Alternative scales in reliability models for a repairable system

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Abstract

In an industry, the lifetime of a technical system is often assessed according to its accumulated throughput or usage. Performance of Blast Furnace, in a steel making factory, is assessed in terms of the accumulated quantity of its product (i.e. liquid iron); the lifetime of a vehicle in the transport industry, can be assessed in terms of the accumulated number of miles it has travelled or the accumulated amount of load it has transported. Most of these systems are repairable systems. The failure process of a system is conventionally measured in the time domain. Nevertheless, the lifetime of some repairable systems and their failures may be measured in terms of their throughput/usage. Therefore, it makes sense to quantify their failure processes in terms of accumulated throughput or usage. These accumulated usages may be better indicators than time, of system failure and reliability and hence can form better scales for quantifying the failure process of the system. Such scales, individually or in combination with time, may be used as alternative scales of measurement in modelling the failure process. This paper proposes alternative scales, considering usage along with time, to measure the failure process of a repairable system. A method is devised in the paper to

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identify better scales to model the failure process and the appropriate scale to assess reliability identified. Industrial failure data are used to illustrate the proposed method.

Key words: System condition, time scale, alternative scale, imperfect repair, repairable system

Notation table

t^Z	Alternative scale for modelling the failure process of a repairable system
$Z(t)$	Usage scale for modelling the failure process of a repairable system
$\phi\,(t, Z(t))$	Alternative scale, which is a function of the primary measure t and concomitant measure $Z(t)$
t	Global time scale for modelling the failure process of a repairable system
x	Local time scale for modelling the failure process of a repairable system
m	Global mileage scale for modelling the failure process of a repairable system
w	Local mileage scale for modelling the failure process of a repairable system
u	Local mileage rate for modelling the failure process of a repairable system
γ	Weightage parameter of the individual scales in a combined alternative scale
N_{t^-}	Number of failures before t
$T_{N_{t^-}}$	Time of the previous failure before t
$M_{N_{t^-}}$	Usage at the previous failure before t
H_{t^-}	History of the failure process prior to t. Includes the number of failures, failure times and any other covariate information on the failure process
$R(t \mid H_{t^-})$	Reliability of the repairable system given the history of the failure process prior to time t

$\lambda(t \mid H_{t^-})$	Intensity of the failure process of the repairable system given the history of the failure process prior to time t
$G(t^Z)$	Function of the alternative scale t^Z
$W(t)$	Covariate information on the failure process
η	Weightage parameter of the covariate

1. Introduction

1.1 Motivation

In an industry, the lifetime of a technical system is often assessed according to its accumulated throughput or usage. For example, in a steel making factory, the performance of a blast furnace is assessed in terms of the accumulated quantity of its product (i.e. liquid iron); in the transport industry, the lifetime of a vehicle can be assessed in terms of the accumulated number of miles it has travelled or the accumulated amount of load it has transported. Most of these systems are repairable systems. The lifetime of these systems can be assessed in terms of their total throughput/usage, and maintenance policies can be formulated based on this. For example, a blast furnace is taken for a capital repair of categories 1, 2 or 3 based on the tonnage of iron it has produced, similarly maintenance on a transport vehicle can be performed based on mileages accrued.

In the reliability literature, mathematical models, which are used to depict the failure process of systems, are conventionally functions of *time* domain. That is, the reliability of a system is typically measured with respect to the *time domain*. Nevertheless, the lifetime of some repairable systems and their failures may be measured in terms of their throughput/usage. Therefore, it makes sense to quantify their failure processes in terms of accumulated throughput or usage. This leads to the development of alternative domains for quantifying the reliability and/or the failure intensity function of such systems. Such domains need not be uni-dimensional i.e., either time or usage. These can also be a combination of both time and usage to create bi/multi-dimensional domains.

The time domain of the reliability function and failure intensity of a repairable system is often termed as a time scale in the literature (Farewell and Cox (1979), Kordonsky and Gertsbakh (1995a, 1995b, 1997), Duchesne and Lawless (2000), and Finkelstein (2004), to name a few). Since other domains/scales are being considered in this paper for modelling the reliability and failure intensity, all domains/scales have been termed as alternative scales, which can be time, usage or a combination of time and usage as a scale.

1.2 Related work

The use of alternative scales to quantify the failure process and reliability have been first investigated by Kordonsky and Gertsbakh (1995a, 1995b, 1997) in the context of airline industry. They proposed a linear / additive combination of a number of scales such as calendar time, flight time, the number of flights or landings and take offs for aircraft, to model the reliability of aircraft. Though these systems are repairable they treat the systems as non-repairable in their analysis. They use co-efficient of variation to distinguish between scales to identify the better scale for modelling their failure process and reliability. Their ideas of additive combination of scales and least variation with respect to failure times as the criteria for selection of better scale can be extended to develop similar concepts for a repairable system.

Duchesne and Lawless (2000) carried out an exhaustive study on alternative scales for non-repairable systems. They have stated that the qualities required for a good alternative scale are, *relevance* in scientific terms, which captures most of the variation in failure times under varying usage measures. These hold good in the context of repairable systems also and can be used define the criteria to select the better alternative scale to model the system's reliability and failure intensity. They also indicated that the effects of varying environmental conditions can also be considered while formulating the alternative scales. These can also be extended to the case of a repairable system.

Alternative scales were first proposed for repairable systems in the context of cars where two dimensional data, times and mileages at failure are available. Lawless et al. (1995) proposed a single combined alternative scale consisting of a multiplicative combination of time and mileage to address the first failure times of multiple cars. Lawless et al. (2009) applied this alternative scale to model all the failure times of multiple cars in the context of automotive warranties, where multiple failure data is available at each failure time. They do not address the use of this scale for a single / individual car where the sparse data available makes the estimation of the parameters more difficult. Ahn et al. (1998) redressed this by using the alternative scale proposed by Lawless (1995) for six individual cars. However, they did not carry out simultaneous estimation of the parameters while using the method of maximum likelihood. The above papers have not considered an imperfect repair process which is the more general process for modelling repairable systems. Krivstov and Frankstein (2006) use the alternative scale developed by Lawless et al. (1995) and conclude that the most important criterion in deciding the alternative scale is the engineering relevance of the failure mode, random failures being reflected through time and wear out / deterioration being reflected through mileage. None of the above papers have provided a distinct methodology to identify the better scale for a car.

1.3 Novelty and contribution

In the existing literature, alternatives scales, other than the one proposed by Ahn et al. (1998), have not been proposed for an individual repairable system. No attempt has been made to develop and apply different alternative scales to the failure data of a repairable system. The proposed scales have not been used in modelling imperfect repair, which are more applicable to repairable systems. In the literature, little research has been found on how to identify and choose which alternative scale is a better scale in assessing the failure process of a repairable system.

These issues are dealt with in this paper. The paper proposes different alternative scales for an individual/single repairable system as opposed to multiple repairable systems. It then proposes a method to determine which alternative scale is a better one to model its failure process and reliability. The paper also extends the concepts associated with such alternative scales as proposed by Duchesne and Lawless (2000) to a repairable system. This methodology can in turn be easily extended to the case of multiple repairable systems with or without heterogeneity and also for non-repairable systems.

1.4 Overview

The remainder of this paper is structured as follows. Various alternative scales for modelling the failure process and reliability of a repairable system are proposed in section 2. Section 3 incorporates these alternative scales into the reliability and intensity functions of a repairable system. Section 4 provides methods for parameter inference. Section 5 applies the developed models to the failure data of individual repairable systems to identify the better alternative scale. Section 6 concludes the paper.

2. Alternative scales

As discussed above, the reliability and failure intensity of a repairable system can be modelled in terms of a time scale, usage scale or their combination, which may form alternative scales in reliability models for a repairable system.

The usage measures chosen here are external to the system in which failure takes place and are dynamic i.e., they vary in time. Internal measures of system condition / deterioration are not considered here. These lead to joint distributions with failure times and the convolution of distributions and are to be dealt with separately.

The measures can be considered as global or local measures. A global measure is defined as the time or the cumulative usage since time zero (when the system is new). A local measure is defined as the time or the cumulative usage since its last failure. Time here is considered as working or operating and repair times are ignored (Lindqvist 2006).

T_i is the time to the ith failure and X_i is the time duration between the $(i-1)$th and the ith failures. Hence. T_i is a global scale and X_i is a local scale.

An alternative scale can thus be proposed in terms of a single measure, time or usage or their combination. When considered in terms of a single measure, the measure is a primary measure.

When considered in terms of two measures, one measure is treated as a primary measure and the second as a concomitant measure. This has an advantage when comparing the models for performance measures, as the models can be compared in terms of the primary measure. Usually time is taken as the primary measure.

An alternative scale can thus be represented in terms of two measures, time and usage as:

$$t^{Z} = \phi(t,\ Z(t)), \tag{1}$$

where t^{Z} is the alternative scale, which is a measure of the system condition and is a function ϕ of the primary measure t and concomitant measure $Z(t)$.

The function $\phi(.)$ may take different forms basic, combined additive form or combined multiplicative form, as shown below leading to different alternative scales.

A basic alternative scale is given by:

$$t^{Z} = \phi_1(t) \tag{2}$$

or

$$t^{Z} = \phi_2(Z(t)) \tag{3}$$

A combined additive alternative scale is given by:

$$t^{Z} = \phi_3(\alpha t + \beta Z(t + \omega)), \tag{4}$$

where α, β, and ω are parameters.

A combined multiplicative alternative scale is given by:

$$t^{Z} = \phi_4(t \times Z(t)). \tag{5}$$

2.1 Basic alternative scales

A basic alternative scale is proposed below, by considering time or usage as the primary scale forming a one dimensional scale.

Consider that the failures of a system take place at times $t_i, i = 1,2,3,.....,n$ or usage measure $m_i, i = 1,2,3,.....,n$, given n failures

Four basic alternative scales based on global time t, local time $(t - T_{N_{t^-}}) = x$, global usage measure m, and local usage measure $(m - M_{N_{t^-}}) = w$, are proposed here respectively as:

$$t^{Z_1} = t, \tag{6}$$

$$t^{Z_2} = \sum_{i=1}^{n} x_i + x, \tag{7}$$

$$t^{Z_3} = m, \tag{8}$$

and

$$t^{Z_4} = \sum_{i=1}^{n} w_i + w. \tag{9}$$

2.2 Combined alternative scales

A combined alternative scale is proposed with time as the primary measure and usage / usage rate as the concomitant measure i.e., forming a two/multi-dimensional scale by assigning a weightage parameter γ to each of the scales. The concomitant measure, is considered as a collapsible measure, which is described by its end value only and the path taken to reach this value is not considered. Thus if usage is considered as a collapsible measure, the usage $Z(t)$ is considered as the value of z at t only i.e., z_i at t_i. This will provide flexibility in the use of alternative scales, providing easy tractability without affecting model properties.

2.2.1Usage Rate as a concomitant measure

The usage rate instead of usage is considered as the concomitant measure for combined multiplicative alternative scales. This has an advantage when using the reliability models with these scales for prediction purposes.

When the collapsibility of the usage measure is considered, i.e., its values m_i at t_i and m_{i-1} at t_{i-1} are considered, then the local usage rate u_i is given by:

$$u_i = \frac{m_i - m_{i-1}}{t_i - t_{i-1}} = \frac{w_i}{x_i} . \tag{10}$$

This leads to averaging out of the fluctuations in the usage between failures and arriving at an average linear usage between failures, which is a reasonable assumption for repairable system failures.

The combined scale has two components time and usage rate. To obtain the expected time to next failure using a combined scale the value of usage rate i.e., the usage at this expected time to next failure is to be known a-priori which is not possible. To overcome this problem, we make use of the available information on usage / usage rate prior to this failure. If at any given time that has already elapsed prior to the future failure, the value of m at t is known, i.e., the usage at that time is known, the usage rate can be obtained from Eq. (10). It can be reasonably assumed that this value will be the same at the next failure and used to estimate the expected time to next failure.

2.2.2 Additive combinations of basic alternative scales

Two combined alternative scales, which are of additive forms, are proposed here. Usage is considered as the concomitant measure. An additive combination of time and usage form the scale with γ as the, weight of the age, parameter.

Considering global time t as the primary measure and global usage m as the concomitant measure, the combined alternative additive scale is proposed as:

$$t^{Z_5} = \sum_{i=1}^{n}\left((1-\gamma)(t_i - t_{i-1}) + \gamma\, m_i\right) + (1-\gamma)(t - t_n) + \gamma\, m, \tag{11}$$

Similarly, considering local time $(t - T_{N_{t^-}}) = x$ as the primary measure and local usage $(m - M_{N_{t^-}}) = w$ as the concomitant measure, the combined alternative additive scale is proposed as:

$$t^{Z_6} = \sum_{i=1}^{n}\left((1-\gamma)\, x_i + \gamma\, w_i\right) + (1-\gamma)\, x + \gamma\, w \ . \tag{12}$$

In Eqs. (11) and (12), if $\gamma = 0$, t^Z reduces to a time scale. If $\gamma = 1$, t^Z reduces to a usage scale. For any other value of γ, t^Z gives a combined alternative additive scale of time and usage.

These scales have an inherent disadvantage while they are used for a prediction purpose. This is because the values of usage at the time to next failure will not be known a priori. Hence these need to be extrapolated, considering the usage rate at the previous failure or at any elapsed time prior to the next failure where the usage rate value can be obtained. It can be reasonably assumed that this usage rate will prevail at the next failure and its mileage arrived at by multiplying this usage rate with the time to next failure.

2.2.3Multiplicative combinations of basic alternative scales

Four combined alternative scales, which are of multiplicative forms, are proposed here. In this sub section, usage rate is considered as the concomitant measure. A multiplicative combination of time and usage rate forms the scale with γ as the, weight of the age, parameter.

Considering global time t as the primary measure and local usage rate u as the concomitant measure, two combined alternative multiplicative scales are proposed as:

$$\begin{aligned} t^{Z_7} &= \sum_{i=1}^{n}(t_i - t_{i-1})\, u_i^{\gamma} + (t - t_n)\, u^{\gamma} \\ &= \sum_{i=1}^{n}(t_i - t_{i-1})^{1-\gamma}\, (m_i - m_{i-1})^{\gamma} + (t - t_n)^{1-\gamma}\, (m - m_n)^{\gamma} \end{aligned} \quad , \tag{13}$$

and

$$t^{Z_8} = \sum_{i=1}^{n} (t_i - t_{i-1}) \left(\exp(\gamma u_i) \right) + (t - t_n) \left(\exp(\gamma u) \right) \\ = \sum_{i=1}^{n} (t_i - t_{i-1}) \left(\exp(u_i) \right)^{\gamma} + (t - t_n) \left(\exp(u) \right)^{\gamma} \quad , \qquad (14)$$

Similarly, if we consider local time $(t - T_{N_{t^-}}) = x$ as the primary measure and the local usage rate u as the concomitant measure, two combined alternative multiplicative scale are proposed as:

$$t^{Z_9} = \sum_{i=1}^{n} x_i \, u_i^{\gamma} + x \, u^{\gamma} = \sum_{i=1}^{n} x_i^{1-\gamma} w_i^{\gamma} + x^{1-\gamma} w^{\gamma} \quad , \qquad (15)$$

and

$$t^{Z_{10}} = \sum_{i=1}^{n} x_i \left(\exp(\gamma u_i) \right) + x \left(\exp(\gamma u) \right) = \sum_{i=1}^{n} x_i \left(\exp(u_i) \right)^{\gamma} + x \left(\exp(u) \right)^{\gamma} \quad . \qquad (16)$$

The combined alternative multiplicative scales proposed in Eqs. (13) and (15), and Eqs. (14) and (16) are essentially the same scales. Eqs. (13) and (14) are formulated in terms of global times to facilitate the use of global time intensity and reliability functions. Eqs. (15) and (16) are formulated in terms of local times.

In addition to its usefulness for prediction purposes, the advantage of using usage rate in Eqs. (13) and (15) also includes: if $\gamma = 0$, t^Z reduces to a time scale; if $\gamma = 1$, t^Z reduces to a usage scale. For any other value of γ, it gives a combined alternative additive scale of time and usage together.

In Eqs. (14) and (16) exponential function values of the usage rate are considered as the concomitant measure. These will be usefull when the usage measure values are very high compared to the time measure values. At the same time the values shall lie in the positive quadrant only.

For Eqs. (14) and (16) for a value of weight parameter $\gamma = 0$ the combined alternative multiplicative scale collapses to a time scale and for any other value of γ it gives a combined alternative additive scale of time and usage together.

Combined alternative scales can be formulated with global usage rate also as a concomitant measure. Such alternative scales are not considered here.

3. Modelling of reliability and intensity functions with alternative scales

Having developed alternative scales for measuring the failures in a repairable system, the failure intensity function and reliability function of the repairable system is defined in terms of these scales in this section.

The reliability of a repairable system can be defined as a function of the alternative scales:

$$R(t \mid H_{t^-}) = G(t^Z). \tag{17}$$

As such, the conditional intensity process $\lambda(t \mid H_{t^-})$ is given by:

$$\lambda(t \mid H_{t^-}) = -\frac{dt^Z}{dt} dG(t^Z). \tag{18}$$

Point processes are generally used for modelling the reliability of repairable systems. Ascher (2008) states that the most plausible first order model to deal with the reliability of repairable systems is the non-homogeneous Poison process (NHPP). This process considers that repair has no effect on the failure intensity. The NHPP with a power law process is considered here for modelling the failure process of a repairable system.

This, however, is an extreme case. Repair has some effect on the failure intensity and this effect is captured by a factor known as maintenance effectiveness in general/imperfect repair models. Times between failures of a system with imperfect repair may be the virtual age model such as Kijima 1 and 2 models of Kijima (1989), and times to failures of a system with imperfect repair can be Arithmetic Reduction of Intensity (ARI) models of Doyen and Gaudoin (2004). There are a large number of other general/imperfect repair models which can also be considered for modelling these alternative scales, for example, Syamsundar et al. (2011), Doyen et al. (2017), Wu and Scarf (2017), Wu (2019), among others.

In all the above point process models, the time scale is replaced with alternative scales to form failure intensity, reliability functions, and models for a repairable system with alternative scales.

In the following sub-sections, modelling of the failure process of a repairable system using the above-proposed alternative scales is developed.

3.1 Minimal repair model with an alternative scale

Minimal repair, whose repair effectiveness is as bad as old (ABAO), restores a system under repair to the same state or condition of the system, immediately before it failed. Minimal repair is modelled by a non-homogeneous Poisson process (NHPP) with its conditional intensity function being a function of the global time of the system given by $\lambda_0(t)$, Incorporating an alternative scale t^Z in the place of the usual time scale, the conditional intensity function as per Eq. (18) of the minimal repair model is given by:

$$\lambda(t \mid H_{t^-}) = \lambda_0(t^Z)\frac{dt^Z}{dt}, \tag{19}$$

where the intensity failure function $\lambda_0(t^Z)$ can be a power law process or a log-linear process.

The intensity function of the NHPP in Eq. (19), with the alternative scale $t^{Z_7} = \sum_{i=1}^{n}(t_i - t_{i-1})u_i^{\gamma} + (t - t_n)u^{\gamma}$ as proposed in Eq. (13), and power law process, is given by:

$$\lambda(t \mid t_i) = u_{i+1}^{\gamma}\,\alpha\,\beta\left(\sum_{k=1}^{i} u_k^{\gamma}\,(t_k - t_{k-1}) + u_{i+1}^{\gamma}\,(t - t_i)\right)^{\beta-1}, \tag{20}$$

and

$$\lambda(t_{i+1} \mid t_i) = u_{i+1}^{\gamma}\, \alpha\, \beta \left(\sum_{k=1}^{i+1} u_k^{\gamma}\, (t_k - t_{k-1}) \right)^{\beta-1} = u_{i+1}^{\gamma}\, \alpha\, \beta\, (t_{i+1}^{z})^{\beta-1}. \tag{21}$$

The cumulative intensity function is given by:

$$\Lambda(t_{i+1} \mid t_i) = u_{i+1}^{\gamma} \int_{t_i}^{t_{i+1}} \alpha\, \beta \left(\sum_{k=1}^{i} u_k^{\gamma}\, (t_k - t_{k-1}) + u_{i+1}^{\gamma}\, (t - t_i) \right)^{\beta-1} du = \alpha \left(\left(\sum_{k=1}^{i+1} u_k^{\gamma} (t_k - t_{k-1}) \right)^{\beta} - \left(\sum_{k=1}^{i} u_k^{\gamma} (t_k - t_{k-1}) \right)^{\beta} \right) = \alpha \left(\left(t_{i+1}^{z}\right)^{\beta} - \left(t_i^{z}\right)^{\beta} \right). \tag{22}$$

The conditional failure density function is given by:

$$f\,(t_{i+1} \mid t_i) = u_{i+1}^{\gamma}\, \alpha\, \beta \left(t_{i+1}^{z}\right)^{\beta-1} \exp\left(-\alpha \left(\left(t_{i+1}^{z}\right)^{\beta} - \left(t_i^{z}\right)^{\beta} \right) \right). \tag{23}$$

3.2 Models with alternative scales and covariates

Repairable systems are subjected to varying levels of environmental conditions in the form of, stress, temperature, pressure or other factors related to their design, operation or maintenance, all of which may affect the failure process of the systems. This additional information that affects the failure process of the systems can be incorporated as covariates to the alternative scales of the intensity function of the failure process of a repairable system. These covariates are deemed to act multiplicatively on the system's failure intensity using a suitable link function.

An alternative scale with other covariates can be represented as:

$$t^{Z} = \phi(t,\ (Z(t), W(t)), \tag{24}$$

where $W(t)$ is a function of the covariates influencing the failure process of the system.

The conditional intensity function with covariates using a multiplicative exponential link function can then be given by:

$$\lambda(t \mid H_{t^-}) = \frac{dt^{Z}}{dt}\, \lambda(t^{Z}) \exp(\eta\ W(t)), \tag{25}$$

where η is a weight parameter of the covariates.

Such models are not considered further in this paper and would be the subject of future work in this area.

4. Parameter Inference

The most common and widely used method of inferring the parameters of the failure process of a repairable system is the method of the maximum likelihood estimation, see Lindqvist (2006), for example.

The likelihood function of a minimal repair model with alternative scale $t^{Z_7} = \sum_{i=1}^{n} (t_i - t_{i-1}) u_i^{\gamma} + (t - t_n) u^{\gamma}$ as proposed in Eq. (13) for a failure truncated process, and power law process is given by:

$$L(\theta \mid data) = \alpha^n \beta^n u_1^{\gamma} (t_1^z)^{\beta-1} \prod_{i=1}^{n-1} \left(u_{i+1}^{\gamma} (t_{i+1}^z)^{\beta-1} \right) \exp(-\alpha (t_1^z)^{\beta}) \exp\left(-\alpha \sum_{i=1}^{n-1} \left((t_{i+1}^z)^{\beta} - (t_i^z)^{\beta} \right) \right). \tag{26}$$

The likelihood function of an Arithmetic Reduction of Intensity model with memory 1 (ARI_1) model with the alternative scale $t^{Z_7} = \sum_{i=1}^{n} (t_i - t_{i-1}) u_i^{\gamma} + (t - t_n) u^{\gamma}$ as proposed in Eq. (13) for a failure truncated process, and the power law process is given by:

$$L(\theta \mid data) = \alpha^n \beta^n u_1^{\gamma} (t_1^z)^{\beta-1} \prod_{i=1}^{n-1} u_{i+1}^{\gamma} \left(\left((t_{i+1}^z)^{\beta-1} - \rho (t_i^z)^{\beta-1} \right) \right) \exp(-\alpha (t_1^z)^{\beta}) \exp\left(-\alpha \sum_{i=1}^{n-1} \left(\left((t_{i+1}^z)^{\beta} - (t_i^z)^{\beta} \right) - \beta \rho (t_i^z)^{\beta-1} \left(t_{i+1}^z - t_i^z \right) \right) \right). \tag{27}$$

The likelihood functions of other models can be developed in a similar way.

Through maximizing $L(\theta|\text{data})$ in Eq. (27), one can obtain the parameters.

The usual criteria for checking the model with the better fit to the failure data is to look at the log likelihood values. The model with the maximum log likelihood estimate is considered as the model with the better fit.

A better check for models will be the Akaike likelihood criterion (AIC), which penalises the log likelihood of a model with more parameters in the model. This is done to avoid adjusting for the problem of better fit when the model has more parameters and thus provides a better criterion for comparison of the models. The criterion (Akaike, 1973) is given by;

$$AIC(k) = -2\ln L + 2k\,, \tag{28}$$

where k is the number of parameters of the model

The model with the minimum AIC estimate is considered as the model with the better fit.

This methodology cannot be considered for comparing models using two different measures or more for the scale of the failure process of a system. Here a check can be made based on the scale which captures most of the variation in the failure times. For this a check for fit is made by looking at the variation between failure numbers and the estimated cumulative intensity, by comparing the sum of squared distances $\sum(\hat{\Lambda}(t) - i)^2$ values for all the models. The model with the least variation as indicated by a lower value of the sum of squared distance is deemed to be the model with the better fit.

A check for a better fit can also be obtained graphically by plotting the estimated cumulative intensity of the model along with failure numbers versus alternative scale. The model giving the closest fit to the failure numbers versus alternative scale provides the better fit.

Another graphical check for fit can be obtained by plotting the normalised alternative scales vs number of failures. The global alternative scales are normalised as given below:

$$t_{i\,[N]}^{N} = t_i / t_n\,, \tag{29}$$

$$m_{i\,[N]}^{N} = m_i / m_n\,, \tag{30}$$

and

$$t_{i\,[N]}^{Z} = t_i^Z \,/\, t_n^Z \,, \tag{31}$$

where $t_{i\,[N]}^{N}$, $m_{i\,[N]}^{N}$, and $t_{i\,[N]}^{Z}$ are the normalised alternative scales.

This will indicate whether there is a variation between the primary and concomitant scale or not. Apart from this, it will indicate whether the combined alternative scale is closer to the primary or the concomitant scale.

5. Applications of the proposed alternative scales

Failure data of single repairable systems are studied to determine which alternative scale, time or usage or a combined alternative scale incorporating both these measures, is a better one to assess the reliability of these systems. Those failure data include Excavator Engines from Yang et al. (2016) in calendar time and working time, AMC Ambassador Cars from Ahn et al. (1998) in time and mileage, and Trucks from Fuqing et al. (2017) in time and loading x distance.

5.1 Analysis of the failure data of Excavator Engines

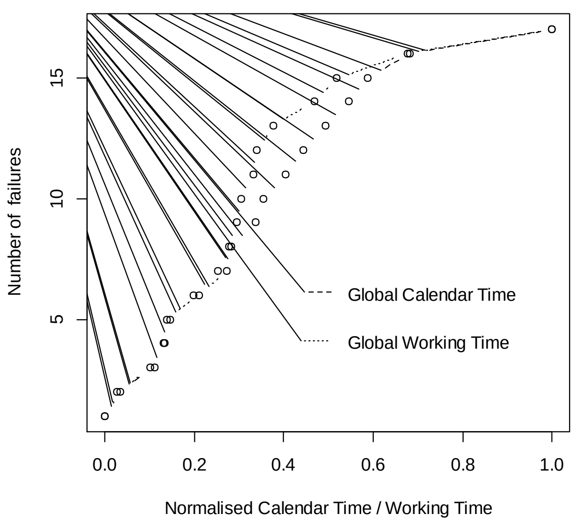
Yang et al. (2016) provide data on the times between failures and working hours at failure for three Excavator Engines. Excavator Engine 1 has long working time, Engine 2 has medium working time and Engine 3 has short working time. Times between failures are in calendar time and may include maintenance or idle times. Usually when one applies models to the failure times, the maintenance times are ignored and the working or operational time is considered as the time scale.

It is proposed to study this data set of failure times of the Excavator Engines using the ten alternative scales given in Eqs, (6)-(9), and (11)-(16) with the usage rate as considered in Eq. (10).

The best fit model in each of the alternative scales shown in Eqs. (6)-(9) are fitted to the Excavator Engines failure data with log likelihood, AIC and the sum of squared distances $\sum(\hat{\Lambda}(t)-i)^2$ values are shown in Table 1. Based on this, it can be seen that calendar time provides a better fit to the failure data for Excavator Engines 1 and 3 and either scale can provide a better fit to the failure data of Excavator Engine 2. This goes against the conventional wisdom that working or operational time is the best alternative scale for all systems.

Table 1 – Values of log likelihood, AIC for models with different alternative scales fitted to the excavator engine failure times data.

Excavator Engine	Time/Usage	Model	Alternative Time Scale	ln L	AIC	$\sum(\hat{\Lambda}(t)-i)^2$
1	Calendar Time	ARI_∞ with PLP baseline	$t^{Z_1}=t$	-109.45	224.90	**24.80**
	Working Time	ARI_∞ with PLP baseline	$t^{Z_3}=m$	-89.6	185.20	39.20
2	Calendar Time	Kijima II with PLP baseline in local time	$t^{Z_2}=\sum_{i=1}^{n}x_i+x$	-88.70	183.40	**45.06**
	Working Time	Kijima II with PLP baseline in local time	$t^{Z_4}=\sum_{i=1}^{n}w_i+w$	-70.31	146.62	**45.47**
3	Calendar Time	Kijima II with PLP baseline in local time	$t^{Z_2}=\sum_{i=1}^{n}x_i+x$	-60.86	127.72	**13.04**
	Working Time	Kijima II with PLP baseline in local time	$t^{Z_4}=\sum_{i=1}^{n}w_i+w$	-45.27	96.54	14.63



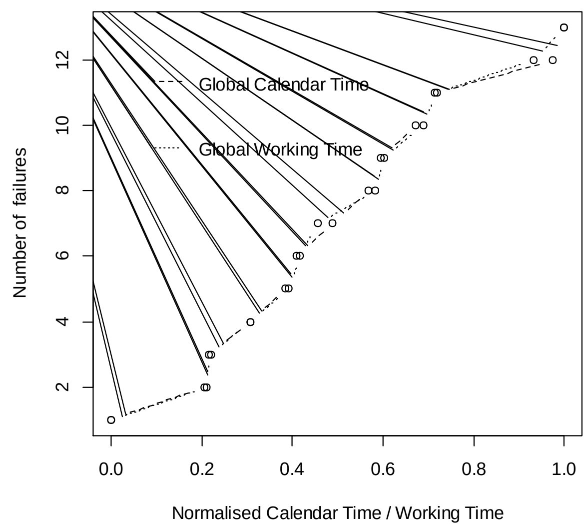
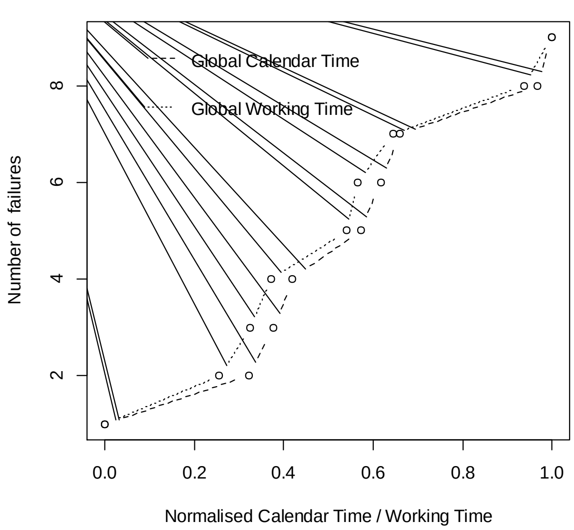


Fig. 1 – Plot of Normalised Alternative scales– Global Calendar Time / Global Working Time vs Number of Failures for Excavator Engine 1.

Fig. 2 – Plot of Normalised Alternative scales– Global PLP Calendar Time / Global Working Time vs Number of Failures for Excavator Engine 2.



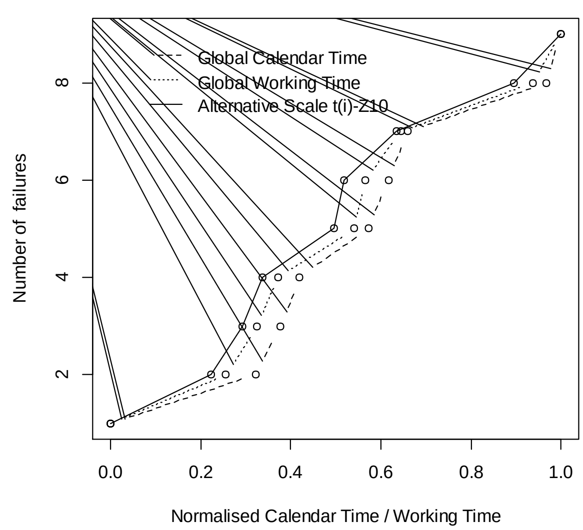


Fig. 3 – Plot of Normalised Alternative scales– Global Calendar Time / Global Working Time vs

Fig. 4 – Plot of Normalised Alternative scales– Global PLP Calendar Time / Global Working

Number of Failures for Excavator Engine 3.

Time / Alternative scale $t_i^{Z_{10}}$ vs Number Failures for Excavator Engine 3.

Normalised alternative scales given in Eqs. (29)-(31) for the Excavator Engine failure data are plotted vs number of failures in Figs. 1 to 4. For Excavator Engine 1, there is a small variation between both the scales after the sixth failure. This is reflected in the difference between the sum of squares values as can be seen from Table 1. The calendar time alternative scale provides the better fit having the lower sum of squares value.

For Excavator Engine 2 it can be seen that both the scales are identical. This is also reflected in the sum of squares values as seen in Table 1. In this case either scale can provide a better fit to the failure data. For Excavator Engine 3 there is variation between both the scales however they are close to each other. This is reflected in the small difference between the sum of squares values as can be seen from Table 1. The calendar time alternative scale provides the better fit having the lower sum of squares value.

Now the alternative scales given in Eqs. (11)-(16) with usage rate as considered in Eq. (10) are fitted to the failure data of all the Excavator Engines. Based on AIC, it can be seen that in the case of Excavator Engine 3, a combined model comes close to providing a good fit to the failure data with calendar time as the primary scale and local usage i.e., local working time as the concomitant scale. These values are shown in Table 2. This indicates that there is a possibility that both calendar time and working or operational time together can form a combined scale in case of failure time data for repairable systems. In this case as there are only eight failures, being a very small number is probably the reason a combined alternative scale does not provide a better fit.

Table 2 – Values of log likelihood, AIC for models with different alternative scales fitted to Excavator Engine 3 Failure Times Data.

Excavator Engine	Time/Usage	Model	Alternative Time Scale	ln L	AIC	$\sum(\hat{\Lambda}(t)-i)^2$
3	Calendar Time	Kijima II with PLP baseline in local time	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-60.86	127.72	**13.04**
	Working Time	Kijima II with PLP baseline in local time	$t^{Z_4} = \sum_{i=1}^{n} w_i + w$	-45.27	96.54	14.63
	Combined	Kijima II with PLP baseline in local time	$t^{Z_{10}} = \sum_{i=1}^{n} x_i \left(\exp(u_i)\right)^{\gamma} + x\left(\exp(u)\right)^{\gamma}$	-59.90	127.80	17.31

The analysis of Excavator Engine failure data indicates that it is not necessary that working or operational time is the best alternative scale for modelling the failure process. Calendar Time or even a combined alternative scale that uses both calendar time and working time, with one as primary and the other concomitant can provide a better alternative scale for modelling the failure process.

5.2 Analysis of the failure data of AMC Ambassador Cars

Ahn et al. (1998) provide data on times between failures and the mileages accumulated by AMC Ambassador Cars at each of the failure times. They stated that these form two measures of the time index, dependent on each other, but with the stochastic relation between them possibly having considerable variation. To incorporate this, they suggested a functional form of synthesising mileage and failure times into a single time index as:

$$t_i^Z = m_i^{\gamma}\, t_i^{1-\gamma} \qquad (32)$$

where m_i and t_i, $i = 1, 2, \ldots, n$ are the ith failure mileage and ith failure time respectively. In this case if $\gamma = 0$ then the model is a failure time only model and $\gamma = 1$, then the model is a mileage only model.

They then used an NHPP with the power law model for all the six cars to obtain estimates of the parameters. They use two procedures for estimating the parameters. First they use a log likelihood procedure to estimate the parameters of the NHPP for various assumed values of γ and obtained the parameter set for a better fit model. Then they used a least squares procedure for fitting the mean value function of the NHPP to obtain the estimates of the parameters for the better fit model. They found that for Cars 1, 2 and 4, the failure time model forms the better fit. For other cars the combined mileage – failure time model forms a better fit.

They however, ignored the dt^Z / dt term in the conditional intensity of the NHPP model given by $\lambda(t \mid H_{t^-}) = \lambda_0(t^Z)\dfrac{dt^Z}{dt}$. This has resulted in a non-identifiability problem for estimating all the parameters together using the MLE procedure directly. Both the procedures used for estimation are not very efficient and the use of only NHPP as a model may not have provided the desired results.

Table 3 – Values of log likelihood, AIC for models with different alternative scales fitted to AMC Ambassador Cars Failure Times Data.

Car	Time/Usage	Model	Alternative Time Scale	ln L	AIC	$\sum(\hat{\Lambda}(t) - i)^2$
1	Time	ARI_∞ with PLP baseline	$t^{Z_1} = t$	-86.63	179.26	**44.20**
	Mileage	ARI_1 with PLP baseline	$t^{Z_3} = m$	-133.31	272.62	105.48

2	Time	Kijima I with PLP baseline	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-69.91	145.82	**24.27**
	Mileage	NHPP-PLP Global Time	$t^{Z_3} = m$	-116.03	236.06	30.93
3	Time	ARI∞ with PLP baseline	$t^{Z_1} = t$	-96.78	199.56	138.62
	Mileage	ARI∞ with PLP baseline	$t^{Z_3} = m$	-183.35	372.70	**26.68**
4	Time	ARI∞ with PLP baseline	$t^{Z_1} = t$	-90.90	187.80	**13.18**
	Mileage	NHPP-PLP Local Time	$t^{Z_4} = \sum_{i=1}^{n} w_i + w$	-161.56	327.12	24.08
5	Time	Kijima I with PLP baseline	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-80.47	166.94	**18.90**
	Mileage	NHPP-PLP Global Time	$t^{Z_3} = m$	-127.90	259.80	51.49
6	Time	Kijima II with PLP baseline	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-83.01	172.02	43.66
	Mileage	Kijima II with PLP baseline	$t^{Z_4} = \sum_{i=1}^{n} w_i + w$	- 151.80	309.60	**8.59**

In place of the alternative scale considered by Ahn et al. (1998) the ten alternative scales at Eqs. (6)-(9) and Eqs. (11)-(16) are considered with usage rate as considered at Eq. (10).

The best model in each of the alternative scales defined in Eqs. (6)-(9) fitted to the Ambassador Cars failure data with log likelihood, AIC and the sum of squared distances $\sum (\hat{\Lambda}(t) - i)^2$ values are shown in Table 3. It can be seen from the table that time provides a better fit to the failure data of cars 1, 2, 4, and 5 and mileage better fit for cars 3, and 6.

Now the alternative scales defined in Eqs. (11)-(16) with usage rate as considered in Eq. (10) are fitted to the failure data of all the cars. The log likelihood, the AIC and the sum of squared distances

$\sum(\hat{\Lambda}(t)-i)^2$ values are shown in Table 4. Based on the AIC values, it can be seen that in the case of Cars 2 and 6, a combined model is seen to provide a better fit to the failure data with time as the primary scale and local usage as the concomitant scale.

These are then compared using the sum of squared distances $\sum(\hat{\Lambda}(t)-i)^2$ values. It can be seen that the combined alternative scale $t^{Z_9} = \sum_{i=1}^{n} x_i\, u_i^{\gamma} + x\, u^{\gamma} = \sum_{i=1}^{n} x_i\, u_i^{-0.093} + x\, u^{-0.093}$ forms a better scale for failure data of Car 2 and local failure mileage scale $t^{Z_4} = \sum_{i=1}^{n} w_i + w$ for failure data of Car 6. For Car 2, the alternative scale $t^{Z_9} = \sum_{i=1}^{n} x_i\, u_i^{-0.093} + x\, u^{-0.093} = \sum_{i=1}^{n} x_i^{1.093} w_i^{-0.093} + x^{1.093} w^{-0.093}$ is closer to the time scale, as also evidenced by the sum of squares value.

Table 4 – Values of log likelihood, AIC for models with different alternative scales fitted to AMC Ambassador Cars Failure Times Data.

Car	Time/Usage	Model	Alternative Time Scale	ln L	AIC	$\sum(\hat{\Lambda}(t)-i)^2$
2	Time	Kijima I with PLP baseline	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-69.91	145.82	24.27
	Mileage	NHPP-PLP Global Time	$t^{Z_3} = m$	-116.03	236.06	30.93
	Combined	Kijima I with PLP baseline	$t^{Z_9} = \sum_{i=1}^{n} x_i\, u_i^{\gamma} + x\, u^{\gamma}$	- 62.71	131.42	**7.91**
6	Time	Kijima II with PLP baseline	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-83.01	172.02	43.66

	Mileage	Kijima II with PLP baseline	$t^{Z_4} = \sum_{i=1}^{n} w_i + w$	-151.8	309.60	**8.59**
	Combined	Kijima II with PLP baseline	$t^{Z_{10}} = \sum_{i=1}^{n} x_i \left(\exp(u_i)\right)^{\gamma} + x\left(\exp(u)\right)^{\gamma}$	-80.90	169.80	32.31

Estimated values of the parameters for failure data of Car 2 with alternative scale $t_i^{Z_9}$ are provided in Table 5.

Table 5 – Estimated values of the parameters of the alternative scale model Kijima I with PLP baseline and alternative scale $t_i^{Z_9}$ used for AMC Ambassador Car 2 failure data.

Parameter	Value
$\hat{\alpha}$	9.19e-09
$\hat{\beta}$	6.97
$\hat{\rho}$	1.54e-01
$\hat{\gamma}$	-9.29e-01
ln L	- 62.71
AIC	131.42
$\sum (\hat{\Lambda}(t) - i)^2$	7.91

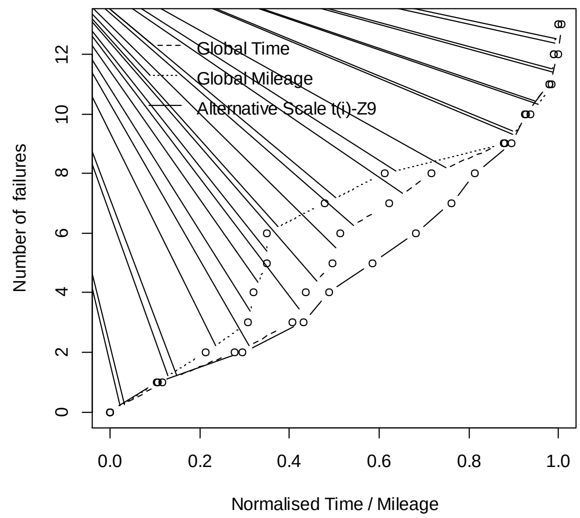


Fig. 5 – Plot of Normalised Alternative scales– Global Time / Global Mileage / Scale $t_i^{Z_9}$ vs Number of Failures for Car 2.

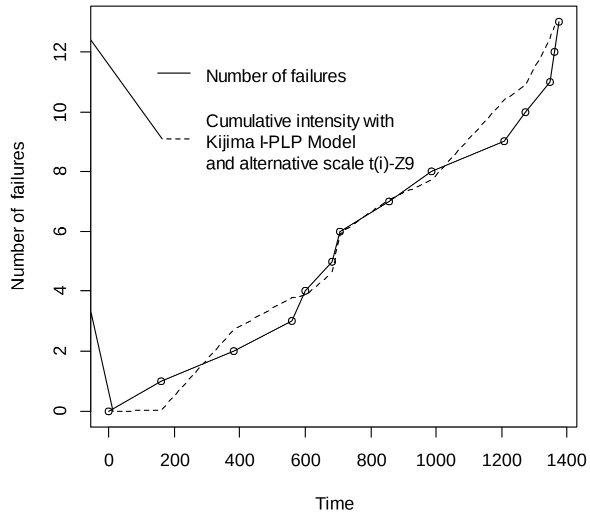


Fig. 6 – Cumulative intensity of the NHPP PLP model used for AMC Ambassador Car 2 failure data with Kijima I-PLP model and Alternative scale $t_i^{Z_9}$ as the alternative scale.

The best fit alternative scale values of Car 2 are normalised as given at Eqs. (29)-(31) and are plotted vs number of failures in Fig. 5. As can be seen from the figure, the combined alternative scale with the failure time as the primary scale and the local usage as the concomitant scale provides a clear indicator of a deteriorating system with respect to the failure data of car 2, compared to the two original scales time and mileage.

Plots of the cumulative intensity and the number of failures versus better fit alternative scales are given in Fig. 6 for failure data of Car 2. Though it does not provide a very close fit, it provides a better fit to the failure data than any other scale.

5.3 Analysis of the failure data of Trucks

Fuqing et al. (2017) provide failure data in terms of times between failures and loading as tons x kilometres accumulated at each of the failure times for two trucks. Here the usage itself is a two dimensional scale formed by multiplying load with distance. These trucks were used to move ore rock and waste rock from Jajaram open-pit Bauxite mine to allocated deposition places.

This data set of failure times of Trucks are studied using the ten alternative scales at Eqs. (6)-(9) and Eqs. (11)-(16) with the usage rate defined by Eq. (10).

The best fit model in each of the alternative scales Eqs. (6)-(9) fitted to the Trucks failure data with the log likelihood, the AIC and the sum of squared distances $\sum(\hat{\Lambda}(t)-i)^2$ values are shown in Table 6, based on which it can be seen that the alternative scale with load provides a better fit to the failure data of both the trucks.

Table 6 – Values of log likelihood, AIC for models with different alternative scales fitted to Trucks Failure Times Data.

Truck	Time/ Usage	Model	Alternative Time Scale	ln L	AIC	$\sum(\hat{\Lambda}(t)-i)^2$
1	Time	NHPP-PLP Global Time	$t^{Z_1} = t$	253.78	511.56	264.93
	Tons x Kms	NHPP-PLP Global Time	$t^{Z_3} = m$	497.67	999.34	**258.49**
2	Time	Kijima II with PLP baseline	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	177.99	361.98	62.77
	Tons x Kms	Kijima II with PLP baseline	$t^{Z_4} = \sum_{i=1}^{n} w_i + w$	335.19	676.38	**59.96**

Now the alternative scales in Eqs. (11)-(16) with the usage rate defined in Eq. (10) are fitted to the failure data of all the trucks. The log likelihood, the AIC and the sum of squared distances $\sum(\hat{\Lambda}(t)-i)^2$ values are shown in Table 7. Based on AIC values, it can be seen that in the case of both the trucks a combined model is seen to provide a better fit to the failure data with time as the primary scale and local usage as the concomitant scale.

Based on the sum of squared distances $\sum(\hat{\Lambda}(t)-i)^2$ values the alternative scale

$$t^{Z_7} = \sum_{i=1}^{n}(t_i - t_{i-1})u_i^{\gamma} + (t - t_n)u^{\gamma} = \sum_{i=1}^{n}(t_i - t_{i-1})u_i^{0.092} + (t - t_n)u^{0.092} = \sum_{i=1}^{n} x_i^{0.08}\, w_i^{0.92} + x^{0.08}\, w^{0.092}$$

forms a better scale for truck 1 failure data. This is closer to the loading alternative scale as is also evidenced by the closer sum of squares values.

The timescale

$$\begin{aligned} t^{Z_{10}} &= \sum_{i=1}^{n} x_i \exp(\gamma u_i) + x \exp(\gamma u) = \sum_{i=1}^{n} x_i \exp(0.00553\, u_i) + x \exp(0.00553\, u) \\ &= \sum_{i=1}^{n} x_i \left(\exp(u_i)\right)^{0.00553} + x\left(\exp(u)\right)^{0.00553} \end{aligned}$$

forms a better scale for truck 2 failure data. This is closer to the loading alternative scale as is evidenced by the closer sum of squares values.

Table 7 – Values of log likelihood, AIC for models with different alternative scales fitted to Trucks Failure Times Data.

Truck	Time/ Usage	Model	Alternative Time Scale	ln L	AIC	$\sum(\hat{\Lambda}(t)-i)^2$

1	Time	NHPP-PLP Global Time	$t^{Z_1} = t$	-253.78	511.56	264.93
	Tons x Kms	NHPP-PLP Global Time	$t^{Z_3} = m$	-497.67	999.34	258.49
	Combined	NHPP-PLP Global Time	$t^{Z_7} = \sum_{i=1}^{n}(t_i - t_{i-1})u_i^{\gamma} + (t - t_n)u^{\gamma}$	-251.02	**508.04**	**256.06**
2	Time	Kijima II with PLP baseline in local time	$t^{Z_2} = \sum_{i=1}^{n} x_i + x$	-177.99	361.98	62.77
	Tons x Kms	Kijima II with PLP baseline in local time	$t^{Z_4} = \sum_{i=1}^{n} w_i + w$	-335.19	676.38	59.96
	Combined	Kijima II with PLP baseline in local time	$t^{Z_{10}} = \sum_{i=1}^{n} x_i \left(\exp(u_i)\right)^{\gamma} + x\left(\exp(u)\right)^{\gamma}$	-176.89	361.78	**51.40**

Estimated values of the parameters for the trucks with better fit alternative scales are provided in Table 8.

The best fit alternative scale values of truck 1 are normalised as given at Eqs. (29)-(31) and are plotted vs number of failures in Fig. 7. As can be seen from the Fig. 7, the combined alternative scale with failure time as the primary scale and local usage as the concomitant scale provides a clear indicator of an improving system with respect to the failure data of truck 1 as compared to the two original scales time and mileage.

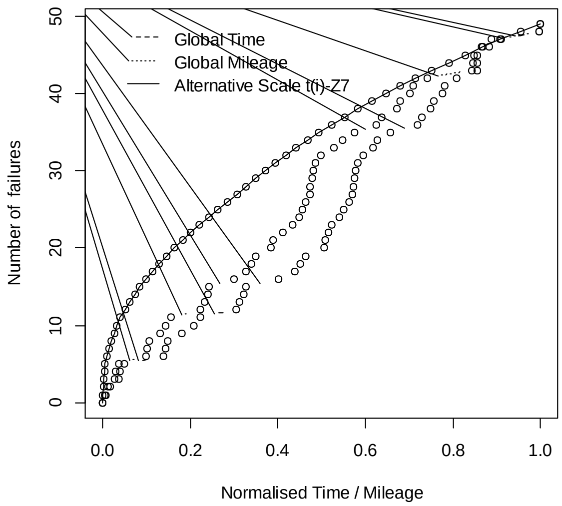
The plot of the cumulative intensity and the number of failures versus the better fit alternative scale is given in Fig. 8 for failure data of truck 1. It shows a good fit to the failure data and from Fig. 7.

The best fit alternative scale values of truck 2 are normalised as given at Eqs. (29)-(31) and are plotted vs number of failures in Fig. 9. As can be seen from the figure, the combined alternative scale with failure time as the primary scale and local usage as the concomitant scale provides a better fit to the failure data of truck 2 as compared to the two original scales time and loading.

Plot of cumulative intensity and number of failures versus better fit alternative scale is presented in Fig. 10 for failure data of truck 2. It shows a good fit to the failure data. From Fig. 9, it can be seen that this is closer to the usage scale as is also evidenced by the sum of square values.

Table 8 – Estimated values of the parameters of the alternative scale model NHPP-PLP baseline and scale $t_i^{Z_7}$ used for Truck 1 and Kijima II with PLP baseline and scale $t_i^{Z_{10}}$ used for Truck 2 failure data.

Trucks	1	2
Model	NHPP-PLP Global Time	Kijima II with PLP baseline in local time
Alternative scale	$t_i^{Z_7}$	$t_i^{Z_{10}}$
Parameter	Value	Value
$\hat{\alpha}$	1.12e-03	1.35e-07
$\hat{\beta}$	0.84	2.43
$\hat{\rho}$	---	7.52e-01
$\hat{\gamma}$	0.92	5.53e-03
ln L	-251.02	-176.89
AIC	508.04	361.78
$\sum(\hat{\Lambda}(t)-i)^2$	256.06	51.40



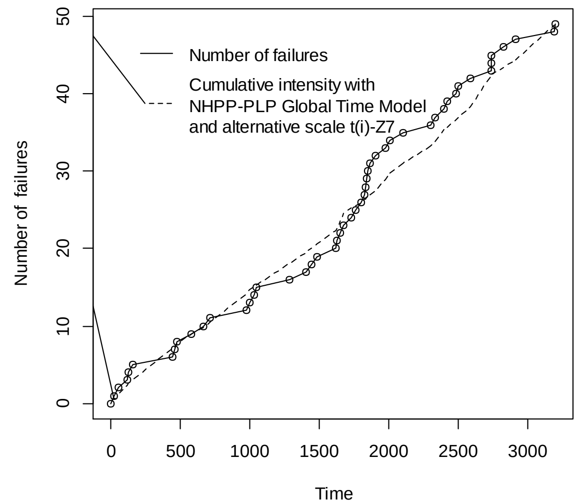
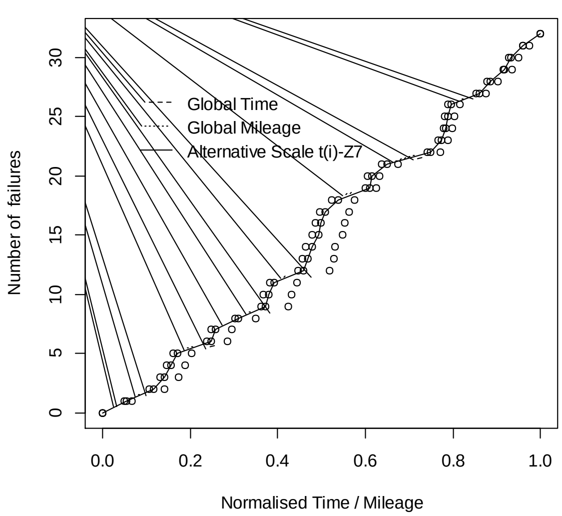


Fig. 7 – Plot of Normalised Alternative scales– Global Time / Global Mileage / Scale $t_i^{Z_7}$ vs Number of Failures for truck 1.

Fig. 8 – Cumulative intensity of the NHPP PLP model used for truck 1 failure data with NHPP-PLP global time model and alternative scale $t_i^{Z_7}$ as the alternative scale.



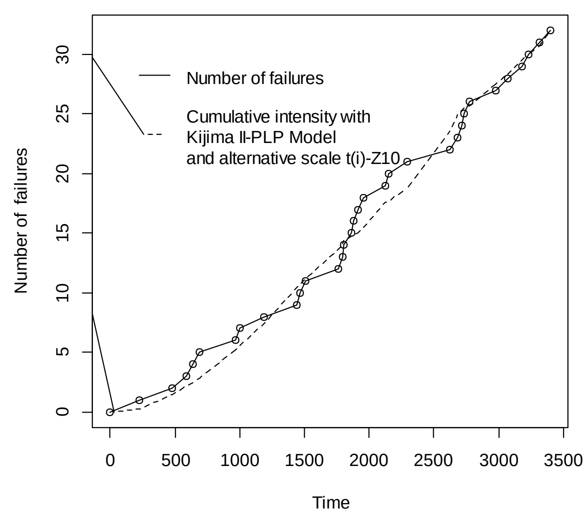


Fig. 9 – Plot of Normalised Alternative scales– Global Time / Global Mileage / Scale $t_i^{Z_{10}}$ vs Number of

Fig. 10 – Cumulative intensity of the NHPP PLP model used for truck 2 failure data with

Failures for truck 2.	Kijima II-PLP model and alternative time Scale $t_i^{Z_{10}}$ as the alternative scale.

5.4 Findings

The failure process of a repairable system is usually defined as a function of time in the reliability literature. For systems of the same type during the same time period, their usage can vary from system to system and their failure processes differ. To assess whether this really happens and how this affects the failure process of the system, alternative scales have been developed to take into consideration both usage rates and time. The failure process of a repairable system has been redefined as a function of the alternative scales. The method to choose better alternative scales to fit a given failure dataset is suggested.

It can be seen from the applications of the alternative scales that usage plays a role in the failure process. An analysis of the failure data of the excavator engines by Yang et al. (2016) shows that the calendar time rather than operational time offers a better fit to the data of engines 1 and 3. For engine 2 both the scales provide the same result. This goes against the conventional wisdom that both the operating time and calendar time can be equated. An analysis of the failure data of ambassador cars from Ahn et al. (1998) shows that for Car 2 a combined time and mileage scale provides a better fit to the failure data as compared to either time or mileage based models. For cars 1, 4 and 5 time is a better indicator of the failure process while for cars 3 and 6, mileage is a better indicator of the failure process. This is in variance to the results obtained by Ahn et al. (1998) who have indicated that for cars 1, 2 and 4 time is a better indicator of the failure process and for cars 3, 5 and 6 a combined time and mileage scale is a better indicator of the failure process. This is probably because their estimation processes are not very robust and that they have not considered imperfect repair processes for modelling these scales. For the failure data of trucks from Fuqing et al. (2017), it can be seen that the combined model of time and load distance proves to be better scale for both the trucks.

The results indicate that this is probably due to different failure modes occurring on account of usage and time, as indicated in Kordonsky and Gertsbakh (1995a) and Krivstov and Frankstein (2006). For some systems the failures due to usage caused by more rapid deterioration dominate the failures solely on account of time. For such systems usage may be a better indicator of system condition and will form a better alternative scale to model the failure process and provide a better indicator of assessing its reliability. For some systems, multiplicative combinations of scales with time as the primary measure and usage as the concomitant measure provide better scales to model the failure process of a system where two different failure modes, both random failures and failures on account of deterioration, may take place.

It has been observed that the additive alternative scales at Eqs. (11) and (12) do not work with single repairable systems. They only provide a monotonic increase or decrease in log likelihood values as obtained with the time scale to that obtained with the usage scale and beyond. Also issues have been observed which hinder the convergence of the log likelihood function. Hence such scales may not be useful.

It has also been observed that the sum of squares values is comparable across models with either global time or models with local time separately / independently.

6. Conclusion

In this paper, alternative scales and the method for choice of better alternative scale for a given set of failure data were developed for a repairable system. It has been observed that the alternative scales based models proposed in this paper outperform the time scaled based models.

There is a large scope for future work in this area. For example, the asymptotic convergence of the parameters and the properties of the models need investigating. Models with global usage rates as a concomitant measure can also be considered.

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