtained in Boutin, [11].

Theorem 2.4.22 (Gonçalves and Mansfield, [32])**.** *Let $\int L(\kappa_1,\kappa_2,...)\ \mathrm{d}\mathbf{x}$ be invariant under the Lie group action $G\times M \to M$ where $M = J(X\times U^{(n)})$, with generating invariants κ_j and $g\cdot x_i = x_i$. Introduce a dummy variable t to effect the variation. Using integration by parts*

$$\mathcal{D}_t\int L(\kappa_1,\kappa_2,...)\ \mathrm{d}\mathbf{x} = \int\left[\sum_{j,\alpha}\mathcal{H}^*_{j,\alpha}E^{\kappa_j}(L)I^{\alpha}_t + \sum_i \mathcal{D}_i\left(\sum_{J,\alpha} I^{\alpha}_{tJ}C_{i,J}\right)\right]\mathrm{d}\mathbf{x}, \tag{2.64}$$

where this defines the coefficients $C^{\alpha}_{i,J}$. Recall that $I^{\alpha}_{tJ} = I(u^{\alpha}_{tJ})$, where J is an index of differentiation with respect to x_i. Let $(a_1,...,a_R)$ be the coordinates of G near the identity e, and $\mathbf{v}_i$, for $i=1,...,R$ be the infinitesimal vector fields associated to each parameter defining G. Moreover, let $\mathcal{A}d$ be the Adjoint representation of G with respect to these vector fields. Let

$\Omega^{\alpha}(I)$ *for* $\alpha = 1, ..., q$ *be the invariantized form of the matrix of infinitesimals.*

Then the R *conservation laws obtained via Noether's First Theorem can be rewritten in the form*

$$\sum_i \mathcal{D}_i \left(\mathcal{A}d(\rho)^{-1} V_i(I) \right) = 0 \tag{2.65}$$

where

$$V_i(I) = \sum_{\alpha} \Omega^{\alpha}(I) \mathcal{C}_i^{\alpha} \tag{2.66}$$

and where

$$\mathcal{C}_i^{\alpha} = (C_{i,J}^{\alpha}).$$

Remark 2.4.23. *For the one–dimensional case we have that the conservation laws can be written of the form*

$$\left(\mathcal{A}d(\rho)^{-1} \mathbf{V}(I) \right) = \mathbf{c} \tag{2.67}$$

where

$$\mathbf{V}(I) = \sum_{\alpha} \Omega^{\alpha}(I) \mathcal{C}^{\alpha}.$$

Remark 2.4.24. *Note that as this Theorem concerns the smooth case, we are adopting the convention* (2.2.1) *for the Adjoint matrix.*

Example 2.4.25. *In order to compute the conservation laws we first need to keep track of the boundary terms. We have that the boundary terms are of the form*

$$\mathcal{D}_x \left(I_2^u \mathcal{D}_x^2 \mathrm{E}^{\kappa}(L) + \mathrm{E}^{\kappa}(L) \mathcal{D}_x^2 I_2^u + (\mathcal{D}_x \mathrm{E}^{\kappa}(L)) \mathcal{D}_x I_2^u \right)$$

and therefore, taking into account that $\mathcal{D}_x I_2^u = I_{12}^u$ *and* $\mathcal{D}_x^2 I_2^u = I_{112}^u$ *we have that*

$$\mathbf{C} = \begin{pmatrix} \mathcal{D}_x^2 \mathrm{E}^{\kappa}(L) + \kappa \mathrm{E}^{\kappa}(L) \\ \mathcal{D}_x \mathrm{E}^{\kappa}(L) \\ \mathrm{E}^{\kappa}(L) \end{pmatrix}.$$

The invariantized form of (2.27) *is*

$$\mathbf{\Omega}(I) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} u & u_x & u_{xx} \end{array} \\ \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array}.$$

Writing (2.33) *in the original variables and using* (2.65) *the conservation laws are*

$$\begin{pmatrix} 1-\dfrac{uu_{xx}}{u_x^2} & \dfrac{2u}{u_x} & \dfrac{u_{xx}}{u_x}-\dfrac{uu_{xx}^2}{2u_x^3} \\ -\dfrac{u_{xx}}{2u_x^2} & \dfrac{1}{u_x} & -\dfrac{u_{xx}^2}{4u_x^3} \\ -u+\dfrac{u^2u_{xx}}{2u_x^2} & -\dfrac{u^2}{u_x} & u_x-\dfrac{uu_{xx}}{u_x}+\dfrac{u^2u_{xx}^2}{4u_x^3} \end{pmatrix} \begin{pmatrix} -2\mathcal{D}_x\mathrm{E}^\kappa(L) \\ \kappa\mathrm{E}^\kappa(L)+\mathcal{D}_x^2\mathrm{E}^\kappa(L) \\ -2\mathrm{E}^\kappa(L) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

as appearing in Gonçalves and Mansfield, [32].

Conservation Laws for Semi-simple Lie groups

In the case where the Killing form is invertible, one can always obtain a first integral of the Euler–Lagrange equation. This is the case for semisimple Lie groups. Let us denote $\mathfrak{g}_s$ the semisimple Lie algebra of infinitesimal vector fields of a Lie group G. In the following, we will consider one dimensional problems.

Note that from (2.67), multiplying both sides by $\mathbf{c}^T B^{-1}$ we obtain

$$\mathbf{c}^T B^{-1}\mathcal{A}d(\rho)^{-1}\mathbf{V}(I) = \mathbf{c}^T\mathrm{B}^{-1}\mathbf{c}.$$

Substituting $\mathbf{c}^T$ by $\mathbf{V}(I)^T\mathcal{A}d(\rho)^{-T}$ we obtain

$$\mathbf{V}(I)^T\mathcal{A}d(\rho)^{-T}B^{-1}\mathcal{A}d(\rho)^{-1}\mathbf{V}(I) = \mathbf{c}^T\mathrm{B}^{-1}\mathbf{c}.$$

Using (2.36), i.e, $\mathrm{B} = \mathcal{A}d(\rho)\mathrm{B}\mathcal{A}d(\rho)^T$ we obtain the first integral

$$\mathbf{V}(I)^T\mathrm{B}^{-1}\mathbf{V}(I) = \mathbf{c}^T\mathrm{B}^{-1}\mathbf{c}.$$

Theorem 2.4.26 (Gonçalves and Mansfield, [32])**.** *Consider a semi-simple Lie algebra $\mathfrak{g}_s$. Let $\mathbf{V}(I)$ be the vector of invariants and let* B *be the Killing form of $\mathfrak{g}_s$. Let $L(\kappa^\alpha, \kappa^\alpha_x, ...)\,\mathrm{d}\mathbf{x}$ be invariant under the group action G. Then*

$$\mathbf{V}(I)^T\mathrm{B}^{-1}\mathbf{V}(I) = \mathbf{c}^T\mathrm{B}^{-1}\mathbf{c}$$

is a first integral for the Euler–Lagrange equations

$$\mathrm{E}^u(L) = \mathcal{H}^*\mathrm{E}^{\boldsymbol{\kappa}}(L)$$

where $\mathbf{V}(I)$ *is given in* (2.66) *and* $\mathbf{c}$ *is a constant vector.*

Example 2.4.27. *In order to obtain the first integral of the Euler–Lagrange equation* $\mathrm{E}^\kappa(L)$

we use (2.4.26) *and* (2.20) *to get*

$$4(\mathcal{D}_x \mathrm{E}^\kappa(L))^2 - 8\mathrm{E}^\kappa(L)\mathcal{D}_x^2 \mathrm{E}^\kappa(L) - 8\kappa(\mathrm{E}^\kappa(L))^2 = c_1^2 + 4c_2c_3.$$

The conservation law

$$-2\mathrm{E}^\kappa(L)u_x - c_1 u + c_2 u^2 - c_3 = 0 \tag{2.68}$$

is obtained by making use of (2.67) *and it is a first order ODE as shown in Gonçalves and Mansfield, [32]. By setting*

$$\tau = \int \frac{1}{2\mathrm{E}^\kappa(L)} \mathrm{d}x$$

the authors also show that (2.68) *can be transformed into a Riccati equation with constant coefficients as follows*

$$u_\tau = -c_1 u + c_2 u^2 - c_3.$$

Hence the solution of (2.68) *is*

$$u(x) = \frac{c_1}{2c_2} - \frac{\sqrt{c_2^2 + 4c_2c_3}}{2c_2} \tanh\left(\frac{1}{2}\sqrt{c_2^2 + 4c_2c_3} \int \frac{1}{2\mathrm{E}^\kappa(L)} \mathrm{d}x + c_4\right)$$

after solving for κ.

Discrete Moving Frames and Noether's Finite Difference Conservation Laws

This chapter is based on the results presented in [74], which is a joint work with my supervisor Elizabeth Mansfield and the authors Peter Hydon (University of Kent) and Linyu Peng (Waseda Institute for Advanced Study). My contribution to this paper was the development of the running example as well as the application to Euler's elastica, which will be presented in §4.1 as well as checking the theory and providing comments and observations. In this chapter, some of the results and examples have been extended and where Proposition 3.5.3 and Theorem 3.5.10 have been included.

3.1 Introduction

Discrete moving frames, which are essentially a sequence of moving frames with overlapping domains, arise with the need to use moving frames in discrete spaces. In order to adapt discrete moving frames to prolongation spaces for the study of difference equations and their conservation laws, the authors of [74] derive the difference moving frame. This adaptation allows to write the Euler–Lagrange equations and conservation laws in terms of the invariant variables in an appropriate space.

Conservation laws play an important role in the study of the solution of differential and difference equations. Emmy Noether proved in 1918 (see Noether, [83]) that every Lie group of symmetries of a physical system acting on the space of independent and dependent variables has a corresponding conservation law. The equivalent theorems for difference equations as well as other results by Noether have been developed in Dorodnitsyn, [19], Hydon, [48], Hydon and Mansfield, [50] and Peng, [92]. For some complicated problems it is easier to work in terms of the invariant variables rather than in the original variables. Once this invariant problem has been solved, one can express the solution in the original variables. This can be achieved for difference systems using discrete moving frames theory.

Let $\mathbf{u} = (u^1, \dots, u^q) \in \mathbb{R}^q$ denote a not necessarily finite set of dependent variables, and let S denote the forwards shift operator defined as follows

$$\mathrm{S} : n \mapsto n+1, \qquad \mathrm{S} : f(n) \mapsto f(n+1),$$

for all functions f whose domain includes n and $n+1$. In particular,

$$\mathrm{S} : u_j^\alpha \mapsto u_{j+1}^\alpha$$

on any domain where both u_j^α and u_{j+1}^α are defined. The forward difference operator is $\mathrm{S} - \mathrm{id}$, where id is the identity operator defined by

$$\mathrm{id} : n \mapsto n, \qquad \mathrm{id} : f(n) \mapsto f(n), \qquad \mathrm{id} : u_j^\alpha \mapsto u_j^\alpha.$$

We consider discrete Lagrangians of the form $\mathcal{L}[\mathbf{u}] = \sum_n \mathsf{L}(n, \mathbf{u}_0, \dots, \mathbf{u}_J)$ where the Lagrangian L depends on only a finite number of arguments. We seek sequences which extremise the sum, in the sense that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \sum_n \mathsf{L}(n, \mathbf{u}_0 + \epsilon \mathbf{w}_0, \dots, \mathbf{u}_J + \epsilon \mathbf{w}_J) = 0$$

for all functions $\mathbf{w} : \mathbb{Z} \to \mathbb{R}^q$. It is well known that the extremising sequences satisfy the recurrence relation known as the discrete Euler–Lagrange equation, (see Hydon and Mansfield, [49], and Kupershmidt, [64])

$$\mathrm{E}_{u^\alpha}(\mathsf{L}) := \sum_{j=0}^{J} \mathrm{S}_{-j}\left(\frac{\partial \mathsf{L}}{\partial u_j^\alpha}\right) = 0, \qquad \text{where } \mathrm{S}_{-j} = (\mathrm{S}_{-1})^j. \tag{3.1.1}$$

Each $\mathrm{E}_{u^\alpha}(L)$ depends only on n and $\mathbf{u}_{-J}, \dots, \mathbf{u}_J$, so the Euler–Lagrange equations are of order at most $2J$.

It is usual to suppress the n in the indices, and we follow that convention in this thesis. For example, the expression $u_{n+2}u_n - 2u_{n+3}^2$ will be written as $u_2u_0 - 2u_3^2$.

In §3.2, the concept of difference prolongation space as an analogue of the jet space in the case of differential equations is introduced.

In §3.3, the finite difference Calculus of Variations is briefly reviewed.

In §3.4, the discrete moving frame and the difference moving frame is introduced, which gives the geometric framework for the results.

In §3.5, it is shown how a difference moving frame can be used to calculate the difference

Euler–Lagrange equations directly in terms of the invariants. This calculation yields boundary terms that can be transformed into the conservation laws, which require both invariants and the frame for their expression. In §3.6, the Adjoint representation of the frame and the matrix of infinitesimals is recalled. In §3.7, key results on the difference conservation laws that arise via the difference analogue of Noether's Theorem are formulated.

In §3.8, it is shown how the difference moving frame may be used to integrate a difference system which is invariant under a Lie group action. Further, we show how the conservation laws and the frame together may be used to ease the integration process, in cases where one can solve for the frame, and in cases where one cannot.

The running example is a scaling and translation group invariant Lagrangian, with two dependent variables defined on a one dimensional suitable discrete subgroup.

3.2 Difference prolongation spaces

In order to work with difference equations, the concept of difference prolongation space is useful. A difference prolongation space is basically the discrete equivalent of the jet space for differential equations. From now on equations that may have singularities will not be considered.

The difference prolongation spaces are obtained from the space of independent and dependent variables, $\mathbb{Z} \times \mathbb{R}^q$. Over each base point $n \in \mathbb{Z}$, the dependent variables take values in a continuous fibre $U \subset \mathbb{R}^q$, which has the coordinates $\mathbf{u} = (u^1, \ldots, u^q)$. It is assumed that all structures on each fibre are the same.

Let $\mathbf{u}_j$ denote $\mathbf{u}(n+j)$, for all sequences $\big(\mathbf{u}(m)\big)_{m \in \mathbb{Z}}$. The fibre over n is the prolongation space $P_n^{(0,0)}(U) \simeq U$, and it has coordinates $\mathbf{u}_0$. The first forward prolongation space over n is $P_n^{(0,1)}(U) \simeq U \times U$ with coordinates $z = (\mathbf{u}_0, \mathbf{u}_1)$. The J^{th} forward prolongation space over n is the product space $P_n^{(0,J)}(U) \simeq U \times \cdots \times U$ ($J+1$ copies) with coordinates $z = (\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_J)$. Including both forward and backward shifts, one can obtain the prolongation spaces $P_n^{(J_0,J)}(U) \simeq U \times \cdots \times U$ ($J - J_0 + 1$ copies) with coordinates $z = (\mathbf{u}_{J_0}, \ldots, \mathbf{u}_J)$, where $J_0 \leq 0$ and $J \geq 0$.

The total prolongation space over n, denoted by $P_n^{(-\infty,\infty)}(U)$, has coordinates $z = (\ldots, \mathbf{u}_{-2}, \mathbf{u}_{-1}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots)$. Every prolongation space $P_n^{(J_0,J)}(U)$ is a submanifold of the total prolongation space over n. The same structures are repeated over each n as n is a free variable. This yields the natural map

$$\pi : P_n^{(-\infty,\infty)}(U) \longrightarrow P_{n+1}^{(-\infty,\infty)}(U), \qquad \pi : z \mapsto \hat{z}, \tag{3.2.1}$$

where the coordinates on $P_{n+1}^{(-\infty,\infty)}(U)$ are denoted by a caret, so $\widehat{\mathbf{u}}_j$ refers to $\mathbf{u}(n+1+j)$.

For difference equations, it is enough to use the restriction of S to finite prolongation spaces. To adapt difference equations on a finite or semi-infinite interval, the constraint that $\mathbf{u}_{j+1} = \mathrm{S}\mathbf{u}_j$ is defined only if $n+j$ and $n+1+j$ are in the interval it is added.

The variable n will be treated as fixed, using powers of the shift operator S to represent structures on prolongation spaces over any base point m as equivalent structures on all sufficiently large prolongation spaces over n. This will allow difference moving frames to be constructed. Throughout, we work formally, without considering convergence of sums or integrals.

3.3 The difference variational calculus

The methods developed in this chapter will emulate the difference variational calculus as far as possible, but using the invariant difference calculus.

Consider a discrete Lagrangian of the form

$$\mathcal{L}[\mathbf{u}] = \sum \mathsf{L}(n, \mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_J), \tag{3.3.1}$$

where $\mathbf{u}_j = (u_j^1, \ldots, u_j^q) \in \mathbb{R}^q$. From now on the unadorned summation symbol denotes summation over n and the range of this summation is a given interval in $\mathbb{Z}$, which can be unbounded. For sums over all other variables, the Einstein summation convention will be used as far as possible. The *variation* of $\mathcal{L}[\mathbf{u}]$ in the direction $\mathbf{w}$ is taken to be

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{L}[\mathbf{u}+\epsilon\mathbf{w}] = \sum w_j^\alpha \frac{\partial \mathsf{L}}{\partial u_j^\alpha}. \tag{3.3.2}$$

Making repeated summation by parts, specifically

$$(\mathrm{S}_j f)\, g = f\, \mathrm{S}_{-j} g + (\mathrm{S}_j - \mathrm{id})\,(f\, \mathrm{S}_{-j} g) \tag{3.3.3}$$

and pulling out the factor $(\mathrm{S}-\mathrm{id})$ from $(\mathrm{S}_j - \mathrm{id})$, it follows that

$$w_j^\alpha \frac{\partial \mathsf{L}}{\partial u_j^\alpha} = w_0^\alpha\, \mathrm{S}_{-j} \frac{\partial \mathsf{L}}{\partial u_j^\alpha} + (\mathrm{S}-\mathrm{id}) A_{\mathbf{u}}(n,\mathbf{w}), \quad \text{where} \quad A_{\mathbf{u}}(n,\mathbf{w}) = \sum_{j=1}^{J} \sum_{l=0}^{j-1} \mathrm{S}_l \left\{ w_0^\alpha\, \mathrm{S}_{-j} \frac{\partial \mathsf{L}}{\partial u_j^\alpha} \right\}. \tag{3.3.4}$$

This defines the boundary terms. A formula to compute this boundary terms is given in (3.5). The sum over n of the differences $(\mathrm{S}-\mathrm{id})A_{\mathbf{u}}$ telescopes, contributing only boundary terms to the variation. If the variation is zero for every $\mathbf{w}$ we say that $\mathbf{u}$ is an extremal for the

Euler–Lagrange system of difference equations

$$\mathrm{E}_{u^\alpha}(\mathsf{L}) := \mathrm{S}_{-j}\frac{\partial \mathsf{L}}{\partial u_j^\alpha} = 0, \qquad \alpha = 1, \dots, q \tag{3.3.5}$$

which is a set of recurrence equations for $\mathbf{u}$. The boundary terms will, in general be the discrete analogue of the natural boundary terms. Moreover, the boundary terms yield natural boundary conditions that must be satisfied if $\mathbf{u}$ is not fully constrained at the boundary.

Example 3.3.6. *Consider the variational problem*

$$\mathcal{L}[x, u] = \sum \mathsf{L}(x_0, u_0, x_1, u_1, u_2). \tag{3.3.7}$$

The variation of $\mathcal{L}[x, u]$ in the direction of $\mathbf{w} = (w^x, w^u)$ is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\mathcal{L}[x + \epsilon w^x, u + \epsilon w^u] &= \sum\left\{w_0^x\,\frac{\partial \mathsf{L}}{\partial x_0} + w_0^u\,\frac{\partial \mathsf{L}}{\partial u_0} + w_1^x\,\frac{\partial \mathsf{L}}{\partial x_1} + w_1^u\,\frac{\partial \mathsf{L}}{\partial u_1} + w_2^u\,\frac{\partial \mathsf{L}}{\partial u_2}\right\}\\ &= \sum\left\{w_0^u\,\mathrm{E}_x(\mathsf{L}) + w_0^u\,\mathrm{E}_u(\mathsf{L}) + (\mathrm{S} - \mathrm{id})A_{\mathbf{u}}(n, \mathbf{w})\right\}.\end{aligned}$$

The Euler–Lagrange equations, for the variables x and u are

$$\mathrm{E}_x(\mathsf{L}) := \frac{\partial \mathsf{L}}{\partial x_0} + \mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial x_1} = 0, \qquad \mathrm{E}_u(\mathsf{L}) := \frac{\partial \mathsf{L}}{\partial u_0} + \mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial u_1} + \mathrm{S}_{-2}\frac{\partial \mathsf{L}}{\partial u_2} = 0$$

respectively. The expression of $A_{\mathbf{u}}(n, \mathbf{w}) = A^x + A^u$ is of the form

$$A^x = w_0^x\,\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial x_1}, \qquad A^u = w_0^u\,\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial u_1} + (\mathrm{S} + \mathrm{id})\left(w_0^u\,\mathrm{S}_{-2}\frac{\partial \mathsf{L}}{\partial u_2}\right).$$

Those variations that leave the Lagrangian invariant, up to a total difference term are now considered.

Definition 3.3.8. *Suppose that a non-zero function $\boldsymbol{\phi} = (\phi^1(n, \mathbf{u}), \dots, \phi^q(n, \mathbf{u}))^T$ satisfies*

$$\phi_j^\alpha(n, \mathbf{u})\,\frac{\partial \mathsf{L}}{\partial u_j^\alpha} = (\mathrm{S} - \mathrm{id})B(n, \mathbf{u}), \qquad \textit{where} \quad \phi_j^\alpha = \mathrm{S}_j\phi_0^\alpha, \tag{3.3.9}$$

for some $B(n, \mathbf{u})$ which may be zero. Then the Lagrangian L is said to have a one-parameter local Lie group of variational symmetries with characteristic $\boldsymbol{\phi}$. *The Lagrangian is invariant under this symmetry if $B = 0$. If $B \neq 0$, this symmetry is called* divergence symmetry.

The meaning of the word infinitesimal and its relation to symmetries, will made clear in §3.6. The importance of symmetries is given in the next Theorem.

Theorem 3.3.10 (Difference Noether's Theorem)**.** *Suppose that a Lagrangian* L *has a variational symmetry with characteristic* $\boldsymbol{\phi} \neq \mathbf{0}$*. If* $\mathbf{u} = \bar{\mathbf{u}}$ *is a solution of the Euler–Lagrange system for* L *then*

$$\left((\mathrm{S} - \mathrm{id})\{A_{\mathbf{u}}(n, \boldsymbol{\phi}) - B(n, \mathbf{u})\} \right)\Big|_{\mathbf{u}=\bar{\mathbf{u}}} = 0. \tag{3.3.11}$$

Proof. Substituting $\boldsymbol{\phi}$ for $\mathbf{w}$ in (3.3.4) and using (3.3.9) it follows that

$$\phi^{\alpha}(n, \mathbf{u})\mathrm{E}_{u^{\alpha}}(\mathsf{L}) + (\mathrm{S} - \mathrm{id})A_{\mathbf{u}}(n, \boldsymbol{\phi}) = \phi^{\alpha}_{j}(n, \mathbf{u})\,\frac{\partial \mathsf{L}}{\partial u^{\alpha}_{j}} = (\mathrm{S} - \mathrm{id})B(n, \mathbf{u}).$$

Therefore

$$\phi^{\alpha}(n, \mathbf{u})\mathrm{E}_{u^{\alpha}}(\mathsf{L}) = (\mathrm{S} - \mathrm{id})\left(B(n, \mathbf{u}) - A_{\mathbf{u}}(n, \boldsymbol{\phi})\right).$$

If $\mathbf{u} = \bar{\mathbf{u}}$ is a solution of the Euler–Lagrange system for L then $\mathrm{E}_{u^{\alpha}}(\mathsf{L}) = 0$. Hence

$$\left((\mathrm{S} - \mathrm{id})\{A_{\mathbf{u}}(n, \boldsymbol{\phi}) - B(n, \mathbf{u})\} \right)\Big|_{\mathbf{u}=\bar{\mathbf{u}}} = 0.$$

□

The expression in Equation (3.3.11) is a *conservation law* for the Euler–Lagrange system. As there is only one independent variable, the expression in brackets is a first integral, so every solution of the Euler–Lagrange system satisfies

$$\{A_{\mathbf{u}}(n, \boldsymbol{\phi}) - B(n, \mathbf{u})\}\Big|_{\mathbf{u}=\bar{\mathbf{u}}} = c,$$

where c is a constant.

Example 3.3.12. *The Lagrangian*

$$\mathsf{L}(x_0, u_0, x_1, u_1, u_2) = \frac{x_1 - x_0}{\{(u_2 - u_1)(u_1 - u_0)\}^{3/2}} \tag{3.3.13}$$

is of the form (3.3.7)*. Therefore the Euler–Lagrange equations are*

$$\left((u_1 - u_0)\,(u_0 - u_{-1}) \right)^{-\frac{3}{2}} - \left((u_2 - u_1)\,(u_1 - u_0) \right)^{-\frac{3}{2}} = 0,$$

$$\frac{(x_1 - x_0)\,(u_1 - u_2)}{\left((u_2 - u_1)\,(u_1 - u_0) \right)^{\frac{5}{2}}} + \frac{(x_0 - x_{-1})\,(-2\,u_0 + u_{-1} + u_1)}{\left((u_1 - u_0)\,(u_0 - u_{-1}) \right)^{\frac{5}{2}}} + \frac{(x_{-1} - x_{-2})\,(u_{-1} - u_{-2})}{\left((u_0 - u_{-1})\,(u_{-1} - u_{-2}) \right)^{\frac{5}{2}}} = 0.$$

One can construct 3 *first integrals by using* (3.3.10)*:*

1. *For $\phi^x = 1$ and $\phi^u = 0$ we have the first integral*

$$\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial x_1} = c_1.$$

2. *For $\phi^x = 0$ and $\phi^u = 1$ we have the first integral*

$$\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial u_1} + (\mathrm{S}+\mathrm{id})\mathrm{S}_{-2}\frac{\partial \mathsf{L}}{\partial u_2} = c_2.$$

3. *For $\phi^x = 3x$ and $\phi^u = u$ we have the first integral*

$$3x_0\,\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial x_1} + u_0\,\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial u_1} + (\mathrm{S}+\mathrm{id})\left(u_0\,\mathrm{S}_{-2}\frac{\partial \mathsf{L}}{\partial u_2}\right) = c_3$$

where c_1, c_2 and c_3 are constants.

It has three variational symmetries, all with $B = 0$.

The first symmetry comes from the invariance of the Lagrangian under translations in x, that is, $x \mapsto x + \epsilon_1$ for all $\epsilon_1 \in \mathbb{R}$, the second symmetry comes from from invariance under translations in u, that is, $u \mapsto u + \epsilon_2$, $\epsilon_2 \in \mathbb{R}$ and the third symmetry comes from the invariance of the Lagrangian under the scalings of the form $(x, u) \mapsto (\lambda^3 x, \lambda u)$, for $\lambda \in \mathbb{R}^+$.

Note that (3.3.13) *has three first integrals for the system of Euler–Lagrange equations. However, these first integrals are really complicated which makes the system tedious to solve. Using coordinates adapted to the three symmetries, one can ease the calculations and deal simultaneously with all Lagrangians which have these symmetries.*

3.4 Discrete moving frames

We now turn our attention to discrete moving frames.

A discrete moving frame is a discrete analogue of a moving frame. The discrete moving frame is adapted to discrete base points and it amounts to a sequence of frames defined on a product manifold. Details on discrete moving frames and their applications can be found in Beffa and Mansfield, [5] and Beffa, Mansfield and Wang, [6].

From now on, the manifold where G acts will be the product manifold $\mathcal{M} = M^N$. It is assumed that the action on $\mathcal{M}$ is free, taking the number of copies N of the manifold M to be as high as necessary. For a discussion of this see Boutin, [11] and see Olver, [88] for an example where the product action is not free for any N. Questions like the regularity and freeness of the action will refer to the diagonal action on the product, specifically, given the action $(g, z_j) \mapsto g \cdot z_j$ for $z_j \in M$, the diagonal action of G on $z = (z_1, z_2, \dots, z_N) \in \mathcal{M}$ is

$$g \cdot (z_1, z_2, \dots, z_N) \mapsto (g \cdot z_1, g \cdot z_2, \dots, g \cdot z_N).$$

Throughout this subsection, no assumptions are made about any relationship between the elements $z_1, \dots, z_N$.

The definition of discrete moving frame is now given.

Definition 3.4.1 (Discrete Moving Frames: Beffa and Mansfield, [5] and Beffa, Mansfield and Wang, [6]). *Let G^N denote the Cartesian product of N copies of the group G. A map*

$$\rho : M^N \to G^N, \qquad \rho(z) = (\rho_1(z), \dots, \rho_N(z))$$

is a right discrete moving frame if

$$\rho_k(g \cdot z) = \rho_k(z) g^{-1}, \qquad k = 1, \dots, N,$$

and a left discrete moving frame if

$$\rho_k(g \cdot z) = g \rho_k(z), \qquad k = 1, \dots, N.$$

As in the smooth case, obtaining a discrete frame via the use of normalization equations yields a right discrete frame. As the theory for right and left frames is parallel, only right frames will be considered.

A discrete moving frame is a sequence of moving frames (ρ_k) with a nontrivial intersection of domains which, locally, are uniquely determined by the cross-section $\mathcal{K} = (\mathcal{K}_1, \dots, \mathcal{K}_N)$ to the group orbit through z. The right moving frame component ρ_k is the unique element of the group G that takes z to the cross section $\mathcal{K}_k$. We also define for a right frame, the invariants

$$I_{k,j} := \rho_k(z) \cdot z_j. \tag{3.4.2}$$

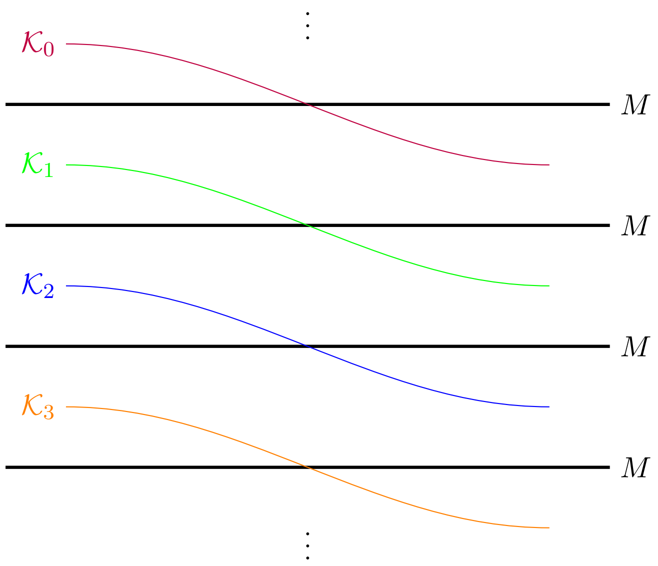
If M is q-dimensional, so that z_j has components $z_j^1, \dots, z_j^q$, the q components of $I_{k,j}$ are the invariants

$$I_{k,j}^\alpha := \rho_k(z) \cdot z_j^\alpha, \qquad \alpha = 1, \dots q. \tag{3.4.3}$$

Let ι_k denote the invariantization operator with respect to the frame $\rho_k(z)$, so that

$$I_{k,j} = \iota_k(z_j), \qquad I_{k,j}^\alpha = \iota_k(z_j^\alpha).$$

Figure 3.1: Replication of the cross-secction over n where $\mathcal{K}_k = \mathrm{S}_k\mathcal{K}_0$.



3.4.1 Difference moving frames

The construction of the discrete moving frame allows us to adapt the moving frame to any discrete domain. Usually, $\mathcal{M}$ represents the fibres M over a sequence of N discrete points where the geometric context may determine additional structures on $\mathcal{M}$.

From §3.2, as n is a free variable, we can replicate the same structures over each base point m, using powers of the natural map π, see Equation (3.2.1). Thanks to the shift operator these structures can be represented on prolongation spaces over any given n. This indicates that the natural moving frame for a given OΔE has $\mathcal{M} = P_n^{(J_0,J)}(U)$ for some suitable $J_0 \leq 0$ and $J \geq 0$. Therefore, $N = J - J_0 + 1$. From now on, the indices $1, \ldots, N$ will be replaced by $J_0, \ldots, J$ and we will use $\mathcal{K}_k$ and ρ_k to denote the cross-sections and frames on $\mathcal{M}$ respectively.

The cross-section over n, denoted $\mathcal{K}_0$, is replicated for all other base points $n + k$ if and only if the cross-section over $n + k$ is represented on $\mathcal{M}$ by

$$\mathcal{K}_k = \mathrm{S}_k\mathcal{K}_0 \tag{3.4.4}$$

for all k, see (3.4.1). If this condition holds, then by definition it follows that $\rho_k = \mathrm{S}_k\rho_0$ for all k. Consequently, $\mathcal{K}_{k+1} = \mathrm{S}\mathcal{K}_k$ and $\rho_{k+1} = \mathrm{S}\rho_k$.

A difference moving frame is defined as follows:

Definition 3.4.5. *A* difference moving frame *is a discrete moving frame such that $\mathcal{M}$ is a prolongation space $P_n^{(J_0,J)}(U)$ and* (3.4.4) *holds for all $J_0 \leq k \leq J$.*

By definition, the invariants $I_{k,j}$ given by a difference moving frame satisfy

$$\mathrm{S}I_{k,j} = I_{k+1,j+1}. \tag{3.4.6}$$

Therefore, every invariant $I_{k,j}$ can be expressed as a shift of $I_{0,j-k}$.

Example 3.4.7. *Consider the scaling and translation group action on* $\mathbb{R}^2$ *given by*

$$(x,u) \mapsto (\lambda^3 x + a, \lambda u + b), \qquad \lambda \in \mathbb{R}^+,\ a,b \in \mathbb{R}. \tag{3.4.8}$$

The Lie group is the semi-direct product, $\mathbb{R}^+ \ltimes \mathbb{R}^2$*. For the variables* x_0, u_0, x_1, u_1, u_2 *we have*

$$\widetilde{x_0} = \lambda^3 x_0 + a, \quad \widetilde{u_0} = \lambda u_0 + b, \quad \widetilde{x_1} = \lambda^3 x_1 + a, \quad \widetilde{u_1} = \lambda u_1 + b, \quad \widetilde{u_2} = \lambda u_2 + b.$$

Therefore

$$\begin{aligned}
\mathsf{L}(\widetilde{x_0}, \widetilde{u_0}, \widetilde{x_1}, \widetilde{u_1}, \widetilde{u_2}) &= \frac{\widetilde{x_1} - \widetilde{x_0}}{\{(\widetilde{u_2} - \widetilde{u_1})(\widetilde{u_1} - \widetilde{u_0})\}^{3/2}} \\
&= \frac{\lambda^3 (x_1 - x_0)}{\{\lambda^2 (u_2 - u_1)(u_1 - u_0)\}^{3/2}} \\
&= \frac{(x_1 - x_0)}{\{(u_2 - u_1)(u_1 - u_0)\}^{3/2}}.
\end{aligned}$$

Hence the Lagrangian (3.3.13) *is invariant under* (3.4.8)*. However, the action is not free on the space* $\mathbb{R}^2$ *over* n *with coordinates* (x_0, u_0)*. In order to achieve freeness, the action is extended to the first forward prolongation space* $P_n^{(0,1)}(\mathbb{R}^2)$ *which has coordinates* (x_0, u_0, x_1, u_1)*. The action is given by*

$$(x_0, u_0, x_1, u_1) \mapsto \left(\lambda^3 x_0 + a, \lambda u_0 + b, \lambda^3 x_1 + a, \lambda u_1 + b\right).$$

Choosing the normalization equations

$$\widetilde{x_0} = 0, \qquad \widetilde{u_0} = 0, \qquad \widetilde{u_1} = 1$$

and solving for the group parameters we obtain

$$\lambda = \frac{1}{u_1 - u_0}, \qquad a = -\frac{x_0}{(u_1 - u_0)^3}, \qquad b = -\frac{u_0}{u_1 - u_0}. \tag{3.4.9}$$

A representation of a generic group element is given by

$$g(\lambda,a,b)=\begin{pmatrix}\lambda^3 & 0 & a\\ 0 & \lambda & b\\ 0 & 0 & 1\end{pmatrix}\qquad \textit{with}\qquad \begin{pmatrix}g\cdot x\\ g\cdot u\\ 1\end{pmatrix}=g(\lambda,a,b)\begin{pmatrix}x\\ u\\ 1\end{pmatrix}. \tag{3.4.10}$$

Note that this representation is faithful, which means that

$$\begin{pmatrix}\lambda^3 & 0 & a\\ 0 & \lambda & b\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\qquad \textit{implies that}\qquad g=e$$

where e *is the identity of* G*. Substituting* (3.4.9) *into* (3.4.10) *we obtain a matrix representation of the moving frame*

$$\rho_0(x_0,u_0,x_1,u_1)=\begin{pmatrix}\dfrac{1}{(u_1-u_0)^3} & 0 & -\dfrac{x_0}{(u_1-u_0)^3}\\ 0 & \dfrac{1}{u_1-u_0} & -\dfrac{u_0}{u_1-u_0}\\ 0 & 0 & 1\end{pmatrix}. \tag{3.4.11}$$

Note that the frame satisfies

$$\begin{aligned}\rho_0(\widetilde{x_0},\widetilde{u_0},\widetilde{x_1},\widetilde{u_1}) &= \begin{pmatrix}\dfrac{1}{\lambda^3(u_1-u_0)^3} & 0 & -\dfrac{x_0+a/\lambda^3}{(u_1-u_0)^3}\\ 0 & \dfrac{1}{\lambda(u_1-u_0)} & -\dfrac{u_0+b/\lambda}{u_1-u_0}\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}\dfrac{1}{(u_1-u_0)^3} & 0 & -\dfrac{x_0}{(u_1-u_0)^3}\\ 0 & \dfrac{1}{u_1-u_0} & -\dfrac{u_0}{u_1-u_0}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\lambda^{-3} & 0 & -\dfrac{a}{\lambda^3}\\ 0 & \lambda^{-1} & -\dfrac{b}{\lambda}\\ 0 & 0 & 1\end{pmatrix}\\ &= \rho_0(x_0,u_0,x_1,u_1)g(\lambda,a,b)^{-1}\end{aligned}$$

so it is equivariant. Note that

$$\iota_0\begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix} = \rho_0 \cdot \begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix} \tag{3.4.12}$$

$$= \begin{pmatrix} \dfrac{1}{(u_1-u_0)^3} & 0 & -\dfrac{x_0}{(u_1-u_0)^3} \\ 0 & \dfrac{1}{u_1-u_0} & -\dfrac{u_0}{u_1-u_0} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{x_j-x_0}{(u_1-u_0)^3} \\ \dfrac{u_j-u_0}{u_1-u_0} \\ 1 \end{pmatrix}. \tag{3.4.13}$$

Therefore

$$\iota_0(x_j) := \rho_0 \cdot x_j = \frac{x_j - x_0}{(u_1-u_0)^3}, \qquad \iota_0(u_j) := \rho_0 \cdot u_j = \frac{u_j-u_0}{u_1-u_0}, \qquad j \in \mathbb{Z}.$$

Setting

$$\kappa = \iota_0(u_2) = \rho_0 \cdot u_2, \qquad \eta = \iota_0(x_1) = \rho_0 \cdot x_1, \tag{3.4.14}$$

for each $j \in \mathbb{Z}$, it follows that

$$\mathrm{S}\{\iota_0(x_j)\} = \frac{x_{j+1}-x_1}{(u_2-u_1)^3} = \frac{\dfrac{x_{j+1}-x_0}{(u_1-u_0)^3} - \dfrac{x_1-x_0}{(u_1-u_0)^3}}{\left(\dfrac{u_2-u_0}{u_1-u_0} - \dfrac{u_1-u_0}{u_1-u_0}\right)^3} = \frac{\iota_0(x_{j+1})-\eta}{(\kappa-1)^3}$$

or using the Replacement Rule (2.4.6)

$$\mathrm{S}\{\iota_0(x_j)\} = \frac{x_{j+1}-x_1}{(u_2-u_1)^3} = \iota_0\left(\frac{x_{j+1}-x_1}{(u_2-u_1)^3}\right) = \frac{\iota_0(x_{j+1})-\iota_0(x_1)}{(\iota_0(u_2)-\iota_0(u_1))^3} = \frac{\iota_0(x_{j+1})-\eta}{(\kappa-1)^3}.$$

Also

$$\mathrm{S}\{\iota_0(u_j)\} = \frac{u_{j+1}-u_1}{u_2-u_1} = \frac{\dfrac{u_{j+1}-u_0}{u_1-u_0} - \dfrac{u_1-u_0}{u_1-u_0}}{\dfrac{u_2-u_0}{u_1-u_0} - \dfrac{u_1-u_0}{u_1-u_0}} = \frac{\iota_0(u_{j+1})-1}{\kappa-1}$$

or using the Replacement Rule (2.4.6)

$$\mathrm{S}\{\iota_0(u_j)\} = \frac{u_{j+1}-u_1}{u_2-u_1} = \iota_0\left(\frac{u_{j+1}-u_1}{u_2-u_1}\right) = \frac{\iota_0(u_{j+1})-\iota_0(u_1)}{\iota_0(u_2)-\iota_0(u_1)} = \frac{\iota_0(u_{j+1})-1}{\kappa-1}.$$

Therefore

$$\iota_0(u_{j+1}) = (\kappa-1)\,\mathrm{S}\{\iota_0(u_j)\} + 1, \qquad \iota_0(x_{j+1}) = (\kappa-1)^3\,\mathrm{S}\{\iota_0(x_j)\} + \eta. \tag{3.4.15}$$

This shows that the invariants with positive j *can be written in terms of* κ, η *and their forward shifts. Let us now set* $\kappa_j = \mathrm{S}_j\kappa$ *and* $\eta_j = \mathrm{S}_j\eta$ *for all* $j \in \mathbb{Z}$*. In order to find the expression for the invariants with negative indices* j*, we do the following: We first apply* S_{-1} *to both sides of* (3.4.15)*, obtaining*

$$\mathrm{S}_{-1}\big\{\iota_0(u_{j+1})\big\} = (\kappa_{-1} - 1)\,\iota_0(u_j) + 1, \qquad \mathrm{S}_{-1}\big\{\iota_0(x_{j+1})\big\} = (\kappa_{-1} - 1)^3\,\iota_0(x_j) + \eta_{-1}.$$

Now, we send j *to* $j-1$ *and consequently,* $j+1$ *to* j*. We obtain*

$$\mathrm{S}_{-1}\big\{\iota_0(u_j)\big\} = (\kappa_{-1} - 1)\,\iota_0(u_{j-1}) + 1, \qquad \mathrm{S}_{-1}\big\{\iota_0(x_j)\big\} = (\kappa_{-1} - 1)^3\,\iota_0(x_{j-1}) + \eta_{-1}.$$

Isolating the invariantization of u_{j-1} *and* x_{j-1} *we get*

$$\iota_0(u_{j-1}) = \frac{\mathrm{S}_{-1}\{\iota_0(u_j)\} - 1}{\kappa_{-1} - 1}, \qquad \iota_0(x_{j-1}) = \frac{\mathrm{S}_{-1}\{\iota_0(x_j)\} - \eta_{-1}}{(\kappa_{-1} - 1)^3}.$$

It is important to note that $\mathrm{S}\{\iota_0(u_j)\} \neq \iota_0(u_{j+1})$ *as*

$$\mathrm{S}\{\iota_0(u_j)\} = \mathrm{S}\{\rho_0 \cdot u_j\} = \rho_1 \cdot u_{j+1} \neq \rho_0 \cdot u_{j+1} = \iota_0(u_{j+1}).$$

However, it is possible to write the shift of the generating invariants in terms of other generating invariants.

The discrete Maurer–Cartan group elements allow us to find relationships between invariants and their shifts.

Definition 3.4.16 (Discrete Maurer–Cartan invariants)**.** *Given a right discrete moving frame* ρ*, the* right discrete Maurer–Cartan group elements *are*

$$K_k = \rho_{k+1}\rho_k^{-1} \tag{3.4.17}$$

for $J_0 \leq k \leq J - 1$*.*

These relationships are an example of syzygies.

Definition 3.4.18 (Syzygy)**.** *A syzygy on a set of invariants is a identity between invariants that expresses functional dependency.*

The equivariance of the frames yields that K_k is invariant under the action of G and the components of the Maurer–Cartan elements are called the *Maurer–Cartan invariants* or *curvature invariants* .

Since ρ_k is a frame for each k, the components of $\rho_k(z) \cdot z$ generate the set of all invariants by the Replacement Rule (2.4.6).

Essentially the Maurer–Cartan group elements, are well-adapted to studying difference equations. One can express *all* invariants in terms of a small generating set. Using (3.4.2) and (3.4.17)

$$K_k \cdot I_{k,j} = \rho_{k+1}\rho_k^{-1} \cdot \rho_k \cdot z_j = \rho_{k+1} \cdot z_j = I_{k+1,j}, \tag{3.4.19}$$

and iterating this, $K_{k+1}K_k \cdot I_{k,j} = I_{k+2,j}$, and so on. This leads to the following result:

Theorem 3.4.20 (See Proposition 3.11 in Beffa, Mansfield and Wang, [6])**.** *Given a right discrete moving frame ρ, the components of K_k, together with the set of all diagonal invariants, $I_{j,j} = \rho_j(z) \cdot z_j$, generate all other invariants.*

The notion of a generating set from can be extended as follows:

Definition 3.4.21. *A set of invariants is a generating set for an algebra of difference invariants if any difference invariant in the algebra can be written as a function of elements of the generating set and their shifts.*

For a right difference moving frame, the identities $I_{j,j} = \mathrm{S}_j I_{0,0}$ and $K_k = \mathrm{S}_k K_0$ hold, so Theorem (3.4.20) reduces to the following result:

Theorem 3.4.22. *Given a right difference moving frame ρ, the set of all invariants is generated by the set of components of $K_0 = \rho_1\rho_0^{-1}$ and $I_{0,0} = \rho_0(z) \cdot z_0$.*

As K_0 is invariant, by the Replacement Rule, it follows that

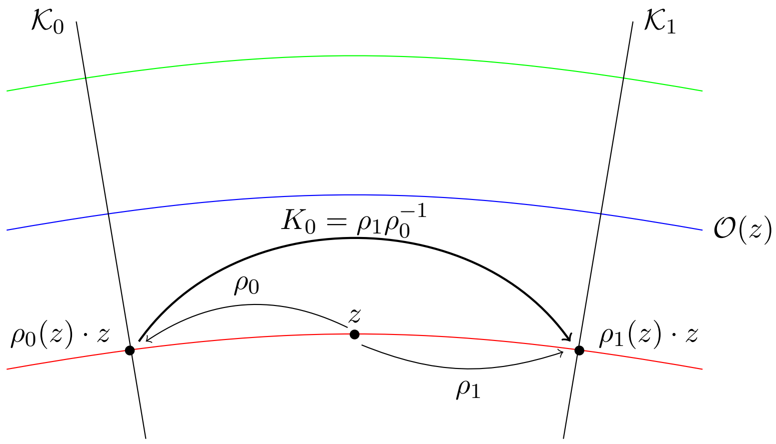
$$K_0 = \iota_0(\rho_1) \tag{3.4.23}$$

where ß$_0(\rho_1)$ denotes the invariantization of ρ_1 using ρ_0. In matrix form, the elements of ρ_1 of the form z_j are replaced by $\rho_0(z) \cdot z_j$.

Example 3.4.24. *The Euler–Lagrange equations associated to* (3.3.13) *define a subspace of the prolongation space $\mathcal{M} = P_n^{(-2,2)}(\mathbb{R}^2)$, due to the fact that* (3.3.13) *is a second-order Lagrangian. Therefore, we will be working on this space for the rest of this example. A difference moving frame in $\mathcal{M}$ coming from* (3.4.11) *is constructed by considering the sequence of frames $\rho_k = \mathrm{S}_k\rho_0$. Recall* (3.4.12)

$$\begin{pmatrix} I^x_{0,j} \\ I^u_{0,j} \\ 1 \end{pmatrix} = \iota_0 \begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix} = \rho_0 \begin{pmatrix} x_j \\ u_j \\ 1 \end{pmatrix}.$$

Figure 3.2: Assuming a left action, in this way, the action by the Maurer-Cartan element provides a change of coordinates from one set of generating invariants to another.



Taking the forward shift we obtain

$$\begin{pmatrix} \mathrm{S}I^x_{0,j} \\ \mathrm{S}I^u_{0,j} \\ 1 \end{pmatrix} = \rho_1 \begin{pmatrix} x_{j+1} \\ u_{j+1} \\ 1 \end{pmatrix} = \left(\rho_1\rho_0^{-1}\right)\rho_0 \begin{pmatrix} x_{j+1} \\ u_{j+1} \\ 1 \end{pmatrix} = K_0 \begin{pmatrix} I^x_{0,j+1} \\ I^u_{0,j+1} \\ 1 \end{pmatrix}$$

where the matrix $K_0 = \rho_1\rho_0^{-1} = \iota_0(\rho_1)$ is of the form

$$K_0 = \begin{pmatrix} \dfrac{1}{(\kappa-1)^3} & 0 & -\dfrac{\eta}{(\kappa-1)^3} \\ 0 & \dfrac{1}{\kappa-1} & -\dfrac{1}{\kappa-1} \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.4.25}$$

where η and κ are defined in (3.4.14). *Explicitly it follows that*

$$I^u_{0,j+1} = (\kappa-1)\,\mathrm{S}I^u_{0,j} + 1, \qquad I^x_{0,j+1} = (\kappa-1)^3\,\mathrm{S}I^x_{0,j} + \eta. \tag{3.4.26}$$

Note that equations (3.4.15) *and* (3.4.26) *are consistent.*

The Maurer–Cartan invariants for this example are the components of K_0 and their shifts. By Theorem (3.4.22)*, the algebra of invariants is generated by η, κ and their shifts, because both components of $I_{0,0} = \rho_0 \cdot (x_0, u_0)$ are zero.*

For a complete discussion of Maurer–Cartan invariants for discrete moving frames, with their recurrence relations and discrete syzygies, see Beffa and Mansfield, [5].

3.4.2 Differential–difference invariants and the differential–difference syzygy

The introduction of a dummy variable t will be key to obtain the Euler–Lagrange equations in terms of the invariants.

Consider now a smooth path $t \mapsto z(t)$ in the space $\mathcal{M} = M^N$ and consider the induced group action on the path and its tangent. The group action is extended to the dummy variable t trivially, so that t is invariant and the action to the first-order jet space of $\mathcal{M}$ as follows:

$$g \cdot \frac{\mathrm{d}z(t)}{\mathrm{d}t} = \frac{\mathrm{d}\left(g \cdot z(t)\right)}{\mathrm{d}t}.$$

If the action is free and regular on $\mathcal{M}$, it will also be free and regular on the jet space and the same frame may be used to find the first-order differential invariants, specifically

$$I_{k,j;\,t}(t) := \rho_k(z(t)) \cdot \frac{\mathrm{d}z_j(t)}{\mathrm{d}t}. \tag{3.4.27}$$

Let $I_{k,j}(t)$ denote the restriction of $I_{k,j}$ to the path $z(t)$. Since the frame depends on $z(t)$, we have in general that

$$I_{k,j;\,t}(t) \neq \frac{\mathrm{d}}{\mathrm{d}t} I_{k,j}(t). \tag{3.4.28}$$

For the computation of the invariantized form of the Euler–Lagrange equations, the evolution of the curvature invariants are required to be written in terms of the first order differential invariants and a linear differential operator, specifically

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\kappa} = \mathcal{H}\boldsymbol{\sigma}, \tag{3.4.29}$$

where $\boldsymbol{\kappa}$ is a vector of generating invariants, $\mathcal{H}$ is a linear difference operator with coefficients that are functions of κ and its shifts, and $\boldsymbol{\sigma}$ is a vector of generating first order differential invariants of the form (3.4.27). There are two methods for finding (3.4.29):

Method 1 If the explicit formulae in the original variables of the curvature invariants are known, (3.4.29) can be found by direct differentiation followed by the Replacement Rule, (2.4.6).

Method 2 By differentiating Maurer–Cartan matrix as follows

$$\frac{\mathrm{d}}{\mathrm{d}t} K_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_{k+1}\rho_k^{-1}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_{k+1}\right)\rho_{k+1}^{-1} K_k - K_k \left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_k\right)\rho_k^{-1}. \tag{3.4.30}$$

and equating components.

This motivates the following definition:

Definition 3.4.31 (Curvature Matrix)**.** *The curvature matrix N_k is given by*

$$N_k = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,\rho_k\right)\rho_k^{-1} \tag{3.4.32}$$

when ρ_k is in matrix form.

It can be seen that for a right frame, N_k is an invariant matrix that involves the first order differential invariants. The above derivation applies to all discrete moving frames. For a difference frame, moreover, $N_k = \mathrm{S}_k N_0$ and (3.4.30) simplifies to the set of shifts of a generating syzygy,

$$\frac{\mathrm{d}}{\mathrm{d}t}K_0 = (\mathrm{S}N_0)K_0 - K_0N_0. \tag{3.4.33}$$

As N_0 is invariant, the Replacement Rule (2.4.6) yields the following:

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\rho_0\right). \tag{3.4.34}$$

Finally, the differential–difference syzygies for the diagonal invariants are needed (see Theorem (3.4.22)). For a linear (matrix) action,

$$\frac{\mathrm{d}}{\mathrm{d}t}I_{0,0}(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right)\rho_0^{-1}\cdot(\rho_0\cdot z_0(t)) + \rho_0\cdot\frac{\mathrm{d}}{\mathrm{d}t}z_0(t) = N_0I_{0,0}(t) + I_{0,0;\,t}(t). \tag{3.4.35}$$

For nonlinear actions, the techniques described in Mansfield, [70], may be modified to accommodate difference moving frames, as we will show in more detail in §5.

In all the examples in this thesis, the diagonal invariants $I^{\alpha}_{0,0}$ are normalized to be constants. However, this does not hold in general as sometimes it is necessary to chose a normalization that makes off-diagonal invariants constants, in which case some diagonal invariants may depend on $z(t)$.

Remark 3.4.36. *Equation* (3.4.29) *will be called the reduced form or canonical form of* (3.4.33).

Example 3.4.37. *Suppose $x_j = x_j(t)$ and $u_j = u_j(t)$, etc. The aim is to compute the expressions on the original variables of differential invariants and also obtain recurrence relations. The action on the derivatives $x'_j = \mathrm{d}x_j/\mathrm{d}t$, $u'_j = \mathrm{d}u_j/\mathrm{d}t$ is induced by the chain rule (also known as implicit differentiation), as follows:*

$$g\cdot x'_j = \frac{\mathrm{d}\,(g\cdot x_j)}{\mathrm{d}\,(g\cdot t)} = \frac{\mathrm{d}\,(g\cdot x_j)}{\mathrm{d}t} = \lambda^3 x'_j, \qquad g\cdot u'_j = \frac{\mathrm{d}\,(g\cdot u_j)}{\mathrm{d}\,(g\cdot t)} = \frac{\mathrm{d}\,(g\cdot u_j)}{\mathrm{d}t} = \lambda u'_j.$$

Define

$$\begin{pmatrix} I^x_{0,j;\,t} \\ I^u_{0,j;\,t} \\ 0 \end{pmatrix} = \rho_0 \begin{pmatrix} x'_j \\ u'_j \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{(u_1-u_0)^3} & 0 & -\dfrac{x_0}{(u_1-u_0)^3} \\ 0 & \dfrac{1}{u_1-u_0} & -\dfrac{u_0}{u_1-u_0} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x'_j \\ u'_j \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{x'_j}{(u_1-u_0)^3} \\ \dfrac{u'_j}{u_1-u_0} \\ 0 \end{pmatrix}.$$

Therefore

$$I^x_{0,j;\,t} = \rho_0 \cdot x'_j = \frac{x'_j}{(u_1-u_0)^3}, \qquad I^u_{0,j;\,t} = \rho_0 \cdot u'_j = \frac{u'_j}{u_1-u_0}. \tag{3.4.38}$$

Taking the forward shift we obtain

$$\begin{pmatrix} \mathrm{S}I^x_{0,j;\,t} \\ \mathrm{S}I^u_{0,j;\,t} \\ 0 \end{pmatrix} = \rho_1 \begin{pmatrix} x'_{j+1} \\ u'_{j+1} \\ 0 \end{pmatrix} = \left(\rho_1\rho_0^{-1}\right)\rho_0 \begin{pmatrix} x'_{j+1} \\ u'_{j+1} \\ 0 \end{pmatrix} = K_0 \begin{pmatrix} I^x_{0,j+1;\,t} \\ I^u_{0,j+1;\,t} \\ 0 \end{pmatrix}$$

obtaining

$$\begin{pmatrix} \mathrm{S}I^x_{0,j;\,t} \\ \mathrm{S}I^u_{0,j;\,t} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{(\kappa-1)^3} & 0 & -\dfrac{\eta}{(\kappa-1)^3} \\ 0 & \dfrac{1}{\kappa-1} & -\dfrac{1}{\kappa-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I^x_{0,j+1;\,t} \\ I^u_{0,j+1;\,t} \\ 0 \end{pmatrix}.$$

It follows that

$$\mathrm{S}I^x_{0,j;\,t} = \frac{I^x_{0,j+1;\,t}}{(\kappa-1)^3}, \qquad \mathrm{S}I^u_{0,j;\,t} = \frac{I^u_{0,j+1;\,t}}{\kappa-1}. \tag{3.4.39}$$

In the same way, one can use the shift operator and $\rho_k\rho_0^{-1} = K_{k-1}K_{k-2}\cdots K_0$ *to obtain all* $I^x_{k,j;\,t}$, $I^u_{k,j;\,t}$ *in terms of the generating Maurer–Cartan invariants,*

$$\sigma^x := I^x_{0,0;\,t} = \iota_0(x'_0) \;= \rho_0 \cdot x'_0 = \frac{x'_0}{(u_1-u_0)^3}, \qquad \sigma^u := I^u_{0,0;\,t} = \iota_0(u'_0) \;= \rho_0 \cdot u'_0 = \frac{u'_0}{u_1-u_0},$$

and their shifts. The differential–difference syzygies (3.4.33) *are now obtained where* (3.4.34) *is used to calculate* N_0. *It follows that*

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right) = \begin{pmatrix} -3(\iota_0(u'_1)-\iota_0(u'_0)) & 0 & -\iota_0(x'_0) \\ 0 & -(\iota_0(u'_1)-\iota_0(u'_0)) & -\iota_0(u'_0) \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4.40}$$

From (3.4.39) *for* $j = 0$ *it follows that*

$$I^x_{0,1;\,t} = (\kappa - 1)^3\,\mathrm{S}I^x_{0,0;\,t}, \qquad I^u_{0,1;\,t} = (\kappa - 1)\,\mathrm{S}I^u_{0,0;\,t}$$

Substituting this into (3.4.40) N_0 *is obtained in terms of* σ^x, σ^u *and their shifts:*

$$N_0 = \begin{pmatrix} -3\left((\kappa-1)\mathrm{S}\sigma^u - \sigma^u\right) & 0 & -\sigma^x \\ 0 & -\left((\kappa-1)\mathrm{S}\sigma^u - \sigma^u\right) & -\sigma^u \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4.41}$$

Inserting (3.4.25) *and* (3.4.41) *into* (3.4.33) *yields, after equating components and simplifying,*

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}t} &= \left[(\kappa-1)^3\,\mathrm{S} - \mathrm{id}\right]\sigma^x + 3\eta\left[\mathrm{id} - (\kappa-1)\,\mathrm{S}\right]\sigma^u, \\ \frac{\mathrm{d}\kappa}{\mathrm{d}t} &= (\kappa-1)\left[\mathrm{id} - \kappa\,\mathrm{S} + (\kappa_1 - 1)\,\mathrm{S}_2\right]\sigma^u. \end{aligned} \tag{3.4.42}$$

Therefore, the differential–difference syzygy between the generating difference invariants, η *and* κ*, and the generating differential invariants,* σ^x *and* σ^u*, can be put into the reduced form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \eta \\ \kappa \end{pmatrix} = \mathcal{H}\begin{pmatrix} \sigma^x \\ \sigma^u \end{pmatrix},$$

where $\mathcal{H}$ *is a linear difference operator whose coefficients depend only on the generating difference invariants and their shifts.*

3.5 The Euler–Lagrange equations for a Lie group invariant Lagrangian

In this section the calculation of the Euler–Lagrange equations is presented, in terms of invariants, for a Lie group invariant difference Lagrangian.

First, we make the following definition and propositions, which we will prove.

Definition 3.5.1. *Given a linear difference operator* $\sum_j \mathcal{H} = c_j\mathrm{S}_j$*, the adjoint operator* $\mathcal{H}^*$ *is defined by*

$$\mathcal{H}^*(F) = \sum_j \mathrm{S}_{-j}(c_jF)$$

and the associated boundary term $A_{\mathcal{H}}$ *is defined by*

$$F\mathcal{H}(G) - \mathcal{H}^*(F)G = (\mathrm{S} - \mathrm{id})(A_{\mathcal{H}}(F, G)),$$

for all appropriate expressions F and G.

Remark 3.5.2. *Note that in the above definition c_j denote the coefficients of S_j for each j, not the j-shift of c_0.*

We now make the following remark:

Proposition 3.5.3. *The equality*

$$(\mathrm{S}_k - \mathrm{id}) = (\mathrm{S} - \mathrm{id}) \sum_{j=0}^{k-1} \mathrm{S}_j$$

holds.

Proof. We prove this equality by induction. For $k = 1$ we have

$$(\mathrm{S} - \mathrm{id}) \sum_{j=0}^{0} \mathrm{S}_j = (\mathrm{S} - \mathrm{id}).$$

Let us suppose the equality holds for k. For $k + 1$ we have

$$\begin{aligned}
(\mathrm{S}_{k+1} - \mathrm{id}) &= (\mathrm{S}_{k+1} - \mathrm{S}_k + \mathrm{S}_k - \mathrm{id}) \\
&= \mathrm{S}_k (\mathrm{S} - \mathrm{id}) + \mathrm{S}_k - \mathrm{id} \\
&= \mathrm{S}_k (\mathrm{S} - \mathrm{id}) + (\mathrm{S} - \mathrm{id}) \sum_{j=0}^{k-1} \mathrm{S}_j \\
&= (\mathrm{S} - \mathrm{id}) \left(\mathrm{S}_k + \sum_{j=0}^{k-1} \mathrm{S}_j \right) \\
&= (\mathrm{S} - \mathrm{id}) \sum_{j=0}^{k} \mathrm{S}_j
\end{aligned}$$

so it holds for $k + 1$ and therefore, by induction it holds for all k. □

Proposition 3.5.4. *For $\mathcal{H} = \sum_{k=0}^{m} c_k \mathrm{S}_k$ where $\mathcal{H}^* = \sum_{k=0}^{m} (\mathrm{S}_{-k} c_k) \mathrm{S}_{-k}$ it follows that*

$$F\mathcal{H}(G) - \mathcal{H}^*(F)G = (\mathrm{S} - \mathrm{id}) A_{\mathcal{H}}(F, G)$$

where

$$A_{\mathcal{H}}(F, G) = \sum_{k-1}^{m} \left(\sum_{j=0}^{k-1} \mathrm{S}_j \right) (\mathrm{S}_{-k} (c_k F) G)$$

for all appropriate expressions F and G.

Proof.

$$F\mathcal{H}(G) - \mathcal{H}^*(F)G = F\sum_{k=0}^{m} c_k\left(\mathrm{S}_k G\right) - \sum_{k=0}^{m}\left(\mathrm{S}_{-k}(c_k F)\right)G \tag{3.5.5}$$

$$= \sum_{k=0}^{m}\left(Fc_k\left(\mathrm{S}_k G\right) - \left(\mathrm{S}_{-k}(c_k F)\right)G\right) \tag{3.5.6}$$

$$= \sum_{k=0}^{m}\left(\left(\mathrm{S}_k - \mathrm{id}\right)\left(\mathrm{S}_{-k}(c_k F)\right)G\right) \tag{3.5.7}$$

$$= \sum_{k=0}^{m}\left(\left(\mathrm{S} - \mathrm{id}\right)\left(\sum_{j=0}^{k-1}\mathrm{S}_j\right)\left(\mathrm{S}_{-k}\left(c_k F\right)G\right)\right) \tag{3.5.8}$$

$$= \left(\mathrm{S} - \mathrm{id}\right)\sum_{k=0}^{m}\left(\left(\sum_{j=0}^{k-1}\mathrm{S}_j\right)\left(\mathrm{S}_{-k}\left(c_k F\right)G\right)\right) \tag{3.5.9}$$

where we have used Proposition 3.5.3. □

Suppose a group action $G \times M \to M$ is given and that a difference frame for this action has been found. Any Lie group invariant Lagrangian $\mathsf{L}(n, \mathbf{u}_0, \ldots, \mathbf{u}_J)$ can be written, in terms of the generating invariants $\boldsymbol{\kappa}$ and their shifts $\boldsymbol{\kappa}_j = \mathrm{S}_j\boldsymbol{\kappa}$, as $L(n, \boldsymbol{\kappa}_0, \ldots, \boldsymbol{\kappa}_{J_1})$ for some J_1. The argument from the associated functional is dropped, setting

$$\mathcal{L} = \sum \mathsf{L}(n, \mathbf{u}_0, \ldots, \mathbf{u}_J) = \sum L(n, \boldsymbol{\kappa}_0, \ldots, \boldsymbol{\kappa}_{J_1}).$$

The discrete version of the Fundamental Lema of Calculus of Variations is as follows:

Theorem 3.5.10. *Consider the inner product*

$$< f, g > := \sum f_n g_n$$

on the space ℓ_2

$$\ell_2 = \left\{\sum f_n \middle| \sum f_n^2 < \infty\right\}.$$

If

$$< f, g > = 0 \qquad \textit{for all } g$$

then $f = 0$.

Now the Invariant Euler–Lagrange Equations theorem is given:

Theorem 3.5.11 (Invariant Euler–Lagrange Equations)**.** *(See Mansfield, R–E, Hydon and Peng, [74]). Let* $\mathcal{L}$ *be a Lagrangian functional whose invariant Lagrangian is given in terms of* *the generating invariants as*

$$\mathcal{L} = \sum L(n, \boldsymbol{\kappa}_0, \dots, \boldsymbol{\kappa}_{J_1}),$$

and suppose that the differential–difference syzygies are

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t} = \mathcal{H}\boldsymbol{\sigma}.$$

Then, it follows that

$$\mathrm{E}_{\mathbf{u}}(\mathsf{L}) \cdot \mathbf{u}_0' = \left(\mathcal{H}^* \mathrm{E}_{\boldsymbol{\kappa}}(L)\right) \cdot \boldsymbol{\sigma}, \tag{3.5.12}$$

where $\mathrm{E}_{\boldsymbol{\kappa}}(L)$ *is the difference Euler operator with respect to* $\boldsymbol{\kappa}$ *and where here* $\cdot$ *denotes the inner product. Consequently, the invariantization of the original Euler–Lagrange equations is*

$$\iota_0\left(\mathrm{E}_{\mathbf{u}}(\mathsf{L})\right) = \mathcal{H}^* \mathrm{E}_{\boldsymbol{\kappa}}(L). \tag{3.5.13}$$

Proof. Set $\mathbf{u} = \mathbf{u}(t)$. In order to effect the variation the calculation of

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L} = \sum \left\{\mathrm{E}_{\mathbf{u}}(\mathsf{L}) \cdot \mathbf{u}_0' + (\mathrm{S} - \mathrm{id})(A_{\mathbf{u}})\right\} \tag{3.5.14}$$

is replicated but computing it in terms of the invariants. This gives $\mathrm{d}\mathcal{L}/\mathrm{d}t = \sum \mathrm{d}L/\mathrm{d}t$, where

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial \kappa_j^\alpha} \frac{\mathrm{d}\kappa_j^\alpha}{\mathrm{d}t} \\
&= \frac{\partial L}{\partial \kappa_j^\alpha} \mathrm{S}_j \frac{\mathrm{d}\kappa^\alpha}{\mathrm{d}t} \\
&= \left(\mathrm{S}_{-j} \frac{\partial L}{\partial \kappa_j^\alpha}\right) \frac{\mathrm{d}\kappa^\alpha}{\mathrm{d}t} + (\mathrm{S} - \mathrm{id})(A_{\boldsymbol{\kappa}}) \\
&= \mathrm{E}_{\boldsymbol{\kappa}}(L) \cdot \frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t} + (\mathrm{S} - \mathrm{id})(A_{\boldsymbol{\kappa}}) \\
&= \mathrm{E}_{\boldsymbol{\kappa}}(L) \cdot \mathcal{H}\boldsymbol{\sigma} + (\mathrm{S} - \mathrm{id})(A_{\boldsymbol{\kappa}}) \\
&= \left(\mathcal{H}^* \mathrm{E}_{\boldsymbol{\kappa}}(L)\right) \cdot \boldsymbol{\sigma} + (\mathrm{S} - \mathrm{id})\{A_{\boldsymbol{\kappa}} + A_{\mathcal{H}}\}.
\end{aligned} \tag{3.5.15}$$

The boundary terms arising from the first and second summations by parts are $(\mathrm{S} - \mathrm{id})A_{\boldsymbol{\kappa}}$ and $(\mathrm{S} - \mathrm{id})A_{\mathcal{H}}$ respectively where $A_{\boldsymbol{\kappa}}$ is linear in the $\mathrm{d}\kappa^\alpha/\mathrm{d}t$ and their shifts, while $A_{\mathcal{H}}$ is linear in the σ^α and their shifts. Also note that $\boldsymbol{\sigma}$ is the invariantized variation. By (3.5.10), the identity (3.5.12) holds. Now note that

$$\iota_0(\mathbf{u}_0') = \boldsymbol{\sigma}.$$

Therefore, applying ι_0 to (3.5.12) and comparing components of $\boldsymbol{\sigma}$ we obtain (3.5.13). □

Hence, the original Euler–Lagrange equations, in invariant form, are equivalent to

$$\mathcal{H}^*\mathrm{E}_{\boldsymbol{\kappa}}(L) = 0.$$

Example 3.5.16. *Consider an invariant Lagrangian of the form*

$$\mathcal{L} = \sum L(\eta, \kappa, \mathrm{S}\kappa).$$

Making use of $\eta_j = \mathrm{S}_j\eta$ *and* $\kappa_j = \mathrm{S}_j\kappa$ *we can write* (3.4.42) *as*

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \mathcal{H}_{11}\,\sigma^x + \mathcal{H}_{12}\,\sigma^u, \qquad \frac{\mathrm{d}\kappa}{\mathrm{d}t} = \mathcal{H}_{22}\,\sigma^u, \tag{3.5.17}$$

where

$$\begin{aligned} \mathcal{H}_{11} &= (\kappa-1)^3\,\mathrm{S} - \mathrm{id}, \qquad \mathcal{H}_{12} = 3\eta\{\mathrm{id} - (\kappa-1)\,\mathrm{S}\}, \\ \mathcal{H}_{22} &= (\kappa-1)\,\{\mathrm{id} - \kappa\mathrm{S} + (\kappa_1 - 1)\,\mathrm{S}_2\}. \end{aligned}$$

The invariantized Euler–Lagrange equations are by Theorem 3.5.11

$$\mathcal{H}^*_{11}\mathrm{E}_\eta(L) = 0, \qquad \mathcal{H}^*_{12}\mathrm{E}_\eta(L) + \mathcal{H}^*_{22}\mathrm{E}_\kappa(L) = 0,$$

where

$$\begin{aligned} &\mathcal{H}^*_{11} = (\kappa_{-1}-1)^3\,\mathrm{S}_{-1} - \mathrm{id}, \qquad \mathcal{H}^*_{12} = 3\eta\,\mathrm{id} - 3\eta_{-1}(\kappa_{-1}-1)\,\mathrm{S}_{-1}, \\ &\mathcal{H}^*_{22} = (\kappa-1)\mathrm{id} - \kappa_{-1}(\kappa_{-1}-1)\mathrm{S}_{-1} + (\kappa_{-2}-1)(\kappa_{-1}-1)\mathrm{S}_{-2}. \end{aligned}$$

Note that (3.3.13) *can be written in terms of the invariants as follows*

$$\iota_0\left(\frac{x_1 - x_0}{\{(u_2-u_1)(u_1-u_0)\}^{3/2}}\right) = \frac{\iota_0(x_1) - \iota_0(x_0)}{\{(\iota_0(u_2)-\iota_0(u_1))(\iota_0(u_1)-\iota_0(u_0))\}^{3/2}} = \eta(\kappa-1)^{-3/2}.$$

Therefore

$$\mathrm{E}_\eta = (\kappa-1)^{-3/2}, \qquad \mathrm{E}_\kappa = -\tfrac{3}{2}\,\eta(\kappa-1)^{-5/2}.$$

Hence, the invariantized Euler–Lagrange equations are

$$(\kappa_{-1}-1)^{3/2} - (\kappa-1)^{-3/2} = 0, \tag{3.5.18}$$

$$\tfrac{3}{2}\big\{\eta(\kappa-1)^{-3/2} - \eta_{-1}(\kappa_{-1}-1)^{-1/2} + \eta_{-1}(\kappa_{-1}-1)^{-3/2} - \eta_{-2}(\kappa_{-1}-1)(\kappa_{-2}-1)^{-3/2}\big\} = 0. \tag{3.5.19}$$

From (3.5.18)

$$\kappa = \frac{\kappa_{-1}}{\kappa_{-1}-1}.$$

Therefore

$$\kappa_1 = \frac{\kappa}{\kappa - 1} = \frac{\frac{\kappa_{-1}}{\kappa_{-1}-1}}{\frac{\kappa_{-1}}{\kappa_{-1}-1} - 1} = \kappa_{-1}.$$

It follows that

$$\kappa_j = \begin{cases} \dfrac{\kappa_{-1}}{\kappa_{-1} - 1}, & \text{if } j \text{ is even,} \\ \kappa_{-1}, & \text{if } j \text{ is odd.} \end{cases}$$

Shifting backwards by S_{-j} *and setting* $\kappa_{-1} - 1$ *to be* k_1 *where* k_1 *is an arbitrary nonzero constant, assuming that* L *is real-valued (*$\kappa > 1$*), the general solution of* (3.5.18) *is*

$$\kappa = 1 + \tfrac{1}{4}\left[k_1 + k_1^{-1} + (k_1 - k_1^{-1})(-1)^n\right]^2. \tag{3.5.20}$$

Therefore (3.5.19) *simplifies to*

$$k_1^{3(-1)^{n+1}}\eta + \left(k_1^{3(-1)^n} - k_1^{(-1)^n}\right)\eta_{-1} - k_1^{5(-1)^{n+1}}\eta_{-2} = 0,$$

whose general solution is

$$\eta = k_1^{3(-1)^n}\left\{k_2\left((n+1)k_1^{(-1)^{n+1}} - nk_1^{(-1)^n}\right) + k_3(-1)^n\right\}, \tag{3.5.21}$$

where k_2 *and* k_3 *are arbitrary constants.*

3.6 On infinitesimals and the Adjoint action

In §2.2, we introduced the matrix of infinitesimals and the Adjoint action as presented in the series of papers by Gonçalves and Mansfield, [32, 33, 34]. Now we present the same concept as derived in Mansfield, R–E, Hydon and Peng, [74] and Mansfield and R–E, [75] and we adopt this form for the discrete case as stated in (2.2).

Recall (2.1.29). The infinitesimal generator is extended to the prolongation space $\mathcal{M} = P_n^{(J_0,J)}(U)$ by the prolongation formula

$$\mathbf{v}(u_j^\alpha) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t)\cdot u_j^\alpha = \phi_j^\alpha = \mathrm{S}_j\phi_0^\alpha, \qquad J_0 \le j \le J,$$

see Hydon, [48]. In coordinates, the prolonged infinitesimal generator is

$$\mathbf{v} = \phi_j^\alpha \frac{\partial}{\partial u_j^\alpha}.$$

Lemma 3.6.1. *If a Lagrangian* $\mathsf{L}[\mathbf{u}]$ *is invariant under the group action* $G \times \mathcal{M} \to \mathcal{M}$*, the* *components of the infinitesimal generator of the group action given by definition* (2.1.29) *form the characteristic of a variational symmetry of* $\mathsf{L}[\mathbf{u}]$, *as defined in definition 3.3.8.*

Proof. Since the Lagrangian L is invariant, it follows that

$$\mathsf{L}(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_J) = \mathsf{L}(g \cdot \mathbf{u}_0, g \cdot \mathbf{u}_1, \ldots, g \cdot \mathbf{u}_J)$$

for all g. Thus

$$0 = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathsf{L}\left(\gamma(t) \cdot \mathbf{u}_0, \gamma(t) \cdot \mathbf{u}_1, \ldots\right) = \mathbf{v}(\mathsf{L}) = \phi_j^\alpha \frac{\partial \mathsf{L}}{\partial u_j^\alpha}$$

where

$$\phi_j^\alpha = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \gamma(t) \cdot u_j^\alpha.$$

By Definition 3.3.8, the components ϕ^α of the infinitesimal generator are the components of the characteristic of a variational symmetry of L. □

Each infinitesimal generator is determined by $\gamma'(0) \in T_eG$. Recall form (2.1.4) that T_eG is isomorphic to the Lie algebra $\mathfrak{g}$, which is the set of right-invariant vector fields on G. Right-invariance yields a Lie algebra homomorphism from $\mathfrak{g}$ to the set $\mathcal{X}$ of infinitesimal generators of symmetries (see Olver, [84] for details). If the group action is faithful, this is an isomorphism.

Also recall that the R-dimensional Lie group G has coordinates $\mathbf{a} = (a^1, \ldots, a^R)$ in a neighbourhood of the identity, e, so that the general group element is $\Gamma(\mathbf{a})$, where $\Gamma(\mathbf{0}) = e$. Given local coordinates $\mathbf{u} = (u^1, \ldots, u^q)$ on U, let $\widehat{\mathbf{u}} = \Gamma(\mathbf{a}) \cdot \mathbf{u}$. By varying each independent parameter a^r in turn, the process above yields R infinitesimal generators,

$$\mathbf{v}_r = \xi_r^\alpha(\mathbf{u})\partial_{u^\alpha}, \quad \text{where} \quad \xi_r^\alpha = \left.\frac{\partial \hat{u}^\alpha}{\partial a^r}\right|_{\mathbf{a}=\mathbf{0}}. \tag{3.6.2}$$

These form a basis for $\mathcal{X}$.

As $\mathcal{X}$ is homomorphic to $\mathfrak{g}$, the Adjoint representation of G on $\mathfrak{g}$ gives rise to the Adjoint representation of G on $\mathcal{X}$. Given $g \in G$, recall from (2.1.4) that the Adjoint representation Ad_g is the tangent map on $\mathfrak{g}$ induced by the conjugation $h \mapsto ghg^{-1}$. The corresponding Adjoint representation on $\mathcal{X}$ is expressed by a matrix, $\mathcal{A}d(g)$, which is obtained as follows. Having calculated a basis for $\mathcal{X}$,

$$\mathbf{v}_r = \xi_r^\alpha(\mathbf{u})\, \partial_{u^\alpha}, \qquad r = 1, \ldots, R,$$

let $\widetilde{\mathbf{u}} = g \cdot \mathbf{u}$ and define

$$\widetilde{\mathbf{v}}_r = \xi_r^\alpha(\widetilde{\mathbf{u}})\, \partial_{\widetilde{u}^\alpha}, \qquad r = 1, \ldots, R.$$

Now express each $\mathbf{v}_r$ in terms of $\widetilde{\mathbf{v}}_1, \ldots, \widetilde{\mathbf{v}}_R$ and determine $\mathcal{A}d(g)$ from the identity

$$(\mathbf{v}_1 \ \cdots \ \mathbf{v}_R) = (\widetilde{\mathbf{v}}_1 \ \cdots \ \widetilde{\mathbf{v}}_R)\mathcal{A}d(g). \tag{3.6.3}$$

Regarding the infinitesimal generators as differential operators and applying the identity (3.6.3) to each $\tilde{u}^\alpha$ in turn, one obtains

$$(\mathbf{v}_1(\tilde{u}^\alpha) \ \cdots \ \mathbf{v}_R(\tilde{u}^\alpha)) = (\xi_1^\alpha(\tilde{\mathbf{u}}) \ \cdots \ \xi_R^\alpha(\tilde{\mathbf{u}}))\mathcal{A}d(g). \tag{3.6.4}$$

Example 3.6.5. *Recall the action* (3.4.8)

$$\widetilde{x} = \lambda^3 x + a, \qquad \widetilde{u} = \lambda u + b.$$

The group parameters are λ, a *and* b *with identity* $(\lambda, a, b) = (1, 0, 0)$. *It follows that*

$$\left.\frac{\partial \widetilde{x}}{\partial \lambda}\right|_{g=e} = 3x, \quad \left.\frac{\partial \widetilde{u^\alpha}}{\partial \lambda}\right|_{g=e} = u, \quad \left.\frac{\partial \widetilde{x}}{\partial a}\right|_{g=e} = 1, \quad \left.\frac{\partial \widetilde{u^\alpha}}{\partial a}\right|_{g=e} = 0, \quad \left.\frac{\partial \widetilde{x}}{\partial b}\right|_{g=e} = 0, \quad \left.\frac{\partial \widetilde{u^\alpha}}{\partial b}\right|_{g=e} = 1.$$

Hence, the table of infinitesimals is of the form

	x	u
λ	$3x$	u
a	1	0
b	0	1

.

Therefore the vector of infinitesimals are of the form

$$\mathbf{v}_\lambda = 3x\partial_x + u\partial_u, \qquad \mathbf{v}_a = \partial_x, \qquad \mathbf{v}_b = \partial_u.$$

Note that

$$\partial_x = \frac{\partial \widetilde{x}}{\partial x}\partial_{\widetilde{x}} + \frac{\partial \widetilde{u}}{\partial x}\partial_{\widetilde{u}}, \qquad \partial_u = \frac{\partial \widetilde{x}}{\partial u}\partial_{\widetilde{x}} + \frac{\partial \widetilde{u}}{\partial u}\partial_{\widetilde{u}}$$

so

$$\partial_x = \lambda^3 \partial_{\widetilde{x}}, \qquad \partial_u = \lambda \partial_{\widetilde{u}}.$$

It also follows from (3.4.8)

$$x = \lambda^{-3}(\widetilde{x} - a), \qquad u = \lambda^{-1}(\widetilde{u} - b).$$

Therefore

$$\begin{aligned}
\mathbf{v}_\lambda &= 3x\partial_x + u\partial_u \\
&= 3\lambda^{-3}(\widetilde{x} - a)\lambda^3\partial_{\widetilde{x}} + \lambda^{-1}(\widetilde{u} - b)\lambda\partial_{\widetilde{u}} \\
&= 3(\widetilde{x} - a)\partial_{\widetilde{x}} + (\widetilde{u} - b)\partial_{\widetilde{u}} \\
&= 3\widetilde{x}\partial_{\widetilde{x}} + \widetilde{u}\partial_{\widetilde{u}} - 3a\partial_{\widetilde{x}} - b\partial_{\widetilde{u}}, \\
\mathbf{v}_a &= \partial_x = \lambda^3\partial_{\widetilde{x}}, \\
\mathbf{v}_b &= \partial_u = \lambda\partial_{\widetilde{u}}.
\end{aligned}$$

Hence

$$\mathbf{v}_\lambda = \widetilde{\mathbf{v}_\lambda} - 3a\widetilde{\mathbf{v}_a} - b\widetilde{\mathbf{v}}_b, \qquad \mathbf{v}_a = \lambda^3\widetilde{\mathbf{v}_a} \qquad \textit{and} \qquad \mathbf{v}_b = \lambda\widetilde{\mathbf{v}}_b.$$

Consequently,

$$(\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3) = (\widetilde{\mathbf{v}}_1\ \widetilde{\mathbf{v}}_2\ \widetilde{\mathbf{v}}_3)\mathcal{A}d(g), \quad \textit{where} \quad \mathcal{A}d(g) = \begin{pmatrix} 1 & 0 & 0 \\ -3a & \lambda^3 & 0 \\ -b & 0 & \lambda \end{pmatrix}.$$

The matrix of infinitesimals introduced already in (2.2.1) is called in [74] matrix of characteristics and it is given by the following definition:

Definition 3.6.6. *The matrix of characteristics is defined to be the $q \times R$ matrix*

$$\Phi(\mathbf{u}) = \big(\xi_r^\alpha(\mathbf{u})\big). \tag{3.6.7}$$

The equivalent lemma to (2.2.4) is as follows:

Lemma 3.6.8. *The follow identity holds*

$$\left(\frac{\partial\widetilde{\mathbf{u}}}{\partial\mathbf{u}}\right)\Phi(\mathbf{u}) = \Phi(\widetilde{\mathbf{u}})\mathcal{A}d(g), \tag{3.6.9}$$

where $(\partial\widetilde{\mathbf{u}}/\partial\mathbf{u})$ is the Jacobian matrix.

The equation (3.6.9) can be extended to prolongation spaces with coordinates $z = (\mathbf{u}_{J_0}, \ldots, \mathbf{u}_J)$, where $J_0 \leq 0$ and $J \geq 0$; the matrix of prolonged infinitesimals is defined to be

$$\Phi(z) = \begin{pmatrix} \Phi(\mathbf{u}_{J_0}) \\ \vdots \\ \Phi(\mathbf{u}_J) \end{pmatrix}.$$

The infinitesimal generators $\mathbf{v}_r$, prolonged to all variables in z, satisfy (3.6.3), where the tilde now denotes replacement of z by $g \cdot z$. Applying this identity to $g \cdot z$ gives

$$\left(\frac{\partial(g \cdot z)}{\partial z}\right) \Phi(z) = \Phi(g \cdot z)\mathcal{A}d(g). \tag{3.6.10}$$

Example 3.6.11. *It is easily checked that Equation* (2.26) *holds. Indeed, setting* $z = (x_0, u_0, x_1, u_1)$ *it follows that*

$$\begin{pmatrix} \lambda^3 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^3 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} 3x_0 & 1 & 0 \\ u_0 & 0 & 1 \\ 3x_1 & 1 & 0 \\ u_1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3\widetilde{x_0} & 1 & 0 \\ \widetilde{u_0} & 0 & 1 \\ 3\widetilde{x_1} & 1 & 0 \\ \widetilde{u_1} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -3a & \lambda^3 & 0 \\ -b & 0 & \lambda \end{pmatrix}.$$

3.7 Conservation laws

In general, the conservation laws are not invariant. However, they are equivariant as they can be written in terms of invariants and the frame.

In the non invariantized version of the calculation of the Euler–Lagrange equations and boundary terms, the dummy variable t is taken to effect the variation to be a group parameter for G, under which the Lagrangian is invariant. Then the resulting boundary terms yield conservation laws, which gives the differential–difference version of Noether's theorem. For more details about this version of Noether's theorem see Peng, [92]. It is then useful to identify t with a group parameter by considering the following path in G:

$$t \mapsto \gamma_r(t) = \Gamma\left(a^1(t), \ldots, a^R(t)\right), \quad \text{where} \quad a^r(t) = t \quad \text{and} \quad a^l(t) = 0,\ l \neq r. \tag{3.7.1}$$

Recall from §3.6 that $\mathbf{a} \mapsto \Gamma(\mathbf{a})$ expresses the general group element in terms of the coordinates $\mathbf{a}$. On this path, each $(\mathbf{u}_0)'$ at $t = 0$ is an infinitesimal generator, from (3.6.2).

For the invariantized calculation, the dummy variable effecting the variation is identified with each group parameter in turn. Recall the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} L\left(n, \boldsymbol{\kappa}, \ldots, \mathrm{S}_{J_1}(\boldsymbol{\kappa})\right) = \left(\mathcal{H}^* \mathrm{E}_{\boldsymbol{\kappa}}(L)\right) \cdot \boldsymbol{\sigma} + (\mathrm{S} - \mathrm{id})\{A_{\boldsymbol{\kappa}} + A_{\mathcal{H}}\} \tag{3.7.2}$$

from the proof of Theorem 3.5.11. Also recall that $A_{\boldsymbol{\kappa}}$ is linear in $\mathrm{d}\kappa^\alpha/\mathrm{d}t$ and their shifts, while $A_{\mathcal{H}}$ is linear in the σ^α and their shifts. As t is a group parameter and each κ^α is invariant,

$\mathrm{d}\kappa^\alpha/\mathrm{d}t = 0$. Hence, (3.7.2) reduces to

$$\big(\mathcal{H}^*\mathrm{E}_{\boldsymbol{\kappa}}(L)\big)\cdot\boldsymbol{\sigma} + (\mathrm{S}-\mathrm{id})A_{\mathcal{H}} = 0, \tag{3.7.3}$$

so $(\mathrm{S}-\mathrm{id})A_{\mathcal{H}} = 0$ on all solutions of the invariantized Euler–Lagrange equations $\mathcal{H}^*\mathrm{E}_{\boldsymbol{\kappa}}(L) = 0$. From this condition, the conservation laws can be derived.

Theorem 3.7.4 (See Mansfield, R–E, Hydon and Peng, [74]). *Suppose that the conditions of Theorem 3.5.11 hold. Write*

$$A_{\mathcal{H}} = \mathcal{C}^j_\alpha \mathrm{S}_j(\sigma^\alpha),$$

where each $\mathcal{C}^j_\alpha$ *depends only on* $n, \boldsymbol{\kappa}$ *and its shifts. Let* $\Phi^\alpha(\mathbf{u}_0)$ *be the row of the matrix of characteristics corresponding to the dependent variable* u^α_0 *and denote its invariantization by* $\Phi^\alpha_0(I) = \Phi^\alpha(\rho_0\cdot\mathbf{u}_0)$. *Then the* R *conservation laws in row vector form amount to*

$$\mathcal{C}^j_\alpha \mathrm{S}_j\{\Phi^\alpha_0(I)\mathcal{A}d\,(\rho_0)\} = 0. \tag{3.7.5}$$

That is, to obtain the conservation laws, it is sufficient to make the replacement

$$\sigma^\alpha \mapsto \{\Phi^\alpha(g\cdot\mathbf{u}_0)\mathcal{A}d(g)\}\big|_{g=\rho_0} \tag{3.7.6}$$

in $A_{\mathcal{H}}$.

Proof. Recall that

$$\sigma^\alpha = \rho_0\cdot(u^\alpha_0)' = \left(\frac{\mathrm{d}}{\mathrm{d}t}\,g\cdot u^\alpha_0\right)\Big|_{g=\rho_0}. \tag{3.7.7}$$

To obtain the conservation laws, conflate t with the group parameter a^r, making the replacement

$$\rho_0\cdot(u^\alpha_0)' \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\rho_0\cdot\gamma_r(t)\cdot u^\alpha_0 \tag{3.7.8}$$

in the boundary terms $A_{\mathcal{H}}$, where $\gamma_r(t)$ is the path defined in (3.7.1). Using the chain rule, it follows that for any $g\in G$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(g\cdot\gamma_r(t)\cdot u^\alpha_0) &= \left(\frac{\partial\,(g\cdot\gamma_r(t)\cdot u^\alpha_0)}{\partial\left(\gamma_r(t)\cdot u^\beta_j\right)}\right)\Bigg|_{t=0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_r(t)\cdot u^\beta_j\right)\\ &= \frac{\partial\,(g\cdot u^\alpha_0)}{\partial u^\beta_j}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\gamma_r(t)\cdot u^\beta_j\right).\end{aligned} \tag{3.7.9}$$

In matrix form, (3.7.9) amounts to the following

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(g\cdot\gamma_r(t)\cdot u_0^\alpha)=\left(\frac{\partial(g\cdot z)}{\partial z}\,\Phi(z)\right)_{(u_0^\alpha,r)}=\big(\Phi(g\cdot z)\mathcal{A}d(g)\big)_{(u_0^\alpha,r)},$$

where (3.6.10) has been taken into account and where (u_0^α,r) denotes the entry in the row corresponding to u_0^α and the r^{th} column. Setting $g=\rho_0$, the required replacement is

$$\sigma^\alpha\mapsto\big(\Phi(\rho_0\cdot z)\mathcal{A}d(\rho_0)\big)_{(u_0^\alpha,r)}=\big(\Phi(\rho_0\cdot\mathbf{u}_0)\mathcal{A}d(\rho_0)\big)_r^\alpha.$$

By using each parameter a^r in turn, σ^α is replaced by a row vector,

$$\sigma^\alpha\mapsto\Phi_0^\alpha(I)\mathcal{A}d(\rho_0),$$

as required. □

Note that $\mathrm{S}_j\rho_0=\rho_j$, so the conservation laws amount to

$$(\mathrm{S}-\mathrm{id})\left(\mathcal{C}_j^\alpha\left(\mathrm{S}_j\Phi_0^\alpha(I)\right)\mathcal{A}d\left(\rho_j\right)\right)=0. \tag{3.7.10}$$

Also $\mathcal{A}d(\rho_j)\mathcal{A}d(\rho_0)^{-1}=\mathcal{A}d(\rho_j\rho_0^{-1})$ is invariant, which leads to the following corollary:

Corollary 3.7.11. *The conservation laws for a difference frame may be written in the form*

$$(\mathrm{S}-\mathrm{id})\{\mathbf{V}(I)\mathcal{A}d(\rho_0)\}=0 \tag{3.7.12}$$

where $\mathbf{V}(I)=(V_1\ \cdots\ V_R)$ *is an invariant row vector. Specifically,*

$$\mathbf{V}(I)=\mathcal{C}_\alpha^j\left(\mathrm{S}_j\Phi_0^\alpha(I)\right)\mathcal{A}d\left(\rho_j\rho_0^{-1}\right). \tag{3.7.13}$$

Corollary 3.7.14. *On any solution of the invariantized Euler–Lagrange equations,*

$$\mathbf{V}(I)\mathcal{A}d\left(\rho_0\right)=\mathbf{c}, \tag{3.7.15}$$

for some constant row vector $\mathbf{c}=(c_1\ \cdots\ c_R)$.

As the conservation laws depend only on the terms arising from $A_{\mathcal{H}}$, they can be calculated for all Lagrangians in the relevant invariance class, in terms of the $\mathrm{E}_{\boldsymbol{\kappa}}(L)$, independently of the precise form that the Lagrangian takes.

Example 3.7.16. *In order to compute the conservation laws, the boundary terms coming from performing Calculus of Variations need to be performed first. To do this we use* (3.5.4).

Doing the calculation (3.5.15) *while keeping track of the terms in* $A_{\mathcal{H}}$ *we obtain*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(\eta,\kappa,\mathrm{S}\kappa) &= \iota_0\{\mathrm{E}_x(L)\}\,\sigma^x + \iota_0\{\mathrm{E}_u(L)\}\,\sigma^u + (\mathrm{S}-\mathrm{id})A_{\boldsymbol{\kappa}}\\ &\quad+(\mathrm{S}-\mathrm{id})\left(\mathrm{S}_{-1}\{(\kappa-1)^3\,\mathrm{E}_\eta(L)\}\,\sigma^x\right)\\ &\quad+(\mathrm{S}-\mathrm{id})\left(-\,\mathrm{S}_{-1}\{3\eta(\kappa-1)\mathrm{E}_\eta(L)+\kappa(\kappa-1)\mathrm{E}_\kappa(L)\}\,\sigma^u\right)\\ &\quad+(\mathrm{S}_2-\mathrm{id})\left(\mathrm{S}_{-2}\{(\kappa-1)(\kappa_1-1)\mathrm{E}_\kappa(L)\}\,\sigma^u\right),\end{aligned}$$

where

$$A_{\boldsymbol{\kappa}} = \frac{\mathrm{d}\kappa}{\mathrm{d}t}\,\mathrm{S}_{-1}\left(\frac{\partial L}{\partial \mathrm{S}\kappa}\right)$$

and

$$A_{\mathcal{H}} = \mathcal{C}_x^0\,\sigma^x + \mathcal{C}_u^0\,\sigma^u + \mathcal{C}_u^1\,\mathrm{S}(\sigma^u), \tag{3.7.17}$$

where

$$\begin{aligned}\mathcal{C}_x^0 &= \mathrm{S}_{-1}\{(\kappa-1)^3\mathrm{E}_\eta(L)\},\\ \mathcal{C}_u^0 &= -\,\mathrm{S}_{-1}\{3\eta(\kappa-1)\mathrm{E}_\eta(L)+\kappa(\kappa-1)\mathrm{E}_\kappa(L)\}+\mathrm{S}_{-2}\{(\kappa-1)(\kappa_1-1)\mathrm{E}_\kappa(L)\},\\ \mathcal{C}_u^1 &= \mathrm{S}_{-1}\left\{(\kappa-1)(\kappa_1-1)\mathrm{E}_\kappa(L)\right\}.\end{aligned}$$

For this example the Adjoint representation evaluated on the frame is

$$\mathcal{A}d(\rho_0) = \begin{pmatrix} 1 & 0 & 0\\ \dfrac{3x_0}{(u_1-u_0)^3} & \dfrac{1}{(u_1-u_0)^3} & 0\\ \dfrac{u_0}{u_1-u_0} & 0 & \dfrac{1}{u_1-u_0}\end{pmatrix}$$

and the invariantized form of the matrix of infinitesimals restricted to the variables x_0 *and* u_0 *is*

$$\Phi_0(I) = \begin{pmatrix}\Phi_0^x\\ \\ \Phi_0^u\end{pmatrix} = \iota_0\begin{pmatrix}3x_0 & 1 & 0\\ \\ u_0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ \\ 0 & 0 & 1\end{pmatrix}.$$

Therefore, by (3.7.5)*, the conservation laws are of the form* $(\mathrm{S}-\mathrm{id})A_{\mathcal{H}}=0$*, where*

$$\begin{aligned}A_{\mathcal{H}} &= \mathcal{C}_x^0(0\ 1\ 0)\mathcal{A}d(\rho_0)+\mathcal{C}_u^0(0\ 0\ 1)\mathcal{A}d(\rho_0)+\mathcal{C}_u^1\mathrm{S}\big\{(0\ 0\ 1)\mathcal{A}d(\rho_0)\big\}\\ &= \left[\mathcal{C}_x^0(0\ 1\ 0)+\mathcal{C}_u^0(0\ 0\ 1)+\mathcal{C}_u^1(0\ 0\ 1)\,\mathcal{A}d(\rho_1\rho_0^{-1})\right]\mathcal{A}d(\rho_0).\end{aligned}$$

Note that the last equality is written in the form (3.7.12). *Taking into account that*

$$\mathcal{A}d(\rho_1\rho_0^{-1}) = \mathcal{A}d(K_0) = \iota_0(\mathcal{A}d(\rho_1)) = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{3\eta}{(\kappa-1)^3} & \dfrac{1}{(\kappa-1)^3} & 0 \\ \dfrac{1}{\kappa-1} & 0 & \dfrac{1}{\kappa-1} \end{pmatrix}$$

it follows that $A_{\mathcal{H}} = \mathbf{V}(I)\mathcal{A}d(\rho_0)$, *where*

$$\mathbf{V}(I) = \begin{pmatrix} \mathrm{S}_{-1}\big\{(\kappa-1)\mathrm{E}_\kappa(L)\big\} \\ \mathrm{S}_{-1}\big\{(\kappa-1)^3\,\mathrm{E}_\eta(L)\big\} \\ -\mathrm{S}_{-1}\big\{3\eta(\kappa-1)\mathrm{E}_\eta(L) + (\kappa-1)^2\,\mathrm{E}_\kappa(L)\big\} + \mathrm{S}_{-2}\big\{(\kappa-1)(\kappa_1-1)\mathrm{E}_\kappa(L)\big\} \end{pmatrix}^T.$$

For the particular Lagrangian (3.3.13), *the solutions* (3.5.20), (3.5.21) *of the invariantized Euler–Lagrange equations yield*

$$\begin{aligned} V_1 &= -\tfrac{3}{2}\,\eta_{-1}(\kappa_{-1}-1)^{-3/2} = -\tfrac{3}{4}k_2\big[k_1+k_1^{-1}+(k_1-k_1^{-1})(2n-1)(-1)^n\big]+\tfrac{3}{2}k_3(-1)^n, \\ V_2 &= (\kappa_{-1}-1)^{3/2} = k_1^{3(-1)^{n+1}}, \\ V_3 &= -\tfrac{3}{2}\left[\eta(\kappa-1)^{-3/2}+\eta_{-1}(\kappa_{-1}-1)^{-3/2}\right] = -3k_2k_1^{(-1)^{n+1}}. \end{aligned} \tag{3.7.18}$$

In the used coordinates, the first element of $(\mathrm{S}-\mathrm{id})A_{\mathcal{H}} = 0$ *is the conservation law due to the scaling invariance, the second is due to invariance under translation of* x, *and the third is due to translation of* u.

Corollary (3.7.11) can be used to write an alternate form of the Euler–Lagrange equations. Equation (3.7.12) yields

$$\mathrm{S}\mathbf{V}(I)\mathcal{A}d(\rho_1) = \mathbf{V}(I)\mathcal{A}d(\rho_0)$$

and therefore

$$\mathbf{V}(I) = \mathrm{S}\mathbf{V}(I)\mathcal{A}d(K_0). \tag{3.7.19}$$

Corollary 3.7.20. *If the components of* $\mathbf{V}(I)$ *are not all zero, the components of the vector equation* (3.7.19) *are equivalent to the Euler–Lagrange equations.*

Proof. Using (3.7.3) and (3.7.4) it follows that

$$0 = \big(\mathcal{H}^*\mathrm{E}_{\boldsymbol{\kappa}}(L)\big)\cdot\boldsymbol{\sigma} + (\mathrm{S}-\mathrm{id})\mathbf{V}(I)\mathcal{A}d(\rho_0)$$

from where the result follows. □

Remark 3.7.21. *There is another way to calculate the laws for difference frames. By Corollary* (3.7.11)*, one can use symbolic software to calculate the conservation laws in the original variables, and then use the Replacement Rule* (2.4.6)*, to obtain the invariantized first integrals* $\mathbf{V}(I) = \iota_0\{A_{\mathbf{u}}(n, \boldsymbol{\phi})\}$.

This follows from the fact that the Replacement Rule (2.4.6) *sends* ρ_0 *to the identity matrix. The recurrence formulae can then be used to write* $\mathbf{V}(I)$ *in terms of the generating invariants, namely, the methods to solve for the extremals in the original variables, given in the next section, can still be used without having to perform the more complex, invariantized summation by parts computation.*

Example 3.7.22. *For our running example, the invariantized first integrals are*

1. *For* $\phi^x = 1$ *and* $\phi^u = 0$ *we have the first integral*

$$\iota_0\left(\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial x_1}\right) = c_1.$$

2. *For* $\phi^x = 0$ *and* $\phi^u = 1$ *we have the first integral*

$$\iota_0\left(\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial u_2} - \frac{\partial \mathsf{L}}{\partial u_0}\right) = c_2.$$

3. *For* $\phi^x = 3x$ *and* $\phi^u = u$ *we have the first integral*

$$\iota_0\left(\mathrm{S}_{-1}\frac{\partial \mathsf{L}}{\partial u_2}\right) c_3.$$

Note that c_1, c_2 *and* c_3 *are constants of integration.*

3.8 Solving for the original dependent variables $\mathbf{u}_0$, once the generating invariants are known

In the one dimensional case the solutions $\mathbf{u}_0$ to the original Euler–Lagrange equations, can be obtained from the conservation laws once the invariant Euler–Lagrange equations have been solved for the generating invariants κ^α. The starting-point is that $\boldsymbol{\kappa}$ is a known function of n and some arbitrary constants, which are determined if initial data are specified. There are three methods, depending on what it is known. The running example is used in order to illustrate each method. Some applications will be shown in the next chapter.

3.8.1 How to solve for $\mathbf{u}_0$ from the invariants, knowing only the Maurer–Cartan matrix.

This method can be used for any invariant difference system. Indeed, when the Adjoint representation of the Lie group is trivial, it is the only available method.

Assume that the Maurer–Cartan matrix $K_0 = \rho_1\rho_0^{-1}$ is known in terms of the generating invariants, so that it can be written in terms of n and some arbitary constants. This gives the system of recurrence relations for ρ_0

$$\rho_1 = K_0\rho_0. \tag{3.8.1}$$

Definition 3.8.2. *The system* (3.8.1) *is known as the set of* Maurer–Cartan equations *for the frame* ρ.

Once the Maurer–Cartan equations for ρ_0 have been solved, $\mathbf{u}_0$ can be obtained from

$$u_0^\alpha = \rho_0^{-1}\left(\rho_0 \cdot u_0^\alpha\right) = \rho_0^{-1} I_{0,0}^\alpha \tag{3.8.3}$$

where the invariant $I_{0,0}^\alpha$ is known, either from the normalization equations or from the set of generating invariants already determined.

Example 3.8.4. *From equation* (3.4.25)*, the Maurer–Cartan matrix is*

$$K_0 = \begin{pmatrix} (\kappa-1)^{-3} & 0 & -\eta(\kappa-1)^{-3} \\ 0 & (\kappa-1)^{-1} & -(\kappa-1)^{-1} \\ 0 & 0 & 1 \end{pmatrix}.$$

Setting λ_k, a_k and b_k to be the parameter values for the group element ρ_k, the set of Maurer–Cartan equations is

$$\begin{pmatrix} \lambda_1^3 & 0 & a_1 \\ 0 & \lambda_1 & b_1 \\ 0 & 0 & 1 \end{pmatrix} = K_0 \begin{pmatrix} \lambda_0^3 & 0 & a_0 \\ 0 & \lambda_0 & b_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This gives three recurrence relations for the group parameters:

$$\begin{aligned} \lambda_1 &= (\kappa-1)^{-1}\lambda_0, \\ a_1 &= (\kappa-1)^{-3}\left(a_0 - \eta\right), \\ b_1 &= (\kappa-1)^{-1}\left(b_0 - 1\right). \end{aligned} \tag{3.8.5}$$

Now suppose that the general solution of these recurrence relations is known. It follows that

$$\begin{aligned} x_0 &= \rho_0^{-1}\cdot(\rho_0\cdot x_0) &= \lambda_0^{-3}(\rho_0\cdot x_0 - a_0) &= -\lambda_0^{-3}a_0,\\ u_0 &= \rho_0^{-1}\cdot(\rho_0\cdot u_0) &= \lambda_0^{-1}(\rho_0\cdot u_0 - b_0) &= -\lambda_0^{-1}b_0 \end{aligned}$$

where the normalization equations $\rho_0\cdot x_0 = 0$ and $\rho_0\cdot u_0 = 0$ have been used.

3.8.2 Solving for $\mathbf{u}_0$ from the invariants and conservation laws when the Adjoint representation is nontrivial

This method will be used when the Adjoint representation is not the identity representation. Recall

$$\mathbf{V}(I)\mathcal{A}d(\rho_0) = \mathbf{c} \tag{3.8.6}$$

where c is a constant row vector. The components of $\mathbf{V}(I)$ depend only on $\boldsymbol{\kappa}$, and they are therefore known functions of n. As $\mathcal{A}d(g)$ is known in terms of the group parameters, equation (3.8.6) yields equations for these parameters.

If the Adjoint action of the group on its Lie algebra is not transitive, the algebraic system of equations for the parameters may be under-determined. To complete the solution, it is then necessary to add the Maurer-Cartan equations (3.8.1) to this system. Even so, the algebraic equations coming from the conservation laws can ease considerably the problem of solving the Maurer–Cartan equations alone. Once ρ_0 is known as a function of n, Equation (3.8.3) yields $\mathbf{u}_0$, as before.

Example 3.8.7. *For the running example,* (3.8.6) *is*

$$(V_1\ V_2\ V_3)\begin{pmatrix} 1 & 0 & 0\\ -3a_0 & \lambda_0^3 & 0\\ -b_0 & 0 & \lambda_0 \end{pmatrix} = (c_1\ c_2\ c_3).$$

From the third column, $\lambda_0 = c_3/V_3$. Therefore, a first integral of the Euler–Lagrange equations is

$$\frac{V_2}{(V_3)^3} = \frac{c_2}{(c_3)^3}\,. \tag{3.8.8}$$

The remaining equation is a linear expression for a_0 and b_0,

$$3a_0V_2 + b_0V_3 - V_1 + c_1 = 0. \tag{3.8.9}$$

If one of the second and third equations of (3.8.5) *can be solved,* (3.8.9) *yields the remaining parameter.*

3.8.3 Solving for $\mathbf{u}_0$ from $\boldsymbol{\kappa}$ from the conservation laws, and with a nontrivial Adjoint representation of ρ which is known as a function of $\mathbf{u}_0$

Consider the conservation laws $\mathbf{V}(I)\mathcal{A}d(\rho_0) = \mathbf{c}$ and suppose that $\rho_0(\mathbf{u})$ is known as a function of the dependent variables. Sometimes deriving explicit equations for $\mathbf{u}$ which are simple to solve is possible.

Example 3.8.10. *The conservation laws amount to*

$$(V_1\ V_2\ V_3)\begin{pmatrix} 1 & 0 & 0 \\ \dfrac{3x_0}{(u_1-u_0)^3} & \dfrac{1}{(u_1-u_0)^3} & 0 \\ \dfrac{u_0}{u_1-u_0} & 0 & \dfrac{1}{u_1-u_0} \end{pmatrix} = (c_1\ c_2\ c_3). \tag{3.8.11}$$

The first integral (3.8.8) *is obtained once more and the simple recurrence relation from the third column*

$$u_1 - u_0 = V_3/c_3. \tag{3.8.12}$$

Solving for u_0, one can obtain x_0 from the first column of (3.8.11).

For the Lagrangian (3.3.13)*, each V_r is given* (3.7.18) *in terms of n and k_i, $i = 1,2,3$. The first integral* (3.8.8) *yields $c_3 = -3k_2 c_2^{1/3}$. Assuming that k_2 is nonzero and defining $k_4 = c_2^{-1/3}$, the general solution of* (3.8.12) *is*

$$u_0 = \tfrac{1}{4}k_4\left[2(k_1+k_1^{-1})n + (k_1-k_1^{-1})(-1)^n + k_5\right],$$

where k_5 is an arbitrary constant. Finally, the first column of (3.8.11) *gives*

$$x_0 = k_4^3\left[k_2 n k_1^{(-1)^n} - \tfrac{1}{2}k_3(-1)^n + k_6\right],$$

where $k_6 = c_1/3 + k_2(k_1+k_1^{-1}+k_5)/4$ is the remaining arbitrary constant.

Applications for Finite Difference Noether's Conservation Laws

In this chapter, we present applications for difference moving frames and finite difference Noether's conservation laws for some particular Lie groups. We first show another use of difference moving frames: to create symmetry-preserving numerical approximations. §4.1 illustrates this for the Euler elastica, which is invariant under the Euclidean group action in $\mathbb{R}^2$. We extend the calculations appearing in Mansfield, R–E, Hydon and Peng, [74]. For this example, we demonstrate how to obtain discrete invariants that have the correct continuum limit to their smooth counterparts. The specific difference Lagrangian we consider is the discrete analogue of that for Euler's elastica, and we show how our results compare with that of the smooth case. We also show that the discrete Euler–Lagrange system is a variational integrator that has the analogues of all three conservation laws. In §4.2, we consider a complex Lie group, specifically the special unitary group in $\mathbb{C}^2$. We obtain a difference moving frame in two ways and perform invariant Calculus of Variations using the latest one. Further, we obtain the conservation laws and obtain the moving frame for the conjugate action. In §4.3, we consider three different semisimple Lie group actions and we extend the computations in Mansfield and R–E, [75]. We show how to solve the integration problem taking advantage of the properties of these groups after obtaining the Euler–Lagrange equations and the conservation laws for Lagrangians that are invariant under these Lie group actions.

4.1 Study of the discrete Euler's elastica

Let us consider the smooth variational problem commonly known as Euler's elastica, namely

$$\mathcal{L} = \int \kappa^2 \mathrm{d}s, \qquad \kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \mathrm{d}s = \sqrt{1+u_x^2}\,\mathrm{d}x, \tag{4.1.1}$$

where κ is the Euclidean curvature and s is the Euclidean arc length. This problem was studied by Euler in 1744 [20] in which he obtains the Euler–Lagrange equation

$$\kappa_{ss} + \frac{1}{2}\kappa^3 = 0,$$

and a first integral. For a mathematical history of the problem see Leiven, [66].

In this section, we study a discrete variational problem analogous to this one.

The aim is to design the discrete Lagrangian such that the discrete Euler–Lagrange equations and the discrete conservation laws become the smooth Euler–Lagrange equations and conservation laws when taking an appropriate continuum limit. This allows us to construct a variational integrator whose discrete conservation laws approximate the smooth ones.

In the smooth cases, the conservation of energy is achieved when a Lagrangian is invariant under translations in the independent variable. In order to obtain the difference analogue, the independent variable needs to appear as a discrete dependent variable and the difference Lagrangian needs to be invariant under translation in this dependent variable. In this way, the conservation of energy in the smooth case becomes a conservation of a linear momentum in the difference analogue.

Note: That our method works in general is an open conjecture. In order to evidence this conjecture, we calculate all the relevant quantities in detail.

Review of the smooth Euler's elastica

This example was studied by Gonçalves and Mansfield in [34]. The Euclidean group of rotations and translations in the plane acts on curves $(x, u(x))$ as

$$\begin{pmatrix} x \\ u \end{pmatrix} \mapsto R_\theta \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}, \qquad R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.1.2}$$

For the normalization equations

$$\tilde{x} = 0, \qquad \tilde{u} = 0, \qquad \widetilde{u_x} = 0, \tag{4.1.3}$$

solving by the group parameters, we obtain the smooth moving frame

$$\widehat{\rho} = \begin{pmatrix} R_\theta & -R_\theta \begin{pmatrix} x \\ u \end{pmatrix} \\ 0 & 1 \end{pmatrix}, \tag{4.1.4}$$

where R_θ is the 2 by 2 rotation matrix with $\sin\theta = -u_x/\sqrt{1+u_x^2}$ and $\cos\theta = 1/\sqrt{1+u_x^2}$. One can compute the curvature matrix with respect to s which has this form

$$\widehat{\rho}_s\widehat{\rho}^{-1} = \frac{1}{\sqrt{1+u_x^2}}\widehat{\rho}_x\widehat{\rho}^{-1} = \begin{pmatrix} 0 & \kappa & -1 \\ -\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1.5}$$

It was shown in Gonçalves and Mansfield, [34], and Mansfield [70] that the conservation laws for the Lagrangian (4.1.1) are, in terms of the moving frame $\widehat{\rho}$ derivatives with respect to the arc length s, of the form

$$(-\kappa^2 \ \ -2\kappa_s \ \ 2\kappa)\underbrace{\begin{pmatrix} x_s & u_s & xu_s - ux_s \\ -u_s & x_s & xx_s + uu_s \\ 0 & 0 & 1 \end{pmatrix}}_{\mathcal{A}d(\widehat{\rho})} = (c_1 \ c_2 \ c_3). \tag{4.1.6}$$

Remark 4.1.7. *Note that in* (4.1.6) *we have used the convention* (3.7.15) *as appearing in [74].*

Using the identity $x_s^2 + u_s^2 = 1$ in (4.1.6) we obtain

$$(-\kappa^2 \ \ -2\kappa_s \ \ 2\kappa) = (c_1 \ \ c_2 \ \ c_3)\begin{pmatrix} x_s & -u_s & u \\ u_s & x_s & -x \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1.8}$$

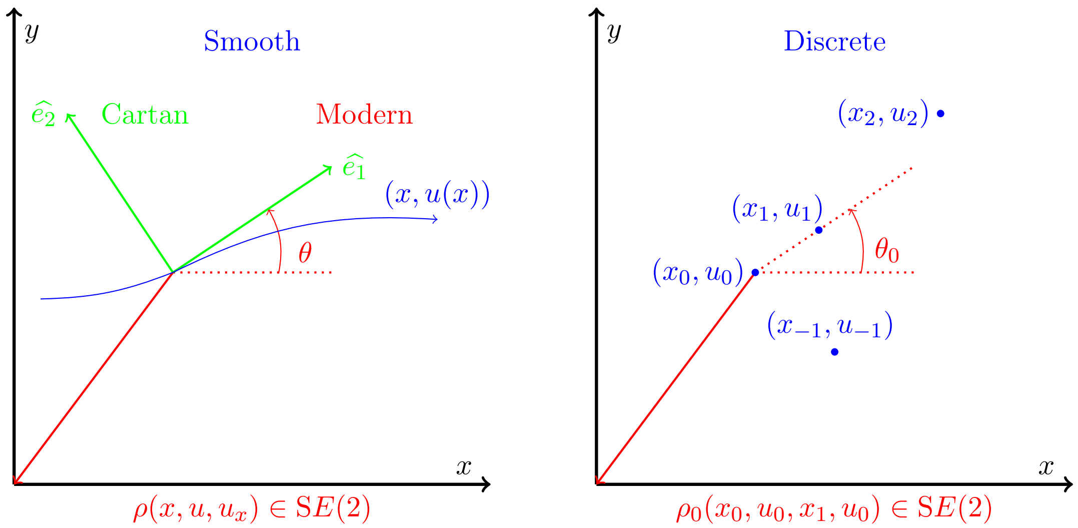
Using the same identity, from the first of second column, we have that

$$\kappa^4 + 4\kappa_s^2 = c_1^2 + c_2^2 \tag{4.1.9}$$

which gives a first integral for the Euler–Lagrange equation. Eliminating x_s from the first two columns of (4.1.8) we obtain

$$u_s = \frac{1}{c_1^2 + c_2^2}\left(2c_1\kappa_s - c_2\kappa^2\right). \tag{4.1.10}$$

Figure 4.1: Smooth moving frame and discrete moving frame for $SE(2)$



By solving (4.1.9), (4.1.10) and the third column of (4.1.8) in order to determine x, we obtain the smooth solution in Figure (4.2) once the constants of integration c_1 and c_2 are determined.

Discrete Euler's elastica

We want to take a difference frame with matching normalization equations and to take the discrete analogues of the curvature and the arc length. First, we consider the action of $SE(2)$ in the plane where the points $\mathbf{u}_j$ have coordinates (x_j, u_j)

$$\begin{pmatrix} x_j \\ u_j \end{pmatrix} \mapsto R_\theta \begin{pmatrix} x_j \\ u_j \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \widetilde{x_j} \\ \widetilde{u_j} \end{pmatrix}. \tag{4.1.11}$$

We take the analogous normalization equations to (4.1.3) to be

$$\rho_0 \cdot x_0 = 0, \qquad \rho_0 \cdot u_0 = 0, \qquad \rho_0 \cdot u_1 = 0.$$

Solving for the parameters of the Lie group, we obtain the moving frame

$$\rho_0 = \begin{pmatrix} R_{\theta_0} & -R_{\theta_0}\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \\ 0 & 1 \end{pmatrix},$$

using the standard representation (2.2) for $n = 3$. Note that R_{θ_0} is the 2 by 2 rotation matrix that sends $\mathbf{u}_1 - \mathbf{u}_0$ to a row vector with a zero second component, so that $\sin\theta_0 = -(u_1 - u_0)/\ell$ and $\cos\theta_0 = (x_1 - x_0)/\ell$, where $\ell = |\mathbf{u}_1 - \mathbf{u}_0|$.

The Maurer-Cartan martix is of the form

$$K_0 = \rho_1\rho_0^{-1} = \begin{pmatrix} R_{h_\theta} & -R_{\theta_1}\begin{pmatrix} x_1 - x_0 \\ u_1 - u_0 \end{pmatrix} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_{h_\theta} & -R_{h_\theta}\begin{pmatrix} \ell \\ 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix},$$

where $h_\theta = \theta_1 - \theta_0$. Hence the generating invariants are h_θ and ℓ.

In order to obtain the discrete analogues of curvature and arc length, we approximate $\widehat{\rho}_x\widehat{\rho}^{-1}$ by $(\widehat{\rho}(x+h_x) - \widehat{\rho}(x))\,\widehat{\rho}^{-1}/h_x = \left(\widehat{\rho}(x+h_x)\widehat{\rho}^{-1} - \mathrm{Id}_3\right)/h_x$, where Id_3 is the 3 by 3 identity matrix, and $\widehat{\rho}(x+h_x)\widehat{\rho}^{-1}$ to be approximated by K_0 when $x = x_0$ and $h_x = x_1 - x_0$.

One can observe that the component of the first row and second column of the matrix $K_0 - \mathrm{Id}$ is $-\sin h_\theta$ and that, to first order in h_θ, the component of the first row and third column of the matrix $K_0 - \mathrm{Id}$ is $-\ell$. Therefore, we can take the discrete analogue of $\mathrm{d}s$ to be ℓ and the discrete analogue of κ to be

$$\overline{\kappa} = \ell^{-1}\sin h_\theta.$$

Hence, we consider the variational problem

$$\mathcal{L} = \sum \ell^{-1}\sin^2 h_\theta,$$

which is the discrete analogue to (4.1.1).

It is possible to compute the evolution of the curvature invariants h_θ and ℓ without computing the curvature matrices and the differential–difference syzygy as mentioned in (3.4.2). One can differentiate the expression in the original variables of the invariants and then use the Replacement Rule, 2.4.6. This is done as follows:

First of all, the evolution of ℓ is computed by taking the derivative of $|\mathbf{u}_1 - \mathbf{u}_0|$ with respect to t, i,e

$$\begin{aligned}\frac{\mathrm{d}\ell}{\mathrm{d}t} &= \frac{\mathrm{d}|\mathbf{u}_1 - \mathbf{u}_0|}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{(x_1 - x_0)^2 + (u_1 - u_0)^2} \\ &= \frac{(x_1 - x_0)(x_1' - x_0') + (u_1 - u_0)(u_1' - u_0')}{|\mathbf{u}_1 - \mathbf{u}_0|} \\ &= I^x_{0,1;t} - I^x_{0,0;t}.\end{aligned}$$

In order to compute the evolution of $h_\theta = \theta_1 - \theta_0$ we first compute the evolution of

θ_0 and then we apply the forward difference operator $(\mathrm{S}-\mathrm{id})$ to the obtained result as $\theta_1-\theta_0=(\mathrm{S}-\mathrm{id})\theta_0$. We make use of the expression of $\cos\theta_0$ and $\sin\theta_0$ in terms of the original variables.

We have, on one hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos\theta_0=-\sin\theta_0\frac{\mathrm{d}\theta_0}{\mathrm{d}t}=\frac{u_1-u_0}{\ell}\frac{\mathrm{d}\theta_0}{\mathrm{d}t}$$

and on the other hand

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\cos\theta_0&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{x_1-x_0}{\ell}\\&=\frac{x_1'-x_0'}{\ell}-\frac{x_1-x_0}{\ell^2}\frac{\mathrm{d}\ell}{\mathrm{d}t}\\&=\frac{x_1'-x_0'}{\ell}-\frac{x_1-x_0}{\ell^2}\frac{(x_1-x_0)(x_1'-x_0')+(u_1-u_0)(u_1'-u_0')}{\ell}\\&=\frac{x_1'-x_0'}{\ell}-\frac{1}{\ell^3}[(x_1-x_0)^2(x_1'-x_0')+(x_1-x_0)(u_1-u_0)(u_1'-u_0')]\\&=(x_1'-x_0')\left[\frac{(x_1-x_0)^2+(u_1-u_0)^2}{\ell^3}-\frac{(x_1-x_0)^2}{\ell^3}\right]\\&\quad-\frac{1}{\ell^3}[(x_1-x_0)(u_1-u_0)(u_1'-u_0')]\\&=\frac{(x_1'-x_0')(u_1-u_0)^2-(u_1'-u_0')(x_1-x_0)(u_1-u_0)}{\ell^3}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}\theta_0}{\mathrm{d}t}&=\frac{(x_1'-x_0')(u_1-u_0)-(u_1'-u_0')(x_1-x_0)}{\ell^2}\\&=\frac{I^u_{0,0;t}-I^u_{0,1;t}}{\ell}.\end{aligned}$$

Hence

$$\frac{\mathrm{d}h_\theta}{\mathrm{d}t}=(\mathrm{S}-\mathrm{id})\left(\frac{I^u_{0,0;t}-I^u_{0,1;t}}{\ell}\right).$$

The next step is to write $I_{0,1;t}$ in terms of $I_{0,0;t}$. We have that

$$\begin{pmatrix}I^x_{0,1;t}\\I^u_{0,1;t}\end{pmatrix}=R_{\theta_0}\begin{pmatrix}x_1'\\u_1'\end{pmatrix}=R_{\theta_0}R^{-1}_{\theta_1}R_{\theta_1}\begin{pmatrix}x_1'\\u_1'\end{pmatrix}=R_{-\Delta\theta_0}\mathrm{S}\begin{pmatrix}I^x_{0,0;t}\\I^u_{0,0;t}\end{pmatrix}.$$

Therefore, the differential–difference syzygies are

$$\begin{aligned}\ell'&=\cos h_\theta\,\mathrm{S}\sigma^x+\sin h_\theta\,\mathrm{S}\sigma^u-\sigma^x,\\h_\theta'&=(\mathrm{S}-\mathrm{id})\left(\ell^{-1}[\sin h_\theta\,\mathrm{S}\sigma^x-\cos h_\theta\,\mathrm{S}\sigma^u+\sigma^u]\right)\end{aligned}$$

where we have set $\sigma^x = I^x_{0,0;t}$ and $\sigma^u = I^u_{0,0;t}$. These syzygies can be written in canonical form as follows:

$$\begin{pmatrix} \ell' \\ h'_\theta \end{pmatrix} = \mathcal{H} \begin{pmatrix} \sigma^x \\ \sigma^u \end{pmatrix} \tag{4.1.12}$$

where

$$\mathcal{H} = \begin{pmatrix} \cos h_\theta \mathrm{S} - \mathrm{id} & \sin h_\theta \mathrm{S} \\ (\mathrm{S}-\mathrm{id})(\ell^{-1}\sin h_\theta \mathrm{S}) & -(\mathrm{S}-\mathrm{id})(\cos h_\theta \mathrm{S} + \mathrm{id}) \end{pmatrix}.$$

Using (3.5.11), the invariantized Euler–Lagrange equations are

$$\begin{aligned}
&\{\mathrm{S}_{-1}(\cos h_\theta)\,\mathrm{S}_{-1} - \mathrm{id}\}\mathrm{E}_\ell(L) + \left\{\mathrm{S}_{-1}\left(\ell^{-1}\sin h_\theta\right)(\mathrm{S}_{-2}-\mathrm{S}_{-1})\right\}\mathrm{E}_{h_\theta}(L) = 0,\\
&\{\mathrm{S}_{-1}(\sin h_\theta)\,\mathrm{S}_{-1}\}\mathrm{E}_\ell(L) + \left\{\ell^{-1}(\mathrm{S}_{-1}-\mathrm{id}) - \mathrm{S}_{-1}\left(\ell^{-1}\cos h_\theta\right)(\mathrm{S}_{-2}-\mathrm{S}_{-1})\right\}\mathrm{E}_{h_\theta}(L) = 0
\end{aligned}$$

where

$$\mathrm{E}_{h_\theta}(L) = \frac{\partial L}{\partial h_\theta} = \ell^{-1}\sin(2h_\theta), \qquad \mathrm{E}_\ell(L) = \frac{\partial L}{\partial \ell} = -\ell^{-2}\sin^2 h_\theta.$$

These equations are then solved for ℓ and h_θ. Using (3.5.4) the boundary terms can be written in the form

$$A_{\mathcal{H}} = \mathcal{C}^0_x\, I^x_{0,0;t} + \mathcal{C}^0_u\, I^u_{0,0;t} + \mathcal{C}^1_x\,\mathrm{S}(I^x_{0,0;t}) + \mathcal{C}^1_u\,\mathrm{S}(I^u_{0,0;t}),$$

where

$$\begin{aligned}
\mathcal{C}^0_x &= \mathrm{S}_{-1}\left\{\cos h_\theta\,\mathrm{E}_\ell(L) - \ell^{-1}\sin h_\theta \mathrm{E}_{h_\theta}(L) + \ell^{-1}\sin h_\theta\,\mathrm{S}_{-1}\left(\mathrm{E}_{h_\theta}(L)\right)\right\},\\
\mathcal{C}^0_u &= \mathrm{S}_{-1}\left\{\sin h_\theta \mathrm{E}_\ell(L) + \left(\mathrm{S}\left(\ell^{-1}\right) + \ell^{-1}\cos h_\theta - \ell^{-1}\cos h_\theta\,\mathrm{S}_{-1}\right)\mathrm{E}_{h_\theta}(L)\right\},\\
\mathcal{C}^1_x &= \ell^{-1}\sin h_\theta\,\mathrm{S}_{-1}\left\{\mathrm{E}_{h_\theta}(L)\right\},\\
\mathcal{C}^1_u &= -\ell^{-1}\cos h_\theta\,\mathrm{S}_{-1}\left\{\mathrm{E}_{h_\theta}(L)\right\}.
\end{aligned}$$

In order to obtain the conservation laws, we first need to compute the vector fields, the matrix of infinitesimals and the Adjoint action. Recall the action (4.1.11). We have that

$$\widetilde{x_0} = x_0\cos\theta - u_0\sin\theta + a, \qquad \widetilde{u_0} = x_0\sin\theta + u_0\cos\theta + b,$$

$$\widetilde{x_1} = x_1\cos\theta - u_1\sin\theta + a, \qquad \widetilde{u_1} = x_1\sin\theta + u_1\cos\theta + b.$$

Therefore the table of infinitesimals is given by

	x_0	u_0	x_1	u_1
a	1	0	1	0
b	0	1	0	1
θ	u_0	$-x_0$	u_1	$-x_1$

.

Hence the infinitesimal vector fields are

$$\mathbf{v}_\theta = -u_0\partial_{x_0} + x_0\partial_{u_0}, \qquad \mathbf{v}_a = \partial_{x_0}, \qquad \mathbf{v}_b = \partial_{u_0}$$

and the matrix of infinitesimals and its invariantized form restricted to the variables x_0 and u_0 are

$$\Phi(\mathbf{u}_0) = \begin{pmatrix} 1 & 0 & u_0 \\ 0 & 1 & -x_0 \end{pmatrix}, \qquad \Phi(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now we are going to calculate the Adjoint matrix. We have that

$$\frac{\partial}{\partial x_0} = \frac{\partial \widetilde{x_0}}{\partial x_0}\frac{\partial}{\partial \widetilde{x_0}} + \frac{\partial \widetilde{u_0}}{\partial x_0}\frac{\partial}{\partial \widetilde{u_0}}, \qquad \frac{\partial}{\partial u_0} = \frac{\partial \widetilde{x_0}}{\partial u_0}\frac{\partial}{\partial \widetilde{x_0}} + \frac{\partial \widetilde{u_0}}{\partial u_0}\frac{\partial}{\partial \widetilde{u_0}},$$

so

$$\nabla = \begin{pmatrix} \dfrac{\partial \widetilde{x_0}}{\partial x_0} & \dfrac{\partial \widetilde{u_0}}{\partial x_0} \\ \dfrac{\partial \widetilde{x_0}}{\partial u_0} & \dfrac{\partial \widetilde{u_0}}{\partial u_0} \end{pmatrix} \widetilde{\nabla},$$

and then

$$\widetilde{\nabla} = \frac{1}{\dfrac{\partial \widetilde{x_0}}{\partial x_0}\dfrac{\partial \widetilde{u_0}}{\partial u_0} - \dfrac{\partial \widetilde{u_0}}{\partial x_0}\dfrac{\partial \widetilde{x_0}}{\partial u_0}} \begin{pmatrix} \dfrac{\partial \widetilde{u_0}}{\partial u_0} & -\dfrac{\partial \widetilde{u_0}}{\partial x_0} \\ -\dfrac{\partial \widetilde{x_0}}{\partial u_0} & \dfrac{\partial \widetilde{x_0}}{\partial x_0} \end{pmatrix} \nabla,$$

so in our case we have that

$$\widetilde{\nabla} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \nabla.$$

Now we can compute $\widetilde{u_0}\partial_{\widetilde{x_0}} - \widetilde{x_0}\partial_{\widetilde{u_0}}$. We have that

$$\begin{aligned} \widetilde{u_0}\partial_{\widetilde{x_0}} - \widetilde{x_0}\partial_{\widetilde{u_0}} &= (x_0\sin\theta + u_0\cos\theta + b)\cdot(\cos\theta\partial_{x_0} - \sin\theta\partial_{u_0}) - \\ &- (x_0\cos\theta - u_0\sin\theta + a)\cdot(\sin\theta\partial_{x_0} + \cos\theta\partial_{u_0}) = \\ &= u_0\partial_{x_0} - x_0\partial_{u_0} + (b\cos\theta - a\sin\theta)\partial_{x_0} + (-a\cos\theta - b\sin\theta)\partial_{u_0}. \end{aligned}$$

Therefore

$$\begin{pmatrix} \widetilde{\mathbf{v}_\theta} & \widetilde{\mathbf{v}_a} & \widetilde{\mathbf{v}_b} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_\theta & \mathbf{v}_a & \mathbf{v}_b \end{pmatrix} \mathcal{A}d(g(\theta,a,b))^{-1}$$

where

$$\mathcal{A}d\big(g(\theta,a,b)\big) = \begin{pmatrix} \cos\theta & -\sin\theta & b \\ \sin\theta & \cos\theta & -a \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying the replacement (3.7.6), simplifying and collecting terms, the conservation laws

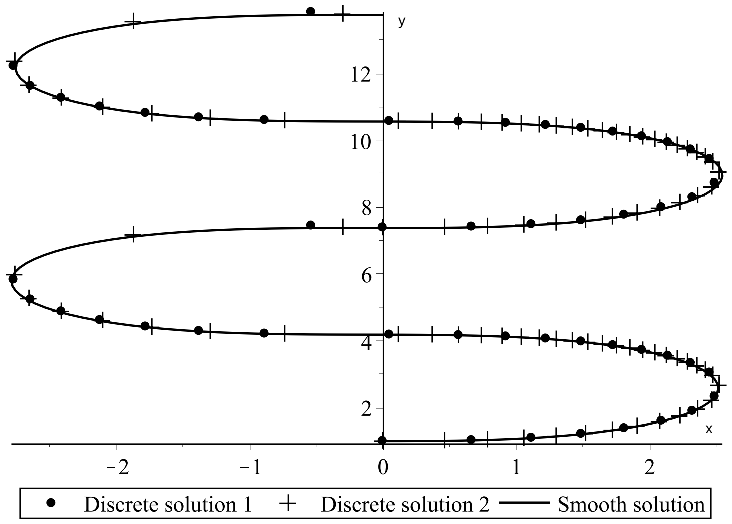


Figure 4.2: A plot of an extract of 847 points of the discrete solution for certain initial data and an extract of 507 points of the discrete solution for a variation of the previous initial data. This is compared with an accurate numerical solution of the third column of (4.1.8), (4.1.9) and (4.1.10), and using a Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant, with uniform step 0.1. The conservation laws are used in the solution in order to match the initial data.

can be written in terms of the row vector of invariants as follows

$$(V_1\ V_2\ V_3)\begin{pmatrix} R_{\theta_0} & JR_{\theta_0}\mathbf{u}_0 \\ 0 & 1 \end{pmatrix} = (c_1\ c_2\ c_3), \qquad \text{where} \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.1.13}$$

and where

$$\begin{aligned} V_1 &= \mathrm{S}_{-1}(\cos h_\theta \mathrm{E}_\ell(L)) + \left\{\mathrm{S}_{-1}\left(\ell^{-1}\sin h_\theta\right)(\mathrm{S}_{-2} - \mathrm{S}_{-1})\right\}\mathrm{E}_{h_\theta}(L), \\ V_2 &= \mathrm{S}_{-1}(\sin h_\theta \mathrm{E}_\ell(L)) + \left\{\ell^{-1}(\mathrm{S}_{-1} - \mathrm{id}) - \mathrm{S}_{-1}\left(\ell^{-1}\cos \mathrm{h}_\theta\right)(\mathrm{S}_{-2} - \mathrm{S}_{-1})\right\}\mathrm{E}_{h_\theta}(L), \\ V_3 &= -\mathrm{S}_{-1}\left(\mathrm{E}_{h_\theta}(L)\right). \end{aligned}$$

Using Maple , we solve the discrete Euler–Lagrange equations for the invariants as an initial data problem. Note that from (4.1.13) we have that

$$(V_1\ V_2)R_{\theta_0} = (c_1\ c_2). \tag{4.1.14}$$

Taking transposes

$$R_{\theta_0}^T(V_1\ V_2)^T = (c_1\ c_2)^T.$$

Therefore, multiplying by $R_{\theta_0}^T(V_1\ V_2)^T$ on the left hand side of (4.1.14) and $(c_1\ c_2)^T$ on the right hand side of (4.1.14), and taking into account that R_{θ_0} is in $SO(2)$ so $R_{\theta_0}^T = R_{\theta_0}^{-1}$ we

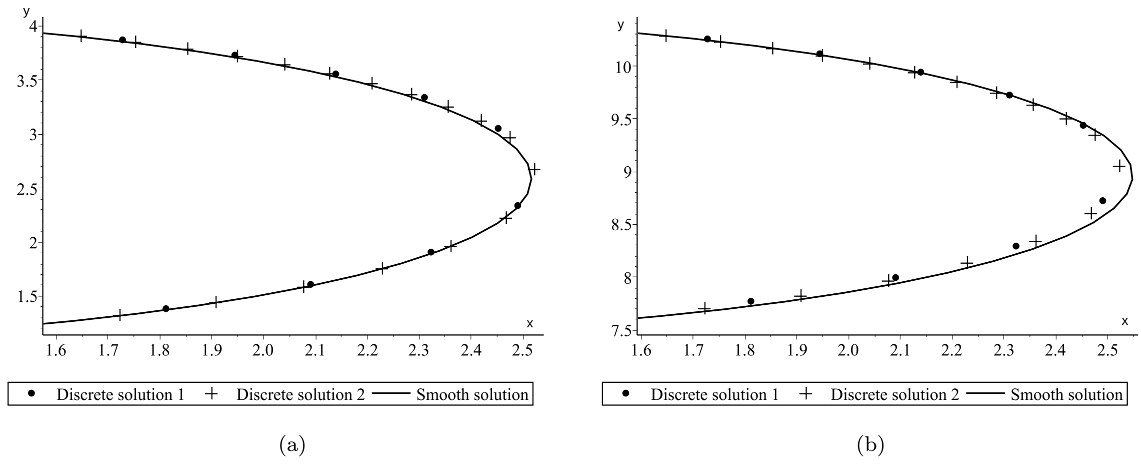


Figure 4.3: Plots (a) and (b) magnify two regions of Figure 2.

have that

$$(V_1)^2 + (V_2)^2 = (c_1)^2 + (c_2)^2\,,$$

which gives a first integral of the discrete Euler–Lagrange equations.

It is possible to obtain the solution in terms of the original variables by using the methods of §(3.8.2). The initial data give the values of the constants c_1, c_2 and c_3. We have used these constants and the initial values $(x_0, u_0) = (0, 1)$ to obtain the initial data for the smooth solution. The discrete equations require one more initial datum than the smooth equations, so that more than one discrete solution will have the same constants and starting point, and hence more than one discrete solution can approximate a given smooth solution. In Figure (4.2), we compare two discrete solutions with different initial step sizes, both approximating the single smooth solution.

More sophisticated methods to derive discrete Lagrangians using interpolation are also being explored in Beffa and Mansfield, [5].

Even though this numerical method is not very efficient, it shows that one can get the discrete conservation laws as close as desired to the smooth ones.

4.2 Study of SU(2)

Consider the special unitary group

$$\mathrm{S}U(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

and the linear action on (z_0, z_1) given by

$$\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} \widetilde{z_0} \\ \widetilde{z_1} \end{pmatrix} \tag{4.2.1}$$

that can be easily extended to (z_j, z_{j+1}) for $j \in \mathbb{N}$. We take the normalization equations to be

$$\Im(\widetilde{z_0}) = \Re(\widetilde{z_1}) = \Im(\widetilde{z_1}) = 0 \tag{4.2.2}$$

which yields the following moving frame

$$\rho_0 = \frac{1}{\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} \overline{z_0} & \overline{z_1} \\ -z_1 & z_0 \end{pmatrix}.$$

The invariants are of the form

$$\begin{pmatrix} I_{0,j} \\ I_{0,j+1} \end{pmatrix} = \rho_0 \cdot \begin{pmatrix} z_j \\ z_{j+1} \end{pmatrix} = \frac{1}{\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} \overline{z_0} z_j + \overline{z_1} z_{j+1} \\ \overline{z_0} z_{j+1} - \overline{z_1} z_j \end{pmatrix}$$

and the first order differential invariants are of the form

$$\begin{pmatrix} I_{0,j;t} \\ I_{0,j+1;t} \end{pmatrix} = \rho_0 \cdot \begin{pmatrix} z_j' \\ z_{j+1}' \end{pmatrix} = \frac{1}{\sqrt{|z_0|^2 + |z_1|^2}} \begin{pmatrix} \overline{z_0} z_j' + \overline{z_1} z_{j+1}' \\ \overline{z_0} z_{j+1}' - \overline{z_1} z_j' \end{pmatrix}$$

where we assume that $z_j = z_j(t)$ for all j in $\mathbb{Z}$.

The Maurer–Cartan matrix is

$$K_0 = \iota_0(\rho_1) = \frac{1}{|I_{0,2}|} \begin{pmatrix} 0 & \overline{I_{0,2}} \\ -I_{0,2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\kappa} \\ -\kappa & 0 \end{pmatrix} \in \mathrm{S}U(2)$$

where we have set κ to be $\dfrac{I_{0,2}}{|I_{0,2}|}$, and the curvature matrix

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right) = \frac{1}{\eta}\begin{pmatrix} -\mathfrak{i}\Im(I_{0,0;t}) & \overline{I_{0,1;t}} \\ -I_{0,1;t} & \mathfrak{i}\Im(I_{0,0;t}) \end{pmatrix} \in \mathfrak{su}(2)$$

where we have set $I_{0,0}$ to be η and where we have denote the imaginary number $\sqrt{-1}$ by $\mathfrak{i}$. Therefore using (3.4.33) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa = \mathfrak{i}\frac{\kappa}{\eta\eta_1}(\eta\mathrm{S} + \eta_1)\Im(I_{0,0;t}).$$

Also note that

$$\eta = \sqrt{|z_0|^2 + |z_1|^2} = \sqrt{\Re(z_0)^2 + \Im(z_0)^2 + \Re(z_1)^2 + \Im(z_1)^2}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = \frac{\Re(z_0)\Re(z_0)' + \Im(z_0)\Im(z_0)' + \Re(z_1)\Re(z_1)' + \Im(z_0)\Im(z_0)'}{\sqrt{\Re(z_0)^2 + \Im(z_0)^2 + \Re(z_1)^2 + \Im(z_1)^2}}.$$

Using the Replacement Rule 2.4.6

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = \Re(I_{0,0;t})$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \eta \\ \kappa \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathfrak{i}\dfrac{\kappa}{\eta\eta_1}(\eta\mathrm{S} + \eta_1) \end{pmatrix}\begin{pmatrix} \Re(I_{0,0;t}) \\ \Im(I_{0,0;t}) \end{pmatrix}. \tag{4.2.3}$$

Note that a complex number z can be expressed in terms of its modulus r and its argument θ as follows

$$z = re^{\mathfrak{i}\theta} = r(\cos\theta + \mathfrak{i}\sin\theta)$$

where

$$\Re(z) = r\cos\theta \qquad \text{and} \qquad \Im(z) = r\sin\theta.$$

Therefore the normalization equations (4.2.2) yield

$$\widetilde{r_0}\sin\widetilde{\theta_0} = \widetilde{r_1}\cos\widetilde{\theta_1} = \widetilde{r_1}\sin\widetilde{\theta_1} = 0.$$

We can take the normalization equations to be

$$\widetilde{\theta_0} = \widetilde{r_1} = 0$$

which yields the following moving frame

$$\rho_0 = \frac{1}{\sqrt{r_0^2 + r_1^2}} \begin{pmatrix} r_0 e^{-\mathrm{i}\theta_0} & r_1 e^{-\mathrm{i}\theta_1} \\ -r_1 e^{\mathrm{i}\theta_1} & r_0 e^{\mathrm{i}\theta_0} \end{pmatrix}.$$

The invariants are of the form

$$\begin{pmatrix} I_{0,j} \\ I_{0,j+1} \end{pmatrix} = \rho_0 \cdot \begin{pmatrix} z_j \\ z_{j+1} \end{pmatrix} = \frac{1}{\sqrt{r_0^2 + r_1^2}} \begin{pmatrix} r_0 r_j e^{\mathrm{i}(\theta_j - \theta_0)} + r_1 r_{j+1} e^{\mathrm{i}(\theta_1 - \theta_{j+1})} \\ r_0 r_{j+1} e^{\mathrm{i}(\theta_{j+1} + \theta_0)} + r_1 r_j e^{\mathrm{i}(\theta_1 + \theta_{j+1})} \end{pmatrix}$$

and similarly, the first order differential invariants are of the form

$$\begin{pmatrix} I_{0,j;t} \\ I_{0,j+1;t} \end{pmatrix} = \rho_0 \cdot \begin{pmatrix} z_j'(t) \\ z_{j+1}'(t) \end{pmatrix} = \frac{1}{\sqrt{r_0^2 + r_1^2}} \begin{pmatrix} r_0 r_j' e^{\mathrm{i}(\theta_j' - \theta_0)} + r_1 r_{j+1}' e^{\mathrm{i}(\theta_1 - \theta_{j+1}')} \\ r_0 r_{j+1}' e^{\mathrm{i}(\theta_{j+1}' + \theta_0)} + r_1 r_j' e^{\mathrm{i}(\theta_1 + \theta_{j+1}')} \end{pmatrix}.$$

Note that

$$I_{0,j} = I^r_{0,j} e^{\mathrm{i} I^\theta_{0,j}} \quad \text{and} \quad I_{0,j;t} = I^r_{0,j;t} e^{\mathrm{i} I^\theta_{0,j;t}}$$

where we are denoting $I^r_{0,j}$ and $I^\theta_{0,j}$ the invariantized forms of r_j and θ_j respectively and $I^r_{0,j;t}$ and $I^\theta_{0,j;t}$ the invariantized forms of r_j' and θ_j' respectively. The Maurer–Cartan matrix is

$$K_0 = \iota_0(\rho_1) = \begin{pmatrix} 0 & e^{-\mathrm{i} I^\theta_{0,2}} \\ -e^{\mathrm{i} I^\theta_{0,2}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{\tau} \\ -\tau & 0 \end{pmatrix} \in \mathrm{SU}(2)$$

where we are setting $e^{\mathrm{i} I^\theta_{0,2}}$ to be τ. The curvature matrix has the form

$$N_0 = \iota_0 \left(\frac{\mathrm{d}}{\mathrm{d}t} \rho_0 \right) = \begin{pmatrix} -\mathrm{i} I^\theta_{0,0;t} & \dfrac{I^r_{0,1;t}}{I^r_{0,0}} \\ -\dfrac{I^r_{0,1;t}}{I^r_{0,0}} & \mathrm{i} I^\theta_{0,0;t} \end{pmatrix} \in \mathfrak{su}(2).$$

Therefore using (3.4.33) we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \kappa = i\tau(\mathrm{id} + \mathrm{S}) I^\theta_{0,0;t}.$$

Also note that

$$I^r_{0,0} = \sqrt{r_0^2 + r_1^2}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} I^r_{0,0}(t) = \frac{r_0 r_0' + r_1 r_1'}{\sqrt{r_0^2 + r_1^2}}.$$

Using the Replacement Rule 2.4.6

$$\frac{\mathrm{d}}{\mathrm{d}t} I^r_{0,0}(t) = I^r_{0,0;t}$$

and therefore denoting $I^r_{0,0}$ by η we have the differential–difference syzygy

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \eta \\ \tau \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i\tau(\mathrm{id}+\mathrm{S}) \end{pmatrix}\begin{pmatrix} I^r_{0,0;t} \\ I^\theta_{0,0;t} \end{pmatrix}$$

which has a simpler form than (4.2.3) which makes it more suitable for the Calculus of Variations.

Consider the Lagrangian

$$\mathcal{L}[\eta, \tau, \tau_1, ..., \tau_{J_1}] = \sum L(\tau, \tau_1, ..., \tau_{J_1}) + \lambda(\eta - 1).$$

Setting $I^r_{0,0;t}$ to be σ^r and $I^\theta_{0,0;t}$ to be σ^θ and applying Calculus of Variations we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}[\eta,\tau] &= \lambda\frac{\mathrm{d}\eta}{\mathrm{d}t} + \frac{\mathrm{d}L}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} + \frac{\mathrm{d}L}{\mathrm{d}\tau_1}\frac{\mathrm{d}\tau_1}{\mathrm{d}t} + ... + \frac{\mathrm{d}L}{\mathrm{d}\tau_{J_1}}\frac{\mathrm{d}\tau_{J_1}}{\mathrm{d}t} \\ &= \mathrm{E}_\tau(L)\frac{\mathrm{d}\tau}{\mathrm{d}t} + \lambda\frac{\mathrm{d}\eta}{\mathrm{d}t} \\ &= \mathrm{E}_\tau(L)\mathrm{i}\tau(\mathrm{id}+\mathrm{S})\sigma^\theta + \lambda\sigma^r \\ &= (\mathrm{i}\tau + \tau_{-1}\mathrm{S}_{-1})\,\mathrm{E}_\tau(L)\sigma^\theta + \lambda\sigma^r + (\mathrm{S}-\mathrm{id})\left(\tau_{-1}(\mathrm{S}_{-1}\mathrm{E}_\tau(L))\sigma^\theta\right).\end{aligned}$$

Therefore, the Euler–Lagrange equation is of the form

$$(\mathrm{i}\tau + \tau_{-1}\mathrm{S}_{-1})\,\mathrm{E}_\tau(L) = 0$$

and the boundary terms are of the form

$$(\mathrm{S}-\mathrm{id})\left(\tau_{-1}(\mathrm{S}_{-1}\mathrm{E}_\tau(L))\sigma^\theta\right).$$

Hence

$$\mathcal{C}^\theta_0 = \tau_{-1}(\mathrm{S}_{-1}\mathrm{E}_\tau(L)).$$

In order to compute the conservation laws, we first need to compute the matrix of infinitesimals and the adjoint representation.

Recall the action (4.2.1). We have that

$$\widetilde{z_0} = \alpha z_0 + \beta z_1, \qquad \widetilde{z_1} = -\bar{\beta} z_0 + \bar{\alpha} z_1. \tag{4.2.4}$$

The table of infinitesimals is

	z_0	z_1
α	z_0	$-z_1$
β	z_1	0
$\bar{\beta}$	0	$-z_0$

.

Hence, the infinitesimal vector fields for this action are

$$\mathbf{v}_\alpha = z_0\partial_{z_0} - z_1\partial_{z_1}, \quad \mathbf{v}_\beta = z_1\partial_{z_0}, \quad \mathbf{v}_{\bar{\beta}} = -z_0\partial_{z_1}.$$

Note that

$$\partial_{z_0} = \frac{\partial\widetilde{z_0}}{\partial z_0}\partial_{\widetilde{z_0}} + \frac{\partial\widetilde{z_1}}{\partial z_0}\partial_{\widetilde{z_1}}, \qquad \partial_{z_1} = \frac{\partial\widetilde{z_0}}{\partial z_1}\partial_{\widetilde{z_0}} + \frac{\partial\widetilde{z_1}}{\partial z_1}\partial_{\widetilde{z_1}},$$

so

$$\partial_{z_0} = \alpha\partial_{\widetilde{z_0}} - \bar{\beta}\partial_{\widetilde{z_1}}, \qquad \partial_{z_1} = \beta\partial_{\widetilde{z_0}} + \bar{\alpha}\partial_{\widetilde{z_1}}.$$

We also have from (4.2.4)

$$z_0 = \bar{\alpha}\widetilde{z_0} - \beta\widetilde{z_1}, \qquad z_1 = \bar{\beta}\widetilde{z_0} + \alpha\widetilde{z_1}.$$

Therefore

$$\begin{aligned}
\mathbf{v}_\alpha &= z_0\partial_{z_0} - z_1\partial_{z_1} \\
&= (\bar{\alpha}\widetilde{z_0} - \beta\widetilde{z_1})(\alpha\partial_{\widetilde{z_0}} - \bar{\beta}\partial_{\widetilde{z_1}}) - (\bar{\beta}\widetilde{z_0} + \alpha\widetilde{z_1})(\beta\partial_{\widetilde{z_0}} + \bar{\alpha}\partial_{\widetilde{z_1}}) \\
&= (\alpha\bar{\alpha} - \beta\bar{\beta})(\widetilde{z_0}\partial_{\widetilde{z_0}} - \widetilde{z_1}\partial_{\widetilde{z_1}}) - 2\bar{\alpha}\bar{\beta}\widetilde{z_0}\partial_{\widetilde{z_1}} - 2\alpha\beta\widetilde{z_1}\partial_{\widetilde{z_0}} \\
&= (\alpha\bar{\alpha} - \beta\bar{\beta})\widetilde{\mathbf{v}_\alpha} - 2\alpha\beta\widetilde{\mathbf{v}_\beta} + 2\bar{\alpha}\bar{\beta}\widetilde{\mathbf{v}_{\bar{\beta}}}, \\
\mathbf{v}_\beta &= z_1\partial_{z_0} \\
&= (\bar{\beta}\widetilde{z_0} + \alpha\widetilde{z_1})(\alpha\partial_{\widetilde{z_0}} - \bar{\beta}\partial_{\widetilde{z_1}}) \\
&= \alpha\bar{\beta}(\widetilde{z_0}\partial_{\widetilde{z_0}} - \widetilde{z_1}\partial_{\widetilde{z_1}}) - \bar{\beta}^2\widetilde{z_0}\partial_{\widetilde{z_1}} + \alpha^2\widetilde{z_1}\partial_{\widetilde{z_0}} \\
&= \alpha\bar{\beta}\widetilde{\mathbf{v}_\alpha} + \alpha^2\widetilde{\mathbf{v}_\beta} + \bar{\beta}^2\widetilde{\mathbf{v}_{\bar{\beta}}},
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_{\bar\beta} &= -z_0\partial_{z_1}\\
&= -(\bar\alpha\widetilde{z_0} - \beta\widetilde{z_1})(\beta\partial_{\widetilde{z_0}} + \bar\alpha\partial_{\widetilde{z_1}})\\
&= -\bar\alpha\beta(\widetilde{z_0}\partial_{\widetilde{z_0}} - \widetilde{z_1}\partial_{\widetilde{z_1}}) + \beta^2\widetilde{z_1}\partial_{\widetilde{z_0}} - \bar\alpha^2\widetilde{z_0}\partial_{\widetilde{z_1}}\\
&= -\bar\alpha\beta\widetilde{\mathbf{v}_\alpha} + \beta^2\widetilde{\mathbf{v}_\beta} + \bar\alpha^2\widetilde{\mathbf{v}_{\bar\beta}}.
\end{aligned}$$

We have that the induced action on these are

$$\begin{pmatrix} \widetilde{\mathbf{v}_\alpha} & \widetilde{\mathbf{v}_\beta} & \widetilde{\mathbf{v}_{\bar\beta}} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_\alpha & \mathbf{v}_\beta & \mathbf{v}_{\bar\beta} \end{pmatrix} \mathcal{A}d(g)^{-1}$$

where

$$\mathcal{A}d(g) = \begin{matrix} & \alpha & \beta & \bar\beta \\ \alpha & \alpha\bar\alpha - \beta\bar\beta & \alpha\bar\beta & -\bar\alpha\beta \\ \beta & -2\alpha\beta & \alpha^2 & \beta^2 \\ \bar\beta & 2\bar\alpha\bar\beta & \bar\beta^2 & \bar\alpha^2 \end{matrix}. \tag{4.2.5}$$

The invariantized form of the matrix of infinitesimals restricted to the variables z_0 and z_1 is

$$\Phi_0(I) = \begin{matrix} & \alpha & \beta & \bar\beta \\ z_0 & 1 & 0 & 0 \\ z_1 & 0 & 0 & -1 \end{matrix}$$

and then the replacement required by (3.7.6) is given by

$$\mathrm{S}_k\sigma^{z_0} \mapsto \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathrm{S}_k\mathcal{A}d(\rho_0) \quad \text{and} \quad \mathrm{S}_k\sigma^{z_1} \mapsto \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathrm{S}_k\mathcal{A}d(\rho_0)$$

where we have denoted $I^z_{0,0;t}$ by σ^{z_0} and $I^z_{0,1;t}$ by σ^{z_1}. However, in this case we are interested in replacing σ^{θ_0}. Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}z_0 = \frac{\mathrm{d}}{\mathrm{d}t}r_0e^{\mathrm{i}\theta_0} + \mathrm{i}r_0e^{\mathrm{i}\theta_0}\frac{\mathrm{d}}{\mathrm{d}t}\theta_0.$$

Therefore using the Replacement Rule 2.4.6 and taking into account that we are setting η to be 1 because of the constraint imposed when performing Calculus of Variations we have that

$$\sigma^{z_0} = \sigma^{r_0} + \mathrm{i}\sigma^{\theta_0}.$$

Hence we can conclude that σ^{θ_0} is the imaginary part of σ^{z_0}, and therefore

$$\sigma^{\theta_0} \mapsto \Im\Big(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathcal{A}d(\rho_0)\Big).$$

We obtain Noether's Conservation Law in the form

$$\mathbf{k} = \tau_{-1}(\mathrm{S}_{-1}\mathrm{E}_\tau(L))\Im\left(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathcal{A}d(\rho_0)\right) \tag{4.2.6}$$

where the vector $\mathbf{k} = (k_1, k_2, k_3)$ is a vector of constants and where

$$\mathcal{A}d(\rho_0) = \frac{1}{r_0^2 + r_1^2}\begin{pmatrix} r_0^2 - r_1^2 & r_0 r_1 e^{\mathrm{i}(\theta_1 - \theta_0)} & -r_0 r_1 e^{\mathrm{i}(\theta_1 + \theta_0)} \\ -2 r_0 r_1 e^{-\mathrm{i}(\theta_1 + \theta_0)} & r_0^2 e^{-2\mathrm{i}\theta_0} & r_1^2 e^{-2\mathrm{i}\theta_1} \\ 2 r_0 r_1 e^{\mathrm{i}(\theta_1 + \theta_0)} & r_1^2 e^{2\mathrm{i}\theta_1} & r_0^2 e^{2\mathrm{i}\theta_0} \end{pmatrix}.$$

Explicitly, we have that the conservation law (4.2.6) is of the form

$$\mathbf{k} = \tau_{-1}(\mathrm{S}_{-1}\mathrm{E}_\tau(L))\frac{r_0 r_1}{r_0^2 + r_1^2}\begin{pmatrix} 0 & \sin(\theta_1 - \theta_0) & -\sin(\theta_1 + \theta_0) \end{pmatrix}.$$

Moving frame for the conjugate action of $SU(2)$ on $\mathfrak{su}(2)$

There exists an isomorphism between the quaternions and $SU(2)$ given by

$$q = a + b\mathfrak{i} + c\mathfrak{j} + d\mathfrak{k} \leftrightarrow \begin{pmatrix} a + b\mathrm{i} & -c + d\mathrm{i} \\ c + d\mathrm{i} & a - b\mathrm{i} \end{pmatrix} = g$$

where

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Note that the lie algebra $\mathfrak{su}(2)$ is spanned by

$$\left\{\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\right\}$$

and therefore an element of $\mathfrak{su}(2)$ can be represented as

$$\begin{pmatrix} \mathrm{i}x_0 & y_0 + \mathrm{i}z_0 \\ -y_0 + \mathrm{i}z_0 & \mathrm{i}x_0 \end{pmatrix}.$$

Let us consider the conjugate action of $\mathrm{S}U(2)$ on $\mathfrak{su}(2)$

$$g \mapsto gAg^{-1}$$

where $A \in \mathfrak{su}(2)$. Therefore the action is given by

$$\begin{pmatrix} \mathrm{i}\widetilde{x_0} & \widetilde{y_0} + \mathrm{i}\widetilde{z_0} \\ -\widetilde{y_0} + \mathrm{i}\widetilde{z_0} & \mathrm{i}\widetilde{x_0} \end{pmatrix} = g \begin{pmatrix} \mathrm{i}x_0 & y_0 + \mathrm{i}z_0 \\ -y_0 + \mathrm{i}z_0 & \mathrm{i}x_0 \end{pmatrix} g^{-1}. \tag{4.2.7}$$

Therefore, for all $(x_0, y_0, z_0) \in \mathbb{R}^3$ we have that

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \mapsto \mathfrak{C}(A) \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} \widetilde{x_0} \\ \widetilde{y_0} \\ \widetilde{z_0} \end{pmatrix}$$

where $\mathfrak{C}(A)$ is a rotation in $\mathrm{S}O(3)$ explicitly given by

$$\begin{pmatrix} a^2+b^2+c^2+d^2 & -2(ac+bd) & 2(bc-ad) \\ 2(ac-bd) & a^2-b^2-c^2+d^2 & -2(ab+cd) \\ 2(ad+bc) & 2(ab-cd) & a^2-b^2+c^2-d^2 \end{pmatrix}.$$

The homomorphism

$$\mathfrak{C} : \mathrm{S}U(2) \to SO(3),$$
$$A \mapsto \mathfrak{C}(A).$$

is known as the Caley map. Let us consider now the normalization equations

$$\widetilde{y_0} = \widetilde{z_0} = \widetilde{z_1} = 0,$$

i.e,

$$gAg^{-1} = \begin{pmatrix} \mathrm{i}\widetilde{x_0} & 0 \\ 0 & -\mathrm{i}\widetilde{x_0} \end{pmatrix}.$$

The action is nonlinear but one can easily linearise the equations in order to get the explicit frame as follows

$$gA = \begin{pmatrix} \mathrm{i}\widetilde{x_0} & 0 \\ 0 & -\mathrm{i}\widetilde{x_0} \end{pmatrix} g. \tag{4.2.8}$$

Solving for the parameters a, b, c, d after a lengthy computation, we can get the frame

$$\rho_0 = \begin{pmatrix} \mathrm{i}Fe^{\mathrm{i}\theta} & FGe^{\mathrm{i}\theta} \\ -F\bar{G}e^{-\mathrm{i}\theta} & -\mathrm{i}Fe^{-\mathrm{i}\theta} \end{pmatrix}$$

where

$$F = \frac{\sqrt{(I^x_{0,0} + x_0)}}{\sqrt{2I^x_{0,0}}}, \quad G = \frac{y_0 + \mathrm{i}z_0}{2(I^x_{0,0} + x_0)}, \quad I^x_{0,0} = \sqrt{x_0^2 + {y_0}^2 + {z_0}^2} \quad \text{and} \quad \theta = \frac{1}{2}\arctan\left(\frac{\alpha_1}{\alpha_2}\right)$$

and

$$\alpha_1 = z_1(I^x_{0,0}+x_0)-y_0(v+x_1), \;\; \alpha_2 = y_1(I^x_{0,0}+x_0)-y_0I^x_{0,0}(v+x_1), \;\; v = \frac{\bar{x}\cdot(\mathrm{S}\bar{x})}{I^x_{0,0}}, \;\; \bar{x} = (x_0, y_0, z_0).$$

The method extends to all the Spin group actions. Examples of Spin groups are for instance S$U(n)$ or S$p(n)$ - see (2.1.4) for their description. Future work would include to find applications to Spin group invariant Lagrangians appearing in quantum physics.

4.3 Study of S$L(2)$ actions

In this section we show the finite difference analogue for the smooth variational problems with an S$L(2)$ and S$L(2) \ltimes \mathbb{R}^2$ symmetry that were considered using moving frame techniques in Gonçalves and Mansfield, [32, 34] and Mansfield, [70].

4.3.1 The linear action of S$L(2)$ in the plane

We consider the action of S$L(2)$ on the prolongation space $P_n^{(0,0)}(\mathbb{R}^2)$, which has coordinates (x_0, y_0). This action is given by

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \widetilde{x_0} \\ \widetilde{y_0} \end{pmatrix}, \qquad ad - bc = 1. \tag{4.3.1}$$

The infinitesimal vector fields and the adjoint action

For our calculations we need the adjoint representation of S$L(2)$ relative to this group action. From (4.3.1) we have that $\widetilde{x_0} = ax_0 + by_0$ and $\widetilde{y_0} = cx_0 + dy_0$ where $d = \dfrac{1+bc}{a}$.

Therefore the table of infinitesimals is of the form

	x_0	y_0
a	x_0	$-y_0$
b	y_0	0
c	0	x_0

and the infinitesimal vector fields for this action are

$$\mathbf{v}_a = x_0\partial_{x_0} - y_0\partial_{y_0}, \quad \mathbf{v}_b = y_0\partial_{x_0}, \quad \mathbf{v}_c = x_0\partial_{y_0}.$$

Note that

$$\partial_{x_0} = \frac{\partial\widetilde{x_0}}{\partial x_0}\partial_{\widetilde{x_0}} + \frac{\partial\widetilde{y_0}}{\partial x_0}\partial_{\widetilde{y_0}}, \qquad \partial_{y_0} = \frac{\partial\widetilde{x_0}}{\partial y_0}\partial_{\widetilde{x_0}} + \frac{\partial\widetilde{y_0}}{\partial y_0}\partial_{\widetilde{y_0}}$$

so

$$\partial_{x_0} = a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}}, \qquad \partial_{y_0} = b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}}.$$

We also have from (4.3.1)

$$x_0 = d\widetilde{x_0} - b\widetilde{y_0}, \qquad y_0 = -c\widetilde{x_0} + a\widetilde{y_0}.$$

Therefore

$$\begin{aligned}
\mathbf{v}_a &= x_0\partial_{x_0} - y_0\partial_{y_0}\\
&= (d\widetilde{x_0} - b\widetilde{y_0})(a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}}) - (-c\widetilde{x_0} + a\widetilde{y_0})(b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}})\\
&= (ad+cb)(\widetilde{x_0}\partial_{\widetilde{x_0}} - \widetilde{y_0}\partial_{\widetilde{y_0}}) + 2cd\widetilde{x_0}\partial_{\widetilde{y_0}} - 2ab\widetilde{y_0}\partial_{\widetilde{x_0}}\\
&= (ad+cb)\widetilde{\mathbf{v}_a} - 2ba\widetilde{\mathbf{v}}_b + 2cd\widetilde{\mathbf{v}}_c,\\
\mathbf{v}_b &= y_0\partial_{x_0}\\
&= (-c\widetilde{x_0} + a\widetilde{y_0})(a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}})\\
&= -ac(\widetilde{x_0}\partial_{\widetilde{x_0}} - \widetilde{y_0}\partial_{\widetilde{y_0}}) - c^2\widetilde{x_0}\partial_{\widetilde{y_0}} + a^2\widetilde{y_0}\partial_{\widetilde{x_0}}\\
&= -ac\widetilde{\mathbf{v}_a} + a^2\widetilde{\mathbf{v}}_b - c^2\widetilde{\mathbf{v}}_c,\\
\mathbf{v}_c &= x_0\partial_{y_0}\\
&= (d\widetilde{x_0} - b\widetilde{y_0})(b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}})\\
&= bd(\widetilde{x_0}\partial_{\widetilde{x_0}} - \widetilde{y_0}\partial_{\widetilde{y_0}}) + d^2\widetilde{x_0}\partial_{\widetilde{y_0}} - b^2\widetilde{y_0}\partial_{\widetilde{x_0}}\\
&= bd\widetilde{\mathbf{v}_a} - b^2\widetilde{\mathbf{v}}_b + d^2\widetilde{\mathbf{v}}_c.
\end{aligned}$$

We have that the induced action on these are

$$\begin{pmatrix} \widetilde{\mathbf{v}_a} & \widetilde{\mathbf{v}_b} & \widetilde{\mathbf{v}_c} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c \end{pmatrix} \mathcal{A}d(g)^{-1}$$

where

$$\mathcal{A}d(g) = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix} \end{matrix}. \tag{4.3.2}$$

The discrete frame, the generating invariants and differential invariants and their syzygies

We consider the normalization equations

$$\widetilde{x}_0 = 1, \quad \widetilde{x}_1 = \widetilde{y}_0 = 0. \tag{4.3.3}$$

Solving for a, b and c, we obtain the moving frame

$$\rho_0(x_0, y_0, x_1, y_1) = \begin{pmatrix} \dfrac{y_1}{\tau} & -\dfrac{x_1}{\tau} \\ -y_0 & x_0 \end{pmatrix} \in \mathrm{S}L(2) \tag{4.3.4}$$

where we have set $\tau = x_0y_1 - x_1y_0$. Then $\rho_k = \mathrm{S}_k\rho_0$ gives the discrete moving frame (ρ_k).

The Maurer–Cartan matrix is

$$K_0 = \iota_0(\rho_1) = \begin{pmatrix} \kappa & \dfrac{1}{\tau} \\ -\tau & 0 \end{pmatrix} \tag{4.3.5}$$

where we have set $\kappa = \dfrac{x_0y_2 - x_2y_0}{x_1y_2 - x_2y_1}$.

By (3.4.22) the algebra of invariants is generated by τ, κ and their shifts.

We now consider $x_j = x_j(t)$, $y_j = y_j(t)$ and we define some first order differential invariants by setting

$$I^x_{k,j;t}(t) := \rho_k \cdot x'_j \qquad \text{and} \qquad I^y_{k,j;t}(t) := \rho_k \cdot y'_j, \tag{4.3.6}$$

where $x'_j = \dfrac{\mathrm{d}}{\mathrm{dt}}x_j(t)$ and $y'_j = \dfrac{\mathrm{d}}{\mathrm{dt}}y_j(t)$. We set the notation

$$\sigma^x := I^x_{0,0;t}(t) \qquad \text{and} \qquad \sigma^y := I^y_{0,0;t}(t). \tag{4.3.7}$$

For our calculations, we need to know $I^x_{0,2;t}(t)$, $I^x_{0,1;t}(t)$ and $I^y_{0,1;t}(t)$ in terms of σ^x and σ^y.

We have

$$\begin{pmatrix} I^x_{0,1;t}(t) \\ I^y_{0,1;t}(t) \end{pmatrix} = \rho_0 \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \rho_0\rho_1^{-1}\rho_1 \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = K_0^{-1} \begin{pmatrix} \mathrm{S}\sigma^x \\ \mathrm{S}\sigma^y \end{pmatrix} = \begin{pmatrix} -\dfrac{\mathrm{S}\sigma^y}{\tau} \\ \tau\mathrm{S}\sigma^x + \kappa\mathrm{S}\sigma^y \end{pmatrix}. \tag{4.3.8}$$

Setting $\tau_j = \mathrm{S}_j\tau$ and $\kappa_j = \mathrm{S}_j\kappa$ we have that

$$\begin{pmatrix} I^x_{0,2;t}(t) \\ I^y_{0,2;t}(t) \end{pmatrix} = K_0^{-1}(\mathrm{S}K_0^{-1}) \begin{pmatrix} \mathrm{S}_2\sigma^x \\ \mathrm{S}_2\sigma^y \end{pmatrix} = \begin{pmatrix} -\dfrac{\tau_1}{\tau} & -\dfrac{\kappa_1}{\tau} \\ \kappa\tau_1 & \kappa\kappa_1 - \dfrac{\tau}{\tau_1} \end{pmatrix} \begin{pmatrix} \mathrm{S}_2\sigma^x \\ \mathrm{S}_2\sigma^y \end{pmatrix}. \tag{4.3.9}$$

The curvature matrix is of the form

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right) = \begin{pmatrix} -\sigma^x & -\dfrac{I^x_{0,1;t}(t)}{\tau} \\ -\sigma^y & \sigma^x \end{pmatrix} \in \mathfrak{sl}(2). \tag{4.3.10}$$

From (3.4.33), using (4.3.8) and (4.3.9) equating components and simplifying we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa = \kappa(\mathrm{id}-\mathrm{S})\sigma^x + \left(\frac{1}{\tau} - \frac{\tau}{\tau_1^2}\mathrm{S}_2\right)\sigma^y, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\tau = \tau(\mathrm{S}+\mathrm{id})\sigma^x + \kappa\mathrm{S}\sigma^y \tag{4.3.11}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \kappa \\ \tau \end{pmatrix} = \mathcal{H}\begin{pmatrix} \sigma^x \\ \sigma^y \end{pmatrix}$$

where

$$\mathcal{H} = \begin{pmatrix} \kappa(\mathrm{id}-\mathrm{S}) & \frac{1}{\tau} - \frac{\tau}{\tau_1^2}\mathrm{S}_2 \\ \tau(\mathrm{id}+\mathrm{S}) & \kappa\mathrm{S} \end{pmatrix}. \tag{4.3.12}$$

The Euler–Lagrange equations and conservation laws

We now consider a Lagrangian of the form

$$\mathcal{L}[x,y] = \sum L(\tau, \tau_1, \dots \tau_{J_1}, \kappa, \kappa_1, \dots, \kappa_{J_2}).$$

Using (3.5.11) , we have that the Euler–Lagrange equations are

$$\begin{aligned} 0 &= (\mathrm{id} - \mathrm{S}_{-1})\, \kappa \mathrm{E}_\kappa(L) + (\mathrm{id} + \mathrm{S}_{-1})\, \tau \mathrm{E}_\tau(L), \\ 0 &= -\mathrm{S}_{-2}\left(\frac{\tau}{\tau_1^2}\mathrm{E}_\kappa(L)\right) + \frac{1}{\tau}\mathrm{E}_\kappa(L) + \mathrm{S}_{-1}\left(\kappa \mathrm{E}_\tau(L)\right). \end{aligned} \tag{4.3.13}$$

To obtain the conservation laws we need only the boundary terms arising from $\mathrm{E}(L)\mathcal{H}\left(\sigma^x\ \sigma^y\right)^T - \mathcal{H}^*(\mathrm{E}(L))\left(\sigma^x\ \sigma^y\right)^T$. Using (3.5.4) these boundary terms are $(\mathrm{S} - \mathrm{id})A_{\mathcal{H}}$ where

$$\begin{aligned} A_{\mathcal{H}} &= \mathcal{C}_0^x \sigma^x + \mathcal{C}_0^y \sigma^y + \mathcal{C}_1^y \mathrm{S}\sigma^y \\ &= \left[-\mathrm{S}_{-1}\left(\kappa \mathrm{E}_\kappa(L)\right) + \mathrm{S}_{-1}\left(\tau \mathrm{E}_\tau(L)\right)\right]\sigma^x \\ &\quad + \left[\mathrm{S}_{-1}\left(\kappa \mathrm{E}_\tau(L)\right) - \mathrm{S}_{-2}\left(\frac{\tau}{\tau_1^2}\mathrm{E}_\kappa(L)\right)\right]\sigma^y \\ &\quad - \mathrm{S}_{-1}\left(\frac{\tau}{\tau_1^2}\mathrm{E}_\kappa(L)\right)\mathrm{S}\sigma^y, \end{aligned} \tag{4.3.14}$$

where this defines $\mathcal{C}_0^x$, $\mathcal{C}_0^y$ and $\mathcal{C}_1^y$.

To find the conservation laws, we first calculate the invariantized form of the matrix of infinitesimals restricted to the variables x_0 and y_0

$$\Phi_0(I) = \begin{array}{c} \\ x_0 \\ y_0 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The replacement (3.7.4) is given by

$$\mathrm{S}_k\sigma^x \mapsto \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \mathrm{S}_k \mathcal{A}d(\rho_0)$$

and

$$\mathrm{S}_k\sigma^y \mapsto \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \mathrm{S}_k \mathcal{A}d(\rho_0).$$

Since $\mathrm{S}\mathcal{A}d(\rho_0) = \mathcal{A}d\left(K_0\right)\mathcal{A}d(\rho_0)$, after collecting terms and simplifying we obtain the Noether's Conservation Laws in the form

$$\begin{aligned}\mathbf{k} &= \left[\mathcal{C}_0^x\begin{pmatrix}1 & 0 & 0\end{pmatrix}+\mathcal{C}_0^y\begin{pmatrix}0 & 0 & 1\end{pmatrix}+\mathcal{C}_1^y\begin{pmatrix}0 & 0 & 1\end{pmatrix}\mathcal{A}d(K_0)\right]\mathcal{A}d(\rho_0)\\ &= \mathbf{V}(I)\mathcal{A}d(\rho_0)\end{aligned} \tag{4.3.15}$$

where

$$\mathcal{A}d(\rho_0)=\begin{pmatrix}\dfrac{x_0y_1+x_1y_0}{\tau} & \dfrac{y_0y_1}{\tau} & -\dfrac{x_0x_1}{\tau}\\ 2\dfrac{x_1y_1}{\tau^2} & \dfrac{y_1^2}{\tau^2} & -\dfrac{x_1^2}{\tau^2}\\ -2x_0y_0 & -y_0^2 & x_0^2\end{pmatrix}$$

and

$$\mathcal{A}d(K_0)=\begin{pmatrix}-1 & \kappa\tau & 0\\ -2\dfrac{\kappa}{\tau} & \kappa^2 & -\dfrac{1}{\tau^2}\\ 0 & -\tau^2 & 0\end{pmatrix}$$

and where $\mathcal{C}_0^x$, $\mathcal{C}_0^y$ and $\mathcal{C}_1^y$ are defined in Equation (4.3.14), the vector $\mathbf{k}=(k_1,k_2,k_3)$ is a vector of constants and where this equation defines $\mathbf{V}(I)=(V_0^1\ \ V_0^2\ \ V_0^3)$. Explicitly, the vector of invariants $\mathbf{V}(I)$ is of the form

$$\mathbf{V}(I)=\mathrm{S}_{-1}\begin{pmatrix}\tau\mathrm{E}_\tau(L)-\kappa\mathrm{E}_\kappa(L) & \mathrm{E}_\kappa(L) & \kappa\mathrm{E}_\tau(L)-\mathrm{S}_{-1}\left(\dfrac{\tau}{\tau^2}\mathrm{E}_\kappa(L)\right)\end{pmatrix}.$$

Recall that from $(\mathrm{S}-\mathrm{id})(\mathbf{V}(I)\mathcal{A}d(\rho_0))=0$ we obtain the discrete Euler–Lagrange equations in the form $\mathrm{S}\mathbf{V}(I)\mathcal{A}d(\rho_1\rho_0^{-1})=\mathbf{V}(I)$ which yields the equations

$$\begin{pmatrix}V_0^1 & V_0^2 & V_0^3\end{pmatrix}=\begin{pmatrix}V_1^1 & V_1^2 & V_1^3\end{pmatrix}\begin{pmatrix}-1 & \kappa\tau & 0\\ -2\dfrac{\kappa}{\tau} & \kappa^2 & -\dfrac{1}{\tau^2}\\ 0 & -\tau^2 & 0\end{pmatrix}. \tag{4.3.16}$$

The general solution

Suppose that we can solve for the discrete frame (ρ_k). Then taking into account that the normalization equations for ρ_k are $\rho_k\cdot(x_k,y_k)^T=(1,0)^T$ we have that

$$\begin{pmatrix}x_k\\ y_k\end{pmatrix}=\rho_k^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}d_k & -b_k\\ -c_k & a_k\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}d_k\\ -c_k\end{pmatrix}.$$

We present the following Theorem:

Theorem 4.3.17. *Given a solution (κ_k), (τ_k) to the Euler–Lagrange equations, so that the vector of invariants $\mathrm{S}_k\mathbf{V}(I)=(V_k^1\ \ V_k^2\ \ V_k^3)$ appearing in the conservation laws are known and satisfy $V_k^2\neq 0$ for all k, (4.3.15), and that three constants $\mathbf{k}=(k_1,k_2,k_3)^T$ satisfying*

$k_3(k_1^2+4k_2k_3)\neq 0$ *are given, then the general solution to the Euler–Lagrange equations, in terms of* (x_k, y_k) *is*

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Q \begin{pmatrix} \prod_{l=0}^{k} \zeta_l \lambda_{1,l} & 0 \\ 0 & \prod_{l=0}^{j} k\zeta_l \lambda_{2,l} \end{pmatrix} Q^{-1} \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}$$

where here, c_0 *and* d_0 *are two further arbitrary constants of integration,*

$$Q = \begin{pmatrix} k_1 - \sqrt{k_1^2+4k_2k_3} & k_1 + \sqrt{k_1^2+4k_2k_3} \\ 2k_3 & 2k_3 \end{pmatrix}, \tag{4.3.18}$$

and where

$$\lambda_{1,l} = V_l^1 - \sqrt{k_1^2+4k_2k_3}, \qquad \lambda_{2,l} = V_l^1 + \sqrt{k_1^2+4k_2k_3}, \qquad \zeta_l = -\frac{\tau_l}{2V_l^2}. \tag{4.3.19}$$

Proof. If we set

$$\rho_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \qquad a_0d_0 - b_0c_0 = 1 \tag{4.3.20}$$

and write (4.3.15) in the form $\mathbf{k}\mathcal{A}d(\rho_0)^{-1} = \mathbf{V}(I)$ as three equations for $\{a_0, b_0, c_0, d_0\}$, we obtain

$$\begin{aligned} (a_0d_0+b_0c_0)k_1 + 2b_0d_0k_2 - 2a_0c_0k_3 &= V_0^1, \\ c_0d_0k_1 + d_0^2k_2 - c_0^2k_2 &= V_0^2, \\ -a_0b_0k_1 - b_0^2k_2 + a_0^2k_3 &= V_0^3. \end{aligned} \tag{4.3.21}$$

Computing a Gröebner basis associated to these equations, together with the equation $a_0d_0 - b_0c_0 = 1$, using the lexicographic ordering $k_3 < k_2 < k_1 < c_0 < b_0 < a_0$, we obtain

$$k_1^2 + 4k_2k_3 - \left(V_0^1\right)^2 - 4V_0^2V_0^3 = 0, \tag{4.3.22a}$$

$$k_3c_0^2 - k_1c_0d_0 - k_2d_0^2 + V_0^2 = 0, \tag{4.3.22b}$$

$$2b_0V_0^2 - 2c_0k_3 + (k_1 - V_0^1)d_0 = 0, \tag{4.3.22c}$$

$$2a_0V_0^2 - c_0(k_1 + V_0^1) - 2k_2d_0 = 0. \tag{4.3.22d}$$

We note that (4.3.22a) is a first integral of the Euler–Lagrange equations, (4.3.22b) is a conic equation for (c_0, d_0) while (4.3.22c) and (4.3.22d) are linear for (a_0, b_0) in terms of (c_0, d_0). We have

$$\rho_1 = \begin{pmatrix} \kappa & \frac{1}{\tau} \\ -\tau & 0 \end{pmatrix} \rho_0$$

where $\rho_1 = \mathrm{S}\rho_0$. Hence

$$c_1 = -\tau a_0, \quad \text{and} \quad d_1 = -\tau b_0.$$

Back-substituting for a_0 and b_0 from (4.3.22c) and (4.3.22d) yields, assuming $V_0^2 \neq 0$

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \frac{-\tau}{2V_0^2}\left(V_0^1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} k_1 & 2k_2 \\ 2k_3 & -k_1 \end{pmatrix}\right)\begin{pmatrix} c_0 \\ d_0 \end{pmatrix}. \tag{4.3.23}$$

Now, setting

$$\mathbf{c_0} = \begin{pmatrix} c_0 \\ d_0 \end{pmatrix}, \quad \mathbf{c_1} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \quad \zeta_0 = \frac{-\tau}{2V_0^2} \quad \text{and} \quad X_0 = V_0^1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} k_1 & 2k_2 \\ 2k_3 & -k_1 \end{pmatrix}$$

equation (4.3.23) can be written as

$$\mathbf{c_1} = \zeta_0 X_0 \mathbf{c_0}. \tag{4.3.24}$$

Diagonalising X_0 we obtain Λ_0 diagonal such that

$$\Lambda_0 = Q^{-1}X_0Q = \begin{pmatrix} \lambda_0^1 & 0 \\ 0 & \lambda_0^2 \end{pmatrix}$$

where

$$\lambda_0^1 = V_0^1 - \sqrt{k_1^2 + 4k_2k_3} \quad \text{and} \quad \lambda_0^2 = V_0^1 + \sqrt{k_1^2 + 4k_2k_3}$$

and

$$Q = \begin{pmatrix} k_1 - \sqrt{k_1^2 + 4k_2k_3} & k_1 + \sqrt{k_1^2 + 4k_2k_3} \\ 2k_3 & 2k_3 \end{pmatrix}. \tag{4.3.25}$$

Note that Q is a constant matrix. Therefore it is now simple to solve the recurrence relation. From (4.3.24), supposing $k_3\sqrt{k_1^2 + 4k_2k_3} \neq 0$ so Q^{-1} exists, we obtain

$$\mathbf{c_{k+1}} = Q\begin{pmatrix} \prod_{l=0}^{k} \zeta_l \lambda_{1,l} & 0 \\ 0 & \prod_{l=0}^{k} \zeta_l \lambda_{2,l} \end{pmatrix} Q^{-1}\mathbf{c_0}$$

where here, $\mathbf{c_0}$ is the initial data. Using the normalization equations

$$\begin{pmatrix} x_k \\ y_k \end{pmatrix} = \rho_k^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d_k & -b_k \\ -c_k & a_k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} d_k \\ -c_k \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c_k \\ d_k \end{pmatrix}$$

the result follows. □

Remark 4.3.26. *This proof does not make use of* (4.3.22b). *However, it is consistent with* *the second component of* (4.3.16) *as we show now. We have that* (4.3.22b) *can be written as*

$$\begin{pmatrix} c_0 & d_0 \end{pmatrix} \begin{pmatrix} k_3 & -\frac{k_1}{2} \\ -\frac{k_1}{2} & k_2 \end{pmatrix} \begin{pmatrix} c_0 \\ d_0 \end{pmatrix} = -V_0^2$$

and therefore

$$\begin{pmatrix} c_1 & d_1 \end{pmatrix} \begin{pmatrix} k_3 & -\frac{k_1}{2} \\ -\frac{k_1}{2} & k_2 \end{pmatrix} \begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = -V_1^2. \tag{4.3.27}$$

Substituting (4.3.27) *into* (4.3.23) *yields after simplification the equation*

$$V_1^2 = -\tau^2 V_0^3$$

as stated.

4.3.2 The S$A(2) = $ S$L(2) \ltimes \mathbb{R}^2$ linear action

A general element of the equi-affine group S$A(2) =$ S$L(2) \ltimes \mathbb{R}^2$, is given by (g, α, β) where $g \in \mathrm{S}L(2)$ and α, $\beta \in \mathbb{R}$. The standard representation of this group is given by

$$(g, \alpha, \beta) \mapsto \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{pmatrix}.$$

We consider the equi-affine group action on $P_n^{(0,0)}(\mathbb{R}^2)$ with coordinates (x_0, y_0) given by

$$(g, \alpha, \beta) \cdot (x_0, y_0) = (\widetilde{x}_0, \widetilde{y}_0) = (ax_0 + by_0 + \alpha, cx_0 + dy_0 + \beta), \qquad ad - bc = 1. \tag{4.3.28}$$

The infinitesimal vector fields and the Adjoint action

The table of infinitesimals for (4.3.28) is of the form

	x_0	y_0
a	x_0	$-y_0$
b	y_0	0
c	0	x_0
α	1	0
β	0	1

Therefore the infinitesimal vector fields are of the form

$$\mathbf{v}_a = x\partial_x - y\partial_y, \quad \mathbf{v}_b = y\partial_x, \quad \mathbf{v}_c = x\partial_y, \quad \mathbf{v}_\alpha = \partial_x, \quad \mathbf{v}_\beta = \partial_y.$$

Note that

$$\partial_{x_0} = \frac{\partial\widetilde{x_0}}{\partial x_0}\partial_{\widetilde{x_0}} + \frac{\partial\widetilde{y_0}}{\partial x_0}\partial_{\widetilde{y_0}}, \qquad \partial_{y_0} = \frac{\partial\widetilde{x_0}}{\partial y_0}\partial_{\widetilde{x_0}} + \frac{\partial\widetilde{y_0}}{\partial y_0}\partial_{\widetilde{y_0}},$$

so

$$\partial_{x_0} = a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}}, \qquad \partial_{y_0} = b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}}.$$

We also have from (4.3.1)

$$x_0 = d\widetilde{x_0} - b\widetilde{y_0} - \alpha d + \beta b, \qquad y_0 = -c\widetilde{x_0} + a\widetilde{y_0} - \beta a + \alpha c.$$

Therefore

$$\begin{aligned}
\mathbf{v}_a &= x_0\partial_{x_0} - y_0\partial_{y_0} \\
&= (d\widetilde{x_0} - b\widetilde{y_0} - \alpha d + \beta b)(a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}}) - (-c\widetilde{x_0} + a\widetilde{y_0} - \beta a + \alpha c)(b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}}) \\
&= (ad + cb)(\widetilde{x_0}\partial_{\widetilde{x_0}} - \widetilde{y_0}\partial_{\widetilde{y_0}}) + 2cd\widetilde{x_0}\partial_{\widetilde{y_0}} - 2ab\widetilde{y_0}\partial_{\widetilde{x_0}} \\
&\qquad - (\alpha(ad + bc) + 2ab\beta)\partial_{\widetilde{x_0}} + (\beta(ad + bc) - 2cd\alpha)\partial_{\widetilde{y_0}} \\
&= (ad + cb)\widetilde{\mathbf{v}_a} - 2ab\widetilde{\mathbf{v}_b} + 2cd\widetilde{\mathbf{v}_c} - (\alpha(ad + bc) + 2ab\beta)\widetilde{\mathbf{v}_\alpha} + (\beta(ad + bc) - 2cd\alpha)\widetilde{\mathbf{v}_\beta},
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_b &= y_0\partial_{x_0} \\
&= (-c\widetilde{x_0} + a\widetilde{y_0} - \beta a + \alpha c)(a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}}) \\
&= -ac(\widetilde{x_0}\partial_{\widetilde{x_0}} - \widetilde{y_0}\partial_{\widetilde{y_0}}) - c^2\widetilde{x_0}\partial_{\widetilde{y_0}} + a^2\widetilde{y_0}\partial_{\widetilde{x_0}} + a(c\alpha - a\beta)\partial_{\widetilde{x_0}} + c(c\alpha - a\beta)\partial_{\widetilde{y_0}} \\
&= -ac\widetilde{\mathbf{v}_a} + a^2\widetilde{\mathbf{v}_b} - c^2\widetilde{\mathbf{v}_c} + a(c\alpha - a\beta)\widetilde{\mathbf{v}_\alpha} + c(c\alpha - a\beta)\widetilde{\mathbf{v}_\beta},
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_c &= x_0\partial_{y_0} \\
&= (d\widetilde{x_0} - b\widetilde{y_0} - \alpha d + \beta b)(b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}}) \\
&= bd(\widetilde{x_0}\partial_{\widetilde{x_0}} - \widetilde{y_0}\partial_{\widetilde{y_0}}) + d^2\widetilde{x_0}\partial_{\widetilde{y_0}} - b^2\widetilde{y_0}\partial_{\widetilde{x_0}} + b(b\beta - d\alpha)\partial_{\widetilde{x_0}} + d(b\beta - a\alpha)\partial_{\widetilde{y_0}} \\
&= bd\widetilde{\mathbf{v}_a} - b^2\widetilde{\mathbf{v}_b} + d^2\widetilde{\mathbf{v}_c} + b(b\beta - d\alpha)\widetilde{\mathbf{v}_\alpha} + d(b\beta - a\alpha)\widetilde{\mathbf{v}_\beta},
\end{aligned}$$

$$\mathbf{v}_\alpha = \partial_{x_0} = a\partial_{\widetilde{x_0}} + c\partial_{\widetilde{y_0}} = a\widetilde{\mathbf{v}_\alpha} + c\widetilde{\mathbf{v}_\beta},$$

$$\mathbf{v}_\beta = \partial_{y_0} = b\partial_{\widetilde{x_0}} + d\partial_{\widetilde{y_0}} + b\widetilde{\mathbf{v}_\alpha} + d\widetilde{\mathbf{v}_\beta}.$$

We have that the induced action on these vector fields is

$$\begin{pmatrix} \widetilde{\mathbf{v}_a} & \widetilde{\mathbf{v}_b} & \widetilde{\mathbf{v}_c} & \widetilde{\mathbf{v}_\alpha} & \widetilde{\mathbf{v}_\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c & \mathbf{v}_\alpha & \mathbf{v}_\beta \end{pmatrix} \mathcal{A}d(g,\alpha,\beta)^{-1}$$

where

$$\mathcal{A}d(g,\alpha,\beta)^{-1} = \begin{array}{c} \\ a \\ b \\ c \\ \alpha \\ \beta \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & \alpha & \beta \end{array} \\ \begin{pmatrix} ad+bc & cd & -ab & 0 & 0 \\ 2bd & d^2 & -b^2 & 0 & 0 \\ -2ac & -c^2 & a^2 & 0 & 0 \\ \alpha d + b\beta & \beta d & -b\alpha & d & -b \\ -a\beta - \alpha c & -c\beta & \alpha a & -c & a \end{pmatrix} \end{array}. \tag{4.3.29}$$

and where

$$\mathcal{A}d(g,\alpha,\beta) = \begin{array}{c} \\ a \\ b \\ c \\ \alpha \\ \beta \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & \alpha & \beta \end{array} \\ \begin{pmatrix} ad+bc & -ac & bd & 0 & 0 \\ -2ab & a^2 & -b^2 & 0 & 0 \\ 2cd & -c^2 & d^2 & 0 & 0 \\ -\alpha(ad+bc)+2ab\beta & a(c\alpha - a\beta) & b(b\beta - d\alpha) & a & b \\ \beta(ad+bc) - 2cd\alpha & c(c\alpha - a\beta) & d(b\beta - d\alpha) & c & d \end{pmatrix} \end{array}. \tag{4.3.30}$$

Remark 4.3.31. *We note that* (4.3.30) *can be written as*

$$\mathcal{A}d(g,\alpha,\beta) = \left(\begin{array}{c|c} Id_3 & 0 \\ \hline \alpha \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} & Id_2 \end{array} \right) \left(\begin{array}{c|c} \mathcal{A}d(g) & 0 \\ \hline 0 & g \end{array} \right) \tag{4.3.32}$$

where Id_2 and Id_3 are the 2×2 *and* 3×3 *identity matrices respectively.*

The discrete frame, the generating invariants and difference invariants and their syzygies

We consider the normalization equations

$$\widetilde{x_0} = \widetilde{y_0} = \widetilde{y_1} = \widetilde{x_2} = 0 \quad \text{and} \quad \widetilde{x_2} = 0. \tag{4.3.33}$$

Solving for the group parameters a, b, c, d, α and β we obtain the following standard matrix representation of the moving frame

$$\rho_0 = \begin{pmatrix} \dfrac{y_2 - y_0}{\kappa} & \dfrac{x_0 - x_2}{\kappa} & \dfrac{x_2 y_0 - x_0 y_2}{\kappa} \\ y_0 - y_1 & x_1 - x_0 & x_0 y_1 - x_1 y_0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\kappa = (y_1 - y_2)\, x_0 + (y_2 - y_0)\, x_1 + (y_0 - y_1)\, x_2$$

is an invariant as $\kappa = \rho_0 \cdot y_2$.

We define the discrete moving frame to be (ρ_k) where $\rho_k = \mathrm{S}_k \rho_0$. The Maurer–Cartan matrix is

$$K_0 = \iota_0(\rho_1) = \begin{pmatrix} \tau & \dfrac{1+\tau}{\kappa} & -\tau \\ -\kappa & -1 & \kappa \\ 0 & 0 & 1 \end{pmatrix} \tag{4.3.34}$$

where κ is given above, and

$$\tau = \frac{x_0(y_1 - y_3) + x_1(y_3 - y_0) + x_3(y_0 - y_1)}{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)} = \frac{\rho_0 \cdot y_3}{\kappa_1}$$

where we have used the Replacement Rule 2.4.6, and where $\kappa_k = \mathrm{S}_k \kappa$. By (3.4.22) the algebra of invariants is generated by τ, κ and their shifts.

Computing the curvature matrix, we obtain

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right) = \begin{pmatrix} \sigma^x - I^x_{0,1;t}(t) & \dfrac{\sigma^x - I^x_{0,2;t}(t)}{\kappa} & -\sigma^x \\ \sigma^y - I^y_{0,1;t}(t) & I^x_{0,1;t}(t) - \sigma^x & -\sigma^y \\ 0 & 0 & 0 \end{pmatrix} \tag{4.3.35}$$

where we have set $\sigma^x := I^x_{0,0;t}(t)$ and $\sigma^y := I^y_{0,0;t}(t)$.

To obtain $\rho_0 \cdot x'_j = I^x_{0,j;t}(t)$, $\rho_0 \cdot y'_j = I^y_{0,j;t}(t)$, $j = 1, 2$ in terms of σ^x, σ^y, τ, κ and their shifts, we have, since the translation part of the group plays no role in the action on the derivatives,

$$I_{0,1;t} = \rho_0 \begin{pmatrix} x'_1 \\ y'_1 \\ 0 \end{pmatrix} = \rho_0 \rho_1^{-1} \rho_1 \begin{pmatrix} x'_1 \\ y'_1 \\ 0 \end{pmatrix} = K_0^{-1} \begin{pmatrix} \mathrm{S}\sigma^x \\ \mathrm{S}\sigma^y \\ 0 \end{pmatrix}$$

and similarly

$$I_{0,2;t} = \rho_0 \begin{pmatrix} x_2' \\ y_2' \\ 0 \end{pmatrix} = \rho_0\rho_1^{-1}\rho_1\rho_2^{-1}\rho_2 \begin{pmatrix} x_2' \\ y_2' \\ 0 \end{pmatrix} = K_0^{-1}(\mathrm{S}K_0^{-1}) \begin{pmatrix} \mathrm{S}_2\sigma^x \\ \mathrm{S}_2\sigma^y \\ 0 \end{pmatrix}.$$

Finally from (3.4.33) and the relations above, we have the differential-difference syzygy in reduced form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tau \\ \kappa \end{pmatrix} = \mathcal{H}\begin{pmatrix} \sigma^x \\ \sigma^y \end{pmatrix}, \qquad \text{where} \qquad \mathcal{H} = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix} \tag{4.3.36}$$

with

$$\begin{aligned} \mathcal{H}_{11} &= -\tau + \left(1 + \frac{\kappa}{\kappa_1}(1+\tau)\right)\mathrm{S} + \tau\mathrm{S}_2 - \frac{\kappa}{\kappa_1^2}\left[\kappa_2(1+\tau_1) - \kappa_1\right]\mathrm{S}_3, \\ \mathcal{H}_{12} &= -\frac{1+\tau}{\kappa} + \frac{\tau(1+\tau_1)}{\kappa_1}\mathrm{S}_2 - \frac{\kappa}{\kappa_1^2\kappa_2}\left[\kappa_2\tau_2(1+\tau_1) - \kappa_1(1+\tau_2)\right]\mathrm{S}_3, \\ \mathcal{H}_{21} &= -\kappa - \kappa\mathrm{S} + (\tau\kappa_1 - \kappa)\mathrm{S}_2, \\ \mathcal{H}_{22} &= -1 - (1+\tau)\mathrm{S} + \left(\tau\tau_1 - \frac{\kappa(1+\tau_1)}{\kappa_1}\right)\mathrm{S}_2. \end{aligned} \tag{4.3.37}$$

The Euler–Lagrange equations and the conservation laws.

We consider a invariant Lagrangian of the form $L(\tau, \dots, \tau_{J_1}, \kappa, \dots, \kappa_{J_2})$. Then by (3.5.11) we have that the Euler–Lagrange equations are

$$\begin{aligned} 0 &= \mathcal{H}_{11}^*\mathrm{E}_\tau(L) + \mathcal{H}_{21}^*\mathrm{E}_\kappa(L), \\ 0 &= \mathcal{H}_{12}^*\mathrm{E}_\tau(L) + \mathcal{H}_{22}^*\mathrm{E}_\kappa(L) \end{aligned} \tag{4.3.38}$$

where the $\mathcal{H}_{ij}$ are given in Equation (4.3.37).

The boundary terms contributing to the conservation laws are

$$\begin{aligned} A_{\mathcal{H}} &= A_{\mathcal{H}_{11}}(\mathrm{E}_\tau(L), \sigma^x) + A_{\mathcal{H}_{21}}(\mathrm{E}_\kappa(L), \sigma^x) + A_{\mathcal{H}_{12}}(\mathrm{E}_\tau(L), \sigma^y) + A_{\mathcal{H}_{22}}(\mathrm{E}_\kappa(L), \sigma^y) \\ &= \sum_{k=0}^{2} \mathcal{C}_k^x\mathrm{S}_k\sigma^x + \mathcal{C}_k^y\mathrm{S}_k\sigma^y \end{aligned} \tag{4.3.39}$$

where this defines the $\mathcal{C}^x_k$, $\mathcal{C}^y_k$. Explicitly

$$
\begin{aligned}
\mathcal{C}^x_0 =& \mathrm{S}_{-1}(1+\frac{\kappa}{\kappa_1}(1+\tau)\mathrm{E}_\tau(L)) + \mathrm{S}_{-2}(\tau\mathrm{E}_\tau(L)) + \mathrm{S}_{-3}\left(-\frac{\kappa}{\kappa_1^2}(\kappa_2(1+\tau_1-\kappa_1)\right)\mathrm{E}_\tau(L))\\
&+\mathrm{S}_{-1}(-\kappa\mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}((\tau\kappa_1-\kappa)\mathrm{E}_\kappa(L)),\\
\mathcal{C}^x_1 =& \mathrm{S}_{-1}(\tau\mathrm{E}_\tau(L)) + \mathrm{S}_{-2}\left(-\frac{\kappa}{\kappa_1^2}(\kappa_2(1+\tau_1-\kappa_1)\right)\mathrm{E}_\tau(L)) + \mathrm{S}_{-1}((\tau\kappa_1-\kappa)\mathrm{E}_\kappa(L)),\\
\mathcal{C}^x_2 =& \mathrm{S}_{-1}\left(-\frac{\kappa}{\kappa_1^2}(\kappa_2(1+\tau_1-\kappa_1)\right)\mathrm{E}_\tau(L)),\\
\mathcal{C}^y_0 =& \mathrm{S}_{-2}(\frac{\tau(1+\tau_1)}{\kappa_1}\mathrm{E}_\tau(L)) + \mathrm{S}_{-3}(-\frac{\kappa}{\kappa_1^2\kappa_2}(\kappa_2\tau_2(1-\tau_1)-\kappa_1(1+\tau_2))\mathrm{E}_\tau(L))\\
&-\mathrm{S}_{-1}(1+\tau)\mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\tau\tau_1-\frac{\kappa(1-\tau_1)}{\kappa_1}\mathrm{E}_\kappa(L)),\\
\mathcal{C}^y_1 =& \mathrm{S}_{-1}(\frac{\tau(1+\tau_1)}{\kappa_1}\mathrm{E}_\tau(L)) + \mathrm{S}_{-2}(-\frac{\kappa}{\kappa_1^2\kappa_2}(\kappa_2\tau_2(1-\tau_1)-\kappa_1(1+\tau_2))\mathrm{E}_\tau(L))\\
&+\mathrm{S}_{-1}(\tau\tau_1-\frac{\kappa(1-\tau_1)}{\kappa_1}\mathrm{E}_\kappa(L)),\\
\mathcal{C}^y_2 =& \mathrm{S}_{-1}(-\frac{\kappa}{\kappa_1^2\kappa_2}(\kappa_2\tau_2(1-\tau_1)-\kappa_1(1+\tau_2))\mathrm{E}_\tau(L)).
\end{aligned}
\tag{4.3.40}
$$

To obtain the conservation laws we need the invariantized form of the matrix of infinitesimals restricted to the variables x_0 and y_0

$$
\Phi_0(I) = \begin{matrix} \\ x_0 \\ y_0 \end{matrix}\begin{matrix} \begin{matrix} a & b & c & \alpha & \beta \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{matrix}
$$

and then using (3.7.6) the replacements required to obtain the conservation laws from $A_{\mathcal{H}}$ are

$$
\mathrm{S}_k\sigma^x \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}\mathrm{S}_k\mathcal{A}d(\rho_0), \qquad \mathrm{S}_k\sigma^y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}\mathrm{S}_k\mathcal{A}d(\rho_0).
$$

Hence, the conservation laws are given by $(\mathrm{S}-\mathrm{id})A = 0$ where

$$
\begin{aligned}
A =& \Big[\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}(\mathcal{C}^x_0 + \mathcal{C}^x_1\mathcal{A}d(K_0) + \mathcal{C}^x_2\mathcal{A}d(K_0(\mathrm{S}K_0)))\\
&+\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}(\mathcal{C}^y_0 + \mathcal{C}^y_1\mathcal{A}d(K_0) + \mathcal{C}^y_2\mathcal{A}d((\mathrm{S}K_0)K_0))\Big]\mathcal{A}d(\rho_0)\\
=& \mathbf{V}(I)\mathcal{A}d(\rho_0)
\end{aligned}
\tag{4.3.41}
$$

and

$$\mathcal{A}d(K_0) = \left(\begin{array}{c|c} \mathrm{Id}_3 & 0 \\ \hline \begin{pmatrix} \tau & -\kappa & 0 \\ \kappa & 0 & -\tau \end{pmatrix} & \mathrm{Id}_2 \end{array}\right) \begin{pmatrix} -2\tau & \tau\kappa & -\dfrac{1+\tau}{\kappa} & 0 & 0 \\ \dfrac{-2\tau(1+\tau)}{\kappa} & \tau^2 & -\dfrac{(1+\tau)^2}{\kappa^2} & 0 & 0 \\ 2\kappa & -\kappa^2 & 1 & 0 & 0 \\ 0 & 0 & 0 & \tau & \dfrac{1+\tau}{\kappa} \\ 0 & 0 & 0 & -\kappa & -1 \end{pmatrix}.$$

This defines the vector of invariants, $\mathbf{V}(I) = \left(V_0^1, V_0^2, V_0^3, V_0^4, V_0^5\right)^T$ where the $\mathcal{C}_j^x$, $\mathcal{C}_j^y$ are defined in Equation (4.3.39) and (4.3.40).

Therefore we can write the conservation laws in the form

$$\mathbf{k} = \mathbf{V}(I)\mathcal{A}d(\rho_0) \tag{4.3.42}$$

where $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5)$ is a vector of constants and where

$$\mathcal{A}d(\rho_0) = \left(\begin{array}{c|c} \mathrm{Id}_3 & 0 \\ \hline \begin{pmatrix} \dfrac{x_0y_2 - x_2y_0}{\kappa} & x_1y_0 - x_0y_1 & 0 \\ x_0y_1 - x_1y_0 & 0 & \dfrac{x_2y_0 - x_0y_2}{\kappa} \end{pmatrix} & \mathrm{Id}_2 \end{array}\right) \left(\begin{array}{c|c} (\mathcal{A}d(g))\big|_{\rho_0} & 0 \\ \hline 0 & g\big|_{\rho_0} \end{array}\right)$$

and

$$\mathcal{A}d(g))\big|_{\rho_0} = \begin{pmatrix} \dfrac{(2y_0 - y_1 - y_2)x_0 - (x_1 + x_2)y_0 + y_2x_1 + x_2y_1}{\kappa} & \dfrac{(y_0 - y_2)(y_0 - y_1)}{\kappa} & \dfrac{(x_0 - x_2)(x_1 - x_0)}{\kappa} \\ \dfrac{2(y_0 - y_2)(x_0 - x_2)}{\kappa^2} & \dfrac{(y_0 - y_2)^2}{\kappa^2} & -\dfrac{(x_0 - x_2)^2}{\kappa^2} \\ 2(y_0 - y_1)(x_1 - x_0) & -(y_0 - y_1)^2 & (x_0 - x_1)^2 \end{pmatrix}$$

with

$$g\big|_{\rho_0} = \begin{pmatrix} \dfrac{y_2 - y_0}{\kappa} & \dfrac{x_0 - x_2}{\kappa} \\ y_0 - y_1 & x_1 - x_0 \end{pmatrix}.$$

We will show in the next section that a first integral of the Euler–Lagrange equations is given by

$$k_1k_4k_5 + k_2k_5^2 - k_3k_4^2 = V_0^1V_0^4V_0^5 + V_0^2(V_0^5)^2 - V_0^3(V_0^4)^2. \tag{4.3.43}$$

The general solution

Given the vector of invariants and the constants in the conservation laws (4.3.42) we now show how to obtain the solution to the Euler–Lagrange equations in terms of the original variables.

Theorem 4.3.44. *Suppose a solution* (τ_k), (κ_k) *to the Euler–Lagrange equations* (4.3.38), *is given, so that the vector of invariants* $(\mathrm{S}_k\mathbf{V}(I))$ *appearing in the conservation laws* (4.3.41) *is known, and that* $V_0^4V_0^5 \neq 0$. *Suppose further that a vector of constants* $\mathbf{k} = (k_1, k_2, k_3, k_4, k_5)$ *satisfying* $k_4k_5 \neq 0$ *is given. Then the general solution to the Euler–Lagrange equations, in terms of* (x_k, y_k) *is given by*

$$\begin{pmatrix} x_k \\ y_k \\ 1 \end{pmatrix} = \rho_k^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha_k d_k + \beta_k b_k \\ \alpha_k c_k - \beta_k c_k \\ 1 \end{pmatrix} \tag{4.3.45}$$

where, setting $\mu := k_1k_4k_5 + k_2k_5^2 - k_3k_4^2$,

$$\begin{aligned}
a_0 &= -\frac{V_0^5}{V_0^4}c_0 + \frac{k_4}{V_0^4}, \\
b_0 &= -\frac{V_0^5k_5}{V_0^4k_4}c_0 + \frac{k_4k_5 - V_0^4V_0^5}{V_0^4k_4}, \\
d_0 &= \frac{k_5}{k_4}c_0 + \frac{V_0^4}{k_4}, \\
\alpha_0 &= \frac{\mu V_0^5}{\left(V_0^4k_4\right)^2}c_0^2 + \frac{\left(k_2k_5^2 + k_3k_4^2 + \mu\right)V_0^4(V_0^5)^2 - 2\mu k_4k_5V_0^5}{\left(V_0^4k_4\right)^2V_0^5k_5}c_0 \\
&\quad + \frac{1}{\left(V_0^4k_4\right)^2V_0^5k_5}\left(k_2k_5\left(V_0^4V_0^5\right)^2 - \left(k_2k_5^2 + k_3k_4^2 + \mu\right)V_0^4V_0^5k_4 + k_4^2k_5\left(V_0^3\left(V_0^4\right)^2 + \mu\right)\right), \\
\beta_0 &= -\frac{\mu}{k_4^2V_0^4}c_0^2 - \frac{V_0^4\left(k_2k_5^2 + k_3k_4^2 + \mu\right)}{k_4^2k_5V_0^4}c_0 + \frac{k_4^2V_0^2 - k_2(V_0^4)^2}{k_4^2V_0^4}
\end{aligned} \tag{4.3.46}$$

and where

$$c_k = \prod_{l=0}^{k-1}\left(\frac{\kappa_lV_l^5}{V_l^4} - 1\right)c_0 - \sum_{l=0}^{k-1}\prod_{m=l+1}^{k-1}\left(\frac{\kappa_lV_m^5}{V_m^4} - 1\right)\frac{k_4\kappa_l}{V_l^4} \tag{4.3.47}$$

where in this last equation, c_0 *is the initial datum, or constant of integration.*

Proof. If we can solve for the discrete frame (ρ_k)

$$\rho_k = \begin{pmatrix} a_k & b_k & \alpha_k \\ c_k & d_k & \beta_k \\ 0 & 0 & 1 \end{pmatrix},$$

then we have by the normalization equations (4.3.33) that (4.3.45) holds. We consider (4.3.42) as five equations for $\{a_0, b_0, c_0, d_0, \alpha_0, \beta_0\}$, which can be written in the form

$$
\begin{aligned}
0 =&(a_0d_0 + b_0c_0)k_1 + 2b_0d_0k_2 - 2a_0c_0k_3 + (b\beta_0 + d_0\alpha_0)k_4 - (a_0\beta_0 + c_0\alpha_0) - V_0^1,\\
0 =& - c_0^2k_3 + c_0k_1d_0 - c_0k_5\beta_0 + k_2d_0^2 + k_4d_0k_2 - V_0^2,\\
0 =&a_0^2k_3 - a_0b_0k_1 + a_0k_5\alpha_0 - b_0^2k_2 - b_0k_4\alpha_0 - V_0^3,\\
0 =& - c_0k_5 + k_4d_0 - V_0^4,\\
0 =&a_0k5 - b_0k_4 - V_0^5.
\end{aligned}
$$

Computing a Gröbner basis associated to these equations with the lexicographic ordering $k_2 < k_1 < a_0 < b_0 < d_0 < \beta_0 < \alpha_0$, we obtain the first integral noted in Equation (4.3.43), and the expressions for a_0, b_0, d_0, α_0 and β_0 in terms of c_0 given in (4.3.46), provided V_0^4, V_0^5, k_4 and k_5 are all non zero.

From $\rho_1 = K_0\rho_0$ we can obtain a recurrence equation for (c_k), specifically,

$$c_1 = -\kappa a_0 - c_0 = \left(\frac{\kappa V_0^5}{V_0^4} - 1\right) c_0 - \frac{k_4\kappa}{V_0^4}$$

where we have back substituted for a_0 from (4.3.46). This is linear and can be easily solved to obtain the expression for c_k given in (4.3.47). Substituting this into the shifts of (4.3.46) yields (a_k), (b_k), (d_k), (α_k) and (β_k) and substituting these into (4.3.45) yields the desired result. □

4.3.3 The S$L(2)$ projective action

In this example, we study the S$L(2)$ projective action acting on discrete variables given by

$$\widetilde{x_0} = g \cdot x_0 = \frac{ax_0 + b}{cx_0 + d}, \qquad ad - bc = 1. \tag{4.3.48}$$

We show how to calculate the recurrence relations when the action is nonlinear. We detail the calculations for a class of one-dimensional S$L(2)$ Lagrangians, which are invariant under (4.3.48).

The Adjoint action

The infinitesimal vector fields for this action were previously computed in (2.15). They are of the form

$$\mathbf{v}_a = 2x\partial_x, \quad \mathbf{v}_b = \partial_x, \quad \mathbf{v}_c = -x^2\partial_x. \tag{4.3.49}$$

We have (see Example (2.2.6) for the full calculation) that the induced action on these are

$$\begin{pmatrix} \widetilde{\mathbf{v}_a} & \widetilde{\mathbf{v}_b} & \widetilde{\mathbf{v}_c} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_a & \mathbf{v}_b & \mathbf{v}_c \end{pmatrix} \mathcal{A}d(g)^{-1}$$

where

$$\mathcal{A}d(g) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} ad+bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix} \end{array} \tag{4.3.50}$$

which matches with (4.3.2) as expected.

The discrete frame, the generating invariants and their syzygies

We consider the normalization equations

$$\widetilde{x_0} = \frac{1}{2}, \quad \widetilde{x_1} = 0, \quad \widetilde{x_2} = -\frac{1}{2}. \tag{4.3.51}$$

Solving these for the group parameters, together with $ad - bc - 1$, we find the moving frame

$$\rho_0 = \frac{\sqrt{x_0 - x_2}}{\sqrt{(x_0 - x_1)(x_1 - x_2)}} \begin{pmatrix} \frac{1}{2} & -\frac{x_1}{2} \\ \frac{x_2 - 2x_1 + x_0}{x_0 - x_2} & \frac{x_0x_1 - 2x_0x_2 + x_1x_2}{x_0 - x_2} \end{pmatrix} \tag{4.3.52}$$

and we take $\rho_k = \mathrm{S}_k\rho_0$.

The generating discrete invariants

The famous, historical invariant for this action, given four points, is the cross ratio,

$$\kappa = \frac{(x_0 - x_1)(x_2 - x_3)}{(x_0 - x_3)(x_2 - x_1)}. \tag{4.3.53}$$

Using the Replacement Rule 2.4.6 we have that

$$\kappa = \frac{1 + 2I^x_{0,3}}{1 - 2I^x_{0,3}}.$$

The Maurer–Cartan matrix is then,

$$K_0 = \iota_0(\rho_1) = \sqrt{\frac{\kappa-1}{4\kappa}} \begin{pmatrix} 1 & \frac{1}{2} \\ -\dfrac{6\kappa+2}{\kappa-1} & 1 \end{pmatrix}. \tag{4.3.54}$$

By (3.4.22), the discrete invariants are generated by κ and its shifts.

We now show how to obtain the recurrence relations for this non-linear action.

The generating differential invariants

We now consider $x_j = x_j(t)$ where t is an invariant parameter. In order to compute the generating differential invariants we first need to compute the induced action on the derivatives with respect to t of $x_j(t)$.

We have that

$$g \cdot x_j' = g \cdot \frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\mathrm{d}(g\cdot x_j)}{\mathrm{d}(g\cdot t)} = \frac{\dfrac{\mathrm{d}(g\cdot x_j)}{\mathrm{d}t}}{\dfrac{\mathrm{d}(g\cdot t)}{\mathrm{d}t}} = \frac{\mathrm{d}}{\mathrm{d}t}(g\cdot x_j) = \frac{x_j'}{(cx_j+d)^2}.$$

Hence we have for

$$\rho_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}$$

that

$$\rho_k \cdot x_j' = \frac{x_j'}{(c_k x_j + d_k)^2}.$$

We define

$$\sigma_j^x := \rho_0 \cdot x_{j,t} = \frac{x_j'}{(c_0 x_j + d_0)^2} = \frac{x_j'(x_1 - x_0)}{(x_0 - x_2)(x_0 - x_1)} \tag{4.3.55}$$

where c_0 and d_0 are given in (4.3.52). In terms of the σ_j^x, the curvature matrix is given by

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right) = \begin{pmatrix} \frac{1}{2}\sigma_2^x - \frac{1}{2}\sigma_0^x & -\sigma_1^x \\ 2\sigma_0^x - 4\sigma_1^x + 2\sigma_2^x & -\frac{1}{2}\sigma_2^x + \frac{1}{2}\sigma_0^x \end{pmatrix}. \tag{4.3.56}$$

We now obtain the recurrence relations for the σ_j^x. We have for all k and j that

$$\rho_k \cdot x_j' = \left.\frac{\widetilde{x_j'}}{(\widetilde{c_k}\widetilde{x_j} + \widetilde{d_k})^2}\right|_{\rho_0} = \frac{\rho_0 \cdot x_j'}{(\widetilde{c_k}|_{\rho_0}\rho_0 \cdot x_j + \widetilde{d_k}|_{\rho_0})^2}.$$

Note that $\widetilde{\rho_k} = \rho_k g^{-1}$ due to the fact that the frames ρ_k are equivariant. Hence

$$\left.\begin{pmatrix} \widetilde{a_k} & \widetilde{b_k} \\ \widetilde{c_k} & \widetilde{d_k} \end{pmatrix}\right|_{\rho_0} = \rho_k \rho_0^{-1} = \rho_k \rho_{m-1}^{-1} \cdots \rho_1 \rho_0^{-1} = (\mathrm{S}^{m-1} K_0) \cdots K_0.$$

In particular, we have

$$\mathrm{S}\sigma_0^x = \rho_1 \cdot x_1' = \frac{\rho_0 \cdot x_1'}{((K_0)_{2,1}\rho_0 \cdot x_1 + (K_0)_{2,2})^2} = \frac{4\kappa}{\kappa - 1}\sigma_1^x \tag{4.3.57}$$

since $\rho_0 \cdot x_1 = 0$ and $\rho_0 \cdot x_1' = \sigma_1^x$. Next,

$$\mathrm{S}_2\sigma_0^x = \rho_2 \cdot x_2' = \frac{\rho_0 \cdot x_2'}{(((\mathrm{S}K_0)K_0)_{2,1}\rho_0 \cdot x_2 + ((\mathrm{S}K_0)K_0)_{2,2})^2} = \frac{\kappa_1(\kappa - 1)}{(\kappa_1 - 1)\kappa}\sigma_2^x \tag{4.3.58}$$

where we have used the normalization equations, $\rho_0 \cdot x_1 = 0$ and $\rho_0 \cdot x_2 = -1/2$.

Similarly, one can prove that

$$\mathrm{S}\sigma_1^x = \frac{\kappa - 1}{4\kappa}\sigma_2^x. \tag{4.3.59}$$

We can now calculate the differential difference syzygy. Calculating (3.4.33) and equating components and using the syzygies (4.3.57), (4.3.57) and (4.3.59), we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\kappa &= \frac{\kappa(\kappa-1)\kappa_1(\kappa_2-1)}{\kappa_2(\kappa_1-1)}\mathrm{S}_3\sigma_0^x + \frac{\kappa(\kappa_1-1)}{\kappa_1}\mathrm{S}_2\sigma_0^x - (\kappa-1)\mathrm{S}\sigma_0^x - \kappa(\kappa-1)\sigma_0^x \\ &= \mathcal{H}\sigma_0^x \end{aligned} \tag{4.3.60}$$

where this defines the linear difference operator $\mathcal{H}$.

The Euler–Lagrange equations and the conservation laws

We consider a Lagrangian of the form

$$\mathcal{L}[x] = \sum L(\kappa, \kappa_1, \ldots, \kappa_J).$$

From (3.5.11) we have that the Euler–Lagrange equation is

$$0 = \mathcal{H}^*(\mathrm{E}_\kappa(L)) = \mathrm{S}_{-3}(\alpha\mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\beta\mathrm{E}_\kappa(L)) + \mathrm{S}_{-1}(\gamma\mathrm{E}_\kappa(L)) + \delta\mathrm{E}_\kappa(L)$$

where

$$\alpha = \frac{\kappa(\kappa-1)\kappa_1(\kappa_2-1)}{\kappa_2(\kappa_1-1)}, \quad \beta = \frac{\kappa(\kappa_1-1)}{\kappa_1}, \quad \gamma = -(\kappa-1), \quad \delta = -\kappa(\kappa-1). \tag{4.3.61}$$

In order to calculate the conservation law, we need the matrix of infinitesimals, which is

$$\Phi_0 = \begin{array}{c} \\ x_0 \end{array} \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 2x_0 & 1 & -x_0^2 \end{pmatrix} \end{array}$$

and so its invariantized form

$$\Phi_0(I) = \begin{array}{c} \\ x_0 \end{array} \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 1 & 1 & -\frac{1}{4} \end{pmatrix} \end{array}.$$

From (3.5.4) we have that the boundary terms are of the form

$$\begin{aligned} A_{\mathcal{H}}(\mathrm{E}_\kappa(L), \sigma_0^x) = &\left(\mathrm{S}_{-3}(\gamma \mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\beta \mathrm{E}_\kappa(L)) + \mathrm{S}_{-1}(\alpha \mathrm{E}_\kappa(L))\right) \sigma_0^x \\ &+ \left(\mathrm{S}_{-1}(\beta \mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\alpha \mathrm{E}_\kappa(L))\right) \mathrm{S}\sigma_0^x + \mathrm{S}_{-1}(\alpha \mathrm{E}_\kappa(L)) \mathrm{S}_2 \sigma_0^x. \end{aligned} \tag{4.3.62}$$

Hence by (3.7.4) the conservation law is

$$\begin{aligned} \mathbf{k} = &\left(\mathrm{S}_{-1}(\gamma \mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\beta \mathrm{E}_\kappa(L)) + \mathrm{S}_{-1}(\alpha \mathrm{E}_\kappa(L))\right) \Phi_0(I) \mathcal{A}d(\rho_0) \\ &+ \left(\mathrm{S}_{-3}(\beta \mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\alpha \mathrm{E}_\kappa(L))\right) \Phi_0(I) \left(\mathrm{S}\mathcal{A}d(\rho_0)\right) \\ &+ \mathrm{S}_{-1}(\alpha \mathrm{E}_\kappa(L)) \Phi_0(I) \left(\mathrm{S}_2 \mathcal{A}d(\rho_0)\right). \end{aligned} \tag{4.3.63}$$

Using

$$\begin{aligned} \mathrm{S}\mathcal{A}d(\rho_0) = &\mathcal{A}d(\rho_1) = \mathcal{A}d(K_0)\mathcal{A}d(\rho_0), \\ \mathrm{S}_2\mathcal{A}d(\rho_0) = &\mathcal{A}d(\mathrm{S}(K_0))\mathcal{A}d(K_0)\mathcal{A}d(\rho_0) \end{aligned}$$

and collecting terms, we obtain the conservation law of the form

$$\mathbf{k} = \mathbf{V}(I)\mathcal{A}d(\rho_0) \tag{4.3.64}$$

where this defines the vector $\mathbf{V}(I) = (V_0^1 \;\; V_0^2 \;\; V_0^3)$ and where

$$\mathcal{A}d(\rho_0) = \begin{pmatrix} \frac{x_1^2 - x_0x_2}{(x_0-x_1)(x_1-x_2)} & \frac{2x_1 - x_2 - x_0}{2(x_0-x_1)(x_1-x_2)} & \frac{x_1(2x_0x_2 - x_1(x_0+x_2))}{2(x_0-x_1)(x_1-x_2)} \\ \frac{(x_0-x_2)x_1}{2(x_0-x_1)(x_1-x_2)} & \frac{x_0-x_2}{4(x_0-x_1)(x_1-x_2)} & \frac{x_1^2(x_2-x_0)}{4(x_0-x_1)(x_1-x_2)} \\ \frac{2(x_2-2x_1+x_0)((x_1-2x_2)x_0+x_1x_2)}{(x_0-x_2)(x_0-x_1)(x_1-x_2)} & -\frac{(x_2-2x_1+x_0)^2}{(x_0-x_2)(x_0-x_1)(x_1-x_2)} & \frac{((x_1-2x_2)x_0+x_1x_2)^2}{(x_0-x_2)(x_0-x_1)(x_1-x_2)} \end{pmatrix}.$$

Explicitly, $\mathbf{V}(I)$ is given by

$$\begin{aligned}\mathbf{V}(I) = \begin{pmatrix} 1 & 1 & \frac{1}{4} \end{pmatrix} \{ & (\mathrm{S}_{-1}(\gamma \mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\beta \mathrm{E}_\kappa(L)) + \mathrm{S}_{-1}(\alpha \mathrm{E}_\kappa(L))) \\ & + (\mathrm{S}_{-3}(\beta \mathrm{E}_\kappa(L)) + \mathrm{S}_{-2}(\alpha \mathrm{E}_\kappa(L)))\mathcal{A}d(K_0) \\ & + \mathrm{S}_{-1}(\alpha \mathrm{E}_\kappa(L))\mathcal{A}d(\mathrm{S}(K_0))\mathcal{A}d(K_0)\}\end{aligned}$$

where

$$\mathcal{A}d(K_0) = \begin{pmatrix} -\dfrac{\kappa+1}{2\kappa} & \dfrac{3\kappa+1}{2\kappa} & \dfrac{\kappa-1}{8\kappa} \\ \dfrac{-\kappa+1}{4\kappa} & \dfrac{\kappa-1}{4\kappa} & \dfrac{-\kappa+1}{16\kappa} \\ -\dfrac{3\kappa+1}{\kappa} & -\dfrac{(3\kappa+1)^2}{(\kappa-1)\kappa} & \dfrac{\kappa-1}{4\kappa} \end{pmatrix}$$

and where

$$\mathcal{A}d(\mathrm{S}(K_0))\mathcal{A}d(K_0) = \frac{1}{\kappa\kappa_1} \begin{pmatrix} \frac{(1-\kappa_1)\kappa+\kappa_1+1}{2} & \frac{(1-3\kappa_1)\kappa^2+2(1-\kappa_1)\kappa+1}{2(\kappa-1)} & \frac{(\kappa_1+1)(1-\kappa)}{8} \\ \frac{(\kappa_1-1)(\kappa+1)}{4} & \frac{(\kappa_1-1)(\kappa+1)^2}{4(\kappa-1)} & \frac{(\kappa_1+1)(1-\kappa)}{16} \\ \frac{(3\kappa-1)\kappa_1^2+2(\kappa-1)\kappa_1-\kappa-1}{\kappa_1-1} & -\frac{(\kappa_1(3\kappa-1)-\kappa-1)^2}{(\kappa_1-1)(\kappa-1)} & \frac{(\kappa-1)(\kappa_1+1)^2}{4(\kappa_1-1)} \end{pmatrix}.$$

The general solution

If we can solve for the discrete frame (ρ_k) then we have

$$x_k = \rho_k^{-1} \cdot \frac{1}{2} = \frac{d_k - 2b_k}{2a_k - c_k} \tag{4.3.65}$$

since $\rho_k \cdot x_k = \dfrac{1}{2}$ is the normalization equation.

Recall that the Adjoint representation in this example matches the one for (4.3.1) as is the adjoint representation of the same Lie group. Therefore we make use of the simplification of the algebraic equations for the group parameters in (4.3.21). However, the Maurer–Cartan matrix is different, and so the recurrence relations needed to compute the solution are different. Nevertheless, we again find that the remaining recurrence relations are diagonalisable, and are therefore easily solved.

We again have equations (4.3.22a)–(4.3.22d), where now the V_0^i are those of equation

(4.3.64). Recall that these equations are of the form,

$$k_1^2 + 4k_2k_3 - \left(V_0^1\right)^2 - 4V_0^2V_0^3 = 0, \tag{4.3.66a}$$

$$k_3c_0^2 - k_1c_0d_0 - k_2d_0^2 + V_0^2 = 0, \tag{4.3.66b}$$

$$2b_0V_0^2 - 2c_0k_3 + (k_1 - V_0^1)d_0 = 0, \tag{4.3.66c}$$

$$2a_0V_0^2 - c_0(k_1 + V_0^1) - 2k_2d_0 = 0. \tag{4.3.66d}$$

The recurrence relation is $\rho_1 = K_0\rho_0$, explicitly:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \sqrt{\frac{\kappa-1}{4\kappa}} \begin{pmatrix} 1 & \frac{1}{2} \\ -\dfrac{6\kappa+2}{\kappa-1} & 1 \end{pmatrix} \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}.$$

Therefore

$$c_1 = \sqrt{\frac{\kappa-1}{4\kappa}}\left(-\frac{6\kappa+2}{\kappa-1}a_0 + c_0\right), \qquad d_1 = -\sqrt{\frac{\kappa-1}{4\kappa}}\left(-\frac{6\kappa+2}{\kappa-1}b_0 + d_0\right).$$

Using these to eliminate a_0 and b_0 from (4.3.66c) and (4.3.66d), leads to the linear system,

$$\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = Q\Lambda_0Q^{-1}\begin{pmatrix} c_0 \\ d_0 \end{pmatrix}$$

where Q is a constant matrix

$$Q = \frac{1}{2\mu}\begin{pmatrix} \mu + k_1 & \mu - k_1 \\ 2k_3 & -2k_3 \end{pmatrix}$$

and $\Lambda_0 = (\lambda_0^1, \lambda_0^2)$ where

$$\begin{aligned} \lambda_0^1 &= \frac{1}{2\sqrt{\kappa-1}\sqrt{\kappa}\,V_0^2}\left(-(3\kappa+1)\left(\mu + V_0^1\right) + (\kappa-1)V_0^2\right), \\ \lambda_0^2 &= \frac{1}{2\sqrt{\kappa-1}\sqrt{\kappa}\,V_0^2}\left((3\kappa+1)\left(\mu - V_0^1\right) + (\kappa-1)V_0^2\right) \end{aligned}$$

where we have set

$$\mu = \sqrt{k_1^2 + 4k_2k_3}.$$

We have then that

$$\begin{pmatrix} c_k \\ d_k \end{pmatrix} = Q\begin{pmatrix} \prod_{l=0}^{k-1}\lambda_l^1 & 0 \\ 0 & \prod_{l=0}^{k-1}\lambda_l^2 \end{pmatrix}Q^{-1}\begin{pmatrix} c_0 \\ d_0 \end{pmatrix}$$

where in this last, c_0 and d_0 are the initial data and $\lambda_k^l = \mathrm{S}_k\lambda_0^l$.

Substituting these into the k^{th} shifts of (4.3.66c) and (4.3.66d), specifically,

$$2b_kV_k^2 - 2c_kk_3 + (k_1 - V_k^1)d_k = 0, \tag{4.3.67a}$$

$$2a_kV_k^2 - c_k(k_1 + V_k^1) - 2k_2d_k = 0 \tag{4.3.67b}$$

yields the expressions for a_k and b_k needed to obtain, finally, x_k given in (4.3.65).

Commuting Flows on the Curvature Invariants

In this chapter, we first show how to construct the correction matrix in the discrete case. We also compare the evolutions on the Lie group and on the Lie algebra in the smooth framework with the discrete one and we prove that the relationship between a flow and its induced curvature flow is in terms of a linear shift operator depending only on curvature invariants. We analyse the condition for discrete curve evolutions to commute in terms of a discrete moving frame and give an alternate proof of Theorem 11 in Mansfield and Van der Kamp, [73] for the smooth case and prove the theorem for the discrete case. We use a very simple Lie group action as a running example. Finally, we exhibit an example in order to illustrate the theory developed in this chapter and relate this examples to discrete integrable systems.

5.1 Introduction

Discrete moving frames have been proven useful for the study of discrete integrable systems, which arise as analogues of curvature flows for polygon evolutions in homogeneous spaces (see Beffa and Wang, [6]). Most integrable systems have a natural discretization that preserves the integrability features, described by difference equations. For instance, the Toda Lattice (see Toda, [105])

$$\frac{\mathrm{d}^2 u_s}{\mathrm{d}t^2} = \exp(u_{s-1} - u_s) - \exp(u_s - u_{s+1}) \tag{5.1.1}$$

and the Volterra Lattice (see Manakov, [68])

$$\frac{\mathrm{d}^2 q_s}{\mathrm{d}t^2} = q_s(q_{s+1} - q_{s-1}) \tag{5.1.2}$$

are the most famous discretizations of the well known Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} - 6uu_x = 0.$$

Using Flaschka coordinates (see Flaschka, [28], [29])

$$q_s = \frac{\mathrm{d}u_s}{\mathrm{d}t}, \qquad p_s = \exp(u_s - u_{s+1})$$

equation (5.1.1) can be re-written of the form

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = p_s(q_s - q_{s+1}), \qquad \frac{\mathrm{d}q_s}{\mathrm{d}t} = p_{s-1} - p_s$$

which is a complete discrete integrable system (see Flaschka, [28],[29], and Manakov, [68]). Further, there exists a relationship between equation (5.1.2) and

$$\frac{\mathrm{d}p_s}{\mathrm{d}t} = p_s^2(p_{s+1} - p_{s-1})$$

by the Miura transformation $q_s = p_s p_{s-1}$ which is an integrable discretization of the modified KdV equation

$$u_t + u_{xxx} + 6\sigma u^2 u_x = 0, \qquad \sigma = \pm 1.$$

In this chapter we understand integrability as the existence of an infinite set of commuting evolutions. Our results inform the discussion on when discrete equivariant flows and their invariantization are both integrable as is commonly observed.

The smooth case was previously studied in Mansfield and van de Kamp, [73] where a method that provides the evolution equation for the curvature invariants of a curve as in (2.59) is presented. It is shown that it derives from a syzygy between sets of invariants. For instance, in the case of the linear action of $\mathrm{S}L(2)$ on (x, u), the syzygy (2.58) has the form

$$\mathcal{D}_t Q^x - \mathcal{D}_x Q^t = -2I_t Q^x + [Q^t, Q^x]$$

provides the relation

$$\mathcal{D}_t \kappa = (\mathcal{D}_x^2 - 4\kappa) I_t.$$

The study in Mansfield and Van der Kamp, [73] further makes a comparison between the symmetry condition of the curve evolutions and the curvature evolutions. Given two invariant curve evolutions it is shown that the symmetry condition for curvature evolutions to commute appears as a differential consequence of the syzygy between different evolution invariants. For the same example, it can be verified that

$$\mathcal{D}_s(\mathcal{D}_x^2 - 4\kappa)I_t - \mathcal{D}_t(\mathcal{D}_x^2 - 4\kappa)I_s = (\mathcal{D}_x^2 - 4\kappa)(\mathcal{D}_s F_t[\kappa] - \mathcal{D}_t F_s[\kappa])$$

where

$$I_t = F_t[\kappa] \quad \text{and} \quad I_s = F_s[\kappa]$$

are constraints imposed in order to describe the curve moving in different time directions.

In this chapter we derive the discrete analogue of the results appearing in Mansfield and Van der Kamp, [73] and show that the condition for two curvature evolution to commute is a differential consequence of the condition for two curve evolutions to commute.

In §5.2, we present our running example in both smooth and discrete formats. In §5.3, we explore the invariant differentiation, we present the correction terms and the correction matrix for the discrete case and prove their construction. In §5.4, we compare the evolutions on the Lie algebra and the evolutions on the Lie group in the continuous and in the discrete case, as well as the differential syzygy (2.58) with the differential–difference syzygy (3.4.33). We also present one of the main theorems of this chapter regarding the reduced form of the differential – difference syzygy. In §5.5, we show that the condition for two curvature evolution to commute is a differential consequence of the condition for two curve evolutions to commute. In §5.7, we illustrate the theory using the $SL(2)$ linear action and we relate it to discrete integrable systems.

5.2 Presentation of our running example: linear transformations

We first introduce the linear transformations group action in the continuous and discrete case, which will be our running example throughout the chapter.

5.2.1 The smooth case

Example 5.2.1. *Consider the group of linear transformations acting on curves $(x, u(x))$ such that*

$$x \to x = \widetilde{x}, \qquad u \to \lambda u + \epsilon = \widetilde{u}.$$

Since x is invariant, the prolongation action is simple to calculate. We obtain

$$\widetilde{u_J} = \lambda u_J$$

where J is the index of differentiation.

Let us take the cross section $\mathcal{K}$ to be the coordinate plane $u = 0$ and $u_x = 1$. Thus the *normalization equations are*

$$\widetilde{u} = 0, \qquad \widetilde{u_s} = 1. \tag{5.2.2}$$

Solving equations (5.2.2) *in terms of* $u, u_x, \ldots$ *yields*

$$\lambda = \frac{1}{u_x}, \qquad \epsilon = -\frac{u}{u_x}.$$

In matrix form, the frame is obtained by substituting the values of the parameters on the frame into a matrix representation of the generic group element. For a standard representation of the group of linear transformations

$$g = \begin{pmatrix} \lambda & \epsilon \\ 0 & 1 \end{pmatrix} \tag{5.2.3}$$

we obtain

$$\rho = \begin{pmatrix} \dfrac{1}{u_x} & -\dfrac{u}{u_x} \\ 0 & 1 \end{pmatrix}.$$

Note that

$$\rho(\widetilde{u}, \widetilde{u_x}) = \begin{pmatrix} \dfrac{1}{\widetilde{u_x}} & -\dfrac{\widetilde{u}}{\widetilde{u_x}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\lambda u_x} & -\dfrac{\lambda u + \epsilon}{\lambda u_x} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{u_x} & -\dfrac{u}{u_x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{\lambda} & -\dfrac{\epsilon}{\lambda} \\ 0 & 1 \end{pmatrix} = \rho(u, u_x) g^{-1}$$

which is the equivariance of a right frame for a left action.

For the linear transformation group (5.2.1) *the invariants are of the form*

$$\iota(u) = \rho \cdot u = 0, \qquad \iota(u_x) = \rho \cdot u_x = 1, \qquad \iota(u_J) = \rho \cdot u_J = \frac{u_J}{u_x}.$$

5.2.2 The discrete case

Example 5.2.4. *Consider the group* G *of linear transformations and its action on the prolongation space* $P_n^{(0)}(\mathbb{R})$ *with coordinate* u_0*. On this prolongation space, the action is given by*

$$u_0 \mapsto \lambda u_0 + \epsilon = \widetilde{u_0}. \tag{5.2.5}$$

Taking the normalization equations $\widetilde{u_0} = 0$, $\widetilde{u_1} = 1$, *and solving for the parameters* λ *and* ϵ *yields the following moving frame*

$$\rho_0 = \begin{pmatrix} -\dfrac{1}{u_0 - u_1} & \dfrac{u_0}{u_0 - u_1} \\ 0 & 1 \end{pmatrix}$$

where the solutions for λ and ϵ have been substituted into (5.2.3)*. The invariants are of the form*

$$I^u_{0,0} = \rho_0 \cdot u_0 = 0, \qquad I^u_{0,1} = \rho_0 \cdot u_1 = 1, \qquad I^u_{0,j} = \rho_0 \cdot u_j = \frac{u_j - u_0}{u_1 - u_0}.$$

The Maurer-Cartan matrix is

$$K_0 = \iota_0 \left(\begin{pmatrix} -\dfrac{1}{u_1 - u_2} & \dfrac{u_1}{u_1 - u_2} \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} -\dfrac{1}{1 - I^u_{0,2}} & \dfrac{1}{1 - I^u_{0,2}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\kappa & \kappa \\ 0 & 1 \end{pmatrix} \in G$$

where we have set κ to be $\dfrac{1}{1 - I^u_{0,2}}$. Suppose now that $u_j = u_j(t)$. The first order differential invariants are of the form

$$I^u_{0,j;t} = \rho_0 \cdot u'_j = \begin{pmatrix} -\dfrac{1}{u_0 - u_1} & \dfrac{u_0}{u_0 - u_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u'_j \\ 0 \end{pmatrix} = \frac{u'_j}{u_1 - u_0}.$$

For the next calculation we need the invariant $I^u_{0,1;t}$ expressed in terms of $I^u_{0,0;t}$ which we will denote σ_t from now on. We have that

$$\mathrm{S}\sigma_t = \rho_1 \cdot u'_1 = \rho_1 \rho_0 \rho_0^{-1} \cdot u'_1 = K_0 I^u_{0,1;t} = -\kappa I^u_{0,1;t}$$

and therefore $I^u_{0,1;t} = -\dfrac{\mathrm{S}\sigma_t}{\kappa}$.

The curvature matrix is

$$\begin{aligned} N_{0;t} &= \iota_0 \begin{pmatrix} \dfrac{u'_0 - u'_1}{(u_0 - u_1)^2} & \dfrac{u'_0(u_0 - u_1) - u_0\,(u'_0 - u'_1)}{(u_0 - u_1)^2} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I^u_{0,0;t} - I^u_{0,1;t} & -I^u_{0,0;t} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_t + \dfrac{\mathrm{S}\sigma_t}{\kappa} & -\sigma_t \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \end{aligned} \tag{5.2.6}$$

where $\mathfrak{g}$ is the Lie algebra of G.

5.3 Invariant differentiation

In this section we introduce the discrete analogue of §2.4.2. In the continuous case (definition 4.5.3, [70]) a set of distinguished invariant operators is defined by evaluating the transformed total differential operators of the frame. In the discrete case, the total derivative with respect to t plays the role of the linear derivations. We have

$$\mathcal{D}_t = \mathrm{D}_t = \frac{\mathrm{d}}{\mathrm{d}t} = \widetilde{\frac{\mathrm{d}}{\mathrm{d}t}}\Big|_{g=\rho_k}.$$

Recall from (3.4.2) and (3.4.27)

$$I_{k,j} = \rho_k \cdot z_j \qquad \text{and} \qquad I_{k,j;t} = \rho_k \cdot z'_j$$

and from (3.4.28) recall that the invariantization and derivation do not commute, i.e,

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{k,j} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\widetilde{z}_j\Big|_{g=\rho_k}\right) \neq \left(\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{z}_j\right)\Big|_{g=\rho_k} = \widetilde{z}'_j\Big|_{g=\rho_k} = I_{k,j;t}$$

where $I_{k,j} = I_{k,j}(t)$ and $I_{k,j;t} = I_{k,j;t}(t)$. We define the time-correction terms $M_{k,j;t}$ by

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{k,j} = M_{k,j;t} + I_{k,j;t}. \tag{5.3.1}$$

Note that differentiating $I_{k,j} = \rho_k \cdot z_j$ with respect to t and using (3.4.27) and (2.4.6) we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I_{k,j} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_k \cdot z_j\right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_k\right) \cdot z_j + \rho_k \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} z_j\right) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_k\right) \rho_k^{-1}\rho_k \cdot z_j + \rho_k\rho_k^{-1}\rho_k \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} z_j\right) = \iota_k\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_k\right) \cdot z_j\right) + I_{k,j;t}.
\end{aligned}$$

Isolating $\iota_k\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_k\right) \cdot z_j\right)$ we obtain the expression for $M_{k,j;t}$.

Example 5.3.2. *The correction terms are of the form*

$$\begin{aligned} M_{0,j;t} &= \iota_0\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right)u_j\right) \\ &= \iota_0\left(\left(\begin{array}{cc} \dfrac{u_0'-u_1'}{(u_0-u_1)^2} & \dfrac{u_0'(u_0-u_1)-u_0\,(u_0'-u_1')}{(u_0-u_1)^2} \\ 0 & 0 \end{array}\right)\left(\begin{array}{c} u_j \\ 1 \end{array}\right)\right) \\ &= \iota_0\left(\begin{array}{c} \dfrac{(u_j-u_0)\,(u_0'-u_1')+u_0'(u_0-u_1)}{(u_0-u_1)^2} \\ 0 \end{array}\right) \\ &= \left(\begin{array}{c} I^u_{0,j}(\sigma_t - I^u_{0,1;t})-\sigma_t \\ 0 \end{array}\right). \end{aligned}$$

For instance, the correction terms $M_{0,0;t}$ and $M_{0,1;t}$ are of the form

$$M_{0,0;t} = -\sigma_t \qquad \textit{and} \qquad M_{0,1;t} = -I^u_{0,1;t} \tag{5.3.3}$$

as expected. Note that we have ignored the last component of the vector.

Proposition 5.3.4. *Assuming that t is the only smooth parameter and that the normalization equations do not involve t, there exist a $1\times R$ correction row $\mathbf{K}=\{K_l\}$ where $l=1,...,R$ such that*

$$M_{k,j;t} = \sum_{l=1}^{R} K_l \phi_{k,j;l} \tag{5.3.5}$$

where

$$\phi_{k,j;l} = \iota_k\left(\frac{\partial \widetilde{u_j}}{\partial a^l}\right). \tag{5.3.6}$$

Proof. By definition we have that

$$I_{k,j} = \rho_k \cdot z_j = g\cdot z_j\Big|_{g=\rho_k} = \widetilde{z_j}\Big|_{g=\rho_k}.$$

Note that we can write $\widetilde{z_j}$ as a function depending on the variables z_m and the group parameters a^l. We set $\widetilde{z_j} = f_j(z_m, a^l)$. Therefore

$$I_{k,j} = f_j(z_m, a^l)\Big|_{g=\rho_k}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{k,j} = \frac{\mathrm{d}f_j(z_m,a^l)}{\mathrm{d}t}\Big|_{g=\rho_k} + \sum_l \frac{\partial f_j(z_m,a^l)}{\partial a^l}\frac{\mathrm{d}a^l}{\mathrm{d}t}\Big|_{g=\rho_k} = I_{k,j;t} + \sum_l \phi_{k,j;l}\frac{\mathrm{d}a^l}{\mathrm{d}t}\Big|_{g=\rho_k}.$$

Setting

$$\left.\frac{\mathrm{d}a^l}{\mathrm{d}t}\right|_{g=\rho_k} = K_l \tag{5.3.7}$$

we obtain the required result. □

The correction row $\mathbf{K}$ with respect to the moving frame ρ_k can be calculated without explicit knowledge of the frame as follows: Suppose the ϑ ordered variables appearing in the normalization equations are ζ_j, where $j = 1, ..., \vartheta$. The row $\mathbf{T} = \{T_j\}$ is the invariant $1 \times \vartheta$ matrix

$$T_j = \iota_k \left(\frac{\mathrm{d}}{\mathrm{d}t}\zeta_j(t)\right). \tag{5.3.8}$$

We denote

$$\mathbf{\Phi} = \{\Phi_{jl}\} = \iota_k \left(\frac{\partial \widetilde{\zeta_j}}{\partial a^l}\right)$$

the $R \times \vartheta$ matrix of invariant generators. Let ψ_λ, $\lambda = 1, ..., R$ be the normalization equations, the matrix $\mathbf{J} = \{J_{j\lambda}\}$ is the invariant $\vartheta \times R$ matrix such that

$$J_{j\lambda} = \iota_k \left(\frac{\partial \psi_\lambda}{\partial \zeta_j}\right). \tag{5.3.9}$$

Theorem 5.3.10. *The correction matrix which provides the error terms in the process of invariant differentiation in* (5.3.5) *is given by*

$$\mathbf{K} = -\mathbf{TJ}(\mathbf{\Phi J})^{-1}. \tag{5.3.11}$$

Proof. Recall that the normalization equations are of the form

$$\psi_\lambda(g \cdot z_m) = 0, \qquad \text{for} \qquad \lambda = 1, ..., R. \tag{5.3.12}$$

They depend on the variables z_m, but also depend on the parameters of the group a^l. Therefore, we can rewrite the normalization equations of the form $\Psi_\lambda(\zeta_j, a^l) = 0$, where $\lambda = 1, ..., R$ and where we have denoted the variables appearing in the normalization equations by ζ_j . Differentiating this equation with respect to t we obtain

$$0 = \sum_j \frac{\mathrm{d}\zeta_j}{\mathrm{d}t}\frac{\partial \Psi_\lambda}{\partial \zeta_j} + \sum_l \frac{\mathrm{d}a^l}{\mathrm{d}t}\frac{\partial \Psi_\lambda}{\partial a^l} \tag{5.3.13}$$

Now, from (5.3.12)

$$\frac{\partial \psi_\lambda}{\partial a^l} = \frac{\partial g \cdot z_m}{\partial a^l}\frac{\partial \psi_\lambda}{\partial g \cdot z_m}. \tag{5.3.14}$$

Taking into account that $\psi_\lambda(g \cdot z_m) = \Psi_\lambda(\zeta_j, a^l)$, substituting (5.3.14) into (5.3.13) and using the invariantizing operator ι_k

$$0 = \mathbf{TJ} + \mathbf{K}(\mathbf{\Phi J}).$$

Isolating $\mathbf{K}$ we obtain the required result. □

Example 5.3.15. *The variables appearing in the normalization equations* (5.2.2) *are* u_0 *and* u_1*. Therefore,*

$$\zeta_1 = u_0 \qquad and \qquad \zeta_2 = u_1.$$

Thus $\vartheta = 2$*. Note that as the group action depends on two paramenters we have that* $R = 2$*. Hence,*

$$\mathbf{T} = \begin{pmatrix} \iota_0\left(\frac{\mathrm{d}}{\mathrm{dt}}\zeta_1\right) & \iota_0\left(\frac{\mathrm{d}}{\mathrm{dt}}\zeta_2\right) \end{pmatrix} = \begin{pmatrix} I^u_{0,0;t} & I^u_{0,1;t} \end{pmatrix}.$$

The matrix of invariant generators has the form

$$\mathbf{\Phi} = \begin{pmatrix} \iota_0\left(\frac{\partial\widetilde{\zeta_1}}{\partial a_1}\right) & \iota_0\left(\frac{\partial\widetilde{\zeta_1}}{\partial a_2}\right) \\ \iota_0\left(\frac{\partial\widetilde{\zeta_2}}{\partial a_1}\right) & \iota_0\left(\frac{\partial\widetilde{\zeta_2}}{\partial a_2}\right) \end{pmatrix} = \begin{pmatrix} \iota_0(u_0) & 1 \\ \iota_0(u_1) & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

where $a_1 = \lambda$ *and* $a_2 = \epsilon$ *and finally*

$$\mathbf{J} = \begin{pmatrix} \iota_0\left(\frac{\partial\psi_1}{\partial\zeta_1}\right) & \iota_0\left(\frac{\partial\psi_1}{\partial\zeta_2}\right) \\ \iota_0\left(\frac{\partial\psi_2}{\partial\zeta_1}\right) & \iota_0\left(\frac{\partial\psi_2}{\partial\zeta_2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore

$$\mathbf{K} = -\mathbf{TJ}(\mathbf{\Phi J})^{-\mathbf{1}} = \begin{pmatrix} I^u_{0,0;t} - I^u_{0,1;t} & -I^u_{0,0;t} \end{pmatrix}.$$

Thus the correction terms can be calculated as follow

$$\begin{aligned} M_{0,0;t} &= K_1\phi_{0,0;1} + K_2\phi_{0,0;2} = (I^u_{0,0;t} - I^u_{0,1;t})\cdot 0 + (-I^u_{0,0;t})\cdot 1 = -\sigma_t, \\ M_{0,1;t} &= K_1\phi_{0,1;1} + K_2\phi_{0,1;2} = (I^u_{0,0;t} - I^u_{0,1;t})\cdot 1 + (-I^u_{0,0;t})\cdot 1 = -I^u_{0,1;t} \end{aligned}$$

that match with those ones calculated before in (5.3.3).

Theorem 5.3.16. *The curvature matrix can also be given by*

$$N_k = \sum_l K_l \mathfrak{a}_l \tag{5.3.17}$$

where $\{\mathfrak{a}_l\}$, $j = 1, ..., dim(\mathfrak{g})$ *is the basis of the Lie algebra* $\mathfrak{g}$ *of the group* G.

Proof. On one hand we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho_k(\widetilde{z_m})\Big|_{g=\rho_k} = \frac{\mathrm{d}}{\mathrm{dt}}\rho_k(g\cdot z_m)\Big|_{g=\rho_k} = \frac{\mathrm{d}}{\mathrm{dt}}\rho_k(z_m)\cdot g^{-1}\Big|_{g=\rho_k} = N_k.$$

On the other hand

$$\frac{\mathrm{d}}{\mathrm{dt}}\rho_k(\widetilde{z_m})\Big|_{g=\rho_k} = \sum_{l=1}^{r}\frac{\partial \rho_k}{\partial a^l}\frac{\mathrm{d}a^l}{\mathrm{dt}}\Big|_{g=\rho_k} = \sum_{j=1}^{r} K_l\mathfrak{a}_l.$$

□

Example 5.3.18. *The Lie algebra of the Lie group of linear transformations is spanned by the basis*

$$\left\{\mathfrak{a}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad \mathfrak{a}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right\}.$$

Therefore the curvature matrix can be computed as follows:

$$\begin{aligned} N_0 &= K_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + K_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= (I^u_{0,0;t} - I^u_{0,1;t})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - I^u_{0,0;t}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} I^u_{0,0;t} - I^u_{0,1;t} & -I^u_{0,0;t} \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Note that this matches the matrix obtained in (5.2.6)*.*

Remark 5.3.19. *It is important to note that the order of the elements of the Lie algebra have to match the order of the infinitesimal vector fields in the sense that the Lie bracket multiplication for the* $-\mathfrak{a}_j$ *is the same as the bracket multiplication for the infinitesimal vector fields (see Remark* 5.2.5 *in Mansfield, [70]).*

5.4 Evolutions on the Lie group and on the Lie algebra

In the smooth case, the evolution of the curvature invariants is easily understood in terms of an evolution on the Lie algebra, (see Section 3 in Mansfield and Van der Kamp, [73]). Considering the $1+1$ dimensional case $(x,t)\mapsto z(x,t)$, the maps

$$x \mapsto Q^x := (\mathrm{D}_x\rho(z))\rho(z)^{-1}, \qquad t \mapsto Q^t := (\mathrm{D}_t\rho(z))\rho(z)^{-1} \tag{5.4.1}$$

are curves in the Lie algebra $\mathfrak{g}$ of G.

In the discrete case, discrete curves evolve in the space by the shift operator and inducing a path which allow us to differentiate with respect to the invariant t the discrete curves evolve in the time (see (3.4.2)). In the discrete case N_k plays the role of Q^t, and while both of them are in the Lie algebra, K_k playing the role of Q^x is in the group while Q^x is in the algebra. Therefore, discrete curves evolve in the space in the group and not in the algebra, while the smooth curves evolve in the space in the algebra. Recall that Q^x and Q^t satisfy the syzygy (2.58). We can say that the analogue discrete to (2.58) is (3.4.33).

Remark 5.4.2. *Recall* (3.4.33)

$$\frac{\mathrm{d}}{\mathrm{d}t} K_0 = (\mathrm{S}N_0)K_0 - K_0 N_0.$$

Note that multiplying both sides by K_0^{-1}

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} K_0\right) K_0^{-1} = (\mathrm{S}N_0) - K_0 N_0 K_0^{-1} = (\mathrm{S} - \mathcal{A}d_{K_0}(N_0))N_0$$

we obtain an element in $\mathfrak{g}$ *where* $\mathcal{A}d$ *is the left Adjoint action.*

Theorem 5.4.3. *If the normalization equations do not involve time-derivative invariants then it is always possible to rewrite the syzygy* (3.4.33)

$$\frac{\mathrm{d}}{\mathrm{d}t} K_0 = (SN_0)K_0 - K_0 N_0$$

in its reduced form (3.4.29)

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\kappa} = \mathcal{H}\boldsymbol{\sigma}$$

where $\mathcal{H}$ *is an invariant linear shift operator involving curvature invariants only.*

Proof. On the left hand side the entries of K_0 are the curvature invariants, so the entries of $\frac{\mathrm{d}}{\mathrm{d}t} K_0$ are the derivatives of the curvature invariants, which are the components of $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\kappa}$. On the right hand side, the entries of N_0 will depend on the components of $\sigma_t = I_{0,0;t}, I_{0,1;t}, ..., I_{0,j;t}$.

Notice that we can always write

$$\begin{aligned}I_{0,j;t} &= \rho_0 \cdot z_j' \\ &= \rho_0\rho_1^{-1}\rho_1 \cdot z_j' \\ &= K_0^{-1}\rho_1 \cdot z_j' \\ &= K_0^{-1}\rho_1\rho_2{}^{-1}\rho_2 \cdot z_j' \\ &= K_0^{-1}\mathrm{S}(\rho_0\rho_1{}^{-1})\rho_2 \cdot z_j' \\ &= K_0^{-1}(\mathrm{S}K_0^{-1})\rho_2 \cdot z_j' \\ &= ... \\ &= K_0^{-1}(\mathrm{S}K_0^{-1})(\mathrm{S}_2K_0^{-1})...(\mathrm{S}_{j-1}K_0^{-1})\rho_j \cdot z_j' \\ &= K_0^{-1}(\mathrm{S}K_0^{-1})(\mathrm{S}_2K_0^{-1})...(\mathrm{S}_{j-1}K_0^{-1})\mathrm{S}_j(\rho_0 \cdot z_0') = P\sigma_t,\end{aligned}$$

where P is the invariant linear shift operator matrix of the form

$$K_0^{-1}(\mathrm{S}K_0^{-1})(\mathrm{S}_2K_0^{-1})...(\mathrm{S}_{j-1}K_j^{-1})\mathrm{S}_j$$

involving curvature invariants only. Therefore, we can write every entry of N_0 and $\mathrm{S}N_0$ as linear combinations of $P_j\sigma_t$. Therefore, as the entries of K_0 are the curvature invariants, by equating components and reorganising we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa_\alpha = \sum_\beta P_j^{\alpha,\beta}\sigma_t^\beta.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\kappa} = \mathcal{H}\boldsymbol{\sigma_t}$$

where $\mathcal{H}$ is an invariant matrix shift operator involving curvature invariants only of the form $\mathcal{H} = \{\mathcal{H}\}_{\alpha\beta} = \{P_j^{\alpha,\beta}\}$. □

In (5.4.1) we define the curvature matrices with respect to the parameters x and t. In the case that $\mathcal{D}_x = \mathrm{D}_x$ and $\mathcal{D}_t = \mathrm{D}_t$ we have that

$$\begin{aligned}\mathrm{D}_tQ^x - \mathrm{D}_xQ^t = \mathrm{D}_x\mathrm{D}_t(\rho)\rho^{-1} - \mathrm{D}_x(\rho)\rho^{-1}\mathrm{D}_t(\rho)\rho^{-1} - \mathrm{D}_t\mathrm{D}_x(\rho)\rho^{-1} \\ + \mathrm{D}_t(\rho)\rho^{-1}\mathrm{D}_x(\rho)\rho^{-1} = [Q^t, Q^x]\end{aligned}$$

where we have used the fact that

$$[\mathrm{D}_x, \mathrm{D}_t] = 0.$$

Hence we have the following syzygy

$$\mathrm{D}_tQ^x = \mathrm{D}_xQ^t + [Q^t, Q^x]. \tag{5.4.4}$$

This syzygy along with (3.4.33) motivate the following definitions:

Definition 5.4.5. *Let us define the* $\mathcal{F}$ *operator acting on* $\mathfrak{g}$ *as*

$$\mathcal{F}_{Q^i} = \mathrm{D}_i - ad_{Q^i} \tag{5.4.6}$$

where ad is given by (2.18).

Definition 5.4.7. *We define the discrete* $\mathcal{F}^{\Delta}$ *operator acting on* G *as*

$$\mathcal{F}^{\Delta}_{N_0} = (\mathrm{S} - \mathrm{id})N_0 + ad_{N_0}. \tag{5.4.8}$$

Note that for $i = x$ applying (5.4.6) to Q^t we obtain

$$\mathcal{F}_{Q^x}(Q^t) = \mathrm{D}_x(Q^t) + [Q^t, Q^x]$$

and that applying (5.4.8) to K_0 we obtain

$$\mathcal{F}^{\Delta}_{N_0}(K_0) = (\mathrm{S}N_0)K_0 - K_0N_0.$$

Remark 5.4.9. *It follows from* (5.4.6), (5.4.8) *and* (5.4.3) *that given an expression of the form*

$$C = \mathcal{F}^{\Delta}_{A}(B)$$

where $A \in \mathfrak{g}$ *and* $B \in G$*, after equating components we can always get an expression of the form*

$$C(\sigma_t, \sigma_s) = \mathcal{H}\mathbf{a}.$$

where $\mathbf{a}$ *is is a vector containing the components of* A*,* $\mathcal{H}$ *is the linear shift operator appearing in* (5.4.3) *depending on the components of* B *and their shifts. In the smooth case, it follows from Remark 10 in Mansfield and Van der Kamp, [73] that given an expression of the form*

$$C = \mathcal{F}_A(B)$$

where $A, B \in \mathfrak{g}$*, after equating components we can always get an expression of the form*

$$C(\sigma_t, \sigma_s) = H\mathbf{a}$$

where H *is a linear differential operator.*

Example 5.4.10. *Using the syzygy* (5.4.4) *we can write the evolution of the curvature invariants* κ *in terms of the evolution invariant* σ_t *as follows*

$$\kappa_t = \frac{\kappa\,\mathrm{S}_2\sigma_t}{\mathrm{S}\kappa} + (\kappa-1)\,\mathrm{S}\sigma_t - \kappa\,\sigma_t.$$

Therefore there exists a linear shift operator $\mathcal{H}$ *such that*

$$\kappa_t = \mathcal{H}\sigma_t \tag{5.4.11}$$

where

$$\mathcal{H} = \frac{\kappa\,\mathrm{S}_2}{\mathrm{S}\kappa} + (\kappa-1)\,\mathrm{S} - \kappa.$$

5.5 Lifting integrability

Following the theory developed in Mansfield and Van der Kamp, [73], in this section we answer the question whether integrability of a curvature evolution does lift to the motion of its curve in the discrete case. We also understand integrability as existence of infinitely many generalized symmetries and we prove in the discrete framework that a symmetry of the curvature evolution gives rise to a symmetry of the curve evolution. Suppose now that $z_j = z_j(s,t)$ for all j in $\mathbb{Z}$. The lowest order syzygy between time derivatives of evolution variables is

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}z_0(s,t) - \frac{\partial}{\partial s}\frac{\partial}{\partial t}z_0(s,t) = 0. \tag{5.5.1}$$

Given two evolutions of the discrete curve

$$\frac{\partial}{\partial t}z_0(s,t) = P_t[z_0] \qquad \text{and} \qquad \frac{\partial}{\partial s}z_0(s,t) = P_s[z_0]$$

where $[z_0]$ denotes the dependence of z_0 and its shifts, we say that the curve evolutions commute if

$$\left(\frac{\partial}{\partial t}P_s[z_0]\right)\Big|_{\frac{\partial}{\partial t}z_0(s,t) = P_t[z_0]} - \left(\frac{\partial}{\partial s}P_t[z_0]\right)\Big|_{\frac{\partial}{\partial s}z_0(s,t) = P_s[z_0]} = 0.$$

We will call this condition the *symmetry condition.* Now let us consider curve evolutions that are invariant under a group action. The lowest order syzygy involving invariant time derivatives of the fundamental evolution invariants is

$$C(\sigma_t,\sigma_s) = \frac{\partial}{\partial t}\sigma_s - \frac{\partial}{\partial s}\sigma_t + M_{0;s,0;t} - M_{0;t,0;s} = 0, \tag{5.5.2}$$

where

$$\sigma_t = I_{0,0;t} = \rho_0 \cdot \frac{\partial}{\partial t} z_0(s,t) \qquad \text{and} \qquad \sigma_s = I_{0,0;s} = \rho_0 \cdot \frac{\partial}{\partial s} z_0(s,t)$$

and where

$$M_{0;t,0;s} := \frac{\partial}{\partial t}\rho_0 \left(\frac{\partial}{\partial s} z_0(s,t) \right) \qquad \text{and} \qquad M_{0;t,0;s} := \frac{\partial}{\partial s}\rho_0 \left(\frac{\partial}{\partial t} z_0(s,t) \right).$$

Further, we will call (5.5.2) fundamental syzygy.

Example 5.5.3. *The correction terms are of the form*

$$M_{0;s,0;t} = \iota_0 \left(\frac{\partial}{\partial s}\rho_0 \left(\frac{\partial}{\partial t} u_0(s,t) \right) \right) = \sigma_t \sigma_s + \frac{\sigma_t \mathrm{S} \sigma_s}{\kappa}.$$

Analogously

$$M_{0;t,0;s} = \sigma_s \sigma_t + \frac{\sigma_s \mathrm{S} \sigma_t}{\kappa}.$$

Therefore the fundamental syzygy is

$$C(\sigma_t, \sigma_s) = \frac{\partial}{\partial t}\sigma_s - \frac{\partial}{\partial s}\sigma_t + \frac{\sigma_t \mathrm{S}\sigma_s}{\kappa} - \frac{\sigma_s \mathrm{S} \sigma_t}{\kappa} = 0.$$

Now suppose that two invariant evolution of a curve are given by

$$\sigma_t = F_t[\kappa] \qquad \text{and} \qquad \sigma_s = F_s[\kappa] \tag{5.5.4}$$

where $[\kappa]$ denotes the dependence of the curvature invariants and their shifts. Recall that under the conditions of (5.4.3) we have

$$\frac{\partial}{\partial t}\kappa = \mathcal{H}\sigma_t \qquad \text{and} \qquad \frac{\partial}{\partial s}\kappa = \mathcal{H}\sigma_s.$$

Therefore using (5.5.4) we have

$$\frac{\partial}{\partial t}\kappa = \mathcal{H}F_t \qquad \text{and} \qquad \frac{\partial}{\partial s}\kappa = \mathcal{H}F_s.$$

The *invariant symmetry condition* is given by

$$\left(\frac{\partial}{\partial t} F_s \right) \Big|_{\frac{\mathrm{d}}{\mathrm{d}t}\kappa = \mathcal{H}F_t} - \left(\frac{\partial}{\partial s} F_t \right) \Big|_{\frac{\mathrm{d}}{\mathrm{d}s}\kappa = \mathcal{H}F_s} + (M_{0;s,0;t} - M_{0;t,0;s}) \Big|_{\sigma_t = F_t, \sigma_s = F_s}.$$

In other words

$$C(\sigma_t, \sigma_s) \Big|_{\sigma_t = F_t, \sigma_s = F_s} = 0.$$

Remark 5.5.5. *When the action of the Lie group neither depends nor acts on the variables t and s and no evolution variables appear in the normalization equations, the identities* (5.5.1) *and* (5.5.2) *are related by*

$$\rho_0\left(\frac{\partial}{\partial t}\frac{\partial}{\partial s}z_0(s,t)-\frac{\partial}{\partial s}\frac{\partial}{\partial t}z_0(s,t)\right)=\frac{\partial}{\partial t}\sigma_s-\frac{\partial}{\partial s}\sigma_t+M_{0;s,0;t}-M_{0;t,0;s}. \tag{5.5.6}$$

Consider for the smooth case the curvature matrices with respect to the parameters s and t. From (5.4) interchanging variables and rearranging the terms of the equation we obtain the following compatibility condition

$$C(Q^t,Q^s)=\mathrm{D}_tQ^s-\mathrm{D}_sQ^t+[Q^s,Q^t]=0. \tag{5.5.7}$$

Let us now consider the curvature matrices with respect to the parameters s and t in the discrete case

$$N_{0;t}=\left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1} \qquad \text{and} \qquad N_{0;s}=\left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}.$$

We have that

$$\frac{\partial}{\partial s}N_{0;t}=\left(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}-\left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}\left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}$$

and analogously

$$\frac{\partial}{\partial t}N_{0;s}=\left(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}-\left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}\left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}N_{0;s}-\frac{\partial}{\partial s}N_{0;t}=&\left(\frac{\partial}{\partial t}\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}-\left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}\left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}-\left(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}\\&-\left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}\left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}=[N_{0;t},N_{0;s}].\end{aligned}$$

Therefore the compatibility condition in the discrete case is

$$C(N_{0;t},N_{0;s})=\frac{\partial}{\partial t}N_{0;s}-\frac{\partial}{\partial s}N_{0;t}+[N_{0;s},N_{0;t}]=0. \tag{5.5.8}$$

Note that both compatibility conditions (5.5.7) and (5.5.8) have the same structure.

Remark 5.5.9. *Also note that the expression*

$$C(\sigma_t,\sigma_s)=\frac{\partial}{\partial t}\sigma_s-\frac{\partial}{\partial s}\sigma_t+M_{0;s,0;t}-M_{0;t,0;s} \tag{5.5.10}$$

is equivalent to

$$C(N_{0;t}, N_{0;s}) = \frac{\partial}{\partial t}\sigma_s - \frac{\partial}{\partial s}\sigma_t + N_{0;s}\sigma_t - N_{0;t}\sigma_s$$

as

$$N_{0;s}\sigma_t = \left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}\rho_0\left(\frac{\partial}{\partial t}z_0(s,t)\right) = \frac{\partial}{\partial s}\rho_0\left(\frac{\partial}{\partial t}z_0(s,t)\right) = M_{0;s,0;t}$$

and

$$N_{0;t}\sigma_s = \left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}\rho_0\left(\frac{\partial}{\partial s}z_0(s,t)\right) = \frac{\partial}{\partial t}\rho_0\left(\frac{\partial}{\partial s}z_0(s,t)\right) = M_{0;t,0;s}.$$

In general, the condition on the functions F_s and F_t for the discrete curve evolution to commute is

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}z_0(s,t) - \frac{\partial}{\partial s}\frac{\partial}{\partial t}z_0(s,t)\\
&= \frac{\partial}{\partial t}(\rho_0^{-1}\cdot F_s) - \frac{\partial}{\partial s}(\rho_0^{-1}\cdot F_t)\\
&= -\left(\rho_0^{-1}\left(\frac{\partial}{\partial t}\rho_0\right)\rho_0^{-1}\right)\cdot F_s - \rho_0^{-1}\cdot\left(\frac{\partial}{\partial t}F_s\right) + \left(\rho_0^{-1}\left(\frac{\partial}{\partial s}\rho_0\right)\rho_0^{-1}\right)\cdot F_t - \rho_0^{-1}\cdot\left(\frac{\partial}{\partial s}F_t\right)\\
&= \rho_0^{-1}\cdot\left(N_{0;t}\ F_s - N_{0;s}\ F_t + \frac{\partial}{\partial t}F_s - \frac{\partial}{\partial s}F_t\right).
\end{aligned} \tag{5.5.11}$$

Proposition 5.5.12. *The $\mathcal{F}$ operator satisfies*

$$[\mathcal{F}_{Q^i}, \mathcal{F}_{Q^k}] = ad_{C(Q^i,Q^k)}.$$

Proof. Let us consider

$$\mathcal{F}_{Q^i}(Q^j) = \mathrm{D}_i Q^j + [Q^j, Q^i], \qquad \mathcal{F}_{Q^k}(Q^j) = \mathrm{D}_k Q^j + [Q^j, Q^k].$$

Therefore we have

$$\mathcal{F}_{Q^i}(\mathcal{F}_{Q^k}(Q^j)) = \mathrm{D}_i\mathrm{D}_k Q^j + \mathrm{D}_i[Q^j, Q^k] + [\mathrm{D}_k Q^j + [Q^j, Q^k], Q^i] \tag{5.5.13}$$

and

$$\mathcal{F}_{Q^k}(\mathcal{F}_{Q^i}(Q^j)) = \mathrm{D}_k\mathrm{D}_i Q^j + \mathrm{D}_k[Q^j, Q^i] + [\mathrm{D}_i Q^j + [Q^j, Q^i], Q^k]. \tag{5.5.14}$$

Substracting (5.5.14) to (5.5.13) we obtain

$$\begin{aligned}
\mathcal{F}_{Q^i}(\mathcal{F}_{Q^k}(Q^j)) - \mathcal{F}_{Q^k}(\mathcal{F}_{Q^i}(Q^j)) &= [Q^j, \mathrm{D}_i Q^k] - [Q^j, \mathrm{D}_k Q^i] + [Q^j, [Q^k, Q^i]]\\
&= ad_{C(Q^i,Q^k)}Q^j
\end{aligned}$$

and therefore

$$[\mathcal{F}_{Q^i}, \mathcal{F}_{Q^k}] = ad_{C(Q^i,Q^k)}.$$

□

Proposition 5.5.15. *The $\mathcal{F}^{\Delta}$ operator satisfies*

$$[\mathcal{F}^{\Delta}_{A}, \mathcal{F}^{\Delta}_{B}] = \mathcal{F}^{\Delta}_{[A,B]}$$

for appropriate expressions A and B.

Proof. Now let us consider

$$\mathcal{F}^{\Delta}_{A}(C) = (\mathrm{S}A)C - CA, \quad \text{and} \quad \mathcal{F}^{\Delta}_{B}(C) = (\mathrm{S}B)C - CB$$

for appropriate expressions A, B and C. Therefore we have

$$\begin{aligned} \mathcal{F}^{\Delta}_{A}(\mathcal{F}^{\Delta}_{B}(C)) &= (\mathrm{S}A)((\mathrm{S}B)C - CB) - ((\mathrm{S}B)C - CB)A \\ &= \mathrm{S}(AB)C - (\mathrm{S}A)CB - (\mathrm{S}B)CA + CBA \end{aligned} \tag{5.5.16}$$

and

$$\begin{aligned} \mathcal{F}^{\Delta}_{B}(\mathcal{F}^{\Delta}_{A}(C)) &= (\mathrm{S}B)((\mathrm{S}A)C - CA) - ((\mathrm{S}A)C - CA)B \\ &= \mathrm{S}(BA)C - (\mathrm{S}B)CA - (\mathrm{S}A)CB + CAB. \end{aligned} \tag{5.5.17}$$

Substracting (5.5.17) to (5.5.16) we obtain

$$\mathcal{F}^{\Delta}_{A}(\mathcal{F}^{\Delta}_{B}(C)) - \mathcal{F}^{\Delta}_{B}(\mathcal{F}^{\Delta}_{A}(C)) = \mathrm{S}[A,B]C - C[A,B] = \mathcal{F}^{\Delta}_{[A,B]}(C)$$

and therefore

$$[\mathcal{F}^{\Delta}_{A}, \mathcal{F}^{\Delta}_{B}] = \mathcal{F}^{\Delta}_{[A,B]}.$$

□

Remark 5.5.18. *Note that $\mathcal{F}$ is a derivation of Lie Algebras while $\mathcal{F}^{\Delta}$ is an homomorphism of Lie Algebras. This is expected due to the nature of the derivative operator and shift operator.*

	Smooth	**Discrete**
Product	$\mathrm{D}(f \cdot g) = \mathrm{D}(f) \cdot g + f \cdot \mathrm{D}(g)$	$\mathrm{S}(f \cdot g) = (\mathrm{S}f) \cdot (\mathrm{S}g)$
Bracket	$\partial[x,y] = [\partial x, y] + [x, \partial y]$	$\phi[x,y] = [\phi x, \phi y]$

Here ∂ is a derivation and ϕ is an homomorphism. Note that in the smooth case, the Liebnitz law is satisfied whereas in the discrete case it is not. Recall that $ad_{Q^i} \in \mathrm{Der}(L)$ and therefore is clear that $\mathcal{F}_{Q^i}$ is a derivation.

Proposition 5.5.19. *The following identity*

$$[\mathrm{D}_t, \mathrm{D}_s]Q^x = \mathcal{F}_{Q^x}(C(Q^t, Q^s))$$

is satisfied.

Proof. Recall that

$$\mathrm{D}_t Q^x = \mathrm{D}_x Q^t + [Q^t, Q^x], \tag{5.5.20a}$$

$$\mathrm{D}_s Q^x = \mathrm{D}_x Q^s + [Q^s, Q^x]. \tag{5.5.20b}$$

Differentiating (5.5.20a) with respect to s we obtain

$$\begin{aligned} \mathrm{D}_s\mathrm{D}_t Q^x &= \mathrm{D}_s\mathrm{D}_x Q^t + [\mathrm{D}_s Q^t, Q^x] + [Q^t, \mathrm{D}_s Q^x] \\ &= \mathrm{D}_s\mathrm{D}_x Q^t + [\mathrm{D}_s Q^t, Q^x] + [Q^t, \mathrm{D}_x Q^s] + [Q^t, [Q^s, Q^x]]. \end{aligned} \tag{5.5.21}$$

Analogously differentiating (5.5.20b) with respect to t we obtain

$$\mathrm{D}_t\mathrm{D}_s Q^x = \mathrm{D}_t\mathrm{D}_x Q^s + [\mathrm{D}_t Q^s, Q^x] + [Q^s, \mathrm{D}_x Q^t] + [Q^s, [Q^t, Q^x]]. \tag{5.5.22}$$

Substrating (5.5.22) to (5.5.21) we obtain

$$\begin{aligned} [\mathrm{D}_t, \mathrm{D}_s]Q^x &= \mathrm{D}_x\left(\mathrm{D}_t Q^s - \mathrm{D}_s Q^t + [Q^s, Q^t]\right) + [\mathrm{D}_t Q^s - \mathrm{D}_s Q^t + [Q^s, Q^t], Q^x] \\ &= \mathcal{F}_{Q^x}(\mathrm{D}_t Q^s - \mathrm{D}_s Q^t + [Q^s, Q^t]) \\ &= \mathcal{F}_{Q^x}(C(Q^t, Q^s)) \end{aligned}$$

we obtain the required result. □

Proposition 5.5.23. *The following identity*

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] K_0 = \mathcal{F}_{C(N_{0;t}, N_{0;s})}(K_0)$$

is satisfied.

Recall that

$$\frac{\partial}{\partial t} K_0 = (\mathrm{S}N_{0;t})K_0 - K_0 N_{0;t}, \tag{5.5.24a}$$

$$\frac{\partial}{\partial s}K_0 = (\mathrm{S}N_{0;s})K_0 - K_0N_{0;s}. \tag{5.5.24b}$$

Differentiating (5.5.24a) with respect to s we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial}{\partial t}K_0 &= \frac{\partial}{\partial s}(\mathrm{S}N_{0;t})K_0 + (\mathrm{S}N_{0;t})\frac{\partial}{\partial s}K_0 - \left(\frac{\partial}{\partial s}K_0\right)N_{0;t} - K_0\left(\frac{\partial}{\partial s}N_{0;t}\right)\\ &= \mathrm{S}\left(\frac{\partial}{\partial s}N_{0;t}\right)K_0 + \mathrm{S}(N_{0;t}N_{0;s})K_0 - (\mathrm{S}N_{0;t})K_0N_{0;s}\\ &\qquad - (\mathrm{S}N_{0;s})K_0N_{0;t} + K_0N_{0;s}N_{0;t} - K_0\frac{\partial}{\partial s}N_{0;t}.\end{aligned} \tag{5.5.25}$$

Analogously differentiating (5.5.24b) with respect to t we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s}K_0 &= \frac{\partial}{\partial t}(\mathrm{S}N_{0;s})K_0 + (\mathrm{S}N_{0;s})\frac{\partial}{\partial t}K_0 - \left(\frac{\partial}{\partial t}K_0\right)N_{0;s} - K_0\left(\frac{\partial}{\partial t}N_{0;s}\right)\\ &= \mathrm{S}\left(\frac{\partial}{\partial t}N_{0;s}\right)K_0 + \mathrm{S}(N_{0;s}N_{0;t})K_0 - (\mathrm{S}N_{0;s})K_0N_{0;t}\\ &\qquad - (\mathrm{S}N_{0;t})K_0N_{0;s} + K_0N_{0;t}N_{0;s} - K_0\frac{\partial}{\partial t}N_{0;s}.\end{aligned} \tag{5.5.26}$$

Substrating (5.5.26) to (5.5.25) we obtain

$$\begin{aligned}\left[\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right]K_0 &= \mathrm{S}\left(\frac{\partial}{\partial t}N_{0;s} - \frac{\partial}{\partial s}N_{0;t} + [N_{0;s},N_{0;t}]\right)K_0 - K_0\left(\frac{\partial}{\partial t}N_{0;s} - \frac{\partial}{\partial s}N_{0;t} + [N_{0;s},N_{0;t}]\right)\\ &= \mathcal{F}^{\Delta}_{\frac{\partial}{\partial t}N_{0;s} - \frac{\partial}{\partial s}N_{0;t} + [N_{0;s},N_{0;t}]}(K_0)\\ &= \mathcal{F}_{C(N_{0;t},N_{0;s})}(K_0)\end{aligned}$$

obtaining the required result.

Theorem 5.5.27. *The symmetry condition for two curvature evolutions is a differential consequence of the symmetry condition on the curve evolutions. We have*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\kappa - \frac{\partial}{\partial s}\frac{\partial}{\partial t}\kappa = \mathcal{H}C(\sigma_t,\sigma_s)$$

where

$$C(\sigma_t,\sigma_s) = \frac{\partial}{\partial t}\sigma_s - \frac{\partial}{\partial s}\sigma_t + N_{0;s}\sigma_t - N_{0;t}\sigma_s.$$

Proof. From (5.4.9), (5.5.9) and (5.5.23) the proof is straightforward. □

Remark 5.5.28. *In the continuous case the authors of [73] show that*

$$\mathcal{D}_{t_1}HI_{t_2} - \mathcal{D}_{t_2}HI_{t_1} - [\mathcal{D}_{t_1},\mathcal{D}_{t_2}]\kappa = HC(I_{t_1},I_{t_2}) \tag{5.5.29}$$

and they state that this implies that integrability does not necessarily lift from the curvature

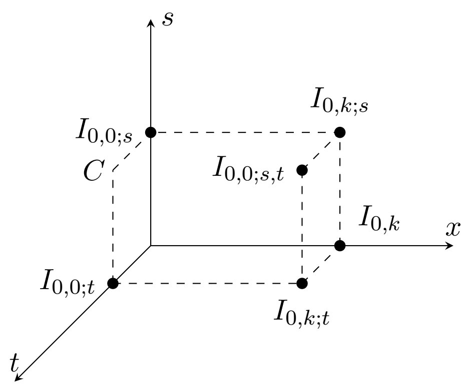


Figure 5.1: A graphic explanation of $\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right]\kappa = \mathcal{H}C$ being the syzygy of a syzygy.

evolution to the curve evolution. The same occurs in the discrete case. However, most commonly studied integrable curvature equations are homogeneous polynomials or rational functions of the differential invariants. Since in these classes the kernel of the differential operator H is empty, pairs of integrable equations result (see Langer and Perline, [65]). The same occurs for discrete integrable curvature equations. The authors also give an outline of the proof of (5.5.29). *Here we give an alternative proof, which is now straightforward using* (5.4.9) *and* (5.5.19).

Smooth	**Discrete**
Syzygy	
$\mathrm{D}_t Q^x = \mathrm{D}_x Q^t + [Q^t, Q^x]$	$\frac{\mathrm{d}}{\mathrm{dt}} K_0 = (\mathrm{S} N_{0;t}) K_0 - K_0 N_{0;t}$
Compatibility condition	
$\mathrm{D}_t Q^s - \mathrm{D}_s Q^t + [Q^s, Q^t] = 0$	$\frac{\partial}{\partial t} N_{0;s} - \frac{\partial}{\partial s} N_{0;t} + [N_{0;s}, N_{0;t}] = 0$
$\mathcal{F}$ and $\mathcal{F}^\Delta$ operator	
$\mathcal{F}_{Q^i} = \mathrm{D}_i - ad_{Q^i}$	$\mathcal{F}^\Delta_{N_{0;t_i}} = (\mathrm{S} - \mathrm{id}) N_{0;t_i} + ad_{N_{0;t_i}}$
$\mathcal{F}$ operator bracket	
$\left[\mathcal{F}_{Q^i}, \mathcal{F}_{Q^k}\right] = ad_{C(Q^i, Q^k)}$	$\left[\mathcal{F}^\Delta_{N_{0;t}}, \mathcal{F}^\Delta_{N_{0;s}}\right] = \mathcal{F}^\Delta_{[N_{0;t}, N_{0;s}]}$
Evolution of curvature matrix / Maurer–Cartan matrix	
$[\mathrm{D}_t, \mathrm{D}_s] Q^x = \mathcal{F}_{Q^x}(C([Q^t, Q^s]))$	$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] K_0 = \mathcal{F}^\Delta_{C([N_{0;t}, N_{0;s}])}(K_0)$

Example 5.5.30. *We show for this example that the symmetry condition for two curvature evolutions is a differential consequence of the symmetry condition on the curve evolutions.*

We show using MAPLE *(See Appendix) that*

$$\frac{\partial}{\partial t}\mathcal{H}\sigma_s - \frac{\partial}{\partial s}\mathcal{H}\sigma_t = \mathcal{H}C(\sigma_t, \sigma_s).$$

5.6 Integrable differential–difference equations

In order to relate our examples to discrete integrable systems, we will make use of the theory appearing in Khanizadeh, Mikhailov and Wang, [54] as well as the list of Integrable differential–difference equations appearing in such paper. We introduce some essential concepts first.

Consider the differential–difference equation

$$\mathbf{u}_t = K[\mathbf{u}] \tag{5.6.1}$$

where $K[\mathbf{u}]$ is a smooth vector-valued function depending of $\mathbf{u}$ and its shifts. Suppose that a is a function of $\mathbf{u}$. The *Frechét derivative* is defined as

$$a_\star = \sum_k \left(\frac{\partial a}{\partial u_k^1}, ..., \frac{\partial a}{\partial u_k^N}\right) \mathrm{S}^k.$$

The *variational derivative* of a is defined as

$$\delta_{\mathbf{u}}(a) = \left(\frac{\partial a}{\partial u_k^1}, ..., \frac{\partial a}{\partial u_k^N}\right) \sum_k \mathrm{S}_k(a).$$

If (5.6.1) is Hamiltonian, then we can write it in the form

$$\mathbf{u}_t = \mathcal{H}(\delta_{\mathbf{u}}(f))$$

where here $\mathcal{H}$ denotes a Hamiltonian (pseudo)–difference operator - so it might include backward shifts - and f the Hamiltonian function.

Example 5.6.2. *It is possible to relate the previous results to integrable systems as follows:*

Recall $I^u_{0,1;t} = \dfrac{\mathrm{S}\sigma_t}{\kappa}$ *and therefore* $\sigma_t = \mathrm{S}_{-1}(\kappa I^u_{0,1;t})$. *Hence* (5.4.11) *is equivalent to*

$$\kappa_t = (-\kappa\mathrm{S} + \kappa - \kappa^2 + \kappa\mathrm{S}_{-1}\kappa)I^u_{0,1;t} = \kappa(-\mathrm{S} + id - \kappa + \mathrm{S}_{-1}\kappa)I^u_{0,1;t}.$$

Setting $I^u_{0,1;t} = \kappa$, we obtain

$$\kappa_t = -\kappa\kappa_1 + \kappa^2 - \kappa^3 + \kappa\kappa_{-1}^2 = \kappa(\kappa - \kappa_1) + \kappa(\kappa_{-1}^2 - \kappa^2).$$

which is a Volterra type equation

$$u_t = f(u_{-1}, u, u_1)$$

as listed in Khanizadeh, Mikhailov and Wang, [54].

5.7 The S$L(2)$ linear action

In this example we consider the S$L(2)$ linear action previously studied in §4.3.1. After computing the correction terms and verifying (5.3.5), (5.3.11) and (5.3.17), we show that the symmetry condition of the discrete curve evolutions is a differential consequence of the symmetry condition of the curvature evolutions. Furthermore, we relate this example to the Toda lattice.

The first three correction terms with respect to the moving frame ρ_0 (4.3.4) are

$$\begin{aligned} M_{0,0;t} &= \begin{pmatrix} -\sigma^x_t \\ -\sigma^y_t \end{pmatrix}, \\ M_{0,1;t} &= \begin{pmatrix} -I^x_{0,1;t} \\ \sigma^y_t I^y_{0,1;t} \end{pmatrix}, \\ M_{0,2;t} &= \begin{pmatrix} -\sigma^x_t I^x_{0,2} - \dfrac{I^x_{0,1;t} I^y_{0,2}}{\tau} \\ -I^x_{0,2}\sigma^y_t + I^y_{0,2}\sigma^x_t \end{pmatrix}. \end{aligned} \tag{5.7.1}$$

The variables appearing in the normalization equations (4.3.3) are x_0, y_0 and x_1. Therefore,

$$\zeta_1 = x_0, \qquad \zeta_2 = y_0 \qquad \text{and} \qquad \zeta_3 = x_1.$$

Thus $\vartheta = 3$. Note that as the group action depends on three paramenters we have that $R = 3$. Hence,

$$\mathbf{T} = \begin{pmatrix} \iota_0\left(\dfrac{\mathrm{d}}{\mathrm{dt}}\zeta_1\right) & \iota_0\left(\dfrac{\mathrm{d}}{\mathrm{dt}}\zeta_2\right) & \iota_0\left(\dfrac{\mathrm{d}}{\mathrm{dt}}\zeta_3\right) \end{pmatrix} = \begin{pmatrix} \sigma^x_t & \sigma^y_t & I^x_{0,1;t} \end{pmatrix}.$$

The matrix of invariant generators has the form

$$\mathbf{\Phi} = \begin{pmatrix} \iota_0\left(\dfrac{\partial\widetilde{\zeta_1}}{\partial a_1}\right) & \iota_0\left(\dfrac{\partial\widetilde{\zeta_1}}{\partial a_2}\right) & \iota_0\left(\dfrac{\partial\widetilde{\zeta_1}}{\partial a_3}\right) \\ \iota_0\left(\dfrac{\partial\widetilde{\zeta_2}}{\partial a_1}\right) & \iota_0\left(\dfrac{\partial\widetilde{\zeta_2}}{\partial a_2}\right) & \iota_0\left(\dfrac{\partial\widetilde{\zeta_2}}{\partial a_3}\right) \\ \iota_0\left(\dfrac{\partial\widetilde{\zeta_3}}{\partial a_1}\right) & \iota_0\left(\dfrac{\partial\widetilde{\zeta_3}}{\partial a_2}\right) & \iota_0\left(\dfrac{\partial\widetilde{\zeta_3}}{\partial a_3}\right) \end{pmatrix} = \begin{pmatrix} \iota_0(x_0) & 0 & \iota_0(x_1) \\ \iota_0(y_0) & 0 & \iota_0(y_1) \\ 0 & \iota_0(x_0) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & 1 & 0 \end{pmatrix}$$

where $a_1 = a, a_2 = b$ and $a_3 = c$. Finally

$$\mathbf{J} = \begin{pmatrix} \iota_0\left(\dfrac{\partial\psi_1}{\partial\zeta_1}\right) & \iota_0\left(\dfrac{\partial\psi_1}{\partial\zeta_2}\right) & \iota_0\left(\dfrac{\partial\psi_1}{\partial\zeta_3}\right) \\ \iota_0\left(\dfrac{\partial\psi_2}{\partial\zeta_1}\right) & \iota_0\left(\dfrac{\partial\psi_2}{\partial\zeta_2}\right) & \iota_0\left(\dfrac{\partial\psi_2}{\partial\zeta_3}\right) \\ \iota_0\left(\dfrac{\partial\psi_3}{\partial\zeta_1}\right) & \iota_0\left(\dfrac{\partial\psi_3}{\partial\zeta_2}\right) & \iota_0\left(\dfrac{\partial\psi_3}{\partial\zeta_3}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\mathbf{K} = -\mathbf{TJ}(\mathbf{\Phi J})^{-\mathbf{1}} = \begin{pmatrix} -\sigma^x_t & -\dfrac{I^x_{0,1;t}}{\tau} & -\sigma^y_t \end{pmatrix}. \tag{5.7.2}$$

Hence the correction terms can be calculated as follow

$$\begin{pmatrix} K_{1,1}\cdot 1 \\ K_{1,3}\cdot 1 \end{pmatrix} = \begin{pmatrix} -I^x_{0,0;t} \\ -I^y_{0,0;t} \end{pmatrix} = M_{0,0;t},$$
$$\begin{pmatrix} K_{1,2}\cdot I^y_{0,1} \\ -K_{1,1}\cdot I^y_{0,1} \end{pmatrix} = \begin{pmatrix} -I^u_{0,1;t} \\ I^u_{0,0;t}I^y_{0,1} \end{pmatrix} = M_{0,1;t},$$
$$\begin{pmatrix} K_{1,1}\cdot I^x_{0,2} + K_{1,2}\cdot I^y_{0,1} \\ -K_{1,1}\cdot I^y_{0,2} + K_{1,2}\cdot I^x_{0,2} \end{pmatrix} = \begin{pmatrix} -I^x_{0,0;t}I^x_{0,2} - \dfrac{I^x_{0,1;t}I^y_{0,2}}{I^y_{0,1}} \\ -I^x_{0,2}I^y_{0,0;t} + I^y_{0,2}I^x_{0,0;t} \end{pmatrix} = M_{0,2;t}$$

which matches with the correction terms (5.7.1) calculated using (**??**).

Recall the Lie algebra basis of the Lie group (2.16). Therefore the curvature matrix can be computed as follows

$$\begin{aligned} N_{0;t} &= -\sigma^x_t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{I^x_{0,1;t}}{\tau}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \sigma^y_t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\sigma^x_t & -\dfrac{I^x_{0,1;t}}{\tau} \\ -\sigma^y_t & \sigma^x_t \end{pmatrix} \end{aligned}$$

matching that one obtain in (4.3.10).

Now we show for this example that the symmetry condition for two curvature evolutions is a differential consequence of the symmetry condition on the curve evolutions.

The fundamental syzygy is

$$C(\sigma_t,\sigma_s)=\begin{pmatrix} C_1\\ C_2\end{pmatrix}=\begin{pmatrix} \dfrac{\sigma_t^y S\sigma_s^y-\sigma_s^y S\sigma_t^y}{\tau^2}+\dfrac{\partial}{\partial t}\sigma_s^x-\dfrac{\partial}{\partial s}\sigma_t^x\\ 2\sigma_s^x\sigma_t^y-2\sigma_s^y\sigma_t^x+\dfrac{\partial}{\partial t}\sigma_s^y-\dfrac{\partial}{\partial s}\sigma_t^y\end{pmatrix}.$$

We have

$$\frac{\partial}{\partial t}\mathcal{H}\sigma_s=\left(\frac{\partial}{\partial t}\mathcal{H}\right)\sigma_s+\mathcal{H}\frac{\partial}{\partial t}\sigma_s$$

where

$$\left(\frac{\partial}{\partial t}\mathcal{H}\right)\sigma_s=\begin{pmatrix} \dfrac{\partial}{\partial t}\kappa(\mathrm{id}-\mathrm{S}) & -\dfrac{\frac{\partial}{\partial t}\tau}{\tau^2}-\left(\dfrac{\frac{\partial}{\partial t}\tau}{\tau_1^2}-2\dfrac{\tau\frac{\partial}{\partial t}\tau_1}{\tau_1^3}\right)\mathrm{S}_2\\ \dfrac{\partial}{\partial t}\tau(\mathrm{id}+\mathrm{S}) & \dfrac{\partial}{\partial t}\kappa\mathrm{S}\end{pmatrix}\begin{pmatrix}\sigma_s^x\\ \sigma_s^y\end{pmatrix}.$$

Analogously

$$\frac{\partial}{\partial s}\mathcal{H}\sigma_t=\left(\frac{\partial}{\partial s}\mathcal{H}\right)\sigma_t+\mathcal{H}\frac{\partial}{\partial s}\sigma_t$$

where

$$\left(\frac{\partial}{\partial s}\mathcal{H}\right)\sigma_t=\begin{pmatrix} \dfrac{\partial}{\partial s}\kappa(\mathrm{id}-\mathrm{S}) & -\dfrac{\frac{\partial}{\partial s}\tau}{\tau^2}-\left(\dfrac{\frac{\partial}{\partial s}\tau}{\tau_1^2}-2\dfrac{\tau\frac{\partial}{\partial s}\tau_1}{\tau_1^3}\right)\mathrm{S}_2\\ \dfrac{\partial}{\partial s}\tau(\mathrm{id}+\mathrm{S}) & \dfrac{\partial}{\partial s}\kappa\mathrm{S}\end{pmatrix}\begin{pmatrix}\sigma_t^x\\ \sigma_t^y\end{pmatrix}.$$

And therefore, simplifying both expressions using MAPLE (See Appendix)

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{H}\sigma_s-\frac{\partial}{\partial s}\mathcal{H}\sigma_t &= \begin{pmatrix}\kappa(\mathrm{id}-\mathrm{S}) & \dfrac{1}{\tau}-\dfrac{\tau}{\tau_1^2}\mathrm{S}_2\\ \tau(\mathrm{id}+\mathrm{S}) & \kappa\mathrm{S}\end{pmatrix}\begin{pmatrix}C_1\\ C_2\end{pmatrix}\\ &= \mathcal{H}C(\sigma_t,\sigma_s).\end{aligned}$$

We now consider the change of variable $\eta=-\kappa\tau$. We obtain

$$\begin{aligned}\eta_t&=-\frac{\eta^2\mathrm{S}\sigma_t^y}{\tau^2}-\sigma_t^y+2\,\eta\sigma_t^x+\frac{\tau^2\mathrm{S}_2\sigma_t^y}{\mathrm{S}\tau^2},\\ \tau_t&=-\frac{\eta\mathrm{S}\sigma_t^y}{\tau}+\tau\mathrm{S}\sigma_t^x+\tau\,\sigma_t^x.\end{aligned}$$

Therefore there exists a linear shift operator $\mathcal{H}(\eta,\tau)$ such that

$$\begin{pmatrix} \eta_t \\ \tau_t \end{pmatrix} = \mathcal{H}(\eta,\tau) \begin{pmatrix} \sigma_t^x \\ \sigma_t^y \end{pmatrix} \quad \text{where} \quad \mathcal{H}(\eta,\tau) = \begin{pmatrix} 2\eta & -\dfrac{\eta^2 \mathrm{S}}{\tau^2} + \dfrac{\tau^2 \mathrm{S}_2}{\mathrm{S}\tau^2} - \mathrm{id} \\ \tau(\mathrm{S}+\mathrm{id}) & -\dfrac{\eta}{\tau}\mathrm{S} \end{pmatrix}.$$

Let us set $\gamma := -\dfrac{\mathrm{S}\sigma_t^y}{\tau^2}$. We can write the evolution of the curvature invariants η and τ in terms of the evolution invariants σ_t^x and γ as follows

$$\eta_t = \kappa(\mathrm{S}-\mathrm{id})\sigma_t^x + \eta\tau\gamma, \qquad \tau_t = 2\tau\sigma_t^x + \left((\mathrm{S}_{-1}\eta^2)\mathrm{S}_{-1} - \eta^2\mathrm{S} + \tau^2\right)\gamma.$$

Therefore

$$\begin{pmatrix} \eta_t \\ \tau_t \end{pmatrix} = \begin{pmatrix} \eta(\mathrm{S}-\mathrm{id}) & \eta\tau \\ 2\tau & \mathrm{S}_{-1}\eta^2 - \eta^2\mathrm{S} + \tau^2 \end{pmatrix} \begin{pmatrix} \sigma_t^x \\ \gamma \end{pmatrix}$$

where we have used the notation $\mathrm{S}_{-1}\eta^2 = \mathrm{S}_{-1}(\eta^2)\mathrm{S}_{-1}$. Let us set

$$\widehat{\mathcal{H}} = \begin{pmatrix} \eta(\mathrm{S}-\mathrm{id}) & \eta\tau \\ 2\tau & \mathrm{S}_{-1}\eta^2 - \eta^2\mathrm{S} + \tau^2 \end{pmatrix}$$

and let us define the matrix

$$\mathcal{P} = \begin{pmatrix} (\mathrm{id}-\mathrm{S}_{-1})\eta & 2\tau \\ 0 & -4 \end{pmatrix}$$

and compute the pseudo-difference operator

$$\widehat{\mathcal{H}}\mathcal{P} = \begin{pmatrix} \eta(\mathrm{S}-\mathrm{S}_{-1})\eta & 2\eta(\mathrm{S}-\mathrm{id})\tau \\ 2\tau(\mathrm{id}-\mathrm{S}_{-1})\eta & -4(\mathrm{S}_{-1}\eta^2 - \eta^2\mathrm{S}) \end{pmatrix} \tag{5.7.3}$$

which is clearly symmetric. Let us set $\mathcal{H}_2 := \widehat{\mathcal{H}}\mathcal{P}$.

Theorem 5.7.4. *The operator $\mathcal{H}_2$, given by* (5.7.3) *is a Hamiltonian operator. It forms a Hamiltonian pair with Hamiltonian operator*

$$\mathcal{H}_1 = \begin{pmatrix} 0 & 2\eta(\mathrm{S}-\mathrm{id}) \\ 2(\mathrm{id}-\mathrm{S}_{-1})\eta & 0 \end{pmatrix}.$$

Proof. Let us introduce the following transformation

$$p = \eta^2, \qquad q = \tau. \tag{5.7.5}$$

Its Frechét derivative is

$$D(p,q) = \begin{pmatrix} 2\eta & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $D(p,q) = D(p,q)^*$.

Under this transformation the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ become

$$\widetilde{\mathcal{H}_1} = D(p,q)\mathcal{H}_1 D(p,q)^* = 4 \begin{pmatrix} 0 & p(\mathrm{S}-\mathrm{id}) \\ (\mathrm{id}-\mathrm{S}_{-1})p & 0 \end{pmatrix},$$

$$\widetilde{\mathcal{H}_2} = D(p,q)\mathcal{H}_2 D(p,q)^* = 4 \begin{pmatrix} p(\mathrm{S}-\mathrm{S}_{-1})p & p(\mathrm{S}-\mathrm{id})q \\ q(\mathrm{id}-\mathrm{S}_{-1})p & p\mathrm{S}-\mathrm{S}_{-1}p \end{pmatrix}.$$

Setting

$$\overline{\mathcal{H}_1} := \begin{pmatrix} 0 & p(\mathrm{S}-\mathrm{id}) \\ (\mathrm{id}-\mathrm{S}_{-1})p & 0 \end{pmatrix} \quad \text{and} \quad \overline{\mathcal{H}_2} := \begin{pmatrix} p(\mathrm{S}-\mathrm{S}_{-1})p & p(\mathrm{S}-\mathrm{id})q \\ q(\mathrm{id}-\mathrm{S}_{-1})p & p\mathrm{S}-\mathrm{S}_{-1}p \end{pmatrix}$$

we have that these two operators form a hamiltonian pair for the well-known Toda-Lattice in Flaschka coordinates (see Adler, [1], Khanizadeh, Mikhailov and Wang, [54] and Suris, [103])

$$p_t = p(q_1 - q), \qquad q_t = p - p_{-1} \tag{5.7.6}$$

where $q_1 = \mathrm{S}q$ and $p_{-1} = \mathrm{S}_{-1}p$. □

Theorem 5.7.7. *The evolution of the curvature invariants for the* S$L(2)$ *linear action induces a completely integrable system in its curvatures* η *and* τ *equivalent to the Toda-Lattice (see Toda* (5.7.6)*).*

Proof. Recall

$$\begin{pmatrix} \eta_t \\ \tau_t \end{pmatrix} = \widehat{\mathcal{H}} \begin{pmatrix} \sigma_t^x \\ \gamma \end{pmatrix}.$$

and suppose we can write

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \mathcal{P} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} \eta_t \\ \tau_t \end{pmatrix} = \mathcal{H}_2 \begin{pmatrix} a \\ b \end{pmatrix}.$$

For $a = 0$, $b = 1$ we get

$$\eta_t = 2\eta(\tau_1 - \tau), \qquad \tau_t = 4(\eta^2 - \eta_{-1}^2) \tag{5.7.8}$$

where $\tau_1 = \mathrm{S}\tau$ and $\eta_{-1} = \mathrm{S}_{-1}\eta$. Using (5.7.5) the system (5.7.8) is converted to the system

$$p_t = 4p(q_1 - q), \qquad q_t = 4(p - p_{-1}) \tag{5.7.9}$$

which is equivalent to (5.7.6) (notice that (5.7.9) just differs to (5.7.6) by a constant factor). □

Remark 5.7.10. *For the system* (5.7.8) *we have the following hamiltonian structure*

$$\mathcal{H}_1 = \begin{pmatrix} 0 & 2\eta(\mathrm{S} - \mathrm{id}) \\ 2(\mathrm{id} - \mathrm{S}_{-1})\eta & 0 \end{pmatrix}, \qquad f_1 = \eta^2 + \frac{\tau^2}{2},$$

$$\mathcal{H}_2 = \begin{pmatrix} \eta(\mathrm{S} - \mathrm{S}_{-1})\eta & 2\eta(\mathrm{S} - \mathrm{id})\tau \\ 2\tau(\mathrm{id} - \mathrm{S}_{-1})\eta & -4(\mathrm{S}_{-1}\eta^2 - \eta^2\mathrm{S}) \end{pmatrix}, \qquad f_2 = \tau,$$

i.e,

$$\begin{pmatrix} \eta_t \\ \tau_t \end{pmatrix} = \mathcal{H}_1\delta(f_1) = \mathcal{H}_2\delta(f_2).$$

Application of Multispaces for some Lie Groups

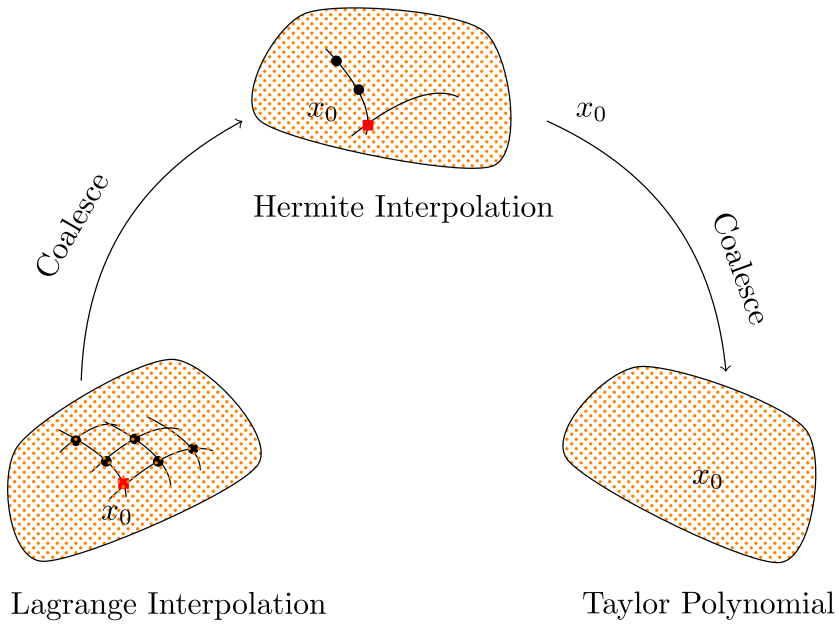
It is possible to construct a discrete moving frame as the limit of a continuous one, and vice-versa, by coordinating the transverse sections that determine them. This was achieved by Beffa and Mansfield in [5], where the authors define the concept of multispace, a manifold including the jet bundle and cartesian products of the base space simultaneously. A frame on a multispace contains the smooth and the discrete frame and one can be obtained from the other by taking an appropriate continuum limit. In this paper, the authors also show that the discrete invariants converge to differential invariants and local discrete syzygies converge to differential syzygies. In this chapter, we give a very brief introduction to multispaces and we study the $\mathrm{S}E(2)$ case where we explore the convergence of the discrete frame to the smooth frame and discrete curvature invariants to the smooth ones. For all these examples we show convergence of the Maurer–Cartan matrix to the curvature matrix with the respect to the space independent variable. We also study the projective $\mathrm{S}L(2)$ action and show convergence of the discrete action to the smooth one as well as the convergence of the discrete frame and discrete infinitesimals to the smooth ones.

6.1 A very brief introduction to multispaces

The concept of multispace arises as a consequence of creating a manifold that is smooth and discrete at the same time. A multispace looks like the jet space, but also includes discrete versions of the jet space where a frame is simultaneously a smooth frame and a frame on a discrete space. The equivariance is successfully maintained in the continuum limit and the discrete Maurer–Cartan invariants and discrete syzygy coalesce to the smooth ones. In order to achieve this, the process starts making use of interpolation methods, where the coefficients are given by the solution of a linear system of equations.

Under coalescence of the points at which the interpolation is calculated, Lagrange interpolation becomes Hermite interpolation, ending with the Taylor approximation to a surface when all the interpolation points coalesce, as shown in Figure (6.1).

Figure 6.1: Hyperplane coalescence.



6.2 The Lie group action $\mathrm{S}E(2)$ acting on multispaces

Recall the Lie group action of rotations and translations of curves in the plane (4.1.2). The aim is to choose an interpolation polynomial that will allow us to construct a moving frame in the multispace. We will show that this frame encodes the discrete and smooth information and that taking an appropriate continuum limit, the discrete curvature invariants converge to the smooth ones and that the Maurer–Cartan matrix converge to the corresponding curvature matrix in the smooth case.

6.2.1 Action and moving frame

Recall that the curvature invariants for (4.1.2) are the arc length

$$\mathrm{d}s = \sqrt{1+u_x^2}$$

and the curvature

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$$

which is an invariant of order two. Therefore, as we want to show that the discrete curvature invariants obtained using multispace theory converge to the smooth ones, we need a polynomial interpolator of at least order two.

We choose the order two polynomial interpolator of the points t_0, t_1, t_2 with base point

$t_0 = 0$. For the variable $x(t)$ we have

$$p(x(t)) = A(x(t)) + B(x(t))t + \frac{1}{2}C(x(t))t^2. \tag{6.2.1}$$

In order to obtain an expression for the coefficients $A(x(t)), B(x(t))$ and $C(x(t))$ we solve the equations

$$\begin{aligned}
x_0 &= A(x(t)),\\
x_1 &= A(x(t)) + B(x(t))t_1 + \frac{1}{2}C(x(t))t_1^2,\\
x_2 &= A(x(t)) + B(x(t))t_2 + \frac{1}{2}C(x(t))t_2^2
\end{aligned}$$

for $A(x(t))$, $B(x(t))$ and $C(x(t))$ where we have set $x(t_i)$ to be x_i for $i = 0, 1, 2$. Using Cramer's rule we obtain

$$\begin{aligned}
A(x(t)) &= x_0,\\
B(x(t)) &= \frac{\begin{vmatrix} 1 & x_0 & 0 \\ 1 & x_1 & \frac{1}{2}t_1^2 \\ 1 & x_2 & \frac{1}{2}t_2^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & t_1 & \frac{1}{2}t_1^2 \\ 1 & t_2 & \frac{1}{2}t_2^2 \end{vmatrix}} = \frac{(x_2 - x_0)t_1^2 - (x_1 - x_0)t_2^2}{t_1 t_2 (t_1 - t_2)},\\
C(x(t)) &= \frac{\begin{vmatrix} 1 & 0 & x_0 \\ 1 & t_1 & x_1 \\ 1 & t_2 & x_2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & t_1 & \frac{1}{2}t_1^2 \\ 1 & t_2 & \frac{1}{2}t_2^2 \end{vmatrix}} = 2\frac{(x_2 - x_0)t_1 - (x_1 - x_0)t_2}{t_1 t_2 (t_2 - t_1)}.
\end{aligned}$$

For the variable $u(t)$, the order two interpolator of the points t_0, t_1, t_2 with base point $t_0 = 0$ will be

$$p(u(t)) = A(u(t)) + B(u(t))t + \frac{1}{2}C(u(t))t^2. \tag{6.2.2}$$

Solving the equations

$$u_0 = A(u(t)) + B(u(t))t_0 + \frac{1}{2}C(u(t))t_0^2,$$
$$u_1 = A(u(t)) + B(u(t))t_1 + \frac{1}{2}C(u(t))t_1^2,$$
$$u_2 = A(u(t)) + B(u(t))t_2 + \frac{1}{2}C(u(t))t_2^2$$

for $A(u(t))$, $B(u(t))$ and $C(u(t))$ where we have set $u(t_i)$ to be u_i for $i = 0, 1, 2$ and using Cramer's rule once more we obtain

$$A(u(t)) = u_0,$$
$$B(u(t)) = \frac{(u_2 - u_0)t_1^2 - (u_1 - u_0)t_2^2}{t_1 t_2 (t_1 - t_2)},$$
$$C(u(t)) = 2\frac{(u_2 - u_0)t_1 - (u_1 - u_0)t_2}{t_1 t_2 (t_2 - t_1)}.$$

Making the group action (4.1.11) acting on the coeffients $A(x(t)), A(u(t))$ and $B(u(t))$, we can construct the normalization equations which are of the form

$$\widetilde{A(x(t))} = \widetilde{A(u(t))} = \widetilde{B(u(t))} = 0.$$

Solving for the group parameters θ, a, b, we obtain the frame

$$\rho_{\mathcal{M}} = \begin{pmatrix} R_{\theta_{\mathcal{M}}} & -R_{\theta_{\mathcal{M}}} z(t_0) \\ 0 & 1 \end{pmatrix} \qquad (6.2.3)$$

where

$$R_{\theta_{\mathcal{M}}} = \begin{pmatrix} \cos\theta_{\mathcal{M}} & -\sin\theta_{\mathcal{M}} \\ \sin\theta_{\mathcal{M}} & \cos\theta_{\mathcal{M}} \end{pmatrix}$$

with

$$\theta_{\mathcal{M}} = -\arctan\frac{(u_1 - u_0)t_2^2 - (u_2 - u_0)t_1^2}{(x_1 - x_0)t_2^2 - (x_2 - x_0)t_1^2}, \qquad \text{and} \qquad z(t_0) = (x_0, u_0).$$

We now consider the Taylor series

$$t_1 = h, \quad t_2 = 2h, \quad x_0 = x, \quad u_0 = u,$$
$$x_1 = x + hx_t + \frac{1}{2}h^2 x_{tt}, \quad x_2 = x + 2hx_t + 2h^2 x_{tt}, \qquad (6.2.4)$$
$$u_1 = u + hu_t + \frac{1}{2}h^2 u_{tt}, \quad u_2 = u + 2hu_t + 2h^2 u_{tt}.$$

Substituting this into (6.2.3) and taking the limit when h tends to 0 we obtain the smooth moving frame

$$\rho = \begin{pmatrix} \frac{x_t}{\sqrt{{u_t}^2+{x_t}^2}} & \frac{u_t}{\sqrt{{u_t}^2+{x_t}^2}} & -\frac{u_t\,u + x_t\,x}{\sqrt{{u_t}^2+{x_t}^2}} \\ -\frac{u_t}{\sqrt{{u_t}^2+{x_t}^2}} & \frac{x_t}{\sqrt{{u_t}^2+{x_t}^2}} & \frac{u_t\,x - u\,x_t}{\sqrt{{u_t}^2+{x_t}^2}} \\ 0 & 0 & 1 \end{pmatrix}$$

which matches the smooth one obtained in (4.1.4). Hence we have shown that the discrete moving frame (6.2.3) obtained via multispace theory converges to the smooth moving frame (4.1.4).

6.2.2 Curvature invariants and Maurer–Cartan matrix

Inducing the action (4.1.11) on $B(x(t))$, $C(x(t))$ and $C(u(t))$ and using the Taylor series (6.2.4), we can see that in the limit, these are the arclength, the dot product and the crossproduct of (x_t, u_t) and (x_{tt}, u_{tt}) respectively, both divided by the arc length, which are invariants. In order words, taking the limit when h tends to zero of

$$\widetilde{A(u(t))} = \widetilde{u_0},$$

$$\widetilde{B(u(t))} = \frac{(\widetilde{u_2}-\widetilde{u_0})t_1^2 - (\widetilde{u_1}-\widetilde{u_0})t_2^2}{t_1t_2(t_1-t_2)},$$

$$\widetilde{C(u(t))} = 2\frac{(\widetilde{u_2}-\widetilde{u_0})t_1 - (\widetilde{u_1}-\widetilde{u_0})t_2}{t_1t_2(t_2-t_1)}$$

where $\widetilde{u_0}$, $\widetilde{u_1}$ and $\widetilde{u_2}$ are given by (4.1.11). We get that

$$\widetilde{A(u(t))} \longrightarrow \sqrt{x_t^2+u_t^2}, \qquad \widetilde{B(u(t))} \longrightarrow \frac{x_tx_{tt}+u_tu_{tt}}{\sqrt{x_t^2+u_t^2}}, \qquad \widetilde{C(u(t))} \longrightarrow \frac{x_tu_{tt}-u_tx_{tt}}{\sqrt{x_t^2+u_t^2}}.$$

These results were obtained by Beffa and Mansfield in [5] using another (but similar) Taylor approximation strategy.

Computing $K_{\mathcal{M}} = (\mathrm{S}\rho_{\mathcal{M}})\rho_{\mathcal{M}}^{-1}$, we obtain the multispace Maurer-Cartan matrix. Taking the Taylor series

$$\begin{aligned} &t_1 = h, \quad t_2 = 2h, \quad t_3 = 6h, \quad x_0 = x, \quad u_0 = u, \\ &x_1 = x + hx_t + \frac{1}{2}h^2x_{tt}, \quad x_2 = x + 2hx_t + 2h^2x_{tt}, \quad x_3 = x + 4hx_t + 8h^2x_{tt}, \\ &u_1 = u + hu_t + \frac{1}{2}h^2u_{tt}, \quad u_2 = u + 2hu_t + 2h^2u_{tt}, \quad u_3 = u + 4hu_t + 8h^2u_{tt} \end{aligned}$$

we have that

$$\lim_{h\to 0}\ \frac{\mathrm{d}}{\mathrm{d}h}\left(K_{\mathcal{M}}\Big|_{Taylor}\right)=\begin{pmatrix} 0 & \dfrac{x_t\,u_{tt}-x_{tt}\,u_t}{{u_t}^2+{x_t}^2} & -\sqrt{{u_t}^2+{x_t}^2} \\ -\dfrac{x_t\,u_{tt}-x_{tt}\,u_t}{{u_t}^2+{x_t}^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\lim_{h\to 0}\frac{K_{\mathcal{M}}\Big|_{Taylor}-\mathrm{Id}_3}{-K_{\mathcal{M}[1,3]}\Big|_{Taylor}}=\begin{pmatrix} 0 & \dfrac{x_t\,u_{tt}-x_{tt}\,u_t}{({u_t}^2+{x_t}^2)^{3/2}} & 1 \\ -\dfrac{x_t\,u_{tt}-x_{tt}\,u_t}{({u_t}^2+{x_t}^2)^{3/2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & \kappa & 1 \\ -\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.2.5}$$

where Id_3 denotes the 3×3 identity matrix and where we have used the approximation method presented in §4.1. Note that (6.2.5) matches (4.1.5). Hence, the discrete Maurer–Cartan matrix obtain via multispace theory converges to the smooth curvature matrix with respect to s.

Remark 6.2.6. *In practice, the Taylor approximation used in §4.1 was*

$$\begin{aligned} &x_0=x, \quad x_1=x+h, \quad x_2=x+2h,\\ &u_0=u, \quad u_1=u+hu_x+\frac{1}{2}h^2u_{xx}, \quad u_2=u+2hu_x+2h^2u_{xx}. \end{aligned}$$

As explained in §4.1, taking an appropriate continuum limit we obtain the equivalent to the smooth curvature and the smooth arc–length in the discrete case. Explicitly

$$\lim_{h\to 0}\ \frac{\mathrm{d}}{\mathrm{d}h}\left(K_0\Big|_{Taylor}\right)=\begin{pmatrix} 0 & \dfrac{u_{xx}}{(1+u_{xx}^2)} & -\sqrt{1+u_x^2} \\ -\dfrac{u_{xx}}{(1+u_{xx}^2)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\lim_{h\to 0}\frac{K_0\Big|_{Taylor}-\mathrm{Id}_3}{-K_{0[1,3]}\Big|_{Taylor}}=\begin{pmatrix} 0 & \dfrac{u_{xx}}{(1+u_{xx}^2)^{\frac{3}{2}}} & -1 \\ -\dfrac{u_{xx}}{(1+u_{xx}^2)^{\frac{3}{2}}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

6.3 The projective S$L(2)$ action acting on multispaces

Recall the action (2.6)

$$(x, u(x)) \to g \cdot (x, u(x)) = \left(x, \frac{au(x)+b}{cu(x)+d}\right).$$

In this example, we will consider prolongations of order 2. Recall from (2.8)

$$\widetilde{x} = x, \quad \widetilde{u} = \frac{au+b}{cu+d}, \quad \widetilde{u_x} = \frac{u_x}{(cu+d)^2}, \quad \widetilde{u_{xx}} = \frac{u_{xx}}{(cu+d)^2} - 2\frac{cu_x^2}{(cu+d)^3}.$$

Therefore, we choose an order 2 interpolator to the points t_0, t_1 and t_2 with base point $t_0 = 0$. Our interpolator will be

$$p(u(t)) = A(u(t)) + B(u(t))t + \frac{1}{2}C(u(t))t^2.$$

In order to obtain an expression for the coefficients $A(u(t)), B(u(t))$ and $C(u(t))$ we solve the equations

$$\begin{aligned} u_0 &= A(u(t)) + B(u(t))t_0 + \frac{1}{2}C(u(t))t_0^2, \\ u_1 &= A(u(t)) + B(u(t))t_1 + \frac{1}{2}C(u(t))t_1^2, \\ u_2 &= A(u(t)) + B(u(t))t_2 + \frac{1}{2}C(u(t))t_2^2 \end{aligned}$$

for $A(u(t))$, $B(u(t))$ and $C(u(t))$. Using Cramer's rule we obtain

$$\begin{aligned} A(u(t)) &= u_0, \\ B(u(t)) &= \frac{(u_2-u_0)t_1^2 - (u_1-u_0)t_2^2}{t_1t_2(t_1-t_2)}, \\ C(x(t)) &= 2\frac{(u_2-u_0)t_1 - (u_1-u_0)t_2}{t_1t_2(t_2-t_1)}. \end{aligned}$$

Let us set

$$\begin{aligned} \mathcal{M}(u) &= u_0, \\ \mathcal{M}(u_x) &= \frac{(u_2-u_0)t_1^2 - (u_1-u_0)t_2^2}{t_1t_2(t_1-t_2)}, \\ \mathcal{M}(u_{xx}) &= 2\frac{(u_2-u_0)t_1 - (u_1-u_0)t_2}{t_1t_2(t_2-t_1)}. \end{aligned} \tag{6.3.1}$$

Consider the Taylor's series

$$\begin{aligned} &t_1 = h, t_2 = 2h, u_0 = u, \\ &u_1 = u + hu_x + \frac{1}{2}h^2u_{xx}, u_2 = u + 2hu_x + 2h^2u_{xx}. \end{aligned} \tag{6.3.2}$$

Substituting (6.3.2) into (6.3.1) and taking the limit when h tends to zero, we have that

$$\mathcal{M}(u) \to u, \qquad \mathcal{M}(u_x) \to u_x, \qquad \mathcal{M}(u_{xx}) \to u_{xx}.$$

Recall the prolongation action of the S$L(2)$ projective action (2.8) and the action in the discrete case (4.3.48).

For $\mathcal{M}(u), \mathcal{M}(u_x)$ and $\mathcal{M}(u_{xx})$ we have

$$\begin{aligned}
\widetilde{\mathcal{M}(u)} &= \widetilde{u_0}, \\
\widetilde{\mathcal{M}(u_x)} &= \frac{(\widetilde{u_2} - \widetilde{u_0})t_1^2 - (\widetilde{u_1} - \widetilde{u_0})t_2^2}{t_1 t_2 (t_1 - t_2)}, \\
\widetilde{\mathcal{M}(u_{xx})} &= 2\frac{(\widetilde{u_2} - \widetilde{u_0})t_1 - (\widetilde{u_1} - \widetilde{u_0})t_2}{t_1 t_2 (t_1 - t_2)}
\end{aligned} \tag{6.3.3}$$

where $\widetilde{u_0}, \widetilde{u_1}$ and $\widetilde{u_2}$ are given by

$$\widetilde{u_0} = \frac{au_0 + b}{cu_0 + d}, \qquad \widetilde{u_1} = \frac{au_1 + b}{cu_1 + d}, \qquad \widetilde{u_2} = \frac{au_2 + b}{cu_2 + d}.$$

One can check that substituting (6.3.2) into (6.3.3) and taking the limit when h tends to 0 results in

$$\begin{aligned}
\mathcal{M}(u) &\to \frac{au + b}{cu + d}, \\
\mathcal{M}(u_x) &\to \frac{u_x}{(cu + d)^2}, \\
\mathcal{M}(u_{xx}) &\to \frac{u_{xx}}{(cu + d)^2} - 2\frac{cu_x^2}{(cu + d)^3}
\end{aligned}$$

which matches (2.8).

Recall in the smooth case, for the normalization equations (2.49)

$$\widetilde{u} = 0, \qquad \widetilde{u_x} = 1 \qquad \text{and} \qquad \widetilde{u_{xx}} = 0$$

we obtained (2.50)

$$a = \frac{1}{\sqrt{u_x}}, \qquad b = -\frac{u}{\sqrt{u_x}}, \qquad c = \frac{u_{xx}}{2u_x^{\frac{3}{2}}}.$$

For the normalization equations

$$\widetilde{\mathcal{M}(u)} = 0, \qquad \widetilde{\mathcal{M}(u_x)} = 1 \qquad \text{and} \qquad \widetilde{\mathcal{M}(u_{xx})} = 0$$

we obtain

$$
\begin{aligned}
a &= -\sqrt{\frac{t_1t_2(u_1-u_2)}{(u_0-u_2)(u_0-u_1)(t_1-t_2)}} := \mathcal{M}(a),\\
b &= u_0\sqrt{\frac{t_1t_2(u_1-u_2)}{(u_0-u_2)(u_0-u_1)(t_1-t_2)}} := \mathcal{M}(b),\\
c &= -\frac{(t_1-t_2)u_0-t_1u_2+t_2u_1}{\sqrt{(u_0-u_2)(u_0-u_1)(u_1-u_2)(t_1-t_2)t_1t_2}} := \mathcal{M}(c).
\end{aligned}
$$

Again, using (6.3.2) and taking the limit when h tends to zero we have that

$$
\mathcal{M}(a)\to\frac{1}{\sqrt{u_x}},\qquad \mathcal{M}(b)\to-\frac{u}{\sqrt{u_x}},\qquad \mathcal{M}(c)\to\frac{u_{xx}}{2u_x^{\frac{3}{2}}}.
$$

The table of infinitesimals for the multispace action is

	$\mathcal{M}(u)$	$\mathcal{M}(u_x)$	$\mathcal{M}(u_{xx})$
a	$2u_0$	$-2\dfrac{(u_0-u_2)t_1^2-(u_0-u_1)t_2^2}{t_1t_2(t_1-t_2)}$	$4\dfrac{(u_0-u_2)t_1-(u_0-u_1)t_2}{t_1t_2(t_1-t_2)}$
b	1	0	0
c	$-u_0^2$	$\dfrac{(u_0^2-u_2^2)t_1^2-(u_0^2-u_1^2)t_2^2}{t_1t_2(t_1-t_2)}$	$2\dfrac{(t_2-t_1)u_0+2t_1u_2^2-2t_2u_1^2}{t_1t_2(t_1-t_2)}$

Note that the first column is the table of infinitesimals for the discrete case. Also, substituting (6.3.2) and taking the limit when h tends to zero, one can check that the table of infinitesimals for the multispace action converges to the one of the smooth case (2.44).

The convergence of the smooth curvature invariants to the discrete ones and the Maurer-Cartan matrix to the curvature matrix with respect to x requires further research.

Variational Systems with a Euclidean Symmetry using the Rotation Minimizing Frame

In this chapter, we study variational systems for space curves, for which the Lagrangian or action principle has a Euclidean symmetry, using the Rotation Minimizing frame, also known as the Normal, Parallel or Bishop frame (see Bishop, [8] and Wang and Joe, [110]). Such systems have previously been studied using the Frenet–Serret frame. However, the Rotation Minimizing frame has many advantages and can be used to study a wider class of examples.

We achieve our results by extending the powerful symbolic invariant calculus for Lie group based moving frames, to the Rotation Minimizing frame case. To date, the invariant calculus has been developed for frames defined by algebraic equations. By contrast, the Rotation Minimizing frame is defined by a differential equation.

We derive the recurrence formulae for the symbolic invariant differentiation of the symbolic invariants. We then derive the syzygy operator needed to obtain Noether's conservation laws as well as the Euler–Lagrange equations directly in terms of the invariants, for variational problems with a Euclidean symmetry. We show how to use the six Noether laws to ease the integration problem for the minimizing curve, once the Euler–Lagrange equations have been solved for the generating differential invariants. Our applications include variational problems used in the study of strands of proteins, nucleic acids and polymers.

7.1 Introduction

The study of variational problems with Euclidean symmetry is an old problem, indeed, Euler's 1744 study of elastic beams is such a case. However, methods to analyse such problems efficiently and effectively, are still of interest.

In this chapter, we consider variational problems for curves in 3-space for which the Lagrangian is invariant under the special Euclidean group $\mathrm{S}E(3) = \mathrm{S}O(3) \ltimes \mathbb{R}^3$ acting linearly in the standard way, that is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto R \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \qquad R \in \mathrm{S}O(3). \tag{7.1.1}$$

The Euler–Lagrange equations satisfied by the extremising curves have $\mathrm{S}E(3)$ as a Lie symmetry group, and can therefore be written in terms of the differential invariants of the action, and their derivatives with respect to arc-length. Further, the six dimensional space of Noether's laws are key to analysing the space of extremals.

To date, the Frenet–Serret frame has been used to analyse Euclidean invariant variational problems, and this requires that the Lagrangian can be written in terms of the Euclidean curvature and torsion. Because the Frenet–Serret frame can be derived using *algebraic* equations (at each point) on the relevant jet bundle, the powerful symbolic calculus of invariants can be used, to obtain not only the Euler–Lagrange equations directly in terms of the curvature and torsion, but the full set of Noether's laws can also be written down directly using both the invariants and the frame (Gonçalvez and Mansfield, [33]).

Let us denote the space curve as $s \mapsto P(s) \in \mathbb{R}^3$, where s is arc-length, and the tangent vector to this curve by P', so that $' = \mathrm{d}/\mathrm{d}s$. By the definition of arc-length, $|P'|^2 = P' \cdot P' = 1$. Then provided $P'' \neq 0$, the left Frenet–Serret frame is given by

$$\sigma^{\ell}_{FS} = \left(P'(s) \;\; \frac{P''(s)}{||P''(s)||} \;\; \frac{P'(s) \times P''(s)}{||P''(s)||} \right) \in \mathrm{S}O(3). \tag{7.1.2}$$

From a computational point of view, the Frenet–Serret frame is convenient as it can be computed straightforwardly at arbitrary points along the curve. However, it is undefined wherever the curvature is degenerate, such as at inflection points or along straight sections of the curve. The left Frenet–Serret frame is *left equivariant*, that is, if at any point $z = P(s)$ on the curve, since $R \in \mathrm{S}O(3)$ acts linearly in the standard way on the tangent space $T_z\mathbb{R}^3$, then it is readily seen that

$$\sigma^{\ell}_{FS} \mapsto \left(RP'(s) \;\; \frac{RP''(s)}{||RP''(s)||} \;\; \frac{RP'(s) \times RP''(s)}{||RP''(s)||} \right) = R\sigma^{\ell}_{FS}.$$

The Euclidean curvature κ and the torsion τ at the point $P(s)$ are then the nonzero components of the invariant so-called curvature matrix, specifically,

$$\sigma_{FS}^{-1}\sigma_{FS}' = \begin{pmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}. \tag{7.1.3}$$

In contrast to this frame, *relatively parallel* frames were described by Bishop, [8] who detailed what is now known variously as the Normal, Parallel, Bishop or Rotation Minimizing frame. The Rotation Minimizing frame has many advantages over the Frenet–Serret frame. First of all, unlike the Frenet–Serret frame, the Rotation Minimizing frame is defined at all points of a smooth curve. The Rotation Minimizing frame may be used to study a larger class of variational problems, because while the generating invariants for the symbolic invariant calculus given by the Frenet–Serret frame, curvature and torsion, are of order 2 and 3 respectively, those given by the Rotation Minimizing frame are both of order only 2. Finally, for the Rotation Minimizing frame, its computation, approximation and its applications, have been extensively used and studied in the Computer Aided Design literature,(see Bloomenthal and Riensenfeld, [9], Pottmann and Wagner, [96], Siltanen and Woodward, [100], Han, [38], Farouki,[28], Farouki and Sakkalis, [29], Farouki, Gentili, Giannelli, Sestini and Stoppato, [23], Klok, [62], Poston, Fang and Lawton, [95], Guggenheimer, [37], Wang, Jüttler, Zheng and Liu, [111]).

One reason is that the sweep surfaces they generate are, in general, superior, (see Wang and Joe, [110]); as illustrated in Figure 7.1, sweep surfaces generated from the Frenet–Serret frame can exhibit strong twisting at inflection points.

Bishop, [8] defines a normal vector field along a curve to be *relatively parallel* if its derivative is proportional to the tangent vector. The equation used in the Computer Aided Design literature for the relatively parallel normal vector $V = V(s)$ to the curve $s \mapsto P(s)$ is (see Wang and Joe, [110]),

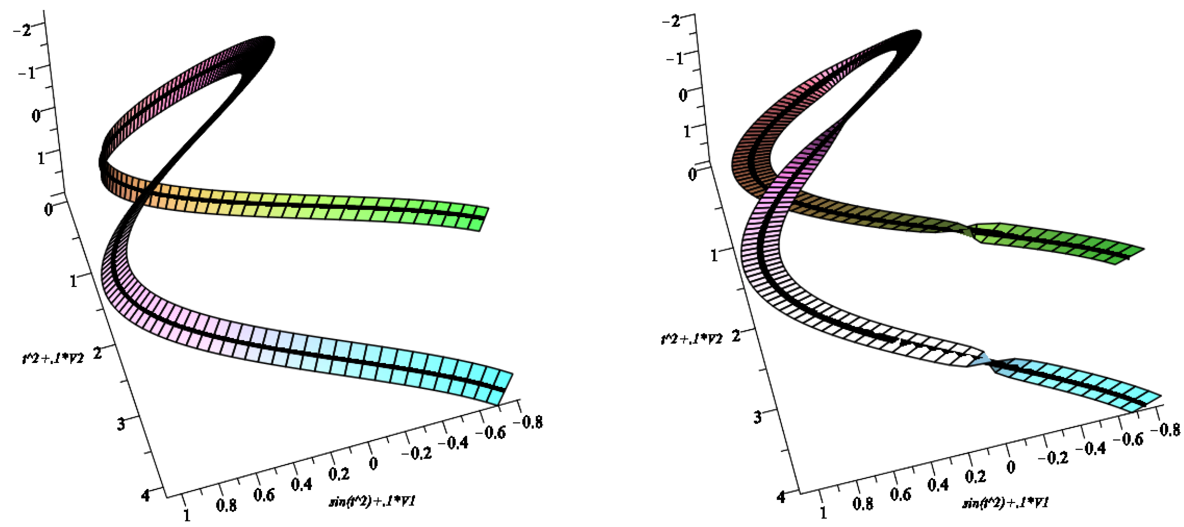
$$V' = -(P'' \cdot V)P'. \tag{7.1.4}$$

The function of proportionality between V' and P' is chosen to guarantee that, without loss of generality, we may suppose that $|V| \equiv 1$ and $P' \cdot V \equiv 0$, see Proposition 7.2.4. Then the left Rotation Minimizing frame is

$$\sigma_{RM}^{\ell} = (P' \quad V \quad P' \times V). \tag{7.1.5}$$

We have that σ_{RM}^{ℓ} is left equivariant and, as shown by Bishop, [8], the invariant curvature

Figure 7.1: Given a curve in space, we compare the sweeping surface generated by the Frenet–Serret frame with the one generated by the Rotation Minimizing frame along the curve. In this case, the curved plotted is $(\sin t^2, t^2, t)$. We can see that the Rotation Minimizing frame gives a less abrupt surface so it is more preferable than the Frenet-Serret frame for computer design purposes.



Surface sweeping given by V using the Rotation Minimizing frame

Surface sweeping given by P'' using the Frenet–Serret frame

matrix $\left(\sigma^\ell_{RM}\right)^{-1}\left(\sigma^\ell_{RM}\right)'$ takes the form

$$\left(\sigma^\ell_{RM}\right)^{-1}\left(\sigma^\ell_{RM}\right)' = \begin{pmatrix} 0 & -\kappa_1 & -\kappa_2 \\ \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & 0 \end{pmatrix}, \tag{7.1.6}$$

that is, where the $(2,3)$-component is guaranteed to be zero.

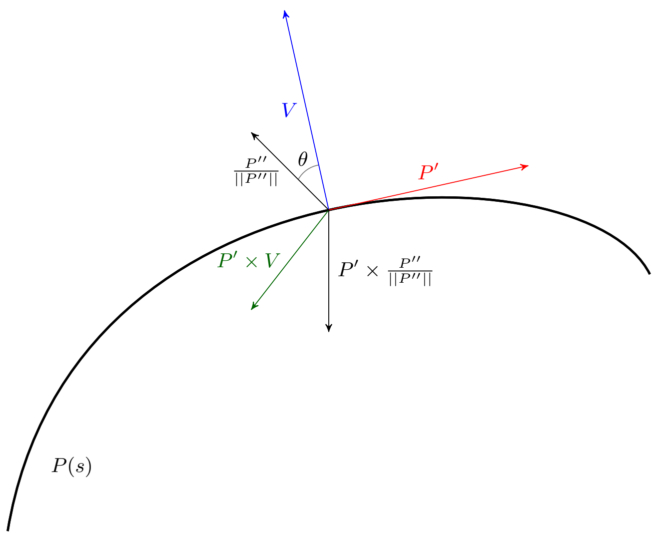
Since both the Rotation Minimizing and the Frenet–Serret frames share the same first column, we have for some angle $\theta = \theta(s)$, (see Figure (7.2)),

$$\sigma^\ell_{RM} = \sigma^\ell_{FS} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{7.1.7}$$

Calculating $\left(\sigma^\ell_{RM}\right)^{-1}\left(\sigma^\ell_{RM}\right)'$, using (7.1.3), and (7.1.7), and comparing the result to (7.1.6) leads to the well known relations,

$$\kappa_1 = \kappa\cos\theta, \qquad \kappa_2 = \kappa\sin\theta, \qquad \theta_s = \tau. \tag{7.1.8}$$

Figure 7.2: Diagram of a Rotation Minimizing frame and a Frenet–Serret frame of a curve $P(s)$ in $\mathbb{R}^3$. Note that $P'(s)$ is common in both frames.



Treating the Rotation Minimizing frame as a gauge transformation of the Frenet–Serret frame, together with

$$\theta(s) - \theta_0 = \int_{s_0}^{s} \tau(s)\ \mathrm{d}s$$

has been proven to lack numerical robustness for a general space curve, (see Guggenheimer, [37]). This makes the use of the Rotation Minimizing frame defined in terms of the normal vector V, as in (7.1.5), to be a better choice in the application literature, and is our choice here.

As shown in §2.5, the formulae for the recurrence relations in the symbolic invariant calculus require the equations defining the frame to be algebraic at each point in the domain of the frame, and indeed, the equations defining the Frenet–Serret frame, despite involving the components of $P(s)$, $P'(s)$ and $P''(s)$, are algebraic at each point of the relevant jet bundle. However, the recurrence formulae for the invariant derivatives defined using the Rotation Minimizing frame need to be derived in another way, because the equations defining the frame are not algebraic in the jet variables. Indeed, considering (7.1.8), it would seem that the Rotation Minimizing frame is defined by a relation on the invariants, τ and θ_s, or, a differential equation on an extended space, one which includes either θ, or V.

Our approach is to extend the manifold on which the group acts, to include the vector V and its derivatives, in such a way that the differential equation defining V is a simple constraint for our variational problem. Because the group acts linearly on P', V and their derivatives, it turns out to be straightforward to write down a set of generating invariants, the recurrence formulae for their invariant differentiation and their differential syzygies. With these to hand, the methods used by Gonçalvez and Mansfield, [33] can be adapted to obtain Euler–Lagrange equations directly in terms of the invariants and to write down the six Noether conservation laws.

In §7.2, the symbolic invariantized form of the curvature matrices for the Rotation Minimizing frame are found, and we derive the recurrence formulae for the symbolic differential invariants and the syzygy operator we will need in the sequel.

In §7.3, we obtain the Euler–Lagrange equations and Noether's laws for a Lagrangian with a Euclidean symmetry, using the results of §7.2.

In §7.4, the use of Noether's laws to ease the integration problem is carried out.

In §7.5, some examples and applications are presented.

7.2 The extended right Rotation Minimizing frame

We will consider derivatives with respect to arc-length s of our curve $s \mapsto P(s)$, where we note that arc-length is a Euclidean invariant, and we will also consider the evolution of this curve with respect to a 'time' parameter t, which we declare to be invariant under our $\mathrm{S}E(3)$ action.

Since the symbolic invariant calculus is standardly carried out for a right frame, we consider a right Rotation Minimizing frame, ρ_{RM}, which we need for our application to include the translation component of the Special Euclidean group $\mathrm{S}E(3)$.

We consider the Lie group $\mathrm{S}E(3)$ to act on an enlarged manifold (jet bundle) having local coordinates to be the components of

$$P, P', P'', \ldots, P^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}P, \ldots, V, V', V'', \ldots V^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}s^n}V, \ldots$$

where the left action is, for $g = (R, \mathbf{a}) \in \mathrm{S}E(3) = \mathrm{S}O(3) \ltimes \mathbb{R}^3$,

$$P \mapsto RP + \mathbf{a}, \qquad P^{(n)} \mapsto RP^{(n)}, n > 0, \qquad V^{(n)} \mapsto RV^{(n)}, n \geq 0.$$

In the standard representation of $\mathrm{S}E(3)$ in $GL(4, \mathbb{R})$,

$$g = (R, \mathbf{a}) \mapsto \begin{pmatrix} R & \mathbf{a} \\ 0 & 1 \end{pmatrix},$$

our *extended right Rotation Minimizing frame* for this action is defined to be,

$$\rho_{RM} = \begin{pmatrix} \sigma_{RM} & -\sigma_{RM}P \\ 0 & 1 \end{pmatrix} \qquad (7.2.1)$$

where

$$\sigma_{RM} = \left(\sigma^{\ell}_{RM}\right)^T \in \mathrm{SO}(3). \tag{7.2.2}$$

The curvature matrix is, by direct calculation and noting that $\sigma_{RM} P' = (1\ 0\ 0)^T$,

$$Q^s = \rho'_{RM}\rho^{-1}_{RM} = \left(\begin{array}{c|c} & -1 \\ \sigma'_{RM}\sigma^{-1}_{RM} & 0 \\ & 0 \\ \hline 0 & 0 \end{array}\right). \tag{7.2.3}$$

To obtain the complete set of normalized invariants and the (reduced) curvature matrix $\sigma'_{RM}\sigma^{-1}_{RM}$, we first consider solutions of the defining equation for V.

Proposition 7.2.4. *Given a curve $s \mapsto P(s) \in \mathbb{R}^3$ such that $P' \cdot P' = |P'|^2 = 1$, and suppose that $V = V(s)$ satisfies equation* (7.1.4)*, which for convenience we give again here,*

$$V' = -(P'' \cdot V)P' \tag{7.2.5}$$

together with the initial conditions $V(s_0) = 1$, $V(s_0) \cdot P'(s_0) = 0$. Then

1. $V \cdot P' \equiv 0$
2. $V \cdot V \equiv 1$
3. *For any constant $\psi \in \mathbb{R}$,*

   $$W = \cos\psi\, V + \sin\psi\, P' \times V$$

   also solves equation (7.2.5) *with $|W| \equiv 1$ and $W \cdot P' \equiv 0$.*

Proof. 1. By direct calculation, the scalar product $V \cdot P'$ is constant with respect to s. The result follows from the assumption on the initial data.

2. Equation (7.2.5) implies $V' \cdot V = -(P'' \cdot V)(P' \cdot V) = 0$ by 1. above. Hence $V \cdot V$ is constant with respect to s. The result follows from the assumption on the initial data.

3. Since (7.2.5) is linear in V, it suffices to prove that $W = P' \times V$ also solves Equation (7.2.5). We have by the orthogonality of both V and P'' to P that $V = b(s)P'' + c(s)P' \times P''$ for some coefficients $b(s)$, $c(s)$. Then $P' \times V = b(s)P' \times P'' - c(s)P''$ and

$$\begin{aligned} (P' \times V)' &= P'' \times V + P' \times V' \\ &= P'' \times V \\ &= c(s)(P'' \cdot P'')P'. \end{aligned}$$

But $P'' \cdot (P' \times V) = -c(s)P'' \cdot P''$ and hence

$$W' = -(P'' \cdot W) \cdot P'$$

as required.

□

The proposition shows that if V solves (7.2.5) and for some s_0, $V(s_0)$ has unit length and is orthogonal to $P(s_0)$, then $\sigma_{RM} \in \mathrm{SO}(3)$ for all s, and this we now assume. In the applications, it is necessary to ensure the initial data for V holds when integrating for the frame. The proposition shows further that in fact there is a one-parameter family of Rotation Minimizing frames, determined by the initial data for V.

Let $\mathfrak{so}(3)$ denote the set of 3×3 skew-symmetric matrices, the Lie algebra of $\mathrm{S}O(3)$. We have by direct calculation that

$$\sigma'_{RM}\sigma_{RM}^{-1} = \begin{pmatrix} 0 & P'' \cdot V & P'' \cdot (P' \times V) \\ -P'' \cdot V & 0 & 0 \\ -P'' \cdot (P' \times V) & 0 & 0 \end{pmatrix} \in \mathfrak{so}(3). \tag{7.2.6}$$

We now write down the symbolic normalized invariants, and obtain $\sigma'_{RM}\sigma_{RM}^{-1}$ in terms of them. We denote the components of $P(s)$ as $P(s) = (X(s), Y(s), Z(s))$ and that of the n-th derivative with respect to s as $P^{(n)} = (X^{(n)}, Y^{(n)}, Z^{(n)})$.

By construction,

$$\rho_{RM} \cdot P = 0$$

and by definition of the action,

$$\rho_{RM} \cdot P^{(n)} = \sigma_{RM} P^{(n)}$$

where $n > 0$. We now recall the standard symbolic names of these normalized invariants, as

$$\sigma_{RM} P^{(n)} = (\iota(X^{(n)}), \iota(Y^{(n)}), \iota(Z^{(n)}))^T. \tag{7.2.7}$$

Since $((\iota(X'), \iota(Y'), \iota(Z'))^T = \sigma_{RM} P' = (P' \cdot P', V \cdot P', (P' \times V) \cdot P')^T = (1, 0, 0)^T$, we make the following definition:

Definition 7.2.8 (Arc-length constraint)**.** *The equation $\iota(X') = 1$ is denoted as the* arc-length constraint.

Differentiating (7.2.7) with respect to s, yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\iota(X^{(n)})\\ \iota(Y^{(n)})\\ \iota(Z^{(n)})\end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}s}(\sigma_{RM})\sigma_{RM}^{-1}\begin{pmatrix}\iota(X^{(n)})\\ \iota(Y^{(n)})\\ \iota(Z^{(n)})\end{pmatrix} + \begin{pmatrix}\iota(X^{(n+1)})\\ \iota(Y^{(n+1)})\\ \iota(Z^{(n+1)})\end{pmatrix}. \tag{7.2.9}$$

Setting $n = 1$ and recalling

$$\sigma_{RM}P' = (1, 0, 0)^T,$$

we have from (7.2.6) and (7.2.9) that

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}\iota(X'')\\ \iota(Y'')\\ \iota(Z'')\end{pmatrix} + \begin{pmatrix}0\\ -P''\cdot V\\ -P''\cdot(P'\times V)\end{pmatrix}.$$

Therefore we can write down $\frac{\mathrm{d}}{\mathrm{d}s}(\sigma_{RM})\sigma_{RM}^{-1}$ in terms of the normalized invariants, specifically,

$$\frac{\mathrm{d}}{\mathrm{d}s}(\sigma_{RM})\sigma_{RM}^{-1} = \begin{pmatrix}0 & \iota(Y'') & \iota(Z'')\\ -\iota(Y'') & 0 & 0\\ -\iota(Z'') & 0 & 0\end{pmatrix}. \tag{7.2.10}$$

Inserting this into Equation (7.2.9) yields the all important recurrence formulae for the symbolic invariant differentiation of the normalized invariants of the $P^{(n)}$.

We next consider the normalized invariants of the $V^{(n)}$, which are

$$\sigma_{RM}V^{(n)} = (\iota(V_1^{(n)}), \iota(V_2^{(n)}), \iota(V_3^{(n)}))^T, \quad n \geq 0. \tag{7.2.11}$$

Differentiating both sides of (7.2.11) with respect to s yields the recurrence formula for the invariant differentiation of the symbolic normalized invariants of the components of $V^{(n)}$,

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\iota(V_1^{(n)})\\ \iota(V_2^{(n)})\\ \iota(V_3^{(n)})\end{pmatrix} = \frac{\mathrm{d}}{\mathrm{d}s}(\sigma_{RM})\sigma_{RM}^{-1}\begin{pmatrix}\iota(V_1^{(n)})\\ \iota(V_2^{(n)})\\ \iota(V_3^{(n)})\end{pmatrix} + \begin{pmatrix}\iota(V_1^{(n+1)})\\ \iota(V_2^{(n+1)})\\ \iota(V_3^{(n+1)})\end{pmatrix}. \tag{7.2.12}$$

Setting $n = 0$ into this, and since $\sigma_{RM}V = (0, 1, 0)^T$ we have that

$$\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}\iota(V_1')\\ \iota(V_2')\\ \iota(V_3')\end{pmatrix} + \begin{pmatrix}\iota(Y'')\\ 0\\ 0\end{pmatrix}. \tag{7.2.13}$$

Finally, taking a right orthonormal frame $\sigma_{RM} = (P' \; V \; P' \times V)^T$, where we have momentarily relaxed the differential equation condition on V, calculate $\sigma_{RM}'\sigma_{RM}^{-1}$ and write the components in terms of the normalized invariants using the Replacement Rule, (2.4.6), we obtain

$$\sigma_{RM}'\sigma_{RM}^{-1} = \begin{pmatrix} 0 & \iota(Y'') & \iota(Z'') \\ -\iota(Y'') & 0 & \iota(V_3') \\ -\iota(Z'') & -\iota(V_3') & 0 \end{pmatrix}. \tag{7.2.14}$$

We thus see that $(2,3)$-component of $\sigma_{RM}'\sigma_{RM}^{-1}$ being zero, which is what makes σ_{RM} a Rotation Minimizing frame, yields a constraint on the symbolic invariant $\iota(V_3')$. The invariantization of the differential equation for V yields

$$\begin{pmatrix} \iota(V_1') \\ \iota(V_2') \\ \iota(V_3') \end{pmatrix} = -\iota(Y'') \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Using calculations similar to those above, it can be seen that the first two components of this equation relate to the orthonormality of V with respect to P'. We thus make the following definition:

Definition 7.2.15 (Rotation Minimizing frame constraint)**.** *The equation* $\iota(V_3') = 0$ *is denoted as the* Rotation Minimizing frame constraint*.*

When deriving the differential syzygy needed in the sequel, we will write the (reduced) curvature matrix with respect to s for the Rotation Minimizing frame as

$$\frac{\mathrm{d}}{\mathrm{d}s}(\sigma_{RM})\sigma_{RM}^{-1} = \begin{pmatrix} 0 & \iota(Y'') & \iota(Z'') \\ -\iota(Y'') & 0 & \iota(V_3') \\ -\iota(Z'') & -\iota(V_3') & 0 \end{pmatrix}, \qquad \iota(V_3') = 0. \tag{7.2.16}$$

This is because we will need to calculate the evolution of $\iota(V_3')$ with respect to time.

7.2.1 The time evolution of the frame

We now suppose that our curve $s \mapsto P(s)$ evolves in time. The time derivatives of our variables are denoted as $\dfrac{\mathrm{d}}{\mathrm{d}t}P^{(n)} = P_t^{(n)}$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}V^{(n)} = V_t^{(n)}$ and the action is, for $g = (R, \mathbf{a}) \in \mathrm{SO}(3) \ltimes \mathbb{R}$, and all $n \geq 0$,

$$P_t^{(n)} \mapsto g \cdot P_t^{(n)} = RP_t^{(n)} \quad \text{and} \quad V_t^{(n)} \mapsto g \cdot V_t^{(n)} = RV_t^{(n)}.$$

The normalized differential invariants are the components of

$$\iota(P_t^{(n)}) = \sigma_{RM} P_t^{(n)}, \qquad \iota(V_t^{(n)}) = \sigma_{RM} V_t^{(n)}, \quad n = 0, 1, 2, \ldots$$

The curvature matrix for the extended Rotation Minimizing frame, with respect to time, is

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{RM}\rho_{RM}^{-1} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_{RM}\,\sigma_{RM}^{-1} & -\sigma_{RM}P_t \\ 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} \frac{\mathrm{d}}{\mathrm{d}t}\sigma_{RM}\,\sigma_{RM}^{-1} & \begin{array}{c} -\iota(X_t) \\ -\iota(Y_t) \\ -\iota(Z_t) \end{array} \\ \hline 0 & 0 \end{array}\right). \tag{7.2.17}$$

Calculating the invariant matrix $\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{RM})\sigma_{RM}^{-1} \in \mathfrak{so}(3)$ yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{RM})\sigma_{RM}^{-1} &= \begin{pmatrix} 0 & P'_t \cdot V & P'_t \cdot (P' \times V) \\ -P'_t \cdot V & 0 & V_t \cdot (P' \times V) \\ -P'_t \cdot (P' \times V) & -V_t \cdot (P' \times V) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \iota(Y'_t) & \iota(Z'_t) \\ -\iota(Y'_t) & 0 & \iota(V'_{3,t}) \\ -\iota(Z'_t) & -\iota(V'_{3,t}) & 0 \end{pmatrix} \end{aligned}$$

where we have used the Replacement Rule, Theorem (2.4.6), recalling $\sigma_{RM}P' = (1\ 0\ 0)^T$ and $\sigma_{RM}V = (0\ 1\ 0)^T$.

Differentiating both sides of $\sigma_{RM}P' = (1, 0, 0)^T$ with respect to t yields

$$\sigma_{RM}P'_t + \left(\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{RM})\sigma_{RM}^{-1}\right)(\sigma_{RM}P') = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so that indeed,

$$\begin{pmatrix} \iota(X'_t) \\ \iota(Y'_t) \\ \iota(Z'_t) \end{pmatrix} = \begin{pmatrix} 0 \\ P'_t \cdot V \\ P'_t \cdot (P' \times V) \end{pmatrix}.$$

Further, differentiating both sides of $\sigma_{RM}V = (0, 1, 0)^T$ with respect to t yields

$$\sigma_{RM}V_t + \left(\frac{\mathrm{d}}{\mathrm{d}t}(\sigma_{RM})\sigma_{RM}^{-1}\right)(\sigma_{RM}V) = (0\ \ 0\ \ 0)^T$$

so that

$$\begin{pmatrix} \iota(V_{1,t}) \\ \iota(V_{2,t}) \\ \iota(V_{3,t}) \end{pmatrix} = \begin{pmatrix} -P_t' \cdot V \\ 0 \\ V_t \cdot (P' \times V) \end{pmatrix}.$$

7.2.2 The syzygy operator $\mathcal{H}$

Recall the extended Rotation Minimizing frame, ρ_{RM}, and the curvature matrices, $Q^s = \rho_{RM}'\rho_{RM}^{-1}$, $Q^t = \frac{\mathrm{d}}{\mathrm{d}t}\rho_{RM}\rho_{RM}^{-1}$. Equations (7.2.1), (7.2.3), (7.2.17) and repeated here for convenience,

$$\rho_{RM} = \begin{pmatrix} \sigma_{RM} & -\sigma_{RM}P \\ 0 & 1 \end{pmatrix} \tag{7.2.18}$$

and

$$Q^s = \rho_{RM}'\rho_{RM}^{-1} = \left(\begin{array}{c|c} & -\iota(X') \\ \sigma_{RM}'\sigma_{RM}^{-1} & 0 \\ & 0 \\ \hline 0 & 0 \end{array}\right). \tag{7.2.19}$$

where we have not yet imposed the arc length constraint $\iota(X') = 1$ since we need its time evolution, and

$$Q^t = \frac{\mathrm{d}}{\mathrm{d}t}\rho_{RM}\rho_{RM}^{-1} = \left(\begin{array}{c|c} & -\iota(X_t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\sigma_{RM}\,\sigma_{RM}^{-1} & -\iota(Y_t) \\ & -\iota(Z_t) \\ \hline 0 & 0 \end{array}\right). \tag{7.2.20}$$

The non-constant components of Q^s are the generating invariants of the algebra of invariants of the form $F = F(P, P', P'', \dots, V, V', V'', \dots)$; every invariant of this form can be written as a function of $\iota(Y'')$, $\iota(Z'')$ and their derivatives with respect to s.

The syzygy operator $\mathcal{H}$ that we need for our calculations in the Calculus of Variations, relates the time derivatives of these generating invariants to the s derivatives of the components of $\iota(P_t)$ and $\iota(V_t)$, occurring in Q^t. In our case here, the syzygy operator $\mathcal{H}$ can be calculated by examining the components of the compatibility condition of the curvature matrices Q^s and Q^t, (5.4.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}Q^s - \frac{\mathrm{d}}{\mathrm{d}s}Q^t = \left[Q^t, Q^s\right] \tag{7.2.21}$$

which follows from the fact the derivatives with respect to t and s commute (see [70], §5.2). We use $\sigma_{RM}'\sigma_{RM}^{-1}$ in the form of Equation (7.2.14), that is, with the Rotation Minimizing constraint not yet imposed, as we will need its variation with respect to time in the sequel.

Calculating the components of Equation (7.2.21) yields,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iota(X') &= \frac{\mathrm{d}}{\mathrm{d}s}\iota(X_t) - \iota(Y'')\iota(Y_t) - \iota(Z'')\iota(Z_t), \\
\frac{\mathrm{d}}{\mathrm{d}t}\iota(Y'') &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\iota(Y_t) + \frac{\mathrm{d}}{\mathrm{d}s}\left(\iota(Y'')\iota(X_t)\right) + \iota(V_{3,t})\iota(Z''), \\
\frac{\mathrm{d}}{\mathrm{d}t}\iota(Z'') &= \frac{\mathrm{d}^2}{\mathrm{d}s^2}\iota(Z_t) + \frac{\mathrm{d}}{\mathrm{d}s}\left(\iota(Z'')\iota(X_t)\right) - \iota(V_{3,t})\iota(Y''), \\
\frac{\mathrm{d}}{\mathrm{d}t}\iota(V_3') &= \frac{\mathrm{d}}{\mathrm{d}s}\iota(V_{3,t}) + \iota(Y'')\frac{\mathrm{d}}{\mathrm{d}s}\iota(Z_t) - \iota(Z'')\frac{\mathrm{d}}{\mathrm{d}s}\iota(Y_t)
\end{aligned} \tag{7.2.22}$$

or in the form we require,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \iota(X') \\ \iota(Y'') \\ \iota(Z'') \\ \iota(V_3') \end{pmatrix} = \mathcal{H}\begin{pmatrix} \iota(X_t) \\ \iota(Y_t) \\ \iota(Z_t) \\ \iota(V_{3,t}) \end{pmatrix}$$

where

$$\mathcal{H} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}s} & -\iota(Y'') & \iota(Z'') & 0 \\ \iota(Y'')\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\iota(Y'') & \frac{\mathrm{d}^2}{\mathrm{d}s^2} & 0 & \iota(Z'') \\ \iota(Z'')\frac{\mathrm{d}}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\iota(Z'') & 0 & \frac{\mathrm{d}^2}{\mathrm{d}s^2} & -\iota(Y'') \\ 0 & -\iota(Z'')\frac{\mathrm{d}}{\mathrm{d}s} & \iota(Y'')\frac{\mathrm{d}}{\mathrm{d}s} & \frac{\mathrm{d}}{\mathrm{d}s} \end{pmatrix}. \tag{7.2.23}$$

We note that $\mathcal{H}$ is an invariant, linear differential operator matrix.

7.3 Invariant Calculus of Variations

We consider an $\mathrm{S}E(3)$ invariant Lagrangian of the form

$$\mathcal{L}[X', Y', Z', X'', Y'', Z'', ...] = \int L(\kappa_1, \kappa_2, \kappa_{1,s}, \kappa_{2,s}, ...) + \mu\zeta + \lambda(\eta - 1)\,\mathrm{d}s$$

where we have set $\zeta = \iota(V_3')$, $\eta = \iota(X')$, $\kappa_1 = \iota(Y'')$ and $\kappa_2 = \iota(Z'')$, and where μ and λ are Lagrange multipliers for the Rotation Minimizing frame constraint (Definition 7.2.15) and the arc-length constraint (Definition 7.2.8) respectively.

Recall the Euler operator with respect to a dependent variable u is defined by

$$\mathrm{E}^u(L) = \sum_n (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\frac{\partial L}{\partial u^{(n)}}$$

where $u^{(n)} = \dfrac{\mathrm{d}^n}{\mathrm{d}s^n}u$. We will denote this operator by just E^u for simplification. We apply the invariantized version of the calculation of the Euler–Lagrange equations presented in §2.5.4, to obtain

$$0 = \begin{pmatrix} \mathrm{E}^X \\ \mathrm{E}^Y \\ \mathrm{E}^Z \\ \mathrm{E}^{V_3} \end{pmatrix} = \mathcal{H}^* \begin{pmatrix} \mathrm{E}^\eta \\ \mathrm{E}^{\kappa_1} \\ \mathrm{E}^{\kappa_2} \\ \mathrm{E}^\zeta \end{pmatrix}$$

that is,

$$0 = \mathrm{E}^X = -\kappa_1 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_1} - \kappa_2 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_2} - \lambda_s, \tag{7.3.1}$$

$$0 = \mathrm{E}^Y = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathrm{E}^{\kappa_1} + \frac{\mathrm{d}}{\mathrm{d}s}(\kappa_2\mu) - \kappa_1\lambda, \tag{7.3.2}$$

$$0 = \mathrm{E}^Z = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathrm{E}^{\kappa_2} - \frac{\mathrm{d}}{\mathrm{d}s}(\kappa_1\mu) - \kappa_2\lambda, \tag{7.3.3}$$

$$0 = \mathrm{E}^{V_3} = \mathrm{E}^{\kappa_1}\kappa_2 - \mathrm{E}^{\kappa_2}\kappa_1 - \mu_s. \tag{7.3.4}$$

Remark 7.3.5. *Note that*

$$-\kappa_1 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_1} - \kappa_2 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_2} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\kappa_1\mathrm{E}^{\kappa_1} + \kappa_2\mathrm{E}^{\kappa_2}\right) + \kappa_{1,s}\mathrm{E}^{\kappa_1} + \kappa_{2,s}\mathrm{E}^{\kappa_2}.$$

Also, by equation (7.17) *in Mansfield, [70] we have that*

$$\begin{aligned} &\kappa_{1,s}\mathrm{E}^{\kappa_1} + \kappa_{2,s}\mathrm{E}^{\kappa_1} \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}s}\left(L - \sum_{m=1}\sum_{k=0}^{m-1}(-1)^k\left(\left(\frac{\mathrm{d}^k}{\mathrm{d}s^k}\frac{\partial L}{\partial \kappa_{1,m}}\right)\kappa_{1,m-k} + \left(\frac{\mathrm{d}^k}{\mathrm{d}s^k}\frac{\partial L}{\partial \kappa_{2,m}}\right)\kappa_{2,m-k}\right)\right). \end{aligned}$$

Therefore, λ_s *is a total derivative and we obtain*

$$\lambda = -\kappa_1\mathrm{E}^{\kappa_1} - \kappa_2\mathrm{E}^{\kappa_2} + L - \sum_{m=1}\sum_{k=0}^{m-1}(-1)^k\left(\left(\frac{\mathrm{d}^k}{\mathrm{d}s^k}\frac{\partial L}{\partial \kappa_{1,m}}\right)\kappa_{1,m-k} - \left(\frac{\mathrm{d}^k}{\mathrm{d}s^k}\frac{\partial L}{\partial \kappa_{2,m}}\right)\kappa_{2,m-k}\right) \tag{7.3.6}$$

where the constant of integration has been absorbed into λ *by Remark* 7.1.9 *of Mansfield, [70]. This result for* λ *relates to the invariance of the Lagrangian under translation in* s*, that is, we have invariance under* $s \mapsto s + \epsilon$ *and hence a corresponding Noether law.*

To obtain the Noether conservation laws, we need to calculate the infinitesimals of our group action, its associated *matrix of infinitesimals*, and the right Adjoint action of the Lie group $\mathrm{S}E(3)$ on the infinitesimal vector fields. For the Lie group $\mathrm{S}E(3)$ and the left linear action, the precise calculations appear in Gonçalvez and Mansfield, [33] with the end results needed for our case here recorded in the proof of the following Theorem. Elements in the Lie group $\mathrm{S}E(3)$ are, in a neighbourhood of the identity element, described by six parameters, three translation parameters, a, b and c, and three rotation parameters, θ_{xy}, θ_{yz} and θ_{xz} where θ_{xy} is the (anticlockwise) rotation in the (x,y)-plane, and similarly for θ_{yz} and θ_{xz}.

We obtain that Noether's laws are as given in the following theorem:

Theorem 7.3.7. *The conservation laws are of the form*

$$\begin{pmatrix} \sigma_{RM}^T & 0 \\ D\mathbf{X}\sigma_{RM}^T & D\sigma_{RM}^T D \end{pmatrix} \begin{pmatrix} \lambda \\ -\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_1} - \mu\kappa_2 \\ -\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_2} + \mu\kappa_1 \\ \mu \\ \mathrm{E}^{\kappa_2} \\ \mathrm{E}^{\kappa_1} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} \tag{7.3.8}$$

where

$$\mathbf{X} = \begin{pmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{pmatrix}$$

$D = diag(1,-1,1)$, *and the* c_i *are constants.*

Proof. In order to compute the conservation laws, we need the boundary terms $\mathcal{A}_{\mathcal{H}}$, the (right) Adjoint representation of the frame ρ_{RM} and the invariantized matrix of infinitesimals, which we defined above. We now consider these in turn.

Let $\mathrm{E}(L) = \left(\mathrm{E}^{\eta}\ \mathrm{E}^{\kappa_1}\ \mathrm{E}^{\kappa_2}\ \mathrm{E}^{\zeta}\right)$ and let $\phi^t = (\iota(X_t)\ \iota(Y_t)\ \iota(Z_t)\ \iota(V_{3,t}))^T$. Then the boundary terms $\mathcal{A}_{\mathcal{H}}$ are defined by

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{A}_{\mathcal{H}} = E(L)\mathcal{H}\phi^t - \mathcal{H}^* E(L)\phi^t.$$

By direct calculation, we obtain

$$\begin{aligned} \mathcal{A}_{\mathcal{H}} &= \lambda\iota(X_t) + \left(-\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_1} - \mu\kappa_2\right)\iota(Y_t) + \left(-\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_2} + \mu\kappa_1\right)\iota(Z_t) \\ &\quad + \mathrm{E}^{\kappa_1}\iota(Y_t') + \mathrm{E}^{\kappa_2}\iota(Z_t') + \mu\iota(V_{3,t}) \\ &= \mathcal{C}^X\iota(X_t) + \mathcal{C}^Y\iota(Y_t) + \mathcal{C}^Z\iota(Z_t) + \mathcal{C}^{Y'}\iota(Y_t') + \mathcal{C}^{Z'}\iota(Z_t') + \mathcal{C}^{V_{3,t}}\iota(V_{3,t}) \end{aligned}$$

where this defines the coefficients $\mathcal{C}$ and where we have used the syzygies

$$\iota(Y_t') = \frac{\mathrm{d}}{\mathrm{d}s}\iota(Y_t) + \kappa_1\iota(X_t), \qquad \iota(Z_t') = \frac{\mathrm{d}}{\mathrm{d}s}\iota(Z_t) + \kappa_2\iota(X_t)$$

to eliminate derivatives of $\iota(Y_t)$ and $\iota(Z_t)$ in the boundary terms.

In Gonçalves and Mansfield, [33], the authors show the (right) Adjoint representation of

$SE(3)$ with respect to the generating infinitesimal vector fields of the action,

$$\mathbf{v}_a = \partial_X,\ \mathbf{v}_b = \partial_Y,\ \mathbf{v}_c = \partial_Z,\ \mathbf{v}_{YZ} = Y\partial_Z - Z\partial_Y,\ \mathbf{v}_{XZ} = X\partial_Z - Z\partial_X,\ \mathbf{v}_{XY} = X\partial_Y - Y\partial_X \tag{7.3.9}$$

is of the form, for $g = (R, \mathbf{a})$,

$$\mathcal{A}d(g) = \begin{pmatrix} R & 0 \\ DAR & DRD \end{pmatrix}$$

where $R \in \mathrm{SO}(3)$, D is the diagonal matrix $D = \mathrm{diag}(1, -1, 1)$ and A is the matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

where $\mathbf{a} = (a, b, c)^T$ is the translation vector component of g.

Hence

$$\mathcal{A}d(\rho_{RM})^{-1} = \begin{pmatrix} \sigma_{RM}^T & 0 \\ D\mathbf{X}\sigma_{RM}^T & D\sigma_{RM}^T D \end{pmatrix} \quad \text{where} \quad \mathbf{X} = \begin{pmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{pmatrix}.$$

The invariantized matrix of infinitesimals with respect to the basis (7.3.9) is

$$\Phi(I) = \begin{array}{c} \\ a \\ b \\ c \\ \theta_{yz} \\ \theta_{xz} \\ \theta_{xy} \end{array} \begin{array}{c} \begin{array}{cccccc} X & Y & Z & Y' & Z' & V_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

Finally, the conservation laws obtained via Noether's theorem for the unidimensional case are (2.67)

$$\mathcal{A}d(\rho)^{-1}\mathbf{V}(I) = \mathbf{c} \tag{7.3.10}$$

where

$$\mathbf{V}(I) = \sum_\alpha \Phi^\alpha(I)\mathcal{C}^\alpha = \begin{pmatrix} \lambda & -\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_1} - \mu\kappa_2 & -\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{E}^{\kappa_2} + \mu\kappa_1 & \mu & \mathrm{E}^{\kappa_2} & \mathrm{E}^{\kappa_1} \end{pmatrix}^T \tag{7.3.11}$$

as required.

□

Remark 7.3.12. *A quick check on this result is obtained by noting the following. Differentiating* (7.3.8) *with respect to* s *and multiplying by* $\mathcal{A}d(\rho_{RM})$, *we get*

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{V}(I) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathcal{A}d(\rho_{RM})\right)\mathcal{A}d(\rho)^{-1}\mathbf{V}(I)$$

i.e,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{V}(I) = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 & 0 & 0 & 0 \\ -\kappa_1 & 0 & 0 & 0 & 0 & 0 \\ -\kappa_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\kappa_1 & \kappa_2 \\ 0 & 0 & -1 & \kappa_1 & 0 & 0 \\ 0 & -1 & 0 & -\kappa_2 & 0 & 0 \end{pmatrix}\mathbf{V}(I). \tag{7.3.13}$$

We observe that the first four rows are equivalent to the Euler-Lagrange equations while last two rows are identically 0, as expected.

7.4 Solution of the integration problem

The conservation laws (7.3.8) can reduce the integration problem. We write these in the form,

$$\begin{pmatrix} \sigma_{RM}^T & 0 \\ D\mathbf{X}\sigma_{RM}^T & D\sigma_{RM}^T D \end{pmatrix}\begin{pmatrix} \mathbf{w}_1(I) \\ \mathbf{w}_2(I) \end{pmatrix} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} \tag{7.4.1}$$

where $\mathbf{V}(I) = (\mathbf{w}_1(I), \mathbf{w}_2(I))^T$, $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2)^T$ and recalling

$$\mathbf{X} = \begin{pmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{pmatrix}.$$

Since $\sigma_{RM} \in \mathrm{SO}(3)$ we have from

$$\sigma_{RM}\mathbf{c}_1 = \mathbf{w}_1(I) \tag{7.4.2}$$

that

$$|\mathbf{c}_1| = |\mathbf{w}_1(I)|. \tag{7.4.3}$$

Further, multiplying the second component of Equation (7.4.1) on the left by $\mathbf{c}_1(I)^T D$, since

$D^2 = I$, we obtain

$$\mathbf{w}_1^T D\mathbf{w}_2 = \mathbf{c}_1^T D\mathbf{c}_2. \tag{7.4.4}$$

In order to solve Equation (7.4.2), as far as we can, for the components of σ_{RM} in terms of the components of $\mathbf{c}_1$ and $\mathbf{w}_1(I)$, we use the Cayley representation $\mathfrak{C}$ of elements of $SO(3)$.

We define

$$\mathfrak{C}(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1^2 + x_2^2 - x_3^3 - x_4^2 & -2(x_1x_4 - x_2x_3) & 2(x_1x_3 + x_2x_4) \\ 2(x_1x_4 + x_2x_3) & x_1^2 - x_2^2 + x_3^3 - x_4^2 & -2(x_1x_2 - x_3x_4) \\ -2(x_1x_3 - x_2x_4) & 2(x_1x_2 + x_3x_4) & x_1^2 - x_2^2 - x_3^3 + x_4^2 \end{pmatrix}.$$

Then provided $x_1^2 + x_2^2 + x_3^3 + x_4^2 = 1$, $\mathfrak{C}(x_1, x_2, x_3, x_3) \in \mathrm{SO}(3)$, has an axis of rotation $(x_2, x_3, x_4)^T$ and the angle of rotation ψ satisfies $2x_1^2 - 1 = \cos\psi$. Hence we may define, for an angle ψ and axis of rotation $\mathbf{a} = (a_1, a_2, a_3)^T \neq 0$,

$$R(\psi, \mathbf{a}) = \mathfrak{C}\left(\cos\left(\frac{\psi}{2}\right), \sin\left(\frac{\psi}{2}\right)\frac{a_1}{|\mathbf{a}|}, \sin\left(\frac{\psi}{2}\right)\frac{a_2}{|\mathbf{a}|}, \sin\left(\frac{\psi}{2}\right)\frac{a_3}{|\mathbf{a}|}\right) \in \mathrm{SO}(3).$$

There are two cases.

Case 1. If $\mathbf{w}_1 + \mathbf{c}_1$ is bounded away from zero, we note that σ_{RM} may be taken to be a product of a rotation about $\mathbf{c}_1 + (0, 0, |\mathbf{c}_1|)^T$ with angle π followed by a rotation about $(0, 0, |\mathbf{c}_1|)^T$ with any angle ψ and a rotation about $\mathbf{w}_1 + (0, 0, |\mathbf{c}_1|)^T$ with angle π, that is,

$$\sigma_{RM} = R(\pi, \mathbf{w}_1 + (0, 0, |\mathbf{c}_1|)^T)R(\psi(s), (0, 0, |\mathbf{c}_1|)^T)R(\pi, \mathbf{c}_1 + (0, 0, |\mathbf{c}_1|)^T).$$

This solves for σ_{RM} up to the angle ψ. If we differentiate this with respect to s, right multiply by σ_{RM}^{-1}

$$\sigma_{RM}^{-1} = R(\pi, \mathbf{c}_1 + (0, 0, |\mathbf{c}_1|)^T)R(-\psi(s), (0, 0, |\mathbf{c}_1|)^T)R(\pi, \mathbf{w}_1 + (0, 0, |\mathbf{c}_1|)^T)$$

using (7.3.13) and taking into account that

$$\frac{\mathrm{d}}{\mathrm{d}s}(\sigma_{RM})\sigma_{RM}^{-1} = \begin{pmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{pmatrix}$$

we obtain a remarkable equation for ψ, specifically,

$$\psi_s = -\kappa_1 + \frac{V_2(I)}{|\mathbf{c}_1| + V_3(I)}\kappa_2 \tag{7.4.5}$$

where recall $V_2(I)$ and $V_3(I)$ are the second and third components of the vector of invariants, $\mathbf{V}(I)$, and also, by definition, the second and third components of $\mathbf{w}_1$.

Case 2. If $\mathbf{w}_1 - \mathbf{c}_1$ is bounded away from zero, we note that σ_{RM} may be taken to be a product of a rotation about $\mathbf{c}_1 + (0, 0, -|\mathbf{c}_1|)^T$ with angle π followed by a rotation about $(0, 0, -|\mathbf{c}_1|)^T$ with any angle ψ and a rotation about $\mathbf{w}_1 + (0, 0, -|\mathbf{c}_1|)^T$ with angle π, that is,

$$\sigma_{RM} = R(\pi, \mathbf{w}_1 + (0, 0, -|\mathbf{c}_1|)^T)R(\psi(s), (0, 0, -|\mathbf{c}_1|)^T)R(\pi, \mathbf{c}_1 + (0, 0, -|\mathbf{c}_1|)^T).$$

Since the matrix on the right and the matrix on the left are constant, we obtain the same equation for ψ as above, but with the signs of $\mathbf{c}_1$ reversed. Hence in this case,

$$\psi_s = \kappa_1 + \frac{V_2(I)}{|\mathbf{c}_1| - V_3(I)} \kappa_2. \tag{7.4.6}$$

In either case, we obtain σ_{RM} up to a quadrature. There is a significant overlap in the domains of the two cases, and matching one to the other, as needed, is not a problem.

Next, we seek P. We note the first row of σ_{RM} is P', and so we can always obtain P by quadrature. However, we note that only one component needs to be calculated this way, as the second component of Equation (7.4.1) provides algebraic equations for two of the components of P, i.e,

$$X = \frac{1}{V_3(I)}(V_4(I) + ZV_2(I) - (\sigma D\mathbf{c_2})_1),$$
$$Y = \frac{1}{V_3(I)}(V_5(I) + ZV_1(I) + (\sigma D\mathbf{c_2})_2)$$

where Z has been solved previously by quadrature.

We conclude by noting that the conservation laws provide two first integrals of the Euler–Lagrange equations. They may be used to solve for P in terms of two quadratures, and they also solve for the normal vector V in terms of one quadrature, that of ψ. Finally, we note that it is easy to obtain the Frenet–Serret frame from our calculations, since it is defined in terms of P' and P''.

7.5 Examples and applications

We examine a Lagrangian which is not possible to study in the Frenet–Serret framework. Secondly, we study functionals used to model some biological structures, invariant under $SE(3)$ and depending on the curvature, torsion and their derivatives, but using our results for the Rotation Minimizing frame.

We first show that every Lagrangian which can be written in terms of the Euclidean curvature κ and torsion τ can be written in terms of the invariants, κ_1 and κ_2. From (7.1.8) we have that

$$\kappa_1 = \kappa\cos\theta, \qquad \kappa_2 = \kappa\sin\theta$$

and therefore, using $\tan\theta = \kappa_2/\kappa_1$ and $\theta_s = \tau$ we have,

$$\kappa = \sqrt{\kappa_1^2+\kappa_2^2}, \qquad \tau = \frac{\kappa_1\kappa_{2,s} - \kappa_{1,s}\kappa_2}{\kappa_1^2+\kappa_2^2}. \tag{7.5.1}$$

But the converse is not true. Lagrangians which depend only on κ_2/κ_1 cannot be written in terms of κ and τ. Our first example is the simplest such Lagrangian, which we study simply because we can.

7.5.1 Invariant Lagrangians involving only κ_2/κ_1

Let us consider the Lagrangian

$$\mathcal{L}\left[\frac{\kappa_2}{\kappa_1}\right] = \int \frac{1}{2}\left(\frac{\kappa_2}{\kappa_1}\right)^2 + \lambda\,(\eta-1) + \mu\zeta \;\mathrm{d}s = \int \tan\theta^2 + \lambda\,(\eta-1) + \mu\zeta \;\mathrm{d}s$$

where recall $\eta = 1$ is the arc-length constraint and $\zeta = 0$ is the Rotation Minimizing frame constraint.

Using the results of the previous section, we obtain the Euler–Lagrange equations

$$\left(-\frac{12\dfrac{\mathrm{d}}{\mathrm{d}s}\kappa_1{}^2}{\kappa_1^5} + \frac{3\dfrac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_1}{\kappa_1^4} - \frac{1}{2\kappa_1}\right)\kappa_2^2 + \left(\frac{12\dfrac{\mathrm{d}}{\mathrm{d}s}\kappa_1\dfrac{\mathrm{d}}{\mathrm{d}s}\kappa_2}{\kappa_1^4} - \frac{2\dfrac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_2}{\kappa_1^3} - \mu_s\right)\kappa_2 \tag{7.5.2}$$
$$- \frac{2\dfrac{\mathrm{d}}{\mathrm{d}s}\kappa_2^2}{\kappa_1^3} + \mu\frac{\mathrm{d}}{\mathrm{d}s}\kappa_2 = 0,$$

$$-\frac{\kappa_2^3}{2\kappa_1^2}+\left(\frac{6\frac{\mathrm{d}}{\mathrm{d}s}\kappa_1^2}{\kappa_1^4}-\frac{2\frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_1}{\kappa_1^3}\right)\kappa_2\frac{\frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_2}{\kappa_1^2}-\frac{4\frac{\mathrm{d}}{\mathrm{d}s}\kappa_1\frac{\mathrm{d}}{\mathrm{d}s}\kappa_2}{\kappa_1^3}-\mu_s\kappa_1-\mu\frac{\mathrm{d}}{\mathrm{d}s}\kappa_1=0, \qquad (7.5.3)$$

$$\mu_s+\frac{\kappa_2^3}{\kappa_1^3}+\frac{\kappa_2}{\kappa_1}=0 \qquad (7.5.4)$$

where $\lambda=\frac{1}{2}\left(\frac{\kappa_2}{\kappa_1}\right)^2$ has been solved using (7.3.6).

Further, the vector of invariants $\mathbf{V}(I)$ needed for the conservation laws is

$$\mathbf{V}(I)=\begin{pmatrix}\frac{1}{2}\left(\frac{\kappa_2}{\kappa_1}\right)^2\\ -\frac{\kappa_2}{\kappa_1^4}(\kappa_1^4\mu-2\kappa_1\frac{\mathrm{d}}{\mathrm{d}s}\kappa_2+3\kappa_2\frac{\mathrm{d}}{\mathrm{d}s}\kappa_1)\\ -\frac{\frac{\mathrm{d}}{\mathrm{d}s}\kappa_2}{\kappa_1^2}+\frac{2\kappa_2\frac{\mathrm{d}}{\mathrm{d}s}\kappa_1}{\kappa_1^3}+\mu\kappa_1\\ \mu\\ \frac{\kappa_2}{\kappa_1^2}\\ -\frac{\kappa_2^2}{\kappa_1^3}\end{pmatrix}.$$

Solving (7.5.2), (7.5.3) along with (7.4.4), (7.4.5) and (7.4.6) for κ_1, κ_2, μ and ψ with initial conditions

$$\begin{aligned}&\kappa_1(0)=1, \quad \kappa_2(0)=\frac{1}{2}, \quad \frac{\mathrm{d}}{\mathrm{d}s}\kappa_1(0)=1, \quad \frac{\mathrm{d}}{\mathrm{d}s}\kappa_2(0)=1,\\ &\lambda(0)=1, \quad \mu(0)=1, \quad Z(0)=1, \quad \psi(0)=0\end{aligned}$$

we obtain the following solutions, see Figures (7.3), (7.4), (7.5).

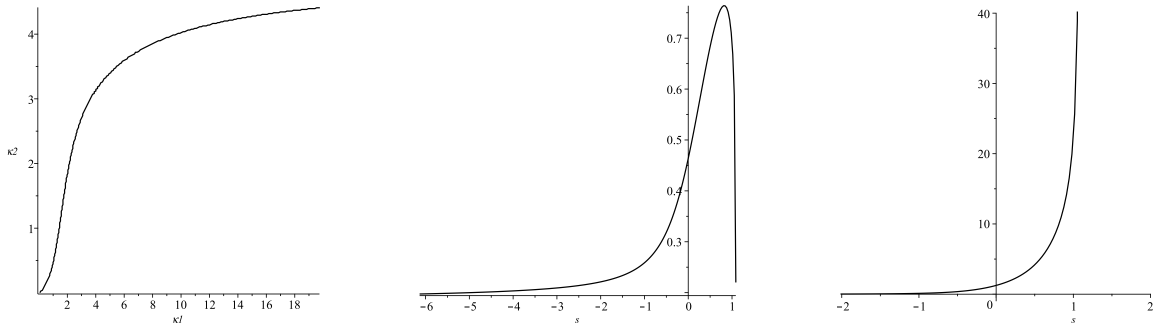
Note: For this example and the following ones, the range is all that MAPLE can do before running into singularities.

7.5.2 Applications in biology

In order to model strands of proteins, nucleic acids and polymers, some authors have made use of the classic Calculus of Variations and studied the Euler–Lagrange equations of an energy functional depending on the curvature, torsion and their first derivatives, of the protein strand. In Thamwattana, McCoy and Hill, [106] and McCoy, [79] the authors consider protein backbones and polymers as a smooth curve in $\mathbb{R}^3$ and use the Frenet–Serret equations in order to compute a variation to the curve. The Euler–Lagrange equations are obtained for these type of functionals. In Feoli, Nesterenko and Scarpetta, [27] the same method is used to obtain the Euler–Lagrange equations for functionals which are linear in the curvature.

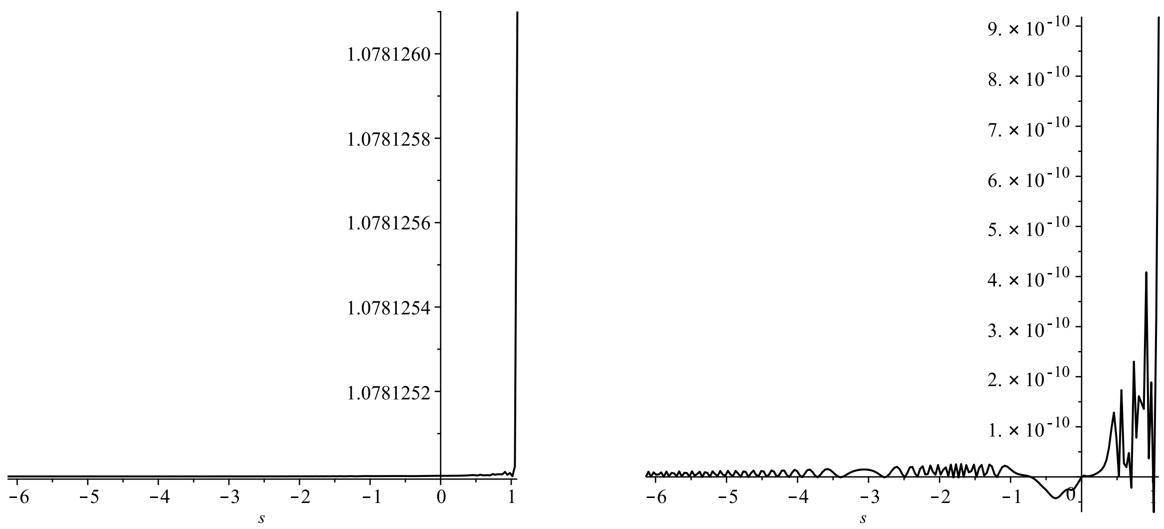
In this section we study two examples of the families of functionals studied, but in terms of

Figure 7.3: Solutions for the invariants κ_1, κ_2, θ and κ^2. From the graphs, we can see that there is a functional dependency between the two normal curvatures that resembles a logarithm. The value of theta reaches a maximum close to $s = 1$ before it reaches a singularity. For κ^2 we also find a singularity when $s = 1$, which is expected from the previous graph.



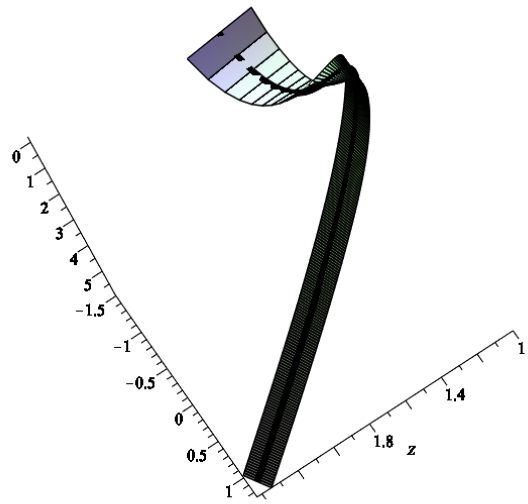
κ_1 vs κ_2 $\qquad$ s vs θ $\qquad$ s vs κ^2

Figure 7.4: Plots of the first integrals. In the following pictures, we check that the conservation laws $V_1(I)^2 + V_2(I)^2 + V_3(I)^2$ and $V_1(I)V_4(I) - V_2(I)V_5(I) + V_3(I)V_6(I)$ are actually conserved along s. The singularity in $s = 1$ shows in these graphs as expected.

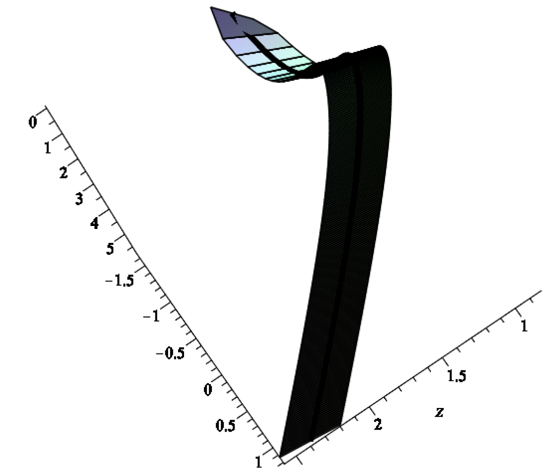


s vs $V_1(I)^2 + V_2(I)^2 + V_3(I)^2$ $\qquad$ s vs $V_1(I)V_4(I) - V_2(I)V_5(I) + V_3(I)V_6(I)$

Figure 7.5: Sweep surfaces using the Rotation Minimizing frame and the Frenet–Serret frame along the extremal curve.



Plot of V along the extremal curve using the Rotation Minimizing frame



Plot of P'' along the extremal curve using the Frenet–Serret frame

the invariants κ_1 and κ_2. The conversion of a functional given in terms of Euclidean curvature and torsion to one given in terms of κ_1 and κ_2 is given in Equation (7.5.1).

The Lagrangian $\int \kappa^2\tau \ \mathrm{d}s = \int \kappa_1\kappa_{2,s} - \kappa_{1,s}\kappa_2 \ \mathrm{d}s$

For the Lagrangian

$$\int \kappa_1\kappa_{2,s} - \kappa_{1,s}\kappa_2 \ \mathrm{d}s$$

the Euler–Lagrange equations are

$$2\kappa_{2,sss} + 3\kappa_{2,s}\kappa^2 = 0, \tag{7.5.5}$$

$$-2\kappa_{1,sss} - 3\kappa_{1,s}\kappa^2 = 0. \tag{7.5.6}$$

The conservation laws are of the form (7.3.10) where

$$\mathbf{V}(I) = \begin{pmatrix} 2(\kappa_{1,s}\kappa_2 - \kappa_1\kappa_{2,s}) & -2\kappa_{2,ss} - \kappa_2\kappa^2 & -2\kappa_{1,ss} + \kappa_1\kappa^2 & \kappa^2 & -2\kappa_{1,s} & 2\kappa_{2,s} \end{pmatrix}^T.$$

Solving (7.5.5), (7.5.6) along with (7.4.5) and (7.4.6) for κ_1, κ_2 and ψ with initial conditions

$$\kappa_1(0) = 1, \qquad \kappa_2(0) = \frac{1}{2}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\kappa_1(0) = 1, \quad \frac{\mathrm{d}}{\mathrm{d}s}\kappa_2(0) = 1,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_1(0) = 1, \quad \frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_2(0) = 1, \quad \psi(0) = 0$$

and integrating to obtain the extremizing curve and its Rotation Minimizing frame, we obtain the following solutions, see Figures (7.6), (7.7), (7.8).

The Lagrangian $\int \kappa^2\tau^3 + \tau(2\kappa_s^2 - \kappa\kappa_{ss}) + \kappa\kappa_s\tau_s \,\mathrm{d}s = \int \kappa_{1,s}\kappa_{2,ss} - \kappa_{1,ss}\kappa_{2,s}\,\mathrm{d}s$

We now consider

$$\int \kappa_{1,s}\kappa_{2,ss} - \kappa_{1,ss}\kappa_{2,s}\,\mathrm{d}s.$$

The Euler–Lagrange equations are

$$-2\kappa_{2,ssss} + \frac{\mathrm{d}}{\mathrm{d}s}(\kappa_2\mu) - \kappa_1\lambda = 0, \tag{7.5.7}$$

$$2\kappa_{1,ssss} - \frac{\mathrm{d}}{\mathrm{d}s}(\kappa_1\mu) - \kappa_2\lambda = 0 \tag{7.5.8}$$

where

$$\lambda = 2\kappa_{2,sss}\kappa_1 - 2\kappa_{1,s}\kappa_{2,ss} + 2\kappa_{2,s}\kappa_{1,ss} - 2\kappa_2\kappa_{1,sss}$$

and

$$\mu = \kappa_{1,s}^2 + \kappa_{2,s}^2 - 2(\kappa_1\kappa_{1,ss} + \kappa_2\kappa_{2,ss}).$$

The conservation laws are of the form (7.3.10) where

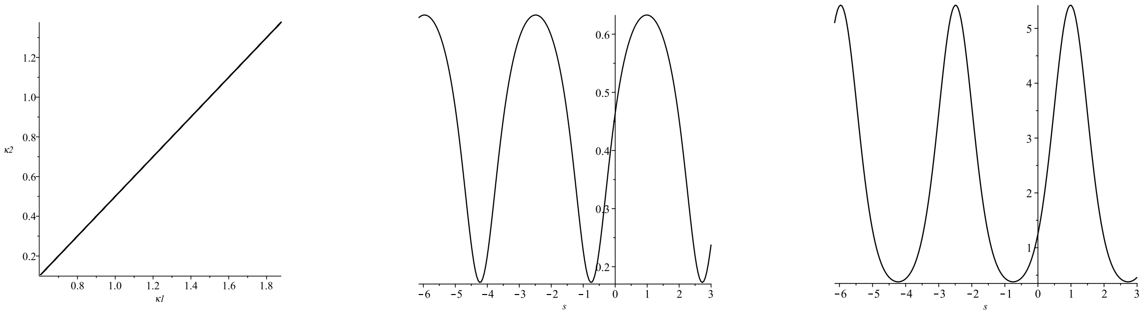
$$\mathbf{V}(I) = (\lambda \quad 2\kappa_{2,ssss} - \mu\kappa_2 \quad -2\kappa_{1,ssss} + \mu\kappa_1 \quad \mu \quad 2\kappa_{1,sss} \quad -2\kappa_{2,sss}).$$

Solving (7.5.7), (7.5.8) along with (7.4.5) and (7.4.6) for κ_1, κ_2 and ψ with initial conditions

$$\begin{array}{lllll} \kappa_1(0) = 1, & \kappa_2(0) = \frac{1}{2}, & \frac{\mathrm{d}}{\mathrm{d}s}\kappa_1(0) = 1, & \frac{\mathrm{d}}{\mathrm{d}s}\kappa_2(0) = 1, & \\ \frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_1(0) = 1, & \frac{\mathrm{d}^2}{\mathrm{d}s^2}\kappa_2(0) = 1, & \frac{\mathrm{d}^3}{\mathrm{d}s^3}\kappa_1(0) = 1, & \frac{\mathrm{d}^3}{\mathrm{d}s^3}\kappa_2(0) = 1, & \psi(0) = 0 \end{array}$$

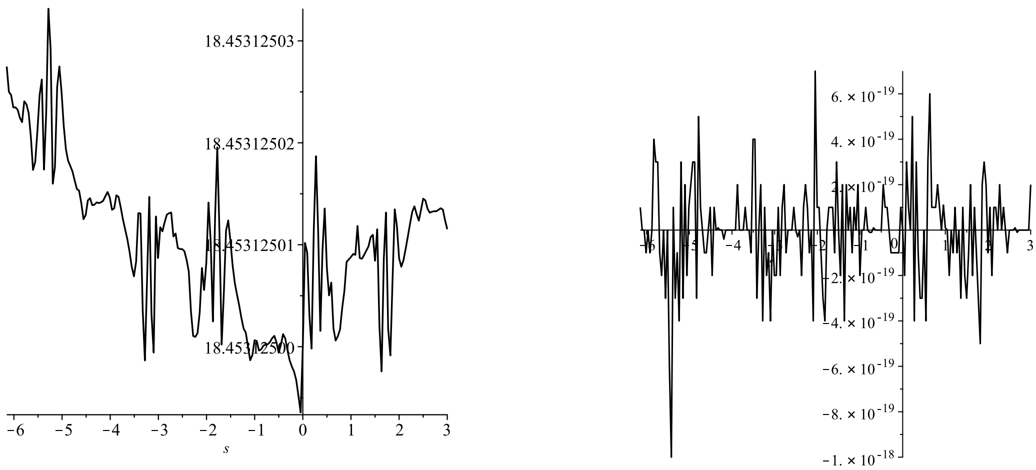
and integrating to obtain the extremizing curve and its Rotation Minimizing frame, we obtain the following solutions, see Figures (7.9), (7.10), (7.11).

Figure 7.6: Solutions for the invariants κ_1, κ_2, θ and κ^2. The plots show that there is a linear dependency between κ_1 and κ_2. We can therefore suppose that $\kappa_2 = \lambda_1\kappa_1 + \lambda_2$ where λ_1 and λ_2 are real numbers. We can also see that both θ and κ^2 have a periodic behaviour along s reaching their maxima and minima at the same s.



κ_1 vs κ_2 s vs θ s vs κ^2

Figure 7.7: Plots of the conservation laws. Again, we check that the conservation laws $V_1(I)^2 + V_2(I)^2 + V_3(I)^2$ and $V_1(I)V_4(I) - V_2(I)V_5(I) + V_3(I)V_6(I)$ are actually conserved along s.



s vs $V_1(I)^2 + V_2(I)^2 + V_3(I)^2$ s vs $V_1(I)V_4(I) - V_2(I)V_5(I) + V_3(I)V_6(I)$

Figure 7.8: Sweep surface using V from the Rotation Minimizing frame along the extremal curve

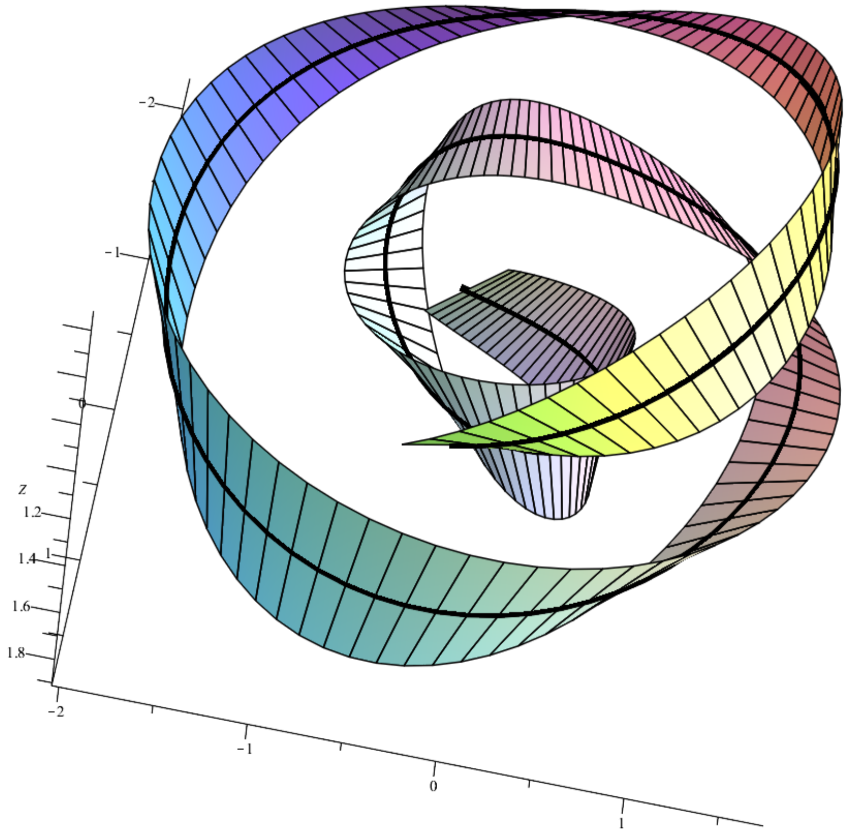
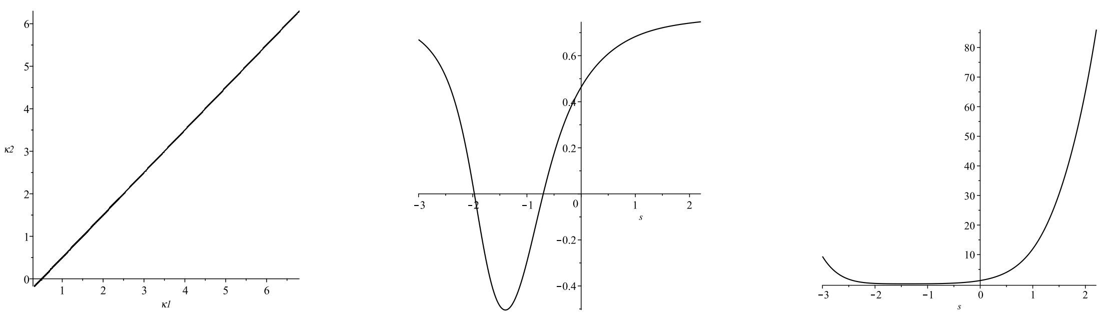
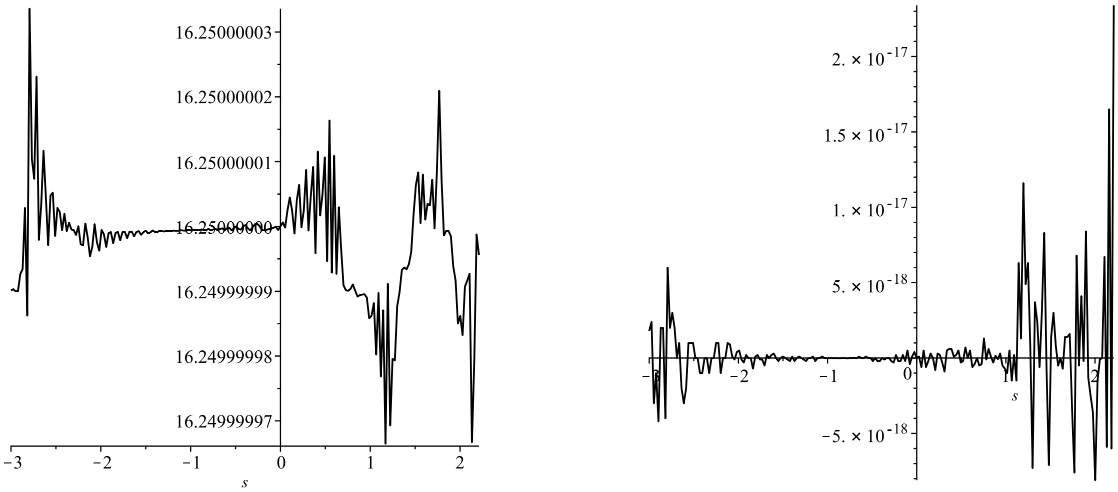


Figure 7.9: Solutions for the invariants κ_1, κ_2, θ and κ^2. In this case, we also find a linear dependency between the curvature invariants. However, now θ and κ^2 don not evolve periodically along s. A minimum can be found for θ and κ^2 for approximately $s = -1.5$.



κ_1 vs κ_2 s vs θ s vs κ^2

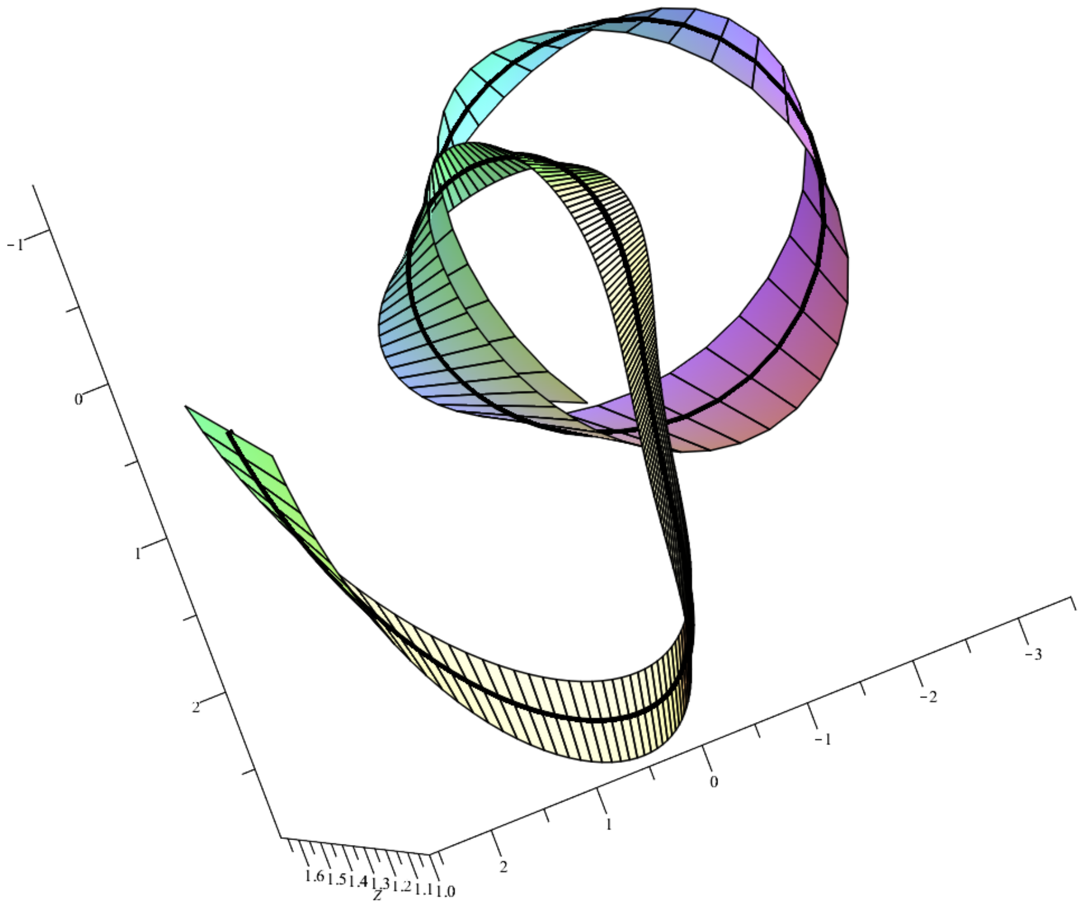
Figure 7.10: Plots of the first integrals. The conservation laws are conserved along s as shown in the following plots.



s vs $V_1(I)^2 + V_2(I)^2 + V_3(I)^2$

s vs $V_1(I)V_4(I) - V_2(I)V_5(I) + V_3(I)V_6(I)$

Figure 7.11: Sweep surface using V from the Rotation Minimizing frame along the extremal curve.



Moving Frames and Gauge Transformations

The aim of this chapter is to study the relationships between two moving frames when one is the gauge transformation of the other. We show that the differential invariants, the curvature matrices and the differential syzygy of one of the frames can be written in terms of the ones coming from the other frame and vice–versa. We use the $\mathrm{S}E(2)$ action as our running example and the $\mathrm{S}L(2)$ projective action as a detailed example in order to illustrate the theory. Some of the results are also illustrated for the linear transformations on curves action.

8.1 Moving frames and gauge transformations

Consider two moving frames

$$\rho^A\colon M \to G \qquad \text{and} \qquad \rho^B\colon M \to G. \tag{8.1.1}$$

If

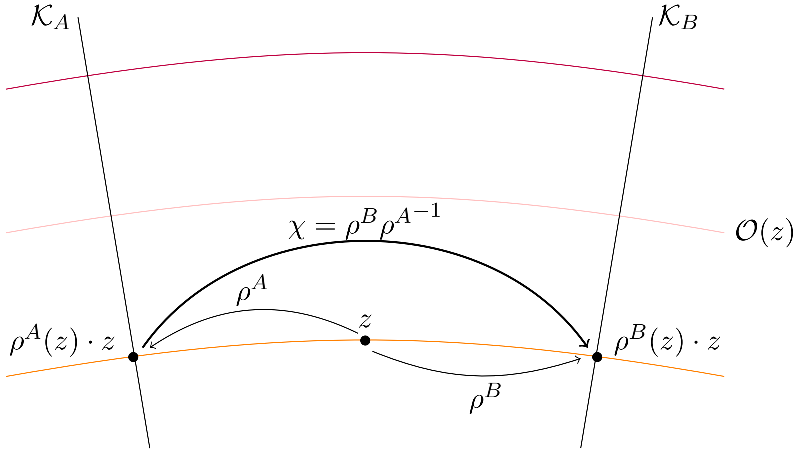
$$\rho^B = \chi \cdot \rho^A \tag{8.1.2}$$

where $\chi \in G$, we will say that χ is a gauge (see Figure (8.1)).

Example 8.1.3. *Consider the special Euclidean group* $\mathrm{S}E(2)$ *of rotations and translations acting on curves* $(u(s), v(s))$ *on the plane parametrized by the arc length*

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}.$$

Figure 8.1: By choosing two different cross-section, we obtain two moving frames related by a gauge.



For the normalization equations $\widetilde{u} = 0$, $\widetilde{v} = 0$ and $\widetilde{v}_s = 0$ we obtain the moving frame (see Gonçalves and Mansfield, [33]) - also obtained previously in (4.1.4)

$$\rho^A = \begin{pmatrix} u_s & v_s & -uu_s - v_s v \\ -v_s & u_s & -vu_s + v_s u \\ 0 & 0 & 1 \end{pmatrix}$$

while for normalization equations $\widetilde{u} = \alpha$, $\widetilde{v} = \beta$ and $\widetilde{v}_s = \delta$ we obtain the moving frame

$$\rho^B = \begin{pmatrix} \delta v_s + u_s\sqrt{1-\delta^2} & -\delta u_s + \sqrt{1-\delta^2} v_s & \alpha + \delta(vu_s - uv_s) - \sqrt{1-\delta^2}(uu_s + vv_s) \\ \delta u_s - \sqrt{1-\delta^2} v_s & \delta v_s + u_s\sqrt{1-\delta^2} & \beta - \delta(uu_s + vv_s) + \sqrt{1-\delta^2}(uv_s - vu_s) \\ 0 & 0 & 1 \end{pmatrix}$$

where α, β and δ are constants. Note that (8.1.2) *is satisfied where*

$$\chi = \begin{pmatrix} \sqrt{1-\delta^2} & -\delta & \alpha \\ \delta & \sqrt{1-\delta^2} & \beta \\ 0 & 0 & 1 \end{pmatrix}.$$

8.1.1 Differential invariants

Let us set I_K^A and I_K^B to be the invariants as defined in (2.4.5) for ρ^A and ρ^B satisfying (8.1.1).

Proposition 8.1.4. *Given two moving frames ρ^A and ρ^B as in* (8.1.1) *such that* (8.1.2) *is satisfied, the differential invariants of the frame ρ^A can be written in terms of the differential invariants of the frame ρ^B and vice-versa as follows*

$$I_K^B = \chi \cdot I_K^A, \qquad I_K^A = \chi^{-1} \cdot I_K^B. \tag{8.1.5}$$

Proof. From (2.4.5) and (8.1.2) we have that

$$I_K^B = \rho^B \cdot z_K = \chi \cdot \rho^A \cdot z_K = \chi \cdot I_K^A.$$

Finally, making the inverse of the gauge transformation act on the left we obtain $I_K^A = \chi^{-1} \cdot I_K^B$. □

Remark 8.1.6. *Note that $\cdot$ is not the standard multiplication but the group product which matches the multiplication of matrices when using representation matrices in linear group actions. At the end of this chapter we give an example of a non-linear action in order to illustrate the theory for these type of actions.*

Example 8.1.7. *For the moving frame ρ^A (see Gonçalves and Mansfield, [33])*

$$I^A = \begin{pmatrix} I^{u,A} \\ I^{v,A} \\ 1 \end{pmatrix} = \rho^A \cdot \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$I_1^A = \begin{pmatrix} I_1^{u,A} \\ I_1^{v,A} \\ 0 \end{pmatrix} = \rho^A \cdot \begin{pmatrix} u_s \\ v_s \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

$$I_{11}^A = \begin{pmatrix} I_{11}^{u,A} \\ I_{11}^{v,A} \\ 0 \end{pmatrix} = \rho^A \cdot \begin{pmatrix} u_{ss} \\ v_{ss} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ u_s v_{ss} - v_s u_{ss} \\ 0 \end{pmatrix},$$

$$I_K^A = \begin{pmatrix} I_K^{u,A} \\ I_K^{v,A} \\ 0 \end{pmatrix} = \rho^A \cdot \begin{pmatrix} u_K \\ v_K \\ 0 \end{pmatrix} = \begin{pmatrix} u_s u_K + v_s v_K \\ u_s v_K - v_s u_K \\ 0 \end{pmatrix}$$

while for the moving frame ρ^B we have that

$$\begin{pmatrix} I^{u,B} \\ I^{v,B} \\ 1 \end{pmatrix} = \rho^B \cdot \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} I_1^{u,B} \\ I_1^{v,B} \\ 0 \end{pmatrix} = \rho^B \cdot \begin{pmatrix} u_s \\ v_s \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{1-\delta^2} \\ \delta \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} I_{11}^{u,B} \\ I_{11}^{v,B} \\ 0 \end{pmatrix} = \rho^B \cdot \begin{pmatrix} u_{ss} \\ v_{ss} \\ 0 \end{pmatrix} = \begin{pmatrix} \delta(u_{ss}v_s - v_{ss}u_s) \\ \sqrt{1-\delta^2}(u_s v_{ss} - v_s u_{ss}) \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} I_k^{u,B} \\ I_k^{v,B} \\ 0 \end{pmatrix} = \rho^B \cdot \begin{pmatrix} u_K \\ v_K \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{1-\delta^2}(u_s u_K + v_s v_K) - \delta(u_K v_s - v_K u_s) \\ \delta(u_K v_s - v_K u_s) + \sqrt{1-\delta^2}(u_s v_K - v_s u_K) \\ 0 \end{pmatrix}.$$

Note for any K (8.1.5) is satisfied as

$$\begin{pmatrix} I_K^{u,B} \\ I_K^{v,B} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1-\delta^2} & -\delta & \alpha \\ \delta & \sqrt{1-\delta^2} & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_K^{u,A} \\ I_K^{v,A} \\ 1 \end{pmatrix}.$$

8.1.2 Curvature matrices

Let us denote Q_A^i and Q_B^i the curvature matrices defined in (2.57) of ρ^A and ρ^B respectively.

Proposition 8.1.8. *Given two moving frames ρ^A and ρ^B such that $\rho^B = \chi \cdot \rho^A$ where χ is a gauge transformation, the curvature matrices for the moving frame ρ^B can be written in terms* *of those ones of the moving frame ρ^A as follows*

$$Q^i_B = \chi_i \chi^{-1} + \chi Q^i_A \chi^{-1} \tag{8.1.9}$$

where $\chi_i = \mathrm{D}_i \chi$.

Proof. Inserting (8.1.2) in $Q^i_B = \mathrm{D}_i(\rho^B)\rho^{B^{-1}}$ and using $Q^i_A = \mathrm{D}_i(\rho^A)\rho^{A^{-1}}$ we obtain

$$Q^i_B = \mathrm{D}_i(\rho^B)(\rho^B)^{-1} = \mathrm{D}_i(\chi\cdot\rho^A)(\chi\cdot\rho^A)^{-1} = (\chi_i\cdot\rho^A + \chi \mathrm{D}_i \rho^A)(\rho^A)^{-1}\chi^{-1} = \chi_i\chi^{-1} + \chi Q^i_A \chi^{-1}.$$

□

Example 8.1.10. *Using (2.57), for the moving frame ρ^A we have (see Mansfield and van der Kamp, [73])*

$$Q^s_A = \begin{pmatrix} 0 & I^{v,A}_{11} & -1 \\ -I^{v,A}_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad Q^t_A = \begin{pmatrix} 0 & I^{v,A}_{12} & -I^{u,A}_{2} \\ -I^{v,A}_{12} & 0 & I^{v,A}_{2} \\ 0 & 0 & 0 \end{pmatrix}$$

and for the moving frame ρ^B we have

$$Q^s_B = \begin{pmatrix} \delta I^{v,B}_{11} + I^{u,B}_{11}\lambda & -\frac{(\delta^2-1)I^{v,B}_{11} + \delta\lambda I^{u,B}_{11} + \delta_s}{\lambda} & \frac{\left(\alpha_s + \left(\beta I^{u,B}_{11} + I^{v,B}_{11}\alpha\right)\delta\right)\lambda + \delta_s\beta + \lambda^2\left(I^{v,B}_{11}\beta - \alpha I^{u,B}_{11} + 1\right)}{\lambda} \\ \frac{(\delta^2-1)I^{v,B}_{11} + \delta I^{u,B}_{11}\lambda + \delta_s}{\lambda} & \delta I^{v,B}_{11} + I^{u,B}_{11}\lambda & \frac{\left(\beta_s + \left(I^{v,B}_{11}\beta - \alpha I^{u,B}_{11} - 1\right)\delta\right)\lambda - \delta_s\alpha - \lambda^2\left(\beta I^{u,B}_{11} + I^{v,B}_{11}\alpha\right)}{\lambda} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Q^t_B = \begin{pmatrix} \delta I^{v,B}_{12} + I^{u,B}_{12}\lambda & -\frac{(\delta^2-1)I^{v,B}_{12} + \delta I^{u,B}_{12}\lambda + \delta_t}{\lambda} & \frac{\left(\alpha_t + \left(\beta I^{u,B}_{12} + \alpha I^{v,B}_{12}\right)\delta - I^{u,B}_{2}\right)\lambda + \beta\delta_t + \lambda^2\left(\beta I^{v,B}_{12} - \alpha I^{u,B}_{12}\right)}{\lambda} \\ \frac{(\delta^2-1)I^{v,B}_{12} + \delta I^{u,B}_{12}\lambda + \delta_t}{\lambda} & \delta I^{v,B}_{12} + I^{u,B}_{12}\lambda & \frac{\left(\delta(\beta I^{v,B}_{12} - \alpha I^{u,B}_{12}) - I^{v,B}_{2} + \beta_t\right)\lambda - \alpha\delta_t - \lambda^2\left(\beta I^{u,B}_{12} + \alpha I^{v,B}_{12}\right)}{\lambda} \\ 0 & 0 & 0 \end{pmatrix}$$

where $\lambda = \sqrt{1-\delta^2}$.

Taking into account that

$$\begin{aligned} I^{u,B}_{2} &= I^{u,A}_{2}\sqrt{1-\delta^2} - I^{v,A}_{2}\delta, \\ I^{v,B}_{2} &= I^{u,A}_{2}\delta + I^{v,A}_{2}\sqrt{1-\delta^2}, \\ I^{u,B}_{12} &= -I^{v,A}_{12}\delta, \\ I^{v,B}_{12} &= I^{v,A}_{12}\sqrt{1-\delta^2} \end{aligned} \tag{8.1.11}$$

and

$$\begin{aligned} I_2^{u,A} &= I_2^{u,B}\sqrt{1-\delta^2} + I_2^{v,B}\delta, \\ I_2^{v,A} &= -I_2^{u,B}\delta + I_2^{v,A}\sqrt{1-\delta^2}, \\ I_{12}^{v,A} &= \frac{I_{12}^{v,B}}{\sqrt{1-\delta^2}} = -\frac{I_{12}^{u,B}}{\delta} \end{aligned} \tag{8.1.12}$$

equations (8.1.9) *can be easily verified.*

8.1.3 Differential syzygy

Recall that the curvature matrices satisfy the relationship (2.58). If

$$\mathcal{D}_s = \mathrm{D}_s \qquad \text{and} \qquad \mathcal{D}_t = \mathrm{D}_t$$

the vanishing commutator $[\mathcal{D}_s, \mathcal{D}_t] = 0$ yields that the syzygy (2.58) can be written as follows

$$\mathrm{D}_s Q_t^i - \mathrm{D}_t Q_s^i = [Q_s^i, Q_t^i]. \tag{8.1.13}$$

Proposition 8.1.14. *The differential syzygy for the moving frame ρ^B can be written in terms of the differential syzygy for the moving frame ρ^A as follows*

$$\mathrm{D}_s Q_B^t - \mathrm{D}_t Q_B^s - [Q_B^s, Q_B^t] = \mathcal{A}d_\chi\left(\mathrm{D}_s Q_A^t - \mathrm{D}_t Q_A^s - [Q_A^s, Q_A^t]\right) \tag{8.1.15}$$

where $\mathcal{A}d$ is left adjoint action.

Proof. Differentiating Q_B^t with respect to s we have that

$$\begin{aligned} \mathrm{D}_s Q_B^t &= \mathrm{D}_s(\chi_t\chi^{-1} + \chi Q_A^t \chi^{-1}) \\ &= \chi_{st}\chi^{-1} - \chi_t\chi^{-1}\chi_s\chi^{-1} + \chi_s Q_A^t\chi^{-1} + \chi\left(\mathrm{D}_s Q_A^t\chi^{-1} - Q_A^t\chi^{-1}\chi_s\chi^{-1}\right). \end{aligned}$$

Analogously, differentiating Q_B^s with respect to t we have that

$$\begin{aligned} \mathrm{D}_t Q_B^s &= \mathrm{D}_t(\chi_s\chi^{-1} + \chi Q_A^s \chi^{-1}) \\ &= \chi_{st}\chi^{-1} - \chi_s\chi^{-1}\chi_t\chi^{-1} + \chi_t Q_A^s\chi^{-1} + \chi\left(\mathrm{D}_t Q_A^s\chi^{-1} - Q_A^s\chi^{-1}\chi_t\chi^{-1}\right). \end{aligned}$$

Hence

$$\begin{aligned} \mathrm{D}_s Q_B^t - \mathrm{D}_t Q_B^s = &-\chi_t\chi^{-1}\chi_s\chi^{-1} + \chi_s\chi^{-1}\chi_t\chi^{-1} + \chi_s Q_A^t\chi^{-1} - \chi_t Q_A^s\chi^{-1} \\ &+ \chi\mathrm{D}_s Q_A^t\chi^{-1} - \chi Q_A^t\chi^{-1}\chi_s\chi^{-1} - \chi\mathrm{D}_t Q_A^s\chi^{-1} + \chi Q_A^s\chi^{-1}\chi_t\chi^{-1}. \end{aligned}$$

We also have

$$\begin{aligned}
[Q_B^s, Q_B^t] &= [\chi_s\chi^{-1} + \chi Q_A^s\chi^{-1}, \chi_t\chi^{-1} + \chi Q_A^t\chi^{-1}] \\
&= \chi_s\chi^{-1}\chi_t\chi^{-1} + \chi_s\chi^{-1}\chi Q_A^t\chi^{-1} + \chi Q_A^s\chi^{-1}\chi_t\chi^{-1} + \chi Q_A^s Q_A^t\chi^{-1} \\
&- \chi_t\chi^{-1}\chi_s\chi^{-1} - \chi_t\chi^{-1}\chi Q_A^s\chi^{-1} - \chi Q_A^t\chi^{-1}\chi_s\chi^{-1} - \chi Q_A^t Q_A^s\chi^{-1}
\end{aligned}$$

and therefore

$$\mathrm{D}_s Q_B^t - \mathrm{D}_t Q_B^s - [Q_B^s, Q_B^t] = \chi\left(\mathrm{D}_s Q_A^t - \mathrm{D}_t Q_A^s - [Q_A^s, Q_A^t]\right)\chi^{-1}$$

obtaining the required result.

□

Example 8.1.16. *Equations* (8.1.9) *and* (8.1.15) *have been checked for this running example with* MAPLE *(see Appendix).*

8.2 Linear transformations action on curves

Recall the group of linear transformations acting on curves $(x, u(x))$

$$x \to x = \widetilde{x}, \qquad u \to \lambda u + \epsilon = \widetilde{u}.$$

given in (5.2.1).

Recall that for the normalization equations

$$\widetilde{u} = 0, \qquad \widetilde{u_s} = 1$$

we obtained the moving frame

$$\rho^A = \begin{pmatrix} \dfrac{1}{u_x} & -\dfrac{u}{u_x} \\ 0 & 1 \end{pmatrix}.$$

Recall also that for the linear transformation group (5.2.1) the invariants are of the form

$$\iota(u) = \rho^A \cdot u = 0, \qquad \iota(u_x) = \rho^A \cdot u_x = 1, \qquad \iota(u_J) = \rho^A \cdot u_J = \frac{u_J}{u_x}.$$

Using (2.57) we obtain the curvature matrices

$$Q_A^x = \begin{pmatrix} -I_{11}^A & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Q_A^t = \begin{pmatrix} -I_{12}^A & -I_2^A \\ 0 & 0 \end{pmatrix}.$$

From (2.58) and setting I^A_{11} to be κ^A and I^A_2 to be σ^A we obtain the syzygy of the form (2.59)

$$\kappa^A_t = (\mathrm{D}_x^2 + \kappa^A \mathrm{D}_x + \kappa^A_x)\sigma^A$$

where

$$\kappa^A = \frac{u_{xx}}{u_x} \quad \text{and} \quad \sigma^A = \frac{u_t}{u_x}.$$

Now we consider the normalization equations $u = \alpha$ and $u_x = \beta$, where α and β are constants. Solving these equations for the group parameters yields

$$\lambda = \frac{\beta}{u_x}, \qquad \epsilon = \alpha - \frac{\beta u}{u_x}.$$

In matrix form,

$$\rho^B = \begin{pmatrix} \dfrac{\beta}{u_x} & \alpha - \dfrac{\beta u}{u_x} \\ 0 & 1 \end{pmatrix}.$$

Note that ρ^A and ρ^B satisfy (8.1.2) where

$$\chi = \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix}.$$

Note that

$$\begin{aligned} \rho^B(\widetilde{u}, \widetilde{u_x}) &= \begin{pmatrix} \dfrac{\beta}{\widetilde{u_x}} & \alpha - \dfrac{\beta \widetilde{u}}{\widetilde{u_x}} \\ 0 & 1 \end{pmatrix} \\ &= \chi \begin{pmatrix} \dfrac{1}{\lambda u_x} & -\dfrac{\lambda u + \epsilon}{\lambda u_x} \\ 0 & 1 \end{pmatrix} \\ &= \chi \begin{pmatrix} \dfrac{1}{\widetilde{u_x}} & -\dfrac{\widetilde{u}}{\widetilde{u_x}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{\lambda} & -\dfrac{\epsilon}{\lambda} \\ 0 & 1 \end{pmatrix} \\ &= \chi \cdot \rho^A(u, u_x) g^{-1} \\ &= \rho^B(u, u_x) g^{-1} \end{aligned}$$

which is the equivariance of a right frame for a left action. The invariants are of the form

$$\iota(u) = \rho^B \cdot u = \alpha, \qquad \iota(u_x) = \rho^B \cdot u_x = \beta, \qquad \iota(u_J) = \rho^B \cdot u_J = \beta \frac{u_J}{u_x}.$$

Using (2.57) we obtain the curvature matrices

$$Q_B^x = \begin{pmatrix} \dfrac{\beta_x - I_{11}^B}{\beta} & \alpha\left(\dfrac{I_{11}^B - \beta_x}{\beta}\right) + \alpha_x - \beta^2 \\ 0 & 0 \end{pmatrix},$$

and

$$Q_B^t = \begin{pmatrix} \dfrac{\beta_t - I_{12}^B}{\beta} & -\alpha\left(\dfrac{I_{12}^B - \beta_t}{\beta}\right) + \alpha_t - \beta_t I_2^B \\ 0 & 0 \end{pmatrix}.$$

Note that

$$\mathrm{D}_x \begin{pmatrix} I_2^B \\ 0 \end{pmatrix} = Q_B^x \begin{pmatrix} I_2^B \\ 0 \end{pmatrix} + \begin{pmatrix} I_{12}^B \\ 0 \end{pmatrix}$$

and therefore

$$I_{12}^B = \mathrm{D}_x I_2^B - \frac{\beta_x - I_{11}^B}{\beta} I_2^B. \tag{8.2.1}$$

Equations (8.1.9) and (8.1.15) have been checked for this example with MAPLE (see Appendix).

8.3 Projective SL(2) case

Recall the projective SL(2) action (2.6) on curves in the plane studied in §2

$$\widetilde{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot u = \frac{au + b}{cu + d}, \qquad \text{where} \qquad ad - bc = 1.$$

For the normalization equations $u = 0, u_x = 1, u_{xx} = 0$ (2.49) we obtained the moving frame (2.50)

$$\rho^A = \begin{pmatrix} \dfrac{1}{\sqrt{u_x}} & -\dfrac{u}{\sqrt{u_x}} \\ \dfrac{u_{xx}}{2u_x^{3/2}} & \dfrac{2u_x^2 - uu_{xx}}{2u_x^{3/2}} \end{pmatrix}$$

while now for the normalization equations $u = \alpha, u_x = \beta, u_{xx} = \gamma$ we obtain the moving frame

$$\rho^B = \begin{pmatrix} \dfrac{u_{xx}\alpha\beta - u_x\alpha\delta + 2\beta^2 u_x}{2u_x^{3/2}\beta^{3/2}} & \dfrac{u_x\alpha\delta u - 2\beta^2 u_x u + 2\beta\alpha u_x^2 - u_{xx}\alpha\beta u}{2u_x^{3/2}\beta^{3/2}} \\ \dfrac{u_{xx}\beta - u_x\delta}{2u_x^{3/2}\beta^{3/2}} & \dfrac{2\,\beta u_x^2 - \beta u + u\delta u_x}{2u_x^{3/2}\beta^{3/2}} \end{pmatrix}$$

where α, β and γ are constants. Note that we have that $\rho^B = \chi \cdot \rho^A$ where

$$\chi = \begin{pmatrix} \dfrac{2\beta^2 - \alpha\delta}{2\beta^{3/2}} & \dfrac{\alpha}{\sqrt{\beta}} \\ \dfrac{\delta}{2\beta^{3/2}} & \dfrac{1}{\sqrt{\beta}} \end{pmatrix}.$$

Using (2.4.5) we computed the invariants (2.4.7)

$$I^A = \rho^A \cdot u = 0, \quad I^A_1 = \rho^A \cdot u_x = 1, \quad I^A_{11} = \rho^A \cdot u_{xx} = 0, \quad I^A_{111} = \rho^A \cdot u_{xxx} = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}.$$

We can also compute the invariants for the frame ρ^B which are of the form

$$\begin{aligned} I^B &= \rho^B \cdot u = \alpha, \quad I^B_1 = \rho^B \cdot u = \beta, \quad I^B_{11} = \rho^B \cdot u = \gamma, \\ I^B_{111} &= \rho^B \cdot u = \beta\left(\frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2}\right) + \frac{3}{2}\frac{\delta^2}{\beta}. \end{aligned}$$

We know from (2.8) that

$$\begin{aligned} \widetilde{u} &= g \cdot u = \frac{au+b}{cu+d}, \\ \widetilde{u_x} &= g \cdot u_x = \frac{u_x}{(cu+d)^2}, \\ \widetilde{u_{xx}} &= g \cdot u_{xx} = \frac{(cu+d)u_{xx} - 2cu_x}{(cu+d)^3}, \\ \widetilde{u_{xxx}} &= g \cdot u_{xxx} = \frac{u_{xxx}}{(cu+d)^2} - \frac{6cu_{xx}u_x}{(cu+d)^3} + \frac{6c^2u_x^3}{(cu+d)^4} \end{aligned}$$

and therefore

$$\begin{aligned} I^B &= \chi \cdot I^A = \widetilde{u}\big|_{g=\chi, u_k = I^A_K} = \alpha, \\ I^B_1 &= \chi \cdot I^A_1 = \widetilde{u_x}\big|_{g=\chi, u_k = I^A_K} = \beta, \\ I^B_{11} &= \chi \cdot I^A_{11} = \widetilde{u_{xx}}\big|_{g=\chi, u_k = I^A_K} = \gamma, \\ I^B_{111} &= \chi \cdot I^A_{111} = \widetilde{u_{xxx}}\big|_{g=\chi, u_k = I^A_K} = \beta I^A_{111} + \frac{3}{2}\frac{\delta^2}{\beta} \end{aligned}$$

verifying (8.1.5). The curvature matrices associated to the frame ρ^A where computed in (2.60) which are of the form

$$Q^s_A = \begin{pmatrix} 0 & -1 \\ \dfrac{I^A_{111}}{2} & 0 \end{pmatrix} \qquad \text{and} \qquad Q^t_A = \begin{pmatrix} -\dfrac{I^A_{12}}{2} & -I^A_2 \\ \dfrac{I^A_{112}}{2} & \dfrac{I^A_{12}}{2} \end{pmatrix}.$$

Using (2.57) we have that the curvature matrices associated to the frame ρ^B are of the form

$$
\begin{aligned}
Q^s_B &= \frac{1}{2}\begin{pmatrix} \frac{\left(\beta^2+\alpha\delta\right)\beta_x-\delta\beta^2+\alpha(\beta(I^B_{111}-\delta_x)-\delta^2)}{\beta^3} & \frac{-2\,\beta^4+2\,\alpha_x\beta^3+2\,\alpha\beta^2(\delta-2\,\beta_x)-\alpha^2(\beta(I^B_{111}-\delta_x)-\delta^2+\delta\beta_x)}{\beta^3} \\ \frac{\beta I^B_{111}-\beta\delta_x-\delta^2+\delta\beta_x}{\beta^3} & \frac{\delta\beta^2-\beta^2\beta_x-\alpha\beta I^B_{111}+\alpha\beta\delta_x+\delta^2\alpha-\alpha\delta\beta_x}{\beta^3} \end{pmatrix}, \\
Q^t_B &= \frac{1}{2}\begin{pmatrix} \frac{\left(\beta^2+\alpha\delta\right)\beta_t-\beta^2 I^B_{12}+\alpha(\beta(I^B_{112}-\delta_t)-I^B_{12}\delta)}{\beta^3} & \frac{-2\,\beta^3 I^B_2+2\,\beta^3\alpha_t+2\,\alpha\beta^2(I^B_{12}-2\,\beta_t)-\alpha^2(\beta(I^B_{112}+\delta_t)+\delta(I^B_{12}-\beta_t))}{\beta^3} \\ \frac{\beta I^B_{112}-\beta\delta_t-I^B_{12}\delta+\beta_t\delta}{\beta^3} & \frac{\beta^2 I^B_{12}-\beta^2\beta_t-\beta I^B_{112}\alpha+\beta\delta_t\alpha+I^B_{12}\delta\alpha-\beta_t\alpha\delta}{\beta^3} \end{pmatrix}.
\end{aligned}
$$

Taking into account that

$$I^B_2=\beta I^A_2,\qquad I^B_{12}=\beta I^A_{12}+\delta I^A_2,\qquad I^B_{112}=\beta I^A_{112}+2\delta+\frac{3\delta^2}{2\beta}I^A_2$$

and that

$$I^A_2=\frac{I^B_2}{\beta},\qquad I^A_{12}=\frac{I^B_{12}}{\beta}-\frac{\delta}{\beta^2}I^B_2,\qquad I^A_{112}=\frac{I^B_{112}}{\beta}-2\frac{\delta}{\beta^2}I^B_{12}+\frac{\delta^2}{2\beta^3}I^B_2$$

equations (8.1.9) and (8.1.15) are easily verified.

Note that I^A_{112} and I^A_{12} can be written in terms of I^A_2 as

$$I^A_{12}=\mathrm{D}_x I^A_2 \qquad \text{and} \qquad I^A_{112}=\mathrm{D}^2_x I^A_2+I^A_{111}I^A_2.$$

Also I^B_{112} and I^B_{12} can be written in terms of I^B_2 as

$$
\begin{aligned}
I^B_{12} &= \left(\frac{\delta-\beta_x}{\beta}+\mathrm{D}_x\right)I^B_2,\\
I^B_{112} &= \left(\mathrm{D}^2_x+\frac{2(\alpha-\beta_x)}{\beta}\mathrm{D}_x+\frac{I^B_{111}\beta-\beta_{xx}\beta+2\beta_x^2-2\beta_x\delta}{\beta}\right)I^B_2.
\end{aligned}
$$

Equations (8.1.9) and (8.1.15) have been checked for this example with Maple (see Appendix).

Conclusions and Future Work

In this thesis, the theory of discrete moving frames and Noether's finite difference conservation laws is discussed. Given a discrete Lagrangian with a Lie group of variational symmetries, a discrete moving frame allows us to express the Euler–Lagrange equations in terms of the invariants and Noether's conservation laws in terms of the discrete frame and a vector of invariants. This makes explicit the equivariance of the conservation laws. The solutions of the Euler–Lagrange equations can then be solved in terms of the original variables.

We apply this theory to three group actions of the semisimple Lie group $\mathrm{S}L(2)$, the special unitary group $\mathrm{S}U(2)$ and the special euclidean group $\mathrm{S}E(2)$, where we study a symmetry preserving discretization of the Euler's elastica.

We show how to construct the correction terms, correction matrix and curvature matrix associated to a discrete frame. We prove that one can always write the evolution of the curvature invariants in terms of the first order differential invariants and a linear shift operator, coming from a differential–difference syzygy between the curvature matrix and the Maurer–Cartan matrix. This is possible when the normalization equations do not involve time derivatives. We also prove that the symmetry condition for two curvature evolutions is a differential consequence of the symmetry condition on the curve evolutions. Some examples are developed and related to discrete integrable systems.

We give a brief introduction to multispaces and construct the multispace moving frame and its invariants for some Lie groups. In these examples, using interpolation in order to define coordinates, we show that the discrete moving frame converges to a smooth one. We also show that the discrete invariants and syzygies approximate their smooth equivalents. In the last example, we construct the multispace prolonged action and the table of infinitesimals and show how taking a continuum limit yields convergence to the smooth results.

We have developed the Calculus of Variations for invariant Lagrangians under the Euclidean action of rotations and translations on curves in 3-space, using the Rotation Minimising frame. We obtain the Euler-Lagrange equations in their invariant form and their corresponding conservation laws. These results yield an easier form than those obtained in Gonçalves and

Mansfield, [34]. We also show how to ease the integration problem using the conservation laws and to recover the extremals in the original variables. We show how to minimize the angle between the normal and binormal vector and give an application in the study of biological problems.

We study the relationship between two moving frames differing by a gauge and how the differential operator linking the curvature invariant with the differential invariants of one of the frames can be expressed in terms of the other.

Future work includes:

- Extending the techniques developed in §3.8 for higher dimensional cases.
- Studying applications of Noether's finite difference conservation laws for other Lie groups such as the spin group and the symplectic group.
- Optimising the use of the difference frame appearing in the example of the discrete Euler's elastica in the approximations of the conservation laws.
- Developing a package in Maple that allows us to compute the invariant form of the Euler–Lagrange equations and conservation laws for particular Lie groups.
- Studying the conjecture of the operator $\mathcal{H}$ to be pre-hamiltonian (see Carpentier, Mikhailov and Wang, [12], [13]).
- Constructing the discrete Rotation Minimising frame and obtaining the invariant Euler-Lagrange equations and conservation laws.
- Generalizing our results to obtain a symbolic calculus of invariants in a broad class of problems for which the frame is not defined in terms of algebraic equations in the coordinates of the manifold on which the Lie group actions.
- Studying joint invariants in problems where two helices appear and interact with each other.
- Investigating the minimization of functionals that are invariant under higher dimensional Euclidean actions.
- Discretizing the results appearing in §8, *Moving frames and gauge transformations* and finding applications to other fields.
- Stuying the relationships between the $\mathcal{H}$ operators coming from two moving frames differing by a gauge.

Indiff Package for Finite Difference Systems

In this appendix, we describe how to adapt the MAPLE package `Indiff` (Mansfield, [78]) for finite difference systems.

Given the independent and dependent variables from a finite system, the group parameters of a Lie group, the matrix of infinitesimals and the normalization equations, the `Indiff` package computes, among other things, the correction matrix as well as syzygies between the invariants.

In order to use the package `Indiff`, it is necessary to open a MAPLE file and then read the `Indiff` package. The independent variables are given in a list denoted `vars`, the dependent variables are given in a list denoted `unks`, and the group parameter names are given in a list denoted `GroupP`.

In the smooth case, we consider derivatives of order K of the variables u^α, i.e, u^α_K, and their invariantized form is denoted in `Indiff` by `In[u[alpha],[K]]`. In the discrete case, we treat each shift of each variable as a different variable. For example, recall (4.3.1) in where we are considering the variables x_0, y_0 and their shifts. We will be treating x_0, its shifts, y_0 and its shifts as different variables.

Variable	Input	MAPLE invariantization syntax
x_0	x0	`In[x0,[]]`
y_0	y0	`In[y0,[]]`
x_1	x1	`In[x1,[]]`
y_1	y0	`In[y1,[]]`

Now recall (3.4.2) in where the induced group action on the path and its tangent was considered and the group action to the dummy variable t was extended trivially. The first order differential invariants with respect to the variable t will be denoted as `In[z,[1]]` where z is a discrete variable.

For example, in (4.3.1) we would have

Invariant	MAPLE syntax
$I^x_{0,0;t}$	`In[x0,[1]]`
$I^y_{0,0;t}$	`In[y0,[1]]`
$I^x_{0,1;t}$	`In[x1,[1]]`
$I^y_{0,1;t}$	`In[y1,[1]]`

In §5, we introduced a second dummy variable. Suppose we have two dummy variables t_1 and t_2 as in (5.7). We would have the following:

Invariant	MAPLE syntax	Invariant	MAPLE syntax
$I^x_{0,0;t_1}$	`In[x0,[1]]`	$I^x_{0,1;t_2}$	`In[x1,[2]]`
$I^y_{0,0;t_1}$	`In[y0,[1]]`	$I^y_{0,1;t_2}$	`In[y1,[2]]`
$I^x_{0,0;t_2}$	`In[x0,[2]]`	$I^x_{0,1;t_1,t_2}$	`In[x1,[1,2]]`
$I^y_{0,0;t_2}$	`In[y0,[2]]`	$I^y_{0,1;t_1,t_2}$	`In[y1,[1,2]]`
$I^x_{0,1;t_1}$	`In[x1,[1]]`	$I^x_{0,1;t_1,t_2}$	`In[x1,[1,2]]`
$I^y_{0,1;t_1}$	`In[y1,[1]]`	$I^y_{0,1;t_1,t_2}$	`In[y1,[1,2]]`

In order to compute the correction matrix and the syzygies between the invariants, we first give MAPLE a list of dummy variables denoted `vars`, a list of as many discrete variables as we are going to use in our computations and at least as many as appearing in the normalization equations denoted by `unks` and a list of the group parameters denoted by `GroupP`. After that, we write the infinitesimal action of the Lie symmetry group in matrix form and we denote it by `XiPhis`.

Note that we have to write as many $0's$ columns in the beginning of the matrix as we have dummy variables. In the discrete cases presented in this thesis is either just one column for t, or two columns for t_1 and t_2. The normalization equations are given using the invariantized syntax of the variables as a list which is denoted as `Neqs`.

In order to compute the correction matrix, MAPLE needs to use the procedure `HNI`, which calculates the highest invariantized derivative terms. This procedure has three arguments. The first one is the index of the derivatives appearing in our calculations, the second one is the variables appearing in the calculations in order, and the third one is the order we are using, which will always be in the examples of this thesis, the total degree ordering, denoted by `ttdeg`. Finally, the command Kmat() give us the **minus** correction matrix.

Note that from (5.3.17) and choosing an appropriate order of the Lie algebra basis (see Remark 5.2.5 in [70]) one can compute the curvature matrix after obtaining the correction matrix by using the command Kmat().

The package `Indiff` also allows one to compute invariant differentiation thanks to the procedure Idiff, which has two arguments: the first one is the invariant we want to differentiate and the second one is the variable we are going to differentiate with respect to.

Recall (5.3.1)

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{k,j} = M_{k,j;t} + I_{k,j;t} \tag{A.0.1}$$

where

$$I_{k,j} = \rho_k \cdot z_j \qquad \text{and} \qquad I_{k,j;t} = \rho_k \cdot z_{j,t}. \tag{A.0.2}$$

One can use the procedure `Idiff` to compute the correction terms, as we will show in the following example.

We illustrate all the above by considering the projective action of $\mathrm{S}L(2)$ on $\mathbb{R}$ studied in section (4.3.3). Recall the curvature matrix associated to the discrete projective $\mathrm{S}L(2)$ action from (4.3.56)

$$N_0 = \iota_0\left(\frac{\mathrm{d}}{\mathrm{d}t}\rho_0\right) = \begin{pmatrix} \frac{1}{2}\sigma_2^x - \frac{1}{2}\sigma_0^x & -\sigma_1^x \\ 2\sigma_0^x - 4\sigma_1^x + 2\sigma_2^x & -\frac{1}{2}\sigma_2^x + \frac{1}{2}\sigma_0^x \end{pmatrix}, \qquad \text{where} \qquad \sigma_j^x := I_{j,0;t}^x.$$

Also recall that a basis of $\mathfrak{sl}(2)$ is (2.16)

$$\left\{ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

with Lie bracket table (2.17)

[,]	h	e	f
h	0	$2e$	$-2f$
e	$-2e$	0	h
f	$2f$	$-h$	0

From (4.3.49) we have that

$$\mathbf{v}_a = 2x\partial_x, \quad \mathbf{v}_b = \partial_x, \quad \mathbf{v}_c = -x^2\partial_x.$$

It is straightforward to check that their Lie bracket table is of the form

[,]	$\mathbf{v}_a$	$\mathbf{v}_b$	$\mathbf{v}_c$
$\mathbf{v}_a$	0	$-2\mathbf{v}_b$	$2\mathbf{v}_c$
$\mathbf{v}_b$	$2\mathbf{v}_b$	0	$-\mathbf{v}_a$
$\mathbf{v}_c$	$-2\mathbf{v}_c$	$\mathbf{v}_a$	0

Note that the Lie bracket table for the **minus** Lie algebra basis matches the Lie bracket table for the infinitesimal vectors, verifying Remark 5.2.5 in [70].

We compute the first few correction terms $M_{0,0;t}, M_{0,1;t}, M_{0,2;t}, M_{0,3;t}$ and $M_{0,4;t}$ with `Indiff` (see MAPLE file at the end of this Appendix).

First of all, from the normalization equations (4.3.51) we have that

$$I_{0,0} = \rho_0 \cdot x_0 = \frac{1}{2}, \quad I_{0,1} = \rho_0 \cdot x_1 = 0, \quad I_{0,2} = \rho_0 \cdot x_2 = -\frac{1}{2},$$

so it is clear that

$$\frac{\mathrm{d}}{\mathrm{d}t}(I_{0,0}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}(I_{0,1}) = \frac{\mathrm{d}}{\mathrm{d}t}(0) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}(I_{0,2}) = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{1}{2}\right) = 0.$$

So from (A.0.1) we get that

$$0 = M_{0,0;t} + I_{0,0}, \quad 0 = M_{0,1;t} + I_{0,1}, \quad 0 = M_{0,2;t} + I_{0,2},$$

and therefore

$$M_{0,0;t} = -I_{0,0}, \quad M_{0,1;t} = -I_{0,1}, \quad M_{0,2;t} = -I_{0,2}$$

as expected. From the MAPLE file attached at the end of this Appendix we can see that

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{0,3} = (4I_{0,1;t} - 2(I_{0,0;t} + I_{0,2;t}))I_{0,3}^2 + (I_{0,2;t} - I_{0,0;t})I_{0,3} - I_{0,1;t} + I_{0,3;t}$$

and therefore by (A.0.1) we can deduce that

$$M_{0,3;t} = (4I_{0,1;t} - 2(I_{0,0;t} + I_{0,2;t}))I_{0,3}^2 + (I_{0,2;t} - I_{0,0;t})I_{0,3} - I_{0,1;t}.$$

Also

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{0,4} = (4I_{0,1;t} - 2(I_{0,0;t} + I_{0,2;t}))I_{0,4}^2 + (I_{0,2;t} - I_{0,0;t})I_{0,4} - I_{0,1;t} + I_{0,4;t}$$

and therefore by (A.0.1)

$$M_{0,4;t} = (4I_{0,1;t} - 2(I_{0,0;t} + I_{0,2;t}))I_{0,4}^2 + (I_{0,2;t} - I_{0,0;t})I_{0,4} - I_{0,1;t}.$$

One can guess that in general

$$\frac{\mathrm{d}}{\mathrm{d}t} I_{0,j} = (4I_{0,1;t} - 2(I_{0,0;t} + I_{0,2;t}))I_{0,j}^2 + (I_{0,2;t} - I_{0,0;t})I_{0,j} - I_{0,1;t}I_{0,j;t}$$

and therefore by (A.0.1) we can deduce that

$$M_{0,j;t} = (4I_{0,1;t} - 2(I_{0,0;t} + I_{0,2;t}))I_{0,j}^2 + (I_{0,2;t} - I_{0,0;t})I_{0,j} - I_{0,1;t}. \tag{A.0.3}$$

```
> restart
> with(LinearAlgebra) :
> read "indiff-src-2" :
Error, (in with) package Groebner does not export normalf
Error, (in with) package Groebner does not export gsolve
Error, (in with) package Groebner does not export inter_reduce
Error, (in with) package Groebner does not export gbasis
Error, (in with) package Groebner does not export termorder
> vars := [t] : ukns := [x0, x1, x2, x3, x4] : GroupP := [a, b, c] :
> XiPhis := Matrix([[0, 2·In[x0, [ ]], 2·In[x1, [ ]], 2·In[x2, [ ]], 2·In[x3, [ ]], 2·In[x4, [ ]]], [0, 1, 1, 1, 1, 1],
    [0,-In[x0, [ ]]^2,-In[x1, [ ]]^2,-In[x2, [ ]]^2,-In[x3, [ ]]^2, -In[x4, [ ]]^2]])
```

$$XiPhis := \begin{bmatrix} 0 & 2\,In_{x0,\,[\,]} & 2\,In_{x1,\,[\,]} & 2\,In_{x2,\,[\,]} & 2\,In_{x3,\,[\,]} & 2\,In_{x4,\,[\,]} \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -In^2_{x0,\,[\,]} & -In^2_{x1,\,[\,]} & -In^2_{x2,\,[\,]} & -In^2_{x3,\,[\,]} & -In^2_{x4,\,[\,]} \end{bmatrix} \quad \textbf{(1)}$$

$> Neqs := \left[In[x0,[\,]] - \frac{1}{2}, In[x1,[\,]], In[x2,[\,]] + \frac{1}{2}\right]:$

```
> HNI([[1], [x0, x1, x2, x3, x4]], ttdeg) :
> Kmat( ) :
```

$> mysubs := \left\{In[x0,[\,]] = \frac{1}{2}, In[x1,[\,]] = 0, In[x2,[\,]] = -\frac{1}{2}\right\}:$

```
> MyKmatrix := subs(mysubs, Kmat( ))
```

$$MyKmatrix := \begin{bmatrix} \frac{In_{x0,\,[1]}}{2} - \frac{In_{x2,\,[1]}}{2} & In_{x1,\,[1]} & -2\,In_{x0,\,[1]} + 4\,In_{x1,\,[1]} - 2\,In_{x2,\,[1]} \end{bmatrix} \quad \textbf{(2)}$$

Here we check Remark 5.2.5 in [68] to make sure we are choosing the correct infinitesimal vector fields and basis of the Lie algebra. The Lie bracket multiplication table for the basis of the Lie algebra has to be the minus Lie bracket multiplication table for the infinitesimal vector fields.

```
> myeij := (i, j) → Matrix(2, 2, (k, l) → if k = i and l = j then 1 else 0 end if);
        myeij := (i, j) ↦ Matrix(2, 2, (k, l) ↦ if k = i and l = j then 1 else 0 end if)      (3)
> a1 := myeij(1, 1) - myeij(2, 2);
```

$$a1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \textbf{(4)}$$

```
> a2 := myeij(1, 2);
```

$$a2 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textbf{(5)}$$

```
> a3 := myeij(2, 1);
```

$$a3 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \textbf{(6)}$$

```
> a1 · a2 − a2 · a1 − 2 a2;
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(7)}$$

```
> a1 · a3 − a3 · a1 + 2 a3;
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(8)}$$

```
> a2 · a3 − a3 · a2 − a1;
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(9)}$$

> $myv1 := f \rightarrow 2\,x \cdot diff(f, x);$

$$myv1 := f \rightarrow 2\,x \left(\frac{\partial}{\partial x} f \right) \tag{10}$$

> $myv2 := f \rightarrow diff(f, x);$

$$myv2 := f \rightarrow \frac{\partial}{\partial x} f \tag{11}$$

> $myv3 := f \rightarrow -x^2 diff(f, x);$

$$myv3 := f \rightarrow -x^2 \left(\frac{\partial}{\partial x} f \right) \tag{12}$$

> $myv1(myv2(f(x, y, z))) - myv2(myv1(f(x, y, z))) + 2 \cdot myv2(f(x, y, z)) : expand(\%);$

$$0 \tag{13}$$

> $myv1(myv3(f(x, y, z))) - myv3(myv1(f(x, y, z))) - 2 \cdot myv3(f(x, y, z)) : expand(\%);$

$$0 \tag{14}$$

> $myv2(myv3(f(x, y, z))) - myv3(myv2(f(x, y, z))) + myv1(f(x, y, z)) : expand(\%);$

$$0 \tag{15}$$

We add a minus sign to the curvature matrix because Indiff gives the minus correction matrix.

> $N0 := -(MyKmatrix[1, 1] \cdot a1 + MyKmatrix[1, 2] \cdot a2 + MyKmatrix[1, 3] \cdot a3)$

$$N0 := \begin{bmatrix} -\frac{In_{x0, [1]}}{2} + \frac{In_{x2, [1]}}{2} & -In_{x1, [1]} \\ 2\,In_{x0, [1]} - 4\,In_{x1, [1]} + 2\,In_{x2, [1]} & \frac{In_{x0, [1]}}{2} - \frac{In_{x2, [1]}}{2} \end{bmatrix} \tag{16}$$

Here we perform a few invariant differentiations that allow us to investigate the formula for the correction terms.

> $Idiff(In[x0, [\,]], 1)$

$$0 \tag{17}$$

> $Idiff(In[x1, [\,]], 1)$

$$0 \tag{18}$$

> $Idiff(In[x2, [\,]], 1)$

$$0 \tag{19}$$

> $Idiff(In[x3, [\,]], 1)$

$$\left(-2\,In_{x0, [1]} + 4\,In_{x1, [1]} - 2\,In_{x2, [1]}\right) In_{x3, [\,]}^2 + \left(-In_{x0, [1]} + In_{x2, [1]}\right) In_{x3, [\,]} - In_{x1, [1]} + In_{x3, [1]} \tag{20}$$

> $Idiff(In[x4, [\,]], 1)$

$$\left(-2\,In_{x0, [1]} + 4\,In_{x1, [1]} - 2\,In_{x2, [1]}\right) In_{x4, [\,]}^2 + \left(-In_{x0, [1]} + In_{x2, [1]}\right) In_{x4, [\,]} - In_{x1, [1]} + In_{x4, [1]} \tag{21}$$

>

Maple Files

1. Running example for §3, Discrete Moving Frames and Noether's Finite Difference Conservation Laws
2. 4.1 - Study of the discrete Euler's elastica
3. 4.2 - Study of $SU(2)$
4. 4.3.1 - The $SL(2)$ linear action
5. 4.3.2 - The $SA(2)$ linear action
6. 4.3.3 - The $SL(2)$ projective action
7. Running example for §5, Commuting Induced Flows on the Curvature Invariants
8. 5.7 - The $SL(2)$ linear action
9. 6.2 - The $SE(2)$ action acting on multispaces
10. 6.3 - The projective $SL(2)$ action acting on multispaces
11. Plots of Figure 7.1
12. 7.5.1 - Invariant Lagrangians involving only $\frac{\kappa_2}{\kappa_1}$
13. 7.5.2 (first example)
14. 7.5.2 (second example)

Running example for Chapter 3, Discrete Moving Frames and Noether's Finite Difference Conservation Laws

> *restart*

> *with*(*LinearAlgebra*) :

Lagrangian

> $L := \frac{(x[n+1]-x[n])}{((u[n+2]-u[n+1])\cdot(u[n+1]-u[n]))^{\frac{3}{2}}}$

$$L := \frac{x_{n+1}-x_n}{((u_{n+2}-u_{n+1})(u_{n+1}-u_n))^{3/2}} \quad \textbf{(1)}$$

Computation of the derivatives of the Lagrangian with respect to the variables x0, x1, u0, u1 and u2 in order to construct the Euler Lagrange equations later on

> *diff*(*L*, *x*[*n*])

$$-\frac{1}{((u_{n+2}-u_{n+1})(u_{n+1}-u_n))^{3/2}} \quad \textbf{(2)}$$

> *diff*(*L*, *x*[*n* + 1])

$$\frac{1}{((u_{n+2}-u_{n+1})(u_{n+1}-u_n))^{3/2}} \quad \textbf{(3)}$$

> *diff*(*L*, *u*[*n*])

$$-\frac{3(x_{n+1}-x_n)(-u_{n+2}+u_{n+1})}{2((u_{n+2}-u_{n+1})(u_{n+1}-u_n))^{5/2}} \quad \textbf{(4)}$$

> *diff*(*L*, *u*[*n* + 1])

$$-\frac{3(x_{n+1}-x_n)(-2u_{n+1}+u_n+u_{n+2})}{2((u_{n+2}-u_{n+1})(u_{n+1}-u_n))^{5/2}} \quad \textbf{(5)}$$

> *diff*(*L*, *u*[*n* + 2])

$$-\frac{3(x_{n+1}-x_n)(u_{n+1}-u_n)}{2((u_{n+2}-u_{n+1})(u_{n+1}-u_n))^{5/2}} \quad \textbf{(6)}$$

Action on the variables x0,x1,u0,u1 and u2

> $X0 := \lambda^3\cdot x[n]+a$:

> $X1 := \lambda^3\cdot x[n+1]+a$:

> $U0 := \lambda\cdot u[n]+b$:

> $U1 := \lambda\cdot u[n+1]+b$:

> $U2 := \lambda\cdot u[n+2]+b$:

We now check that the Lagrangian is invariant under the action

> *subs*({*x*[*n*] = *X0*, *x*[*n* + 1] = *X1*, *u*[*n*] = *U0*, *u*[*n* + 1] = *U1*, *u*[*n* + 2] = *U2*}, *L*) : *simplify*(%, *symbolic*)

$$\frac{x_{n+1}-x_n}{(-u_{n+2}+u_{n+1})^{3/2}(-u_{n+1}+u_n)^{3/2}} \quad \textbf{(7)}$$

Normalisation equations

> *Eq1* := *X0* :

> *Eq2* := *U0* :

> *Eq3* := *U1* − 1 :

> *solve*({*Eq1*, *Eq2*, *Eq3*}, {lambda, *a*, *b*}) : *simplify*(%, *symbolic*)

$$\left\{a = \frac{x_n}{(-u_{n+1}+u_n)^3}, b = \frac{u_n}{-u_{n+1}+u_n}, \lambda = -\frac{1}{-u_{n+1}+u_n}\right\} \quad \textbf{(8)}$$

> *assign*(%)

Frame

> rho[*n*] := *Matrix*([[λ^3, 0, *a*], [0, lambda, *b*], [0, 0, 1]])

$$\rho_n := \begin{bmatrix} -\frac{1}{(-u_{n+1}+u_n)^3} & 0 & \frac{x_n}{(-u_{n+1}+u_n)^3} \\ 0 & -\frac{1}{-u_{n+1}+u_n} & \frac{u_n}{-u_{n+1}+u_n} \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(9)}$$

We now check the equivariance of the frame

> unassign('lambda','a','b') : subs({x[n] = X0, x[n + 1] = X1, u[n] = U0, u[n + 1] = U1, u[n + 2] = U2}, rho[n]) − rho[n].MatrixInverse(Matrix([[λ^3, 0, a], [0, lambda, b], [0, 0, 1]])) : simplify(%, symbolic)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(10)}$$

Invariants

> simplify(rho[n].Matrix([[x[n + j]], [u[n + j]], [1]]), symbolic)

$$\begin{bmatrix} \frac{-x_{n+j}+x_n}{(-u_{n+1}+u_n)^3} \\ \frac{-u_{n+j}+u_n}{-u_{n+1}+u_n} \\ 1 \end{bmatrix} \quad \textbf{(11)}$$

First order differential invariants

> simplify(rho[n].Matrix([[x[n + j, t]], [u[n + j, t]], [0]]), symbolic)

$$\begin{bmatrix} -\frac{x_{n+j,\,t}}{(-u_{n+1}+u_n)^3} \\ -\frac{u_{n+j,\,t}}{-u_{n+1}+u_n} \\ 0 \end{bmatrix} \quad \textbf{(12)}$$

MaurerCartan matrix

> rho[n + 1] := subs(n = n + 1, rho[n]) :

> K[n] := subs({x[n] = 0, u[n] = 0, x[n + 1] = eta[n](t), u[n + 1] = 1, u[n + 2] = kappa[n](t)}, %)

$$K_n := \begin{bmatrix} -\frac{1}{(-\kappa_n(t)+1)^3} & 0 & \frac{\eta_n(t)}{(-\kappa_n(t)+1)^3} \\ 0 & -\frac{1}{-\kappa_n(t)+1} & \frac{1}{-\kappa_n(t)+1} \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(13)}$$

(14)

Relationships between invariants

> simplify(K[n].Matrix([[Inv[x, n + 1](t)], [Inv[u, n + 1](t)], [1]]), symbolic)

(15)

$$\left[\begin{array}{c}\dfrac{Inv_{x,\,n+1}(t)-\eta_n(t)}{\left(\kappa_n(t)-1\right)^3}\\[2ex]\dfrac{Inv_{u,\,n+1}(t)-1}{\kappa_n(t)-1}\\[2ex]1\end{array}\right] \quad \textbf{(15)}$$

Relationships between first order differential invariants

> K[n].Matrix([[Inv[x, n + 1, t](t)], [Inv[u, n + 1, t](t)], [0]])

$$\left[\begin{array}{c}-\dfrac{Inv_{x,\,n+1,\,t}(t)}{\left(-\kappa_n(t)+1\right)^3}\\[2ex]-\dfrac{Inv_{u,\,n+1,\,t}(t)}{-\kappa_n(t)+1}\\[2ex]0\end{array}\right] \quad \textbf{(16)}$$

Curvature matrix

> diff(map(z→z(t), rho[n]), t) :

> subs({ d/dt x_n(t) = sigmax[n](t), d/dt x_{n+1}(t) = Inv[x, n + 1, t](t), d/dt u_n(t) = sigmau[n](t), d/dt u_{n+1}(t) = Inv[u, n + 1, t](t)}, %) :

> subs({x[n](t) = 0, u[n](t) = 0, x[n + 1](t) = eta[n](t), u[n + 1](t) = 1, u[n + 2](t) = kappa[n](t)}, %) :

> N[n] := subs({Inv_{x, n+1, t}(t) = (kappa[n](t) − 1)^3·sigmax[n + 1](t), Inv_{u, n+1, t}(t) = (kappa[n](t) − 1)·sigmau[n + 1](t)}, %)

$$N_n := \left[\left[-3\left(\kappa_n(t)-1\right) sigmau_{n+1}(t)+3\, sigmau_n(t),\ 0,\ -sigmax_n(t)\right],\ \left[0,\ -\left(\kappa_n(t)-1\right) sigmau_{n+1}(t)+sigmau_n(t),\ -sigmau_n(t)\right],\ \left[0,\ 0,\ 0\right]\right] \quad \textbf{(17)}$$

(18)

Syzygy and evolution of curvature invariants

> syzygy := simplify(map(z→diff(z, t), K[n]) − (subs({n = n + 1}, N[n]).K[n] − K[n].N[n]), symbolic) :

> simplify(isolate(syzygy(1, 1), d/dt κ_n(t)), symbolic) : collect(%, sigmau_n(t)) : collect(%, sigmau_{n+1}(t))

$$\frac{d}{dt}\kappa_n(t) = -\kappa_n(t)\left(\kappa_n(t)-1\right) sigmau_{n+1}(t)+\left(\kappa_n(t)-1\right) sigmau_n(t)-\left(-\kappa_{n+1}(t)+1\right) sigmau_{n+2}(t)\left(\kappa_n(t)-1\right) \quad \textbf{(19)}$$

> dtk := % :

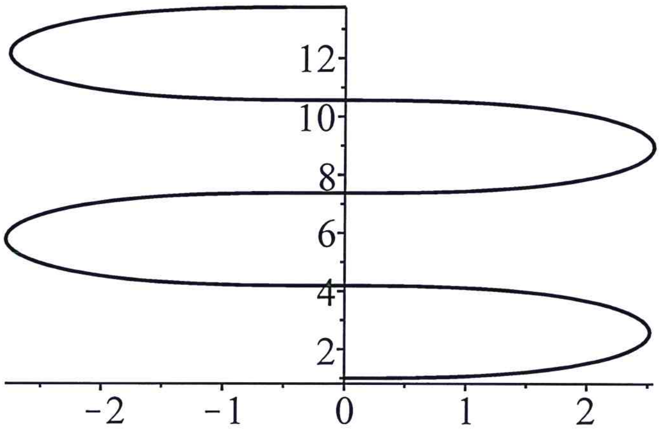
> simplify(isolate(syzygy(1, 3), d/dt η_n(t)), symbolic) : subs(dtk, %) : simplify(%, symbolic) : collect(%, sigmau_n(t)) : collect(%, sigmau_{n+1}(t))

$$\frac{d}{dt}\eta_n(t) = \left(-3\,\kappa_n(t)\,\eta_n(t)+3\,\eta_n(t)\right) sigmau_{n+1}(t)+3\, sigmau_n(t)\,\eta_n(t)+\left(\kappa_n(t)-1\right)^3 sigmax_{n+1}(t)-sigmax_n(t) \quad \textbf{(20)}$$

>

4.1 • *Study of the discrete Euler's elastica*

```
> restart
> with(LinearAlgebra) :
> #Smooth
> Digits := 180 :
> #Constants - matching the discrete case
> c1 :=
    -0.0027634652177898342098779728443661535187662946732511797152043675313513483812619273394860\
    3347433432703962009286825101479769504840417371597210550921864011313985769396159679400129765\
    166 :
> c2 :=
    -0.5654766564636770980474251008971958464313889772407313615969255866397582595379860866182326\
    9377376917169152765748413054787963184294421097183065523234668195962109951176109526386314391\
    3 :
> c3 :=
    0.0880751427714196583964134173265053739158384163204880325620688199141203534158176284295176\
    6325901418322644183109661863048256374289287570647671448225702046187010924526358652518106901
    :
> #Equations
> eq1 := -diff(x(t), t)·(kappa(t))^2 + 2·diff(u(t), t)·diff(kappa(t), t) = c1 :
> eq2 := -diff(u(t), t)·(kappa(t))^2 - 2·diff(x(t), t)·diff(kappa(t), t) = c2 :
> eq3 := -(x(t)·diff(u(t), t) - u(t)·diff(x(t), t))·κ(t)^2 - 2·diff(kappa(t), t)·(u(t)·diff(u(t), t) + x(t)
    ·diff(x(t), t)) + 2·kappa(t) = c3 :
> eq4 := κ(t)^4 + 4·diff(kappa(t), t)^2 - (c1^2 + c2^2) :
> eq5 := diff(u(t), t)·(c1^2 + c2^2) + c2·kappa(t)^2-2·c1·diff(kappa(t), t) :
> eq6 := diff(kappa(t), t) - 1/2·sqrt(c1^2 + c2^2 - kappa(t)^4) :
> eq7 := diff(diff(kappa(t), t), t) + 1/2 κ(t)^3 :
> A := Array(1..300, i→.1·i, datatype = float) :
> Dkappa0 := 1/2·sqrt(c1^2 + c2^2 - ((c1 + c3)/2)^4) :
> #Smooth solution
> sol := dsolve({eq5, eq7, u(0) = 1, kappa(0) = (c1 + c3)/2, D(kappa)(0) = Dkappa0}, numeric) :
> with(plots) :
> newplot := odeplot(sol, [1/c2·(c1·u(t) + c3 - 2·kappa(t)), u(t)], t = 0..27.8328, color = black) :
> display({newplot})
```




```
> #Discrete
> #Initialdata
> Digits := 180 :
```

$$c[-1] := evalf\left(\cos\left(-\frac{\text{Pi}}{1000}\right)\right):$$

$$s[-1] := evalf\left(\sin\left(-\frac{\text{Pi}}{1000}\right)\right):$$

$$c[-2] := evalf\left(\cos\left(\left(-\frac{\text{Pi}}{10000}\right)\right)\right):$$

$$s[-2] := evalf\left(\sin\left(\left(-\frac{\text{Pi}}{10000}\right)\right)\right):$$

$h_{-2} :=$

0.01807026558575580431819934025161951204832629506030904491258477447516011803151326092139987\
385778351278455008778871475170210460564163690743356688656211871214431333151801596192254532 3\
8 :

$h_{-1} :=$

0.07905701725466538502221967992458954865940678225641592729827769302981276087058781105459817\
543071938357980877783968720474119596474759507813659616414515770900262479015994745316811981 8\
4 :

$x[0] := 0 :$

$y[0] := 1 :$

```
>
```

$$> \quad L := \frac{\sin(\text{alpha}[j])^2}{h[j]} \quad :\#Lagrangian$$

$> \quad EulerAlpha[j] := diff(L, \text{alpha}[j]) :$

$> \quad EulerH[j] := diff(L, h[j]) :$

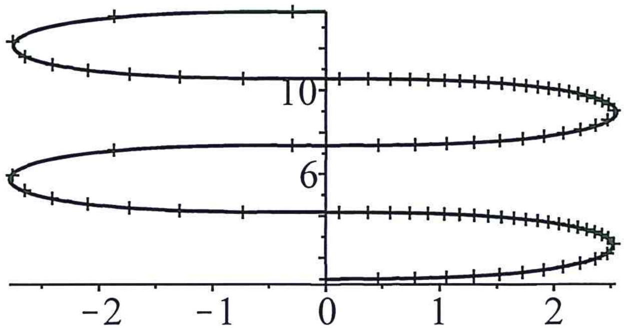
$> \quad newEulerAlpha := subs(\{\sin(\text{alpha}[j]) = s[j], \cos(\text{alpha}[j]) = c[j]\}, EulerAlpha[j]) \;:\#algebraic$

$> \quad newEulerh := subs(\{\sin(\text{alpha}[j]) = s[j], \cos(\text{alpha}[j]) = c[j]\}, EulerH[j]) \;:\#algebraic$

```
> eq1 := c[j-1]·subs({j=j-1}, newEulerh) - newEulerh + s[j-1]/h[j-1]·subs({j=j-2}, newEulerAlpha)
     - s[j-1]/h[j-1]·subs({j=j-1}, newEulerAlpha) :
> eq2 := s[j-1]·subs({j=j-1}, newEulerh) + subs({j=j-1}, newEulerAlpha)/h[j] - newEulerAlpha/h[j]
     - c[j-1]/h[j-1]·subs({j=j-2}, newEulerAlpha) + c[j-1]/h[j-1]·subs({j=j-1}, newEulerAlpha) :
> eq3 := c[j]^2 + s[j]^2 - 1 :
> #Vector of invariants
> v1 := subs(j=j-1, c[j]·newEulerh - s[j]/h[j]·newEulerAlpha + s[j]/h[j]·subs(j=j-1, newEulerAlpha)) :
> v2 := subs(j=j-1, s[j]·newEulerh + c[j]/h[j]·newEulerAlpha - c[j]/h[j]·subs(j=j-1, newEulerAlpha)) :
> v3 := -subs(j=j-1, newEulerAlpha) :
> #Obtaining cosine and sine of the angle from conservation laws
> C[j-1] := ((c1·v1 + v2·c2))/(c1^2 + c2^2) :
> S[j-1] := -(v1·c2 - v2·c1)/(c1^2 + c2^2) :
> C[0] := subs(j=0, C[j-1]) :
> S[0] := subs(j=0, S[j-1]) :
                                                                    (1)
> #Initial conditions for the cosine and sine of the angle theta
> C[0] := evalf(cos(-Pi/3000)) :
> S[0] := evalf(sin(-Pi/3000)) :
>
>
  #MovingFrame procedure
  movingframe := proc(lold, costhetaold, sinthetaold, xold, yold)
  local xnew, ynew;
  xnew := lold·costhetaold + xold;
  ynew := -lold·sinthetaold + yold;
  xnew, ynew;
  end proc:

> #Program
> for i from 0 to 845 do
  fred := subs(j = i, {eq1, eq2, eq3});
  solve(fred union {h[i] > 0} union {-1 ≤ c[i] ≤ 1} union {-1 ≤ s[i] ≤ 1} union {c[i] > 0}, {c[i], s[i],
     h[i]});
  evalf(%);
  assign(%);
  C[i] := subs(j = i, C[j-1]);
  S[i] := subs(j = i, S[j-1]);
  x[i+1] := movingframe(h[i], C[i], S[i], x[i], y[i])[1]; y[i+1] := movingframe(h[i], C[i], S[i], x[i],
     y[i])[2];
  end do:
> with(plots) :
```

```
points1 := {seq([x[i], y[i]], i = 0..846, 10)}
> #Plot of the solution
> dis1 := pointplot(points1, color = black, symbol = cross, symbolsize = 20, titlefont = ["ARIAL", 15]) :
> display({dis1, newplot})
```




```
>
> #Initial data 2
> for i from -2 to 1000 do
  unassign('x[i]','y[i]','c[i]','s[i]','h[i]')
  end do:
```

$> c[-1] := evalf\left(\cos\left(-\frac{Pi}{600}\right)\right):$

$s[-1] := evalf\left(\sin\left(-\frac{Pi}{600}\right)\right):$

$c[-2] := evalf\left(\cos\left(\left(-\frac{Pi}{1000}\right)\right)\right):$

$s[-2] := evalf\left(\sin\left(\left(-\frac{Pi}{1000}\right)\right)\right):$

$h_{-2} := 1.26501189751620900700759078041:$

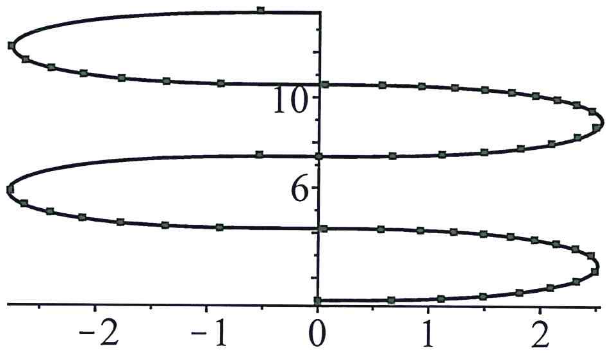
$h_{-1} := 0.131761310110213022802547421744:$

$x[0] := 0:$

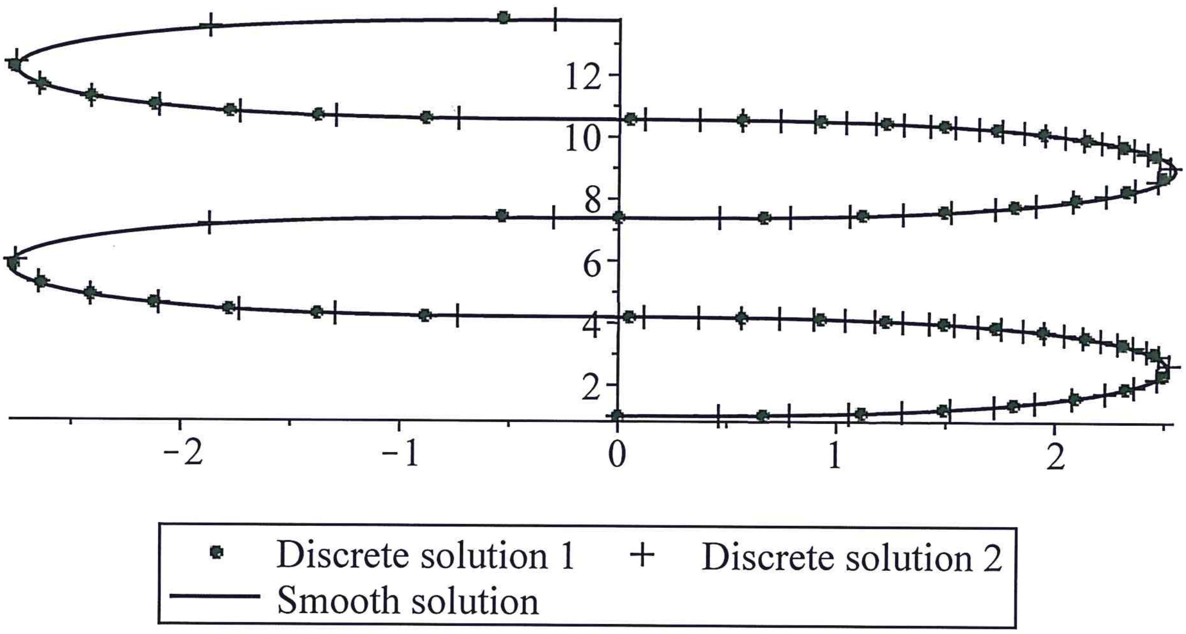
$y[0] := 1:$

```
> #Program
> for i from 0 to 505 do
  fred := subs(j = i, {eq1, eq2, eq3});
  solve(fred union {h[i] > 0} union {-1 ≤ c[i] ≤ 1} union {-1 ≤ s[i] ≤ 1} union {c[i] > 0}, {c[i], s[i],
      h[i]});
  evalf(%);
  assign(%);
  C[i] := subs(j = i, C[j − 1]);
  S[i] := subs(j = i, S[j − 1]);
  x[i + 1] := movingframe(h[i], C[i], S[i], x[i], y[i])[1]; y[i + 1] := movingframe(h[i], C[i], S[i], x[i],
      y[i])[2];
  end do:
> with(plots) :
  points2 := {seq([x[i], y[i]], i = 0..506, 10)} :
```

```
> #Plot of the solution
> dis2 := pointplot(points2, color = black, symbol = solidcircle, symbolsize = 12, titlefont = ["ARIAL", 15]) :
> display({newplot, dis2})
```



```
> #Plot of the smooth solution and both discrete solutions
> display({newplot, dis1, dis2})
```




```
>
```

4.2 - SU(2)

```
> restart
> with(LinearAlgebra) :
```

FRAME in complex form

We first obtain the frame using the normalization equations

$> \alpha := realpha + I \cdot imalpha :$

$> ALPHA := realpha - I \cdot imalpha :$

$> \beta := rebeta + I \cdot imbeta :$

$> BETA := rebeta - I \cdot imbeta :$

$> z[n] := a[n](t) + I \cdot b[n](t) :$

$> z[n+1] := a[n+1](t) + I \cdot b[n+1](t) :$

$> Z[n] := a[n](t) - I \cdot b[n](t) :$

$> Z[n+1] := a[n+1](t) - I \cdot b[n+1](t) :$

$> \alpha \cdot z[n] + \beta \cdot z[n+1] :$

$> expand(\%)$

$$-b_{n+1}(t)\ imbeta - b_n(t)\ imalpha + \mathrm{I}\, a_{n+1}(t)\ imbeta + \mathrm{I}\, b_{n+1}(t)\ rebeta + \mathrm{I}\, a_n(t)\ imalpha + \mathrm{I}\, b_n(t)\ realpha + a_{n+1}(t)\ rebeta + a_n(t)\ realpha \quad (1)$$

$> eq1 := imalpha\, a_n(t) + realpha\, b_n(t) + a_{n+1}(t)\ imbeta + b_{n+1}(t)\ rebeta :$

$> -BETA \cdot z[n] + ALPHA \cdot z[n+1] :$

$> expand(\%)$

$$b_{n+1}(t)\ imalpha - b_n(t)\ imbeta - \mathrm{I}\, a_{n+1}(t)\ imalpha + \mathrm{I}\, b_{n+1}(t)\ realpha + \mathrm{I}\, a_n(t)\ imbeta - \mathrm{I}\, b_n(t)\ rebeta + a_{n+1}(t)\ realpha - a_n(t)\ rebeta \quad (2)$$

$> eq2 := -imbeta\, b_n(t) + b_{n+1}(t)\ imalpha - rebeta\, a_n(t) + a_{n+1}(t)\ realpha :$

$> eq3 := imbeta\, a_n(t) - rebeta\, b_n(t) - a_{n+1}(t)\ imalpha + b_{n+1}(t)\ realpha :$

$> eq4 := realpha^2 + imalpha^2 + rebeta^2 + imbeta^2 - 1 :$

$> solve(\{eq1, eq2, eq3, eq4\}, \{realpha, rebeta, imalpha, imbeta\}) :$

$> allvalues(\%) :$

$> \%[1] :$

$> assign(\%) :$

$> \alpha$

$$a_n(t) \sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} - \mathrm{I}\, b_n(t) \sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} \quad (3)$$

$> \beta$

$$\sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}}\ a_{n+1}(t) - \mathrm{I} \sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}}\ b_{n+1}(t) \quad (4)$$

$> ALPHA$

$$a_n(t) \sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} + \mathrm{I}\, b_n(t) \sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} \quad (5)$$

$> BETA$

$$\sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}}\ a_{n+1}(t) + \mathrm{I} \sqrt{\frac{1}{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}}\ b_{n+1}(t) \quad (6)$$

Moving frame

> rho[n] := simplify(Matrix([[alpha, beta], [-BETA, ALPHA]]), symbolic)

$$\rho_n := \begin{bmatrix} \frac{a_n(t) - \mathrm{I}\, b_n(t)}{\sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} & \frac{a_{n+1}(t) - \mathrm{I}\, b_{n+1}(t)}{\sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} \\ -\frac{a_{n+1}(t) + \mathrm{I}\, b_{n+1}(t)}{\sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} & \frac{a_n(t) + \mathrm{I}\, b_n(t)}{\sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}} \end{bmatrix} \quad \textbf{(7)}$$

INVARIANTS

> mysubs := {a[n](t) = Inv[a[n]](t), b[n](t) = 0, a[n+1](t) = 0, b[n+1](t) = 0, a[n+2](t) = Inv[a[n+2]](t), b[n+2](t) = Inv[b[n+2]](t), a[n+3](t) = Inv[a[n+3]](t), b[n+3](t) = Inv[b[n+3]](t)} :

(8)

> z[n] := a[n](t) + I·b[n](t) :
> z[n+1] := a[n+1](t) + I·b[n+1](t) :
> z[n+2] := a[n+2](t) + I·b[n+2](t) :
> z[n+3] := a[n+3](t) + I·b[n+3](t) :
> Z[n] := a[n](t) − I·b[n](t) :
> Z[n+1] := a[n+1](t) − I·b[n+1](t) :
> Z[n+2] := a[n+2](t) − I·b[n+2](t) :
> Z[n+3] := a[n+3](t) − I·b[n+3](t) :
>
> simplify(rho[n].Matrix([[z[n]], [z[n+1]]]), symbolic)

$$\begin{bmatrix} \sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2} \\ 0 \end{bmatrix} \quad \textbf{(9)}$$

> simplify(subs(mysubs, %), symbolic)

$$\begin{bmatrix} Inv_{a_n}(t) \\ 0 \end{bmatrix} \quad \textbf{(10)}$$

> simplify(simplify(rho[n].Matrix([[z[n+2]], [z[n+3]]]), symbolic), size)

$$\left[\left[\frac{1}{\sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}}\left(\left(b_{n+3}(t) - \mathrm{I}\, a_{n+3}(t)\right) b_{n+1}(t) + \left(a_{n+2}(t) + \mathrm{I}\, b_{n+2}(t)\right) a_n(t) + \left(b_{n+2}(t) - \mathrm{I}\, a_{n+2}(t)\right) b_n(t) + \left(a_{n+3}(t) + \mathrm{I}\, b_{n+3}(t)\right) a_{n+1}(t)\right)\right], \right. \quad \textbf{(11)}$$

$$\left.\left[\frac{1}{\sqrt{a_n(t)^2 + a_{n+1}(t)^2 + b_n(t)^2 + b_{n+1}(t)^2}}\left(\left(b_{n+2}(t) - \mathrm{I}\, a_{n+2}(t)\right) b_{n+1}(t) + \left(a_{n+3}(t) + \mathrm{I}\, b_{n+3}(t)\right) a_n(t) + \left(\mathrm{I}\, a_{n+3}(t) - b_{n+3}(t)\right) b_n(t) - \left(a_{n+2}(t) + \mathrm{I}\, b_{n+2}(t)\right) a_{n+1}(t)\right)\right],\right.$$

> simplify(subs(mysubs, %), symbolic)

$$\begin{bmatrix} Inv_{a_{n+2}}(t) + \mathrm{I}\, Inv_{b_{n+2}}(t) \\ Inv_{a_{n+3}}(t) + \mathrm{I}\, Inv_{b_{n+3}}(t) \end{bmatrix} \quad \textbf{(12)}$$

MAURER-CARTAN MATRIX

> rho[n+1] := subs(n = n+1, rho[n]) :
> K0 := subs(mysubs, rho[n+1])

$$K0 := \begin{bmatrix} 0 & \frac{Inv_{a_{n+2}}(t) - \mathrm{I}\, Inv_{b_{n+2}}(t)}{\sqrt{Inv_{a_{n+2}}(t)^2 + Inv_{b_{n+2}}(t)^2}} \\ -\frac{Inv_{a_{n+2}}(t) + \mathrm{I}\, Inv_{b_{n+2}}(t)}{\sqrt{Inv_{a_{n+2}}(t)^2 + Inv_{b_{n+2}}(t)^2}} & 0 \end{bmatrix} \quad \textbf{(13)}$$

(14)

> $myK0 := Matrix([[0, \mathrm{sigma}(t)], [-\mathrm{kappa}(t), 0]])$

$$myK0 := \begin{bmatrix} 0 & \sigma(t) \\ -\kappa(t) & 0 \end{bmatrix} \quad \textbf{(15)}$$

CURVATURE MATRIX

> $difrho0 := map(z \rightarrow diff(z, t), \mathrm{rho}[n])$:

> $mysubs2 := \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\, a_n(t) = Inv[a[n], t](t), \frac{\mathrm{d}}{\mathrm{d}t}\, b_n(t) = Inv[b[n], t](t), \frac{\mathrm{d}}{\mathrm{d}t}\, a_{n+1}(t) = Inv[a[n+1], t](t), \frac{\mathrm{d}}{\mathrm{d}t}\, b_{n+1}(t) = Inv[b[n+1], t](t) \right\}$:

> $N0 := subs(mysubs2, difrho0)$:

> $N0 := subs(mysubs, \%)$:

> $N0 := simplify(\%, symbolic)$

$$N0 := \begin{bmatrix} \frac{-\mathrm{I}\, Inv_{b_n, t}(t)}{Inv_{a_n}(t)} & \frac{Inv_{a_{n+1}, t}(t) - \mathrm{I}\, Inv_{b_{n+1}, t}(t)}{Inv_{a_n}(t)} \\ \frac{-\mathrm{I}\, Inv_{b_{n+1}, t}(t) - Inv_{a_{n+1}, t}(t)}{Inv_{a_n}(t)} & \frac{\mathrm{I}\, Inv_{b_n, t}(t)}{Inv_{a_n}(t)} \end{bmatrix} \quad \textbf{(16)}$$

Shift of curvature matrix

> $SN0 := subs\Big(\big\{ Inv[a[n], t](t) = SInv[a[n], t](t), Inv[b[n], t](t) = SInv[b[n], t](t), Inv[a[n+1], t](t) = SInv[a[n+1], t](t), Inv[b[n+1], t](t) = SInv[b[n+1], t](t), Inv_{a_n}(t) = SInv_{a_n}(t) \big\}, N0\Big)$

$$SN0 := \begin{bmatrix} \frac{-\mathrm{I}\, SInv_{b_n, t}(t)}{SInv_{a_n}(t)} & \frac{S\,Inv_{a_{n+1}, t}(t) - \mathrm{I}\, SInv_{b_{n+1}, t}(t)}{SInv_{a_n}(t)} \\ \frac{-\mathrm{I}\, SInv_{b_{n+1}, t}(t) - S\,Inv_{a_{n+1}, t}(t)}{SInv_{a_n}(t)} & \frac{\mathrm{I}\, SInv_{b_n, t}(t)}{SInv_{a_n}(t)} \end{bmatrix} \quad \textbf{(17)}$$

Syzygy and evolution of curvature invariants

> $syzygy := simplify(SN0 \cdot myK0 - myK0 \cdot N0, symbolic)$:

> $-syzygy(2, 1)$:

> $collect\left(\%, Inv_{b_n, t}(t)\right)$

$$\frac{\mathrm{I}\, \kappa(t)\, Inv_{b_n, t}(t)}{Inv_{a_n}(t)} + \frac{\mathrm{I}\, SInv_{b_n, t}(t)\, \kappa(t)}{SInv_{a_n}(t)} \quad \textbf{(18)}$$

> $diff\left(\sqrt{a_n(t)^2 + b_n(t)^2 + a_{n+1}(t)^2 + b_{n+1}(t)^2}, t\right)$:

> $subs(mysubs2, \%)$: $subs(mysubs, \%)$: $simplify(\%, symbolic)$

$$Inv_{a_n,t}(t) \quad (19)$$

> restart

FRAME in polar form

We first obtain the frame using the normalization equations

> alpha := realpha + I·imalpha :

> ALPHA := realpha − I·imalpha :

> beta := rebeta + I·imbeta :

> BETA := rebeta − I·imbeta :

> $z[n] := r[n](t)\cdot e^{I\cdot \text{theta}[n](t)}$:

> $z[n+1] := subs(n = n+1, z[n])$:

> $Z[n] := r[n](t)\cdot e^{-I\cdot \text{theta}[n](t)}$:

> $Z[n+1] := subs(n = n+1, Z[n])$:

> alpha·z[n] + beta·z[n + 1] :

> $simplify\left(subs\left(\left\{e^{I\theta_n(t)} = \cos(\text{theta}[n]) + I\cdot\sin(\text{theta}[n]), e^{I\theta_{n+1}(t)} = \cos(\text{theta}[n+1]) + I\cdot\sin(\text{theta}[n+1])\right\}, \%\right), symbolic\right)$:

> expand(%)

$$-\sin(\theta_n)\, r_n(t)\, imalpha - \sin(\theta_{n+1})\, r_{n+1}(t)\, imbeta + \mathrm{I}\cos(\theta_n)\, r_n(t)\, imalpha + \mathrm{I}\sin(\theta_n)\, r_n(t)\, realpha + \mathrm{I}\cos(\theta_{n+1})\, r_{n+1}(t)\, imbeta + \mathrm{I}\sin(\theta_{n+1})\, r_{n+1}(t)\, rebeta + \cos(\theta_n)\, r_n(t)\, realpha + \cos(\theta_{n+1})\, r_{n+1}(t)\, rebeta \quad (20)$$

> $eq1 := \cos(\theta_{n+1})\, r_{n+1}(t)\, imbeta + \sin(\theta_{n+1})\, r_{n+1}(t)\, rebeta + \cos(\theta_n)\, r_n(t)\, imalpha + \sin(\theta_n)\, r_n(t)\, realpha$:

> -BETA·z[n] + ALPHA·z[n + 1] :

> $simplify\left(subs\left(\left\{e^{I\theta_n(t)} = \cos(\text{theta}[n]) + I\cdot\sin(\text{theta}[n]), e^{I\theta_{n+1}(t)} = \cos(\text{theta}[n+1]) + I\cdot\sin(\text{theta}[n+1])\right\}, \%\right), symbolic\right)$:

> expand(%)

$$-\sin(\theta_n)\, r_n(t)\, imbeta + \sin(\theta_{n+1})\, r_{n+1}(t)\, imalpha + \mathrm{I}\cos(\theta_n)\, r_n(t)\, imbeta - \mathrm{I}\sin(\theta_n)\, r_n(t)\, rebeta - \mathrm{I}\cos(\theta_{n+1})\, r_{n+1}(t)\, imalpha + \mathrm{I}\sin(\theta_{n+1})\, r_{n+1}(t)\, realpha - \cos(\theta_n)\, r_n(t)\, rebeta + \cos(\theta_{n+1})\, r_{n+1}(t)\, realpha \quad (21)$$

> $eq2 := -\sin(\theta_n)\, r_n(t)\, imbeta + \sin(\theta_{n+1})\, r_{n+1}(t)\, imalpha - \cos(\theta_n)\, r_n(t)\, rebeta + \cos(\theta_{n+1})\, r_{n+1}(t)\, realpha$:

> $eq3 := -\cos(\theta_{n+1})\, r_{n+1}(t)\, imalpha + \sin(\theta_{n+1})\, r_{n+1}(t)\, realpha + \cos(\theta_n)\, r_n(t)\, imbeta - \sin(\theta_n)\, r_n(t)\, rebeta$:

> $eq4 := realpha^2 + imalpha^2 + rebeta^2 + imbeta^2 - 1$:

> solve({eq1, eq2, eq3, eq4}, {realpha, rebeta, imalpha, imbeta}) :

> allvalues(%) :

> S[1] :

> assign(%)

> $simplify(\text{alpha}, symbolic) : convert(\%, \exp) : subs(\theta_n = \theta_n(t), \%) : \text{alpha} := \%$

$$\alpha := \frac{r_n(t)\, \mathrm{e}^{-\mathrm{I}\theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \quad (22)$$

> $simplify(\text{beta}, symbolic) : convert(\%, \exp) : subs(\theta_{n+1} = \theta_{n+1}(t), \%) : \text{beta} := \%$

$$\beta := \frac{r_{n+1}(t)\, \mathrm{e}^{-\mathrm{I}\,\theta_{n+1}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \quad \textbf{(23)}$$

> *simplify*(*ALPHA*, *symbolic*) : *convert*(%, exp) : *subs*($\theta_n = \theta_n(t)$, %) : *ALPHA* := %

$$ALPHA := \frac{r_n(t)\, \mathrm{e}^{\mathrm{I}\,\theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \quad \textbf{(24)}$$

> *simplify*(*BETA*, *symbolic*) : *convert*(%, exp) : *subs*($\theta_{n+1} = \theta_{n+1}(t)$, %) : *BETA* := %

$$BETA := \frac{r_{n+1}(t)\, \mathrm{e}^{\mathrm{I}\,\theta_{n+1}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \quad \textbf{(25)}$$

Points

> $z[n] := r[n](t) \cdot e^{I \cdot \mathrm{theta}[n](t)}$:

> $z[n+1] := subs(n = n+1, z[n])$:

> $z[n+2] := subs(n = n+2, z[n])$:

> $z[n+3] := subs(n = n+3, z[n])$:

> $Z[n] := r[n](t) \cdot e^{-I \cdot \mathrm{theta}[n](t)}$:

> $Z[n+1] := subs(n = n+1, Z[n])$:

> $Z[n+2] := subs(n = n+2, Z[n])$:

> $Z[n+3] := subs(n = n+3, Z[n])$:

Frame

> rho[*n*] := *simplify*(*Matrix*([[alpha, beta], [-*BETA*, *ALPHA*]]), *symbolic*)

$$\rho_n := \begin{bmatrix} \frac{r_n(t)\, \mathrm{e}^{-\mathrm{I}\,\theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} & \frac{r_{n+1}(t)\, \mathrm{e}^{-\mathrm{I}\,\theta_{n+1}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \\ -\frac{r_{n+1}(t)\, \mathrm{e}^{\mathrm{I}\,\theta_{n+1}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} & \frac{r_n(t)\, \mathrm{e}^{\mathrm{I}\,\theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \end{bmatrix} \quad \textbf{(26)}$$

INVARIANTS

> *mysubs* := { *r*[*n*](*t*) = *Inv*[*r*[*n*]](*t*), theta[*n*](*t*) = 0, *r*[*n* + 1](*t*) = 0, theta[*n* + 1](*t*) = *Inv*[theta[*n* + 1]](*t*), *r*[*n* + 2](*t*) = *Inv*[*r*[*n* + 2]](*t*), theta[*n* + 2](*t*) = *Inv*[theta[*n* + 2]](*t*), *r*[*n* + 3](*t*) = *Inv*[*r*[*n* + 3]](*t*), theta[*n* + 3](*t*) = *Inv*[theta[*n* + 3]](*t*) } :

> *simplify*(rho[*n*].*Matrix*([[*z*[*n*]], [*z*[*n* + 1]]]), *symbolic*)

$$\begin{bmatrix} \frac{r_{n+1}(t)^2\, \mathrm{e}^{-\mathrm{I}\,\theta_{n+1}(t)}\, e^{\mathrm{I}\,\theta_{n+1}(t)} + r_n(t)^2\, \mathrm{e}^{-\mathrm{I}\,\theta_n(t)}\, e^{\mathrm{I}\,\theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \\ \frac{r_{n+1}(t)\, r_n(t) \left(e^{\mathrm{I}\,\theta_{n+1}(t)}\, \mathrm{e}^{\mathrm{I}\,\theta_n(t)} - e^{\mathrm{I}\,\theta_n(t)}\, \mathrm{e}^{\mathrm{I}\,\theta_{n+1}(t)} \right)}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \end{bmatrix} \quad \textbf{(27)}$$

> *simplify*(*subs*(*mysubs*, %), *symbolic*)

$$\begin{bmatrix} Inv_{r_n}(t) \\ 0 \end{bmatrix} \quad \textbf{(28)}$$

> *simplify*(*simplify*(rho[*n*].*Matrix*([[*z*[*n* + 2]], [*z*[*n* + 3]]]), *symbolic*), *size*)

$$\begin{bmatrix} \dfrac{r_{n+1}(t)\, e^{-I\theta_{n+1}(t)}\, r_{n+3}(t)\, e^{I\theta_{n+3}(t)} + r_n(t)\, e^{-I\theta_n(t)}\, r_{n+2}(t)\, e^{I\theta_{n+2}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \\ \dfrac{-r_{n+1}(t)\, e^{I\theta_{n+1}(t)}\, r_{n+2}(t)\, e^{I\theta_{n+2}(t)} + r_n(t)\, e^{I\theta_n(t)}\, r_{n+3}(t)\, e^{I\theta_{n+3}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}} \end{bmatrix} \tag{29}$$

> *simplify*(*subs*(*mysubs*, %), *symbolic*)

$$\begin{bmatrix} Inv_{r_{n+2}}(t)\, e^{I\, Inv_{\theta_{n+2}}(t)} \\ Inv_{r_{n+3}}(t)\, e^{I\, Inv_{\theta_{n+3}}(t)} \end{bmatrix} \tag{30}$$

MAURER-CARTAN MATRIX

> rho[n + 1] := *subs*(n = n + 1, rho[n]) :

> *K0* := *simplify*(*subs*(*mysubs*, rho[n + 1]), *symbolic*)

$$K0 := \begin{bmatrix} 0 & e^{-I\, Inv_{\theta_{n+2}}(t)} \\ -e^{I\, Inv_{\theta_{n+2}}(t)} & 0 \end{bmatrix} \tag{31}$$

(32)

> *myK0* := *Matrix*([[0, sigma(t)], [-kappa(t), 0]])

$$myK0 := \begin{bmatrix} 0 & \sigma(t) \\ -\kappa(t) & 0 \end{bmatrix} \tag{33}$$

CURVATURE MATRIX

> *difrho0* := *map*(z→*diff*(z, t), rho[n]) :

> $mysubs2 := \left\{ \frac{d}{dt} r_n(t) = Inv[r[n], t](t),\ \frac{d}{dt} \text{theta}_n(t) = Inv[\text{theta}[n], t](t),\ \frac{d}{dt} r_{n+1}(t) = Inv[r[n+1], t](t),\ \frac{d}{dt} \text{theta}_{n+1}(t) = Inv[\text{theta}[n+1], t](t) \right\}$:

> *N0* := *subs*(*mysubs2*, *difrho0*) :

> *N0* := *subs*(*mysubs*, %) :

> *N0* := *subs*(ln(e) = 1, *simplify*(%, *symbolic*))

$$N0 := \begin{bmatrix} -I\, Inv_{\theta_n, t}(t) & \dfrac{Inv_{r_{n+1}, t}(t)\, e^{-I\, Inv_{\theta_{n+1}}(t)}}{Inv_{r_n}(t)} \\ -\dfrac{Inv_{r_{n+1}, t}(t)\, e^{I\, Inv_{\theta_{n+1}}(t)}}{Inv_{r_n}(t)} & I\, Inv_{\theta_n, t}(t) \end{bmatrix} \tag{34}$$

Shift of curvature matrix

> $SN0 := subs\big(\{ Inv[r[n], t](t) = SInv[r[n], t](t),\ Inv[\text{theta}[n], t](t) = SInv[\text{theta}[n], t](t),\ Inv[r[n+1], t](t) = SInv[r[n+1], t](t),\ Inv[\text{theta}[n+1], t](t) = SInv[\text{theta}[n+1], t](t),\ Inv_{r_n}(t) = SInv_{r_n}(t) \}, N0 \big)$

$$SN0 := \begin{bmatrix} -\mathrm{I}\, SInv_{\theta_n, t}(t) & \dfrac{S\, Inv_{r_{n+1}, t}(t)\, \mathrm{e}^{-\mathrm{I}\, Inv_{\theta_{n+1}}(t)}}{SInv_{r_n}(t)} \\ -\dfrac{S\, Inv_{r_{n+1}, t}(t)\, \mathrm{e}^{\mathrm{I}\, Inv_{\theta_{n+1}}(t)}}{SInv_{r_n}(t)} & \mathrm{I}\, SInv_{\theta_n, t}(t) \end{bmatrix} \quad (35)$$

SYZYGY and evolution of curvature invariants

> syzygy := simplify(SN0·myK0 − myK0·N0, symbolic) :

> -syzygy(2, 1) :

> $collect\left(\%, Inv_{b_n, t}(t)\right)$

$$\mathrm{I}\,\kappa(t)\,\left(SInv_{\theta_n, t}(t) + Inv_{\theta_n, t}(t)\right) \quad (36)$$

> diff(sqrt((r[n](t))^2 + (r[n + 1](t))^2), t) :

> subs(mysubs2, %) : subs(mysubs, %) : simplify(%, symbolic)

$$Inv_{r_n, t}(t) \quad (37)$$

Moving frame for the conjugate action of SU(2) on su(2)

> restart :
with(LinearAlgebra) :

> $z := \mathrm{sqrt}(x^2 + y1^2 + y2^2)$:

> g := Matrix([[a1 + I·a2, b1 + I·b2], [-b1 + I·b2, a1 - I·a2]]) :

> A := Matrix([[I·x, y1 + I·y2], [-y1 + I·y2, -I·x]]) :

> B := Matrix([[I·z, 0], [0, -I·z]]) :

> M := map(expand, g.A − B.g) :

Normalization equations

> eq1 := M[1, 1] :

> eq2 := M[1, 2] :

> eq3 := M[2, 1] :

> eq4 := M[2, 2] :

> mysols := solve({eq1, eq2, eq3, eq4}, {a1, b1, a2, b2})

$$mysols := \left\{ a1 = \frac{y2\, a2}{y1} - \frac{\left(\sqrt{x^2 + y1^2 + y2^2} + x\right) b2}{y1},\ a2 = a2,\ b1 = \frac{\left(\sqrt{x^2 + y1^2 + y2^2} - x\right) a2}{y1} - \frac{b2\, y2}{y1},\ b2 = b2 \right\} \quad (38)$$

We write the condition $a1^2 + a2^2 + b1^2 + b2^2 - 1$ in *terms of the* $a2, b2, x, y1$ and $y2$· (*called* $x0, y0$ and $z0$ in *the text*)

> $myrel1 := simplify\left(expand\left(subs\left(mysols, a1^2 + a2^2 + b1^2 + b2^2 - 1\right)\right), symbolic\right)$

$$myrel1 := \frac{1}{y1^2}\left(\left(-2\, a2^2\, x - 4\, a2\, b2\, y2 + 2\, b2^2\, x\right)\sqrt{x^2 + y1^2 + y2^2} + \left(2\, x^2 + 2\, y1^2 + 2\, y2^2\right) a2^2 + \left(2\, x^2 + 2\, y1^2 + 2\, y2^2\right) b2^2 - y1^2\right) \quad (39)$$

> $subs\left(x^2 = -y1^2 - y2^2 + Z^2, myrel1\right)$: myrel1 := numer(simplify(%, symbolic));

$$myrel1 := 2\, Z^2\, a2^2 + 2\, Z^2\, b2^2 - 2\, Z\, a2^2\, x - 4\, Z\, a2\, b2\, y2 + 2\, Z\, b2^2\, x - y1^2 \quad (40)$$

> collect(myrel1, {a2, b2}, distributed);

$$\left(2\, Z^2 - 2\, Z\, x\right) a2^2 - 4\, Z\, a2\, b2\, y2 + \left(2\, Z^2 + 2\, Z\, x\right) b2^2 - y1^2 \quad (41)$$

This is a conic, we apply usual theory. We first obtain the change of co-ords to make it an ellipse or a hyperbola

> $subs\left(a2=\frac{X}{4\cdot Z\cdot y1}, b2=Y-\frac{\text{alpha}\cdot X}{4\cdot Z\cdot y1}, myrel1\right) : expand(\%) : collect(\%, \{X, Y\}, distributed);$

$$-y1^2+\left(2Z^2+2Zx\right)Y^2+\left(-\frac{Z\alpha}{y1}-\frac{y2}{y1}-\frac{x\alpha}{y1}\right)XY+\left(\frac{1}{8y1^2}+\frac{\alpha^2}{8y1^2}-\frac{x}{8Zy1^2}+\frac{y2\alpha}{4y1^2Z}+\frac{x\alpha^2}{8Zy1^2}\right)X^2 \quad \textbf{(42)}$$

> $coeff(\%, Y, 1) : solve(\%, \text{alpha});$

$$-\frac{y2}{Z+x} \quad \textbf{(43)}$$

> $subs\left(a2=\frac{X}{4\cdot Z\cdot y1}, b2=Y+\frac{y2\cdot X}{4\cdot Z\cdot(Z+x)\cdot y1}, myrel1\right) : collect(\%, \{X, Y\}, distributed) : map(factor, \%) : subs\left(x^2=Z^2-y1^2-y2^2, \%\right) : map(simplify, \%);$

$$\frac{X^2}{8Z(Z+x)}+2Z(Z+x)Y^2-y1^2 \quad \textbf{(44)}$$

We have the ellipse. We do a final substitution to get it in terms of one angle

> $subs\left(X=y1\cdot\text{sqrt}(8\cdot Z\cdot(Z+x))\cdot\cos(\text{theta}), Y=\frac{y1}{\text{sqrt}(2\cdot Z\cdot(Z+x))}\cdot\sin(\text{theta}), \%\right) : simplify(\%, trig);$

$$0 \quad \textbf{(45)}$$

> $MyA2B2 := map\left(simplify, subs\left(X=y1\cdot\text{sqrt}(8\cdot Z\cdot(Z+x))\cdot\cos(\text{theta}), Y=\frac{y1}{\text{sqrt}(2\cdot Z\cdot(Z+x))}\cdot\sin(\text{theta}), \left\{a2=\frac{X}{4\cdot Z\cdot y1}, b2=Y+\frac{y2\cdot X}{4\cdot Z\cdot(Z+x)\cdot y1}\right\}\right), symbolic\right);$

$$MyA2B2 := \left\{a2=\frac{\sqrt{2}\sqrt{Z+x}\cos(\theta)}{2\sqrt{Z}}, b2=\frac{\sqrt{2}\left(y2\cos(\theta)+y1\sin(\theta)\right)}{2\sqrt{Z}\sqrt{Z+x}}\right\} \quad \textbf{(46)}$$

> $subs\left(x^2=Z^2-y1^2-y2^2, mysols\right) : simplify(\%, symbolic) : select(has, \%, \{a1, b1\}) : subs(MyA2B2, \%) : simplify(\%, symbolic) : subs\left(x^2=Z^2-y1^2-y2^2, \%\right) : simplify(\%, symbolic);$

$$\left\{a1=-\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{2\sqrt{Z}}, b1=\frac{\sqrt{2}\left(\cos(\theta)y1-\sin(\theta)y2\right)}{2\sqrt{Z}\sqrt{Z+x}}\right\} \quad \textbf{(47)}$$

This is now the simplest expression for the frame co-ords, we need only one angle now

> $mygsofar := \%$ **union** $MyA2B2;$

$$mygsofar := \left\{a1=-\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{2\sqrt{Z}}, a2=\frac{\sqrt{2}\sqrt{Z+x}\cos(\theta)}{2\sqrt{Z}}, b1=\frac{\sqrt{2}\left(\cos(\theta)y1-\sin(\theta)y2\right)}{2\sqrt{Z}\sqrt{Z+x}}, b2=\frac{\sqrt{2}\left(y2\cos(\theta)+y1\sin(\theta)\right)}{2\sqrt{Z}\sqrt{Z+x}}\right\} \quad \textbf{(48)}$$

Here we just check that a1^2 + a2^2 + b1^2 + b1^2 is one.

> $simplify\left(\left(-\frac{1}{2}\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{\sqrt{Z}}\right)^2+\left(\frac{1}{2}\frac{\sqrt{2}\sqrt{Z+x}\cos(\theta)}{\sqrt{Z}}\right)^2+\left(\frac{1}{2}\frac{\sqrt{2}\left(\cos(\theta)y1-\sin(\theta)y2\right)}{\sqrt{Z}\sqrt{Z+x}}\right)^2+\left(\frac{1}{2}\frac{\left(y2\cos(\theta)+y1\sin(\theta)\right)\sqrt{2}}{\sqrt{Z}\sqrt{Z+x}}\right)^2, symbolic\right) :$

> $subs\left(y1^2=Z^2-x^2-y2^2, \%\right) :$

> $simplify(\%)$

$$1 \quad \textbf{(49)}$$

> $G := subs(mygsofar, g);$

$$G := \left[\left[-\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{2\sqrt{Z}}+\frac{I\sqrt{2}\sqrt{Z+x}\cos(\theta)}{2\sqrt{Z}}, \frac{\sqrt{2}\left(\cos(\theta)y1-\sin(\theta)y2\right)}{2\sqrt{Z}\sqrt{Z+x}}\right.\right. \quad \textbf{(50)}$$

$$+\frac{I\sqrt{2}\,(y2\cos(\theta)+y1\sin(\theta))}{2\sqrt{Z}\sqrt{Z+x}}\Bigg],$$

$$\Bigg[-\frac{\sqrt{2}\,(\cos(\theta)\,y1-\sin(\theta)\,y2)}{2\sqrt{Z}\sqrt{Z+x}}+\frac{I\sqrt{2}\,(y2\cos(\theta)+y1\sin(\theta))}{2\sqrt{Z}\sqrt{Z+x}},-\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{2\sqrt{Z}}$$

$$-\frac{I\sqrt{2}\sqrt{Z+x}\cos(\theta)}{2\sqrt{Z}}\Bigg]\Bigg]$$

We now convert G into a matrix with exponential entries

> $NewG[1,1] := convert\left(-\frac{1}{2}\,\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{\sqrt{Z}}+\frac{\frac{1}{2}I\sqrt{2}\sqrt{Z+x}\cos(\theta)}{\sqrt{Z}}, \exp\right):$

> $NewG[1,2] := convert\left(\frac{1}{2}\,\frac{\sqrt{2}\,(\cos(\theta)\,y1-\sin(\theta)\,y2)}{\sqrt{Z}\sqrt{Z+x}}+\frac{\frac{1}{2}I\,(y2\cos(\theta)+y1\sin(\theta))\sqrt{2}}{\sqrt{Z}\sqrt{Z+x}}, \exp\right):$

> $NewG[2,1] := convert\left(-\frac{1}{2}\,\frac{\sqrt{2}\,(\cos(\theta)\,y1-\sin(\theta)\,y2)}{\sqrt{Z}\sqrt{Z+x}}+\frac{\frac{1}{2}I\,(y2\cos(\theta)+y1\sin(\theta))\sqrt{2}}{\sqrt{Z}\sqrt{Z+x}}, \exp\right):$

> $NewG[2,2] := convert\left(-\frac{1}{2}\,\frac{\sqrt{2}\sqrt{Z+x}\sin(\theta)}{\sqrt{Z}}-\frac{\frac{1}{2}I\sqrt{2}\sqrt{Z+x}\cos(\theta)}{\sqrt{Z}}, \exp\right):$

> $NewG := simplify(Matrix([[NewG[1,1], NewG[1,2]], [NewG[2,1], NewG[2,2]]]), symbolic)$

$$NewG := \begin{bmatrix} \frac{\frac{I}{2}\sqrt{2}\sqrt{Z+x}\,e^{I\theta}}{\sqrt{Z}} & \frac{\sqrt{2}\,e^{I\theta}\,(y1+I\,y2)}{2\sqrt{Z}\sqrt{Z+x}} \\ \frac{\sqrt{2}\,e^{-I\theta}\,(-y1+I\,y2)}{2\sqrt{Z}\sqrt{Z+x}} & \frac{-\frac{I}{2}\sqrt{2}\sqrt{Z+x}\,e^{-I\theta}}{\sqrt{Z}} \end{bmatrix} \quad \textbf{(51)}$$

Here we check that the determinant is 1

> $simplify(subs(y1^2 = Z^2 - x^2 - y2^2, Determinant(NewG)))$

$$1 \quad \textbf{(52)}$$

Calculating vector of invariants and adjoint matrix

> $restart:$

Condition

> $\alpha\cdot ALPHA + \beta\cdot BETA = 1:$

> $isolate(\%, ALPHA):$

> $assign(\%):$

Action

> $Z0 := \alpha\cdot z0 + \beta\cdot z1:$

> $Z1 := -BETA\cdot z0 + ALPHA\cdot z1:$

> $mysubs := \{\alpha = 1, \beta = 0, BETA = 0\}:$

Here we calculate the infinitesimals

> $diff(Z0, \alpha)$

$$z0 \quad \textbf{(53)}$$

> $diff(Z0, \beta)$

$$z1 \quad \textbf{(54)}$$

> $diff(Z0, BETA)$

$$0 \quad \textbf{(55)}$$

> $diff(Z1, \alpha):$

> $subs(mysubs, \%)$

$$-z1 \quad \textbf{(56)}$$

> $diff(Z1, \mathrm{beta})$:

> $subs(mysubs, \%)$

$$0 \quad \textbf{(57)}$$

> $diff(Z1, BETA)$:

> $subs(mysubs, \%)$

$$-z0 \quad \textbf{(58)}$$

Action on the vector of infintesimals

> *restart*

> $z0 := ALPHA \cdot Z0 - \mathrm{beta} \cdot Z1$:

> $z1 := BETA \cdot Z0 + \mathrm{alpha} \cdot Z1$:

> $dz0 := \mathrm{alpha} \cdot dZ0 - BETA \cdot dZ1$:

> $dz1 := \mathrm{beta} \cdot dZ0 + ALPHA \cdot dZ1$:

> $vALPHA := z0 \cdot dz0 - z1 \cdot dz1$:

> $simplify(\%, symbolic)$

$$\left((Z0\, dZ0 - Z1\, dZ1)\, \alpha - 2\, BETA\, Z0\, dZ1 \right) ALPHA - \left(2\, Z1\, \alpha\, dZ0 + BETA\, (Z0\, dZ0 - Z1\, dZ1) \right) \beta \quad \textbf{(59)}$$

> $vbeta := z1 \cdot dz0$:

> $expand(\%)$

$$-BETA^2\, Z0\, dZ1 + BETA\, Z0\, \alpha\, dZ0 - BETA\, Z1\, \alpha\, dZ1 + Z1\, \alpha^2\, dZ0 \quad \textbf{(60)}$$

> $vBETA = -z0 \cdot dz1$:

> $expand(\%)$

$$vBETA = -ALPHA^2\, Z0\, dZ1 - ALPHA\, Z0\, \beta\, dZ0 + ALPHA\, Z1\, \beta\, dZ1 + Z1\, \beta^2\, dZ0 \quad \textbf{(61)}$$

>

> $Adjoint := Matrix\left(\left[[\mathrm{alpha} \cdot ALPHA - \mathrm{beta} \cdot BETA, \mathrm{alpha} \cdot BETA, -ALPHA \cdot BETA], [-2 \cdot \mathrm{alpha} \cdot \mathrm{beta}, \alpha^2, \beta^2], [2 \cdot ALPHA \cdot BETA, BETA^2, ALPHA^2] \right] \right)$

$$Adjoint := \begin{bmatrix} \alpha\, ALPHA - \beta\, BETA & BETA\, \alpha & -ALPHA\, BETA \\ -2\, \alpha\, \beta & \alpha^2 & \beta^2 \\ 2\, ALPHA\, BETA & BETA^2 & ALPHA^2 \end{bmatrix} \quad \textbf{(62)}$$

We evaluate the frame into the adjoint matrix

> $\mathrm{alpha} := \dfrac{r_n(t)\, \mathrm{e}^{-\mathrm{I}\, \theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}}$:

> $\mathrm{beta} := \dfrac{r_{n+1}(t)\, \mathrm{e}^{-\mathrm{I}\, \theta_{n+1}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}}$:

> $BETA := \dfrac{r_{n+1}(t)\, \mathrm{e}^{\mathrm{I}\, \theta_{n+1}(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}}$:

> $ALPHA := \dfrac{r_n(t)\, \mathrm{e}^{\mathrm{I}\, \theta_n(t)}}{\sqrt{r_n(t)^2 + r_{n+1}(t)^2}}$:

> $Adjointrho := simplify(Adjoint, symbolic)$

$Adjointrho :=$ **(63)**

$$\left[\left[\frac{r_n(t)^2 - r_{n+1}(t)^2}{r_n(t)^2 + r_{n+1}(t)^2}, \frac{r_{n+1}(t)\, \mathrm{e}^{\mathrm{I}\,(\theta_{n+1}(t) - \theta_n(t))}\, r_n(t)}{r_n(t)^2 + r_{n+1}(t)^2}, -\frac{r_n(t)\, \mathrm{e}^{\mathrm{I}\,(\theta_{n+1}(t) + \theta_n(t))}\, r_{n+1}(t)}{r_n(t)^2 + r_{n+1}(t)^2} \right], \right.$$

$$\left[-\frac{2\,r_n(t)\,e^{-I\left(\theta_{n+1}(t)+\theta_n(t)\right)}\,r_{n+1}(t)}{r_n(t)^2+r_{n+1}(t)^2},\ \frac{r_n(t)^2\,e^{-2\,I\,\theta_n(t)}}{r_n(t)^2+r_{n+1}(t)^2},\ \frac{r_{n+1}(t)^2\,e^{-2\,I\,\theta_{n+1}(t)}}{r_n(t)^2+r_{n+1}(t)^2}\right],$$
$$\left[\frac{2\,r_n(t)\,e^{I\left(\theta_{n+1}(t)+\theta_n(t)\right)}\,r_{n+1}(t)}{r_n(t)^2+r_{n+1}(t)^2},\ \frac{r_{n+1}(t)^2\,e^{2\,I\,\theta_{n+1}(t)}}{r_n(t)^2+r_{n+1}(t)^2},\ \frac{r_n(t)^2\,e^{2\,I\,\theta_n(t)}}{r_n(t)^2+r_{n+1}(t)^2}\right]\Bigg]$$

Here we take the imaginary part

> Im(%) : *evalc*(%)

$$\Bigg[\Bigg[\left[0,\ -\frac{r_{n+1}(t)\,\sin\left(-\theta_{n+1}(t)+\theta_n(t)\right)\,r_n(t)}{r_n(t)^2+r_{n+1}(t)^2},\ -\frac{r_n(t)\,\sin\left(\theta_{n+1}(t)+\theta_n(t)\right)\,r_{n+1}(t)}{r_n(t)^2+r_{n+1}(t)^2}\right], \tag{64}$$
$$\left[\frac{2\,r_n(t)\,\sin\left(\theta_{n+1}(t)+\theta_n(t)\right)\,r_{n+1}(t)}{r_n(t)^2+r_{n+1}(t)^2},\ -\frac{r_n(t)^2\sin\left(2\,\theta_n(t)\right)}{r_n(t)^2+r_{n+1}(t)^2},\ -\frac{r_{n+1}(t)^2\sin\left(2\,\theta_{n+1}(t)\right)}{r_n(t)^2+r_{n+1}(t)^2}\right],$$
$$\left[\frac{2\,r_n(t)\,\sin\left(\theta_{n+1}(t)+\theta_n(t)\right)\,r_{n+1}(t)}{r_n(t)^2+r_{n+1}(t)^2},\ \frac{r_{n+1}(t)^2\sin\left(2\,\theta_{n+1}(t)\right)}{r_n(t)^2+r_{n+1}(t)^2},\ \frac{r_n(t)^2\sin\left(2\,\theta_n(t)\right)}{r_n(t)^2+r_{n+1}(t)^2}\right]\quad ,$$

>

4.3.1 - SL(2) discrete linear case

> restart

> $with(LinearAlgebra):$

Normalization equations

> $eq1 := a\cdot x[n](t) + b\cdot y[n](t) = 1:$

> $eq2 := c\cdot x[n](t) + \frac{(1+b\cdot c)}{a}\cdot y[n](t) = 0:$

> $eq3 := a\cdot x[n+1](t) + b\cdot y[n+1](t) = 0:$

Frame

> $solve(\{eq1, eq2, eq3\}, \{a, b, c\})$

$$\left\{a = \frac{y_{n+1}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)},\ b = -\frac{x_{n+1}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)},\ c = -y_n(t)\right\} \tag{1}$$

> $assign(\%)$

> $\text{rho}[n] := Matrix\left(\left[[a, b], \left[c, simplify\left(\frac{(1+b\cdot c)}{a}\right)\right]\right]\right)$

$$\rho_n := \begin{bmatrix} \frac{y_{n+1}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)} & -\frac{x_{n+1}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)} \\ -y_n(t) & x_n(t) \end{bmatrix} \tag{2}$$

Invariants and first order differential invariants

> $\text{rho}[n].Matrix([[x[n+1](t)], [y[n+1](t)]])$

$$\begin{bmatrix} 0 \\ x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t) \end{bmatrix} \tag{3}$$

> $simplify(\text{rho}[n].Matrix([[x[n+2](t)], [y[n+2](t)]]), symbolic)$

$$\begin{bmatrix} \frac{y_{n+1}(t)\, x_{n+2}(t) - x_{n+1}(t)\, y_{n+2}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)} \\ -y_n(t)\, x_{n+2}(t) + x_n(t)\, y_{n+2}(t) \end{bmatrix} \tag{4}$$

> $vk := Matrix([[x[n+j](t)], [y[n+j](t)]]):$

> $v1t := Matrix([[x[n+j, t](t)], [y[n+j, t](t)]]):$

> $simplify(\text{rho}[n].vk, symbolic)$

$$\begin{bmatrix} \frac{y_{n+1}(t)\, x_{n+j}(t) - x_{n+1}(t)\, y_{n+j}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)} \\ -y_n(t)\, x_{n+j}(t) + x_n(t)\, y_{n+j}(t) \end{bmatrix} \tag{5}$$

> $simplify(\text{rho}[n].v1t, symbolic)$

$$\begin{bmatrix} \frac{y_{n+1}(t)\, x_{n+j,t}(t) - x_{n+1}(t)\, y_{n+j,t}(t)}{x_n(t)\, y_{n+1}(t) - y_n(t)\, x_{n+1}(t)} \\ -y_n(t)\, x_{n+j,t}(t) + x_n(t)\, y_{n+j,t}(t) \end{bmatrix} \tag{6}$$

Maurer-Cartan matrix

> $mysubs := (x_n(t) = 1, x_{n+1}(t) = 0, y[n](t) = 0, y[n+1](t) = Inv[y, n+1], y[n+2](t) = Inv[y, n+2], x[n+2](t) = Inv[x, n+2]):$

> $K0 := subs(mysubs, subs(n = n+1, \mathrm{rho}[n]))$

$$K0 := \begin{bmatrix} -\frac{Inv_{y, n+2}}{Inv_{y, n+1}\, Inv_{x, n+2}} & \frac{1}{Inv_{y, n+1}} \\ -Inv_{y, n+1} & 0 \end{bmatrix} \tag{7}$$

Curvature matrix

> $diffrhot := simplify(map(z \rightarrow diff(z, t), \mathrm{rho}[n]))$:

> $mysubs2 := \left(\frac{d}{dt} x_n(t) = Inv[x, n, t], \frac{d}{dt} x_{n+1}(t) = Inv[x, n+1, t], \frac{d}{dt} x_{n+2}(t) = Inv[x, n+2, t], \frac{d}{dt} y_n(t) = Inv[y, n, t], \frac{d}{dt} y_{n+1}(t) = Inv[y, n+1, t], \frac{d}{dt} y_{n+2}(t) = Inv[y, n+2, t] \right)$:

> $subs(mysubs2, diffrhot)$:

> $N0 := simplify(subs(mysubs, \%))$

$$N0 := \begin{bmatrix} -Inv_{x, n, t} & -\frac{Inv_{x, n+1, t}}{Inv_{y, n+1}} \\ -Inv_{y, n, t} & Inv_{x, n, t} \end{bmatrix} \tag{8}$$

> $Trace(\%)$

$$0 \tag{9}$$

Infinitesimals

> $diff(\mathrm{alpha} \cdot x[n] + \mathrm{beta} \cdot y[n], \mathrm{alpha})$

$$x_n \tag{10}$$

> $diff(\mathrm{alpha} \cdot x[n] + \mathrm{beta} \cdot y[n], \mathrm{beta})$

$$y_n \tag{11}$$

> $diff(\mathrm{alpha} \cdot x[n] + \mathrm{beta} \cdot y[n], \mathrm{delta})$

$$0 \tag{12}$$

> $subs\left(\{\mathrm{alpha} = 1, \mathrm{beta} = 0, \mathrm{delta} = 0\}, diff\left(\mathrm{delta} \cdot x[n] + \frac{(1 + \mathrm{beta} \cdot \mathrm{delta})}{\mathrm{alpha}} \cdot y[n], \mathrm{alpha} \right) \right)$

$$-y_n \tag{13}$$

> $subs\left(\{\mathrm{alpha} = 1, \mathrm{beta} = 0, \mathrm{delta} = 0\}, diff\left(\mathrm{delta} \cdot x[n] + \frac{(1 + \mathrm{beta} \cdot \mathrm{delta})}{\mathrm{alpha}} \cdot y[n], \mathrm{beta} \right) \right)$

$$0 \tag{14}$$

> $subs\left(\{\mathrm{alpha} = 1, \mathrm{beta} = 0, \mathrm{delta} = 0\}, diff\left(\mathrm{delta} \cdot x[n] + \frac{(1 + \mathrm{beta} \cdot \mathrm{delta})}{\mathrm{alpha}} \cdot y[n], \mathrm{delta} \right) \right)$

$$x_n \tag{15}$$

Syzygy and evolution of curvature invariants

> $MyK0 := Matrix\left(\left[\left[\mathrm{kappa}[n](t), \frac{1}{\mathrm{tau}[n](t)} \right], [-\mathrm{tau}[n](t), 0] \right] \right)$

$$MyK0 := \begin{bmatrix} \kappa_n(t) & \frac{1}{\tau_n(t)} \\ -\tau_n(t) & 0 \end{bmatrix} \tag{16}$$

> $MyN0 := subs(\{Inv[y, n+1] = \mathrm{tau}[n](t)\}, N0)$

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$$MyN0 := \begin{bmatrix} -Inv_{x,n,t} & -\frac{Inv_{x,n+1,t}}{\tau_n(t)} \\ -Inv_{y,n,t} & Inv_{x,n,t} \end{bmatrix} \quad \textbf{(17)}$$

> $SN0 := subs(\{Inv[x, n, t] = SInv[x, n, t], Inv[x, n+1, t] = SInv[x, n+1, t], Inv[y, n, t] = SInv[y, n, t], Inv[y, n+1, t] = SInv[y, n+1, t], \mathrm{tau}[n](t) = \mathrm{tau}[n+1](t)\}, MyN0)$

$$SN0 := \begin{bmatrix} -SInv_{x,n,t} & -\frac{SInv_{x,n+1,t}}{\tau_{n+1}(t)} \\ -SInv_{y,n,t} & SInv_{x,n,t} \end{bmatrix} \quad \textbf{(18)}$$

> $syzygy := simplify(SN0 \cdot MyK0 - MyK0 \cdot MyN0, symbolic)$

$$syzygy := \left[\left[\frac{\left(\kappa_n(t)\left(Inv_{x,n,t} - SInv_{x,n,t}\right)\tau_n(t) + Inv_{y,n,t}\right)\tau_{n+1}(t) + SInv_{x,n+1,t}\,\tau_n(t)^2}{\tau_{n+1}(t)\,\tau_n(t)}, \frac{\kappa_n(t)\,Inv_{x,n+1,t} - Inv_{x,n,t} - SInv_{x,n,t}}{\tau_n(t)}\right], \left[\left(-Inv_{x,n,t} - SInv_{x,n,t}\right)\tau_n(t) - SInv_{y,n,t}\,\kappa_n(t), \frac{-Inv_{x,n+1,t}\,\tau_n(t) - SInv_{y,n,t}}{\tau_n(t)}\right]\right] \quad \textbf{(19)}$$

> $isolate\left(syzygy(2, 2), Inv_{x,n+1,t}\right)$

$$Inv_{x,n+1,t} = -\frac{SInv_{y,n,t}}{\tau_n(t)} \quad \textbf{(20)}$$

> $collect\left(syzygy(2, 1), Inv_{x,n,t}\right)$

$$-SInv_{y,n,t}\,\kappa_n(t) - SInv_{x,n,t}\,\tau_n(t) - \tau_n(t)\,Inv_{x,n,t} \quad \textbf{(21)}$$

> $simplify\left(subs\left(SInv_{x,n+1,t} = -\frac{SSInv_{y,n,t}}{\tau_{n+1}(t)}, syzygy(1, 1)\right), symbolic\right) :$

> $collect\left(\%, Inv_{x,n,t}\right) : collect\left(\%, SInv_{x,n,t}\right) : collect\left(\%, Inv_{y,n,t}\right)$

$$-SInv_{x,n,t}\,\kappa_n(t) + \kappa_n(t)\,Inv_{x,n,t} + \frac{Inv_{y,n,t}}{\tau_n(t)} - \frac{\tau_n(t)\,SSInv_{y,n,t}}{\tau_{n+1}(t)^2} \quad \textbf{(22)}$$

Computation of the adjoints

> $restart$

> $with(LinearAlgebra) :$

> $Adg := Matrix([[a \cdot d + b \cdot c, -a \cdot c, b \cdot d], [-2 \cdot a \cdot b, a^2, -b^2], [2 \cdot c \cdot d, -c^2, d^2]])$

$$Adg := \begin{bmatrix} a\,d + b\,c & -a\,c & b\,d \\ -2\,a\,b & a^2 & -b^2 \\ 2\,c\,d & -c^2 & d^2 \end{bmatrix} \quad \textbf{(23)}$$

Evaluation of the frame into the adjoint matrix

> $a := \frac{y1}{\mathrm{tau}} :$

> $b := -\frac{x1}{\mathrm{tau}} :$

> $c := -y0 :$

> $d := x0 :$

> $Adrho := simplify(Adg)$

$$Adrho := \begin{bmatrix} \frac{y1\,x0 + x1\,y0}{\tau} & \frac{y1\,y0}{\tau} & -\frac{x1\,x0}{\tau} \\ \frac{2\,y1\,x1}{\tau^2} & \frac{y1^2}{\tau^2} & -\frac{x1^2}{\tau^2} \\ -2\,y0\,x0 & -y0^2 & x0^2 \end{bmatrix} \quad \textbf{(24)}$$

(25)

Evaluation of the Maurer Cartan matrix into the ajdoint matrix

> $a := \text{kappa}:$

> $b := \frac{1}{\text{tau}}:$

> $c := -\text{tau}:$

> $d := 0:$

> $AdK0 := simplify(Adg)$

$$AdK0 := \begin{bmatrix} -1 & \kappa\,\tau & 0 \\ -\frac{2\,\kappa}{\tau} & \kappa^2 & -\frac{1}{\tau^2} \\ 0 & -\tau^2 & 0 \end{bmatrix} \quad \textbf{(26)}$$

The general solution, Groebner basis computation

> $restart$

> $with(LinearAlgebra):$

> $k := Matrix([[k1, k2, k3]]):$

> $V := Matrix([[V1, V2, V3]]):$

> $Adg := Matrix([[a\cdot d + b\cdot c, -a\cdot c, b\cdot d], [-2\cdot a\cdot b, a^2, -b^2], [2\cdot c\cdot d, -c^2, d^2]]):$

> $simplify(MatrixInverse(Adg), symbolic) : Adginv := subs(a\,d - b\,c = 1, \%)$

$$Adginv := \begin{bmatrix} a\,d + b\,c & c\,d & -a\,b \\ 2\,b\,d & d^2 & -b^2 \\ -2\,a\,c & -c^2 & a^2 \end{bmatrix} \quad \textbf{(27)}$$

> $Equations := simplify(k\cdot Adginv, symbolic) - V$

$$Equations := [d\,(a\,k1 + 2\,b\,k2) - 2\,k3\,a\,c + b\,c\,k1 - V1,\ -c^2\,k3 + c\,d\,k1 + d^2\,k2 - V2,\ a^2\,k3 - a\,b\,k1 - b^2\,k2 - V3] \quad \textbf{(28)}$$

> $with(Groebner):$

> $F := [Equations(1,1), Equations(1,2), Equations(1,3), a\cdot d - b\cdot c - 1]$

$$F := [d\,(a\,k1 + 2\,b\,k2) - 2\,k3\,a\,c + b\,c\,k1 - V1,\ -c^2\,k3 + c\,d\,k1 + d^2\,k2 - V2,\ a^2\,k3 - a\,b\,k1 - b^2\,k2 - V3,\ a\,d - b\,c - 1] \quad \textbf{(29)}$$

> $Basis(F, plex(a, b, c, k1, k2, k3))$

$$[-V1^2 - 4\,V2\,V3 + k1^2 + 4\,k2\,k3,\ c^2\,k3 - c\,d\,k1 - d^2\,k2 + V2,\ -V1\,d + 2\,b\,V2 - 2\,c\,k3 + d\,k1,\ -c\,V1 + 2\,a\,V2 - c\,k1 - 2\,k2\,d] \quad \textbf{(30)}$$

4.3.2 - SA(2) linear discrete case

> restart

> with(LinearAlgebra) :

Group action

> $X[n] := a\cdot x[n] + b\cdot y[n] + k1$:

> $Y[n] := c\cdot x[n] + d\cdot y[n] + k2$:

Normalization equations

> $d := \frac{(1 + b\cdot c)}{a}$:

> $eq1 := X[n]$:

> $eq2 := Y[n]$:

> $eq3 := a\cdot x[n+1] + b\cdot y[n+1] + k1 - 1$:

> $eq4 := c\cdot x[n+1] + d\cdot y[n+1] + k2$:

> $eq5 := a\cdot x[n+2] + b\cdot y[n+2] + k1$:

> solve({eq1, eq2, eq3, eq4, eq5}, {a, b, c, k1, k2})

$$\left\{a = -\frac{y_n - y_{n+2}}{x_n y_{n+1} - x_n y_{n+2} - x_{n+1} y_n + x_{n+1} y_{n+2} + x_{n+2} y_n - x_{n+2} y_{n+1}}, b = \frac{x_n - x_{n+2}}{x_n y_{n+1} - x_n y_{n+2} - x_{n+1} y_n + x_{n+1} y_{n+2} + x_{n+2} y_n - x_{n+2} y_{n+1}}, c = y_n - y_{n+1}, k1 = -\frac{x_n y_{n+2} - x_{n+2} y_n}{x_n y_{n+1} - x_n y_{n+2} - x_{n+1} y_n + x_{n+1} y_{n+2} + x_{n+2} y_n - x_{n+2} y_{n+1}}, k2 = x_n y_{n+1} - x_{n+1} y_n\right\} \quad (1)$$

> assign(%)

Moving frame

> rho[n] := simplify(simplify(Matrix([[a, b, k1], [c, d, k2], [0, 0, 1]]), symbolic), size)

$$\rho_n := \left[\left[\frac{y_{n+2} - y_n}{(y_{n+2} - y_n) x_{n+1} + (y_n - y_{n+1}) x_{n+2} + x_n (y_{n+1} - y_{n+2})}, \frac{x_n - x_{n+2}}{(y_n - y_{n+1}) x_{n+2} + x_n (y_{n+1} - y_{n+2}) - x_{n+1} (y_n - y_{n+2})}, \frac{-x_n y_{n+2} + x_{n+2} y_n}{(y_{n+2} - y_n) x_{n+1} + (y_n - y_{n+1}) x_{n+2} + x_n (y_{n+1} - y_{n+2})}\right], \left[y_n - y_{n+1}, -x_n + x_{n+1}, x_n y_{n+1} - x_{n+1} y_n\right], \left[0, 0, 1\right]\right] \quad (2)$$

Shift of the moving frame

> rho[n + 1] := subs(n = n + 1, rho[n])

$$\rho_{n+1} := \left[\left[\frac{y_{n+3} - y_{n+1}}{(y_{n+3} - y_{n+1}) x_{n+2} + (y_{n+1} - y_{n+2}) x_{n+3} + x_{n+1} (y_{n+2} - y_{n+3})}, \frac{x_{n+1} - x_{n+3}}{(y_{n+1} - y_{n+2}) x_{n+3} + x_{n+1} (y_{n+2} - y_{n+3}) - x_{n+2} (y_{n+1} - y_{n+3})}, \frac{-x_{n+1} y_{n+3} + x_{n+3} y_{n+1}}{(y_{n+3} - y_{n+1}) x_{n+2} + (y_{n+1} - y_{n+2}) x_{n+3} + x_{n+1} (y_{n+2} - y_{n+3})}\right], \left[y_{n+1} - y_{n+2}, -x_{n+1} + x_{n+2}, x_{n+1} y_{n+2} - x_{n+2} y_{n+1}\right], \left[0, 0, 1\right]\right] \quad (3)$$

Invariants

> $Points := Matrix([[x[n], x[n+1], x[n+2], x[n+3], x[n+m]], [y[n], y[n+1], y[n+2], y[n+3], y[n+m]], [1, 1, 1, 1, 1]])$

$$Points := \begin{bmatrix} x_n & x_{n+1} & x_{n+2} & x_{n+3} & x_{n+m} \\ y_n & y_{n+1} & y_{n+2} & y_{n+3} & y_{n+m} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (4)$$

> $Invariantsn := simplify(simplify(\mathrm{rho}[n].Points, symbolic), size)$

$$Invariantsn := \left[\left[0, 1, 0, \frac{(y_n - y_{n+3})\, x_{n+2} + (-x_n + x_{n+3})\, y_{n+2} + y_{n+3} x_n - x_{n+3} y_n}{(y_n - y_{n+1})\, x_{n+2} + (-x_n + x_{n+1})\, y_{n+2} + x_n y_{n+1} - x_{n+1} y_n}, \frac{(y_n - y_{n+m})\, x_{n+2} + (-x_n + x_{n+m})\, y_{n+2} + y_{n+m} x_n - x_{n+m} y_n}{(y_n - y_{n+1})\, x_{n+2} + (-x_n + x_{n+1})\, y_{n+2} + x_n y_{n+1} - x_{n+1} y_n}\right], \left[0, 0, (y_{n+2} - y_n)\, x_{n+1} + (y_n - y_{n+1})\, x_{n+2} + x_n (y_{n+1} - y_{n+2}), (-y_n + y_{n+3})\, x_{n+1} + (y_n - y_{n+1})\, x_{n+3} + x_n (y_{n+1} - y_{n+3}), (-y_n + y_{n+m})\, x_{n+1} + (y_n - y_{n+1})\, x_{n+m} + x_n (y_{n+1} - y_{n+m})\right], \left[1, 1, 1, 1, 1\right]\right] \quad (5)$$

> $mysubs := \{x[n] = 0, y[n] = 0, x[n+1] = 1, y[n+1] = 0, x[n+2] = 0, y[n+2] = Inv[y, n+2], x[n+3] = Inv[x, n+3], y[n+3] = Inv[y, n+3], x[n+m] = Inv[x, n+m], y[n+m] = Inv[y, n+m]\}:$

> $Quickcheck1 := subs(mysubs, Invariantsn)$

$$Quickcheck1 := \begin{bmatrix} 0 & 1 & 0 & Inv_{x,n+3} & Inv_{x,n+m} \\ 0 & 0 & Inv_{y,n+2} & Inv_{y,n+3} & Inv_{y,n+m} \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad (6)$$

Maurer Cartan matrix

> $K := subs(mysubs, \mathrm{rho}[n+1])$

$$K := \left[\left[\frac{Inv_{y,n+3}}{-Inv_{x,n+3} Inv_{y,n+2} + Inv_{y,n+2} - Inv_{y,n+3}}, \frac{1 - Inv_{x,n+3}}{-Inv_{x,n+3} Inv_{y,n+2} + Inv_{y,n+2} - Inv_{y,n+3}}, -\frac{Inv_{y,n+3}}{-Inv_{x,n+3} Inv_{y,n+2} + Inv_{y,n+2} - Inv_{y,n+3}}\right], \left[-Inv_{y,n+2}, -1, Inv_{y,n+2}\right], \left[0, 0, 1\right]\right] \quad (7)$$

Quick checks

> $simplify(K[1, 1] \cdot K[2, 2] - K[1, 2] \cdot K[2, 1], symbolic)$

$$1 \quad (8)$$

> $simplify\left(simplify\left(\frac{\left(1 + \frac{Inv_{y,n+3}}{-Inv_{x,n+3} Inv_{y,n+2} + Inv_{y,n+2} - Inv_{y,n+3}}\right)}{Inv_{y,n+2}}, symbolic\right), size\right)$

$$\frac{Inv_{x,n+3} - 1}{Inv_{y,n+2}\,(Inv_{x,n+3} - 1) + Inv_{y,n+3}} \quad (9)$$

> $MyK := Matrix\left(\left[\left[\mathrm{tau}[n], \frac{(1 + \mathrm{tau}[n])}{\mathrm{kappa}[n]}, -\mathrm{tau}[n]\right], [-\mathrm{kappa}[n], -1, \mathrm{kappa}[n]], [0, 0, 1]\right]\right)$

(10)

$$MyK := \begin{bmatrix} \tau_n & \frac{1+\tau_n}{\kappa_n} & -\tau_n \\ -\kappa_n & -1 & \kappa_n \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

> $MyK(1,1) \cdot MyK(2,2) - MyK(2,1) \cdot MyK(1,2)$

$$1 \tag{11}$$

Differential difference invariants relations

> $Sv := Matrix([[SInv_{x,n,t}], [SInv_{y,n,t}], [0]])$:

> $I1 := MatrixInverse(MyK).Sv$

$$I1 := \begin{bmatrix} -SInv_{x,n,t} - \frac{(1+\tau_n)\, SInv_{y,n,t}}{\kappa_n} \\ \kappa_n\, SInv_{x,n,t} + \tau_n\, SInv_{y,n,t} \\ 0 \end{bmatrix} \tag{12}$$

> $SSv := Matrix([[SSInv_{x,n,t}], [SSInv_{y,n,t}], [0]])$:

> $I2 := simplify(MatrixInverse(MyK).subs(n = n+1, MatrixInverse(MyK)).SSv, symbolic) : collect(\%, SSInv_{x,n,t}) : collect(\%, SSInv_{y,n,t})$

$$\begin{bmatrix} \frac{(\kappa_n (1+\tau_{n+1}) - (1+\tau_n)\tau_{n+1}\kappa_{n+1})\, SSInv_{y,n,t}}{\kappa_{n+1}\kappa_n} + \frac{(-(1+\tau_n)\kappa_{n+1}^2 + \kappa_{n+1}\kappa_n)\, SSInv_{x,n,t}}{\kappa_{n+1}\kappa_n} \\ \frac{(-\kappa_n (1+\tau_{n+1}) + \tau_n \tau_{n+1}\kappa_{n+1})\, SSInv_{y,n,t}}{\kappa_{n+1}} + \frac{(\kappa_{n+1}^2 \tau_n - \kappa_{n+1}\kappa_n)\, SSInv_{x,n,t}}{\kappa_{n+1}} \\ 0 \end{bmatrix} \tag{13}$$

> $I2 := \% :$

Curvature matrix

> $rhot[n] := subs(\{x[n] = x[n](t), x[n+1] = x[n+1](t), x[n+2] = x[n+2](t), y[n] = y[n](t), y[n+1] = y[n+1](t), y[n+2] = y[n+2](t)\}, \mathrm{rho}[n])$:

> $diffrhot[n] := map(z \to diff(z, t), rhot[n])$:

> $subs\left(\left\{\frac{d}{dt} x_n(t) = Inv[x, n, t], \frac{d}{dt} y_n(t) = Inv[y, n, t], \frac{d}{dt} x_{n+1}(t) = Inv[x, n+1, t], \frac{d}{dt} y_{n+1}(t) = Inv[y, n+1, t], \frac{d}{dt} x_{n+2}(t) = Inv[x, n+2, t], \frac{d}{dt} y_{n+2}(t) = Inv[y, n+2, t]\right\}, \%\right)$:

> $N := simplify(subs(\{x[n](t) = 0, y[n](t) = 0, x[n+1](t) = 1, y[n+1](t) = 0, x[n+2](t) = 0, y[n+2](t) = \mathrm{kappa}[n], x[n+3](t) = Inv[n, x[n+3]], y[n+3](t) = Inv[n, y[n+3]]\}, \%), symbolic)$

$$N := \begin{bmatrix} Inv_{x,n,t} - Inv_{x,n+1,t} & \frac{Inv_{x,n,t} - Inv_{x,n+2,t}}{\kappa_n} & -Inv_{x,n,t} \\ Inv_{y,n,t} - Inv_{y,n+1,t} & -Inv_{x,n,t} + Inv_{x,n+1,t} & -Inv_{y,n,t} \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

> $N := simplify(subs(\{Inv_{x,n+1,t} = I1(1,1), Inv_{y,n+1,t} = I1(2,1), Inv_{x,n+2,t} = I2(1,1)\}, \%), symbolic)$

$$N := \left[\left[\frac{(Inv_{x,n,t} + SInv_{x,n,t})\kappa_n + (1+\tau_n)\, SInv_{y,n,t}}{\kappa_n}, \frac{1}{\kappa_{n+1}\kappa_n^2}\Big(SSInv_{x,n,t}(1+\tau_n)\kappa_{n+1}^2 + \big((Inv_{x,n,t} - SSInv_{x,n,t})\kappa_n + SSInv_{y,n,t}\tau_{n+1}(1+\tau_n)\big)\kappa_{n+1} - \kappa_n SSInv_{y,n,t}(1+\tau_{n+1})\Big), -Inv_{x,n,t}\right],\right. \tag{15}$$

$$\left[-SInv_{x,n,t}\kappa_n - SInv_{y,n,t}\tau_n + Inv_{y,n,t}, \frac{(-Inv_{x,n,t} - SInv_{x,n,t})\kappa_n - (1+\tau_n) SInv_{y,n,t}}{\kappa_n}, -Inv_{y,n,t}\right],$$

$$\left[0, 0, 0\right]\Big]\Big]$$

> *#Shift of the curvature matrix*

> $SN := subs(\{Inv[x,n,t] = SInv[x,n,t], Inv[y,n,t] = SInv[y,n,t], Inv[x,n+1,t] = SInv[x,n+1,t], Inv[y,n+1,t] = SInv[y,n+1,t], Inv[x,n+2,t] = SInv[x,n+2,t], \text{kappa}[n] = \text{kappa}[n+1], SInv[x,n,t] = SSInv[x,n,t], SInv[y,n,t] = SSInv[y,n,t], SInv[x,n+1,t] = SSInv[x,n+1,t], SInv[y,n+1,t] = SSInv[y,n+1,t], SInv[x,n+2,t] = SSInv[x,n+2,t], \text{kappa}[n+1] = \text{kappa}[n+2], \text{tau}[n] = \text{tau}[n+1], \text{tau}[n+1] = \text{tau}[n+2], SSInv[x,n,t] = SSSInv[x,n,t], SSInv[y,n,t] = SSSInv[y,n,t], SSInv[x,n+1,t] = SSSInv[x,n+1,t], SSInv[y,n+1,t] = SSSInv[y,n+1,t], SSInv[x,n+2,t] = SSSInv[x,n+2,t]\}, N)$

$$SN := \Bigg[\Bigg[\frac{(SInv_{x,n,t} + SSInv_{x,n,t})\kappa_{n+1} + (1+\tau_{n+1}) SSInv_{y,n,t}}{\kappa_{n+1}}, \frac{1}{\kappa_{n+2}\kappa_{n+1}^2}\Big(SSSInv_{x,n,t}(1+\tau_{n+1})\kappa_{n+2}^2 + \big((SInv_{x,n,t} - SSSInv_{x,n,t})\kappa_{n+1} + SSSInv_{y,n,t}\tau_{n+2}(1+\tau_{n+1})\big)\kappa_{n+2} - \kappa_{n+1} SSSInv_{y,n,t}(1+\tau_{n+2})\Big), -SInv_{x,n,t}\Bigg], \quad (16)$$

$$\Bigg[-SSInv_{x,n,t}\kappa_{n+1} - SSInv_{y,n,t}\tau_{n+1} + SInv_{y,n,t}, \frac{(-SInv_{x,n,t} - SSInv_{x,n,t})\kappa_{n+1} - (1+\tau_{n+1}) SSInv_{y,n,t}}{\kappa_{n+1}}, -SInv_{y,n,t}\Bigg],$$

$$\left[0, 0, 0\right]\Big]\Big]$$

Differential--difference Syzygy and evolution of curvature invariants

> $Syzygy := simplify(SN \cdot MyK - MyK \cdot N, symbolic)$:

> $Syzygy(1,1)$:

> $collect(\%, Inv_{x,n,t}) : collect(\%, Inv_{y,n,t}) : collect(\%, SInv_{x,n,t}) : collect(\%, SInv_{y,n,t}) : collect(\%, SSInv_{x,n,t}) : collect(\%, SSInv_{y,n,t}) : collect(\%, SSSInv_{x,n,t}) : collect(\%, SSSInv_{y,n,t})$

$$\frac{\big(-\tau_{n+2}\kappa_n^2(1+\tau_{n+1})\kappa_{n+2} + \kappa_{n+1}\kappa_n^2(1+\tau_{n+2})\big) SSSInv_{y,n,t}}{\kappa_{n+1}^2\kappa_{n+2}\kappa_n} + \frac{\big(-\kappa_n^2(1+\tau_{n+1})\kappa_{n+2}^2 + \kappa_{n+1}\kappa_n^2\kappa_{n+2}\big) SSSInv_{x,n,t}}{\kappa_{n+1}^2\kappa_{n+2}\kappa_n} + \frac{\tau_n(1+\tau_{n+1}) SSInv_{y,n,t}}{\kappa_{n+1}} + \tau_n SSInv_{x,n,t} + \frac{\big(-\kappa_{n+1}\kappa_n^2 + (1+\tau_n)\kappa_{n+1}^2\kappa_n\big) SInv_{x,n,t}}{\kappa_{n+1}^2\kappa_n} + \frac{(-\tau_n - 1) Inv_{y,n,t}}{\kappa_n} - \tau_n Inv_{x,n,t} \quad (17)$$

> $Syzygy(2,3)$:

> $collect(\%, Inv_{x,n,t}) : collect(\%, Inv_{y,n,t}) : collect(\%, SInv_{x,n,t}) : collect(\%, SInv_{y,n,t}) : collect(\%, SSInv_{x,n,t}) : collect(\%, SSInv_{y,n,t}) : collect(\%, SSSInv_{x,n,t}) : collect(\%, SSSInv_{y,n,t})$

$$\frac{\big(-\kappa_n(1+\tau_{n+1}) + \tau_n\tau_{n+1}\kappa_{n+1}\big) SSInv_{y,n,t}}{\kappa_{n+1}} + \frac{\big(\kappa_{n+1}^2\tau_n - \kappa_{n+1}\kappa_n\big) SSInv_{x,n,t}}{\kappa_{n+1}} + (-\tau_n - 1) SInv_{y,n,t} - \kappa_n SInv_{x,n,t} - Inv_{y,n,t} - \kappa_n Inv_{x,n,t} \quad (18)$$

Adjoint computations

Note we just compute the SL(2) part as the rest of the matrix can be done by hand

> *restart*

> *with*(*LinearAlgebra*) :

> $Adg := Matrix([[a \cdot d + b \cdot c, -a \cdot c, b \cdot d], [-2 \cdot a \cdot b, a^2, -b^2], [2 \cdot c \cdot d, -c^2, d^2]])$

$$Adg := \begin{bmatrix} a\,d + b\,c & -a\,c & b\,d \\ -2\,a\,b & a^2 & -b^2 \\ 2\,c\,d & -c^2 & d^2 \end{bmatrix} \qquad \textbf{(19)}$$

Here we evaluate the frame into the adjoint matrix

> $a := \dfrac{(y2 - y0)}{\text{kappa}}$:

> $b := \dfrac{(x0 - x2)}{\text{kappa}}$:

> $c := y0 - y1$:

> $d := x1 - x0$:

> $Adrho := simplify(Adg, symbolic)$

$$Adrho := \left[\left[\frac{(2\,y0 - y1 - y2)\,x0 + (-x1 - x2)\,y0 + y2\,x1 + x2\,y1}{\kappa}, \frac{(-y2 + y0)\,(y0 - y1)}{\kappa}, -\frac{(x0 - x2)\,(-x1 + x0)}{\kappa}\right], \left[\frac{2\,(-y2 + y0)\,(x0 - x2)}{\kappa^2}, \frac{(-y2 + y0)^2}{\kappa^2}, -\frac{(x0 - x2)^2}{\kappa^2}\right], \left[-2\,(y0 - y1)\,(-x1 + x0), -(y0 - y1)^2, (-x1 + x0)^2\right]\right] \qquad \textbf{(20)}$$

Here we evaluate the Maurer Cartan matrix into the adjoint matrix

> $a := \text{tau}$:

> $b := \dfrac{(1 + \text{tau})}{\text{kappa}}$:

> $c := -\text{kappa}$:

> $d := -1$:

> $AdK0 := simplify(Adg, symbolic)$

$$AdK0 := \begin{bmatrix} -2\,\tau - 1 & \tau\,\kappa & \dfrac{-1 - \tau}{\kappa} \\ -\dfrac{2\,\tau\,(1 + \tau)}{\kappa} & \tau^2 & -\dfrac{(1 + \tau)^2}{\kappa^2} \\ 2\,\kappa & -\kappa^2 & 1 \end{bmatrix} \qquad \textbf{(21)}$$

The general solution, Groebner basis computation

> *restart*

> *with*(*LinearAlgebra*) :

> $k := Matrix([[k1, k2, k3, k4, k5]])$:

> $V := Matrix([[V1, V2, V3, V4, V5]])$:

> $Adg := Matrix([[a \cdot d + b \cdot c, -a \cdot c, b \cdot d], [-2 \cdot a \cdot b, a^2, -b^2], [2 \cdot c \cdot d, -c^2, d^2]])$:

(22)

> $simplify(MatrixInverse(Adg), symbolic) : Adginv := subs(a\,d - b\,c = 1, \%)$

(23)

$$Adginv := \begin{bmatrix} a\,d+b\,c & c\,d & -a\,b \\ 2\,b\,d & d^2 & -b^2 \\ -2\,a\,c & -c^2 & a^2 \end{bmatrix} \quad \textbf{(23)}$$

> $A := Matrix([[0,0],[0,0],[0,0]])$

$$A := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(24)}$$

> $B := Matrix([[a\cdot d + b\cdot \text{beta}, \text{beta}\cdot d, -b\cdot \text{alpha}], [-a\cdot \text{beta} - \text{alpha}\cdot c, -c\cdot \text{beta}, \text{alpha}\cdot a]])$

$$B := \begin{bmatrix} a\,d+b\,\beta & \beta\,d & -b\,\alpha \\ -a\,\beta-c\,\alpha & -c\,\beta & \alpha\,a \end{bmatrix} \quad \textbf{(25)}$$

> $C := Matrix([[d,-b],[-c,a]])$

$$C := \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \textbf{(26)}$$

> $Adg := Matrix([[Adginv, A],[B, C]])$

$$Adg := \begin{bmatrix} a\,d+b\,c & c\,d & -a\,b & 0 & 0 \\ 2\,b\,d & d^2 & -b^2 & 0 & 0 \\ -2\,a\,c & -c^2 & a^2 & 0 & 0 \\ a\,d+b\,\beta & \beta\,d & -b\,\alpha & d & -b \\ -a\,\beta-c\,\alpha & -c\,\beta & \alpha\,a & -c & a \end{bmatrix} \quad \textbf{(27)}$$

> $Equations := simplify(k\cdot Adg, symbolic) - V$

$$Equations := \left[\left(-2\,k3\,c + (k1+k4)\,d - \beta\,k5\right)a + \left(\beta\,k4 + c\,k1 + 2\,d\,k2\right)b - \alpha\,c\,k5 - V1,\ k2\,d^2 + \left(\beta\,k4 + c\,k1\right)d - k3\,c^2 - k5\,c\,\beta - V2,\ -k2\,b^2 + \left(-a\,k1 - \alpha\,k4\right)b + k3\,a^2 + k5\,\alpha\,a - V3,\ -k5\,c + k4\,d - V4,\ k5\,a - k4\,b - V5 \right] \quad \textbf{(28)}$$

> $with(Groebner):$

> $F := [Equations(1,1), Equations(1,2), Equations(1,3), Equations(1,4), Equations(1,5), a\cdot d - b\cdot c - 1]$

$$F := \left[\left(-2\,k3\,c + (k1+k4)\,d - \beta\,k5\right)a + \left(\beta\,k4 + c\,k1 + 2\,d\,k2\right)b - \alpha\,c\,k5 - V1,\ k2\,d^2 + \left(\beta\,k4 + c\,k1\right)d - k3\,c^2 - k5\,c\,\beta - V2,\ -k2\,b^2 + \left(-a\,k1 - \alpha\,k4\right)b + k3\,a^2 + k5\,\alpha\,a - V3,\ -c\,k5 + d\,k4 - V4,\ a\,k5 - b\,k4 - V5,\ a\,d - b\,c - 1 \right] \quad \textbf{(29)}$$

> $Basis(F, plex(\text{alpha}, \text{beta}, d, b, a, k1, k2))$

$$\left[k1\left(V5^2\,c^3\,k4\,k5^2 + 2\,V4\,V5^2\,c^2\,k4\,k5 - 2\,V5\,c^2\,k4^2\,k5^2 + V4^2\,V5^2\,c\,k4 - 3\,V4\,V5\,c\,k4^2\,k5 + c\,k4^3\,k5^2 - V4^2\,V5\,k4^2\right) + k2\left(V5^2\,c^3\,k5^3 + 3\,V4\,V5^2\,c^2\,k5^2 - 2\,V5\,c^2\,k4\,k5^3 + 3\,V4^2\,V5^2\,c\,k5 - 4\,V4\,V5\,c\,k4\,k5^2 + c\,k4^2\,k5^3 + V4^3\,V5^2 - 2\,V4^2\,V5\,k4\,k5\right) - V5^2\,c^3\,k3\,k4^2\,k5 - V5^2\,c^2\,k3\,k4^2\,V4 + V5^2\,c^2\,k4^2\,k5\,V4 + 2\,V5\,c^2\,k3\,k4^3\,k5 + V3\,V4^2\,c\,k4^2\,k5 + V4^2\,V5^2\,c\,k4^2 + 2\,c\,k3\,k4^3\,V4\,V5 - c\,k4^3\,k5\,V4\,V5 - c\,k3\,k4^4\,k5 + V1\,k4^2\,V4^2\,V5 + V2\,V4\,V5^2\,k4^2 - V4^2\,k4^3\,V5,\ V4\,a + V5\,c - k4,\ b\,k4\,V4 + V5\,c\,k5 + V4\,V5 - k4\,k5,\ -c\,k5 + d\,k4 - V4,\ \beta\left(V4\,V5^2\,c^3\,k4\,k5^2 + 2\,V4^2\,V5^2\,c^2\,k4\,k5 - 2\,V4\,V5\,c^2\,k4^2\,k5^2 + V4^3\,V5^2\,c\,k4 - 3\,V4^2\,V5\,c\,k4^2\,k5 + V4\,c\,k4^3\,k5^2 - V4^3\,V5\,k4^2\right) + k2\left(-V4\,V5\,c^3\,k5^3 - 3\,V4^2\,V5\,c^2\,k5^2 + V4\,c^2\,k4\,k5^3 - 3\,V4^3\,V5\,c\,k5 + V4^2\,c\,k4\,k5^2 - V4^4\,V5\right) - V4\,V5^2\,c^4\,k4\,k5^2 - V2\,V5^2\,c^3\,k4\,k5^2 - V3\,V4^2\,c^3\,k4\,k5^2 - 2\,V4^2\,V5^2\,c^3\,k4\,k5 - V4\,V5\,c^3\,k3\,k4^2\,k5 + V4\,V5\,c^3\,k4^2\,k5^2 - V1\,V4^2\,V5\,c^2\,k4\,k5 - 3\,V2\,V4\,V5^2\,c^2\,k4\,k5 + 2\,V2\,V5\,c^2\,k4^2\,k5^2 - V3\,V4^3\,c^2\,k4\,k5 - V4^3\,V5^2\,c^2\,k4 - V4^2\,V5\,c^2\,k3\,k4^2 + 2\,V4^2\,V5\,c^2\,k4^2\,k5 + V4\,c^2\,k3\,k4^3\,k5 - V1\,V4^3\,V5\,c\,k4 - 2\,V2\,V4^2\,V5^2\,c\,k4 + 3\,V2\,V4\,V5\,c\,k4^2\,k5 - V2\,c\,k4^3\,k5^2 + V4^3\,V5\,c\,k4^2 + V2\,V4^2\,V5\,k4^2,\ V4^2\,V5^3\,c^2 - c\,k4^3\,k5^2 + V3\,V4^3\,k4 - V4\,k5\,k4^3 - V4\,k3\,k4^3 + V5^3\,c^4\,k5^2 + V4^2\,V5\,k4^2 + \alpha\left(V4\,V5^2\,c^3\,k5^2 + 2\,V4^2\,V5^2\,c^2\,k5 - 2\,V4\,V5\,c^2\,k4\,k5^2 + V4^3\,V5^2\,c - 3\,V4^2\,V5\,c\,k4\,k5 + V4\,c\,k4^2\,k5^2 - V4^3\,V5\,k4\right) \quad \textbf{(30)}$$

$$+ k2 \left(V4\, c\, k5^2\, V5 + V4^2\, k5\, V5 - V4\, k4\, k5^2\right) + V4\, V5\, c\, k3\, k4^2 - 2\, V4^2\, V5^2\, c\, k4 + 2\, V4\, V5^3\, c^3\, k5$$
$$- 3\, V5^2\, c^3\, k4\, k5^2 + V2\, V5^3\, c^2\, k5 + 3\, V5\, c^2\, k4^2\, k5^2 + V1\, V4^2\, V5^2\, c + V2\, V4\, V5^3\, c - V3\, V4^3\, V5\, c$$
$$- V1\, V4^2\, V5\, k4 + V1\, V4\, k4^2\, k5 - V2\, V4\, V5^2\, k4 + V2\, V5\, k4^2\, k5 - 2\, V1\, V4\, V5\, c\, k4\, k5 + V1\, V4\, V5^2\, c^2\, k5$$
$$- V3\, V4^2\, V5\, c^2\, k5 - 5\, V4\, V5^2\, c^2\, k4\, k5 - 2\, V2\, V5^2\, c\, k4\, k5 + 2\, V3\, V4^2\, c\, k4\, k5 + 4\, V4\, V5\, c\, k4^2\, k5 \Big]$$

>

4.3.3 - SL(2) discrete projective case

> $restart$

> $with(LinearAlgebra):$

Normalization equations

> $eq1 := \frac{(a \cdot x[n](t)+b)}{c \cdot x[n](t)+\frac{(1+b \cdot c)}{a}} = \frac{1}{2}:$

> $eq2 := \frac{(a \cdot x[n+1](t)+b)}{c \cdot x[n+1](t)+\frac{(1+b \cdot c)}{a}} = 0:$

> $eq3 := \frac{(a \cdot x[n+2](t)+b)}{c \cdot x[n+2](t)+\frac{(1+b \cdot c)}{a}} = -\frac{1}{2}:$

Frame

> $solve(\{eq1, eq2, eq3\}, \{a, b, c\}):$

> $allvalues(\%) : \%[1]:$

> $assign(\%):$

> $\mathrm{rho}[n] := Matrix\left(\left[[a, b], \left[c, simplify\left(\frac{(1+b \cdot c)}{a}\right)\right]\right]\right)$

$$\rho_n := \left[\left[\sqrt{-\frac{-x_n(t)+x_{n+2}(t)}{4 x_n(t) x_{n+1}(t)-4 x_n(t) x_{n+2}(t)-4 x_{n+1}(t)^2+4 x_{n+2}(t) x_{n+1}(t)}},\right.\right.$$
$$\left.-\sqrt{-\frac{-x_n(t)+x_{n+2}(t)}{4 x_n(t) x_{n+1}(t)-4 x_n(t) x_{n+2}(t)-4 x_{n+1}(t)^2+4 x_{n+2}(t) x_{n+1}(t)}}\, x_{n+1}(t)\right],$$
$$\left[(x_n(t)-2 x_{n+1}(t)+x_{n+2}(t)) \middle/ \left(2 (x_n(t) - x_{n+1}(t)) \sqrt{-\frac{-x_n(t)+x_{n+2}(t)}{4 x_n(t) x_{n+1}(t)-4 x_n(t) x_{n+2}(t)-4 x_{n+1}(t)^2+4 x_{n+2}(t) x_{n+1}(t)}}\, (x_{n+1}(t) - x_{n+2}(t))\right),\right.$$
$$\left.\left.\frac{(x_n(t)+x_{n+2}(t)) x_{n+1}(t)-2 x_n(t) x_{n+2}(t)}{\sqrt{\frac{x_n(t)-x_{n+2}(t)}{(x_{n+1}(t)-x_{n+2}(t))(x_n(t)-x_{n+1}(t))}}\, (x_{n+1}(t)-x_{n+2}(t))(x_n(t)-x_{n+1}(t))}\right]\right] \quad (1)$$

Invariants

> $simplify\left(\frac{(a \cdot x[n](t)+b)}{c \cdot x[n](t)+\frac{(1+b \cdot c)}{a}}, symbolic\right)$

$$\frac{1}{2} \quad (2)$$

> $simplify\left(\frac{(a \cdot x[n+1](t)+b)}{c \cdot x[n+1](t)+\frac{(1+b \cdot c)}{a}}, symbolic\right)$

$$0 \quad (3)$$

> $simplify\left(\frac{(a \cdot x[n+2](t)+b)}{c \cdot x[n+2](t)+\frac{(1+b \cdot c)}{a}}, symbolic\right)$

$$-\frac{1}{2} \quad (4)$$

> $simplify\left(simplify\left(\frac{(a \cdot x[n+3](t)+b)}{c \cdot x[n+3](t)+\frac{(1+b \cdot c)}{a}}, symbolic\right), size\right)$

$$-\frac{(x_n(t)-x_{n+2}(t))(-x_{n+3}(t)+x_{n+1}(t))}{(2x_n(t)+2x_{n+2}(t)-4x_{n+3}(t))x_{n+1}(t)+(-4x_n(t)+2x_{n+3}(t))x_{n+2}(t)+2x_{n+3}(t)x_n(t)} \quad (5)$$

> $simplify\left(\frac{(a \cdot x[n+m](t)+b)}{c \cdot x[n+m](t)+\frac{(1+b \cdot c)}{a}}, symbolic\right)$

$$-\frac{(x_n(t)-x_{n+2}(t))(-x_{n+m}(t)+x_{n+1}(t))}{(2x_n(t)+2x_{n+2}(t)-4x_{n+m}(t))x_{n+1}(t)+(-4x_n(t)+2x_{n+m}(t))x_{n+2}(t)+2x_{n+m}(t)x_n(t)} \quad (6)$$

> $simplify\left(\frac{(a \cdot x[n+m,t](t)+b)}{c \cdot x[n+m,t](t)+\frac{(1+b \cdot c)}{a}}, symbolic\right)$

$$-\frac{(x_n(t)-x_{n+2}(t))(-x_{n+m,t}(t)+x_{n+1}(t))}{(2x_n(t)-4x_{n+1}(t)+2x_{n+2}(t))x_{n+m,t}(t)+(2x_n(t)+2x_{n+2}(t))x_{n+1}(t)-4x_n(t)x_{n+2}(t)} \quad (7)$$

Maurer-Cartan matrix

> $mysubs := \left(x_n(t)=\frac{1}{2}, x_{n+1}(t)=0, x[n+2](t)=-\frac{1}{2}, x[n+3](t)=Inv[x,n+3]\right):$

> $K0 := subs(mysubs, subs(n=n+1, \text{rho}[n]))$

$$K0 := \left[\frac{\frac{\sqrt{-\frac{Inv_{x,n+3}}{-1-2\,Inv_{x,n+3}}}}{1+Inv_{x,n+3}}}{\sqrt{-\frac{Inv_{x,n+3}}{-1-2\,Inv_{x,n+3}}}\left(-\frac{1}{2}-Inv_{x,n+3}\right)} \quad -\frac{\frac{\sqrt{-\frac{Inv_{x,n+3}}{-1-2\,Inv_{x,n+3}}}}{2}\,Inv_{x,n+3}}{\sqrt{-\frac{2\,Inv_{x,n+3}}{-\frac{1}{2}-Inv_{x,n+3}}}\left(-\frac{1}{2}-Inv_{x,n+3}\right)}\right] \quad (8)$$

> $\text{kappa}[n](t) = \frac{(1+2 \cdot Inv_{x,n+3})}{(1-2Inv_{x,n+3})}$

$$\kappa_n(t) = \frac{1+2\,Inv_{x,n+3}}{1-2\,Inv_{x,n+3}} \quad (9)$$

> $isolate(\%, Inv_{x,n+3})$

$$Inv_{x,n+3} = \frac{\kappa_n(t)-1}{2(\kappa_n(t)+1)} \quad (10)$$

> $K0 := simplify(subs(\%, K0), symbolic)$

(11)

$$K0 := \begin{bmatrix} \frac{\sqrt{\kappa_n(t)-1}}{2\sqrt{\kappa_n(t)}} & \frac{\sqrt{\kappa_n(t)-1}}{4\sqrt{\kappa_n(t)}} \\ \frac{-3\kappa_n(t)-1}{\sqrt{\kappa_n(t)-1}\sqrt{\kappa_n(t)}} & \frac{\sqrt{\kappa_n(t)-1}}{2\sqrt{\kappa_n(t)}} \end{bmatrix} \quad \textbf{(11)}$$

Shift of the Maurer Cartan matrix times the Maurer Cartan Matrix

> $SKK := simplify(subs(n = n+1, K0).K0, symbolic)$

$$SKK := \begin{bmatrix} -\frac{\sqrt{\kappa_{n+1}(t)-1}\,(\kappa_n(t)+1)}{2\sqrt{\kappa_{n+1}(t)}\sqrt{\kappa_n(t)-1}\sqrt{\kappa_n(t)}} & \frac{\sqrt{\kappa_{n+1}(t)-1}\sqrt{\kappa_n(t)-1}}{4\sqrt{\kappa_{n+1}(t)}\sqrt{\kappa_n(t)}} \\ \frac{-3\kappa_n(t)\kappa_{n+1}(t)+\kappa_n(t)+\kappa_{n+1}(t)+1}{\sqrt{\kappa_{n+1}(t)-1}\sqrt{\kappa_{n+1}(t)}\sqrt{\kappa_n(t)}\sqrt{\kappa_n(t)-1}} & -\frac{\sqrt{\kappa_n(t)-1}\,(\kappa_{n+1}(t)+1)}{2\sqrt{\kappa_{n+1}(t)-1}\sqrt{\kappa_{n+1}(t)}\sqrt{\kappa_n(t)}} \end{bmatrix} \quad \textbf{(12)}$$

Relationships between invariants

> $SInv[x, n, t] = simplify\left(\frac{Inv[x, n+1, t]}{(K0(2,2))^2}, symbolic\right)$

$$SInv_{x,n,t} = \frac{4\, Inv_{x,n+1,t}\,\kappa_n(t)}{\kappa_n(t)-1} \quad \textbf{(13)}$$

> $isolate(\%, Inv_{x,n+1,t})$

$$Inv_{x,n+1,t} = \frac{SInv_{x,n,t}\,(\kappa_n(t)-1)}{4\kappa_n(t)} \quad \textbf{(14)}$$

> $SSInv[x, n, t] = simplify\left(\frac{Inv[x, n+2, t]}{\left(\left(-\frac{1}{2}SKK\right)(2,1) + (SKK)(2,2)\right)^2}, symbolic\right)$

$$SSInv_{x,n,t} = \frac{(\kappa_n(t)-1)\,\kappa_{n+1}(t)\,Inv_{x,n+2,t}}{\kappa_n(t)\,(\kappa_{n+1}(t)-1)} \quad \textbf{(15)}$$

> $isolate(\%, Inv_{x,n+2,t})$

$$Inv_{x,n+2,t} = \frac{SSInv_{x,n,t}\,\kappa_n(t)\,(\kappa_{n+1}(t)-1)}{(\kappa_n(t)-1)\,\kappa_{n+1}(t)} \quad \textbf{(16)}$$

> $SInv[x, n+1, t] = simplify\left(\frac{Inv[x, n+2, t]}{\left(\left(-\frac{1}{2}K0\right)(2,1) + (K0)(2,2)\right)^2}, symbolic\right)$

$$SInv_{x,n+1,t} = \frac{Inv_{x,n+2,t}\,(\kappa_n(t)-1)}{4\kappa_n(t)} \quad \textbf{(17)}$$

Curvature matrix

> $diffrhot := simplify(map(z \rightarrow diff(z, t), \mathrm{rho}[n]))$:

> $mysubs2 := \left(\frac{d}{dt}x_n(t) = Inv[x, n, t], \frac{d}{dt}x_{n+1}(t) = Inv[x, n+1, t], \frac{d}{dt}x_{n+2}(t) = Inv[x, n+2, t]\right)$:

> $subs(mysubs2, diffrhot)$:

> $N0 := simplify(subs(mysubs, \%))$

(18)

$$N0 := \begin{bmatrix} \frac{Inv_{x,n+2,t}}{2} - \frac{Inv_{x,n,t}}{2} & -Inv_{x,n+1,t} \\ 2\,Inv_{x,n+2,t} + 2\,Inv_{x,n,t} - 4\,Inv_{x,n+1,t} & -\frac{Inv_{x,n+2,t}}{2} + \frac{Inv_{x,n,t}}{2} \end{bmatrix} \quad \textbf{(18)}$$

> $N0 := simplify\left(subs\left(\left\{Inv_{x,n+1,t} = \frac{SInv_{x,n,t}\left(\kappa_n(t)-1\right)}{4\,\kappa_n(t)}, Inv_{x,n+2,t} = \frac{SSInv_{x,n,t}\,\kappa_n(t)\left(\kappa_{n+1}(t)-1\right)}{\left(\kappa_n(t)-1\right)\kappa_{n+1}(t)}\right\}, N0\right), symbolic\right)$

$$N0 := \left[\left[\frac{\left(\left(-Inv_{x,n,t} + SSInv_{x,n,t}\right)\kappa_n(t) + Inv_{x,n,t}\right)\kappa_{n+1}(t) - SSInv_{x,n,t}\,\kappa_n(t)}{2\left(\kappa_n(t)-1\right)\kappa_{n+1}(t)}, -\frac{SInv_{x,n,t}\left(\kappa_n(t)-1\right)}{4\,\kappa_n(t)}\right], \left[\frac{1}{\left(\kappa_n(t)-1\right)\kappa_{n+1}(t)\,\kappa_n(t)}\left(\left(\left(2\,Inv_{x,n,t} - SInv_{x,n,t} + 2\,SSInv_{x,n,t}\right)\kappa_{n+1}(t) - 2\,SSInv_{x,n,t}\right)\kappa_n(t)^2 - 2\,\kappa_{n+1}(t)\left(Inv_{x,n,t} - SInv_{x,n,t}\right)\kappa_n(t) - SInv_{x,n,t}\,\kappa_{n+1}(t)\right), \frac{\left(\left(Inv_{x,n,t} - SSInv_{x,n,t}\right)\kappa_n(t) - Inv_{x,n,t}\right)\kappa_{n+1}(t) + SSInv_{x,n,t}\,\kappa_n(t)}{2\left(\kappa_n(t)-1\right)\kappa_{n+1}(t)}\right]\right] \quad \textbf{(19)}$$

>

Infinitesimals

> $subs\left(\{alpha = 1, beta = 0, delta = 0\}, diff\left(\frac{(alpha \cdot x[n] + beta)}{delta \cdot x[n] + \frac{(1 + beta \cdot delta)}{alpha}}, alpha\right)\right)$

$$2\,x_n \quad \textbf{(20)}$$

> $subs\left(\{alpha = 1, beta = 0, delta = 0\}, diff\left(\frac{(alpha \cdot x[n] + beta)}{delta \cdot x[n] + \frac{(1 + beta \cdot delta)}{alpha}}, beta\right)\right)$

$$1 \quad \textbf{(21)}$$

> $subs\left(\{alpha = 1, beta = 0, delta = 0\}, diff\left(\frac{(alpha \cdot x[n] + beta)}{delta \cdot x[n] + \frac{(1 + beta \cdot delta)}{alpha}}, delta\right)\right)$

$$-x_n^2 \quad \textbf{(22)}$$

Syzygy and evolution of curvature invariants

> $SN0 := subs(\{Inv[x, n, t] = SInv[x, n, t], SInv[x, n, t] = SSInv[x, n, t], SSInv[x, n, t] = SSSInv[x, n, t]\}, N0)$:

> $syzygy := simplify(map(z \rightarrow diff(z, t), K0) - (SN0 \cdot K0 - K0 \cdot N0), symbolic)$:

> $isolate\left(syzygy(1, 1), \frac{d}{dt}\,\kappa_n(t)\right)$:

> $collect(\%, Inv_{x,n,t})$: $collect(\%, SInv_{x,n,t})$: $collect(\%, SSInv_{x,n,t})$: $collect(\%, SSSInv_{x,n,t})$

$$\frac{d}{dt}\,\kappa_n(t) = \frac{\left(\kappa_n(t)^2\,\kappa_{n+1}(t) - \kappa_n(t)^2\right)SSSInv_{x,n,t}}{\kappa_{n+1}(t)} + \frac{\left(-\left(2\,\kappa_n(t)+1\right)\kappa_{n+1}(t) + 3\,\kappa_n(t)^2\right)SSInv_{x,n,t}}{\kappa_{n+1}(t)} + \left(-\kappa_n(t)+1\right)SInv_{x,n,t} + \left(-\kappa_n(t)^2 + \kappa_n(t)\right)Inv_{x,n,t} \quad \textbf{(23)}$$

Computation of the adjoints

> $restart$

> $with(LinearAlgebra)$:

> $Adg := Matrix([[a \cdot d + b \cdot c, -a \cdot c, b \cdot d], [-2 \cdot a \cdot b, a^2, -b^2], [2 \cdot c \cdot d, -c^2, d^2]])$

$$Adg := \begin{bmatrix} a\,d + b\,c & -a\,c & b\,d \\ -2\,a\,b & a^2 & -b^2 \\ 2\,c\,d & -c^2 & d^2 \end{bmatrix} \qquad \textbf{(24)}$$

Here we evaluate the frame into the adjoint matrix

> $A := \frac{\mathrm{sqrt}(x0 - x2)}{\mathrm{sqrt}((x0 - x1) \cdot (x1 - x2))}$:

> $a := \frac{1}{2} \cdot A$:

> $b := -\frac{x1}{2} \cdot A$:

> $c := \frac{(x2 - 2 \cdot x1 + x0)}{x0 - x2} \cdot A$:

> $d := \frac{(x0 \cdot x1 - 2 \cdot x0 \cdot x2 + x1 \cdot x2)}{x0 - x2} \cdot A$:

> $Adrho := simplify(Adg, symbolic)$

$$Adrho := \left[\left[\frac{-x0\,x2 + x1^2}{(x0 - x1)\,(x1 - x2)}, \frac{-x2 + 2\,x1 - x0}{2\,(x0 - x1)\,(x1 - x2)}, -\frac{((x0 + x2)\,x1 - 2\,x0\,x2)\,x1}{2\,(x0 - x1)\,(x1 - x2)}\right],\right.$$
$$\left[\frac{(x0 - x2)\,x1}{2\,(x0 - x1)\,(x1 - x2)}, \frac{x0 - x2}{4\,(x0 - x1)\,(x1 - x2)}, -\frac{x1^2\,(x0 - x2)}{4\,(x0 - x1)\,(x1 - x2)}\right],$$
$$\left[\frac{2\,((x1 - 2\,x2)\,x0 + x1\,x2)\,(x2 - 2\,x1 + x0)}{(x0 - x2)\,(x0 - x1)\,(x1 - x2)}, -\frac{(x2 - 2\,x1 + x0)^2}{(x0 - x2)\,(x0 - x1)\,(x1 - x2)},\right.$$
$$\left.\left.\frac{((x1 - 2\,x2)\,x0 + x1\,x2)^2}{(x0 - x2)\,(x0 - x1)\,(x1 - x2)}\right]\right] \qquad \textbf{(25)}$$

Here we evaluate the Maurer Cartan matrix into the adjoint matrix

> $B := \mathrm{sqrt}\left(\frac{(\mathrm{kappa} - 1)}{4 \cdot \mathrm{kappa}}\right)$:

> $a := B$:

> $b := \frac{1}{2} \cdot B$:

> $c := -\frac{(6 \cdot \mathrm{kappa} + 2)}{\mathrm{kappa} - 1} \cdot B$:

> $d := B$:

> $AdK := simplify(Adg, symbolic)$

$$AdK := \begin{bmatrix} \frac{-\kappa - 1}{2\,\kappa} & \frac{3\,\kappa + 1}{2\,\kappa} & \frac{\kappa - 1}{8\,\kappa} \\ \frac{-\kappa + 1}{4\,\kappa} & \frac{\kappa - 1}{4\,\kappa} & \frac{-\kappa + 1}{16\,\kappa} \\ \frac{-3\,\kappa - 1}{\kappa} & -\frac{(3\,\kappa + 1)^2}{(\kappa - 1)\,\kappa} & \frac{\kappa - 1}{4\,\kappa} \end{bmatrix} \qquad \textbf{(26)}$$

Here we evaluate the shift of the Mauer Cartan matrix into the adjoint matrix

> $B := \mathrm{sqrt}\left(\frac{(Skappa - 1)}{4 \cdot Skappa}\right)$:

> $a := B$:

> $b := \frac{1}{2} \cdot B$:

> $c := -\frac{(6 \cdot Skappa + 2)}{Skappa - 1} \cdot B$:

> $d := B$:

> $AdSK := simplify(Adg, symbolic)$

$$AdSK := \begin{bmatrix} \frac{-Skappa-1}{2\,Skappa} & \frac{3\,Skappa+1}{2\,Skappa} & \frac{Skappa-1}{8\,Skappa} \\ \frac{-Skappa+1}{4\,Skappa} & \frac{Skappa-1}{4\,Skappa} & \frac{-Skappa+1}{16\,Skappa} \\ \frac{-3\,Skappa-1}{Skappa} & -\frac{(3\,Skappa+1)^2}{(Skappa-1)\,Skappa} & \frac{Skappa-1}{4\,Skappa} \end{bmatrix} \quad \textbf{(27)}$$

Here we caluculate the shift of the Maurer Cartan matrix times the Maurer Cartan matrix into the adjoint

> $simplify(AdSK \cdot AdK, symbolic)$

$$\left[\left[\frac{(-Skappa+1)\,\kappa+Skappa+1}{2\,Skappa\,\kappa}, \frac{-3\,Skappa\,\kappa^2-2\,Skappa\,\kappa+\kappa^2+Skappa+2\,\kappa+1}{2\,Skappa\,(\kappa-1)\,\kappa},\right.\right. \quad \textbf{(28)}$$

$$\left.\frac{(Skappa+1)\,(\kappa-1)}{8\,Skappa\,\kappa}\right],$$

$$\left[\frac{(Skappa-1)\,(\kappa+1)}{4\,Skappa\,\kappa}, \frac{(Skappa-1)\,(\kappa+1)^2}{4\,Skappa\,(\kappa-1)\,\kappa}, -\frac{(Skappa-1)\,(\kappa-1)}{16\,Skappa\,\kappa}\right],$$

$$\left[\frac{(3\,\kappa-1)\,Skappa^2+(2\,\kappa-2)\,Skappa-\kappa-1}{(Skappa-1)\,Skappa\,\kappa}, -\frac{(3\,Skappa\,\kappa-Skappa-\kappa-1)^2}{Skappa\,\kappa\,(Skappa-1)\,(\kappa-1)},\right.$$

$$\left.\left.\frac{(\kappa-1)\,(Skappa+1)^2}{4\,(Skappa-1)\,Skappa\,\kappa}\right]\right]$$

(29)

The general solution, Groebner basis computation

> $restart$

> $with(LinearAlgebra)$:

> $k := Matrix([[k1, k2, k3]])$:

> $V := Matrix([[V1, V2, V3]])$:

> $Adg := Matrix([[a\cdot d+b\cdot c, -a\cdot c, b\cdot d], [-2\cdot a\cdot b, a^2, -b^2], [2\cdot c\cdot d, -c^2, d^2]])$:

> $simplify(MatrixInverse(Adg), symbolic)$: $Adginv := subs(a\,d-b\,c=1, \%)$

$$Adginv := \begin{bmatrix} a\,d+b\,c & c\,d & -a\,b \\ 2\,b\,d & d^2 & -b^2 \\ -2\,a\,c & -c^2 & a^2 \end{bmatrix} \quad \textbf{(30)}$$

> $Equations := simplify(k \cdot Adginv, symbolic) - V$

$Equations :=$ **(31)**

$$\begin{bmatrix} d\,(a\,k1+2\,b\,k2)-2\,k3\,a\,c+b\,c\,k1-V1 & -c^2\,k3+c\,d\,k1+d^2\,k2-V2 & a^2\,k3-a\,b\,k1-b^2\,k2-V3 \end{bmatrix}$$

> $with(Groebner)$:

> $F := [Equations(1,1), Equations(1,2), Equations(1,3), a\cdot d-b\cdot c-1]$

$F := [d\,(a\,k1+2\,b\,k2)-2\,k3\,a\,c+b\,c\,k1-V1, -c^2\,k3+c\,d\,k1+d^2\,k2-V2, a^2\,k3-a\,b\,k1-b^2\,k2-V3, a\,d-b\,c-1]$ **(32)**

> $Basis(F, plex(a, b, c, k1, k2, k3))$

$[-V1^2-4\,V2\,V3+k1^2+4\,k2\,k3, c^2\,k3-c\,d\,k1-d^2\,k2+V2, -V1\,d+2\,b\,V2-2\,c\,k3+d\,k1, -c\,V1+2\,a\,V2-c\,k1-2\,k2\,d]$ **(33)**

Running example for Chapter 5, Commuting induced flows on the curvature invariants

> *restart*

> *with*(*LinearAlgebra*) :

Normalization equations

> $Eq1 := \text{lambda}\cdot u[n] + \text{epsilon}:$

> $Eq2 := \text{lambda}\cdot u[n+1] + \text{epsilon} - 1:$

> $solve(\{Eq1, Eq2\}, \{\text{lambda}, \text{epsilon}\})$

$$\left\{ \epsilon = \frac{u_n}{u_n - u_{n+1}}, \lambda = -\frac{1}{u_n - u_{n+1}} \right\} \tag{1}$$

> $assign(\%)$

Frame

> $\text{rho}[n] := Matrix([[\text{lambda}, \text{epsilon}], [0, 1]])$

$$\rho_n := \begin{bmatrix} -\frac{1}{u_n - u_{n+1}} & \frac{u_n}{u_n - u_{n+1}} \\ 0 & 1 \end{bmatrix} \tag{2}$$

Invariants

> $simplify(\text{rho}[n].Matrix([[u[n+j]], [1]]), symbolic)$

$$\begin{bmatrix} \frac{-u_{n+j} + u_n}{u_n - u_{n+1}} \\ 1 \end{bmatrix} \tag{3}$$

First order differential invariants

> $simplify(\text{rho}[n].Matrix([[u[n+j, t]], [0]]), symbolic)$

$$\begin{bmatrix} -\frac{u_{n+j, t}}{u_n - u_{n+1}} \\ 0 \end{bmatrix} \tag{4}$$

MaurerCartan matrix

> $\text{rho}[n+1] := subs(n = n+1, \text{rho}[n]):$

> $subs(\{u[n] = 0, u[n+1] = 1, u[n+2] = Inv[n+2]\}, \%)$

$$\begin{bmatrix} -\frac{1}{1 - Inv_{n+2}} & \frac{1}{1 - Inv_{n+2}} \\ 0 & 1 \end{bmatrix} \tag{5}$$

> $K[n] := subs\left(\frac{1}{1 - Inv_{n+2}} = \text{kappa}[n](t), \%\right)$

$$K_n := \begin{bmatrix} -\kappa_n(t) & \kappa_n(t) \\ 0 & 1 \end{bmatrix} \tag{6}$$

Curvature matrix in terms of sigma_t

> $K[n].Matrix([[Inv[n+1, t]](t), [0]])$

$$\begin{bmatrix} -\kappa_n(t)\, Inv_{n+1, t}(t) \\ 0 \end{bmatrix} \tag{7}$$

> $sigmat[n+1](t) = \%(1, 1)$

$$sigmat_{n+1}(t) = -\kappa_n(t)\, Inv_{n+1, t}(t) \tag{8}$$

> $isolate(\%, Inv_{n+1, t}(t))$

$$Inv_{n+1, t}(t) = -\frac{sigmat_{n+1}(t)}{\kappa_n(t)} \tag{9}$$

Curvature Matrix

> $diff(map(z \to z(t), \text{rho}[n]), t)$:

> $subs\left(\left\{\frac{d}{dt}\, u_n(t) = Inv[n,t](t), \frac{d}{dt}\, u_{n+1}(t) = Inv[n+1,t](t)\right\}, \%\right)$:

> $subs(\{u[n](t) = 0, u[n+1](t) = 1, u[n+2](t) = Inv[n+2]\}, \%)$

$$\begin{bmatrix} Inv_{n,t}(t) - Inv_{n+1,t}(t) & -Inv_{n,t}(t) \\ 0 & 0 \end{bmatrix} \quad \textbf{(10)}$$

> $N[n] := subs\left(\left\{Inv_{n,t}(t) = sigmat[n](t), Inv_{n+1,t}(t) = -\frac{sigmat[n+1](t)}{\kappa_n(t)}\right\}, \%\right)$

$$N_n := \begin{bmatrix} sigmat_n(t) + \frac{sigmat_{n+1}(t)}{\kappa_n(t)} & -sigmat_n(t) \\ 0 & 0 \end{bmatrix} \quad \textbf{(11)}$$

Correction Terms

> $simplify(diff(map(z \to z(t), \text{rho}[n]), t).Matrix([[u[n+j](t), [1]]), symbolic)$:

> $subs\left(\left\{\frac{d}{dt}\, u_n(t) = Inv[n,t](t), \frac{d}{dt}\, u_{n+1}(t) = Inv[n+1,t](t)\right\}, \%\right)$:

> $simplify(subs(\{u[n](t) = 0, u[n+1](t) = 1, u[n+2](t) = Inv[n+2], u[n+j](t) = Inv[n+j](t)\}, \%),$ $symbolic)$

$$\begin{bmatrix} \left(Inv_{n,t}(t) - Inv_{n+1,t}(t)\right) Inv_{n+j}(t) - Inv_{n,t}(t) \\ 0 \end{bmatrix} \quad \textbf{(12)}$$

Correction Matrix

> $T := Matrix([Inv[n,t](t), Inv[n+1,t](t)])$

$$T := \begin{bmatrix} Inv_{n,t}(t) & Inv_{n+1,t}(t) \end{bmatrix} \quad \textbf{(13)}$$

> $\text{Phi} := Matrix([[0,1],[1,1]])$

$$\Phi := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \textbf{(14)}$$

> $J := Matrix([[1,0],[0,1]])$

$$J := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textbf{(15)}$$

> $Kmatrix := -T \cdot J \cdot MatrixInverse(\text{Phi} \cdot J)$

$$Kmatrix := \begin{bmatrix} Inv_{n,t}(t) - Inv_{n+1,t}(t) & -Inv_{n,t}(t) \end{bmatrix} \quad \textbf{(16)}$$

Syzygy and evolution of curvature invariant

> $simplify(map(z \to diff(z,t), K[n]) - (subs(\{n = n+1\}, N[n]).K[n] - K[n].N[n]), symbolic)$:

> $isolate\left(\%(1,1), \frac{d}{dt}\, \kappa_n(t)\right) : collect(\%, sigmat_n(t)) : collect(\%, sigmat_{n+1}(t))$

$$\frac{d}{dt}\, \kappa_n(t) = \left(\kappa_n(t) - 1\right) sigmat_{n+1}(t) - \kappa_n(t)\, sigmat_n(t) + \frac{sigmat_{n+2}(t)\, \kappa_n(t)}{\kappa_{n+1}(t)} \quad \textbf{(17)}$$

> $Hsigmat := rhs(\%)$

$$Hsigmat := \left(\kappa_n(t) - 1\right) sigmat_{n+1}(t) - \kappa_n(t)\, sigmat_n(t) + \frac{sigmat_{n+2}(t)\, \kappa_n(t)}{\kappa_{n+1}(t)} \quad \textbf{(18)}$$

Fundamental syzygy

> $diff(map(z \to z(s,t), \text{rho}[n]), s).Matrix\left(\left[\left[\frac{d}{dt}\, u_n(s,t)\right], [0]\right]\right)$:

> $subs\left(\left\{\frac{d}{dt}\, u_n(s,t) = Inv[n,t](s,t), \frac{d}{dt}\, u_{n+1}(s,t) = Inv[n+1,t](s,t), \frac{d}{ds}\, u_n(s,t) = Inv[n,s](s,t), \frac{d}{ds}\right.$

$u_{n+1}(s,t) = Inv[n+1,s](s,t) \}, \%) :$

> $simplify(subs(\{ u[n](s,t) = 0, u[n+1](s,t) = 1, u[n+2](s,t) = Inv[n+2](s,t), u[n+j](s,t) = Inv[n+j](s,t) \}, \%), symbolic) :$

> $subs\left(\left\{ Inv_{n,t}(s,t) = sigmat[n](s,t), Inv_{n+1,t}(s,t) = -\frac{sigmat[n+1](s,t)}{\kappa_n(s,t)} \right\}, \% \right) :$

> $subs\left(\left\{ Inv_{n,s}(s,t) = sigmas[n](s,t), Inv_{n+1,s}(s,t) = -\frac{sigmas[n+1](s,t)}{\kappa_n(s,t)} \right\}, \% \right)$

$$\begin{bmatrix} \left(sigmas_n(s,t) + \frac{sigmas_{n+1}(s,t)}{\kappa_n(s,t)} \right) sigmat_n(s,t) \\ 0 \end{bmatrix} \quad \textbf{(19)}$$

> $simplify\left(simplify\left(diff(sigmas_n(s,t), t) - diff(sigmat_n(s,t), s) + \left(sigmas_n(s,t) + \frac{sigmas_{n+1}(s,t)}{\kappa_n(s,t)} \right) sigmat_n(s,t) - \left(sigmat_n(s,t) + \frac{sigmat_{n+1}(s,t)}{\kappa_n(s,t)} \right) sigmas_n(s,t), symbolic \right), size \right) :$
$collect\left(\%, \frac{\partial}{\partial t} sigmas_n(s,t) \right) : collect\left(\%, \frac{\partial}{\partial s} sigmat_n(s,t) \right) :$

> $C := \%$

$$C := -\frac{\partial}{\partial s} sigmat_n(s,t) + \frac{\partial}{\partial t} sigmas_n(s,t) + \frac{-sigmas_n(s,t)\, sigmat_{n+1}(s,t) + sigmas_{n+1}(s,t)\, sigmat_n(s,t)}{\kappa_n(s,t)} \quad \textbf{(20)}$$

Commuting evolution flows

> $Hsigmat := subs(\{ sigmat_n(t) = sigmat_n(s,t), sigmat_{n+1}(t) = sigmat_{n+1}(s,t), sigmat_{n+2}(t) = sigmat_{n+2}(s,t),$ $\text{kappa}[n](t) = \text{kappa}[n](s,t), \text{kappa}[n+1](t) = \text{kappa}[n+1](s,t) \}, Hsigmat) :$

> $Hsigmas := subs(\{ sigmat_n(s,t) = sigmas_n(s,t), sigmat_{n+1}(s,t) = sigmas_{n+1}(s,t), sigmat_{n+2}(s,t) = sigmas_{n+2}(s,t) \}, Hsigmat) :$

> $diff(Hsigmas, t) - diff(Hsigmat, s) :$

> $subs\left(\left\{ \frac{\partial}{\partial s} \kappa_n(s,t) = Hsigmas, \frac{\partial}{\partial t} \kappa_n(s,t) = Hsigmat, \frac{\partial}{\partial t} \kappa_{n+1}(s,t) = subs(n = n+1, Hsigmat), \frac{\partial}{\partial s} \kappa_{n+1}(s,t) = subs(n = n+1, Hsigmas) \right\}, \% \right) :$

> $Mylefthandside := simplify(\%, symbolic) :$

> $HC := simplify\left((\kappa_n(s,t) - 1)\, subs(n = n+1, C) - \kappa_n(s,t) \cdot C + \frac{subs(n = n+2, C) \cdot \kappa_n(s,t)}{\kappa_{n+1}(s,t)}, symbolic \right) :$

> $Mylefthandside - HC$

$$0 \quad \textbf{(21)}$$

>

>

>

>

> restart

> with(LinearAlgebra) :

5.7 - The SL(2) LinearAction

Correction Matrix

> T := Matrix([sigmat[n, x], sigmat[n, y], Inv[n + 1, x, t]])

$$T := \begin{bmatrix} sigmat_{n,x} & sigmat_{n,y} & Inv_{n+1,x,t} \end{bmatrix} \quad (1)$$

> Phi := Matrix([[1, 0, 0], [0, 0, tau], [0, 1, 0]])

$$\Phi := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & 1 & 0 \end{bmatrix} \quad (2)$$

> J := Matrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]])

$$J := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

> Kmatrix := -T·J·MatrixInverse(Phi·J)

$$Kmatrix := \begin{bmatrix} -sigmat_{n,x} & -\frac{Inv_{n+1,x,t}}{\tau} & -sigmat_{n,y} \end{bmatrix} \quad (4)$$

Frame

$$> \text{rho}[n] := \begin{bmatrix} \frac{y_{n+1}}{x_n y_{n+1} - y_n x_{n+1}} & -\frac{x_{n+1}}{x_n y_{n+1} - y_n x_{n+1}} \\ -y_n & x_n \end{bmatrix}$$

>

$$\rho_n := \begin{bmatrix} \frac{y_{n+1}}{x_n y_{n+1} - y_n x_{n+1}} & -\frac{x_{n+1}}{x_n y_{n+1} - y_n x_{n+1}} \\ -y_n & x_n \end{bmatrix} \quad (5)$$

Fundamental syzygy

$$> M := diff(map(z \to z(s,t), \text{rho}[n]), s) \, Matrix\left(\left[\left[\frac{d}{dt} x_n(s,t)\right], \left[\frac{d}{dt} y_n(s,t)\right]\right]\right) :$$

$$> mysubs2a := \left(\frac{d}{dt} x_n(s,t) = Inv[x,n,t](s,t), \frac{d}{dt} x_{n+1}(s,t) = Inv[x,n+1,t](s,t), \frac{d}{dt} x_{n+2}(s,t) = Inv[x,n+2, t](s,t), \frac{d}{dt} y_n(s,t) = Inv[y,n,t](s,t), \frac{d}{dt} y_{n+1}(s,t) = Inv[y,n+1,t](s,t), \frac{d}{dt} y_{n+2}(s,t) = Inv[y,n+2, t](s,t) \right) :$$

$$> mysubs2b := \left(\frac{d}{ds} x_n(s,t) = Inv[x,n,s](s,t), \frac{d}{ds} x_{n+1}(s,t) = Inv[x,n+1,s](s,t), \frac{d}{ds} x_{n+2}(s,t) = Inv[x,n + 2,s](s,t), \frac{d}{ds} y_n(s,t) = Inv[y,n,s](s,t), \frac{d}{ds} y_{n+1}(s,t) = Inv[y,n+1,s](s,t), \frac{d}{ds} y_{n+2}(s,t) = Inv[y,n + 2,s](s,t) \right) :$$

> M := subs(mysubs2a, M) :

> M := subs(mysubs2b, M) :

$$> mysubs := (x_n(s,t) = 1, x_{n+1}(s,t) = 0, y[n](s,t) = 0, y[n+1](s,t) = Inv[y,n+1](s,t), y[n+2](s,t) = Inv[y,n+2](s,t), x[n+2](s,t) = Inv[x,n+2](s,t)) :$$

> simplify(subs(mysubs, M), symbolic)

(6

$$\left[\begin{array}{c}\frac{-Inv_{x,n,s}(s,t)\ Inv_{x,n,t}(s,t)\ Inv_{y,n+1}(s,t)-Inv_{x,n+1,s}(s,t)\ Inv_{y,n,t}(s,t)}{Inv_{y,n+1}(s,t)}\\ -Inv_{y,n,s}(s,t)\ Inv_{x,n,t}(s,t)+Inv_{x,n,s}(s,t)\ Inv_{y,n,t}(s,t)\end{array}\right] \quad (6)$$

>

> $simplify\left(subs\left(\left\{Inv_{y,n+1}(s,t)=\mathrm{tau}[n](s,t), Inv_{x,n+1,s}(s,t)=-\frac{sigmas[y,n+1](s,t)}{\mathrm{tau}[n](s,t)}\right\},\%\right), symbolic\right):$

> $subs(\{Inv_{x,n,s}(s,t)=sigmas[x,n](s,t), Inv_{y,n,s}(s,t)=sigmas[y,n](s,t), Inv_{x,n,t}(s,t)=sigmat[x,n](s,t),$
$Inv_{y,n,t}(s,t)=sigmat[y,n](s,t)\},\%)$

$$\left[\begin{array}{c}\frac{-sigmas_{x,n}(s,t)\ sigmat_{x,n}(s,t)\ \tau_n(s,t)^2+sigmas_{y,n+1}(s,t)\ sigmat_{y,n}(s,t)}{\tau_n(s,t)^2}\\ -sigmas_{y,n}(s,t)\ sigmat_{x,n}(s,t)+sigmas_{x,n}(s,t)\ sigmat_{y,n}(s,t)\end{array}\right] \quad (7)$$

> $simplify\left(\left[\begin{array}{c}\frac{-sigmas_{x,n}(s,t)\ sigmat_{x,n}(s,t)\ \tau_n(s,t)^2+sigmas_{y,n+1}(s,t)\ sigmat_{y,n}(s,t)}{\tau_n(s,t)^2}\\ -sigmas_{y,n}(s,t)\ sigmat_{x,n}(s,t)+sigmas_{x,n}(s,t)\ sigmat_{y,n}(s,t)\end{array}\right] - \left[\begin{array}{c}\frac{-sigmat_{x,n}(s,t)\ sigmas_{x,n}(s,t)\ \tau_n(s,t)^2+sigmat_{y,n+1}(s,t)\ sigmas_{y,n}(s,t)}{\tau_n(s,t)^2}\\ -sigmat_{y,n}(s,t)\ sigmas_{x,n}(s,t)+sigmat_{x,n}(s,t)\ sigmas_{y,n}(s,t)\end{array}\right] + \left[\begin{array}{c}\frac{\partial}{\partial t} sigmas_{x,n}(s,t)-\left(\frac{\partial}{\partial s} sigmat_{x,n}(s,t)\right)\\ \frac{\partial}{\partial t} sigmas_{y,n}(s,t)-\left(\frac{\partial}{\partial s} sigmat_{y,n}(s,t)\right)\end{array}\right], symbolic\right) : collect\left(\%, \frac{\partial}{\partial t} sigmas_{x,n}(s,t)\right) : collect\left(\%, \frac{\partial}{\partial s} sigmat_{x,n}(s,t)\right)$

$$\left[\left[-\frac{\partial}{\partial s} sigmat_{x,n}(s,t)+\frac{\partial}{\partial t} sigmas_{x,n}(s,t)\right.\right.$$
$$\left[+\frac{sigmas_{y,n+1}(s,t)\ sigmat_{y,n}(s,t)-sigmat_{y,n+1}(s,t)\ sigmas_{y,n}(s,t)}{\tau_n(s,t)^2},\right],$$
$$\left[-2\,sigmas_{y,n}(s,t)\ sigmat_{x,n}(s,t)+2\,sigmas_{x,n}(s,t)\ sigmat_{y,n}(s,t)+\frac{\partial}{\partial t} sigmas_{y,n}(s,t)-\frac{\partial}{\partial s} sigmat_{y,n}(s,t)\right] ,\ \right] \quad (8)$$

>

> $C := \% :$

Commuting evolution flows

> $Hsigmat := Matrix\left(\left[\left[\mathrm{kappa}[n](s,t)\cdot(sigmat[x,n](s,t)-sigmat[x,n+1](s,t))+\frac{1}{\mathrm{tau}[n](s,t)}sigmat[y,n](s,t)-\frac{\mathrm{tau}[n](s,t)}{\mathrm{tau}[n+1](s,t)^2}sigmat[y,n+2](s,t)\right],[\mathrm{tau}[n](s,t)\cdot sigmat[x,n](s,t)+\mathrm{tau}[n](s,t)sigmat[x,n+1](s,t)+\mathrm{kappa}[n](s,t)sigmat[y,n+1](s,t)]\right]\right):$

> $Hsigmas :=$

$$\left[\begin{array}{c} \kappa_n(s,t)\left(sigmas_{x,n}(s,t) - sigmas_{x,n+1}(s,t)\right) + \frac{sigmas_{y,n}(s,t)}{\tau_n(s,t)} - \frac{\tau_n(s,t)\, sigmas_{y,n+2}(s,t)}{\tau_{n+1}(s,t)^2} \\ \tau_n(s,t)\, sigmas_{x,n}(s,t) + \tau_n(s,t)\, sigmas_{x,n+1}(s,t) + \kappa_n(s,t)\, sigmas_{y,n+1}(s,t) \end{array}\right]:$$

> $simplify(map(z \to diff(z,t), Hsigmas) - map(z \to diff(z,s), Hsigmat), symbolic):$

> $subs\Big(\Big\{\frac{\partial}{\partial s}\kappa_n(s,t) = Hsigmas(1,1), \frac{\partial}{\partial t}\kappa_n(s,t) = Hsigmat(1,1), \frac{\partial}{\partial s}\tau_n(s,t) = Hsigmas(2,1), \frac{\partial}{\partial t}\tau_n(s,t) = Hsigmat(2,1), \frac{\partial}{\partial s}\tau_{n+1}(s,t) = subs(n=n+1, Hsigmas(2,1)), \frac{\partial}{\partial t}\tau_{n+1}(s,t) = subs(n=n+1, Hsigmat(2,1))\Big\}, \%\Big):$

> $Mylefthandside := simplify(\%, symbolic):$

> $HC := simplify\Big(Matrix\Big(\Big[\Big[\kappa_n(s,t)\,(C(1,1) - subs(n=n+1, C(1,1))) + \frac{C(2,1)}{\tau_n(s,t)} - \frac{\tau_n(s,t)\, subs(n=n+2, C(2,1))}{\tau_{n+1}(s,t)^2}\Big], \big[\tau_n(s,t)\, C(1,1) + \tau_n(s,t)\, subs(n=n+1, C(1,1)) + \text{kappa}[n](s,t)\cdot subs(n=n+1, C(2,1))\big]\Big]\Big), symbolic\Big):$

> $simplify(Mylefthandside - HC, symbolic)$

$$\left[\begin{array}{c} 0 \\ 0 \end{array}\right] \quad \textbf{(9)}$$

>

6.2 - Multispaces, SE(2) curvature

> $restart:$

> $with(LinearAlgebra):$

Coefficients

> $Ax := xt0:$

> $Bx := \frac{((xt2 - xt0) \cdot t1^2 - (xt1 - xt0) \cdot t2^2)}{t1 \cdot t2 \cdot (t1 - t2)}:$

> $Cx := -2 \cdot \frac{((xt2 - xt0) \cdot t1 - (xt1 - xt0) \cdot t2)}{t1 \cdot t2 \cdot (t1 - t2)}:$

> $Au := ut0:$

> $Bu := \frac{((ut2 - ut0) \cdot t1^2 - (ut1 - ut0) \cdot t2^2)}{t1 \cdot t2 \cdot (t1 - t2)}:$

> $Cu := -2 \cdot \frac{((ut2 - ut0) \cdot t1 - (ut1 - ut0) \cdot t2)}{t1 \cdot t2 \cdot (t1 - t2)}:$

Action

> $Xtilde := \cos(\text{theta}) \cdot (x - a) + \sin(\text{theta}) \cdot (u - b):$

> $Utilde := -\sin(\text{theta}) \cdot (x - a) + \cos(\text{theta}) \cdot (u - b):$

> $X0tilde := subs(\{x = xt0, u = ut0\}, Xtilde):$

> $U0tilde := subs(\{x = xt0, u = ut0\}, Utilde):$

> $X1tilde := subs(\{x = xt1, u = ut1\}, Xtilde):$

> $U1tilde := subs(\{x = xt1, u = ut1\}, Utilde):$

> $X2tilde := subs(\{x = xt2, u = ut2\}, Xtilde):$

> $U2tilde := subs(\{x = xt2, u = ut2\}, Utilde):$

Normalisation equations

> $norm1 := simplify(subs(\{xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde\}, Ax), symbolic) = 0:$

> $norm1 := simplify(subs(\{xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde\}, Ax), symbolic) = 0:$

> $norm2 := simplify(subs(\{xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde\}, Au), symbolic) = 0:$

> $norm3 := simplify(subs(\{xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde\}, Bu), symbolic) = 0:$

> $solve(\{norm1, norm2, norm3\}, \{a, b, \text{theta}\})$

$$\left\{a = xt0, b = ut0, \theta = \arctan\left(\frac{t1^2\, ut0 - t1^2\, ut2 - t2^2\, ut0 + t2^2\, ut1}{t1^2\, xt0 - t1^2\, xt2 - t2^2\, xt0 + t2^2\, xt1}\right)\right\} \quad (1)$$

> $assign(\%)$

Taylor Series

> $Mysubs := t1 = h, t2 = 2 \cdot h, xt0 = x, ut0 = u, xt1 = x + h \cdot xt + \frac{1}{2} \cdot h^2 \cdot xtt, xt2 = x + 2 \cdot h \cdot xt + 2 \cdot h^2 \cdot xtt, ut1 = u + h \cdot ut + \frac{1}{2} \cdot h^2 \cdot utt, ut2 = u + 2 \cdot h \cdot ut + 2 \cdot h^2 \cdot utt:$

Convergence to the smooth case of the parameters of the frame

> $subs(Mysubs, a):$

> $simplify(\%, symbolic):$

> $map(limit, \%, h = 0):$

> $simplify(\%, symbolic)$

$$x \quad (2)$$

> $subs(Mysubs, b):$

> $simplify(\%, symbolic):$

> $map(limit, \%, h = 0):$

> simplify(%, symbolic)

$$u \quad (3)$$

> subs(Mysubs, theta) :

> simplify(%, symbolic) :

> map(limit, %, h = 0) :

> simplify(%, symbolic)

$$\arctan\left(\frac{ut}{xt}\right) \quad (4)$$

Convergence to the smooth case of the invariants

> arclenght1 := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Bx), symbolic) :

> arclenght2 := subs(Mysubs, arclenght1) :

> arclenght3 := simplify(arclenght2, symbolic) : simplify(map(limit, %, h = 0), symbolic)

$$\sqrt{ut^2 + xt^2} \quad (5)$$

> dotproduct1 := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Cx), symbolic) :

> dotproduct2 := subs(Mysubs, dotproduct1) :

> simplify(%, symbolic)

$$\frac{ut\, utt + xt\, xtt}{\sqrt{ut^2 + xt^2}} \quad (6)$$

> crossproduct1 := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Cu), symbolic) :

> crossproduct2 := subs(Mysubs, crossproduct1) :

> simplify(%, symbolic)

$$\frac{-ut\, xtt + utt\, xt}{\sqrt{ut^2 + xt^2}} \quad (7)$$

Moving frame

> rho := simplify(MatrixInverse(Matrix([[cos(theta), -sin(theta), a], [sin(theta), cos(theta), b], [0, 0, 1]])), symbolic) :

Convergence to the smooth case of the moving frame

> taylorrho := subs({Mysubs}, rho) :

> map(simplify, %, symbolic) :

> map(limit, %, h = 0) :

> simplify(%, symbolic)

$$\begin{bmatrix} \frac{xt}{\sqrt{ut^2 + xt^2}} & \frac{ut}{\sqrt{ut^2 + xt^2}} & -\frac{u\, ut + x\, xt}{\sqrt{ut^2 + xt^2}} \\ -\frac{ut}{\sqrt{ut^2 + xt^2}} & \frac{xt}{\sqrt{ut^2 + xt^2}} & \frac{-u\, xt + ut\, x}{\sqrt{ut^2 + xt^2}} \\ 0 & 0 & 1 \end{bmatrix} \quad (8)$$

> Srho := subs({xt0 = xt1, xt1 = xt2, xt2 = xt3, ut0 = ut1, ut1 = ut2, ut2 = ut3, t0 = t1, t1 = t2, t2 = t3}, rho) :

Maurer Cartan Matrix

> K10 := Srho.MatrixInverse(rho) :

Taylor Expansion for the Maurer Cartan Matrix

> $K10h := subs\Big(t0 = 0, t1 = h, t2 = 2\, h, t3 = 6\, h, xt0 = x, ut0 = u, xt1 = x + h\, xt + \frac{1}{2}\, h^2\, xtt, xt2 = 2\, h^2\, xtt + 2\, h\, xt + x, xt3 = x + 4\, h \cdot xt + 8\, h^2\, xtt, ut1 = u + h\, ut + \frac{1}{2}\, h^2\, utt, ut2 = 2\, h^2\, utt + 2\, h\, ut + u, ut3 = u + 4\, h \cdot ut + 8\, h^2\, utt, K10\Big)$:

> $map(limit, \%, h=0)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(9)}$$

Convergence to the smooth case of the Maurer Cartan matrix

> $diffK10 := map(diff, K10h, h)$:

> $simplify(map(limit, \%, h=0), symbolic)$

$$\begin{bmatrix} 0 & \frac{-ut\,xtt + utt\,xt}{ut^2 + xt^2} & -\sqrt{ut^2 + xt^2} \\ \frac{ut\,xtt - utt\,xt}{ut^2 + xt^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(10)}$$

> $with(Student[LinearAlgebra])$:

> $M := \frac{K10h - IdentityMatrix(3)}{K10h[1,3]}$:

> $simplify(map(limit, \%, h=0), symbolic)$

$$\begin{bmatrix} 0 & \frac{ut\,xtt - utt\,xt}{(ut^2 + xt^2)^{3/2}} & 1 \\ \frac{-ut\,xtt + utt\,xt}{(ut^2 + xt^2)^{3/2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(11)}$$

>

6.3 - Multispaces. Lie group SL(2)

> restart

> with(LinearAlgebra) :

Coefficients

> $Au := ut0$:

> $Bu := \frac{((ut2 - ut0) \cdot t1^2 - (ut1 - ut0) \cdot t2^2)}{t1 \cdot t2 \cdot (t1 - t2)}$:

> $Cu := -2 \frac{((ut2 - ut0) \cdot t1 - (ut1 - ut0) \cdot t2)}{t1 \cdot t2 \cdot (t1 - t2)}$:

Action

> $U0tilde := \frac{(a \cdot ut0 + b)}{(c \cdot ut0 + d)}$:

> $U1tilde := \frac{(a \cdot ut1 + b)}{(c \cdot ut1 + d)}$:

> $U2tilde := \frac{(a \cdot ut2 + b)}{(c \cdot ut2 + d)}$:

Taylor Series

> $Mysubs := t1 = h, t2 = 2 \cdot h, ut0 = u, ut1 = u + h \cdot ut + \frac{1}{2} \cdot h^2 \cdot utt, ut2 = u + 2 \cdot h \cdot ut + 2 \cdot h^2 \cdot utt$:

Multispace action

> MUtilde := simplify(subs({ut0 = U0tilde, ut1 = U1tilde, ut2 = U2tilde}, Au), symbolic)

$$MUtilde := \frac{a\, ut0 + b}{c\, ut0 + d} \tag{1}$$

> MUxtilde := simplify(simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Bu), symbolic), size)

$$MUxtilde := -\frac{1}{(c\, ut2 + d)\,(c\, ut0 + d)\,(c\, ut1 + d)\, t1\, t2\,(t1 - t2)} \Big((a\, d - b\, c)\, \big((ut1\,(ut0 - ut2)\, t1^2 - t2^2\, ut2\,(-ut1 + ut0))\, c + d\,((ut0 - ut2)\, t1^2 - (-ut1 + ut0)\, t2^2) \big) \Big) \tag{2}$$

> MUxxtilde := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Cu), symbolic)

$$MUxxtilde := \frac{2\,\big((ut1\,(ut0 - ut2)\, t1 - t2\, ut2\,(-ut1 + ut0))\, c + d\,((ut0 - ut2)\, t1 - (-ut1 + ut0)\, t2) \big)\,(a\, d - b\, c)}{(c\, ut2 + d)\,(c\, ut0 + d)\,(c\, ut1 + d)\, t1\, t2\,(t1 - t2)} \tag{3}$$

Convergence of the action to the smooth one

> subs(Mysubs, MUtilde)

$$\frac{a\, u + b}{c\, u + d} \tag{4}$$

> subs(Mysubs, MUxtilde) :

> limit(%, h = 0) : subs(a d − b c = 1, %)

$$\frac{ut}{(c\, u + d)^2} \tag{5}$$

> subs(Mysubs, MUxxtilde) :

> limit(%, h = 0) : subs(a d − b c = 1, %)

$$\frac{c\, u\, utt - 2\, c\, ut^2 + d\, utt}{(c\, u + d)^3} \tag{6}$$

> $a \cdot d - b \cdot c - 1$:

> isolate(%, d) :

> assign(%)

Normalisation equations

```
> norm1 := simplify(subs({ut0 = U0tilde, ut1 = U1tilde, ut2 = U2tilde}, Au), symbolic) = 0 :
> norm2 := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Bu),
    symbolic) = 1 :
> norm3 := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Cu),
    symbolic) = 0 :
> solve({norm1, norm2, norm3}, {a, b, c}) :
> allvalues(%) :
> assign(%)
> a := simplify(simplify(a, symbolic), size)
```

$$a := -\frac{\sqrt{t1}\,\sqrt{t2}\,\sqrt{ut1-ut2}}{\sqrt{(ut0-ut2)\,(-ut1+ut0)}\,\sqrt{t1-t2}} \qquad \textbf{(7)}$$

```
> b := simplify(simplify(b, symbolic), size)
```

$$b := \frac{\sqrt{t1}\,\sqrt{t2}\,\sqrt{ut1-ut2}\,ut0}{\sqrt{(ut0-ut2)\,(-ut1+ut0)}\,\sqrt{t1-t2}} \qquad \textbf{(8)}$$

```
> c := simplify(simplify(c, symbolic), size)
```

$$c := -\frac{(t1-t2)\,ut0 - t1\,ut2 + t2\,ut1}{\sqrt{t1-t2}\,\sqrt{t2}\,\sqrt{ut1-ut2}\,\sqrt{-ut1+ut0}\,\sqrt{ut0-ut2}\,\sqrt{t1}} \qquad \textbf{(9)}$$

Moving frame

$$> \text{rho} := simplify\left(Matrix\left(\left[[a, b], \left[c, \frac{(b\cdot c+1)}{a}\right]\right]\right), symbolic\right)$$

$$\rho := \left[\left[-\frac{\sqrt{t1}\,\sqrt{t2}\,\sqrt{ut1-ut2}}{\sqrt{ut0-ut2}\,\sqrt{-ut1+ut0}\,\sqrt{t1-t2}}, \frac{\sqrt{t1}\,\sqrt{t2}\,\sqrt{ut1-ut2}\,ut0}{\sqrt{ut0-ut2}\,\sqrt{-ut1+ut0}\,\sqrt{t1-t2}}\right], \left[-\frac{(t1-t2)\,ut0 - t1\,ut2 + t2\,ut1}{\sqrt{t1-t2}\,\sqrt{t2}\,\sqrt{ut1-ut2}\,\sqrt{-ut1+ut0}\,\sqrt{ut0-ut2}\,\sqrt{t1}}, \frac{\left((-t1+t2)\,ut2 + t1\,ut0\right)ut1 - t2\,ut0\,ut2}{\sqrt{t1-t2}\,\sqrt{t2}\,\sqrt{ut1-ut2}\,\sqrt{-ut1+ut0}\,\sqrt{ut0-ut2}\,\sqrt{t1}}\right]\right] \qquad \textbf{(10)}$$

Convergence of the moving frame to the smooth one

```
> rhoh := subs({Mysubs}, rho) :
> map(simplify, %, symbolic) : map(limit, %, h = 0) :
> simplify(%, symbolic)
```

$$\begin{bmatrix} \frac{1}{\sqrt{ut}} & -\frac{u}{\sqrt{ut}} \\ \frac{utt}{2\,ut^{3/2}} & -\frac{u\,utt - 2\,ut^2}{2\,ut^{3/2}} \end{bmatrix} \qquad \textbf{(11)}$$

Convergence of the coefficients to the variables

```
> subs(Mysubs, Au)
```

$$u \qquad \textbf{(12)}$$

```
> subs(Mysubs, Bu) :
> limit(%, h = 0)
```

$$ut \qquad \textbf{(13)}$$

```
> subs(Mysubs, Cu) :
> limit(%, h = 0)
```

$$utt \qquad \textbf{(14)}$$

Infinitesimal vector fields and convergence

```
> unassign('a','b','c')
> A := simplify(subs({ut0 = U0tilde, ut1 = U1tilde, ut2 = U2tilde}, Au), symbolic) :
```

```
> B := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Bu),
      symbolic) :
> C := simplify(subs({xt0 = X0tilde, ut0 = U0tilde, xt1 = X1tilde, ut1 = U1tilde, xt2 = X2tilde, ut2 = U2tilde}, Cu),
      symbolic) :
> diff(A, a) :
> subs({a = 1, b = 0, c = 0}, %):
```

$$2\, ut0 \qquad (15)$$

```
> subs(Mysubs, %) :
> limit(%, h = 0)
```

$$2\, u \qquad (16)$$

```
> diff(A, b) :
> subs({a = 1, b = 0, c = 0}, %) :
> subs(Mysubs, %) :
> limit(%, h = 0)
```

$$1 \qquad (17)$$

```
> diff(A, c) :
> subs({a = 1, b = 0, c = 0}, %):
```

$$-ut0^2 \qquad (18)$$

```
> subs(Mysubs, %) :
> limit(%, h = 0)
```

$$-u^2 \qquad (19)$$

$$-u^2 \qquad (20)$$

```
> diff(B, a) :
> subs({a = 1, b = 0, c = 0}, %):
```

$$-\frac{2\left((ut0 - ut2)\, t1^2 - (-ut1 + ut0)\, t2^2\right)}{(t1 - t2)\, t1\, t2} \qquad (21)$$

```
> subs(Mysubs, %) :
> limit(%, h = 0)
```

$$2\, ut \qquad (22)$$

```
> diff(B, b) :
> subs({a = 1, b = 0, c = 0}, %):
```

$$0 \qquad (23)$$

```
> subs(Mysubs, %) :
> limit(%, h = 0)
```

$$0 \qquad (24)$$

```
> diff(B, c) :
> simplify(subs({a = 1, b = 0, c = 0}, %), symbolic):
```

$$\frac{(ut0^2 - ut2^2)\, t1^2 + (-ut0^2 + ut1^2)\, t2^2}{t1\, t2\, (t1 - t2)} \qquad (25)$$

```
> subs(Mysubs, %) :
> limit(%, h = 0)
```

$$-2\, u\, ut \qquad (26)$$

```
>
> diff(C, a) :
> simplify(subs({a = 1, b = 0, c = 0}, %), symbolic):
```

$$\frac{(4\, ut0 - 4\, ut2)\, t1 - 4\, (-ut1 + ut0)\, t2}{t1\, t2\, (t1 - t2)} \qquad (27)$$

```
> subs(Mysubs, %) :
```

> $limit(\%, h = 0)$

$$2\, utt \tag{28}$$

> $diff(C, b)$:

> $subs(\{a = 1, b = 0, c = 0\}, \%)$:

$$0 \tag{29}$$

> $subs(Mysubs, \%)$:

> $limit(\%, h = 0)$

$$0 \tag{30}$$

> $diff(C, c)$:

> $simplify(subs(\{a = 1, b = 0, c = 0\}, \%), symbolic)$

$$\frac{(-2\, t1 + 2\, t2)\, ut0^2 + 2\, t1\, ut2^2 - 2\, t2\, ut1^2}{(t1 - t2)\, t1\, t2} \tag{31}$$

> $subs(Mysubs, \%)$:

> $limit(\%, h = 0)$

$$-2\, u\, utt - 2\, ut^2 \tag{32}$$

```
> restart
> with(plots) :
```

Given a spacecurve P(t) we seek the vector V(t) such that
dV/ds = - (P_{ss} . V) P_s
with d/ds = (1/Ds) d/dt
and Ds = sqrt(P_{1,t}^2 + P_{2,t}^2+P_{3,t}^2)
We can divide out by one 1/Ds to simplify the ODE

```
> V := Matrix([[V1(t)], [V2(t)], [V3(t)]]) :
```

$$> \ myDs := \sqrt{\left(\frac{d}{dt} P1(t)\right)^2 + \left(\frac{d}{dt} P2(t)\right)^2 + \left(\frac{d}{dt} P3(t)\right)^2} :$$

P_s

$$> \ myPp := Matrix\left(\left[\left[\frac{diff(P1(t), t)}{myDs}\right], \left[\frac{diff(P2(t), t)}{myDs}\right], \left[\frac{diff(P3(t), t)}{myDs}\right]\right]\right) :$$

d/dt (P_s)

```
> myPpp := map(z→diff(z, t), myPp) :
> with(LinearAlgebra) :
```

We write the RHS of the eqn in matrix form

```
> myA := myPpTranspose(myPpp) :
> myEqRHS := -myA · V :
```

Here are the eqns

```
> myeqs := {diff(V1(t), t) = myEqRHS[1, 1], diff(V2(t), t) = myEqRHS[2, 1], diff(V3(t), t) = myEqRHS[3, 1]} :
```

We convert to being a procedure which inputs the given space curve

```
> myVeq :=proc(Q) global myeqs;subs(P1(t) = Q[1], P2(t) = Q[2], P3(t) = Q[3], myeqs) : map(eval, %) endproc;
          myVeq := proc(Q) global myeqs; subs(P1(t) = Q[1], P2(t) = Q[2], P3(t) = Q[3], myeqs); map(eval, `%`) end proc      (1)
```

We need the initial data to be orthogonal to P_s=myPp at t=0. We try for a particular spacecurve, here called myQ

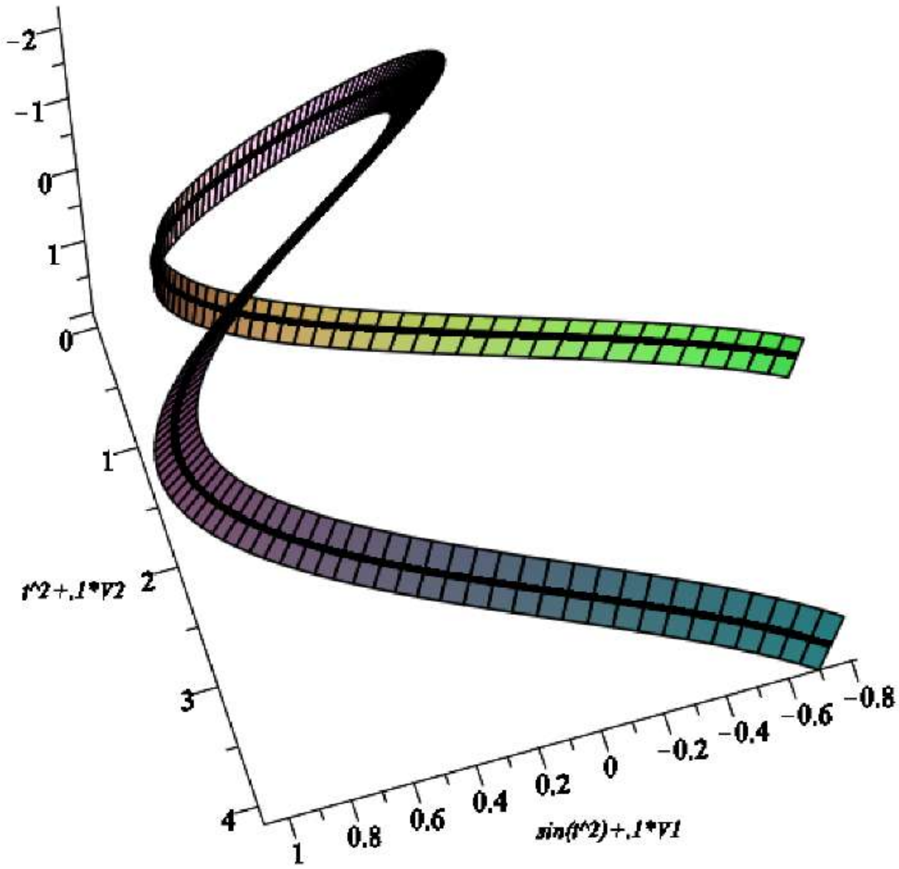
```
> myQ := [sin(t^2), t^2, t] :
> plotCurve := spacecurve(myQ, t=-2 .. 2, color= black, thickness= 7) :
> subs(t=0, map(z→diff(z, t), myQ));
                                   [0, 0, 1]                                  (2)
```

so V(0) orthogonal to P_s(0) is (1,1,1)
We use dsolve,numeric and odeplot in the normal way

```
> mysol := dsolve({op(myVeq(myQ)), V1(0) = 1, V2(0) = 1, V3(0) = 1}, range = -2..2, numeric) :

> plot1 := odeplot(mysol, [myQ[1] + 0.1· V1(t), myQ[2] + 0.1· V2(t), myQ[3] + 0.1· V3(t)], color = black) :
> plot3 := odeplot(mysol, [myQ[1] - 0.1· V1(t), myQ[2] - 0.1· V2(t), myQ[3] - 0.1· V3(t)], color = black) :
>
```

This is the Normal frame.

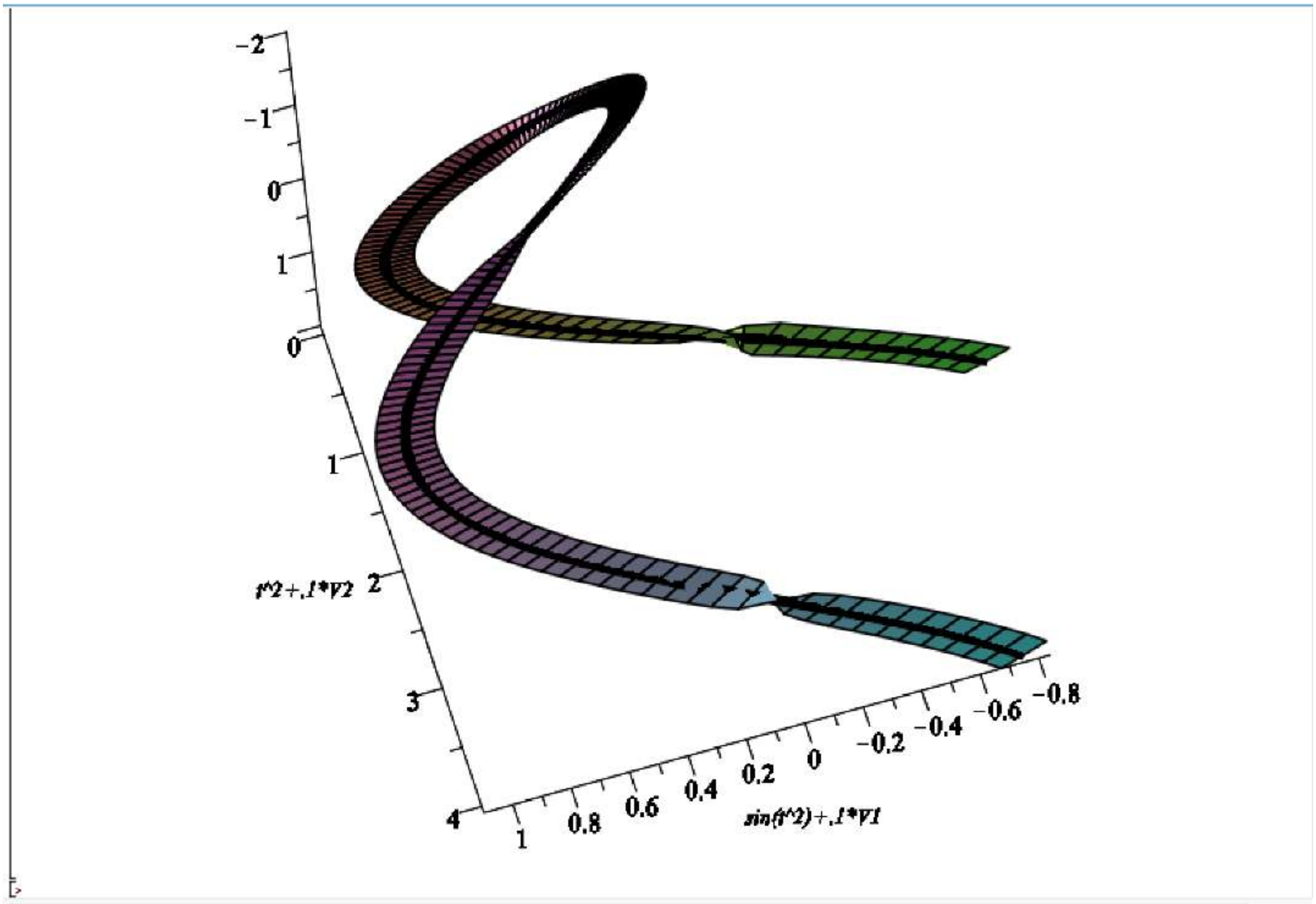
```
> with(plottools) :
> M1 := op(3, getdata(plot1)) : M2 := op(3, getdata(plot3)) :
  A := Array(1..200, 1..4, 1..3, datatype = float) :
  for i to 200 do
    A[i, 1, 1..3] := M1[i, 1..3];
    A[i, 2, 1..3] := M1[i + 1, 1..3];
    A[i, 3, 1..3] := M2[i + 1, 1..3];
    A[i, 4, 1..3] := M2[i, 1..3];
  end do:
  plot2NF := display(seq(polygon(A[i]), i = 1..200), style = patch) :

> display(plotCurve, plot2NF, axes = framed, orientation = [72, -30, 4]);
```



t^2+.1*V2
sin(t^2)+.1*V1

This is the Frenet-Serret frame.

```
> myC :=proc(A, B) [A[2]·B[3] − B[2]·A[3], A[3]·B[1] − A[1]·B[3], A[1]·B[2] − A[2]·B[1]] endproc;
          myC := proc(A, B) [A[2]*B[3] − A[3]*B[2], −A[1]*B[3] + A[3]*B[1], A[1]*B[2] − A[2]*B[1]] end proc        (3)
> mySum :=proc(A, B) [A[1] + B[1], A[2] + B[2], A[3] + B[3]] endproc;
          mySum := proc(A, B) [A[1] + B[1], A[2] + B[2], A[3] + B[3]] end proc        (4)
> DsQ := sqrt(diff(myQ[1], t)^2 + diff(myQ[2], t)^2 + diff(myQ[3], t)^2) :
> Qp := map(z -> diff(z, t)/DsQ, myQ) :
>
> Qpp1 := map(z -> diff(z, t), Qp) :
>
> Qpplength := sqrt(Qpp1[1]^2 + Qpp1[2]^2 + Qpp1[3]^2) :
> Qpp := map(z -> z/Qpplength, Qpp1) :
> hh1 := seq(mySum(lambda·Qpp, myQ), lambda = [0.1]) :
> hh3 := seq(mySum(lambda·Qpp, myQ), lambda = [-0.1]) :
> plotCurvepp1 := spacecurve({hh1}, t = -2..2, color = black) :
> plotCurvepp3 := spacecurve({hh3}, t = -2..2, color = black) :
>
> with(plottools) :
> M1 := op(3, getdata(plotCurvepp1)) : M2 := op(3, getdata(plotCurvepp3)) :
  A := Array(1..199, 1..4, 1..3, datatype = float) :
  for i to 199 do
    A[i, 1, 1..3] := M1[i, 1..3];
    A[i, 2, 1..3] := M1[i + 1, 1..3];
    A[i, 3, 1..3] := M2[i + 1, 1..3];
    A[i, 4, 1..3] := M2[i, 1..3];
  end do:
  plot2NF := display(seq(polygon(A[i]), i = 1..199), style = patch) :
>
> display(plotCurve, plot2NF, axes = framed, orientation = [72, -30, 4]);
```



t^2+.1*V2
sin(t^2)+.1*V1

Lagrangian involving kappa2/kappa1

```
> restart :
> Digits := 20 :
> C :='C':CC2 :='CC2':
```

Lagrangian, euler operator and lambda

```
> L := 1/2·(kappa2/kappa1)^2 :
> E1 := simplify(diff(L, kappa1)) :
> E2 := simplify(diff(L, kappa2)) :
> E1 := subs(kappa1 = kappa1(s), kappa2 = kappa2(s), E1) :
> E2 := subs(kappa1 = kappa1(s), kappa2 = kappa2(s), E2) :
> Lsub := -kappa1(s)·E1 - kappa2(s)·E2 + subs(kappa1 = kappa1(s), kappa2 = kappa2(s), L) :
```

We call the components of the vector of invariants wi so I can set vi(s)=wi in the list of equations

```
> w1 := Lsub :
> w10 := subs(s = 0, Lsub) :
> w2 := simplify(-(diff(E1, s) + mu(s)·kappa2(s))) :
> convert(w2, D) : w20 := subs(s = 0, %) :
> w3 := simplify(-diff(E2, s) + mu(s)·kappa1(s)) :
> convert(w3, D) : w30 := subs(s = 0, %) :
> w4 := mu(s) :
> w40 := mu(0) :
> w5 := E2 :
> w50 := subs(s = 0, w5) :
> w6 := E1 :
> w60 := subs(s = 0, w6) :
```

Euler Lagrange Equations

```
> eq1 := simplify(diff(E1, s, s) + diff(mu(s)·kappa2(s), s) - kappa1(s)·Lsub) :
> eq2 := simplify(diff(E2, s, s) - diff(mu(s)·κ1(s), s) - kappa2(s)·Lsub) :
```

First Integrals

```
> eq5 := simplify(expand((v1(s)^2 + v2(s)^2 + v3(s)^2))) - c1^2 - c2^2 - c3^2 :
> eq6 := v1(s)·v4(s) - v2(s)·v5(s) + v3(s)·v6(s) - 'CC2':
```

We now enter the code to make sigma according to method written up in the thesis

```
> myA := (x1, x2, x3, x4) → Matrix([[x1^2 + x2^2 - x3^2 - x4^2, -2·(x1·x4 - x2·x3), 2·(x1·x3 + x2·x4)], [2·(x1·x4 + x2·x3), x1^2 - x2^2 + x3^2 - x4^2, -2·(x1·x2 - x3·x4)], [-2·(x1·x3 - x2·x4), 2·(x1·x2 + x3·x4), x1^2 - x2^2 - x3^2 + x4^2]]) :
> with(LinearAlgebra) :
```

We do some basic checks

```
> Determinant(myA(x1, x2, x3, x4)) : factor(%);
```

$$(x1^2 + x2^2 + x3^2 + x4^2)^3 \qquad (1)$$

We see the axis is (x2,x3,x4)

```
> myA(x1, x2, x3, x4).Matrix([[x2], [x3], [x4]]) : map(factor, %);
```

$$\begin{bmatrix} x2\,(x1^2 + x2^2 + x3^2 + x4^2) \\ x3\,(x1^2 + x2^2 + x3^2 + x4^2) \\ x4\,(x1^2 + x2^2 + x3^2 + x4^2) \end{bmatrix} \qquad (2)$$

This makes a rotation matrix which has a given angle about a given axis

```
> myR := proc(psi, a1, a2, a3) local C; C := sqrt(a1^2 + a2^2 + a3^2); myA(cos(psi/2), sin(psi/2)·a1/C, sin(psi/2)·a2/C, sin(psi/2)·a3/C) end proc;
```

myR := **proc**(psi, a1, a2, a3) **local** C; C := sqrt(a1^2 + a2^2 + a3^2); myA(cos(1/2*psi), sin(1/2*psi)*a1/C, sin(1/2*psi)*a2/C, sin(1/2*psi)*a3/C) **end proc** (3)

Sigma sends c=(c1,c2,c3) to (0,0,C), and then rotates abut (0,0,C) and then sends (0,0,C) to (v1,v2,v3)

```
> map(simplify, myR(Pi, v1(s), v2(s), v3(s) + C).myR(psi(s), 0, 0, C).myR(Pi, c1, c2, c3 + C), {v1(s)^2 + v2(s)^2 + v3(s)^2 - C^2, c1^2 + c2^2 + c3^2 - C^2}, {v1(s), c1}) : map(simplify, %, symbolic) : map(combine, %, trig) :
> sigma := %:
```

We know that Z'=sigma[1,3]

```
> eqZ := diff(Z(s), s) = sigma[1, 3] :
```

this makes the eqn for psi - no need to d this every time
sigma_s.sigma^{-1}, then subs derivs of the vi's

```
> map(z → diff(z, s), sigma).myR(Pi, c1, c2, c3 + C).myR(-psi(s), 0, 0, C).myR(Pi, v1(s), v2(s), v3(s) + C) : subs(diff(v1(s), s) = kappa1(s)·v2(s) + kappa2(s)·v3(s), diff(v2(s), s) = -kappa1(s)·v1(s), diff(v3(s), s) = kappa2(s)·v1(s), %) :
> fred := %:
> map(simplify, fred, {v1(s)^2 + v2(s)^2 + v3(s)^2 - C^2, c1^2 + c2^2 + c3^2 - C^2}, {v1(s), c1, v2(s)}) : map(combine, %, trig) :
> BigS := %:
> solve(BigS[1, 2] - kappa1(s), diff(psi(s), s)) : simplify(%, symbolic);
```

$$(-v3(s) - C)\,\kappa1(s) + v2(s)\,\kappa2(s)$$

```
> BigS := %:
> solve(BigS[1, 2] - kappa1(s), diff(psi(s), s)) : simplify(%, symbolic);
```

$$\frac{(-v3(s) - C)\,\kappa 1(s) + v2(s)\,\kappa 2(s)}{v3(s) + C} \tag{4}$$

We use this equaiton for sigma, relies on vector (v1,v2,v3) stayng away from (0,0,C)

```
> eq7 := diff(psi(s), s) = -kappa1(s) + (v2(s)·kappa2(s))/(v3(s) + C) :
>
> Ctemp := sqrt(c1^2 + c2^2 + c3^2) :
> CC2temp := c1·c4 - c2·c5 + c3·c6 :
```

This is where we make the choice of initial data

```
> myICsubs := {kappa1(0) = 1, D(kappa1)(0) = 1, kappa2(0) = 1/2, D(kappa2)(0) = 1, mu(0) = 1} :
> myICs := map(eval, subs(op(myICsubs), {v1(0) = w10, v2(0) = w20, v3(0) = w30, v4(0) = w40, v5(0) = w50, v6(0) = w60, psi(0) = 0, Z(0) = 1}))
```

$$myICs := \left\{Z(0) = 1,\ \psi(0) = 0,\ v1(0) = \frac{1}{8},\ v2(0) = -\frac{1}{4},\ v3(0) = 1,\ v4(0) = 1,\ v5(0) = \frac{1}{2},\ v6(0) = -\frac{1}{4}\right\} \tag{5}$$

This guarantees the second first integral condition

```
> mycs := map(eval, subs(op(myICsubs), {c1 = w10, c2 = w20, c3 = w30, c4 = w40, c5 = w50, c6 = w60})) :
> C := simplify(subs(op(mycs), Ctemp));
```

$$C := \frac{\sqrt{69}}{8} \tag{6}$$

```
> CC2 := simplify(subs(op(mycs), CC2temp));
```

$$CC2 := 0 \tag{7}$$

This needs to be far from zero

```
> test := subs(myICs union mycs, v3(0) + C) : evalf(expand(%));
```

$$2.0383279828647593566 \tag{8}$$

We don't need eq5, the first, first integral

```
> AllEqs := subs(op(mycs), {eq1, eq2, eq6, eq7, v1(s) - w1, v2(s) - w2, v3(s) - w3, v4(s) - w4, v5(s) - w5, v6(s) - w6, eqZ}) :
```

The range is all that Maple can do before running into singularities

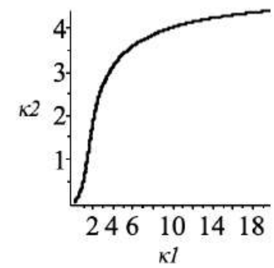
```
> dsolve(AllEqs union myICs union myICsubs, numeric, range = -6.1363..1.087);
```

proc(*x_rkf45_dae*) ... **end proc** (9)

```
> mysol := %:
> with(plots) :
```

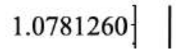
This is (kappa1(s), kappa2(s))

```
> odeplot(mysol, [kappa1(s), kappa2(s)], color = black, refine = 2);
```

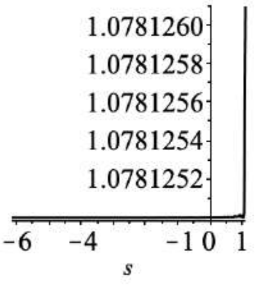


We check the first integral

```
> odeplot(mysol, [s, v1(s)^2 + v2(s)^2 + v3(s)^2], color = black);
```

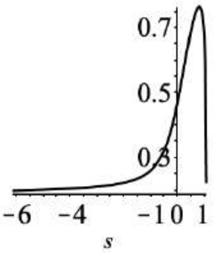


```
> odeplot(mysol, [s, v1(s)^2 + v2(s)^2 + v3(s)^2], color = black);
```



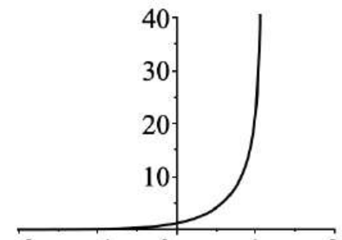
This is (s, theta(s))

```
> odeplot(mysol, [s, arctan(kappa2(s)/kappa1(s))], color = black);
```



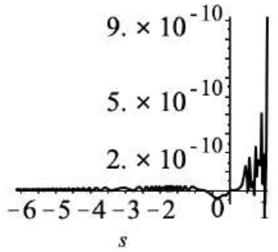
This is (s,kappa(s))

```
> display(odeplot(mysol, [s, kappa1(s)^2 + kappa2(s)^2], color = black), view = [-2..2, -1..40]);
```



We check the second, first integral

```
> odeplot(mysol, [s, v1(s)·v4(s) − v2(s)·v5(s) + v3(s)·v6(s)], color=black);
```

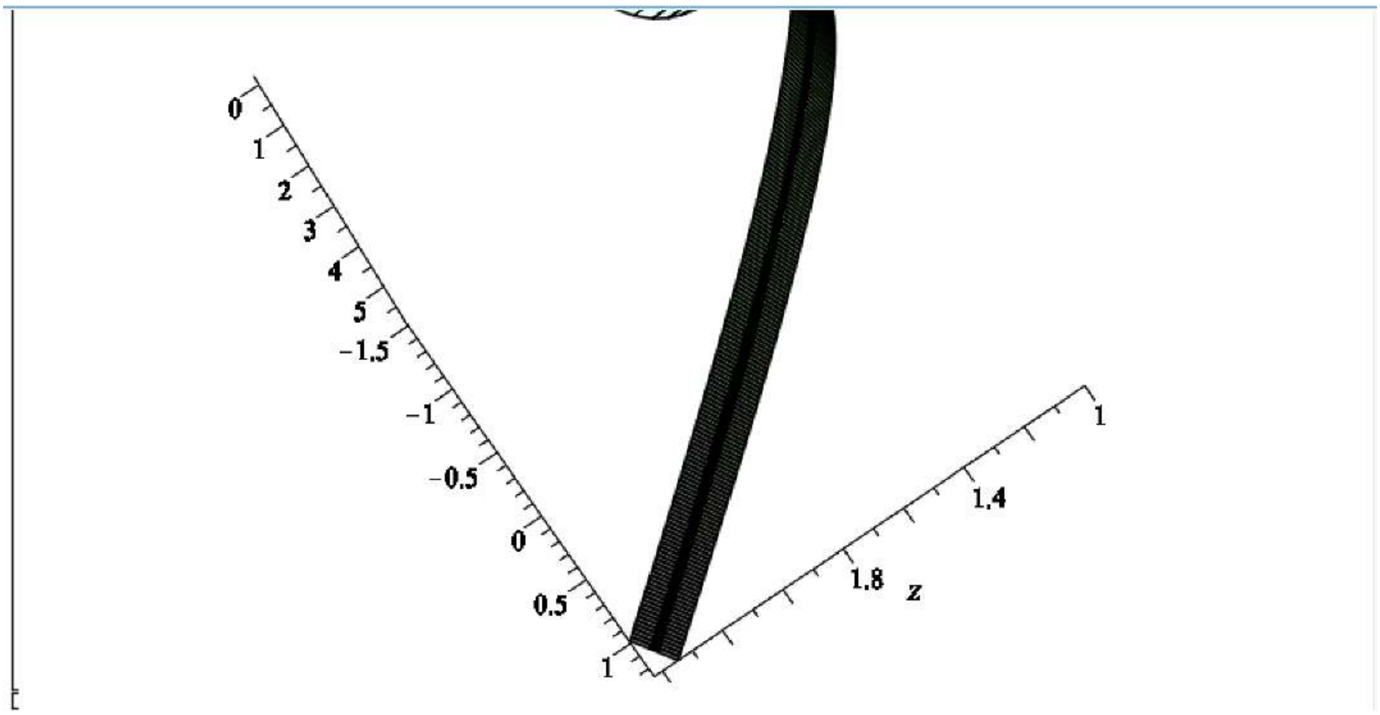


We now want X,Y,Z and the V vector

```
> mySigmaDc2 := subs(op(mycs), sigma·Matrix([[c4], [−c5], [c6]])) :
> myX := 1/v3(s) · (v1(s)·Z(s) + v5(s) + mySigmaDc2[2, 1]) :
> myY := 1/v3(s) · (v2(s)·Z(s) + v4(s) − mySigmaDc2[1, 1]) :
```

We now obtain the V vector from sigma

```
> myV1 := subs(op(mycs), sigma[2, 1]) : myV2 := subs(op(mycs), sigma[2, 2]) : myV3 := subs(op(mycs), sigma[2, 3]) :

> plot1 := odeplot(mysol, [myX, myY, Z(s)], color=black,thickness=7) :
> plot234 := odeplot(mysol, [myX + 0.2·myV1, myY + 0.2·myV2, Z(s) + 0.2·myV3], color=black) :
> plot27 := odeplot(mysol, [myX − 0.2·myV1, myY − 0.2·myV2, Z(s) − 0.2·myV3], color=black) :
> plot24 := odeplot(mysol, [myX + 1/4·myV1, myY + 1/4·myV2, Z(s) + 1/4·myV3], color=black) :
> plot22 := odeplot(mysol, [myX + 1/2·myV1, myY + 1/2·myV2, Z(s) + 1/2·myV3], color=black) :
> with(plottools) :
> M1 := op(3,getdata(plot234)) :M2 := op(3,getdata(plot27)) :
  A := Array(1..201,1..4,1..3,datatype=float) :
  for i to 201 do
   A[i,1,1..3] := M1[i,1..3];
   A[i,2,1..3] := M1[i+1,1..3];
   A[i,3,1..3] := M2[i+1,1..3];
   A[i,4,1..3] := M2[i,1..3];
  end do:
  plot1NF := display(seq(polygon(A[i]),i=1..201),style=patch) :

>
> display(plot1, plot1NF, axes=framed, orientation = [55,−34,−88]);
```


0
1
2
3
4
5
-1.5
-1
-0.5
0
0.5
1
1
1.4
1.8
z

We now make a similar plot of the Frenet Serret frame
Since we know all the derivatives explicitly, we can differentiate exactly

```
> subs(mycs, sigma[1, 1]) : diff(%, s) : subs(eq7, diff(v1(s), s) = kappa1(s)·v2(s) + kappa2(s)·v3(s), diff(v2(s), s) = -kappa1(s)·v1(s), diff(v3(s), s) = -kappa2(s)·v1(s), %) : Xpp := simplify(%) :
> subs(mycs, sigma[1, 2]) : diff(%, s) : subs(eq7, diff(v1(s), s) = kappa1(s)·v2(s) + kappa2(s)·v3(s), diff(v2(s), s) = -kappa1(s)·v1(s), diff(v3(s), s) = -kappa2(s)·v1(s), %) : Ypp := simplify(%) :
> subs(mycs, sigma[1, 3]) : diff(%, s) : subs(eq7, diff(v1(s), s) = kappa1(s)·v2(s) + kappa2(s)·v3(s), diff(v2(s), s) = -kappa1(s)·v1(s), diff(v3(s), s) = -kappa2(s)·v1(s), %) : Zpp := simplify(%) :
```

We choose to plot the third vector in the FS frame, here P' cross P'' is called FSv3

```
> FSv3 := subs(op(mycs), [sigma[1, 2]·Zpp - sigma[1, 3]·Ypp, sigma[1, 3]·Xpp - sigma[1, 1]·Zpp, sigma[1, 1]·Ypp - sigma[1, 2]·Xpp]) :
> lengthFSv3qrd := subs(op(mycs), (FSv3[1]^2 + FSv3[2]^2 + FSv3[3]^2)) :
```

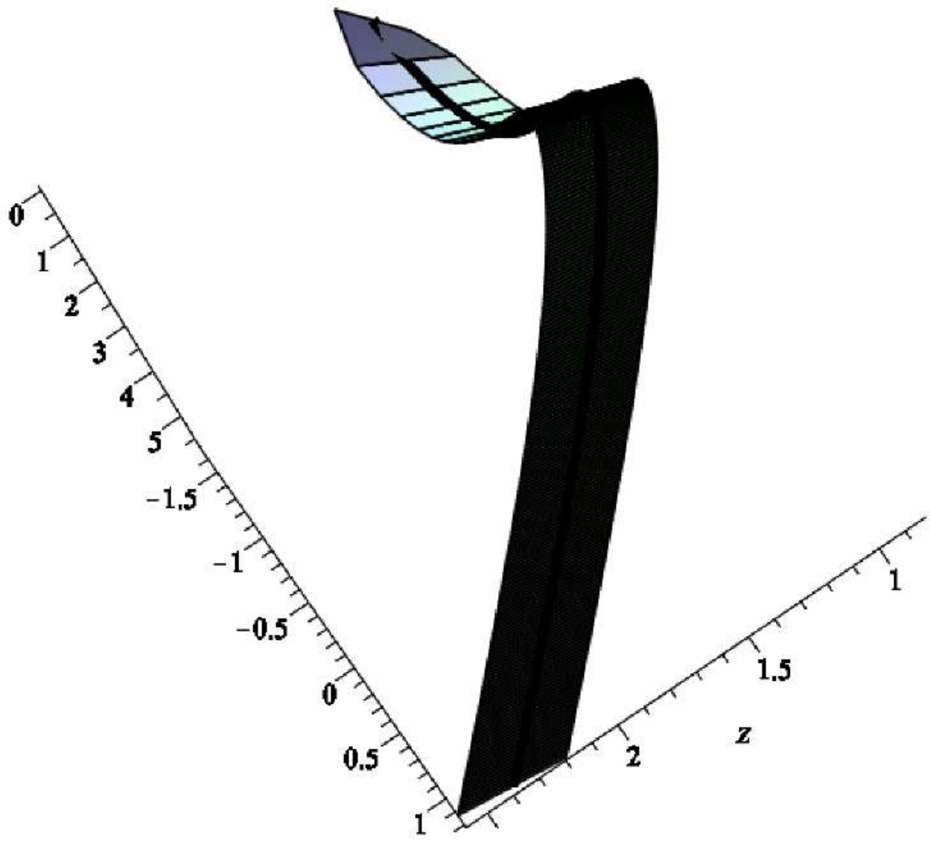
$$> \; plot4FS := odeplot\left(mysol, \left[myX + 0.2\cdot\frac{1}{\mathrm{sqrt}(lengthFSv3qrd)}\cdot FSv3[1], myY + 0.2\cdot\frac{1}{\mathrm{sqrt}(lengthFSv3qrd)}\cdot FSv3[2], Z(s) + 0.2\cdot\frac{1}{\mathrm{sqrt}(lengthFSv3qrd)}\cdot FSv3[3] \right], color = black \right) :$$

$$> \; plot5FS := odeplot\left(mysol, \left[myX - 0.2\cdot\frac{1}{\mathrm{sqrt}(lengthFSv3qrd)}\cdot FSv3[1], myY - 0.2\cdot\frac{1}{\mathrm{sqrt}(lengthFSv3qrd)}\cdot FSv3[2], Z(s) - 0.2\cdot\frac{1}{\mathrm{sqrt}(lengthFSv3qrd)}\cdot FSv3[3] \right], color = black \right) :$$

```
>
```

We now generate the sweep surface for the FS frame

```
> M11 := op(3, getdata(plot4FS)) : M12 := op(3, getdata(plot5FS)) :
  B := Array(1..N, 1..4, 1..3, datatype = float) : N := 201 :
  for i to N do
    B[i, 1, 1..3] := M11[i, 1..3];
    B[i, 2, 1..3] := M11[i + 1, 1..3];
    B[i, 3, 1..3] := M12[i + 1, 1..3];
    B[i, 4, 1..3] := M12[i, 1..3];
  end do:
  plot26FSa := display(seq(polygon(B[i]), i = 1..N), style = patch) :
>
>
> display(plot26FSa, plot1, axes = framed, orientation = [55, -34, -88]);
```


0
1
2
3
4
5
−1.5
−1
−0.5
0
0.5
1
z
2
1.5
1

Application in biology first example. We follow the method used in the example regarding an invariant lagrangian depending on kappa2/kappa1 **-7.5.2 (first example)**

```
> restart
> with(LinearAlgebra) :
> Digits := 20 :
> C :='C': CC2 :='CC2':
```

Euler Lagrange equations

```
> L := kappa1·kappa2s − kappa2·kappa1s :
> Ls := kappa1(s)·diff(kappa2(s), s) − kappa2(s)·diff(kappa1(s), s)
```

$$Ls := \kappa 1(s)\left(\frac{d}{ds}\kappa 2(s)\right) - \kappa 2(s)\left(\frac{d}{ds}\kappa 1(s)\right) \quad (1)$$

```
> EK1 := subs({kappa1s = diff(kappa1(s), s), kappa2s = diff(kappa2(s), s)}, diff(L, kappa1))
    -diff(subs({kappa1 = kappa1(s), kappa2 = kappa2(s)}, diff(L, kappa1s)), s)
```

$$EK1 := 2\frac{d}{ds}\kappa 2(s) \quad (2)$$

```
> EK2 := subs({kappa1s = diff(kappa1(s), s), kappa2s = diff(kappa2(s), s)}, diff(L, kappa2))
    -diff(subs({kappa1 = kappa1(s), kappa2 = kappa2(s)}, diff(L, kappa2s)), s)
```

$$EK2 := -2\frac{d}{ds}\kappa 1(s) \quad (3)$$

```
> mus := EK1·kappa2(s) − EK2·kappa1(s)
```

$$mus := 2\left(\frac{d}{ds}\kappa 2(s)\right)\kappa 2(s) + 2\left(\frac{d}{ds}\kappa 1(s)\right)\kappa 1(s) \quad (4)$$

```
> mu := int(%, s)
```

$$\mu := \kappa 1(s)^2 + \kappa 2(s)^2 \quad (5)$$

```
> lambda := -kappa1(s)·EK1 − kappa2(s)·EK2 + Ls − (subs({kappa1 = kappa1(s), kappa2 = kappa2(s)},
    diff(L, kappa1s))·diff(kappa1(s), s) + subs({kappa1 = kappa1(s), kappa2 = kappa2(s)}, diff(L,
    kappa2s))·diff(kappa2(s), s))
```

$$\lambda := -2\kappa 1(s)\left(\frac{d}{ds}\kappa 2(s)\right) + 2\kappa 2(s)\left(\frac{d}{ds}\kappa 1(s)\right) \quad (6)$$

```
> eq1 := collect(simplify(diff(EK1, s, s) + diff(kappa2(s)·mu, s) − kappa1(s)·lambda, symbolic), d/ds κ2(s))
```

$$eq1 := \left(3\kappa 1(s)^2 + 3\kappa 2(s)^2\right)\left(\frac{d}{ds}\kappa 2(s)\right) + 2\frac{d^3}{ds^3}\kappa 2(s) \quad (7)$$

```
> eq2 := collect(simplify(diff(EK2, s, s) -diff(kappa1(s)·mu, s) -κ2(s)·lambda, symbolic), d/ds κ1(s))
```

$$eq2 := \left(-3\kappa 1(s)^2 - 3\kappa 2(s)^2\right)\left(\frac{d}{ds}\kappa 1(s)\right) - 2\frac{d^3}{ds^3}\kappa 1(s) \quad (8)$$

Vector of invariants

```
> w1 := lambda;
```

$$w1 := -2\kappa 1(s)\left(\frac{d}{ds}\kappa 2(s)\right) + 2\kappa 2(s)\left(\frac{d}{ds}\kappa 1(s)\right) \quad (9)$$

```
> convert(w1, D) : w10 := subs(s = 0, %);
```

$$w10 := -2\kappa 1(0)\,D(\kappa 2)(0) + 2\kappa 2(0)\,D(\kappa 1)(0) \quad (10)$$

```
> w2 := simplify(-(diff(EK1, s) + mu·kappa2(s)));
```

$$w2 := -\kappa 1(s)^2\kappa 2(s) - \kappa 2(s)^3 - 2\frac{d^2}{ds^2}\kappa 2(s) \quad (11)$$

```
> convert(w2, D) : w20 := subs(s = 0, %);
```

$$w20 := -\kappa 1(0)^2\kappa 2(0) - \kappa 2(0)^3 - 2D^{(2)}(\kappa 2)(0) \quad (12)$$

```
> w3 := simplify(-diff(EK2, s) + mu·kappa1(s));
```

$$w3 := \kappa 1(s)^3 + \kappa 1(s)\kappa 2(s)^2 + 2\frac{d^2}{ds^2}\kappa 1(s) \quad (13)$$

> $convert(w3, D) : w30 := subs(s=0, \%);$

$$w30 := \kappa 1(0)^3 + \kappa 1(0)\,\kappa 2(0)^2 + 2\,D^{(2)}(\kappa 1)(0) \quad \textbf{(14)}$$

> $w4 := mu;$

$$w4 := \kappa 1(s)^2 + \kappa 2(s)^2 \quad \textbf{(15)}$$

> $convert(w4, D) : w40 := subs(s=0, \%);$

$$w40 := \kappa 1(0)^2 + \kappa 2(0)^2 \quad \textbf{(16)}$$

> $w5 := EK2;$

$$w5 := -2\,\frac{d}{ds}\,\kappa 1(s) \quad \textbf{(17)}$$

> $convert(w5, D) : w50 := subs(s=0, \%);$

$$w50 := -2\,D(\kappa 1)(0) \quad \textbf{(18)}$$

> $w6 := EK1;$

$$w6 := 2\,\frac{d}{ds}\,\kappa 2(s) \quad \textbf{(19)}$$

> $convert(w6, D) : w60 := subs(s=0, \%);$

$$w60 := 2\,D(\kappa 2)(0) \quad \textbf{(20)}$$

First integrals

> $eq5 := simplify(expand((v1(s)^2 + v2(s)^2 + v3(s)^2))) - c1^2 - c2^2 - c3^2 :$

> $eq6 := v1(s)\cdot v4(s) - v2(s)\cdot v5(s) + v3(s)\cdot v6(s) - 'CC2':$

d/ds psi(s)

> $eq7 := diff(\mathrm{psi}(s), s) = -kappa1(s) + \dfrac{v2(s)\cdot kappa2(s)}{v3(s) + C} :$

Caley Map

> $myA := (x1, x2, x3, x4) \rightarrow Matrix([[x1^2 + x2^2 - x3^2 - x4^2, -2\cdot(x1\cdot x4 - x2\cdot x3), 2\cdot(x1\cdot x3 + x2\cdot x4)], [2\cdot(x1\cdot x4 + x2\cdot x3), x1^2 - x2^2 + x3^2 - x4^2, -2\cdot(x1\cdot x2 - x3\cdot x4)], [-2\cdot(x1\cdot x3 - x2\cdot x4), 2\cdot(x1\cdot x2 + x3\cdot x4), x1^2 - x2^2 - x3^2 + x4^2]]) :$

This makes a rotation matrix which has a given angle about a given axis

> $myR := \mathbf{proc}(\mathrm{psi}, a1, a2, a3)\ \mathbf{local}\ C; C := \mathrm{sqrt}(a1^2 + a2^2 + a3^2); myA\left(\cos\left(\frac{\mathrm{psi}}{2}\right), \frac{\sin\left(\frac{\mathrm{psi}}{2}\right)\cdot a1}{C}, \frac{\sin\left(\frac{\mathrm{psi}}{2}\right)\cdot a2}{C}, \frac{\sin\left(\frac{\mathrm{psi}}{2}\right)\cdot a3}{C}\right)\ \mathbf{end\ proc};$

```
myR := proc(psi, a1, a2, a3)
    local C;
    C := sqrt(a1^2 + a2^2 + a3^2);
    myA(cos(1/2 * psi), sin(1/2 * psi) * a1/C, sin(1/2 * psi) * a2/C, sin(1/2 * psi) * a3/C)
end proc
```
(21)

Sigma sends c=(c1,c2,c3) to (0,0,C), and then rotates abut (0,0,C) and then sends (0,0,C) to (v1,v2,v3)

> $map(simplify, myR(\mathrm{Pi}, v1(s), v2(s), v3(s) + C).myR(\mathrm{psi}(s), 0, 0, C).myR(\mathrm{Pi}, c1, c2, c3 + C), \{v1(s)^2 + v2(s)^2 + v3(s)^2 - C^2, c1^2 + c2^2 + c3^2 - C^2\}, \{v1(s), c1\}) : map(simplify, \%, symbolic) : map(combine, \%, trig) :$

> $\mathrm{sigma} := \% :$

we know that Z'=sigma[1,3]

> $eqZ := diff(Z(s), s) = \mathrm{sigma}[1, 3] :$

> $Ctemp := \mathrm{sqrt}(c1^2 + c2^2 + c3^2) :$

> $CC2temp := c1\cdot c4 - c2\cdot c5 + c3\cdot c6 :$

Initial data

> $myICsubs := \{kappa1(0) = 1, D(kappa1)(0) = 1, D^{(2)}(kappa1)(0) = 1, kappa2(0) = \frac{1}{2}, D(kappa2)(0) = 1,$

$D^{(2)}(kappa2)(0) = 1\}$:

```
> myICs := map(eval, subs(op(myICsubs), {v1(0) = w10, v2(0) = w20, v3(0) = w30, v4(0) = w40, v5(0) = w50,
    v6(0) = w60, psi(0) = 0, Z(0) = 1}));
```

$$myICs := \left\{Z(0) = 1, \psi(0) = 0, v1(0) = -1, v2(0) = -\frac{21}{8}, v3(0) = \frac{13}{4}, v4(0) = \frac{5}{4}, v5(0) = -2, v6(0) = 2\right\} \quad (22)$$

This guarantees the second first integral condition

```
> mycs := map(eval, subs(op(myICsubs), {c1 = w10, c2 = w20, c3 = w30, c4 = w40, c5 = w50, c6 = w60}));
```

$$mycs := \left\{c1 = -1, c2 = -\frac{21}{8}, c3 = \frac{13}{4}, c4 = \frac{5}{4}, c5 = -2, c6 = 2\right\} \quad (23)$$

```
> C := simplify(subs(op(mycs), Ctemp));
```

$$C := \frac{\sqrt{1181}}{8} \quad (24)$$

```
> CC2 := simplify(subs(op(mycs), CC2temp));
```

$$CC2 := 0 \quad (25)$$

This needs to be far from zero

```
> test := subs(myICs union mycs, v3(0) + C) : evalf(expand(%));
```

$$7.5457100693598957086 \quad (26)$$

```
> AllEqs := subs(op(mycs), {eq1, eq2, eq6, eq7, v1(s) - w1, v2(s) - w2, v3(s) - w3, v5(s) - w5, v6(s) - w6,
    eqZ}) :
```

The range is all that Maple can do before running into singularities

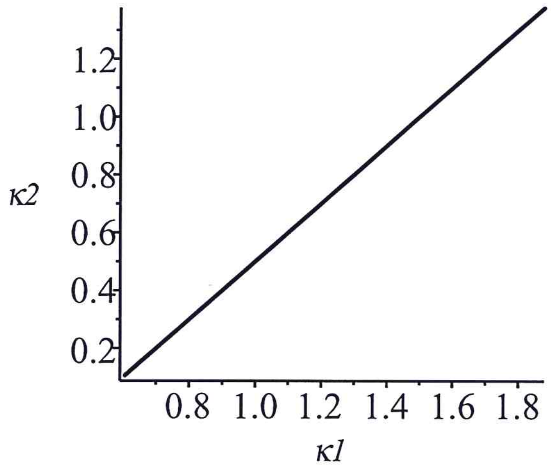
```
> dsolve(AllEqs union myICs union myICsubs, numeric, range =-6.15 ..3);
```

proc(x_rkf45_dae) ... **end proc** (27)

```
> mysol := % :
> with(plots) :
```

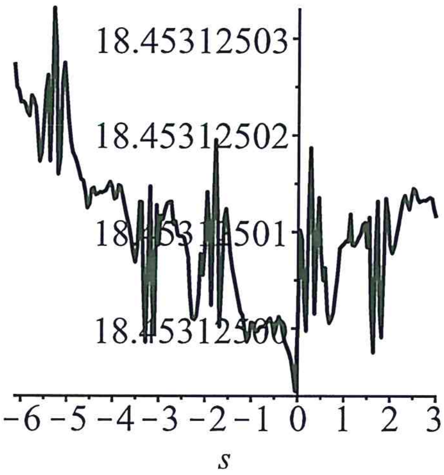
This is (kappa1(s),kappa2(s))

```
> odeplot(mysol, [kappa1(s), kappa2(s)], color = black);
```



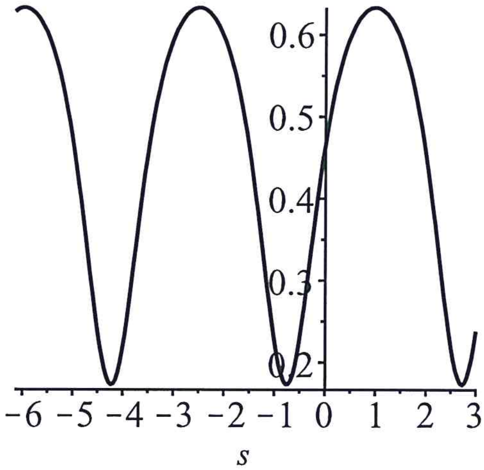
We check the first integral

```
> odeplot(mysol, [s, v1(s)^2 + v2(s)^2 + v3(s)^2], color = black);
```



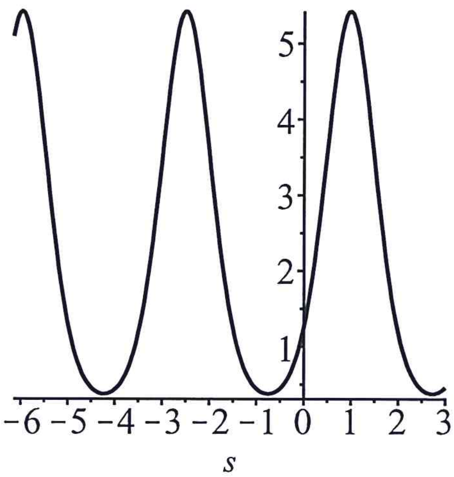
This is (s, theta(s))

> $odeplot\left(mysol, \left[s, \arctan\left(\frac{kappa2(s)}{kappa1(s)}\right)\right], color = black\right);$



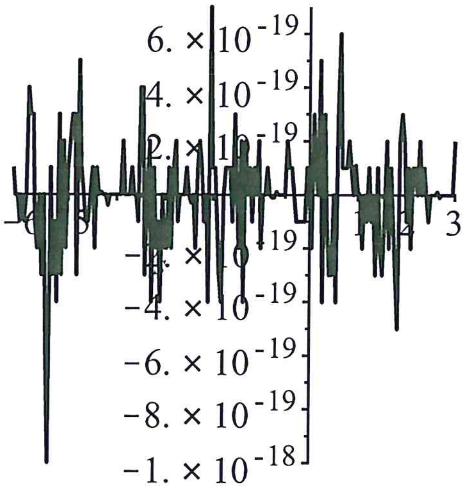
This is (s, kappa(s)^2)

> $display(odeplot(mysol, [s, kappa1(s)^2 + kappa2(s)^2], color = black));$



We check the second, first integral

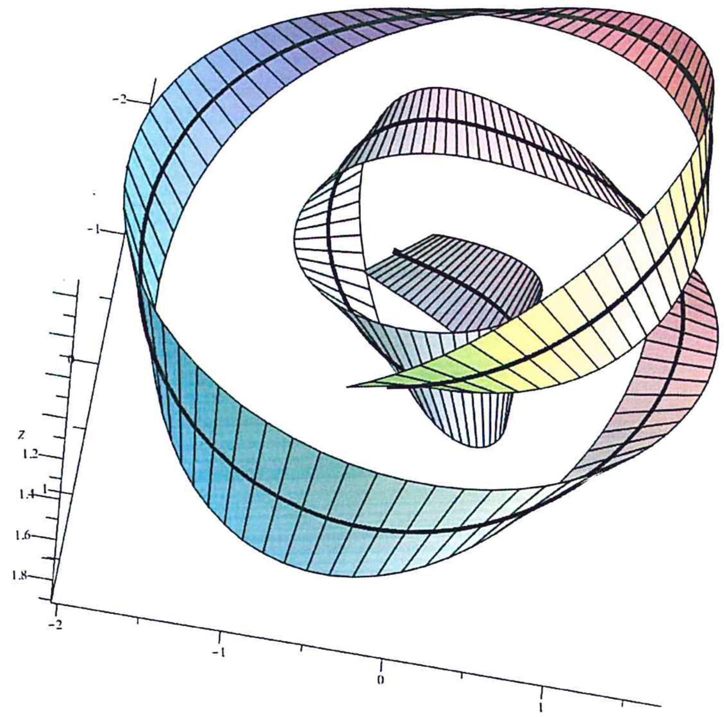
```
> odeplot(mysol, [s, v1(s)·v4(s) − v2(s)·v5(s) + v3(s)·v6(s)], color = black);
```



We now want X,Y,Z and the V vector

```
> mySigmaDc2 := subs(op(mycs), sigma·Matrix([[c4], [-c5], [c6]])) :
> myX := 1/v3(s) · (v1(s)·Z(s) + v5(s) + mySigmaDc2[2, 1]) :
> myY := 1/v3(s) · (v2(s)·Z(s) + v4(s) − mySigmaDc2[1, 1]) :

> myV1 := subs(op(mycs), sigma[2, 1]) : myV2 := subs(op(mycs), sigma[2, 2]) : myV3 := subs(op(mycs),
    sigma[2, 3]) :
> plot1 := odeplot(mysol, [myX, myY, Z(s)], color = black, thickness = 7) :
> plot234 := odeplot(mysol, [myX + 1/4 · myV1, myY + 1/4 · myV2, Z(s) + 1/4 · myV3], color = black) :
> plot27 := odeplot(mysol, [myX− 1/7 · myV1, myY + 1/7 · myV2, Z(s) + 1/7 · myV3], color = black) :
> plot24 := odeplot(mysol, [myX− 1/4 · myV1, myY− 1/4 · myV2, Z(s) − 1/4 · myV3], color = black) :
> plot22 := odeplot(mysol, [myX + 1/2 · myV1, myY + 1/2 · myV2, Z(s) + 1/2 · myV3], color = black) :
> with(plottools) :
> M1 := op(3,getdata(plot1)) :M2 := op(3,getdata(plot234)) :
  A := Array(1 ..200, 1 ..4, 1 ..3, datatype = float) :
  for i to 200 do
    A[i, 1, 1 ..3] := M1[i, 1 ..3];
    A[i, 2, 1 ..3] := M1[i + 1, 1 ..3];
    A[i, 3, 1 ..3] := M2[i + 1, 1 ..3];
    A[i, 4, 1 ..3] := M2[i, 1 ..3];
  end do:
  plot1NF := display(seq(polygon(A[i]), i = 1 ..200), style = patch) :

> M1 := op(3,getdata(plot1)) :M2 := op(3,getdata(plot24)) :
  A := Array(1 ..200, 1 ..4, 1 ..3, datatype = float) :
  for i to 200 do
    A[i, 1, 1 ..3] := M1[i, 1 ..3];
    A[i, 2, 1 ..3] := M1[i + 1, 1 ..3];
    A[i, 3, 1 ..3] := M2[i + 1, 1 ..3];
```

```
  A[i,4,1..3] := M2[i,1..3];
 end do:
 plot2NF := display(seq(polygon(A[i]),i=1..200),style=patch) :
> M1 := op(3,getdata(plot24)) :M2 := op(3,getdata(plot22)) :
 A := Array(1..5850,1..4,1..3,datatype=float) :
 for i to 5850 do
  A[i,1,1..3] := M1[i,1..3];
  A[i,2,1..3] := M1[i+1,1..3];
  A[i,3,1..3] := M2[i+1,1..3];
  A[i,4,1..3] := M2[i,1..3];
 end do:
 plot3NF := display(seq(polygon(A[i]),i=1..200),style=patch) :

> display(plot1, plot1NF, plot2NF, axes=framed, orientation=[77,-31,-176]);
```



Application in biology second example. We follow the method used in the example regarding an invariant lagrangian depending on kappa2/kappa1 - 7.5.2 (second example)

```
> restart
> with(LinearAlgebra) :
> Digits := 20 :
> C :='C': CC2 :='CC2':
```

Euler Lagrange equations

```
> L := kappa1s·kappa2ss − kappa2s·kappa1ss :
> Ls := diff(kappa1(s), s) ·diff(kappa2(s), s, s) − diff(kappa2(s), s) ·diff(kappa1(s), s, s)
```

$$Ls := \left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d^2}{ds^2}\kappa 2(s)\right) - \left(\frac{d}{ds}\kappa 2(s)\right)\left(\frac{d^2}{ds^2}\kappa 1(s)\right) \quad (1)$$

```
> EK1 := subs( {kappa1s = diff(kappa1(s), s), kappa2s = diff(kappa2(s), s)}, diff(L, kappa1))
      -diff(subs( {kappa1ss = diff(kappa1(s), s, s), kappa2ss = diff(kappa2(s), s, s)}, diff(L, kappa1s)), s)
      + diff(subs( {kappa1s = diff(kappa1(s), s), kappa2s = diff(kappa2(s), s)}, diff(L, kappa1ss)), s, s)
```

$$EK1 := -2\frac{d^3}{ds^3}\kappa 2(s) \quad (2)$$

```
> EK2 := subs( {kappa1s = diff(kappa1(s), s), kappa2s = diff(kappa2(s), s)}, diff(L, kappa2))
      -diff(subs( {kappa1ss = diff(kappa1(s), s, s), kappa2ss = diff(kappa2(s), s, s)}, diff(L, kappa2s)), s)
      + diff(subs( {kappa1s = diff(kappa1(s), s), kappa2s = diff(kappa2(s), s)}, diff(L, kappa2ss)), s, s)
```

$$EK2 := 2\frac{d^3}{ds^3}\kappa 1(s) \quad (3)$$

```
> mus := EK1·kappa2(s) − EK2·kappa1(s)
```

$$mus := -2\left(\frac{d^3}{ds^3}\kappa 2(s)\right)\kappa 2(s) - 2\left(\frac{d^3}{ds^3}\kappa 1(s)\right)\kappa 1(s) \quad (4)$$

```
> mu := int(%, s)
```

$$\mu := \left(\frac{d}{ds}\kappa 1(s)\right)^2 - 2\kappa 1(s)\left(\frac{d^2}{ds^2}\kappa 1(s)\right) + \left(\frac{d}{ds}\kappa 2(s)\right)^2 - 2\kappa 2(s)\left(\frac{d^2}{ds^2}\kappa 2(s)\right) \quad (5)$$

```
> eq3 := -kappa1(s) ·diff(EK1, s) − kappa2(s) ·diff(EK2, s) − diff(lambda(s), s) :
```

$$> int\left(2\kappa 1(s)\left(\frac{d^4}{ds^4}\kappa 2(s)\right) - 2\kappa 2(s)\left(\frac{d^4}{ds^4}\kappa 1(s)\right), s\right) :$$

```
> lambda := %
```

$$\lambda := 2\left(\frac{d^3}{ds^3}\kappa 2(s)\right)\kappa 1(s) - 2\left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d^2}{ds^2}\kappa 2(s)\right) + 2\left(\frac{d}{ds}\kappa 2(s)\right)\left(\frac{d^2}{ds^2}\kappa 1(s)\right) - 2\kappa 2(s)\left(\frac{d^3}{ds^3}\kappa 1(s)\right) \quad (6)$$

```
> eq1 := simplify(simplify(diff(EK1, s, s) + diff(kappa2(s) ·mu, s) − kappa1(s) ·lambda, symbolic), size)
```

$$eq1 := \left(\frac{d}{ds}\kappa 1(s)\right)^2\left(\frac{d}{ds}\kappa 2(s)\right) + 2\left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d^2}{ds^2}\kappa 2(s)\right)\kappa 1(s) + \left(\frac{d}{ds}\kappa 2(s)\right)^3 - 2\left(\frac{d}{ds}\kappa 2(s)\right)\left(\frac{d^2}{ds^2}\kappa 2(s)\right)\kappa 2(s) - 4\left(\frac{d}{ds}\kappa 2(s)\right)\left(\frac{d^2}{ds^2}\kappa 1(s)\right)\kappa 1(s) - 2\kappa 2(s)^2\left(\frac{d^3}{ds^3}\kappa 2(s)\right) - 2\kappa 1(s)^2\left(\frac{d^3}{ds^3}\kappa 2(s)\right) - 2\frac{d^5}{ds^5}\kappa 2(s) \quad (7)$$

```
> eq2 := simplify(diff(EK2, s, s) -diff(kappa1(s) ·mu, s) -κ2(s) ·lambda, symbolic)
```

$$eq2 := -\left(\frac{d}{ds}\kappa 1(s)\right)^3 - \left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d}{ds}\kappa 2(s)\right)^2 + 4\left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d^2}{ds^2}\kappa 2(s)\right)\kappa 2(s) + 2\left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d^2}{ds^2}\kappa 1(s)\right)\kappa 1(s) - 2\left(\frac{d}{ds}\kappa 2(s)\right)\left(\frac{d^2}{ds^2}\kappa 1(s)\right)\kappa 2(s) + 2\kappa 2(s)^2\left(\frac{d^3}{ds^3}\kappa 1(s)\right) \quad (8)$$

$$+2\,\kappa 1(s)^2\left(\frac{d^3}{ds^3}\kappa 1(s)\right)+2\frac{d^5}{ds^5}\kappa 1(s)$$

Vector of invariants

> *w1* := lambda;

$$w1 := 2\left(\frac{d^3}{ds^3}\kappa 2(s)\right)\kappa 1(s)-2\left(\frac{d}{ds}\kappa 1(s)\right)\left(\frac{d^2}{ds^2}\kappa 2(s)\right)+2\left(\frac{d}{ds}\kappa 2(s)\right)\left(\frac{d^2}{ds^2}\kappa 1(s)\right)-2\,\kappa 2(s)\left(\frac{d^3}{ds^3}\kappa 1(s)\right) \quad (9)$$

> *convert*(*w1*, D) : *w10* := *subs*(*s* = 0, %);

$$w10 := 2\,D^{(3)}(\kappa 2)(0)\,\kappa 1(0)-2\,D(\kappa 1)(0)\,D^{(2)}(\kappa 2)(0)+2\,D(\kappa 2)(0)\,D^{(2)}(\kappa 1)(0)-2\,\kappa 2(0)\,D^{(3)}(\kappa 1)(0) \quad (10)$$

> *w2* := *simplify*(- (*diff*(*EK1*, *s*) + mu·*kappa2*(*s*)));

$$w2 := 2\frac{d^4}{ds^4}\kappa 2(s)+2\left(\frac{d^2}{ds^2}\kappa 2(s)\right)\kappa 2(s)^2+\kappa 2(s)\left(2\,\kappa 1(s)\left(\frac{d^2}{ds^2}\kappa 1(s)\right)-\left(\frac{d}{ds}\kappa 1(s)\right)^2-\left(\frac{d}{ds}\kappa 2(s)\right)^2\right) \quad (11)$$

> *convert*(*w2*, D) : *w20* := *subs*(*s* = 0, %);

$$w20 := 2\,D^{(4)}(\kappa 2)(0)+2\,D^{(2)}(\kappa 2)(0)\,\kappa 2(0)^2+\kappa 2(0)\left(2\,\kappa 1(0)\,D^{(2)}(\kappa 1)(0)-D(\kappa 1)(0)^2-D(\kappa 2)(0)^2\right) \quad (12)$$

> *w3* := *simplify*(-*diff*(*EK2*, *s*) + mu·*kappa1*(*s*));

$$w3 := -2\frac{d^4}{ds^4}\kappa 1(s)-2\left(\frac{d^2}{ds^2}\kappa 2(s)\right)\kappa 2(s)\,\kappa 1(s)-2\left(\frac{d^2}{ds^2}\kappa 1(s)\right)\kappa 1(s)^2+\kappa 1(s)\left(\left(\frac{d}{ds}\kappa 1(s)\right)^2+\left(\frac{d}{ds}\kappa 2(s)\right)^2\right) \quad (13)$$

> *convert*(*w3*, D) : *w30* := *subs*(*s* = 0, %);

$$w30 := -2\,D^{(4)}(\kappa 1)(0)-2\,D^{(2)}(\kappa 2)(0)\,\kappa 2(0)\,\kappa 1(0)-2\,D^{(2)}(\kappa 1)(0)\,\kappa 1(0)^2+\kappa 1(0)\left(D(\kappa 1)(0)^2+D(\kappa 2)(0)^2\right) \quad (14)$$

> *w4* := mu;

$$w4 := \left(\frac{d}{ds}\kappa 1(s)\right)^2-2\,\kappa 1(s)\left(\frac{d^2}{ds^2}\kappa 1(s)\right)+\left(\frac{d}{ds}\kappa 2(s)\right)^2-2\,\kappa 2(s)\left(\frac{d^2}{ds^2}\kappa 2(s)\right) \quad (15)$$

> *convert*(*w4*, D) : *w40* := *subs*(*s* = 0, %);

$$w40 := D(\kappa 1)(0)^2-2\,\kappa 1(0)\,D^{(2)}(\kappa 1)(0)+D(\kappa 2)(0)^2-2\,\kappa 2(0)\,D^{(2)}(\kappa 2)(0) \quad (16)$$

> *w5* := *EK2*;

$$w5 := 2\frac{d^3}{ds^3}\kappa 1(s) \quad (17)$$

> *convert*(*w5*, D) : *w50* := *subs*(*s* = 0, %);

$$w50 := 2\,D^{(3)}(\kappa 1)(0) \quad (18)$$

> *w6* := *EK1*;

$$w6 := -2\frac{d^3}{ds^3}\kappa 2(s) \quad (19)$$

> *convert*(*w6*, D) : *w60* := *subs*(*s* = 0, %);

$$w60 := -2\,D^{(3)}(\kappa 2)(0) \quad (20)$$

First integrals

> $eq5 := simplify\left(expand\left(\left(v1(s)^2+v2(s)^2+v3(s)^2\right)\right)\right)-c1^2-c2^2-c3^2$:

> $eq6 := v1(s)\cdot v4(s)-v2(s)\cdot v5(s)+v3(s)\cdot v6(s)$ -'CC2':

d/ds psi(s)

```
> eq7 := diff(psi(s), s) = -kappa1(s) + v2(s)·kappa2(s)/(v3(s) + C) :
```

Caley Map

```
> myA := (x1, x2, x3, x4) → Matrix([[x1^2 + x2^2 - x3^2 - x4^2, -2·(x1·x4 - x2·x3), 2·(x1·x3 + x2·x4)], [2·(x1·x4 + x2·x3), x1^2 - x2^2 + x3^2 - x4^2, -2·(x1·x2 - x3·x4)], [-2·(x1·x3 - x2·x4), 2·(x1·x2 + x3·x4), x1^2 - x2^2 - x3^2 + x4^2]]);
```

$$myA := (x1, x2, x3, x4) \mapsto Matrix([[x1^2 + x2^2 - x3^2 - x4^2, -2\,x1\,x4 + 2\,x2\,x3, 2\,x1\,x3 + 2\,x2\,x4], [2\,x1\,x4 + 2\,x2\,x3, x1^2 - x2^2 + x3^2 - x4^2, -2\,x2\,x1 + 2\,x3\,x4], [-2\,x1\,x3 + 2\,x2\,x4, 2\,x2\,x1 + 2\,x3\,x4, x1^2 - x2^2 - x3^2 + x4^2]]) \quad \textbf{(21)}$$

```
> myR := proc(psi, a1, a2, a3) local C; C := sqrt(a1^2 + a2^2 + a3^2); myA(cos(psi/2), sin(psi/2)·a1/C, sin(psi/2)·a2/C, sin(psi/2)·a3/C) end proc;
```

(22)

```
myR := proc(psi, a1, a2, a3)
    local C;
    C := sqrt(a1^2 + a2^2 + a3^2);
    myA(cos(1/2 * psi), sin(1/2 * psi) * a1/C, sin(1/2 * psi) * a2/C, sin(1/2 * psi) * a3/C)
end proc
```

Sigma sends c=(c1,c2,c3) to (0,0,C), and then rotates abut (0,0,C) and then sends (0,0,C) to (v1,v2,v3)

```
> map(simplify, myR(Pi, v1(s), v2(s), v3(s) + C).myR(psi(s), 0, 0, C).myR(Pi, c1, c2, c3 + C), {v1(s)^2 + v2(s)^2 + v3(s)^2 - C^2, c1^2 + c2^2 + c3^2 - C^2}, {v1(s), c1}) : map(simplify, %, symbolic) : map(combine, %, trig) :
> sigma := % :
```

We know that Z'=sigma[1,3]

```
> eqZ := diff(Z(s), s) = sigma[1, 3] :

> Ctemp := sqrt(c1^2 + c2^2 + c3^2) :
> CC2temp := c1·c4 - c2·c5 + c3·c6 :
```

Initial data

```
> myICsubs := {kappa1(0) = 1, D(kappa1)(0) = 1, D^(2)(kappa1)(0) = 1, D^(3)(kappa1)(0) = 1, D^(4)(kappa1)(0) = 1, kappa2(0) = 1/2, D(kappa2)(0) = 1, D^(2)(kappa2)(0) = 1, D^(3)(kappa2)(0) = 1, D^(4)(kappa2)(0) = 1} :
> myICs := map(eval, subs(op(myICsubs), {v1(0) = w10, v2(0) = w20, v3(0) = w30, v4(0) = w40, v5(0) = w50, v6(0) = w60, psi(0) = 0, Z(0) = 1}));
```

$$myICs := \left\{Z(0) = 1, \psi(0) = 0, v1(0) = 1, v2(0) = \frac{5}{2}, v3(0) = -3, v4(0) = -1, v5(0) = 2, v6(0) = -2\right\} \quad \textbf{(23)}$$

Second first integral condition

```
> mycs := map(eval, subs(op(myICsubs), {c1 = w10, c2 = w20, c3 = w30, c4 = w40, c5 = w50, c6 = w60}));
```

$$mycs := \left\{c1 = 1, c2 = \frac{5}{2}, c3 = -3, c4 = -1, c5 = 2, c6 = -2\right\} \quad \textbf{(24)}$$

```
> C := simplify(subs(op(mycs), Ctemp));
```

$$C := \frac{\sqrt{65}}{2} \quad \textbf{(25)}$$

```
> CC2 := simplify(subs(op(mycs), CC2temp));
```

$$CC2 := 0 \quad \textbf{(26)}$$

This needs to be far from zero

```
> test := subs(myICs union mycs, v3(0) + C) : evalf(expand(%));
```

(27)

1.03112887414927482622 **(27)**

> $AllEqs := subs(op(mycs), \{eq1, eq2, eq6, eq7, v1(s) - w1, v2(s) - w2, v3(s) - w3, v5(s) - w5, v6(s) - w6, eqZ\})$:

The range is all that Maple can do before running into singularities

> $dsolve(AllEqs$ **union** $myICs$ **union** $myICsubs, numeric, range = -3 ..2.21)$;

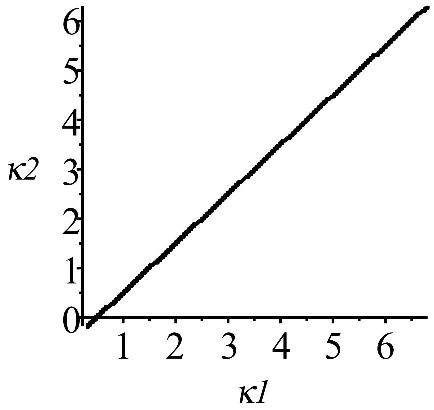
proc(*x_rkf45_dae*) ... **end proc** **(28)**

> $mysol := \%$:

> $with(plots)$:

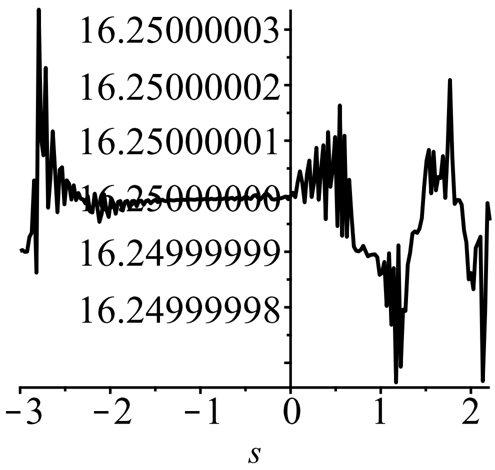
This is (kappa1(s),kappa2(s))

> $odeplot(mysol, [kappa1(s), kappa2(s)], color = black, refine = 2)$;



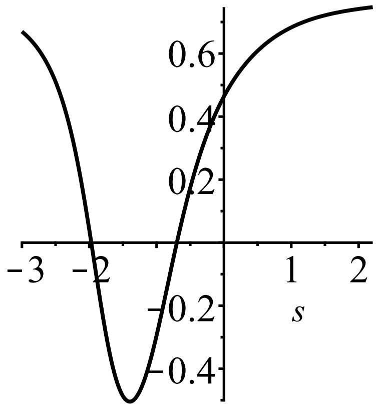
We check the first integral

> $odeplot(mysol, [s, v1(s)^2 + v2(s)^2 + v3(s)^2], color = black)$;



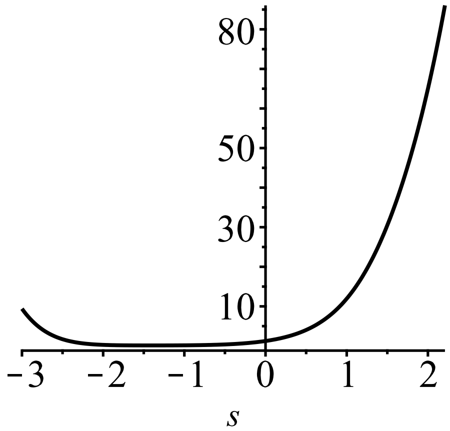
This is (s,theta(s))

> $odeplot\left(mysol, \left[s, \arctan\left(\frac{kappa2(s)}{kappa1(s)}\right)\right], color = black\right)$;



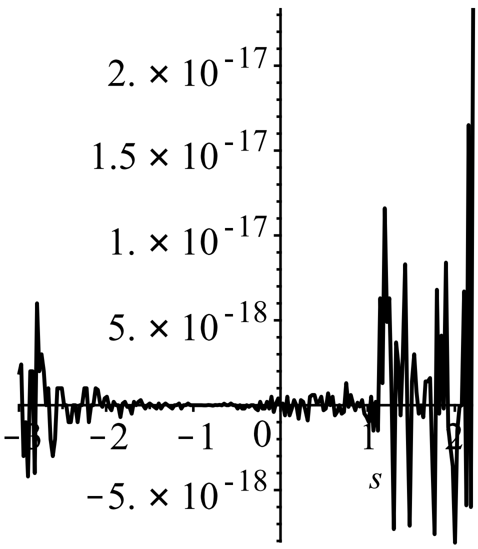
This is (s,kappa(s)^2)

> $display(odeplot(mysol, [s, kappa1(s)^2 + kappa2(s)^2], color = black));$



We check the second, first integral

> $odeplot(mysol, [s, v1(s) \cdot v4(s) - v2(s) \cdot v5(s) + v3(s) \cdot v6(s)], color = black);$

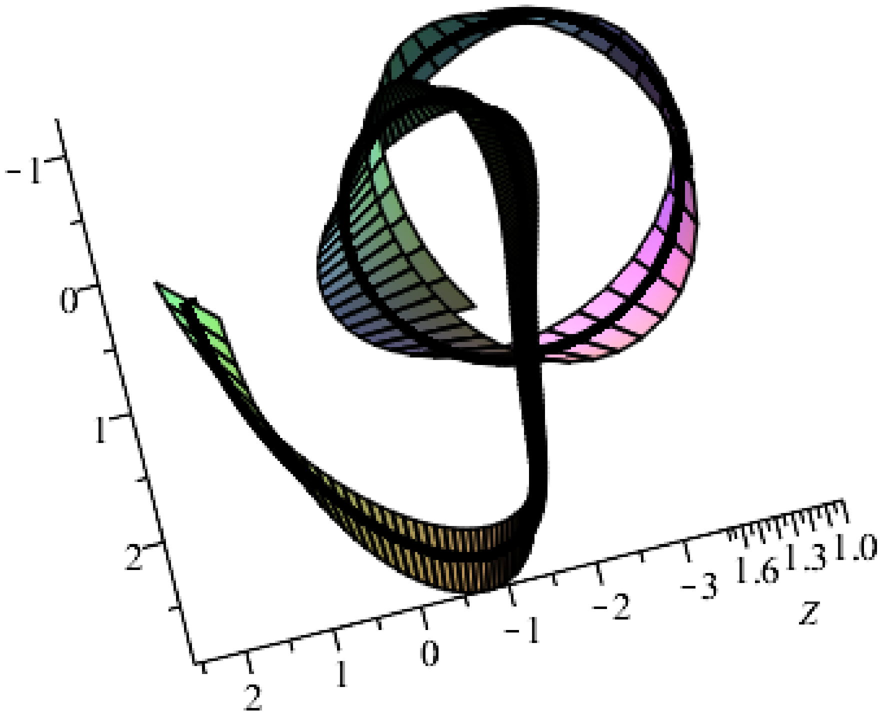


We now want X,Y,Z and the V vector

> $mySigmaDc2 := subs(op(mycs), \text{sigma} \cdot Matrix([[c4], [-c5], [c6]])):$

> $myX := \frac{1}{v3(s)} \cdot (v1(s) \cdot Z(s) + v5(s) + mySigmaDc2[2, 1]):$

```
> myY := 1/v3(s) · (v2(s) · Z(s) + v4(s) − mySigmaDc2[1, 1]) :

> myV1 := subs(op(mycs), sigma[2, 1]) : myV2 := subs(op(mycs), sigma[2, 2]) : myV3 := subs(op(mycs),
      sigma[2, 3]) :

> plot1 := odeplot(mysol, [myX, myY, Z(s)], color = black, thickness = 7) :
> plot234 := odeplot(mysol, [myX + 1/4 · myV1, myY + 1/4 · myV2, Z(s) + 1/4 · myV3], color = black) :
> plot27 := odeplot(mysol, [myX − 1/4 · myV1, myY − 1/4 · myV2, Z(s) − 1/4 · myV3], color = black) :
> with(plottools) :
> M1 := op(3, getdata(plot1)) : M2 := op(3, getdata(plot27)) :
  A := Array(1 ..201, 1 ..4, 1 ..3, datatype = float) :
  for i to 201 do
    A[i, 1, 1 ..3] := M1[i, 1 ..3];
    A[i, 2, 1 ..3] := M1[i + 1, 1 ..3];
    A[i, 3, 1 ..3] := M2[i + 1, 1 ..3];
    A[i, 4, 1 ..3] := M2[i, 1 ..3];
  end do:
  plot1NF := display(seq(polygon(A[i]), i = 1 ..201), style = patch) :

> M1 := op(3, getdata(plot1)) : M2 := op(3, getdata(plot234)) :
  A := Array(1 ..201, 1 ..4, 1 ..3, datatype = float) :
  for i to 201 do
    A[i, 1, 1 ..3] := M1[i, 1 ..3];
    A[i, 2, 1 ..3] := M1[i + 1, 1 ..3];
    A[i, 3, 1 ..3] := M2[i + 1, 1 ..3];
    A[i, 4, 1 ..3] := M2[i, 1 ..3];
  end do:
  plot2NF := display(seq(polygon(A[i]), i = 1 ..201), style = patch) :
>
> display(plot1, plot1NF, plot2NF, axes = framed, orientation = [55, -34, -88]);
```


-1
0
1
2
2
1
0
-1
-2
-3
1.6
1.3
1.0
z

>

>

```
> restart
> with(LinearAlgebra) :
> #Running Example for Chapter 8, Gauge transformations. SE(2)action
> #Moving frame rhoA
> #action
> gu := cos(theta)·u(s, t) − sin(theta)·v(s, t) + a :
> gv := sin(theta)·u(s, t) + cos(theta)·v(s, t) + b :
> gvs := diff(gv, s) :
> gus := diff(gu, s) :
> guss := diff(gus, s) :
> gvss := diff(gvs, s) :
> simplify(solve({gu, gv, gvs}, {theta, a, b}), symbolic)
```

$$\left\{a = -\frac{u(s,t)\left(\frac{\partial}{\partial s}u(s,t)\right)+\left(\frac{\partial}{\partial s}v(s,t)\right)v(s,t)}{\sqrt{\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2}}, b = \frac{\left(\frac{\partial}{\partial s}v(s,t)\right)u(s,t)-v(s,t)\left(\frac{\partial}{\partial s}u(s,t)\right)}{\sqrt{\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2}}, \theta = -\arctan\left(\frac{\frac{\partial}{\partial s}v(s,t)}{\frac{\partial}{\partial s}u(s,t)}\right)\right\} \quad (1)$$

```
> assign(%)
```

Frame

```
> rhoA := simplify(Matrix([[cos(theta),-sin(theta), a], [sin(theta), cos(theta), b], [0, 0, 1]]), symbolic) :
```

$$> rhoA := subs\left(\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2 = 1, \%\right)$$

Invariants

```
> rhoA·Matrix([[u(s, t)], [v(s, t)], [1]])
```

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2)$$

$$> rhoA.Matrix\left(\left[\left[\frac{\partial}{\partial s}u(s,t)\right], \left[\frac{\partial}{\partial s}v(s,t)\right], [0]\right]\right) : subs\left(\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2 = 1, \%\right)$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (3)$$

$$> rhoA.Matrix\left(\left[\left[\frac{\partial^2}{\partial s^2}u(s,t)\right], \left[\frac{\partial^2}{\partial s^2}v(s,t)\right], [0]\right]\right) : subs\left(\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2 = 1, \%\right)$$

$$\begin{bmatrix} \left(\frac{\partial}{\partial s}u(s,t)\right)\left(\frac{\partial^2}{\partial s^2}u(s,t)\right)+\left(\frac{\partial}{\partial s}v(s,t)\right)\left(\frac{\partial^2}{\partial s^2}v(s,t)\right) \\ -\left(\frac{\partial}{\partial s}v(s,t)\right)\left(\frac{\partial^2}{\partial s^2}u(s,t)\right)+\left(\frac{\partial}{\partial s}u(s,t)\right)\left(\frac{\partial^2}{\partial s^2}v(s,t)\right) \\ 0 \end{bmatrix} \quad (4)$$

$$> \#Note\ that\ \left(\frac{\partial}{\partial s}u(s,t)\right)\left(\frac{\partial^2}{\partial s^2}u(s,t)\right)+\left(\frac{\partial}{\partial s}v(s,t)\right)\left(\frac{\partial^2}{\partial s^2}v(s,t)\right) = 0\ as\ \mathbf{D}\left(\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2\right) = 2\left(\left(\frac{\partial}{\partial s}u(s,t)\right)\left(\frac{\partial^2}{\partial s^2}u(s,t)\right)+\left(\frac{\partial}{\partial s}v(s,t)\right)\left(\frac{\partial^2}{\partial s^2}v(s,t)\right)\right) = 2\cdot\mathbf{D}(1) = 0$$

> $rhoA.Matrix\left(\left[\left[\frac{\partial^k}{\partial s^k}u(s,t)\right],\left[\frac{\partial^k}{\partial s^k}v(s,t)\right],[0]\right]\right):subs\left(\left(\frac{\partial}{\partial s}u(s,t)\right)^2+\left(\frac{\partial}{\partial s}v(s,t)\right)^2=1,\%\right)$

$$\left[\begin{array}{c}\left(\frac{\partial}{\partial s}u(s,t)\right)\left(\frac{\partial^k}{\partial s^k}u(s,t)\right)+\left(\frac{\partial}{\partial s}v(s,t)\right)\left(\frac{\partial^k}{\partial s^k}v(s,t)\right)\\ -\left(\frac{\partial}{\partial s}v(s,t)\right)\left(\frac{\partial^k}{\partial s^k}u(s,t)\right)+\left(\frac{\partial}{\partial s}u(s,t)\right)\left(\frac{\partial^k}{\partial s^k}v(s,t)\right)\\ 0\end{array}\right] \quad (5)$$

> ***#CurvatureMatrices***

> $QAs := map(z\to diff(z,s), rhoA).MatrixInverse(rhoA):$

> $subs\left(\left\{\frac{\partial^2}{\partial s^2}v(s,t)=Inv[v,11](s,t),\frac{\partial^2}{\partial s^2}u(s,t)=0\right\},\%\right):$

> $subs\left(\frac{\partial}{\partial s}v(s,t)=0,\%\right):$

> $subs\left(\frac{\partial}{\partial s}u(s,t)=1,\%\right):$

> $subs(u(s,t)=0, v(s,t)=0,\%):$

> $QAs := simplify(\%, symbolic)$

$$QAs := \left[\begin{array}{ccc}0 & Inv_{v,11}(s,t) & -1\\ -Inv_{v,11}(s,t) & 0 & 0\\ 0 & 0 & 0\end{array}\right] \quad (6)$$

> $QAt := map(z\to diff(z,t), rhoA).MatrixInverse(rhoA):$

> $subs\left(\left\{\frac{\partial^2}{\partial t\,\partial s}u(s,t)=0,\frac{\partial^2}{\partial t\,\partial s}v(s,t)=Inv[v,12](s,t)\right\},\%\right):$

> $subs\left(\left\{\frac{\partial^2}{\partial s^2}v(s,t)=Inv[v,11](s,t),\frac{\partial^2}{\partial s^2}u(s,t)=0\right\},\%\right):$

> $subs\left(\frac{\partial}{\partial t}v(s,t)=Inv[v,2](s,t),\%\right):$

> $subs\left(\frac{\partial}{\partial t}u(s,t)=Inv[u,2](s,t),\%\right):$

> $subs\left(\frac{\partial}{\partial s}v(s,t)=0,\%\right):$

> $subs\left(\frac{\partial}{\partial s}u(s,t)=1,\%\right):$

> $subs(u(x,t)=0, v(x,t)=0,\%):$

> $QAt := simplify(\%, symbolic)$

$$QAt := \left[\begin{array}{ccc}0 & Inv_{v,12}(s,t) & -Inv_{u,2}(s,t)\\ -Inv_{v,12}(s,t) & 0 & -Inv_{v,2}(s,t)\\ 0 & 0 & 0\end{array}\right] \quad (7)$$

> ***#Syzygy and H operator***

> $syzygy := simplify(map(z\to diff(z,s), QAt) - map(z\to diff(z,t), QAs) - QAs.QAt + QAt.QAs, symbolic):$

> $isolate(syzygy(2,3), Inv_{v,12}(s,t))$

$$Inv_{v,12}(s,t) = \frac{\partial}{\partial s}Inv_{v,2}(s,t) + Inv_{v,11}(s,t)\,Inv_{u,2}(s,t) \quad (8)$$

> $subs(\%, syzygy(1,2))$

$$\frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}Inv_{v,2}(s,t)+Inv_{v,11}(s,t)\,Inv_{u,2}(s,t)\right)-\frac{\partial}{\partial t}Inv_{v,11}(s,t) \quad (9)$$

> $isolate\left(\%, \frac{\partial}{\partial t} Inv_{v,11}(s,t)\right)$

$$\frac{\partial}{\partial t} Inv_{v,11}(s,t) = \frac{\partial^2}{\partial s^2} Inv_{v,2}(s,t) + \left(\frac{\partial}{\partial s} Inv_{v,11}(s,t)\right) Inv_{u,2}(s,t) + Inv_{v,11}(s,t)\left(\frac{\partial}{\partial s} Inv_{u,2}(s,t)\right) \quad \textbf{(10)}$$

> ***#SE(2)action + gauge***

> ***Moving frame rhoB***

> *unassign*('*a*','*b*','*c*','*d*','theta','alpha(*s*, *t*)','beta(*s*, *t*)','delta(*s*, *t*)')

> *with*(*LinearAlgebra*) :

> ***#Moving frame***

> $solve(\{gu - \alpha(s,t), gv - \beta(s,t), gvs - \delta(s,t)\}, \{\theta, a, b\})$:

> *simplify*(%, *symbolic*) :

> *S* := *allvalues*(%) :

> *S*[1] :

> *assign*(%)

> $rhoB := subs\left(\left\{\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 = 1, \operatorname{sqrt}\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 - \delta(s,t)^2\right) = \operatorname{sqrt}\left(1 - \delta(s,t)^2\right)\right\}, simplify(Matrix([[\cos(\theta), -\sin(\theta), a], [\sin(\theta), \cos(\theta), b], [0, 0, 1]]), symbolic)\right)$

$$rhoB := \left[\left[\left(\frac{\partial}{\partial s} v(s,t)\right)\delta(s,t) + \left(\frac{\partial}{\partial s} u(s,t)\right)\sqrt{1-\delta(s,t)^2}, -\left(\frac{\partial}{\partial s} u(s,t)\right)\delta(s,t) + \left(\frac{\partial}{\partial s} v(s,t)\right)\sqrt{1-\delta(s,t)^2}, \alpha(s,t)\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \alpha(s,t)\left(\frac{\partial}{\partial s} v(s,t)\right)^2 - u(s,t)\left(\frac{\partial}{\partial s} u(s,t)\right)\sqrt{1-\delta(s,t)^2} - u(s,t)\left(\frac{\partial}{\partial s} v(s,t)\right)\delta(s,t) + v(s,t)\left(\frac{\partial}{\partial s} u(s,t)\right)\delta(s,t) - v(s,t)\left(\frac{\partial}{\partial s} v(s,t)\right)\sqrt{1-\delta(s,t)^2}\right], \left[\left(\frac{\partial}{\partial s} u(s,t)\right)\delta(s,t) - \left(\frac{\partial}{\partial s} v(s,t)\right)\sqrt{1-\delta(s,t)^2}, \left(\frac{\partial}{\partial s} v(s,t)\right)\delta(s,t) + \left(\frac{\partial}{\partial s} u(s,t)\right)\sqrt{1-\delta(s,t)^2}, \left(\frac{\partial}{\partial s} v(s,t)\right)^2\beta(s,t) + \left(\sqrt{1-\delta(s,t)^2}\, u(s,t) - v(s,t)\,\delta(s,t)\right)\left(\frac{\partial}{\partial s} v(s,t)\right) - \left(\sqrt{1-\delta(s,t)^2}\, v(s,t) + u(s,t)\,\delta(s,t) - \beta(s,t)\left(\frac{\partial}{\partial s} u(s,t)\right)\right)\left(\frac{\partial}{\partial s} u(s,t)\right)\right], [0, 0, 1]\right] \quad \textbf{(11)}$$

> *simplify*(*rhoB*.*MatrixInverse*(*rhoA*), *symbolic*) :

> $gauge := subs\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 = 1, \%\right)$

> ***#Invariants***

> $simplify(rhoB \cdot Matrix([[u(s,t)], [v(s,t)], [1]]), symbolic) : subs\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 = 1, \%\right)$

$$\begin{bmatrix} \alpha(s,t) \\ \beta(s,t) \\ 1 \end{bmatrix} \quad \textbf{(12)}$$

> $simplify\left(rhoB \cdot Matrix\left(\left[\left[\frac{\partial}{\partial s} u(s,t)\right], \left[\frac{\partial}{\partial s} v(s,t)\right], [0]\right]\right), symbolic\right) : subs\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 = 1, \%\right)$

(13)

$$\begin{bmatrix} \sqrt{1-\delta(s,t)^2} \\ \delta(s,t) \\ 0 \end{bmatrix} \quad \textbf{(13)}$$

> $simplify\left(rhoB \cdot Matrix\left(\left[\left[\frac{\partial^2}{\partial s^2} u(s,t)\right], \left[\frac{\partial^2}{\partial s^2} v(s,t)\right], [0]\right]\right), symbolic\right) : subs\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 = 1, \%\right) : collect(\%, \delta(s,t))$

$$\left[\left[\left(\left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) - \left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\right)\delta(s,t) + \sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) + \sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\right], \left[\left(\left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) + \left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\right)\delta(s,t) - \left(\frac{\partial^2}{\partial s^2} u(s,t)\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s} v(s,t)\right) + \left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s} u(s,t)\right)\right], [0]\right] \quad \textbf{(14)}$$

> **#Note that** $\left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) + \left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right) = 0$ ***as*** $D\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2\right) = 2\left(\left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) + \left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\right) = 2 \cdot D(1) = 0,$
> **and** ***therefore these are*** $\delta(s,t)\left(\left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) - \left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\right)$
> **and** $\sqrt{1-\delta(s,t)^2}\left(\left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^2}{\partial s^2} u(s,t)\right) - \left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^2}{\partial s^2} v(s,t)\right)\right)$

> $rhoB \cdot Matrix\left(\left[\left[\frac{\partial^k}{\partial s^k} u(s,t)\right], \left[\frac{\partial^k}{\partial s^k} v(s,t)\right], [0]\right]\right) : subs\left(\left(\frac{\partial}{\partial s} u(s,t)\right)^2 + \left(\frac{\partial}{\partial s} v(s,t)\right)^2 = 1, \%\right) : collect(\%, \delta(s,t))$

$$\left[\left[\left(\left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^k}{\partial s^k} u(s,t)\right) - \left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^k}{\partial s^k} v(s,t)\right)\right)\delta(s,t) + \left(\frac{\partial}{\partial s} u(s,t)\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial^k}{\partial s^k} u(s,t)\right) + \left(\frac{\partial}{\partial s} v(s,t)\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial^k}{\partial s^k} v(s,t)\right)\right], \left[\left(\left(\frac{\partial}{\partial s} u(s,t)\right)\left(\frac{\partial^k}{\partial s^k} u(s,t)\right) + \left(\frac{\partial}{\partial s} v(s,t)\right)\left(\frac{\partial^k}{\partial s^k} v(s,t)\right)\right)\delta(s,t) - \left(\frac{\partial}{\partial s} v(s,t)\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial^k}{\partial s^k} u(s,t)\right) + \left(\frac{\partial}{\partial s} u(s,t)\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial^k}{\partial s^k} v(s,t)\right)\right], [0]\right] \quad \textbf{(15)}$$

> ***#Invariants B in terms of Invariants A***

> *unassign*('*a*','*b*','*c*','*d*','theta','alpha(*s*, *t*) ','beta(*s*, *t*) ','delta(*s*, *t*) ')

> $gu := \cos(\theta) \cdot u(s,t) - \sin(\theta) \cdot v(s,t) + a$:

> $gv := \sin(\theta) \cdot u(s,t) + \cos(\theta) \cdot v(s,t) + b$:

> $gvs := diff(gv, s)$:

> $gus := diff(gu, s)$:

> $guss := diff(gus, s)$:

> $gvss := diff(gvs, s)$:

> $gust := diff(gus, t)$:

> $gvst := diff(gvs, t)$:

> $gut := diff(gu, t)$:

> $gvt := diff(gv, t)$:

> $gauge$

$$\begin{bmatrix} \sqrt{1-\delta(s,t)^2} & -\delta(s,t) & \alpha(s,t) \\ \delta(s,t) & \sqrt{1-\delta(s,t)^2} & \beta(s,t) \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{(16)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), b = \beta(s,t), gu\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$\alpha(s,t) \quad \textbf{(17)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), b = \beta(s,t), gv\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$\beta(s,t) \quad \textbf{(18)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, gus\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$\sqrt{1-\delta(s,t)^2} \quad \textbf{(19)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, gvs\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$\delta(s,t) \quad \textbf{(20)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, guss\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$-\delta(s,t)\, Inv_{v,11} \quad \textbf{(21)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, gut\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, \frac{\partial}{\partial t} u(s,t) = Inv[u,2], \frac{\partial}{\partial t} v(s,t) = Inv[v,2], u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$\sqrt{1-\delta(s,t)^2}\, Inv_{u,2} - \delta(s,t)\, Inv_{v,2} \quad \textbf{(22)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, gvt\right)$:

> $subs\left(\left\{\frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v,11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, \frac{\partial}{\partial t} u(s,t) = Inv[u,2], \frac{\partial}{\partial t} v(s,t) = Inv[v,2], u(s,t) = 0, v(s,t) = 0\right\}, \%\right)$

$$\delta(s,t)\, Inv_{u,2} + \sqrt{1-\delta(s,t)^2}\, Inv_{v,2} \quad \textbf{(23)}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, gust\right):$

> $subs\left(\left\{ \frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v, 11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, \frac{\partial^2}{\partial t\,\partial s} u(s,t) = Inv[u, 12], \frac{\partial^2}{\partial t\,\partial s} v(s,t) = Inv[v, 12], u(s,t) = 0, v(s,t) = 0 \right\}, \%\right)$

$$\sqrt{1-\delta(s,t)^2}\, Inv_{u,12} - \delta(s,t)\, Inv_{v,12} \tag{24}$$

> $subs\left(\cos(\text{theta}) = \sqrt{1-\delta(s,t)^2}, \sin(\text{theta}) = \delta(s,t), a = \alpha(s,t), \beta(s,t) = 0, gvst\right):$

> $subs\left(\left\{ \frac{\partial^2}{\partial s^2} u(s,t) = 0, \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v, 11], \frac{\partial}{\partial s} u(s,t) = 1, \frac{\partial}{\partial s} v(s,t) = 0, \frac{\partial^2}{\partial t\,\partial s} u(s,t) = Inv[u, 12], \frac{\partial^2}{\partial t\,\partial s} v(s,t) = Inv[v, 12], u(s,t) = 0, v(s,t) = 0 \right\}, \%\right)$

$$\delta(s,t)\, Inv_{u,12} + \sqrt{1-\delta(s,t)^2}\, Inv_{v,12} \tag{25}$$

> ***#CurvatureMatrices***

> $QBs := map(z \rightarrow diff(z, s), rhoB).MatrixInverse(rhoB):$

> $subs\left(\left\{ \frac{\partial^2}{\partial s^2} v(s,t) = Inv[v, 11](s,t), \frac{\partial^2}{\partial s^2} u(s,t) = Inv[u, 11](s,t) \right\}, \%\right):$

> $subs\left(\frac{\partial}{\partial s} v(s,t) = \text{delta}(s,t), \%\right):$

> $subs\left(\frac{\partial}{\partial s} u(s,t) = \sqrt{1-\delta(s,t)^2}, \%\right):$

> $subs(u(x,t) = \text{alpha}(s,t), v(x,t) = \text{beta}(s,t), \%):$

> $QBs := simplify(simplify(\%, symbolic), size)$

$$QBs := \left[\left[Inv_{v,11}(s,t)\,\delta(s,t) + Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2}, -\frac{Inv_{u,11}(s,t)\,\delta(s,t)\sqrt{1-\delta(s,t)^2} + Inv_{v,11}(s,t)\,\delta(s,t)^2 + \frac{\partial}{\partial s}\delta(s,t) - Inv_{v,11}(s,t)}{\sqrt{1-\delta(s,t)^2}}, \frac{1}{\sqrt{1-\delta(s,t)^2}}\left(\left(\frac{\partial}{\partial s}\alpha(s,t) + \left(\beta(s,t)\, Inv_{u,11}(s,t) + Inv_{v,11}(s,t)\,\alpha(s,t)\right)\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} + \left(\frac{\partial}{\partial s}\delta(s,t)\right)\beta(s,t) + (\delta(s,t)-1)(\delta(s,t)+1)\left(Inv_{v,11}(s,t)\,\beta(s,t) - Inv_{u,11}(s,t)\,\alpha(s,t) + 1\right)\right)\right], \left[\frac{Inv_{u,11}(s,t)\,\delta(s,t)\sqrt{1-\delta(s,t)^2} + Inv_{v,11}(s,t)\,\delta(s,t)^2 + \frac{\partial}{\partial s}\delta(s,t) - Inv_{v,11}(s,t)}{\sqrt{1-\delta(s,t)^2}}, Inv_{v,11}(s,t)\,\delta(s,t) + Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2}, \frac{1}{\sqrt{1-\delta(s,t)^2}}\left(\left(\frac{\partial}{\partial s}\beta(s,t) + \left(Inv_{v,11}(s,t)\,\beta(s,t) - Inv_{u,11}(s,t)\,\alpha(s,t) - 1\right)\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} - \left(\frac{\partial}{\partial s}\delta(s,t)\right)\alpha(s,t) - (\delta(s,t)-1)(\delta(s,t)+1)\left(\beta(s,\right.\right.\right. \tag{26}$$

$t) \, Inv_{u,11}(s,t) + Inv_{v,11}(s,t) \, \alpha(s,t) \big) \big) \big] \big],$

$\big[[0, 0, 0] \big] \big]$

> $QBt := map(z \to diff(z,t), rhoB).MatrixInverse(rhoB):$

> $subs\left(\left\{ \frac{\partial^2}{\partial t \partial s} u(s,t) = Inv[u,12](s,t), \frac{\partial^2}{\partial t \partial s} v(s,t) = Inv[v,12](s,t) \right\}, \% \right):$

> $subs\left(\frac{\partial}{\partial t} v(s,t) = Inv[v,2](s,t), \% \right):$

> $subs\left(\frac{\partial}{\partial t} u(s,t) = Inv[u,2](s,t), \% \right):$

> $subs\left(\frac{\partial}{\partial s} v(s,t) = \mathrm{delta}(s,t), \% \right):$

> $subs\left(\frac{\partial}{\partial s} u(s,t) = \sqrt{1 - \delta(s,t)^2}, \% \right):$

> $subs(u(x,t) = \mathrm{alpha}(s,t), v(x,t) = \mathrm{beta}(s,t), \%):$

> $QBt := simplify(simplify(\%, symbolic), size)$

> ***#Syzygy and H operator***

> $syzygy := simplify(map(z \to diff(z,s), QBt) - map(z \to diff(z,t), QBs) - QBs.QBt + QBt.QBs, symbolic):$

> $Eq1 := simplify(simplify(syzygy(1,1), symbolic), size)$

$$Eq1 := \frac{1}{\sqrt{1-\delta(s,t)^2}} \Bigg(\Bigg(Inv_{v,12}(s,t) \left(\frac{\partial}{\partial s} \delta(s,t) \right) - Inv_{v,11}(s,t) \left(\frac{\partial}{\partial t} \delta(s,t) \right) - \delta(s,t) \left(-\frac{\partial}{\partial s} Inv_{v,12}(s,t) + \frac{\partial}{\partial t} Inv_{v,11}(s,t) \right) \Bigg) \sqrt{1-\delta(s,t)^2} + \left(-1+\delta(s,t)^2\right) \left(\frac{\partial}{\partial t} Inv_{u,11}(s,t) \right) - \left(\frac{\partial}{\partial s} Inv_{u,12}(s,t) \right) \delta(s,t)^2 + \left(\frac{\partial}{\partial t} \delta(s,t) \right) Inv_{u,11}(s,t) \, \delta(s,t) - \left(\frac{\partial}{\partial s} \delta(s,t) \right) Inv_{u,12}(s,t) \, \delta(s,t) + \frac{\partial}{\partial s} Inv_{u,12}(s,t) \Bigg) \quad \textbf{(27)}$$

> $Eq2 := simplify(simplify(syzygy(1,2), symbolic), size)$

$$Eq2 := \frac{1}{\sqrt{1-\delta(s,t)^2}} \Bigg(\Bigg(-\left(\frac{\partial}{\partial s} \delta(s,t) \right) Inv_{u,12}(s,t) + \left(\frac{\partial}{\partial t} \delta(s,t) \right) Inv_{u,11}(s,t) + \delta(s,t) \left(\frac{\partial}{\partial t} Inv_{u,11}(s,t) - \frac{\partial}{\partial s} Inv_{u,12}(s,t) \right) \Bigg) \sqrt{1-\delta(s,t)^2} + \left(-1+\delta(s,t)^2\right) \left(\frac{\partial}{\partial t} Inv_{v,11}(s,t) \right) - \left(\frac{\partial}{\partial s} Inv_{v,12}(s,t) \right) \delta(s,t)^2 + Inv_{v,11}(s,t) \, \delta(s,t) \left(\frac{\partial}{\partial t} \delta(s,t) \right) - Inv_{v,12}(s,t) \, \delta(s,t) \left(\frac{\partial}{\partial s} \delta(s,t) \right) + \frac{\partial}{\partial s} Inv_{v,12}(s,t) \Bigg) \quad \textbf{(28)}$$

> $simplify\left(simplify\left(isolate\left(Eq2, \frac{\partial}{\partial t} Inv_{u,11}(s,t) \right), symbolic \right), size \right):$

> $subs(\%, Eq1):$

> $Eq1 := simplify(simplify(\%, symbolic), size):$

> $isolate\left(Eq1, \frac{\partial}{\partial t} Inv_{v,11}(s,t) \right):$

> $simplify(\%, size)$

$$\frac{\partial}{\partial t} Inv_{v,11}(s,t) = -\frac{\left(\left(\frac{\partial}{\partial t} \delta(s,t) \right) Inv_{u,11}(s,t) - \left(\frac{\partial}{\partial s} \delta(s,t) \right) Inv_{u,12}(s,t) \right) \sqrt{1-\delta(s,t)^2}}{(\delta(s,t)-1)(\delta(s,t)+1)} + \frac{\partial}{\partial s} Inv_{v,12}(s,t) \quad \textbf{(29)}$$

#Here we simplify the following coefficient

> $\frac{\sqrt{1-\delta(s,t)^2}}{(\delta(s,t)-1)\,(\delta(s,t)+1)}$

$$\frac{\sqrt{1-\delta(s,t)^2}}{(\delta(s,t)-1)\,(\delta(s,t)+1)} \quad (30)$$

> $simplify(\%, symbolic)$

$$-\frac{1}{\sqrt{1-\delta(s,t)^2}} \quad (31)$$

> $\frac{\partial}{\partial t} Inv_{v,11}(s,t) = \frac{\left(Inv_{u,11}(s,t)\left(\frac{\partial}{\partial t}\delta(s,t)\right) - \left(\frac{\partial}{\partial s}\delta(s,t)\right) Inv_{u,12}(s,t)\right)}{\sqrt{1-\delta(s,t)^2}} + \frac{\partial}{\partial s} Inv_{v,12}(s,t)$

$$\frac{\partial}{\partial t} Inv_{v,11}(s,t) = \frac{\left(\frac{\partial}{\partial t}\delta(s,t)\right) Inv_{u,11}(s,t) - \left(\frac{\partial}{\partial s}\delta(s,t)\right) Inv_{u,12}(s,t)}{\sqrt{1-\delta(s,t)^2}} + \frac{\partial}{\partial s} Inv_{v,12}(s,t) \quad (32)$$

Relationship between invariants

> $expre1 := simplify\left(Matrix\left(\left[\left[\frac{\partial}{\partial s} Inv_{u,2}(s,t)\right], \left[\frac{\partial}{\partial s} Inv_{v,2}(s,t)\right], [0]\right]\right) - QBs.Matrix([[Inv_{u,2}(s,t)], [Inv_{v,2}(s,t)], [0]]), size\right):$

> $simplify\left(simplify\left(subs\left(\{Inv_{u,12}(s,t) = expre1(1,1), Inv_{v,12}(s,t) = expre1(2,1)\}, \frac{\partial}{\partial t} Inv_{v,11}(s,t) = \frac{\left(Inv_{u,11}(s,t)\left(\frac{\partial}{\partial t}\delta(s,t)\right) - \left(\frac{\partial}{\partial s}\delta(s,t)\right) Inv_{u,12}(s,t)\right)}{\sqrt{1-\delta(s,t)^2}} + \frac{\partial}{\partial s} Inv_{v,12}(s,t)\right), symbolic\right), size\right):$

Evolution of Inv[v,[1,1]] in terms of the first order differential invariants

> $collect(\%, Inv_{v,2}(s,t)) : collect(\%, Inv_{u,2}(s,t)) : collect\left(\%, \frac{\partial}{\partial s} Inv_{v,2}(s,t)\right) : collect\left(\%, \frac{\partial}{\partial s} Inv_{u,2}(s,t)\right) : collect\left(\%, \frac{\partial^2}{\partial s^2} Inv_{v,2}(s,t)\right) : collect\left(\%, \frac{\partial^2}{\partial s^2} Inv_{u,2}(s,t)\right)$

$$\frac{\partial}{\partial t} Inv_{v,11}(s,t) = -\frac{(\delta(s,t)+1)\,(\delta(s,t)-1)\left(\frac{\partial^2}{\partial s^2} Inv_{v,2}(s,t)\right)}{1-\delta(s,t)^2} - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\bigg((-2\,\delta(s,t)^2 + 2)\left(\frac{\partial}{\partial s}\delta(s,t)\right) + \left(-Inv_{u,11}(s,t)\,\delta(s,t)\sqrt{1-\delta(s,t)^2} - Inv_{v,11}(s,t)\,\delta(s,t)^2 + Inv_{v,11}(s,t)\right)(\delta(s,t)+1)\,(\delta(s,t)-1)\bigg)\left(\frac{\partial}{\partial s} Inv_{u,2}(s,t)\right)\Bigg) - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\bigg(-\sqrt{1-\delta(s,t)^2}\,\delta(s,t)\,Inv_{v,11}(s,t) + \delta(s,t)^2\,Inv_{u,11}(s,t) - Inv_{u,11}(s,t)\bigg)(\delta(s,t)+1)\,(\delta(s,t)-1)\left(\frac{\partial}{\partial s} Inv_{v,2}(s,t)\right)\Bigg) - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\bigg(\delta(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2 + \left(-\sqrt{1-\delta(s,t)^2}\,\delta(s,t)\left(\frac{\partial}{\partial s} Inv_{u,11}(s,t)\right) + (1-\delta(s,t)^2)\left(\frac{\partial}{\partial s} Inv_{v,11}(s,t)\right) - \frac{\partial^2}{\partial s^2}\delta(s,t)\right)(\delta(s,t)+1)\,(\delta(s,t)-1)\bigg) Inv_{u,2}(s,t)\Bigg) - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\bigg(\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2 + \left((-1+\delta(s,t)^2)\left(\frac{\partial}{\partial s} Inv_{u,11}(s,t)\right)\right. \quad (33)$$

$$-\sqrt{1-\delta(s,t)^2}\,\delta(s,t)\left(\frac{\partial}{\partial s} Inv_{v,11}(s,t)\right)\Big)\,(\delta(s,t)+1)\,(\delta(s,t)-1)\Big)\,Inv_{v,2}(s,t)\Big)$$

$$-\frac{\left(\frac{\partial}{\partial t}\delta(s,t)\right) Inv_{u,11}(s,t)\,(\delta(s,t)+1)\,(\delta(s,t)-1)}{\left(1-\delta(s,t)^2\right)^{3/2}}$$

> ***#Here we simplify the coefficients***

> $simplify\left(\frac{(\delta(s,t)-1)\,(\delta(s,t)+1)}{-1+\delta(s,t)^2}, symbolic\right)$

$$1 \tag{34}$$

> $simplify\left(\frac{1}{\left(-1+\delta(s,t)^2\right)^2}\left(\left(2\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}\,(\delta(s,t)-1)\,(\delta(s,t)+1)+(\delta(s,t)-1)^2\left(\sqrt{1-\delta(s,t)^2}\,Inv_{v,11}(s,t)-\delta(s,t)\,Inv_{u,11}(s,t)\right)(\delta(s,t)+1)^2\right)\right), symbolic\right)$

$$\frac{1}{-1+\delta(s,t)^2}\left(\left(Inv_{v,11}(s,t)\,\delta(s,t)^2+2\frac{\partial}{\partial s}\delta(s,t)-Inv_{v,11}(s,t)\right)\sqrt{1-\delta(s,t)^2}-Inv_{u,11}(s,t)\,\delta(s,t)^3+Inv_{u,11}(s,t)\,\delta(s,t)\right) \tag{35}$$

> $collect\left(\%, Inv_{u,11}(s,t)\right) : collect\left(\%, Inv_{v,11}(s,t)\right)$

$$Inv_{v,11}(s,t)\sqrt{1-\delta(s,t)^2}+\frac{\left(-\delta(s,t)^3+\delta(s,t)\right)Inv_{u,11}(s,t)}{-1+\delta(s,t)^2}+\frac{2\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}}{-1+\delta(s,t)^2} \tag{36}$$

> $simplify\left(-\frac{\left(-\delta(s,t)^2\sqrt{1-\delta(s,t)^2}+\sqrt{1-\delta(s,t)^2}\right)}{-1+\delta(s,t)^2}, symbolic\right)$

$$\sqrt{1-\delta(s,t)^2} \tag{37}$$

> $simplify\left(-\frac{\left(\delta(s,t)^3-\delta(s,t)\right)}{-1+\delta(s,t)^2}, symbolic\right)$

$$-\delta(s,t) \tag{38}$$

> $simplify\left(+\frac{2\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}}{-1+\delta(s,t)^2}, symbolic\right)$

$$-\frac{2\left(\frac{\partial}{\partial s}\delta(s,t)\right)}{\sqrt{1-\delta(s,t)^2}} \tag{39}$$

> $simplify\left(simplify\left(-\frac{\left(\delta(s,t)\,Inv_{v,11}(s,t)+Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2}\right)(\delta(s,t)-1)^2\,(\delta(s,t)+1)^2}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right), size\right)$

$$-Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2}-Inv_{v,11}(s,t)\,\delta(s,t) \tag{40}$$

$$1 \tag{41}$$

$$> simplify\left(\frac{1}{\left(-1+\delta(s,t)^2\right)^2}\left(\left(\left(-\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t)\sqrt{1-\delta(s,t)^2}+\left(-\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)\delta(s,t)\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)+\left(\delta(s,t)-1\right)\sqrt{1-\delta(s,t)^2}\left(\delta(s,t)+1\right)\left(\frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)+\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}\right)\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)\right)\right), symbolic\right)$$

$$\frac{1}{\left(-1+\delta(s,t)^2\right)^2}\left(\left(\left(\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)\left(-1+\delta(s,t)^2\right)+\left(\delta(s,t)^4-2\,\delta(s,t)^2+1\right)\left(\frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)-\delta(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\right)\sqrt{1-\delta(s,t)^2}-\delta(s,t)\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)\left(\delta(s,t)-1\right)^2\left(\delta(s,t)+1\right)^2\right)\right) \quad \textbf{(42)}$$

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$$> collect\left(\%, \frac{\partial}{\partial s}Inv_{u,11}(s,t)\right) : collect\left(\%, \frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)$$

$$\frac{\left(\delta(s,t)^4-2\,\delta(s,t)^2+1\right)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)}{\left(-1+\delta(s,t)^2\right)^2} - \frac{\delta(s,t)\left(\delta(s,t)-1\right)^2\left(\delta(s,t)+1\right)^2\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)}{\left(-1+\delta(s,t)^2\right)^2} + \frac{\left(\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)\left(-1+\delta(s,t)^2\right)-\delta(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\right)\sqrt{1-\delta(s,t)^2}}{\left(-1+\delta(s,t)^2\right)^2} \quad \textbf{(44)}$$

$$> simplify\left(-\frac{\left(-\delta(s,t)^4\sqrt{1-\delta(s,t)^2}+2\,\delta(s,t)^2\sqrt{1-\delta(s,t)^2}-\sqrt{1-\delta(s,t)^2}\right)}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$$

$$\sqrt{1-\delta(s,t)^2} \quad \textbf{(45)}$$

$$> simplify\left(-\frac{\left(\delta(s,t)^5-2\,\delta(s,t)^3+\delta(s,t)\right)\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$$

$$-\delta(s,t)\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right) \quad \textbf{(46)}$$

$$> simplify\left(-\frac{1}{\left(-1+\delta(s,t)^2\right)^2}\left(-\delta(s,t)^2\sqrt{1-\delta(s,t)^2}\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)+\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t)\sqrt{1-\delta(s,t)^2}+\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}\right), symbolic\right)$$

$$\frac{-\delta(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2+\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)\left(\delta(s,t)+1\right)\left(\delta(s,t)-1\right)}{\left(1-\delta(s,t)^2\right)^{3/2}} \quad \textbf{(47)}$$

$$> simplify\left(\frac{1}{\left(-1+\delta(s,t)^2\right)^2}\left(\left(-1+\delta(s,t)^2\right)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2+\left(-\left(\delta(s,t)-1\right)\left(\delta(s,t)\right.\right.\right.\right.$$

$+1)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)-(\delta(s,t)-1)\,\delta(s,t)\,(\delta(s,t)+1)\left(\frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)\Big)\,(\delta(s,t)$
$-1)\,(\delta(s,t)+1)\Big), symbolic\Big)$

$$\frac{(1-\delta(s,t)^2)^{3/2}\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)+(-\delta(s,t)^3+\delta(s,t))\left(\frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)+\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2}{-1+\delta(s,t)^2} \quad (48)$$

> $collect\left(\%, \frac{\partial}{\partial s}Inv_{u,11}(s,t)\right) : collect\left(\%, \frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)$

$$\frac{(-\delta(s,t)^3+\delta(s,t))\left(\frac{\partial}{\partial s}Inv_{v,11}(s,t)\right)}{-1+\delta(s,t)^2}+\frac{(1-\delta(s,t)^2)^{3/2}\left(\frac{\partial}{\partial s}Inv_{u,11}(s,t)\right)}{-1+\delta(s,t)^2}+\frac{\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2}{-1+\delta(s,t)^2} \quad (49)$$

> $simplify\left(-\frac{(\delta(s,t)^3-\delta(s,t))}{-1+\delta(s,t)^2}, symbolic\right)$

$$-\delta(s,t) \quad (50)$$

> $simplify\left(-\frac{\left(\delta(s,t)^2\sqrt{1-\delta(s,t)^2}-\sqrt{1-\delta(s,t)^2}\right)}{-1+\delta(s,t)^2}, symbolic\right)$

$$-\sqrt{1-\delta(s,t)^2} \quad (51)$$

>

> $-\frac{\left(\frac{\partial}{\partial t}\delta(s,t)\right)Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2}\,(\delta(s,t)-1)\,(\delta(s,t)+1)}{(-1+\delta(s,t)^2)^2}$

$$-\frac{\left(\frac{\partial}{\partial t}\delta(s,t)\right)Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2}\,(\delta(s,t)-1)\,(\delta(s,t)+1)}{(-1+\delta(s,t)^2)^2} \quad (52)$$

> $simplify(\%, symbolic)$

$$\frac{\left(\frac{\partial}{\partial t}\delta(s,t)\right)Inv_{u,11}(s,t)}{\sqrt{1-\delta(s,t)^2}} \quad (53)$$

> ***#Here we check Proposition 8.1.8***

> $Theo1 := subs\big(Inv_{v,11}(s,t)=Inv_{vA,11}(s,t), simplify(map(z \rightarrow diff(z,s), gauge).MatrixInverse(gauge)+gauge$
$.QAs.MatrixInverse(gauge), size)\big)$:

> $subs\left(\left\{Inv_{v,11}(s,t)=\sqrt{1-\delta(s,t)^2}\cdot Inv_{vA,11}(s,t), Inv_{u,11}(s,t)=-\delta(s,t)\cdot Inv_{vA,11}(s,t)\right\}, QBs\right)$:

>

> $simplify(\%, size)$:

> $\% - Theo1$:

> $simplify(\%, symbolic)$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (54)$$

> ***#Here we check Proposition 8.1.14***

> $A := simplify(gauge.simplify(map(z \rightarrow diff(z,s), QAt) - map(z \rightarrow diff(z,t), QAs) - QAs.QAt + QAt.QAs,$
$symbolic).MatrixInverse(gauge))$:

>

> $B := simplify(map(z \rightarrow diff(z,s), QBt) - map(z \rightarrow diff(z,t), QBs) - QBs.QBt + QBt.QBs, symbolic)$:

>

> subs({$Inv_{u,11}(s,t) = -\delta(s,t)\,Inv_{v,11}(s,t),\ Inv_{v,11}(s,t) = Inv_{v,11}(s,t)\sqrt{1-\delta(s,t)^2},\ Inv_{u,12}(s,t) = -\delta(s,t)\,Inv_{v,12}(s,t),\ Inv_{v,12}(s,t) = Inv_{v,12}(s,t)\sqrt{1-\delta(s,t)^2},\ Inv_{u,2}(s,t) = Inv_{u,2}(s,t)\sqrt{1-\delta(s,t)^2} - \delta(s,t)\,Inv_{v,2}(s,t),\ Inv_{v,2}(s,t) = Inv_{v,2}(s,t)\sqrt{1-\delta(s,t)^2} + \delta(s,t)\,Inv_{u,2}(s,t)$}, %) :

> simplify(%, symbolic) :

> % − A

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \textbf{(55)}$$

> ***#Here we writeHB in terms of HA and TA***

>

> $$\frac{\partial}{\partial t} Inv_{v,11}(s,t) = -\frac{(\delta(s,t)-1)(\delta(s,t)+1)\left(\frac{\partial^2}{\partial s^2} Inv_{v,2}(s,t)\right)}{1-\delta(s,t)^2} - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\Bigg(\Big((-2\,\delta(s,t)^2+2)\left(\frac{\partial}{\partial s}\delta(s,t)\right) + (\delta(s,t)-1)(\delta(s,t)+1)\Big(-\delta(s,t)\,Inv_{u,11}(s,t)\sqrt{1-\delta(s,t)^2} - \delta(s,t)^2\,Inv_{v,11}(s,t) + Inv_{v,11}(s,t)\Big)\Big)\left(\frac{\partial}{\partial s} Inv_{u,2}(s,t)\right)\Bigg) - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg((\delta(s,t)-1)(\delta(s,t)+1)\Big(-\delta(s,t)\,Inv_{v,11}(s,t)\sqrt{1-\delta(s,t)^2} + \delta(s,t)^2\,Inv_{u,11}(s,t) - Inv_{u,11}(s,t)\Big)\left(\frac{\partial}{\partial s} Inv_{v,2}(s,t)\right)\Bigg) - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\Bigg(\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t) + (\delta(s,t)-1)(\delta(s,t)+1)\Bigg(-\delta(s,t)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s} Inv_{u,11}(s,t)\right) + (1-\delta(s,t)^2)\left(\frac{\partial}{\partial s} Inv_{v,11}(s,t)\right) - \left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)\Bigg)\Bigg)Inv_{u,2}(s,t)\Bigg) - \frac{1}{(1-\delta(s,t)^2)^{3/2}}\Bigg(\Bigg(\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2 + (\delta(s,t)-1)(\delta(s,t)+1)\Bigg((-1+\delta(s,t)^2)\left(\frac{\partial}{\partial s} Inv_{u,11}(s,t)\right) - \delta(s,t)\sqrt{1-\delta(s,t)^2}\left(\frac{\partial}{\partial s} Inv_{v,11}(s,t)\right)\Bigg)\Bigg)Inv_{v,2}(s,t)\Bigg) - \frac{(\delta(s,t)-1)(\delta(s,t)+1)\,Inv_{u,11}(s,t)\left(\frac{\partial}{\partial t}\delta(s,t)\right)}{(1-\delta(s,t)^2)^{3/2}} :$$

> subs({$Inv_{u,11}(s,t) = -\mathrm{delta}(s,t)\cdot Inv_{v,11}(s,t),\ Inv_{v,11}(s,t) = \sqrt{1-\delta(s,t)^2}\cdot Inv_{v,11}(s,t)$}, %) :

> rhs(%) :

> simplify(simplify(%, symbolic), size) :

>

> collect(%, $\frac{\partial^2}{\partial s^2} Inv_{v,2}(s,t)$) : collect(%, $\frac{\partial}{\partial s} Inv_{u,2}(s,t)$) : collect(%, $\frac{\partial}{\partial s} Inv_{v,2}(s,t)$) : collect(%, $Inv_{u,2}(s,t)$) : collect(%, $Inv_{v,2}(s,t)$)

$$\frac{1}{(-1+\delta(s,t)^2)^2}\Bigg(\Bigg(-(\delta(s,t)-1)(\delta(s,t)+1)\,Inv_{v,11}(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} + \left(\frac{\partial}{\partial s}\delta(s,t)\right)^2(\delta(s,t)-1)(\delta(s,t)+1)\Bigg)Inv_{v,2}(s,t)\Bigg) + \frac{1}{(-1+\delta(s,t)^2)^2}\Bigg(\Bigg(\Bigg(\Bigg((-1+\delta(s,t)^2)\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right) - \left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t)\Bigg)\sqrt{1-\delta(s,t)^2} + (\delta(s,t)-1)^2\,(\delta(s,t) \quad \textbf{(56)}$$

$$+1\big)^2 \left(\frac{\partial}{\partial s} Inv_{v,11}(s,t)\right)\Big) Inv_{u,2}(s,t)\Big) + \frac{1}{\left(-1+\delta(s,t)^2\right)^2}\Bigg(\Big(2\,(\delta(s,t)-1)\,(\delta(s,t)$$

$$+1)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} + (\delta(s,t)-1)^2\,(\delta(s,t)+1)^2\, Inv_{v,11}(s,t)\Big)\left(\frac{\partial}{\partial s} Inv_{u,2}(s,t)\right)\Bigg)$$

$$+ \frac{(\delta(s,t)-1)\,(\delta(s,t)+1)\left(\frac{\partial^2}{\partial s^2} Inv_{v,2}(s,t)\right)}{-1+\delta(s,t)^2}$$

$$+ \frac{(\delta(s,t)-1)\,(\delta(s,t)+1)\,\delta(s,t)\, Inv_{v,11}(s,t)\left(\frac{\partial}{\partial t}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}}{\left(-1+\delta(s,t)^2\right)^2}$$

> ***#simplifying coefficients***

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$$> \frac{\left(2\,(\delta(s,t)-1)\,(\delta(s,t)+1)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} + (\delta(s,t)-1)^2\,(\delta(s,t)+1)^2\, Inv_{v,11}(s,t)\right)}{\left(-1+\delta(s,t)^2\right)^2}$$

$$\frac{2\,(\delta(s,t)-1)\,(\delta(s,t)+1)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} + (\delta(s,t)-1)^2\,(\delta(s,t)+1)^2\, Inv_{v,11}(s,t)}{\left(-1+\delta(s,t)^2\right)^2} \quad \textbf{(58)}$$

> *collect*(%, $Inv_{v,11}(s,t)$)

$$\frac{(\delta(s,t)-1)^2\,(\delta(s,t)+1)^2\, Inv_{v,11}(s,t)}{\left(-1+\delta(s,t)^2\right)^2} + \frac{2\,(\delta(s,t)-1)\,(\delta(s,t)+1)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}}{\left(-1+\delta(s,t)^2\right)^2} \quad \textbf{(59)}$$

$$> simplify\left(\frac{2\,(\delta(s,t)-1)\,(\delta(s,t)+1)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$$

$$-\frac{2\left(\frac{\partial}{\partial s}\delta(s,t)\right)}{\sqrt{1-\delta(s,t)^2}} \quad \textbf{(60)}$$

$$> simplify\left(\frac{\left(\left(\left(-1+\delta(s,t)^2\right)\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right) - \left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}\right)}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$$

$$\frac{\delta(s,t)^2\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right) - \left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t) - \left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)}{\left(1-\delta(s,t)^2\right)^{3/2}} \quad \textbf{(61)}$$

> *simplify*(%, *size*)

$$\frac{\delta(s,t)^2\left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right) - \left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\delta(s,t) - \left(\frac{\partial^2}{\partial s^2}\delta(s,t)\right)}{\left(1-\delta(s,t)^2\right)^{3/2}} \quad \textbf{(62)}$$

$$> \frac{1}{\left(-1+\delta(s,t)^2\right)^2}\Bigg(\Big(-(\delta(s,t)-1)\,(\delta(s,t)+1)\, Inv_{v,11}(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2} + (\delta(s,t)$$

$$-1)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2(\delta(s,t)+1)\Big)\Bigg)$$

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$$\frac{1}{\left(-1+\delta(s,t)^2\right)^2}\left(-\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right) Inv_{v,11}(s,t)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}+\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)\right) \quad \textbf{(63)}$$

> $collect(\%, Inv_{v,11}(s,t))$

$$-\frac{\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}\, Inv_{v,11}(s,t)}{\left(-1+\delta(s,t)^2\right)^2}+\frac{\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)}{\left(-1+\delta(s,t)^2\right)^2} \quad \textbf{(64)}$$

> $simplify\left(-\frac{\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)\left(\frac{\partial}{\partial s}\delta(s,t)\right)\sqrt{1-\delta(s,t)^2}}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$

$$\frac{\frac{\partial}{\partial s}\delta(s,t)}{\sqrt{1-\delta(s,t)^2}} \quad \textbf{(65)}$$

> $simplify\left(\frac{\left(\delta(s,t)-1\right)\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2\left(\delta(s,t)+1\right)}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$

$$\frac{\left(\frac{\partial}{\partial s}\delta(s,t)\right)^2}{-1+\delta(s,t)^2} \quad \textbf{(66)}$$

> $simplify\left(\frac{\delta(s,t)\, Inv_{v,11}(s,t)\left(\frac{\partial}{\partial t}\delta(s,t)\right)\left(\delta(s,t)-1\right)\left(\delta(s,t)+1\right)\sqrt{1-\delta(s,t)^2}}{\left(-1+\delta(s,t)^2\right)^2}, symbolic\right)$

$$-\frac{Inv_{v,11}(s,t)\,\delta(s,t)\left(\frac{\partial}{\partial t}\delta(s,t)\right)}{\sqrt{1-\delta(s,t)^2}} \quad \textbf{(67)}$$

>

>

>

```
> restart
> with(LinearAlgebra) :
```
> *#8.1.21 Scaling and translating. Gauge transformations.*

> *#Normalization Equations*

> $Eq1 := \text{lambda}\cdot u(x,t) + \text{epsilon}:$

> $Eq2 := \text{lambda}\cdot \frac{\partial}{\partial x} u(x,t) - 1:$

> $solve(\{Eq1, Eq2\}, \{\text{lambda}, \text{epsilon}\}):$

> $assign(\%)$

Frame

> $RhoA := Matrix([[\text{lambda}, \text{epsilon}], [0, 1]])$

$$RhoA := \begin{bmatrix} \frac{1}{\frac{\partial}{\partial x} u(x,t)} & -\frac{u(x,t)}{\frac{\partial}{\partial x} u(x,t)} \\ 0 & 1 \end{bmatrix} \quad \textbf{(1)}$$

> $unassign('\text{lambda}','\text{epsilon}'):$

> *#Normalisation Equations Gauge*

> $Eq1 := \text{lambda}\cdot u(x,t) + \text{epsilon} - \text{alpha}(x,t):$

> $Eq2 := \text{lambda}\cdot \frac{\partial}{\partial x} u(x,t) - \text{beta}(x,t):$

> $solve(\{Eq1, Eq2\}, \{\text{lambda}, \text{epsilon}\}):$

> $assign(\%)$

> $RhoB := Matrix([[\text{lambda}, \text{epsilon}], [0, 1]])$

$$RhoB := \begin{bmatrix} \frac{\beta(x,t)}{\frac{\partial}{\partial x} u(x,t)} & \frac{\alpha(x,t)\left(\frac{\partial}{\partial x} u(x,t)\right) - \beta(x,t)\, u(x,t)}{\frac{\partial}{\partial x} u(x,t)} \\ 0 & 1 \end{bmatrix} \quad \textbf{(2)}$$

> $gauge := Matrix([[\text{beta}(x,t), \text{alpha}(x,t)], [0, 1]])$

$$gauge := \begin{bmatrix} \beta(x,t) & \alpha(x,t) \\ 0 & 1 \end{bmatrix} \quad \textbf{(3)}$$

Quick check

> $simplify(RhoB - gauge.RhoA, symbolic)$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(4)}$$

> *#Invariants*

> $RhoA \cdot Matrix([[u(x,t)], [1]])$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \textbf{(5)}$$

> $RhoA \cdot Matrix\left(\left[\left[\frac{\partial}{\partial x} u(x,t)\right], [0]\right]\right)$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \textbf{(6)}$$

> $RhoA \cdot Matrix\left(\left[\left[\frac{\partial^2}{\partial x^2} u(x,t)\right], [0]\right]\right)$

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$$\begin{bmatrix} \dfrac{\frac{\partial^2}{\partial x^2} u(x,t)}{\frac{\partial}{\partial x} u(x,t)} \\ 0 \end{bmatrix} \quad \textbf{(7)}$$

> $RhoA \cdot Matrix\left(\left[\left[\frac{\partial^k}{\partial x^k} u(x,t)\right], [0]\right]\right)$

$$\begin{bmatrix} \dfrac{\frac{\partial^k}{\partial x^k} u(x,t)}{\frac{\partial}{\partial x} u(x,t)} \\ 0 \end{bmatrix} \quad \textbf{(8)}$$

> $simplify(RhoB \cdot Matrix([[u(x,t)], [1]]), symbolic)$

$$\begin{bmatrix} \alpha(x,t) \\ 1 \end{bmatrix} \quad \textbf{(9)}$$

> $simplify\left(RhoB \cdot Matrix\left(\left[\left[\frac{\partial}{\partial x} u(x,t)\right], [0]\right]\right), symbolic\right)$

$$\begin{bmatrix} \beta(x,t) \\ 0 \end{bmatrix} \quad \textbf{(10)}$$

> $simplify\left(RhoB \cdot Matrix\left(\left[\left[\frac{\partial^2}{\partial x^2} u(x,t)\right], [0]\right]\right), symbolic\right)$

$$\begin{bmatrix} \dfrac{\beta(x,t)\left(\frac{\partial^2}{\partial x^2} u(x,t)\right)}{\frac{\partial}{\partial x} u(x,t)} \\ 0 \end{bmatrix} \quad \textbf{(11)}$$

> $simplify\left(RhoB \cdot Matrix\left(\left[\left[\frac{\partial^k}{\partial x^k} u(x,t)\right], [0]\right]\right), symbolic\right)$

$$\begin{bmatrix} \dfrac{\beta(x,t)\left(\frac{\partial^k}{\partial x^k} u(x,t)\right)}{\frac{\partial}{\partial x} u(x,t)} \\ 0 \end{bmatrix} \quad \textbf{(12)}$$

> ***#Curvature Matrices***

> $map(z \rightarrow diff(z,x), RhoA).MatrixInverse(RhoA)$:

> $subs\left(\frac{\partial^2}{\partial x^2} u(x,t) = Inv[u, [11]](x,t), \%\right)$:

> $QAx := subs\left(\frac{\partial}{\partial x} u(x,t) = 1, \%\right)$

$$QAx := \begin{bmatrix} -Inv_{u,[11]}(x,t) & -1 \\ 0 & 0 \end{bmatrix} \quad \textbf{(13)}$$

> $map(z \rightarrow diff(z,t), RhoA).MatrixInverse(RhoA)$:

> $subs\left(\frac{\partial^2}{\partial x \partial t} u(x,t) = Inv[u, [12]](x,t), \%\right)$:

> $subs\left(\frac{\partial}{\partial t}u(x,t)=Inv[u,[2]](x,t),\%\right)$:

> $QAt := subs\left(\frac{\partial}{\partial x}u(x,t)=1,\%\right)$

$$QAt := \begin{bmatrix} -Inv_{u,[12]}(x,t) & -Inv_{u,[2]}(x,t) \\ 0 & 0 \end{bmatrix} \quad \textbf{(14)}$$

> ***#Curvature Matrices Gauge***

> $map(z\rightarrow diff(z,x), RhoB).MatrixInverse(RhoB)$:

> $subs\left(\frac{\partial^2}{\partial x^2}u(x,t)=Inv[u,[11]](x,t),\%\right)$:

> $subs\left(\frac{\partial}{\partial x}u(x,t)=\text{beta}(x,t),\%\right)$:

> $QBx := simplify(subs(u(x,t)=\text{alpha}(x,t),\%),size)$

$$QBx := \left[\left[\frac{-Inv_{u,[11]}(x,t)+\frac{\partial}{\partial x}\beta(x,t)}{\beta(x,t)}, \frac{\left(\frac{\partial}{\partial x}\alpha(x,t)\right)\beta(x,t)+\alpha(x,t)\,Inv_{u,[11]}(x,t)-\left(\frac{\partial}{\partial x}\beta(x,t)\right)\alpha(x,t)-\beta(x,t)^2}{\beta(x,t)}\right], \left[0,0\right]\right] \quad \textbf{(15)}$$

> $map(z\rightarrow diff(z,t), RhoB).MatrixInverse(RhoB)$:

> $subs\left(\frac{\partial^2}{\partial x\,\partial t}u(x,t)=Inv[u,[12]](x,t),\%\right)$:

> $subs\left(\frac{\partial}{\partial x}u(x,t)=\text{beta}(x,t),\%\right)$:

> $subs\left(\frac{\partial}{\partial t}u(x,t)=Inv_{u,[2]}(x,t),\%\right)$:

> $QBt := subs(u(x,t)=\text{alpha}(x,t),\%)$

$$QBt := \left[\left[-\frac{Inv_{u,[12]}(x,t)}{\beta(x,t)}+\frac{\frac{\partial}{\partial t}\beta(x,t)}{\beta(x,t)}, \frac{\left(\frac{\partial}{\partial t}\alpha(x,t)\right)\beta(x,t)+\alpha(x,t)\,Inv_{u,[12]}(x,t)-\left(\frac{\partial}{\partial t}\beta(x,t)\right)\alpha(x,t)-\beta(x,t)\,Inv_{u,[2]}(x,t)}{\beta(x,t)}\right], \left[0,0\right]\right] \quad \textbf{(16)}$$

> ***#Here we check proposition 8.1.8***

(17)

> $map(z\rightarrow diff(z,x), gauge).MatrixInverse(gauge)+gauge.QAx.MatrixInverse(gauge)$:

> $subs\left(Inv_{u,[11]}(x,t)=\frac{1}{\text{beta}(x,t)}Inv_{u,[11]}(x,t),\%\right)$:

> $simplify(QBx-\%,symbolic)$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(18)}$$

> $map(z\rightarrow diff(z,t), gauge).MatrixInverse(gauge)+gauge\cdot QAt\cdot MatrixInverse(gauge)$:

> $subs\left(\left\{Inv_{u,[12]}(x,t) = \frac{1}{\mathrm{beta}(x,t)} Inv_{u,[12]}(x,t), Inv_{u,[2]}(x,t) = \frac{1}{\mathrm{beta}(x,t)} Inv_{u,[2]}(x,t)\right\}, \%\right)$:

> $simplify(QBt - \%, symbolic)$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{19}$$

> ***#Syzygy And H Operator***

> $syzygy := simplify(map(z \to diff(z, x), QAt) - map(z \to diff(z, t), QAx) - QAx.QAt + QAt.QAx, symbolic)$

$$syzygy := \left[\left[-\frac{\partial}{\partial x} Inv_{u,[12]}(x,t) + \frac{\partial}{\partial t} Inv_{u,[11]}(x,t), -\frac{\partial}{\partial x} Inv_{u,[2]}(x,t) - Inv_{u,[11]}(x,t)\, Inv_{u,[2]}(x,t) + Inv_{u,[12]}(x,t)\right], \left[0, 0\right]\right] \tag{20}$$

> $isolate(\mathrm{syzygy}(1,2), Inv_{u,[12]}(x,t))$

$$Inv_{u,[12]}(x,t) = \frac{\partial}{\partial x} Inv_{u,[2]}(x,t) + Inv_{u,[11]}(x,t)\, Inv_{u,[2]}(x,t) \tag{21}$$

> $subs(\%, syzygy(1,1))$

$$-\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} Inv_{u,[2]}(x,t) + Inv_{u,[11]}(x,t)\, Inv_{u,[2]}(x,t)\right) + \frac{\partial}{\partial t} Inv_{u,[11]}(x,t) \tag{22}$$

> $isolate\left(\%, \frac{\partial}{\partial t} Inv_{u,[11]}(x,t)\right)$

$$\frac{\partial}{\partial t} Inv_{u,[11]}(x,t) = \frac{\partial^2}{\partial x^2} Inv_{u,[2]}(x,t) + \left(\frac{\partial}{\partial x} Inv_{u,[11]}(x,t)\right) Inv_{u,[2]}(x,t) + Inv_{u,[11]}(x,t)\left(\frac{\partial}{\partial x} Inv_{u,[2]}(x,t)\right) \tag{23}$$

> ***#Syzygy And H Operator Gauge***

> $syzygy := simplify(map(z \to diff(z, x), QBt) - map(z \to diff(z, t), QBx) - QBx.QBt + QBt.QBx, symbolic)$

$$syzygy := \left[\left[\frac{1}{\beta(x,t)^2}\left(-\left(\frac{\partial}{\partial x} Inv_{u,[12]}(x,t)\right)\beta(x,t) + \beta(x,t)\left(\frac{\partial}{\partial t} Inv_{u,[11]}(x,t)\right) + Inv_{u,[12]}(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right) - \left(\frac{\partial}{\partial t}\beta(x,t)\right) Inv_{u,[11]}(x,t)\right), \frac{1}{\beta(x,t)^2}\left(\alpha(x,t)\,\beta(x,t)\left(\frac{\partial}{\partial x} Inv_{u,[12]}(x,t)\right) - \alpha(x,t)\,\beta(x,t)\left(\frac{\partial}{\partial t} Inv_{u,[11]}(x,t)\right) - \alpha(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right) Inv_{u,[12]}(x,t) + \alpha(x,t)\left(\frac{\partial}{\partial t}\beta(x,t)\right) Inv_{u,[11]}(x,t) + \beta(x,t)^2 Inv_{u,[12]}(x,t) - \beta(x,t)^2\left(\frac{\partial}{\partial x} Inv_{u,[2]}(x,t)\right) + \beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right) Inv_{u,[2]}(x,t) - \beta(x,t)\, Inv_{u,[2]}(x,t)\, Inv_{u,[11]}(x,t)\right)\right], \left[0, 0\right]\right] \tag{24}$$

> $isolate\left(syzygy(1,1), \frac{\partial}{\partial t} Inv_{u,[11]}(x,t)\right)$

$$\frac{\partial}{\partial t} Inv_{u,[11]}(x,t) = \frac{\left(\frac{\partial}{\partial x} Inv_{u,[12]}(x,t)\right)\beta(x,t) - Inv_{u,[12]}(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right) + \left(\frac{\partial}{\partial t}\beta(x,t)\right) Inv_{u,[11]}(x,t)}{\beta(x,t)} \tag{25}$$

> $subs\left(Inv_{u,[12]}(x,t) = \frac{\partial}{\partial x} Inv_{u,[2]}(x,t) - QBx(1,1)\cdot Inv_{u,[2]}(x,t), \%\right)$

$$\frac{\partial}{\partial t} Inv_{u,[11]}(x,t) = \frac{1}{\beta(x,t)}\left(\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} Inv_{u,[2]}(x,t) - \frac{\left(-Inv_{u,[11]}(x,t) + \frac{\partial}{\partial x}\beta(x,t)\right) Inv_{u,[2]}(x,t)}{\beta(x,t)}\right)\right)\beta(x,t) - \left(\frac{\partial}{\partial x} Inv_{u,[2]}(x,t) - \frac{\left(-Inv_{u,[11]}(x,t) + \frac{\partial}{\partial x}\beta(x,t)\right) Inv_{u,[2]}(x,t)}{\beta(x,t)}\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right) + \left(\frac{\partial}{\partial t}\beta(x,t)\right) Inv_{u,[11]}(x,t)\right) \quad \mathbf{(26)}$$

> $simplify(\%, symbolic)$:

> $simplify(\%, size)$:

> $collect\left(\%, \frac{\partial}{\partial x} Inv_{u,[2]}(x,t)\right) : collect(\%, Inv_{u,[2]}(x,t))$

$$\frac{\partial}{\partial t} Inv_{u,[11]}(x,t) = \frac{1}{\beta(x,t)^2}\left(\left(\beta(x,t)\left(\frac{\partial}{\partial x} Inv_{u,[11]}(x,t)\right) - \beta(x,t)\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right) - 2\left(\frac{\partial}{\partial x}\beta(x,t)\right) Inv_{u,[11]}(x,t) + 2\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2\right) Inv_{u,[2]}(x,t)\right) + \frac{\left(\beta(x,t)\, Inv_{u,[11]}(x,t) - 2\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)\right)\left(\frac{\partial}{\partial x} Inv_{u,[2]}(x,t)\right)}{\beta(x,t)^2} + \frac{\left(\frac{\partial^2}{\partial x^2} Inv_{u,[2]}(x,t)\right)\beta(x,t)^2 + \left(\frac{\partial}{\partial t}\beta(x,t)\right) Inv_{u,[11]}(x,t)\,\beta(x,t)}{\beta(x,t)^2} \quad \mathbf{(27)}$$

> ***#Here we check proposition 8.1.14***

> $simplify(gauge.simplify(map(z \to diff(z,x), QAt) - map(z \to diff(z,t), QAx) - QAx.QAt + QAt.QAx, symbolic).MatrixInverse(gauge), symbolic)$:

> $subs\left(\left\{Inv_{u,[12]}(x,t) = \frac{1}{\mathrm{beta}(x,t)}\cdot Inv_{u,[12]}(x,t), Inv_{u,[11]}(x,t) = \frac{1}{\mathrm{beta}(x,t)}\cdot Inv_{u,[11]}(x,t), Inv_{u,[2]}(x,t) = \frac{1}{\mathrm{beta}(x,t)}\cdot Inv_{u,[2]}(x,t)\right\}, \%\right)$:

> $simplify(\%, symbolic)$:

> $expression := \% - simplify(map(z \to diff(z,x), QBt) - map(z \to diff(z,t), QBx) - QBx.QBt + QBt.QBx, symbolic)$

$$expression := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{(28)}$$

> ***#Relationship between operators***

> $f := \frac{\mathrm{sigma}(x)}{\mathrm{beta}(x)}$:

> $\mathrm{beta}(x)\cdot(diff(diff(f,x),x) + \mathrm{kappa}(x)\cdot diff(f,x) + diff(\mathrm{kappa}(x),x)\cdot f)$:

> $simplify(\%, symbolic)$:

> $simplify(\%, size)$:

> $collect\left(\%, \frac{d^2}{dx^2}\sigma(x)\right) : collect\left(\%, \frac{d}{dx}\sigma(x)\right) : collect(\%, \sigma(x))$

$$\frac{\left(-\left(\frac{d}{dx}\beta(x)\right)\beta(x)\,\kappa(x) + \beta(x)^2\left(\frac{d}{dx}\kappa(x)\right) + 2\left(\frac{d}{dx}\beta(x)\right)^2 - \left(\frac{d^2}{dx^2}\beta(x)\right)\beta(x)\right)\sigma(x)}{\beta(x)^2} \quad \mathbf{(29)}$$

$$+\frac{\left(\beta(x)^2\,\kappa(x)-2\,\beta(x)\,\left(\frac{d}{dx}\,\beta(x)\right)\right)\left(\frac{d}{dx}\,\sigma(x)\right)}{\beta(x)^2}+\frac{d^2}{dx^2}\,\sigma(x)$$

>

>

> *restart*

> *with(LinearAlgebra) :*

> ***#ProjectiveActionSL2 - 8.2***

Action

> $U := \frac{(a \cdot u(x,t) + b)}{c \cdot u(x,t) + d}$:

> $Ux := subs\left(a\, d - b\, c = 1, simplify(diff(U, x), symbolic)\right)$

$$Ux := \frac{\frac{\partial}{\partial x} u(x,t)}{(c\, u(x,t) + d)^2} \quad (1)$$

> $Uxx := subs\left(a\, d - b\, c = 1, simplify(diff(Ux, x), symbolic)\right)$

$$Uxx := \frac{(c\, u(x,t) + d)\left(\frac{\partial^2}{\partial x^2} u(x,t)\right) - 2\left(\frac{\partial}{\partial x} u(x,t)\right)^2 c}{(c\, u(x,t) + d)^3} \quad (2)$$

> $Uxxx := subs\left(a\, d - b\, c = 1, simplify(diff(Uxx, x), symbolic)\right)$

$$Uxxx := \frac{1}{(c\, u(x,t) + d)^4}\left((c\, u(x,t) + d)^2\left(\frac{\partial^3}{\partial x^3} u(x,t)\right) - 6\left((c\, u(x,t) + d)\left(\frac{\partial^2}{\partial x^2} u(x,t)\right) - \left(\frac{\partial}{\partial x} u(x,t)\right)^2 c\right)\left(\frac{\partial}{\partial x} u(x,t)\right) c\right) \quad (3)$$

> $d := \frac{(1 + b \cdot c)}{a}$:

> ***#Normalization Equations***

> $solve(\{U, Ux - 1, Uxx\}, \{a, b, c\})$:

> $S := allvalues(\%)$:

> $S[1]$

$$\left\{a = \sqrt{\frac{1}{\frac{\partial}{\partial x} u(x,t)}}, b = -\sqrt{\frac{1}{\frac{\partial}{\partial x} u(x,t)}}\, u(x,t), c = \frac{\frac{\partial^2}{\partial x^2} u(x,t)}{2\sqrt{\frac{1}{\frac{\partial}{\partial x} u(x,t)}}\left(\frac{\partial}{\partial x} u(x,t)\right)^2}\right\} \quad (4)$$

> $assign(\%)$

> $rhoA := simplify(simplify(Matrix([[a, b], [c, d]]), symbolic), size)$

$$rhoA := \begin{bmatrix} \frac{1}{\sqrt{\frac{\partial}{\partial x} u(x,t)}} & -\frac{u(x,t)}{\sqrt{\frac{\partial}{\partial x} u(x,t)}} \\ \frac{\frac{\partial^2}{\partial x^2} u(x,t)}{2\left(\frac{\partial}{\partial x} u(x,t)\right)^{3/2}} & -\frac{u(x,t)\left(\frac{\partial^2}{\partial x^2} u(x,t)\right) - 2\left(\frac{\partial}{\partial x} u(x,t)\right)^2}{2\left(\frac{\partial}{\partial x} u(x,t)\right)^{3/2}} \end{bmatrix} \quad (5)$$

(6)

> ***#Checking the invariants***

> U

$$0 \quad (7)$$

> $simplify(Ux, symbolic)$

$$1 \quad (8)$$

> $simplify(Uxx, symbolic)$

(9)

$$0 \quad \textbf{(9)}$$

> simplify(Uxxx, symbolic)

$$\frac{2\left(\frac{\partial}{\partial x}u(x,t)\right)\left(\frac{\partial^3}{\partial x^3}u(x,t)\right)-3\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)^2}{2\left(\frac{\partial}{\partial x}u(x,t)\right)^2} \quad \textbf{(10)}$$

> unassign('a','b','c','d')

> **#Normalization Equations Gauge**

> $d := \frac{(1+b\cdot c)}{a}$:

> solve({U − alpha(x, t), Ux − beta(x, t), Uxx − delta(x, t)}, {a, b, c}) :

> S := allvalues(%) :

> S[1]

$$\left\{a=\frac{\left(\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)\alpha(x,t)\,\beta(x,t)-\left(\frac{\partial}{\partial x}u(x,t)\right)\alpha(x,t)\,\delta(x,t)+2\left(\frac{\partial}{\partial x}u(x,t)\right)\beta(x,t)^2\right)\sqrt{4}}{4\left(\frac{\partial}{\partial x}u(x,t)\right)^2\beta(x,t)^2\sqrt{\frac{1}{\beta(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right)}}}, b\right. \quad \textbf{(11)}$$

$$= -\left(\left(\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)u(x,t)\,\alpha(x,t)\,\beta(x,t)-2\left(\frac{\partial}{\partial x}u(x,t)\right)^2\alpha(x,t)\,\beta(x,t)-\left(\frac{\partial}{\partial x}u(x,t)\right)u(x,t)\,\alpha(x,t)\,\delta(x,t)+2\left(\frac{\partial}{\partial x}u(x,t)\right)u(x,t)\,\beta(x,t)^2\right)\sqrt{4}\right)\Big/\left(4\left(\frac{\partial}{\partial x}u(x,t)\right)^2\beta(x,t)^2\sqrt{\frac{1}{\beta(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right)}}\right), c$$

$$\left.=\frac{\sqrt{4}\sqrt{\frac{1}{\beta(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right)}}\left(\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)\beta(x,t)-\left(\frac{\partial}{\partial x}u(x,t)\right)\delta(x,t)\right)}{4\,\beta(x,t)\left(\frac{\partial}{\partial x}u(x,t)\right)}\right\}$$

> assign(%)

> rhoB := simplify(simplify(Matrix([[a, b], [c, d]]), symbolic), size)

$$rhoB := \left[\left[\frac{\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)\alpha(x,t)\,\beta(x,t)-\left(\frac{\partial}{\partial x}u(x,t)\right)\alpha(x,t)\,\delta(x,t)+2\left(\frac{\partial}{\partial x}u(x,t)\right)\beta(x,t)^2}{2\left(\frac{\partial}{\partial x}u(x,t)\right)^{3/2}\beta(x,t)^{3/2}},\right.\right. \quad \textbf{(12)}$$

$$\frac{1}{2\left(\frac{\partial}{\partial x}u(x,t)\right)^{3/2}\beta(x,t)^{3/2}}\left(-\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)u(x,t)\,\alpha(x,t)\,\beta(x,t)+2\left(\frac{\partial}{\partial x}u(x,t)\right)^2\alpha(x,t)\,\beta(x,t)+\left(\frac{\partial}{\partial x}u(x,t)\right)u(x,t)\,\alpha(x,t)\,\delta(x,t)-2\left(\frac{\partial}{\partial x}u(x,t)\right)u(x,t)\,\beta(x,t)^2\right)\Bigg],$$

$$\left[\frac{\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)\beta(x,t)-\left(\frac{\partial}{\partial x}u(x,t)\right)\delta(x,t)}{2\,\beta(x,t)^{3/2}\left(\frac{\partial}{\partial x}u(x,t)\right)^{3/2}},\ \frac{2\left(\frac{\partial}{\partial x}u(x,t)\right)^2\beta(x,t)-\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)u(x,t)\,\beta(x,t)+\left(\frac{\partial}{\partial x}u(x,t)\right)u(x,t)\,\delta(x,t)}{2\,\beta(x,t)^{3/2}\left(\frac{\partial}{\partial x}u(x,t)\right)^{3/2}}\right]\Bigg]$$

> *gauge* := *simplify*(*rhoB*.*MatrixInverse*(*rhoA*), *symbolic*)

$$gauge := \begin{bmatrix} \frac{-\alpha(x,t)\,\delta(x,t)+2\,\beta(x,t)^2}{2\,\beta(x,t)^{3/2}} & \frac{\alpha(x,t)}{\sqrt{\beta(x,t)}} \\ -\frac{\delta(x,t)}{2\,\beta(x,t)^{3/2}} & \frac{1}{\sqrt{\beta(x,t)}} \end{bmatrix} \quad \textbf{(13)}$$

Quick check

> *simplify*(*gauge*·*rhoA* − *rhoB*, *symbolic*)

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(14)}$$

> ***#Checking the invariants***

> $U := simplify\left(\frac{(a\cdot u(x,t)+b)}{c\cdot u(x,t)+d}, symbolic\right)$

$$U := \alpha(x,t) \quad \textbf{(15)}$$

> $Ux := simplify\left(\frac{diff(u(x,t),x)}{(c\cdot u(x,t)+d)^2}, symbolic\right)$

$$Ux := \beta(x,t) \quad \textbf{(16)}$$

> $Uxx := simplify\left(\frac{(diff(diff(u(x,t),x),x)\cdot(c\cdot u(x,t)+d)-2\cdot c\cdot(diff(u(x,t),x))^2)}{(c\cdot u(x,t)+d)^3}, symbolic\right)$

$$Uxx := \delta(x,t) \quad \textbf{(17)}$$

> *simplify*(*Uxxx*, *symbolic*) :

> $collect\left(\%, \frac{\partial^3}{\partial x^3}u(x,t)\right) : collect\left(\%, \frac{\partial^2}{\partial x^2}u(x,t)\right)$

$$-\frac{3\,\beta(x,t)\left(\frac{\partial^2}{\partial x^2}u(x,t)\right)^2}{2\left(\frac{\partial}{\partial x}u(x,t)\right)^2}+\frac{\beta(x,t)\left(\frac{\partial^3}{\partial x^3}u(x,t)\right)}{\frac{\partial}{\partial x}u(x,t)}+\frac{3\,\delta(x,t)^2}{2\,\beta(x,t)} \quad \textbf{(18)}$$

> *a* := *gauge*(1, 1) :

> *b* := *gauge*(1, 2) :

> *c* := *gauge*(2, 1) :

> *d* := *gauge*(2, 2) :

> U

$$\alpha(x, t) \quad \textbf{(19)}$$

> Ux

$$\beta(x, t) \quad \textbf{(20)}$$

> Uxx

$$\delta(x, t) \quad \textbf{(21)}$$

> $simplify(Uxxx, symbolic)$:

> $simplify\left(subs\left(\left\{\frac{\partial^3}{\partial x^3} u(x, t) = Inv[111](x, t), \frac{\partial^2}{\partial x^2} u(x, t) = 0, \frac{\partial}{\partial x} u(x, t) = 1, u(x, t) = 0\right\}, \%\right), symbolic\right)$

$$\frac{2\,\beta(x, t)^2\, Inv_{111}(x, t) + 3\,\delta(x, t)^2}{2\,\beta(x, t)} \quad \textbf{(22)}$$

> **#Curvature Matrices**

> $map(z \rightarrow diff(z, x), rhoA).MatrixInverse(rhoA)$:

> $simplify(\%, symbolic)$:

> $subs\left(\frac{\partial^3}{\partial x^3} u(x, t) = Inv[111](x, t), \%\right)$:

> $subs\left(\frac{\partial^2}{\partial x^2} u(x, t) = 0, \%\right)$:

> $QAx := subs\left(\frac{\partial}{\partial x} u(x, t) = 1, \%\right)$

$$QAx := \begin{bmatrix} 0 & -1 \\ \frac{Inv_{111}(x, t)}{2} & 0 \end{bmatrix} \quad \textbf{(23)}$$

> $map(z \rightarrow diff(z, t), rhoA).MatrixInverse(rhoA)$:

> $simplify(\%, symbolic)$:

> $subs\left(\frac{\partial^3}{\partial x^2\, \partial t} u(x, t) = Inv[112](x, t), \%\right)$:

> $subs\left(\frac{\partial^2}{\partial x^2} u(x, t) = 0, \%\right)$:

> $subs\left(\frac{\partial^2}{\partial x\, \partial t} u(x, t) = Inv[12](x, t), \%\right)$:

> $subs\left(\frac{\partial}{\partial x} u(x, t) = 1, \%\right)$:

> $QAt := subs\left(\frac{\partial}{\partial t} u(x, t) = Inv[2](x, t), \%\right)$

$$QAt := \begin{bmatrix} -\frac{Inv_{12}(x, t)}{2} & -Inv_{2}(x, t) \\ \frac{Inv_{112}(x, t)}{2} & \frac{Inv_{12}(x, t)}{2} \end{bmatrix} \quad \textbf{(24)}$$

> **#Syzygy and H operator**

> $syzygy := simplify(map(z \rightarrow diff(z, x), QAt) - map(z \rightarrow diff(z, t), QAx) - QAx.QAt + QAt.QAx, symbolic)$

$$syzygy := \left[\left[-\frac{\frac{\partial}{\partial x} Inv_{12}(x, t)}{2} + \frac{Inv_{112}(x, t)}{2} - \frac{Inv_{111}(x, t)\, Inv_{2}(x, t)}{2},\ -\frac{\partial}{\partial x} Inv_{2}(x, t) + Inv_{12}(x, t) \right],\right. \quad \textbf{(25)}$$

$$\left[\frac{\frac{\partial}{\partial x} Inv_{112}(x,t)}{2} - \frac{\frac{\partial}{\partial t} Inv_{111}(x,t)}{2} + \frac{Inv_{111}(x,t)\, Inv_{12}(x,t)}{2}, \frac{\frac{\partial}{\partial x} Inv_{12}(x,t)}{2} + \frac{Inv_{111}(x,t)\, Inv_{2}(x,t)}{2} - \frac{Inv_{112}(x,t)}{2}\right]\Bigg]$$

> *isolate*(*syzygy*(1, 2), $Inv_{12}(x,t)$)

$$Inv_{12}(x,t) = \frac{\partial}{\partial x} Inv_2(x,t) \tag{26}$$

> *subs*(%, *syzygy*(1, 1))

$$-\frac{\frac{\partial^2}{\partial x^2} Inv_2(x,t)}{2} + \frac{Inv_{112}(x,t)}{2} - \frac{Inv_{111}(x,t)\, Inv_2(x,t)}{2} \tag{27}$$

> *isolate*(%, $Inv_{112}(x,t)$)

$$Inv_{112}(x,t) = \frac{\partial^2}{\partial x^2} Inv_2(x,t) + Inv_{111}(x,t)\, Inv_2(x,t) \tag{28}$$

> $subs\left(\left\{Inv_{112}(x,t) = \frac{\partial^2}{\partial x^2} Inv_2(x,t) + Inv_{111}(x,t)\, Inv_2(x,t), Inv_{12}(x,t) = \frac{\partial}{\partial x} Inv_2(x,t)\right\}, syzygy(2,1)\right)$:

> $isolate\left(\%, \frac{\partial}{\partial t} Inv_{111}(x,t)\right)$

$$\frac{\partial}{\partial t} Inv_{111}(x,t) = \frac{\partial^3}{\partial x^3} Inv_2(x,t) + \left(\frac{\partial}{\partial x} Inv_{111}(x,t)\right) Inv_2(x,t) + 2\, Inv_{111}(x,t) \left(\frac{\partial}{\partial x} Inv_2(x,t)\right) \tag{29}$$

> ***#Curvature Matrices Gauge***

> *map*(z→*diff*(z, x), *rhoB*).*MatrixInverse*(*rhoB*) :

> *simplify*(%, *symbolic*) :

> $subs\left(\frac{\partial^3}{\partial x^3} u(x,t) = Inv[111](x,t), \%\right)$:

> $subs\left(\frac{\partial^2}{\partial x^2} u(x,t) = \mathrm{delta}(x,t), \%\right)$:

> $QBx := subs\left(\frac{\partial}{\partial x} u(x,t) = \mathrm{beta}(x,t), \%\right)$

$$QBx := \Bigg[\Bigg[\frac{1}{4\beta(x,t)^5}\Bigg(2\, Inv_{111}(x,t)\, \beta(x,t)^3 \alpha(x,t) - 3\,\delta(x,t)^2 \alpha(x,t)\, \beta(x,t)^2 - 2\,\beta(x,t)^2 \Bigg(\left(-\beta(x,t)^2 - \alpha(x,t)\,\delta(x,t)\right) \left(\frac{\partial}{\partial x}\beta(x,t)\right) + \beta(x,t)^2 \delta(x,t) + \beta(x,t)\,\alpha(x,t) \left(\frac{\partial}{\partial x}\delta(x,t)\right) - \frac{\alpha(x,t)\,\delta(x,t)^2}{2}\Bigg)\Bigg), \frac{1}{4\beta(x,t)^5}\Bigg(-2\, Inv_{111}(x,t)\,\beta(x,t)^3 \alpha(x,t)^2 + 3\,\delta(x,t)^2 \alpha(x,t)^2 \beta(x,t)^2 + 2\,\beta(x,t)^2 \Bigg(\left(-2\,\beta(x,t)^2 \alpha(x,t) - \alpha(x,t)^2 \delta(x,t)\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right) + 2\,\beta(x,t)^3 \left(\frac{\partial}{\partial x}\alpha(x,t)\right) + \left(\frac{\partial}{\partial x}\delta(x,t)\right)\beta(x,t)\,\alpha(x,t)^2 - 2\left(\beta(x,t)^2 - \frac{\alpha(x,t)\,\delta(x,t)}{2}\right)^2\Bigg)\Bigg)\Bigg], \tag{30}$$

$$\Bigg[\frac{1}{4\beta(x,t)^5}\Bigg(2\,\beta(x,t)^3 Inv_{111}(x,t) - 2\,\delta(x,t)^2 \beta(x,t)^2 + 2\,\beta(x,t)^2 \delta(x,t) \left(\frac{\partial}{\partial x}\beta(x,t)\right)$$

$$-2\,\beta(x,t)^3\left(\frac{\partial}{\partial x}\,\delta(x,t)\right)\Bigg),\ \frac{1}{4\,\beta(x,t)^5}\Bigg(-2\,Inv_{111}(x,t)\,\beta(x,t)^3\,\alpha(x,t)+3\,\delta(x,t)^2\,\alpha(x,t)\,\beta(x,t)^2+2\,\beta(x,t)^2\left(\left(-\beta(x,t)^2-\alpha(x,t)\,\delta(x,t)\right)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)+\beta(x,t)^2\,\delta(x,t)+\beta(x,t)\,\alpha(x,t)\left(\frac{\partial}{\partial x}\,\delta(x,t)\right)-\frac{\alpha(x,t)\,\delta(x,t)^2}{2}\right)\Bigg)\Bigg]\Bigg]$$

> map(z→diff(z, t), rhoB).MatrixInverse(rhoB) :

> simplify(%, symbolic) :

> $subs\left(\frac{\partial^3}{\partial x^2\,\partial t}\,u(x,t)=Inv[112](x,t),\%\right)$:

> $subs\left(\frac{\partial^2}{\partial x^2}\,u(x,t)=\mathrm{delta}(x,t),\%\right)$:

> $subs\left(\frac{\partial^2}{\partial x\,\partial t}\,u(x,t)=Inv[12](x,t),\%\right)$:

> $subs\left(\frac{\partial}{\partial x}\,u(x,t)=\mathrm{beta}(x,t),\%\right)$:

> $QBt := simplify\left(simplify\left(subs\left(\frac{\partial}{\partial t}\,u(x,t)=Inv[2](x,t),\%\right),symbolic\right),size\right)$

$$QBt := \Bigg[\Bigg[\frac{1}{2\,\beta(x,t)^3}\Bigg(\left(\beta(x,t)^2+\alpha(x,t)\,\delta(x,t)\right)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)-\beta(x,t)^2\,Inv_{12}(x,t)+\beta(x,t)\,\alpha(x,t)\,Inv_{112}(x,t)-\left(\frac{\partial}{\partial t}\,\delta(x,t)\right)\beta(x,t)\,\alpha(x,t)-\delta(x,t)\,Inv_{12}(x,t)\,\alpha(x,t)\Bigg),\ \frac{1}{2\,\beta(x,t)^3}\Bigg(-2\,Inv_2(x,t)\,\beta(x,t)^3+2\left(\frac{\partial}{\partial t}\,\alpha(x,t)\right)\beta(x,t)^3+2\,Inv_{12}(x,t)\,\beta(x,t)^2\,\alpha(x,t)-2\,\beta(x,t)^2\,\alpha(x,t)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)-\beta(x,t)\,\alpha(x,t)^2\,Inv_{112}(x,t)+\left(\frac{\partial}{\partial t}\,\delta(x,t)\right)\beta(x,t)\,\alpha(x,t)^2+\alpha(x,t)^2\,\delta(x,t)\,Inv_{12}(x,t)-\alpha(x,t)^2\,\delta(x,t)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)\Bigg)\Bigg],\ \Bigg[\frac{\beta(x,t)\,Inv_{112}(x,t)-\beta(x,t)\left(\frac{\partial}{\partial t}\,\delta(x,t)\right)-\delta(x,t)\,Inv_{12}(x,t)+\delta(x,t)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)}{2\,\beta(x,t)^3},\ \frac{1}{2\,\beta(x,t)^3}\Bigg(\beta(x,t)^2\,Inv_{12}(x,t)-\beta(x,t)^2\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)-\beta(x,t)\,\alpha(x,t)\,Inv_{112}(x,t)+\left(\frac{\partial}{\partial t}\,\delta(x,t)\right)\beta(x,t)\,\alpha(x,t)+\delta(x,t)\,Inv_{12}(x,t)\,\alpha(x,t)-\alpha(x,t)\,\delta(x,t)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)\Bigg)\Bigg]\Bigg] \tag{31}$$

> **#syzygyGauge**

> syzygy := simplify(map(z→diff(z, x), QBt) − map(z→diff(z, t), QBx) − QBx.QBt + QBt.QBx, symbolic) :

>

> $simplify\left(isolate\left(syzygy(2,1),\frac{\partial}{\partial t}\,Inv_{111}(x,t)\right),symbolic\right)$

$$\frac{\partial}{\partial t}\,Inv_{111}(x,t)=\frac{1}{\beta(x,t)^2}\Bigg(\left(Inv_{111}(x,t)\,\beta(x,t)-3\,\delta(x,t)^2\right)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)+\left(-\beta(x,t)\,Inv_{112}(x,t)\right. \tag{32}$$

$$+3\,\delta(x,t)\,Inv_{12}(x,t)\Big)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+3\left(\delta(x,t)\left(\frac{\partial}{\partial t}\delta(x,t)\right)-\frac{2\left(\frac{\partial}{\partial x}\delta(x,t)\right)Inv_{12}(x,t)}{3}\right.$$

$$-\frac{\delta(x,t)\left(\frac{\partial}{\partial x}Inv_{12}(x,t)\right)}{3}+\frac{\beta(x,t)\left(\frac{\partial}{\partial x}Inv_{112}(x,t)\right)}{3}-\frac{\delta(x,t)\,Inv_{112}(x,t)}{3}$$

$$\left.+\frac{Inv_{111}(x,t)\,Inv_{12}(x,t)}{3}\right)\beta(x,t)\Bigg)$$

>

> $subs\left(Inv_{112}(x,t)=\frac{\left(-2\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+2\,\beta(x,t)\,\delta(x,t)\right)\left(\frac{\partial}{\partial x}Inv_2(x,t)\right)}{\beta(x,t)^2}\right.$

$$+\frac{1}{\beta(x,t)^2}\left(\left(-\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)+Inv_{111}(x,t)\,\beta(x,t)+2\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2-2\left(\frac{\partial}{\partial x}\beta(x,\right.\right.\right.$$

$$\left.\left.\left.t)\right)\delta(x,t)\right)Inv_2(x,t)\right)+\frac{\partial^2}{\partial x^2}Inv_2(x,t),\%\Bigg)$$

$$\frac{\partial}{\partial t}Inv_{111}(x,t)=\frac{1}{\beta(x,t)^2}\Bigg(\left(Inv_{111}(x,t)\,\beta(x,t)-3\,\delta(x,t)^2\right)\left(\frac{\partial}{\partial t}\beta(x,t)\right)+\Bigg(-\beta(x,\qquad\textbf{(33)}$$

$$t)\left(\frac{\left(-2\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+2\,\beta(x,t)\,\delta(x,t)\right)\left(\frac{\partial}{\partial x}Inv_2(x,t)\right)}{\beta(x,t)^2}\right.$$

$$+\frac{1}{\beta(x,t)^2}\left(\left(-\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)+Inv_{111}(x,t)\,\beta(x,t)+2\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2-2\left(\frac{\partial}{\partial x}\beta(x,\right.\right.\right.$$

$$\left.\left.\left.t)\right)\delta(x,t)\right)Inv_2(x,t)\right)+\frac{\partial^2}{\partial x^2}Inv_2(x,t)\Bigg)+3\,\delta(x,t)\,Inv_{12}(x,t)\Bigg)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+3\left(\delta(x,t)\left(\frac{\partial}{\partial t}\right.\right.$$

$$\left.\delta(x,t)\right)-\frac{2\left(\frac{\partial}{\partial x}\delta(x,t)\right)Inv_{12}(x,t)}{3}-\frac{\delta(x,t)\left(\frac{\partial}{\partial x}Inv_{12}(x,t)\right)}{3}+\frac{1}{3}\Bigg(\beta(x,t)\Bigg(\frac{\partial}{\partial x}$$

$$\left(\frac{\left(-2\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+2\,\beta(x,t)\,\delta(x,t)\right)\left(\frac{\partial}{\partial x}Inv_2(x,t)\right)}{\beta(x,t)^2}\right.$$

$$+\frac{1}{\beta(x,t)^2}\left(\left(-\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)+Inv_{111}(x,t)\,\beta(x,t)+2\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2-2\left(\frac{\partial}{\partial x}\beta(x,t)\right)\delta(x,t)\right)Inv_2(x,t)\right)+\frac{\partial^2}{\partial x^2}Inv_2(x,t)\Bigg)\Bigg)\Bigg)-\frac{1}{3}\Bigg(\delta(x,t)\Bigg(\frac{\left(-2\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+2\,\beta(x,t)\,\delta(x,t)\right)\left(\frac{\partial}{\partial x}Inv_2(x,t)\right)}{\beta(x,t)^2}$$

$$+\frac{1}{\beta(x,t)^2}\left(\left(-\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)+Inv_{111}(x,t)\,\beta(x,t)+2\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2-2\left(\frac{\partial}{\partial x}\beta(x,t)\right)\delta(x,t)\right)Inv_2(x,t)\right)+\frac{\partial^2}{\partial x^2}Inv_2(x,t)\Bigg)\Bigg)+\frac{Inv_{111}(x,t)\,Inv_{12}(x,t)}{3}\Bigg)\beta(x,t)\Bigg)$$

> $subs\left(Inv_{12}(x,t)=\frac{\left(\frac{\partial}{\partial x}Inv_2(x,t)\right)\beta(x,t)-Inv_2(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)+Inv_2(x,t)\,\delta(x,t)}{\beta(x,t)},\%\right)$

> $simplify(\%, size)$:

> $collect(\%, Inv_2(x,t)) : collect\left(\%, \frac{\partial}{\partial x}Inv_2(x,t)\right) : collect\left(\%, \frac{\partial^2}{\partial x^2}Inv_2(x,t)\right) : collect\left(\%, \frac{\partial^3}{\partial x^3}Inv_2(x,t)\right) :$
$collect\left(\%, \frac{\partial}{\partial t}\beta(x,t)\right) : collect\left(\%, \frac{\partial}{\partial t}\delta(x,t)\right) :$

> $RHS := \%$

$$RHS := \frac{\partial}{\partial t}Inv_{111}(x,t)=\frac{\left(-3\,\delta(x,t)^2\beta(x,t)+\beta(x,t)^2\,Inv_{111}(x,t)\right)\left(\frac{\partial}{\partial t}\beta(x,t)\right)}{\beta(x,t)^3}+\frac{\partial^3}{\partial x^3}Inv_2(x,t)-\frac{3\left(\frac{\partial}{\partial x}\beta(x,t)\right)\left(\frac{\partial^2}{\partial x^2}Inv_2(x,t)\right)}{\beta(x,t)}+\frac{1}{\beta(x,t)^3}\left(\left(-3\,\delta(x,t)^2\beta(x,t)-3\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)^2+2\,\beta(x,t)^2\,Inv_{111}(x,t)+6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2\beta(x,t)\right)\left(\frac{\partial}{\partial x}Inv_2(x,t)\right)\right)+\frac{1}{\beta(x,t)^3}\left(\left(6\,\delta(x,t)^2\left(\frac{\partial}{\partial x}\beta(x,t)\right)-3\,\delta(x,t)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\beta(x,t)+6\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right)\beta(x,t)-3\,Inv_{111}(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)\beta(x,t)-6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^3+\left(\frac{\partial}{\partial x}Inv_{111}(x,t)\right)\beta(x,t)^2-\left(\frac{\partial^3}{\partial x^3}\beta(x,t)\right)\beta(x,t)^2\right)Inv_2(x,t)\right)+\frac{3\,\delta(x,t)\left(\frac{\partial}{\partial t}\delta(x,t)\right)}{\beta(x,t)} \quad \textbf{(34)}$$

> ***#Simplifying the coefficients***

> $\frac{\left(-3\,\beta(x,t)^2\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)+6\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2+2\,\beta(x,t)^2\,Inv_{111}(x,t)-3\,\beta(x,t)\,\delta(x,t)^2\right)}{\beta(x,t)^3}$:

> $simplify(\%, size)$

(35)

$$\frac{-3\,\delta(x,t)^2 + 2\,Inv_{111}(x,t)\,\beta(x,t) + 6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^2 - 3\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)}{\beta(x,t)^2} \quad \textbf{(35)}$$

> $\frac{1}{\beta(x,t)^3}\left(6\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right) - \beta(x,t)^2\left(\frac{\partial^3}{\partial x^3}\beta(x,t)\right) + \left(\frac{\partial}{\partial x}Inv_{111}(x,t)\right)\beta(x,t)^2 - 6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^3 - 3\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)Inv_{111}(x,t) + 6\left(\frac{\partial}{\partial x}\beta(x,t)\right)\delta(x,t)^2 - 3\left(\frac{\partial}{\partial x}\delta(x,t)\right)\beta(x,t)\,\delta(x,t)\right)$:

> *simplify*(%, *size*) :

> *collect*$\left(\%, \frac{\partial^2}{\partial x^2}\beta(x,t)\right)$: *collect*$\left(\%, \frac{\partial}{\partial x}\beta(x,t)\right)$: *collect*$(\%, \delta(x,t))$: *collect*$(\%, Inv_{111}(x,t))$

$$-\frac{3\,Inv_{111}(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)}{\beta(x,t)^2} + \frac{6\,\delta(x,t)^2\left(\frac{\partial}{\partial x}\beta(x,t)\right)}{\beta(x,t)^3} - \frac{3\,\delta(x,t)\left(\frac{\partial}{\partial x}\delta(x,t)\right)}{\beta(x,t)^2} - \frac{6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^3}{\beta(x,t)^3} + \frac{6\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right)}{\beta(x,t)^2} + \frac{\left(\frac{\partial}{\partial x}Inv_{111}(x,t)\right)\beta(x,t)^2 - \left(\frac{\partial^3}{\partial x^3}\beta(x,t)\right)\beta(x,t)^2}{\beta(x,t)^3} \quad \textbf{(36)}$$

> $\frac{1}{2}\,\frac{1}{\beta(x,t)^3}\left(\left(-2\,\beta(x,t)^2\left(\frac{\partial^3}{\partial x^3}\beta(x,t)\right) + 12\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right) - 2\,\beta(x,t)\left(\frac{\partial}{\partial x}\beta(x,t)\right)Inv_{111}(x,t) + 2\,\beta(x,t)\,\delta(x,t)\,Inv_{111}(x,t) - 12\left(\frac{\partial}{\partial x}\beta(x,t)\right)^3 + 3\left(\frac{\partial}{\partial x}\beta(x,t)\right)\delta(x,t)^2 - 3\,\delta(x,t)^3\right)\right)$:

> *simplify*(%, *size*) :

> *collect*$\left(\%, \frac{\partial^3}{\partial x^3}\beta(x,t)\right)$: *collect*$\left(\%, \frac{\partial^2}{\partial x^2}\beta(x,t)\right)$: *collect*$\left(\%, \frac{\partial}{\partial x}\beta(x,t)\right)$: *collect*$(\%, \delta(x,t))$: *collect*$(\%, Inv_{111}(x,t))$

$$\left(\frac{\delta(x,t)}{\beta(x,t)^2} - \frac{\frac{\partial}{\partial x}\beta(x,t)}{\beta(x,t)^2}\right)Inv_{111}(x,t) - \frac{3\,\delta(x,t)^3}{2\,\beta(x,t)^3} + \frac{3\,\delta(x,t)^2\left(\frac{\partial}{\partial x}\beta(x,t)\right)}{2\,\beta(x,t)^3} - \frac{6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^3}{\beta(x,t)^3} + \frac{6\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right)}{\beta(x,t)^2} - \frac{\frac{\partial^3}{\partial x^3}\beta(x,t)}{\beta(x,t)} \quad \textbf{(37)}$$

> ***# Here we write HA in terms of HB***

> *subs*$\left(Inv_{111}(x,t) = \frac{1}{2}\,\frac{2\,Inv_{111}(x,t)\,\beta(x,t)^2 + 3\,\delta(x,t)^2}{\beta(x,t)}, RHS\right)$:

> *simplify*(%, *size*) :

> *collect*$(\%, Inv_2(x,t))$: *collect*$\left(\%, \frac{\partial}{\partial x}Inv_2(x,t)\right)$: *collect*$\left(\%, \frac{\partial^2}{\partial x^2}Inv_2(x,t)\right)$: *collect*$\left(\%, \frac{\partial^3}{\partial x^3}Inv_2(x,t)\right)$: *collect*$\left(\%, \frac{\partial}{\partial t}\beta(x,t)\right)$: *collect*$\left(\%, \frac{\partial}{\partial t}\delta(x,t)\right)$

$$\frac{3\,\delta(x,t)\left(\frac{\partial}{\partial t}\,\delta(x,t)\right)}{\beta(x,t)}+\frac{\left(2\,\beta(x,t)^2\,Inv_{111}(x,t)-3\,\delta(x,t)^2\right)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)}{2\,\beta(x,t)^2}+\left(\frac{\partial}{\partial t}\,Inv_{111}(x,t)\right)\beta(x,t)=\frac{\left(2\,\beta(x,t)^3\,Inv_{111}(x,t)-3\,\delta(x,t)^2\,\beta(x,t)\right)\left(\frac{\partial}{\partial t}\,\beta(x,t)\right)}{2\,\beta(x,t)^3}+\frac{\partial^3}{\partial x^3}\,Inv_2(x,t)-\frac{3\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\left(\frac{\partial^2}{\partial x^2}\,Inv_2(x,t)\right)}{\beta(x,t)}+\frac{1}{2\,\beta(x,t)^3}\left(\left(4\,\beta(x,t)^3\,Inv_{111}(x,t)+12\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^2\beta(x,t)-6\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)\beta(x,t)^2\right)\left(\frac{\partial}{\partial x}\,Inv_2(x,t)\right)\right)+\frac{1}{2\,\beta(x,t)^3}\left(\left(2\left(\frac{\partial}{\partial x}\,Inv_{111}(x,t)\right)\beta(x,t)^3-4\,Inv_{111}(x,t)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\beta(x,t)^2-12\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^3+12\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\beta(x,t)-2\left(\frac{\partial^3}{\partial x^3}\,\beta(x,t)\right)\beta(x,t)^2\right)Inv_2(x,t)\right)+\frac{3\,\delta(x,t)\left(\frac{\partial}{\partial t}\,\delta(x,t)\right)}{\beta(x,t)} \quad \textbf{(38)}$$

> *#simplifying coefficients*

> $\frac{1}{2}\,\frac{\left(4\,\beta(x,t)^3\,Inv_{111}(x,t)-6\,\beta(x,t)^2\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)+12\,\beta(x,t)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^2\right)}{\beta(x,t)^3}$

$$\frac{4\,\beta(x,t)^3\,Inv_{111}(x,t)+12\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^2\beta(x,t)-6\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)\beta(x,t)^2}{2\,\beta(x,t)^3} \quad \textbf{(39)}$$

> *simplify*(%, *size*)

$$\frac{2\,\beta(x,t)^2\,Inv_{111}(x,t)+6\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^2-3\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)\beta(x,t)}{\beta(x,t)^2} \quad \textbf{(40)}$$

> *collect*(%, $Inv_{111}(x,t)$) : *collect*(%, $\frac{\partial}{\partial x}\,\beta(x,t)$)

$$2\,Inv_{111}(x,t)+\frac{6\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^2}{\beta(x,t)^2}-\frac{3\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)}{\beta(x,t)} \quad \textbf{(41)}$$

> $\frac{1}{2}\,\frac{1}{\beta(x,t)^3}\left(\left(2\,\beta(x,t)^3\left(\frac{\partial}{\partial x}\,Inv_{111}(x,t)\right)-4\,\beta(x,t)^2\,Inv_{111}(x,t)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)-2\,\beta(x,t)^2\left(\frac{\partial^3}{\partial x^3}\,\beta(x,t)\right)+12\,\beta(x,t)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)-12\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^3\right)\right)$

$$\frac{1}{2\,\beta(x,t)^3}\left(2\left(\frac{\partial}{\partial x}\,Inv_{111}(x,t)\right)\beta(x,t)^3-4\,Inv_{111}(x,t)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\beta(x,t)^2-12\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^3+12\left(\frac{\partial^2}{\partial x^2}\,\beta(x,t)\right)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\beta(x,t)-2\left(\frac{\partial^3}{\partial x^3}\,\beta(x,t)\right)\beta(x,t)^2\right) \quad \textbf{(42)}$$

> *simplify*(%, *size*)

$$\frac{1}{\beta(x,t)^3}\left(\left(\frac{\partial}{\partial x}\,Inv_{111}(x,t)\right)\beta(x,t)^3-2\,Inv_{111}(x,t)\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)\beta(x,t)^2-6\left(\frac{\partial}{\partial x}\,\beta(x,t)\right)^3 \quad \textbf{(43)}$$

$$+6\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right)\beta(x,t)-\left(\frac{\partial^3}{\partial x^3}\beta(x,t)\right)\beta(x,t)^2\Big)$$

> $collect\left(\%, \frac{\partial}{\partial x} Inv_{111}(x,t)\right) : collect\left(\%, \frac{\partial}{\partial x}\beta(x,t)\right)$

$$-\frac{6\left(\frac{\partial}{\partial x}\beta(x,t)\right)^3}{\beta(x,t)^3}+\frac{\left(-2\,\beta(x,t)^2 Inv_{111}(x,t)+6\left(\frac{\partial^2}{\partial x^2}\beta(x,t)\right)\beta(x,t)\right)\left(\frac{\partial}{\partial x}\beta(x,t)\right)}{\beta(x,t)^3}+\frac{\partial}{\partial x} Inv_{111}(x,t)-\frac{\frac{\partial^3}{\partial x^3}\beta(x,t)}{\beta(x,t)} \quad \textbf{(44)}$$

> $\frac{1}{2}\,\frac{\left(2\,\beta(x,t)^3 Inv_{111}(x,t)-3\,\beta(x,t)\,\delta(x,t)^2\right)}{\beta(x,t)^3}$

$$\frac{2\,\beta(x,t)^3 Inv_{111}(x,t)-3\,\delta(x,t)^2\,\beta(x,t)}{2\,\beta(x,t)^3} \quad \textbf{(45)}$$

> $simplify(\%, size)$

$$-\frac{-2\,\beta(x,t)^2 Inv_{111}(x,t)+3\,\delta(x,t)^2}{2\,\beta(x,t)^2} \quad \textbf{(46)}$$

> $collect\left(\%, Inv_{111}(x,t)\right) : collect\left(\%, \frac{\partial}{\partial x}\beta(x,t)\right)$

$$Inv_{111}(x,t)-\frac{3\,\delta(x,t)^2}{2\,\beta(x,t)^2} \quad \textbf{(47)}$$

> ***#Checking proposition 8.1.8***

> $map(z \to diff(z,x), gauge).MatrixInverse(gauge) + gauge.QAx.MatrixInverse(gauge)$:

> $subs\left(Inv_{111}(x,t)=\frac{1}{2}\,\frac{2\,Inv_{111}(x,t)\,\beta(x,t)-3\,\delta(x,t)^2}{\beta(x,t)^2}, \%\right)$:

> $simplify(\% - QBx, symbolic)$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \textbf{(48)}$$

> $unassign('a','b','c','d')$

> $gu := \frac{(a \cdot u(x,t)+b)}{(c \cdot u(x,t)+d)}$:

> $Invt := subs\left(a\,d-b\,c=1, simplify(diff(gu,t), symbolic)\right)$

$$Invt := \frac{\frac{\partial}{\partial t}u(x,t)}{\left(c\,u(x,t)+d\right)^2} \quad \textbf{(49)}$$

> $subs\left(\left\{\frac{\partial^2}{\partial x\,\partial t}u(x,t)=Inv[12](x,t), \frac{\partial}{\partial t}u(x,t)=Inv[2](x,t), \frac{\partial}{\partial x}u(x,t)=1, u(x,t)=0\right\}, Invt\right)$:

> $subs\left(d=\frac{1}{\sqrt{\beta(x,t)}}, c=-\frac{1}{2}\,\frac{\delta(x,t)}{\beta(x,t)^{3/2}}, \%\right)$

$$Inv_2(x,t)\,\beta(x,t) \quad \textbf{(50)}$$

> $Invt := \frac{\frac{\partial}{\partial t}u(x,t)}{\left(c\,u(x,t)+d\right)^2}$:

> $Invtx := diff(\%, x)$

$$Invtx := \frac{\frac{\partial^2}{\partial t \partial x} u(x,t)}{(c\, u(x,t) + d)^2} - \frac{2 \left(\frac{\partial}{\partial t} u(x,t)\right) c \left(\frac{\partial}{\partial x} u(x,t)\right)}{(c\, u(x,t) + d)^3} \tag{51}$$

> $Invtxx := simplify(diff(\%, x), size) :$

> $subs\left(\left\{\frac{\partial^2}{\partial x \partial t} u(x,t) = Inv[12](x,t), \frac{\partial}{\partial t} u(x,t) = Inv[2](x,t), \frac{\partial}{\partial x} u(x,t) = 1\right\}, Invtx\right) :$

> $subs(u(x,t) = 0, \%) :$

> $subs\left(d = \frac{1}{\sqrt{\beta(x,t)}}, c = -\frac{1}{2} \frac{\delta(x,t)}{\beta(x,t)^{3/2}}, \%\right)$

$$Inv_{12}(x,t)\, \beta(x,t) + \delta(x,t)\, Inv_2(x,t) \tag{52}$$

> $subs\left(\left\{\frac{\partial^3}{\partial x^2 \partial t} u(x,t) = Inv[112](x,t), \frac{\partial^2}{\partial x \partial t} u(x,t) = Inv[12](x,t), \frac{\partial}{\partial t} u(x,t) = Inv[2](x,t), \frac{\partial}{\partial x} u(x,t) = 1\right\}, Invtxx\right) :$

> $subs\left(u(x,t) = 0, d = \frac{1}{\sqrt{\beta(x,t)}}, c = -\frac{1}{2} \frac{\delta(x,t)}{\beta(x,t)^{3/2}}, \%\right) :$

> $simplify(\%, symbolic) :$

> $collect(\%, Inv_{112}(x,t)) : collect(\%, Inv_{12}(x,t)) : collect(\%, Inv_2(x,t))$

$$\frac{3\, \delta(x,t)^2\, Inv_2(x,t)}{2\, \beta(x,t)} + 2\, \delta(x,t)\, Inv_{12}(x,t) + \beta(x,t)\, Inv_{112}(x,t) \tag{53}$$

> $-MatrixInverse(gauge).map(z \to diff(z,t), gauge) + MatrixInverse(gauge).QBt.gauge :$

> $subs\left(\left\{Inv_{112}(x,t) = \frac{1}{2} \frac{3\, Inv_2(x,t)\, \delta(x,t)^2 + 4\, \delta(x,t)\, \beta(x,t)\, Inv_{12}(x,t) + 2\, \beta(x,t)^2\, Inv_{112}(x,t)}{\beta(x,t)}, Inv_{12}(x,t) = Inv_{12}(x,t)\, \beta(x,t) + Inv_2(x,t)\, \delta(x,t), Inv_2(x,t) = \mathrm{beta}(x,t) \cdot Inv_2(x,t)\right\}, \%\right) :$

> $simplify(\% - QAt, symbolic)$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{54}$$

> ***#Checking proposition 8.1.14***

> $MatrixInverse(gauge).simplify(map(z \to diff(z,x), QBt) - map(z \to diff(z,t), QBx) - QBx.QBt + QBt.QBx, symbolic).gauge :$

> $subs\left(\left\{Inv_{112}(x,t) = \beta(x,t)\, Inv_{112}(x,t) + 2\, \delta(x,t)\, Inv_{12}(x,t) + \frac{3}{2} \frac{Inv_2(x,t)\, \delta(x,t)^2}{\beta(x,t)}, Inv_{12}(x,t) = Inv_{12}(x,t)\, \beta(x,t) + Inv_2(x,t)\, \delta(x,t), Inv_2(x,t) = \mathrm{beta}(x,t) \cdot Inv_2(x,t), Inv[1](x,t) = \mathrm{beta}(x,t), Inv[11](x,t) = \mathrm{delta}(x,t), Inv[111](x,t) = \mathrm{beta}(x,t) \cdot Inv[111](x,t) + \frac{3}{2} \cdot \frac{\mathrm{delta}(x,t)^2}{\mathrm{beta}(x,t)}\right\}, \%\right) :$

> $EXPRESION := simplify(\% - simplify(map(z \to diff(z,x), QAt) - map(z \to diff(z,t), QAx) - QAx.QAt + QAt.QAx, symbolic), symbolic)$

$$EXPRESION := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{55}$$

>

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