Optimal defence-attack strategies between one defender and two attackers[*]

Di Wu[a], Xiangbin Yan[b], Rui Peng[b,c†], Shaomin Wu[d]

[a]School of Management, Xi'an Jiaotong University, Xian, China

[b]Donlinks School of Economics and Management, University of Science and Technology Beijing, Beijing, China

[c]School of Economics and Management, Beijing University of Technology, Beijing, China

[d]Kent Business School, University of Kent, Canterbury, Kent CT2 7FS, United Kingdom

Abstract: This paper analyses the optimal strategies for one defender and two attackers in an defence-attack game, where a) the defender allocates its resource into defending against and attacking the two attackers, and b) the two attackers, after observing the action of the defender, allocate their resources into attacking and defending against the defender, on either a cooperative or non-cooperative basis. On a cooperative basis, for each of the defender's given strategies, the two attackers work together to maximise the sum of their cumulative prospect values while anticipating the eight possible game outcomes. On a non-cooperative basis, for each of the defender's given strategies, each attacker simultaneously yet independently tries to maximise their own cumulative prospect value. In both cases, the defender maximises its cumulative prospect value while anticipating the attackers' actions. Backward induction is employed to obtain the optimal defence and attack strategies for all scenarios. Numerical examples are performed to illustrate the applications of the strategies. In general, we find two opposing effects considering the attackers' strategies and analyse the alteration of strategies for the participants under two different risk preferences: risk-averse and risk seeking. The reasons for the alteration are also performed to illustrate the practical applications.

Keywords: Reliability; attack–defence game; resource allocation; cumulative prospect; cooperation.

1. Introduction

[*] Suggested citation: Di Wu, Xiangbin Yan, Rui Peng, & Shaomin Wu, 2019, Optimal defence-attack strategies between one defender and two attackers, *Journal of the Operational Research Society*, DOI: 10.1080/01605682.2019.1630332

† Corresponding Author: Rui Peng. Email Address: pengrui1988@bjut.edu.cn, pengruisubmit@163.com.

Recently, analysis of the attack–defence game has gained popularity not only in academic fields but also in practical operations. Existing research usually assumes that there is only one defender against one attacker, where the defender uses methods such as preventive strike, false targets and protection (Guan & Zhuang, 2015; Levitin & Hausken, 2011a). However, in reality, two attackers might try to destroy the defender's object. Here, the defender should allocate its limited resource into attacking and defending against the attackers, and the attackers, in response, should allocate their resources into attacking and defending against the defender. The vulnerability of the defender and attackers is usually characterised by Tullock's contest success function (Tullock, 2001). However, this can always fail to include an important consideration: the risk attitudes of the defender and attackers. Risk attitude is important for many decision problems we encounter in our everyday lives, ranging from insurance take-up to investment choices and medical decisions (Vieider, 2009). To include risk attitudes consideration, one can use the cumulative prospect theory (CPT), which is commonly used for descriptive decisions under risk and uncertainty (Tversky & Kahneman, 1992). By adopting CPT instead of simply evaluating the vulnerability, one can calculate the cumulative prospect values (CPVs) for the defender and attackers, and can therefore consider the different possible outcomes of the defence–attack game. The risk parameters in the CPV functions of both the defender and the attackers can then be tuned to reflect different risk attitudes.

In this paper, we consider the game between one defender and two attackers. The defender moves first and can allocate its resource to four areas: 1) defending against one attacker; 2) defending against a second attacker; 3) attacking one attacker; and 4) attacking a second attacker. Having observed the defender's resource allocation, the attackers can then allocate their resources into attacking the defender and protecting themselves from the defender's attack. For generality, we consider this game under two scenarios: where the two attackers work either cooperatively or non-cooperatively. On a cooperative basis, the two attackers try to maximise the sum of their CPVs while anticipating the different game outcomes. On a non-cooperative basis, each attacker tries to maximise their own CPV while being able to adjust their strategy according to the other attacker's strategy. That is, for any given strategy of the defender, there is a simultaneous non-cooperative game between the two attackers. There are eight possible outcomes in the defence-attack game, as each of the three parties can either survive or be destroyed. Their survivability can be characterised by the contest success function. To adapt to different practical situations, we discuss two different valuations for the attackers: *Vital Target* (VT) and *Vital Base* (VB). Under VT, the attackers regard the destruction of the defender's object as being more important than their own survival. . In fact, people sometimes truly have intention to destroy others even sacrificing themselves, say, the suicide bombers. Under VB, the attackers will take their own survival as the primary goal. In addition, we analyse six different scenarios of the resource relationship between the defender and the attackers (i.e. who has the most, middle and least resource) and we find the optimal defence and attack strategies for each. By altering the risk parameters in the CPV functions of the defender and the attackers, the players' different risk attitudes can be modelled and their respective influences on the optimal strategies can be studied.

This paper contributes to three streams of literature: defence-attack game, intentional impact and reliability modelling. Hausken (1996) analysed self-interest and sympathy in economic behaviour and in this paper, we discuss the issue about self-interest and hatred. When the hatred is high, one may have intention to destroy another even sacrificing itself. Previous literature usually considered the defence-attack game between one attacker and one defender. Hausken (2011) considered the protection of complex infrastructures against multiple strategic attackers and Hausken and Bier (2011) further considered a more generalised model where the defender defends against multiple different attackers. They showed that the presence of one particularly strong attacker can cause other attackers to withdraw from the contest while in our paper the same results are obtained. Nonetheless, we analyse the influence of risk parameters and employ a different method known as CPT to solve the results. Following the literature on preventive strikes, a typical assumption is that both the defender and the attacker can defend and attack (Sandler and Siqueira, 2009; Hausken and Levitin, 2011; Hausken and Zhuang, 2011; Levitin and Hausken, 2011b; Gao, Yan, Liu and Peng, 2019). See Hausken and Levitin (2012) for a systemic review on preventive strike and other defence measures. Bier and Hausken (2011) modelled perverse effects of counterterrorism measures and Bandyopadhyay and Sandler (2011) considered the interplay between preemptive and defensive counterterrorism measure by analysing a two stage game. In our paper, we consider the cooperation and non-cooperation cases between the attackers and come up with some interesting findings.

Intentional impact is where strategic attackers aim to destroy the object of the defender. It is widely studied in the defence-attack game (Peng, Wu, & Zhai, 2018). Many methods have been proposed to obtain the optimal strategies for the attacker and the defender, including genuine object protection, false targets deployment, and the launch of preventive strike (Levitin & Hausken, 2009, 2010a; Shan & Zhuang, 2013). Wu, Xiao and Peng (2018) considered the case where the defender deploys false targets and the attacker deploys false bases, respectively. Hausken, Bier and Zhuang (2009) discussed a similar game involving one defender defending against terrorism (which is the first attacker) and disaster (which is the second attacker). In our paper, we consider two intentional impacts and obtain some insightful conclusions. Levitin and Hausken (2012) analysed the situation where the defender is confronted with two sequential attacks and aims to maximise the system survivability. Kim, Han, Kim and Kang (2017), Monroe, Ramsey and Berglund (2018) and Zhai, Peng and Wang (2017) introduced the attack–defence game in other specific systems, e.g. water distribution systems, common bus systems and cyber systems. Zhuang and Bier (2007) modelled a defence-attack game of resource allocation for simultaneously countering terrorism and natural disasters. Zhang, Ramirez-Marquez and Wang (2015) and Zhang, Ramirez-Marquez and Li (2018) proposed a simultaneous game between a defender and an attacker to study the optimal protection strategies against intentional attacks. Hausken (2008, 2017) conducted a systematic review of defence and attack strategies.

Reliability modelling is another key issue in the attack–defence game. Tullock's contest success model is widely used to characterise the reliability or vulnerability of both the defender and attacker (Tullock, 2001). Trucco, Cagno, Ruggeri and Grande (2008) used a Bayesian belief network to model the maritime transport system, considering its different outcomes and their mutual influences. Saltelli (2010) conducted sensitivity analysis on risk assessment, and Li, Wang, Song and Li (2016) used the analytic hierarchy process to obtain some useful suggestions. Aven (2016) conducted a review of recent advances on risk managements. Backward induction is widely employed in obtaining the equilibrium (Wu, Liu, Yan, Peng and Li, 2019).

This paper is primarily motivated by Levitin and Hausken (2010b) and Cohen (2010). Levitin and Hausken (2010b) assumed that an object is protected by the defender and is attacked by an attacker who launches sequential attacks. For each attack, a contest intensity measures whether the attack has a low or high impact on the object vulnerability. Cohen (2010) assumed that each player has several different types of resource, to be divided in an optimal fashion among a fixed set of objects. The concept of the generalised inverse of a matrix was used to determine the optimal strategies for each player and the value of the game. However, neither Levitin and Hausken (201b) nor Cohen (2010) considered the case where two attackers simultaneously allocate their resources into attacking the defender's object. Moreover, the risk attitude of the players is not discussed in their works. This paper resolves this omission by considering the players' risk attitudes using CPT. The major contributions of this paper are as follows:

- The attack–defence strategy, where one defender defends against two attackers, is investigated. Two scenarios – the cooperation and non-cooperation of the attackers - are discussed.
- The attackers are analysed under six different resource combinations and under two different valuations of the attackers: VT and VB.
- The defender maximises its CPV instead of the survivability of its object. The attackers maximise the sum of their CPVs under a cooperative basis, and they maximise their individual CPVs under a non-cooperative basis. The employment of CPVs helps us to better depict the behaviour of each party and more accurately reflect their risk attitudes.

The rest of this paper is organised as follows. Section 2 describes the model and provides notations. In Section 3, the optimal attack and defence strategies are obtained through backward induction for the scenario where the two attackers cooperate with each other. The influences of the risk parameters in the defender's and attackers' CPVs are also discussed. Section 4 solves the optimal attack and defence strategies for the scenario where the two attackers are non-cooperative. Similarly, sensitivity analysis on the risk parameters is conducted here. Section 5 concludes the paper and discusses several possible extensions.

2. Model Foundation

There are three parties in this model: one defender and two attackers. We calculate the destruction probability of each party and then formulate their respective CPV functions. In this attack–defence game, the defender, D, spends its resource on defending against the two attackers, *A* and *B*, with resources $R_D x_D y_D$ and $R_D x_D(1-y_D)$, respectively, where $0 \le x_D \le 1, 0 \le y_D \le 1$. The defender attacks these two attackers with resources $R_D(1-x_D)y_D$ and $R_D(1-x_D)(1-y_D)$, respectively, where $0 \le z_D \le 1$. The first attacker, *A*, both defends against and attacks *D*, with resources $R_A x_A$ and $R_A(1-x_A)$, respectively. Attacker *B* both defends against and attacks *D* with resources $R_B x_B$ and $R_B(1-x_B)$, respectively. Similarly, $0 \le x_A \le 1$ and $0 \le x_B \le 1$ are held. Note that we assume that the defender's strategy cannot be modified after the selection and the attacker can choose its strategy based on the observation of defender's strategy. The defender, as the underprivileged party, can only make its decision based on the anticipation of attacker's movement. We summarise all notations below.

Notations	
R_D, R_A, R_B	Resource for the defender and the two attackers
x_D, y_D, z_D	Resource allocation parameters of the defender
x_A, x_B	Resource allocation parameters of the two attackers
$m_{DA}, m_{AD}, m_{DB}, m_{BD}$	Contest intensity parameters
P_D, P_A, P_B	The survivability of the defender and the two attackers
u_{ij}	Potential monetary outcomes for party i in outcome j
p_j	Probability of each potential outcome
V_D, V_A, V_B	Prospect values of the defender and the two attackers
$v(u_{ij})$	Value of the potential outcome
π_k^+, π_k^-	Decision weights for the value of the potential gain and loss
g, l, λ	Risk parameters
w^+, w^-	Weighting functions for gains and losses

χ, δ	Weighting function parameters

For simplicity, we assume that the unit cost for any defence effort or attack effort is equal to one. Thus, the survivability of defender's object is

$$P_D = \frac{(R_D x_D y_D)^{m_{DA}}}{(R_D x_D y_D)^{m_{DA}} + (R_A(1-x_A))^{m_{DA}}} \cdot \frac{(R_D x_D (1-y_D))^{m_{DB}}}{(R_D x_D (1-y_D))^{m_{DB}} + (R_B(1-x_B))^{m_{DB}}}. \quad (1)$$

The survivability of attackers *A* and *B* can be represented as

$$P_A = \frac{(R_A x_A)^{m_{AD}}}{(R_A x_A)^{m_{AD}} + (R_D(1-x_D)z_D)^{m_{AD}}}, \quad (2)$$

and

$$P_B = \frac{(R_B x_B)^{m_{BD}}}{(R_B x_B)^{m_{BD}} + (R_D(1-x_D)z_D)^{m_{BD}}}, \quad (3)$$

respectively.

To formulate the CPV of each party, we note that each party can be either destroyed or survive, which ultimately forms eight possible outcomes, as shown in Table 1 below. There should be a specific probability for each outcome under which each party should have certain utility. The CPV for each party can be calculated based on the probabilities of all possible outcomes and their respective utilities. Note that in this paper the destruction of the defender represents the destruction of the defender's object.

Table 1. Eight Possible Game Outcomes

Outcome	*D*	*A*	*B*
1	Survive	Survive	Survive
2	Survive	Survive	Destroyed
3	Survive	Destroyed	Survive
4	Survive	Destroyed	Destroyed
5	Destroyed	Destroyed	Destroyed
6	Destroyed	Destroyed	Survive
7	Destroyed	Survive	Destroyed
8	Destroyed	Survive	Survive

The utility of each party under each outcome is represented by

$u_{ij}, i \in \{D, A, B\}, j \in \{1, 2...., 7, 8\}$ and the probability of each outcome is denoted as $p_j, j \in \{1, 2...., 7, 8\}$. Without loss of generality, the two attackers are assumed to be heterogeneous. In addition, we assume that the defender's primary goal is to survive. Here, we consider the two different valuations of the attackers: VT, where the attackers value the destruction of the defender's object over their own survival; and VB, where the attackers value their own survival over the destruction of the defender's object. The sign of utility under each potential outcome is shown in Table 2 below.

Table 2. Sign of Utility of each Potential Outcome

Utility	u_{D1}	u_{D2}	u_{D3}	u_{D4}	u_{D5}	u_{D6}	u_{D7}	u_{D8}
Sign	>	>	>	>	<	<	<	<
Utility	u_{A1}	u_{A2}	u_{A3}	u_{A4}	u_{A5}	u_{A6}	u_{A7}	u_{A8}
Sign	< (VT) >(VB)	< (VT) >(VB)	<	<	>(VT) <(VB)	>(VT) <(VB)	>	>
Utility	u_{B1}	u_{B2}	u_{B3}	u_{B4}	u_{B5}	u_{B6}	u_{B7}	u_{B8}
Sign	< (VT) >(VB)	<	< (VT) >(VB)	<	>(VT) <(VB)	>	>(VT) <(VB)	>

From CPT, the CPV of each party is given by

$$V_D = \sum_{k=1,2,3,4} v(u_{Dk})\pi_k^+ + \sum_{k=5,6,7,8} v(u_{Dk})\pi_k^-; \tag{4}$$

$$V_A = \begin{cases} \sum_{k=1,2,3,4} v(u_{Ak})\pi_k^- + \sum_{k=5,6,7,8} v(u_{Ak})\pi_k^+ & \text{if VT is taken} \\ \sum_{k=3,4,5,6} v(u_{Ak})\pi_k^- + \sum_{k=1,2,7,8} v(u_{Ak})\pi_k^+ & \text{if VB is taken} \end{cases}; \tag{5}$$

and

$$V_B = \begin{cases} \sum_{k=1,2,3,4} v(u_{Bk})\pi_k^- + \sum_{k=5,6,7,8} v(u_{Bk})\pi_k^+ & \text{if VT is taken} \\ \sum_{k=2,4,5,7} v(u_{Bk})\pi_k^- + \sum_{k=1,3,6,8} v(u_{Bk})\pi_k^+ & \text{if VB is taken} \end{cases}, \tag{6}$$

respectively, where $v(u_{ik}), i \in \{D, A, B\}$, is the value of the potential outcome for each party i under outcome k, π_k^+ is the decision weight for the value of the potential gain, and π_k^- is the decision weight for the value of the potential loss. Like Tversky and Kahneman (1992) and Liu, Fan and Zhang (2014), $v(u_{ik})$ can be represented by

$$x(u_{ik}) = \begin{cases} u_{ik}^g & if \;\; u_{ik} > 0 \\ -\lambda(-u_{ik})^l & otherwise \end{cases}, \tag{7}$$

where both g and l are exponent parameters, λ is the loss parameter, g is the value that exhibits risk-aversion over gains, and l is the value that exhibits risk-seeking over losses. Based on the players' attitudes, the loss-aversion factor λ should be always greater than 1 since the individuals are essentially more sensitive to losses than gains. To better calculate the CPV, the decision weights for the gains and losses can be expressed by

$$\pi_k^+ = w^+(\sum_{j=k}^{n} p_j) - w^+(\sum_{j=k+1}^{n} p_j); \tag{8}$$

and

$$\pi_k^- = w^-(\sum_{j=1}^{k} p_j) - w^-(\sum_{j=1}^{k-1} p_j), \tag{9}$$

respectively, where w^+ and w^- are the respective weighting functions for gains and losses, and are given by

$$w^+(p) = \frac{p^\chi}{[p^\chi + (1-p)^\chi]^{1/\chi}}; \tag{10}$$

and

$$w^-(p) = \frac{p^\delta}{[p^\delta + (1-p)^\delta]^{1/\delta}}, \tag{11}$$

respectively, where χ and δ are weighting parameters and can also be determined through experiments. The probabilities of each potential outcome are modelled using the survivability of each party, as shown in Table 3 below.

Table 3. Outcome Probabilities for All Cases

Case	p_1	p_2	p_3	p_4
Pro	$P_D P_A P_B$	$P_D P_A (1-P_B)$	$P_D (1-P_A) P_B$	$P_D (1-P_A)(1-P_B)$

Case	p_5	p_6	p_7	p_8
Pro	$(1-P_D)(1-P_A)(1-P_B)$	$(1-P_D)(1-P_A)P_B$	$(1-P_D)P_A(1-P_B)$	$(1-P_D)P_AP_B$

3. Optimal Strategies under Cooperation

If the attackers cooperate with each other, for any given resource allocation of the defender, they will maximise their total CPV. Since this is a two-period game, we employ backward induction to solve the problem. For a given combination of (x_D, y_D, z_D), the attackers will choose the optimal resource allocation to maximise their total CPV, which can be represented as $(x_A^*, x_B^*) = \arg\max(V_A(x_D, y_D, z_D) + V_B(x_D, y_D, z_D))$. The defender will choose the resource allocation parameters that maximise its CPV as $(x_D^*, y_D^*, z_D^*) = \arg\max(V_D(x_D, y_D, z_D, x_A^*, x_B^*))$. We consider six different combinations of the three parties' resources:

- Case 1: $R_D = 15, R_A = 10, R_B = 5$ (*D* dominates in the game)
- Case 2: $R_D = 15, R_A = 5, R_B = 10$ (*D* dominates in the game)
- Case 3: $R_A = 15, R_D = 10, R_B = 5$ (*A* dominates in the game)
- Case 4: $R_A = 15, R_D = 5, R_B = 10$ (*A* dominates in the game)
- Case 5: $R_B = 15, R_D = 10, R_A = 5$ (*B* dominates in the game)
- Case 6: $R_B = 15, R_A = 5, R_D = 10$ (*B* dominates in the game)

We also assume that the contest intensity for each battle is the same: $m_{DA} = m_{AD} = m_{DB} = m_{BD} = 2$. Without loss of generality, we set the risk parameters as $g = 0.85, l = 0.85, \lambda = 4.10, \chi = 0.60$ and $\delta = 0.70$. As Abdellaoui (2000), Bleichrodt (2000) and Abdellaoui and Bleichrodt and Paraschiv (2007) pointed out in their well-known works, the specific value of the risk parameters should be determined through some experiments. These researches performed several possible estimations of the loss aversion coefficient and conducted a series of experiments to illustrate the robustness and operability of their settings. Therefore, when it comes for real problems, decision maker can first check the widely employed parameters and make some alteration based on the real situation. We then conduct sensitivity analysis in the extension. The assumed utilities of all parties under all possible outcomes are shown in Table 4 below.

Table 4. Specific Values of Different Participants under All Conditions.

Utility	u_{D1}	u_{D2}	u_{D3}	u_{D4}	u_{D5}	u_{D6}	u_{D7}	u_{D8}
Value	10	15	15	20	-10	-15	-15	-20
Utility	u_{A1}	u_{A2}	u_{A3}	u_{A4}	u_{A5}	u_{A6}	u_{A7}	u_{A8}
Value	-15(VT) 15(VB)	-15(VT) 15(VB)	-20	-20	10(VT) -10(VB)	10(VT) -10(VB)	20	20
Utility	u_{B1}	u_{B2}	u_{B3}	u_{B4}	u_{B5}	u_{B6}	u_{B7}	u_{B8}
Value	-15(VT) 15(VB)	-20	-15(VT) 15(VB)	-20	10(VT) -10(VB)	20	10(VT) -10(VB)	20

3.1 Optimal Attack Strategies

To illustrate the optimal attack strategies for each of the defender's strategies, we consider two different values for each decision parameter of the defender: 0.3 (Low) and 0.7 (High). This gives a total of eight defence strategy combinations. Without loss of generality, we assume that all contest intensity parameters in this paper are equal to 2. As shown in Section 3 on the previous page, the resources of the three parties always form the set {5, 10, 15}, giving cases 1–6. In the following, we use TA1, TA2, TB1, TB2, TC1 and TC2 to represent these six respective cases under the VT setting, and BA1, BA2, BB1, BB2, BC1 and BC2 to represent these six respective cases under the VB setting.

3.1.1 Vital Target Cases

Starting with all VT cases, we first consider TA1, where $R_D = 15, R_A = 10, R_B = 5$. The results are shown in Table 5.

Table 5. Optimal Attack Strategies under TA1 (VT *and* Case 1)

x_D	y_D	z_D	V_A+V_B	x_A^*	x_B^*

0.3	0.3	0.3	10.994	0	0.13
0.3	0.3	0.7	10.681	0	0.03
0.3	0.7	0.3	10.315	0	0.081
0.3	0.7	0.7	10.228	0.022	0
0.7	0.3	0.3	-1.063	0	0.141
0.7	0.3	0.7	-2.213	0	0.097
0.7	0.7	0.3	-7.719	0	0.053
0.7	0.7	0.7	-7.616	0.003	0

From Table 5, we see that the optimal attack strategies follow two trends, which depend on the type of x_D. The attack from the defender can considerably influence the corresponding strategy taken by the two attackers. Nonetheless, regardless of how much resource the defender spends on protecting itself, the optimal strategy for the attackers seems unchanged. For both attackers, they should spend the minority of resource on protecting themselves from the defender's attack and the remainder of their resource on trying to destroy the defender's object after surviving the defender's attack. This is due to the summation of two effects: positive and negative. Because of the positive effect, the higher prospect value of destroying the defender's object (VT) urges the attacker to achieve the goal regardless of its own survival. Because of the negative effect, both attackers must invest a lot into attacking the defender to gain a chance to defeat them, since the defender now owns considerably more resource. Under this specific occasion, the former effect takes the dominating position and makes the attackers unafraid of sacrifice. As for TA2, where $R_D = 15, R_A = 5, R_B = 10$, the results are analogous to the TA1 results in Table 5, so we do not elaborate on them here. Instead, we turn to TB1, where $R_A = 15, R_D = 10, R_B = 5$. The results are shown in Table 6 below.

Table 6. Optimal Attack Strategies under TB1 (VT and Case 3)

x_D	y_D	z_D	V_A+V_B	x_A^*	x_B^*
0.3	0.3	0.3	15.916	0.319	0.511
0.3	0.3	0.7	14.782	0.375	0.322
0.3	0.7	0.3	15.428	0.324	0.386

0.3	0.7	0.7	14.568	0.398	0.128
0.7	0.3	0.3	11.570	0	0.415
0.7	0.3	0.7	10.206	0	0.451
0.7	0.7	0.3	8.685	0	0.183
0.7	0.7	0.7	7.489	0.022	0

From Table 6, the sum of the attackers' CPVs is greatly increased with the augmentation of the resource of one party. The optimal attack strategies obtained for different given defence strategies are not so different to those given in Table 5. Interestingly, the strategy of attacker *A* is greatly changed. When the defender employs an attack, then attacker *A* ignores its own protection and spends almost all its resource on attacking the defender, which again makes the positive effect dominate. When the defender concentrates on protection, then attacker *A* still spends part of its resource on protection. The strategy of attacker *B*, who owns the least resource, also benefits from its ally, adopting a conservative strategy rather than a radical one. Therefore, we gather two insights: 1) when one party in an attack–defence game becomes more conservative, then the other party should act more radically, and vice versa (the sensitivity analysis of the risk parameters in Section 5 again prove this insight); and 2) the participant will benefit from its ally.

We now concentrate on TB2, where $R_A = 15, R_D = 5, R_B = 10$. The results are shown in Table 7 below.

Table 7. Optimal Attack Strategies under TB2 (VT and Case 4)

x_D	y_D	z_D	V_A+V_B	x_A^*	x_B^*
0.3	0.3	0.3	22.255	0.686	0.519
0.3	0.3	0.7	21.569	0.635	0.637
0.3	0.7	0.3	22.257	0.693	0.503
0.3	0.7	0.7	21.554	0.64	0.625
0.7	0.3	0.3	21.039	0.474	0.254
0.7	0.3	0.7	21.073	0.422	0.368
0.7	0.7	0.3	21.033	0.488	0.244

0.7	0.7	0.7	21.023	0.430	0.351

The mutual benefit is more significant in Table 7 than in Table 6. Both attackers now own more resource, leading the CPV to a higher level. Again, when the defender is more radical, the attacker should be more conservative and should spend most of its resource on protection. In addition, as the attacker gains resource it becomes more conservative. The positive effect of VT remains, but the negative effect of the disadvantage in resource reverses itself. Thus, with the two positive effects, the CPV reaches a higher level and the strategy of both attackers becomes conservative. For TC1 and TC2, where $R_B = 15, R_D = 10, R_A = 5$ and $R_B = 15, R_A = 5, R_D = 10$, respectively, all results are the same as in Table 6 and Table 7, respectively.

3.1.2 Vital Base Cases

We now turn to the six VB cases. First, we consider BA1, where $R_D = 15, R_A = 10, R_B = 5$. Results are shown in Table 8 below.

Table 8. Optimal Attack Strategies under BA1 (VB and Case 1)

x_D	y_D	z_D	V_A+V_B	x_A^*	x_B^*
0.3	0.3	0.3	-17.744	0.604	1
0.3	0.3	0.7	-42.786	1	0.696
0.3	0.7	0.3	-19.41	0.604	1
0.3	0.7	0.7	-31.576	1	0.661
0.7	0.3	0.3	-7.946	0.669	1
0.7	0.3	0.7	-7.629	0.637	1
0.7	0.7	0.3	-8.95	1	0.679
0.7	0.7	0.7	-16.979	0.715	1

The situation for the attackers becomes seriously worse here, since the positive effect of VT now changes to VB, which leaves no choice for the attackers. Moreover, the defender owns more resource, which aggravates the situation. Since the base is more important to the attacker than before, no matter what strategy the defender chooses, the optimal strategy for the attackers is always to let one attacker put all its effort into protection while the other attacker concentrates on both protection and attack. The results of BA2, where $R_D = 15, R_A = 5, R_B = 10$, are analogous to BA1, so we do not elaborate on them here.

For BB1, where $R_A = 15, R_D = 10, R_B = 5$, the results are shown in Table 9 below.

Table 9. Optimal Attack Strategies under BB1 (VB and Case 3)

x_D	y_D	z_D	V_A+V_B	x_A^*	x_B^*
0.3	0.3	0.3	-5.161	1	0.895
0.3	0.3	0.7	-13.677	1	0.786
0.3	0.7	0.3	3.349	1	0.9
0.3	0.7	0.7	-5.76	1	0.782
0.7	0.3	0.3	12.58	0.829	1
0.7	0.3	0.7	13.386	0.81	1
0.7	0.7	0.3	12.009	1	0.487
0.7	0.7	0.7	10.422	0.77	1

Like the results in Table 6, the CPV greatly increases because one of the negative effects becomes positive. Moreover, we observe a phenomenon like that in Table 8, where, no matter what strategy the defender chooses, the optimal strategy for the two attackers is always to make one attacker focus on protection and have the other divide its resource into both protection and attack. We regard the attacker who only spends resource on protection as a free-rider on this occasion, since it does not contribute to the destruction of the defender's object and only cares about its own survival. We now turn to BB2, where $R_A = 15, R_D = 5, R_B = 10$. Results are shown in Table 10 below.

Table 10. Optimal Attack Strategies under BB2 (VB and Case 4)

x_D	y_D	z_D	V_A+V_B	x_A^*	x_B^*
0.3	0.3	0.3	19.607	1	0.734
0.3	0.3	0.7	18.856	0.939	1
0.3	0.7	0.3	18.485	1	0.736
0.3	0.7	0.7	18.825	1	0.92

0.7	0.3	0.3	20.764	1	0.552
0.7	0.3	0.7	20.923	1	0.729
0.7	0.7	0.3	20.474	1	0.562
0.7	0.7	0.7	21.710	1	0.765

Because of the VB setting, the CPV for this case does not reach the results obtained in Table 7; however, the attackers' circumstances are still greatly improved. Interestingly, Table 10 shows that attacker *A,* who owns the most resource, spends all its resource on protection, except for the special occasion where the defender allocates most of its resource on attacking attacker *B* and the minority of its resource on attacking attacker A. In other words, the attacker with more resource is more likely to become a free-rider and take advantage of the other attacker's cooperation, whereas the attacker with less resource needs to consider both its own protection and the attack on the defender. We represent the two effects in Fig. 1 below. The brackets between the two sets of opposing effects represent the strategy's attacker preference, and "+" and "-" denote the respective positive and negative effect to the attackers' aggressiveness.

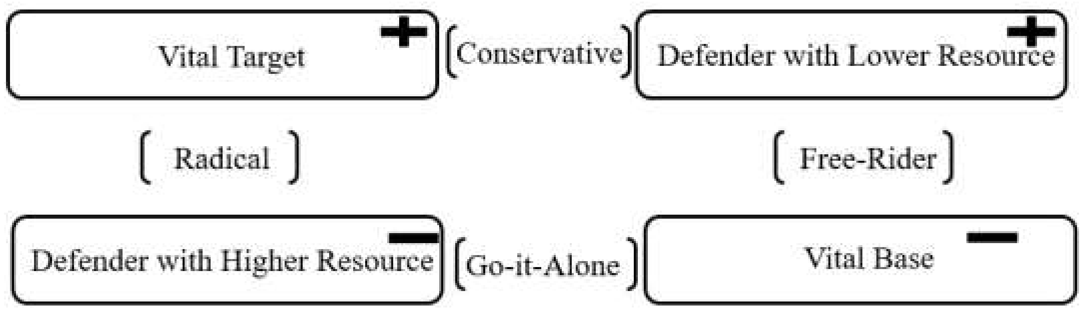

Figure 1. Two Effects Considering the Attackers' Strategies.

Fig. 1 shows the two effects and their influences, and the four segments show the typical characteristics. When the outcome is VT-oriented and the defender owns more resource, the strategy of the attacker becomes more radical. When the defender owns less resource, the opposite is true. When the outcome is VB-oriented and the defender owns less resource, whichever attacker owns more resource becomes a free-rider whose only concern is for its own safety. Thus, it will not want to spare any effort in attacking the defender. The opposite case occurs when the defender owns more resource and the attackers can only fight the enemy separately.

3.2 Optimal Defence Strategies

Having obtained the optimal attack strategies under different conditions, we try to solve the optimal defence strategies through backward induction. Note that there will be differences between the VT scenario and the VB scenario. Results for both are shown in Table 11 below, with the values in brackets representing VB and those in no brackets representing VT.

Table 11. Optimal Defence Strategies under VT and VB

Cases	x_D^*	y_D^*	z_D^*	V_D	x_A^*	x_B^*	V_A+V_B
1	0.918 (0.17)	0.673 (0.5)	0.558 (0.52)	-1.159 (-5.14)	0.19 (1)	0.107 (0)	-21.059 (12.506)
2	0.917 (0.118)	0.321 (0.5)	0.578 (0.52)	-0.397 (-3.221)	0.154 (0)	0.139 (1)	-21.763 (12.517)
3	0.964 (0.096)	0.686 (0.5)	0.576 (0.509)	-10.929 (-16.58)	0.104 (1)	0.195 (0.096)	2.638 (20.495)
4	0.089 (0.086)	0.2 (0.5)	0.1 (0.519)	-31.139 (-12.289)	0.916 (1)	0.744 (0.587)	23.348 (17.594)
5	0.965 (0.092)	0.33 (0.5)	0.593 (0.52)	-9.03 (-13.45)	0.259 (0.132)	0.062 (1)	1.964 (9.57)
6	0.089 (0.088)	0.1 (0.5)	0.2 (0.52)	-30.447 (-14.767)	0.928 (0.749)	0.684 (1)	22.833 (14.024)

The conclusion here is intuitive. Under the VT scenario, if the defender owns relatively more resource than the attackers do, it will choose to protect the object, since its goal is to survive. The only difference here is that the defender may allocate different amounts of resource to defend against attackers *A* and *B*, depending on the respective resources of *A* and *B*. However, if the defender owns less resource than both attackers do, the optimal strategy alters, becoming one of a leap of faith where the defender should spend nearly all its resource on attacking the attackers in the hope that it proves victorious. In addition, the defender must spend more resource on attacking the attacker who owns relatively less resource. Nonetheless, the CPV under this case is far lower than the other cases. In contrast, under the VB scenario, since the key goal of the attackers is to survive rather than to destroy the defender, the free-rider situation occurs again. As Table 11 shows, if the defender is in the dominant position, then the attacker with more resource will prefer to protect itself rather than to attack. However, if the attacker is in the dominant position, then the defender will focus on attacking and thus force the attackers to spend more of their resources on protection. The CPV of the attackers will strictly increase under the VB scenario compared with the VT scenario.

3.3 Influence of Risk Parameters

In the attack–defence game in this paper, the defender is regarded as a party and the attackers are regarded a party. We employ CPT in the model foundation to better depict the players' risk attitudes, and we use the implicit assumption that all parties hold the same risk attitude. For cases where the defender and attackers are confronted with either a more risk-seeking or more risk-averse situation, we assume that $g = 0.9, l = 0.8$ and $g = 0.8, l = 0.9$, respectively. Note that λ represents the greater sensitivity to losses than gains. λ is always greater than one, since most individuals care more about losses. We alter λ from 3 to 5 with the step of 0.5. To avoid repetition, we choose to analyse only cases 1, 3 and 4 (from Section 3), these being the scenarios where the defender owns the most, medium and least resource, respectively, under the VT scenario. All previous conclusions of risk influence remain the same under the VB scenario. Where $g = 0.9$ and $l = 0.8$, the results are shown in Fig. 2 below.

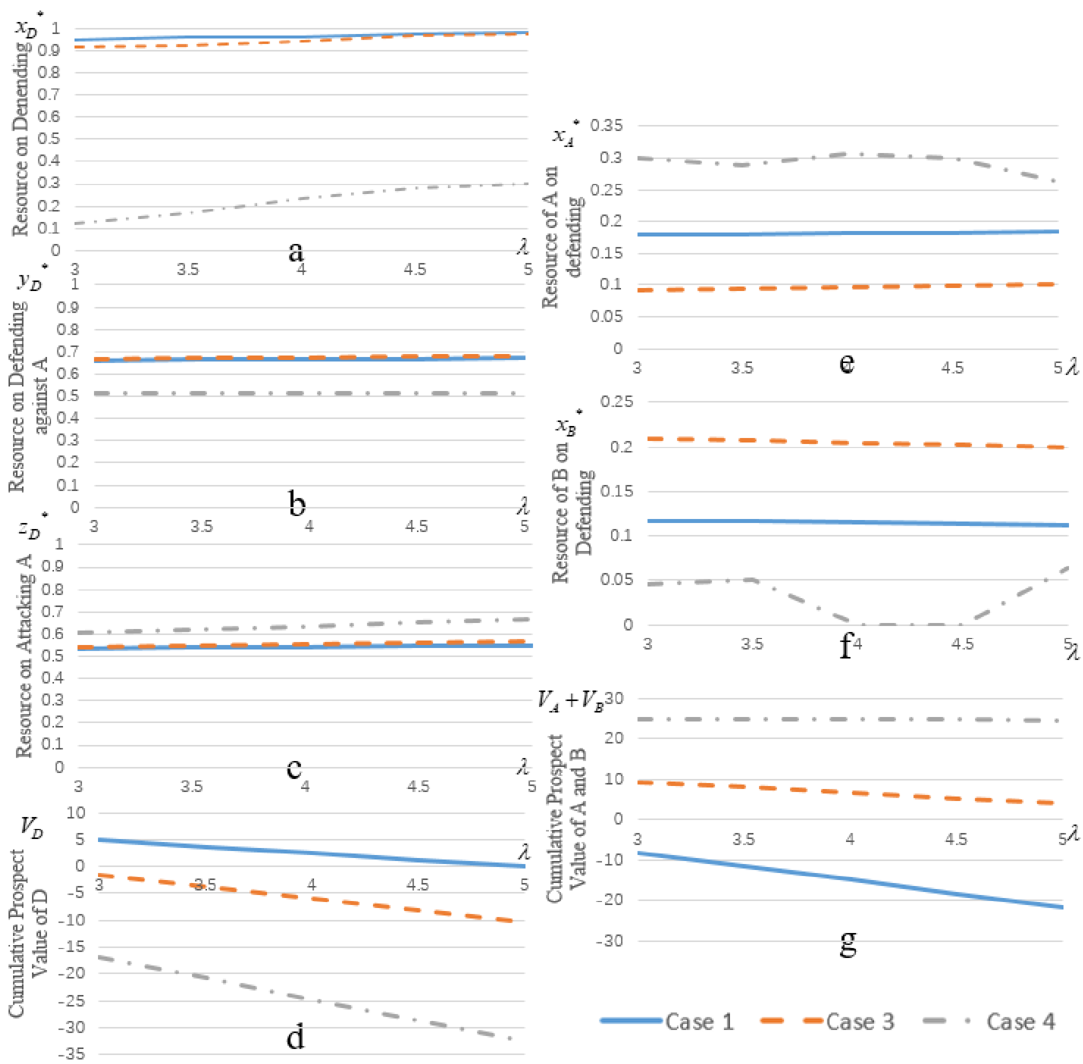

Figure 2. Optimal Strategies and Corresponding CPVs under Cooperation and $(g = 0.9, l = 0.8)$.

From Fig. 2(a), we find the same conclusions that were obtained in the benchmark: (1) the defender will concentrate on protection if its resource is the highest among all participants, and (2) if its resource is the lowest, the defender will take a leap of faith to focus on an attack. Interestingly, the optimal resource spent on attacking and defending against A does not change with the alteration of λ, which implies that these decisions are not sensitive to the losses. Fig. 2(d) illustrates that if the participants become more risk-seeking, then the CPVs increase for both parties. Similarly, if the defender chooses to spend more resource on an attack, then the attackers can only raise the resource allocated to protection, leading to an augmented CPV under cooperation.

Where $g = 0.9$ and $l = 0.8$, the results are shown in Fig. 3 below.

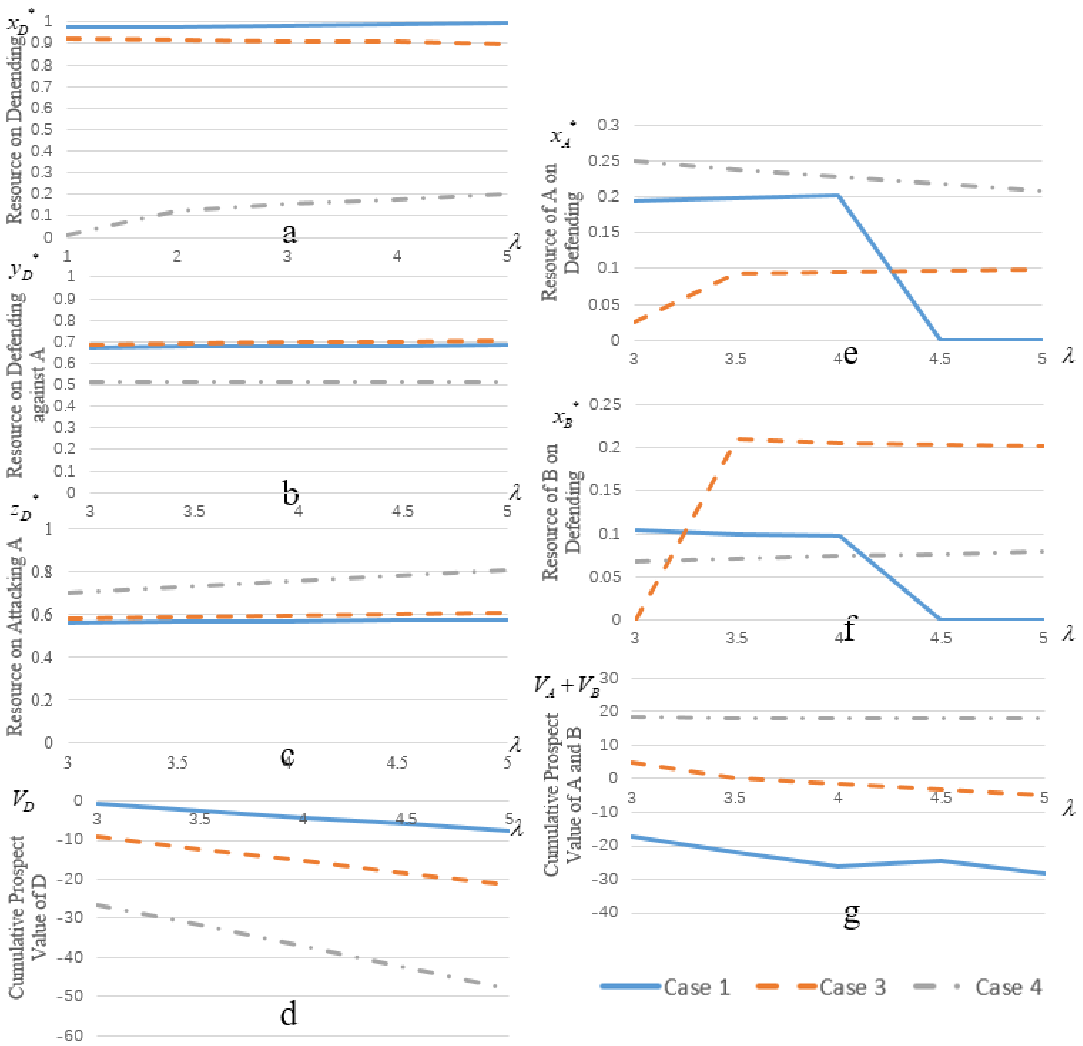

Figure 3. Optimal Strategies and Corresponding CPVs under Cooperation and $(g = 0.8, l = 0.9)$.

Comparing Fig. 2 and Fig. 3, we see that the defender spends more resource on protection if its risk attitude is risk-averse. This allows the attacker to save more resource for an attack, and it fundamentally lowers the CPV of the defender. We can therefore conclude that, if the attackers cooperate with each other, the defender should be more risk-seeking and should take a more radical strategy to raise its CPV.

4. Optimal Strategies under Non-cooperation

Attackers *A* and *B* compete in three cases, where:

- *A and B move simultaneously.* *A* and *B* should be able to seek the optimal solutions based on the equations $(x_A^*) = \arg\max(V_A(x_D, y_D, z_D, x_B))$ and $(x_B^*) = \arg\max(V_B(x_D, y_D, z_D, x_A))$, respectively. The defender will always have

$(x_D^*, y_D^*, z_D^*) = \arg\max(V_D(x_D, y_D, z_D, x_A^*, x_B^*))$.

- *Attacker A moves first*. There will be an alteration in the forerunner, giving $(x_B^*) = \arg\max(V_B(x_D, y_D, z_D, x_A))$ and $(x_A^*) = \arg\max(V_A(x_D, y_D, z_D, x_B^*))$.
- *Attacker B moves first*. There will be an alteration in the forerunner, giving $(x_A^*) = \arg\max(V_A(x_D, y_D, z_D, x_B))$ and $(x_B^*) = \arg\max(V_B(x_D, y_D, z_D, x_A^*))$.

All parameters remain the same as the previous section. Note, we consider the cases of *A* and *B* simultaneously due to space constraint.

4.1 Optimal Attack Strategies

Where the attackers compete, and move simultaneously, it is interesting to study one attacker's response to the other. This scenario is like the duopoly game under non-cooperation in microeconomics where one supplier chooses its optimal price or quantity in response to the other supplier's strategy. By putting the response curves of the two attackers together, the equilibrium of their strategies needs analysis. The equilibrium, if it exists, should be the intersection points of their respective response curves.

The response curves and the corresponding CPVs for both attackers are shown in Fig. 4 below. Again, for simplicity, we choose to analyse only cases 1, 3 and 4 (from Section 3), these being the scenarios where the defender owns the most, medium and least resource, respectively.

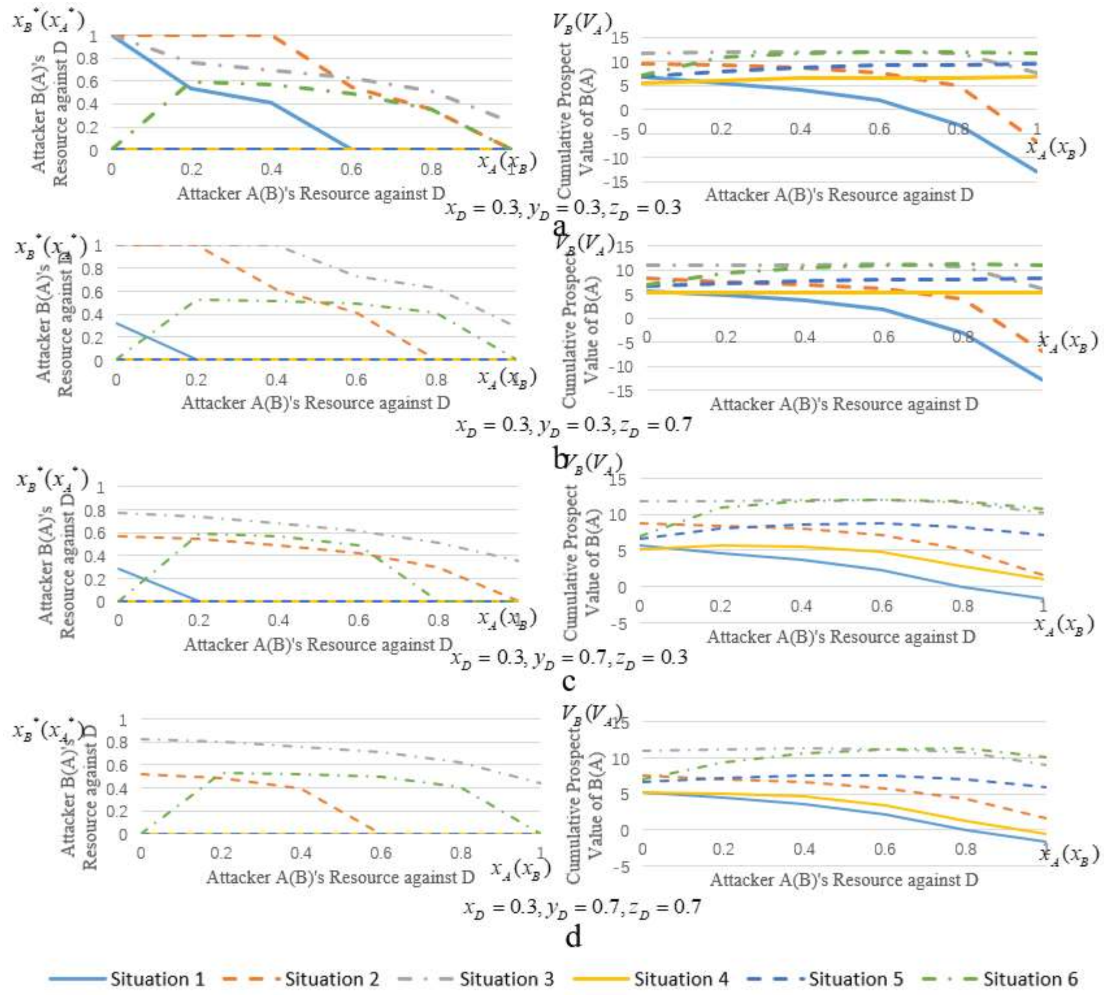

Figure 4. Optimal Responses for *A* and *B* in VT under Different Combinations of

$$(x_D = 0.3, y_D, z_D).$$

In Fig. 4, situations 1–3 represent *B*’s best response to *A* under cases 1, 3 and 4, and situations 4–6 represent *A*’s best response to *B* under cases 1, 3 and 4.

The response curves for the two attackers under the defender’s different strategies are shown in Fig. 4 and Fig. 5. Each of these two figures are divided into four subfigures per the different values of y_D and z_D. For each subfigure, we perform the response curve for each of the two attackers with respect to the other attacker’s strategy, and we also show the attackers’ corresponding CPVs. We see that the intersections of the curves for situations 1–3 and situations 4–6 are the equilibrium of the optimal attack strategies. From Fig. 4 we can conclude that (1) under situation 3, where the attacker owns more resource than the defender, the optimal strategy is always to spend more resource on protection, and (2) for the opposite situation the optimal strategy is to spend nearly all resource on an attack. This phenomenon is clearer in Fig. 4 than in Fig. 3, where we can see the alteration of the curve for situation 1. Like the scenario of attack–defence under cooperation, such a phenomenon occurs because the prospect value here is VT-oriented: if the defender owns more resource, then the strategy becomes more radical, as per the two effects we discussed in Fig. 1. Moreover, in some cases there are no intersection points for the response curves of the two attackers, and in other cases there are multiple intersection points, which implies that, in some cases, the two attackers may not be able to reach an equilibrium, and, in other cases, they may reach different equilibriums. We must analyse the potential equilibrium from three sides:

- *When there is no equilibrium.* When the response curves have no intersection point, then the most conservative strategy of the defender is to anticipate that the attackers should choose strategies that minimise the CPV of the defender: $(x_A^*, \mathrm{x}_B^*) = \arg\min(V_D(x_D, y_D, z_D, x_A, x_B))$ and $(x_D^*, y_D^*, z_D^*) = \arg\max(V_D(x_D, y_D, z_D, x_A^*, x_B^*))$.
- *When there is one equilibrium.* When there is only one intersection point between the two response curves then that point is the most reasonable movement for both attackers. The defender also anticipates that the attackers will make such movement.
- *When there is more than one equilibrium.* When more than one intersection point exists then the attackers may reach different equilibriums. However, it is conservative for the defender to anticipate that the attackers will reach the equilibrium that gives the defender the lowest CPV.

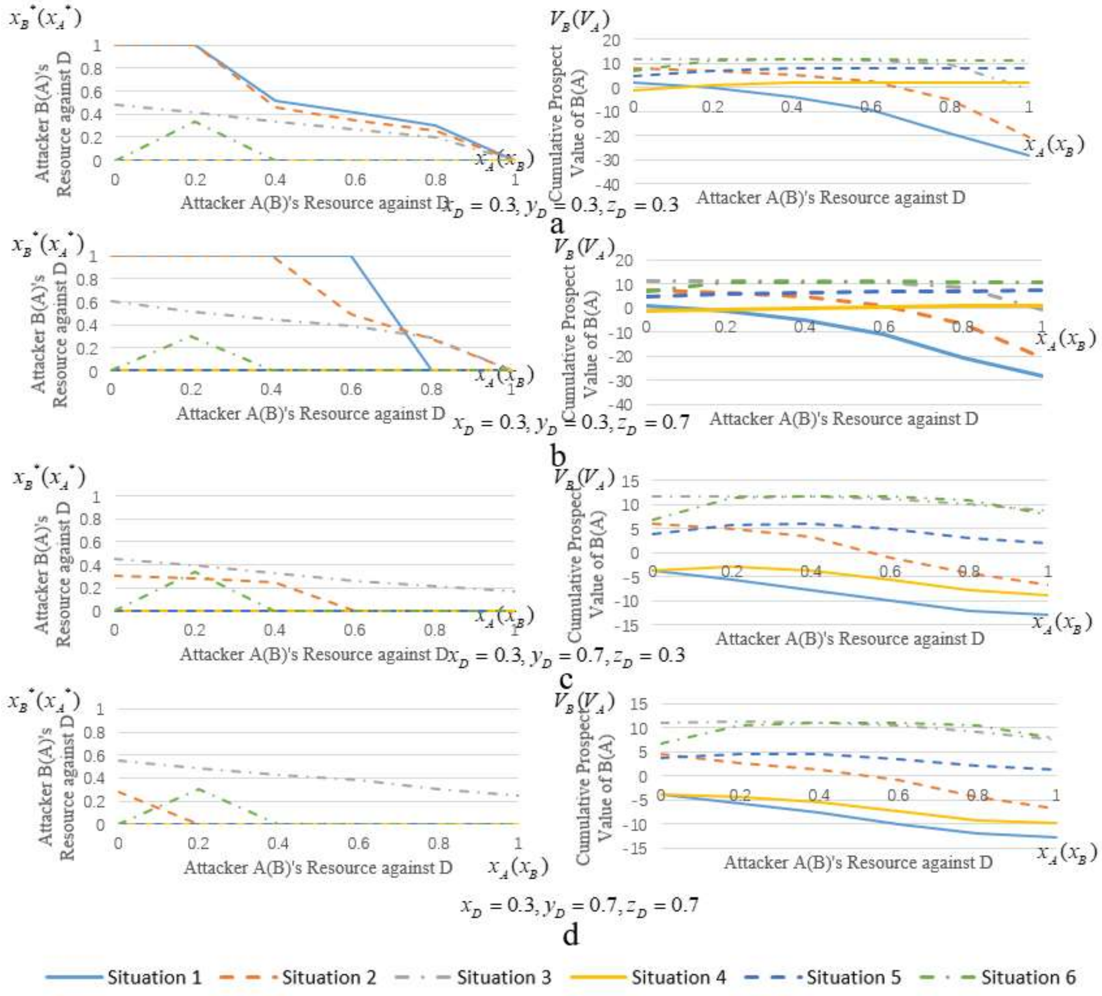

Figure 5. Optimal Responses for A and B in VT under Different Combinations of

$$(x_D = 0.7, y_D, z_D).$$

Comparing Fig. 4 and Fig. 5, we see that the best response for both attackers becomes more polarised. Where the defender owns more resource, the strategies of the attackers become more radical because of the VT effect. In contrast, where the defender owns relatively less resource, the optimal strategies of the attackers become more conservative. Furthermore, the CPV in Fig. 4 is higher than the CPV in Fig. 5. In other words, if more of the defender's resource is spent on protection, the CPV of the attacker decreases because of the more radical action it takes.

We now turn to the scenario of VB, as shown by the results in Fig. 6 and Fig. 7 for different values of x_D.

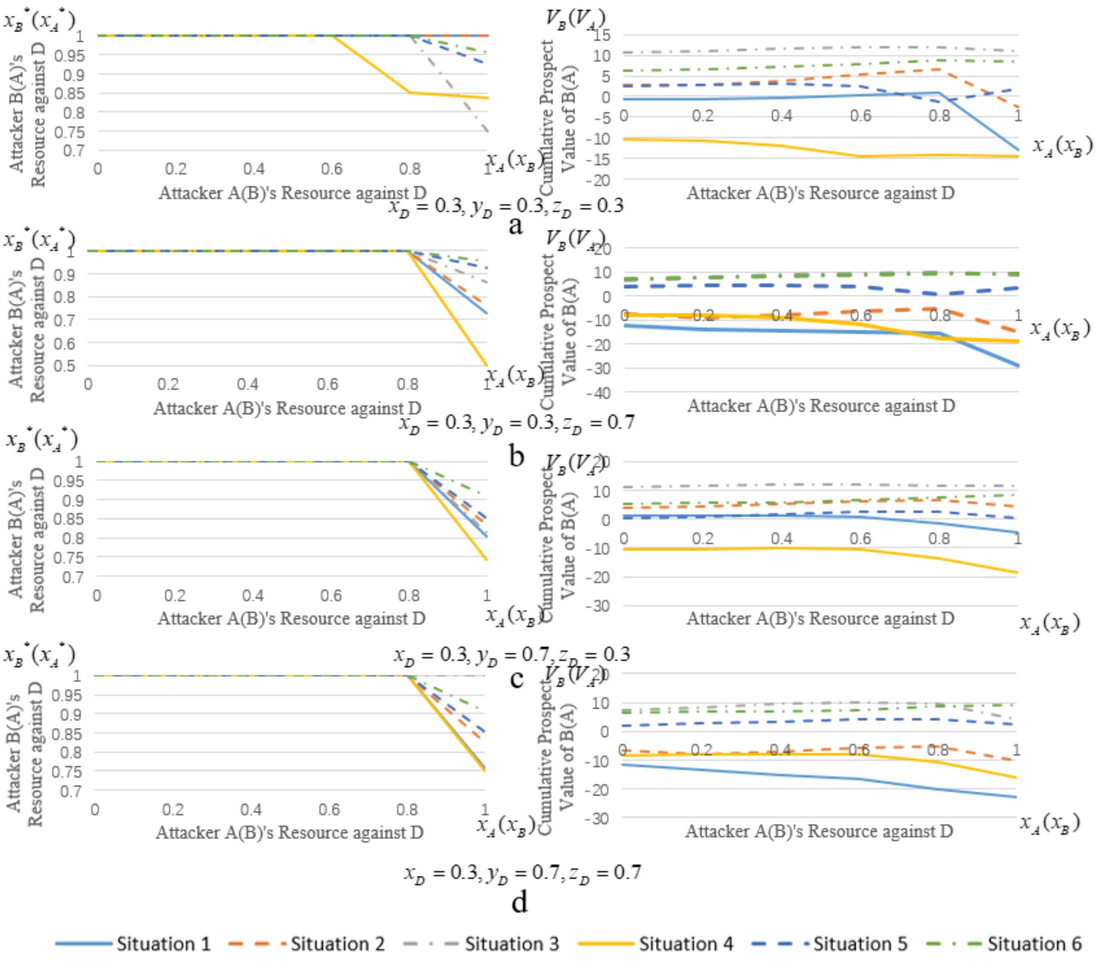

Figure 6. Optimal Responses for A and B in VB under Different Combinations of

$$(x_D = 0.3, y_D, z_D).$$

From Fig. 6 and Fig. 7, the best responses for both attackers seem to remain at a very high level under VB. In contrast to the VT scenario, the attackers cannot cooperate with each other; only one attacker spends resource on an attack while the other concentrates on protection. The non-cooperation between attackers now pushes them to consider only their own CPVs. Since the safety of the base is of greater value, the optimal strategies of both attackers alters to one of protection. Moreover, the summation of CPVs of both parties is far lower than in the scenario under cooperation. In addition, when the defender owns less resource, one of the attackers should spend most of their resource on an attack, which will lead the other to spend nearly all their resource on protection, making them a free-rider. On the contrary, when the defender owns more resource, then the two attackers have to fight on their own without cooperation that again proves the conclusion in Figure 1.

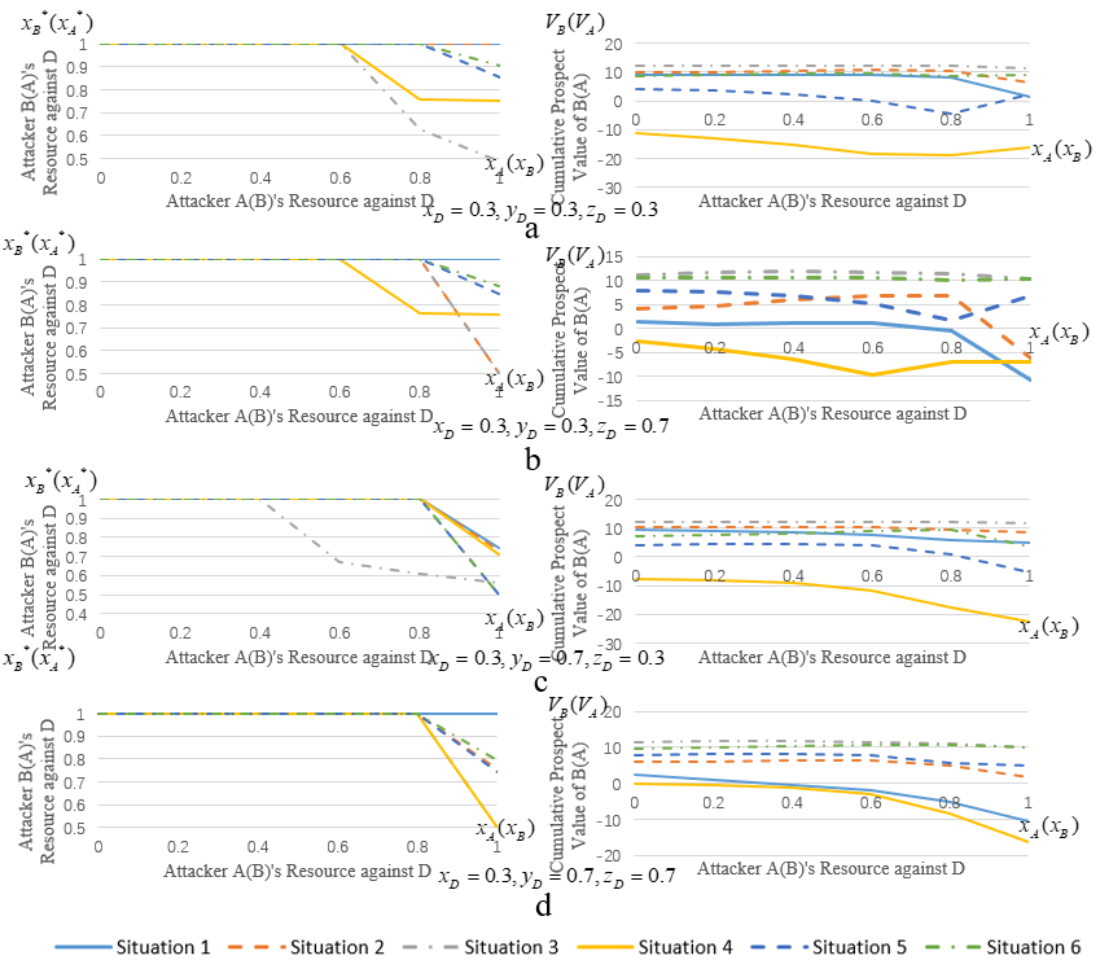

Figure 7. Optimal Responses for A and B in *Vital Base* under Different Combinations of

$$(x_D = 0.7, y_D, z_D).$$

Interestingly, when the defender now decides to spend more resource on protection, the mutual constraint between the two attackers seems to disappear since their bases will not be destroyed by the defender and thus the attackers can spend more efforts in attacking the defender. The cumulative prospect values for both attackers are higher than the case where the defender is more radical. It can be seen that the intersection points are increasing under *Vital Base* since the strategy of both parties is widely overlapping, as can be seen in the figure.

4.2 Optimal Defence Strategies

Having obtained the optimal attack strategies, we analyse the optimal defence strategies through backward induction and obtain the results shown in Table 12. Similarly, the value in brackets represents the case of *Vital Base*. Since the attackers compete with each other, we do not analyse the summation of the two attackers' cumulative prospect value. Instead, we analyse them separately.

Table 12. Optimal Defence Strategies under *Vital Target* and *Vital Base*.

Cases	x_D^*	y_D^*	z_D^*	V_D	x_A^*	x_B^*	V_A	V_B
1	0.805 (0.245)	0.584 (0.504)	0.626 (0.45)	0.706 (-2.13)	0.247 (0.455)	0.428 (0.465)	-5.226 (6.264)	-3.63 (-6.23)
2	0.755 (0.247)	0.425 (0.612)	0.503 (0.54)	0.204 (-2.16)	0.34 (0.451)	0.023 (0.316)	-5.502 (9.34)	-3.954 (-14.49)
3	0.802 (0.187)	0.612 (0.5)	0.618 (0.67)	-8.37 (-10.8)	0.214 (0.535)	0.413 (0.526)	4.587 (9.6)	5.01 (6.39)
4	0.754 (0.034)	0.454 (0.5)	0.648 (0.72)	-27.9 (-19.8)	0.405 (0.456)	0.48 (0.547)	10.13 (11.94)	11.09 (10.46)
5	0.796 (0.165)	0.317 (0.49)	0.6 (0.62)	-8.986 (-10.2)	0.354 (0.471)	0.262 (0.465)	3.499 (10.93)	6.04 (-6.59)
6	0.668 (0.032)	0.48 (0.5)	0.668 (0.71)	-29.9 (-27.6)	0.469 (0.488)	0.513 (0.54)	9.669 (11.97)	11.35 (8.42)

From Table 12, we can have some interesting observations. First, when the attackers now compete with each other, the defender should spend more resource on attacking the attackers instead of protection. The defender's corresponding cumulative prospect value under this case is higher than the case of cooperation but the summation of prospect values of both attackers is decreasing except for two cases where the defender owns more resource. In other words, the split of the attacker makes the defender owns more advantage since the attackers under this case spend more resource on protection instead of attacking. For the attackers, they just fight their own enemy. Nonetheless, when the defender owns more resource, it seems that the attackers prefer competing with each other than cooperation. The reason behind is that when the attackers cooperate with each other, there is a common phenomenon called free-rider, where the one who owns relatively more resource may take advantage of the other attacker and spare all effort in protection, pushing the other attacker to attack the defender. Also, the resource spent on protection will increase under the case of *Vital Base* since the safety of their own base is now the priority of the attackers. Similarly, the defender will change its strategy under two different occasions: dominating or dominated. No matter in which case, under *Vital Base* the defender should always spend the majority of its resource on attacking; and the intensity of the attack will be higher if its resource is not dominating. However, if the defender owns resource advantage, this strategy will lower down its cumulative strategy since the attackers now concentrate more on the safety of the base. On the contrary, if the attackers own advantage, this strategy may benefit the defender as a result of attackers' prudent.

4.3 Influence of Risk Parameters

Similar to Section 3.3, we analyse the influence of the risk parameters when the attackers compete with each other. Still, for the sake of conciseness, we only perform the results under three relationships between the parties, which represents the parameter setting in Cases 1, 3, and 4, respectively. In addition, only the case of *Vital Target* is discussed and we can further prove that all major conclusion of risk parameters remains the same under the case of *Vital Base*. The results when both parties become more risk-seeking are performed in Figure 8.

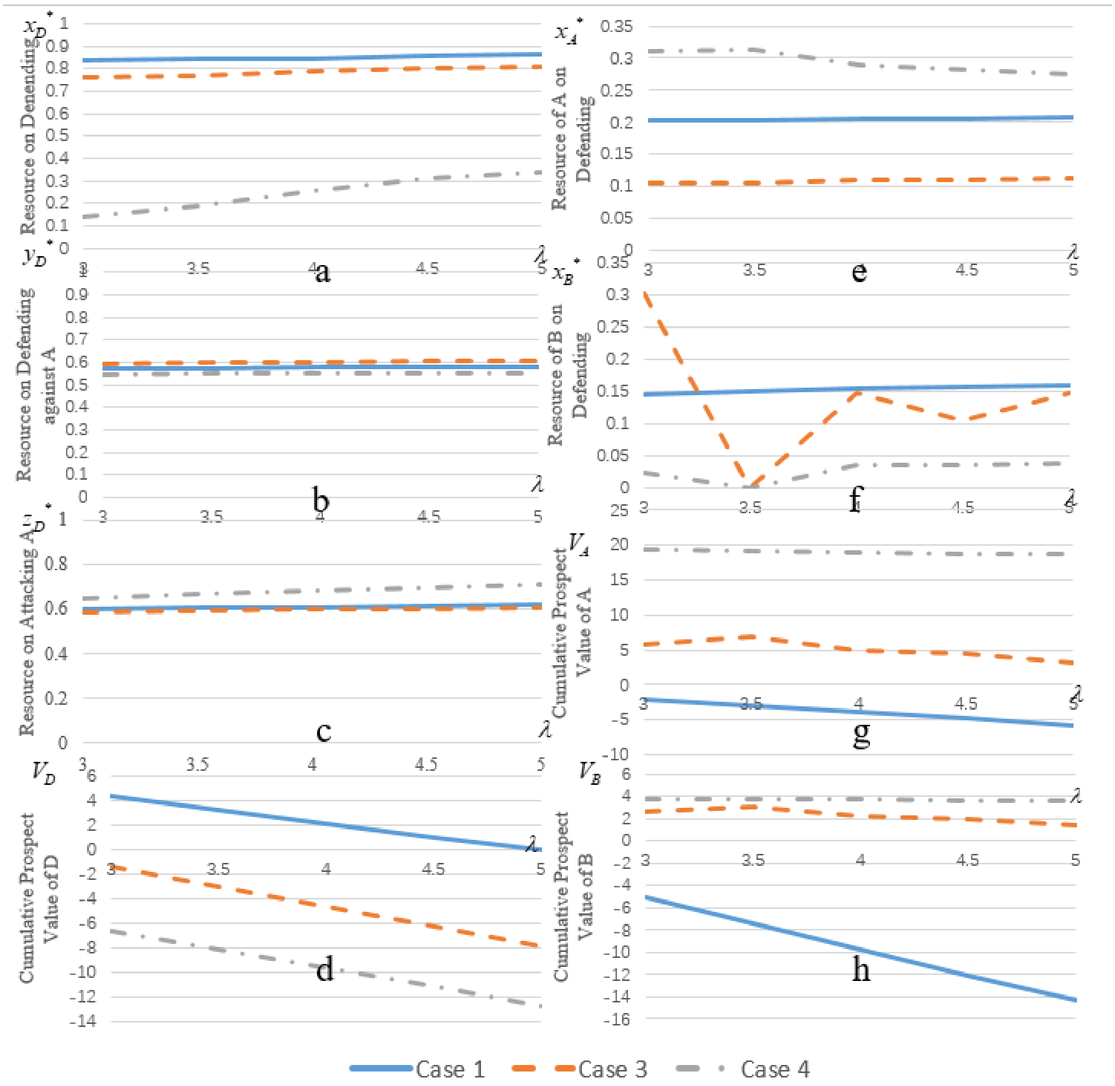

Figure 8. Optimal Strategies and Cumulative Prospect Value under Non-cooperation and

$$(g = 0.9, l = 0.8).$$

From the comparison between Figure 2 and Figure 8, we have the following findings.

- If the defender owns the highest resource, for the attackers, non-cooperation seems to be a better option than cooperation. This is because the attacker with more resource will become a free-rider and therefore makes the summation of the cumulative prospect value to decrease. However, under the case of non-cooperation, the attackers fight the defender on their own and thus make the strategy more harmful, which increases the value.
- If the defender does not own the highest resource, for the attackers, cooperation will become a better option than non-cooperation. This is because the attackers can adjust their strategy based on the action of the defender. Here is so-called late-move advantage.

Moreover, contrary to the case where attackers cooperate with each other, now the defender should be more conservative instead of being radical. This is reasonable since under cooperation, the free-rider may give the defender more opportunity to destroy one of the attackers. In contrast, under non-cooperation the defender should defend against all possible attacks, which makes the defender more conservative.

Now we turn to the case when both parties become risk-averse and perform the results in Figure 9.

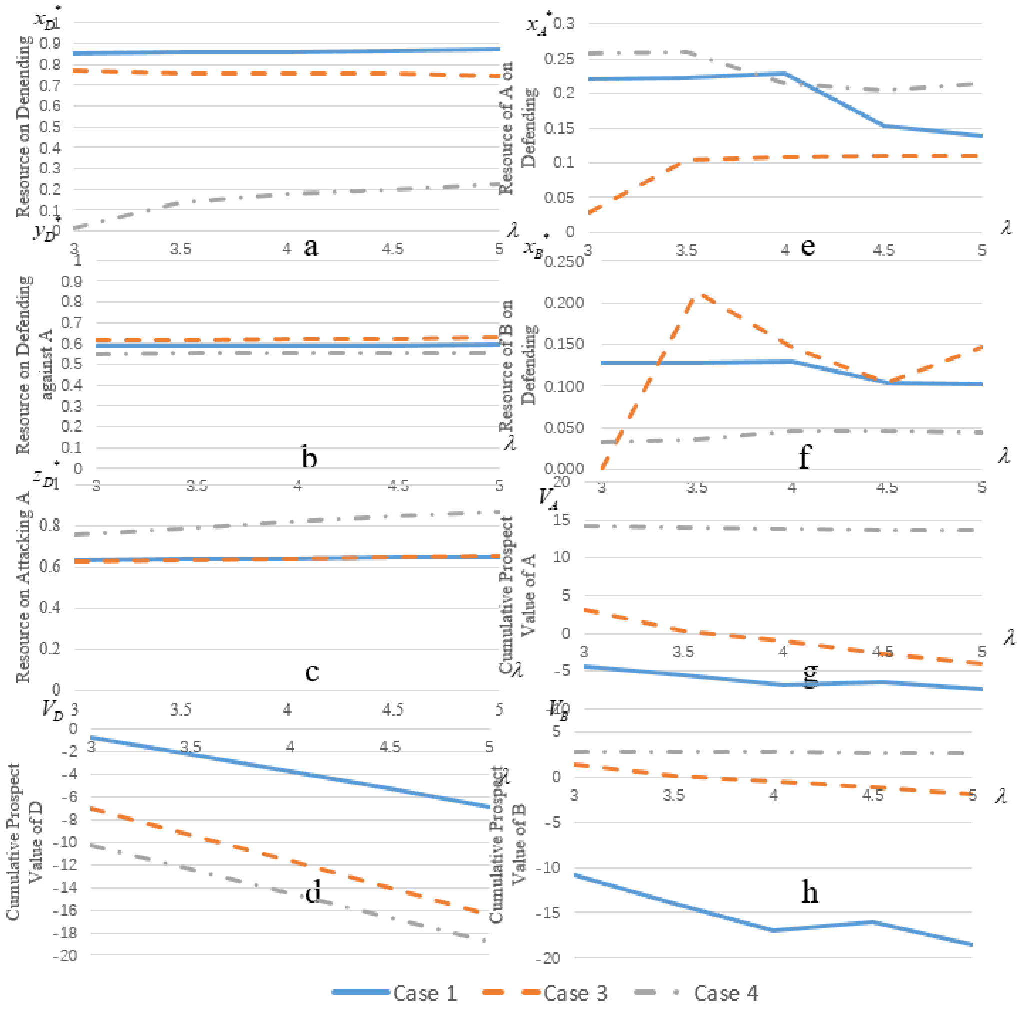

Figure 9. Optimal Strategies and Cumulative Prospect Value under Non-cooperation and

$$(g = 0.8, l = 0.9).$$

From Figure 9, we can obtain the same trend of the curve in each common subfigure as Figure 3. Note that there are seven subfigures in Figure 3 since under cooperation the attackers try to maximise the summation of their cumulative prospect values. Also, the attackers will prefer non-cooperation when the defender owns resource advantages and prefer cooperation in other cases. When we compare results under different combinations of risk attitudes, we can find that the risk attitude will considerably change the optimal strategies of the defender and the attackers. In other words, imagine the case where the defender prefers attacking rather than protection, then two different situations may arise:(1) the resource spent on attack will raise to a higher level if its risk attitude becomes more risk-seeking, and (2) if its risk attitude becomes more risk-averse, the optimal strategy will not change to protection but the optimal resource spent on attack will go down and the resource spared on protection will increase.

5. Conclusion and Future Works

Numerical examples are presented to illustrate the application of the proposed models and sensitivity analysis is conducted to study the influences of different parameters. When attackers cooperate with each other, the attacker with more resource is more likely to become a free-rider and take advantage of the other attacker's cooperation, whereas the attacker with less resource needs to consider both its own protection and the attack on the defender. Moreover, under the VT scenario, if the defender owns relatively more resource than the attackers, it will choose to protect the object. Contrary form that, under the VB scenario, the free-rider situation occurs again. By analysing the risk parameters, we can find that the defender should be more risk-seeking and should take a more radical strategy to raise its CPV. When non-cooperation exists, then equilibrium with different types may occur and the best response for both attackers becomes more polarised, leading to a more complicated case for the defender. Comparing between the cooperation case and non-cooperation case, we can obtain two major conclusions of our paper. If the defender owns the highest resource, for the attackers, non-cooperation seems to be a better option than cooperation. Otherwise, for the attackers, cooperation will become a better option than non-cooperation.

This work can be extended in several directions. Say, it is interesting to consider the cases where the attackers also own intention to attack each other. Another future work is to extend to the case where there are arbitrary numbers of defenders and attackers. In addition, this work modelled the contest between defender and attacks as a sequential game. In the future, simultaneous game can be used to study the contest between an arbitrary number of players, where each owns an object to defend and intention to attack others. Another direction for future work is to incorporate the use of false targets.

Acknowledgement

The research was supported by the NSFC under grant numbers 71671016.

References

Abdellaoui, M. (2000). Parameter-free elicitation of utility and probability weighting functions. Management Science, 46(11), 1497-1512.

Abdellaoui, M., Bleichrodt, H., & Paraschiv, C. (2007). Loss aversion under prospect theory: a parameter-free measurement. Management Science, 53(10), 1659-1674.

Aven, T. (2016). Risk assessment and risk management: review of recent advances on their foundation. European Journal of Operational Research, 253(1), 1-13.

Bandyopadhyay, S., & Sandler, T. (2011). The interplay between preemptive and defensive counterterrorism measures: A two-stage game. Economica, 78(311): 546-564.

Bier, V.M., & Hausken, K. (2011). Endogenizing the Sticks and Carrots: Modeling Perverse Effects of Counterterrorism Measures. Annals of Operations Research, 186(1): 39-59.

Bleichrodt, H. (2000). A parameter-free elicitation of the probability weighting function in medical decision analysis. Management Science, 46(11), 1485-1496.

Cohen, N. D. (2010). An attack-defense game with matrix strategies. Naval Research Logistics, 13(4), 391-402.

Gao, K., Yan, X., Liu, X., & Peng, R., (2019). Object defense of a single object with preventive strike of random effect. Reliability Engineering & System Safety, 186, 209-219.

Guan, P., & Zhuang, J. (2015). Modeling resources allocation in attacker-defender games with "warm up" CSF. Risk Analysis, 36(4), 776-791.

Hausken, K. (1996). Self-interest and sympathy in economic behavior. International Journal of Social Economics, 23(7), 4-24.

Hausken, K. (2008). Strategic defense and attack for reliability systems. Reliability Engineering & System Safety, 93(11), 1740-1750.

Hausken, K. (2011). Protecting Complex Infrastructures Against Multiple Strategic Attackers, International Journal of Systems Science 42, 1, 11-29.

Hausken, K. (2017). Defense and attack for interdependent systems. European Journal of Operational Research, 256(2), 582-591.

Hausken, K., & Bier, V. (2011), Defending Against Multiple Different Attackers, European Journal of Operational Research 211, 2, 370-384.

Hausken, K., & Levitin, G. (2011). Shield vs. Sword Resource Distribution in K-round Duels, Central European Journal of Operations Research 19, 4, 589-603.

Hausken, K., & Levitin, G. (2012). Review of Systems Defense and Attack Models, International Journal of Performability Engineering 8, 4, 355-366.

Hausken, K., & Zhuang, J. (2011). Governments' and Terrorists' Defense and Attack in a T-period Game. Decision Analysis, 8(1): 46-70.

Hausken, K., Bier, V., & Zhuang, J. (2009), Defending Against Terrorism, Natural Disaster, and All Hazards, Game Theoretic Risk Analysis of Security Threats, Springer, New York, 65-97.

Kim, H. E., Han, S. S., Kim, J., & Kang, H. G. (2017). Systematic development of scenarios caused by cyber-attack-induced human errors in nuclear power plants. Reliability Engineering & System Safety, 167, 290-301.

Levitin, G., & Hausken, K. (2009). Parallel systems under two sequential attacks. Reliability Engineering & System Safety, 94(3), 763-772.

Levitin, G., & Hausken, K. (2010a). Defence and attack of systems with variable attacker system structure detection probability. Journal of the Operational Research Society, 61(1), 124-133.

Levitin, G., & Hausken, K. (2010b). Resource distribution in multiple attacks against a single target. Risk Analysis, 30(8), 1231-1239.

Levitin, G., & Hausken, K. (2011a). Defense resource distribution between protection and redundancy for constant resource stockpiling pace. Risk Analysis, 31(10), 1632-1645.

Levitin, G., & Hausken, K (2011b). Preventive strike vs. false targets and protection in defense strategy. Reliability Engineering & System Safety, 96(8): 912-924

Levitin, G., & Hausken, K. (2012). Parallel systems under two sequential attacks with imperfect detection of the first attack outcome. Journal of the Operational Research Society, 63(11), 1545-1555.

Li, Y., Wang, T., Song, X., & Li, G. (2016). Optimal resource allocation for anti-terrorism in protecting overpass bridge based on AHP risk assessment model. KSCE Journal of Civil Engineering, 20(1), 309-322.

Liu, Y., Fan, Z. P., & Zhang, Y. (2014). Risk decision analysis in emergency response: a method based on cumulative prospect theory. Computers & Operations Research, 42(2), 75-82.

Monroe, J., Ramsey, E., & Berglund, E. (2018). Allocating countermeasures to defend water distribution systems against terrorist attack. Reliability Engineering & System Safety, 2018, 179, 37-51.

Peng, R., Wu, D., & Zhai, Q. (2018). Defense resource allocation against sequential unintentional and intentional impacts. IEEE Transactions on Reliability, 68(1), 364-374.

Saltelli, A. (2010). Sensitivity analysis for importance assessment. Risk Analysis, 22(3), 579-590.

Sandler, T., & Siqueira, K. (2009). Games and Terrorism. Simulation & Gaming, 40(2): 164-192.

Shan, X., & Zhuang, J. (2013). Cost of equity in homeland security resource allocation in the face of a strategic attacker. Risk Analysis, 33(6), 1083-1099.

Trucco, P., Cagno, E., Ruggeri, F., & Grande, O. (2008). A bayesian belief network modelling of organisational factors in risk analysis: a case study in maritime transportation. Reliability Engineering & System Safety, 93(6), 845-856.

Tullock G. (2001) Efficient Rent Seeking. Springer.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: cumulative representation of uncertainty. Journal of Risk & Uncertainty, 5(4), 297-323.

Vieider, F. M. (2009). The effect of accountability on loss aversion. Acta Psychol, 132(1), 96-101.

Wu, D., Liu, X., Yan, X., Peng, R., & Li, G. (2019) Equilibrium Analysis of Bitcoin Block Withholding Attack: A Generalized Model. Reliability Engineering and System Safety, 185, 318-328.

Wu, D., Xiao, H., & Peng, R. (2018). Object defense with preventive strike and false targets. Reliability Engineering & System Safety, 169, 76-80.

Zhai, Q., Ye, Z. S., Peng, R., & Wang, W. (2017). Defense and attack of performance-sharing common bus systems. European Journal of Operational Research, 256(3), 962-975.

Zhang, C., Ramirez-Marquez, J. E., & Wang, J. (2015). Critical infrastructure protection using secrecy – a discrete simultaneous game. European Journal of Operational Research, 242(1), 212-221.

Zhang, C., Ramirez-Marquez, J. E., & Li, Q. (2018). Locating and protecting facilities from intentional attacks using secrecy. Reliability Engineering & System Safety, 169, 51-62.

Zhuang, J., & Bier, V. M. (2007). Balancing terrorism and natural disasters: defensive strategy with endogenous attacker effort. Operations Research, 55(5), 976-991.