RIGIDITY OF THE $K(1)$-LOCAL STABLE HOMOTOPY CATEGORY

By

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Abstract

One goal of algebraic topology is to find algebraic invariants that classify topological spaces up to homotopy equivalence. The notion of homotopy is not only restricted to topology. It also appears in algebra, for example as a chain homotopy between two maps of chain complexes. The theory of model categories, introduced by D. Quillen [Qui06], provided us with a powerful common language to represent different notions of homotopy. Quillen's work transformed algebraic topology from the study of topological spaces into a wider setting useful in many areas of mathematics, such as homological algebra and algebraic geometry, where homotopy theoretic approaches led to interesting results.

In brief, a model structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: weak equivalences, fibrations, and cofibrations, satisfying certain axioms. We can pass to the homotopy category $\mathrm{Ho}(\mathcal{C})$ associated to a model category $\mathcal{C}$ by inverting the weak equivalences, i.e. by making them into isomorphisms. While the axioms allow us to define the homotopy relations between classes of morphisms in $\mathcal{C}$, the classes of fibrations and cofibrations provide us with a solution to the set-theoretic issues arising in general localisations of categories. Even though it is sometimes sufficient to work in the homotopy category, looking at the homotopy level alone does not provide us with enough higher order structure information. For example, homotopy (co)limits are not usually a homotopy invariant, and in order to define them we need the tools provided by the model category. This is where the question of *rigidity* may be asked: if we just had the structure of the homotopy category, how much of the underlying model structure can we recover?

This question of rigidity has been investigated during the last decade, and an extremely small list of examples have been studied, which leaves us with a lot of open questions regarding this fascinating subject.

Ou goal is to investigate one of the open questions which have not been answered before. In this thesis, we prove *rigidity* of the $K(1)$-local stable homotopy category $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ at $p = 2$. In other words, we show that recovering higher order structure information, which is meant to be lost on the homotopy level, is possible by just looking at the triangulated structure of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$. This new result does not only add one more example to the list of known examples of rigidity in stable homotopy theory, it is also the first studied case of rigidity in the world of Morava K-theory.

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Chapter 1

Introduction and overview

A Quillen adjunction between two model categories $\mathcal{C}$ and $\mathcal{D}$ is an adjunction that respects the model structure. However, when this Quillen adjunction induces an equivalence of categories on the homotopy level, we say that the model categories $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent. But on the other hand, if there is an equivalence between the homotopy categories of two model categories, can anything be said about the underlying model structures?

Starting with the stable homotopy category Ho(Sp), that is, the homotopy category of spectra, Schwede [Sch07a] showed that if Ho(Sp) is equivalent as a triangulated category to the homotopy category of a stable model category $\mathcal{C}$, then the model category of spectra is Quillen equivalent to $\mathcal{C}$. In other words, Ho(Sp) is *rigid*.

wonder if there is a similar result for Bousfield localisations of the stable homotopy category with respect to certain homology theories. If we look at the part of the stable homotopy category that is readable by a given homology theory, will that structure give us a rigid example? Particularly interesting localisations are the ones with respect to Morava K-theories $K(n)$ with coefficient ring

$$K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}], \ |v_n| = 2p^n - 2,$$

as well as with respect to Johnson-Wilson theories $E(n)$, where

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n, v_n^{-1}], \ |v_i| = 2p^i - 2.$$

Both theories at $n = 1$ are related to complex K-theory in different ways. More precisely, by the Adams splitting [Ada69], the spectrum $E(1)$ is a summand of complex K-theory localised at p

$$K_{(p)} \cong \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1),$$

while $K(1)$ is a summand of mod-p complex K-theory [Rav16, Proposition 1.5.2]. In our case of interest in this thesis, $p = 2$, we have that mod-2 K-theory coincides with $K(1)$ since there is only one such summand.

Starting with the Johnson-Wilson theories $E(n)$ for a fixed prime p, the localisation of spectra with respect to it is denoted $L_n\mathrm{Sp}$ (the prime p is omitted from the notation). This $E(n)$-localisation provides a powerful tool for studying the full stable homotopy category, and much of modern stable homotopy theory is related to studying the chromatic tower

$$\ldots \to L_n(X) \to L_{n-1}(X) \to \ldots \to L_1(X) \to L_0(X).$$

If we look at the case where $n = 1$ and $p = 2$, then it has been shown in [Roi07] that $\mathrm{Ho}(L_1\mathrm{Sp})$ is rigid. However, if we consider the case where $n = 1$ and $p \geq 5$, the situation is different since in [Fra96], Franke constructed an exotic algebraic model for the $E(1)$-local stable homotopy category $\mathrm{Ho}(L_1\mathrm{Sp})$ at $p \geq 5$, i.e. a model category that realises the same homotopy category but is not Quillen equivalent to $L_1\mathrm{Sp}$. For $p = 3$, there is an equivalence, but the question whether it is triangulated remains open, more details about this are discussed in [Pat17b].

Now, if we look at $L_n\mathrm{Sp}$ for other values of n and p, little is known about it. For $2p - 2 > n^2 + n$, it has been shown in [Fra96] that a potential exotic model exists for $E(n)$-local spectra, although what is known so far is that we have a triangulated equivalence only for $n = 1$ and $p \geq 5$. However, for $2p-2 \leq n^2+n$, it is still an open question whether we will have rigidity or an exotic model, except the case $n = 1$ and $p = 2$ which has been shown to be rigid by Roitzheim in [Roi07]. In particular, for $n = 2$ and $p = 2$ or $p = 3$, the question whether we have rigidity or an exotic model remains unanswered. For further examples of exotics models see [Sch02, DS09, Pat17a], and for other cases of rigidity see [Sch01, BR14a, Pat16, PR17].

Another interesting localisation of spectra that we wish to know more about is the localisation with respect to Morava K-theory $K(n)$. In that case, nothing is known about the rigidity $\mathrm{Ho}(L_{K(n)}\mathrm{Sp})$ or whether we have exotic models. For a fixed prime p, $K(n)$-local spectra can be viewed as the difference between $L_n\mathrm{Sp}$ and $L_{n-1}\mathrm{Sp}$. More precisely, we have

$$L_n = L_{K(0)\vee K(1)\vee \ldots \vee K(n)},$$

therefore

$$L_1\mathrm{Sp} = L_{K(0)\vee K(1)}\mathrm{Sp}.$$

For $n = 0$, we have that

$$L_{K(0)} = L_0 = L_{H\mathbb{Q}}$$

is rationalisation.

In this thesis, we investigate one of the open questions mentioned above, which is the rigidity of the $K(1)$-local stable homotopy category $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ at $p = 2$. The fact that the $E(1)$-local stable homotopy category is related to the $K(1)$-local case does not mean in any way that the rigidity of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ can be deduced from that of $\mathrm{Ho}(L_{E(1)}\mathrm{Sp})$. Working in the Morava K-theory setting required coming up with new techniques and strategies to establish this new result. The first challenge faced while working in this setting is the fact that, unlike $E(1)$-localisation, $K(1)$-localisation is not smashing. Losing the smashing property does not only make certain steps harder, it also results in losing the

$K(1)$-local sphere as a compact generator. Therefore, we had to adopt the $K(1)$-local mod-2 Moore spectrum as a compact generator. Adding to that, while $K(1)$-locality implies $E(1)$-locality, the converse is not true, i.e. $E(1)$-locality does not imply $K(1)$-locality. Therefore, a key theorem used in the proof of the rigidity of $\mathrm{Ho}(L_{E(1)}\mathrm{Sp})$, the "$v_1$-periodicity theorem", cannot be used in the $K(1)$-local case. To that end, a major step in proving the $K(1)$-local rigidity was finding a new criterion for when a 2-local spectrum is $K(1)$-local. Hence, another new contribution of the work in this thesis is finding a new characterisation related to v_1-self maps to detect $K(1)$-locality. In literature, this is stated for $E(1)$-local spectra, but we prove here that we can modify it to show that under certain assumptions, a spectrum is $K(1)$-local, which is a stronger statement.

The main result in this thesis tells us that all of the higher homotopy information is already encoded in the triangulated structure of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$, which brings us one step closer towards a global understanding of the behaviour of the rigidity of the stable homotopy category in the world of Morava K-theory. Our main result is thus:

$K(1)$-Local Rigidity Theorem. *Let $\mathcal{C}$ be a stable model category, $p = 2$, and let Φ be an equivalence of triangulated categories*

$$\Phi \quad : \quad \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \quad \to \quad \mathrm{Ho}(\mathcal{C}).$$

Then the underlying model categories $L_{K(1)}\mathrm{Sp}$ and $\mathcal{C}$ are Quillen equivalent.

We now outline the structure of this thesis leading to the proof of the $K(1)$-local rigidity theorem.

Chapter 2 is the opening chapter where we recall some definitions surrounding model categories, and how we pass to the homotopy category associated to a model category. We close this chapter by defining stable model categories, and by giving some examples.

In Chapter 3, we start by talking about spectra and the stable homotopy category. Afterwards, we dedicate a section to tools from triangulated categories.

We define what we mean by *rigidity* in stable homotopy theory in subsection 3.2.1, since at that point we will have all the necessary background. Next, we talk about Bousfield localisation of the stable homotopy category. We finish this chapter by defining an important tool needed later on, that is, homotopy limits and colimits of spectra.

Chapter 4 is the one in which we prove the main theorem. We start by setting up the necessary ingredients in order to construct the desired Quillen equivalence. In the first section, we find a new characterisation related to v_1-self maps to prove that a spectrum is $K(1)$-local. After that, we construct a Quillen functor

$$L_{K(1)}\mathrm{Sp} \to \mathcal{C}$$

by proving that the Quillen functor

$$\mathrm{Sp} \to \mathcal{C},$$

constructed by the Universal Property of Spectra [SS02, 5.1] can be extended to $L_{K(1)}\mathrm{Sp}$ since the right adjoint sends fibrant objects to $K(1)$-local objects. Finally, we prove that the constructed Quillen adjunction is a Quillen equivalence by reducing the argument to endomorphisms of the compact generator of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$.

In the proof of our main theorem, the knowledge of certain homotopy groups in the $K(1)$-local setting is necessary. We choose to write details of such computations and some Toda brackets equalities separately in Chapter 5.

Lastly, in Chapter 6, we discuss possible future research to be carried out in the subject of rigidity in stable homotopy theory. More precisely, we outline what would happen if we consider, 2-locally, the $K(2)$-local stable homotopy category and try to investigate rigidity at this chromatic level.

Chapter 2

Preliminaries

In this opening chapter, we cover all the preliminary material that will be used later. We start by giving an introduction to model categories, and how we pass to the homotopy level. We end this chapter by talking about stable model categories. We intend to give a general exposition of this introductory material. Hence, where appropriate, we will recommend references that provide proofs and more detailed discussions. We assume that the reader is familiar with basic notions regarding categories, a classical reference for category theory is [Mac13].

2.1 Model categories

In order to set up the basic machinery of homotopy theory, Daniel Quillen introduced the language of model categories [Qui06]. Shortly speaking, a model structure on a category $\mathcal{C}$ is a choice of three classes of morphisms, called weak equivalences, fibrations, and cofibrations. A list of axioms should be verified by these classes of morphisms, which provide us with a set-theoretically clean device to describe homotopy between morphisms. In this thesis, the main references for the theory of model categories are [Hov99] and [DS95]. In this section, we introduce basic notions from the theory of model categories. We begin by defining model categories and giving some examples, then we talk about Quillen functors.

Definition 2.1.1. A morphism $g : \mathrm{A} \to \mathrm{B}$ in a category $\mathcal{C}$ is (said to be) a *retract* of $f : X \to Y$ if there exists a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{j} & Y & \xrightarrow{s} & B \end{array}$$

in which the composites $r \circ i$ and $s \circ j$ are the appropriate identities.

Definition 2.1.2. Given a commutative square diagram in a category $\mathcal{C}$ of the following form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y, \end{array}$$

a *lift* or *lifting* in the diagram is a morphism $h : B \to X$ such that the resulting diagram with five arrows commutes, i.e. $h \circ i = f$ and $p \circ h = g$.

Definition 2.1.3. A morphism $i : A \to B$ is said to have the *left lifting property* (LLP) with respect to a morphism $p : X \to Y$ if a lift exists in any commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

Dually, if the lift exists we say that p has the *right lifting property* (RLP) with respect to i.

Definition 2.1.4. A *model category* is a category $\mathcal{C}$ with three distinguished classes of morphisms: weak equivalences ($\xrightarrow{\sim}$), fibrations ($\twoheadrightarrow$), and cofibrations ($\rightarrowtail$). A morphism which is both a fibration (resp. cofibration) and a weak equivalence is called a trivial fibration (resp. trivial cofibration). The three classes are required to be closed under composition and contain all identity morphisms. In addition, the following axioms should hold.

(MC1) Finite limits and colimits exist in $\mathcal{C}$, meaning that any functor

$$F : \mathcal{D} \to \mathcal{C}$$

where $\mathcal{D}$ is a finite category, has a limit and a colimit in $\mathcal{C}$.

(MC2) If f and g are composable morphisms in $\mathcal{C}$, and if any two of the three morphisms $f, g, g \circ f$ are weak equivalences, then so is the third.

(MC3) If f is a retract of g, and g is a fibration, cofibration, or a weak equivalence, then so is f.

(MC4) Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[g]{} & Y \end{array}$$

where i is a cofibration and p a fibration, a lift exists if one of i or p is a weak equivalence.

(MC5) Any morphism $f : X \to Y$ can be factored in two ways:

(i) $X \xrightarrow{j} Z \xrightarrow{q} Y$, where j is a cofibration and q is a trivial fibration,

(ii) $X \xrightarrow{j} Z \xrightarrow{q} Y$, where j is a trivial cofibration and q is a fibration.

Definition 2.1.5. An object $\emptyset$ of a category $\mathcal{C}$ is called an *initial object* if for any object X in $\mathcal{C}$, there is a unique morphism $\emptyset \to X$. Dually, the object $*$ is called a *terminal object* if there is exactly one morphism $X \to *$ for any object $X \in \mathcal{C}$. Clearly, we can see that initial and terminal objects are unique up to canonical isomorphism.

Remark 2.1.6. A model category $\mathcal{C}$ has both an initial object $\emptyset$ and a terminal object $*$. To be more precise, if we apply the axiom (MC1) to the functor $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is the empty category (i.e. the category with no objects), then $\operatorname{colim}(F)$ is an initial object of $\mathcal{C}$, and $\lim(F)$ is a terminal object of $\mathcal{C}$.

Definition 2.1.7. We call a model category (or any category with initial and terminal object) *pointed* if the map from the initial object to the terminal object is an isomorphism.

Definition 2.1.8. An object X in a model category $\mathcal{C}$ is called *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration, and *fibrant* if $X \to *$ is a fibration.

Actually, fibrant and cofibrant objects have nice properties and are our center of interest when passing to the homotopy level. Therefore, even when our object is not fibrant/cofibrant, we would like to "replace" it by another object which is. To that extend, we have the notion of fibrant and cofibrant replacement.

Take the unique morphism $\emptyset \to X$, where X is an object in our model category. If we apply MC5(i) to it, we will obtain the factorisation

$$\emptyset \rightarrowtail X^{\mathrm{c}} \overset{\sim}{\twoheadrightarrow} X,$$

where X^{c} is cofibrant. Similarly, by applying MC5(ii) to $X \to *$, we obtain a fibrant object X^{f} and a trivial cofibration $X \overset{\sim}{\rightarrowtail} X^{\mathrm{f}}$.

Definition 2.1.9. The object X^{f} is called a *fibrant replacement* of X, X^{c} a *cofibrant replacement* of X, and $X^{\mathrm{c,f}}$ a *cofibrant/fibrant replacement* of X. If X was already cofibrant we let $X^{\mathrm{c}} = X$, or if X is fibrant, then $X^{\mathrm{f}} = X$.

Remark 2.1.10. It is worth noticing that a fibrant replacement of a cofibrant object is again cofibrant. To be more precise, if X is a cofibrant object, then the composition of cofibrations

$$\emptyset \rightarrowtail X \overset{\sim}{\rightarrowtail} X^{\mathrm{f}}$$

is itself a cofibration, which tells us that X^{f} is cofibrant.

We now give some examples of a model structure on some categories. Note that one can have multiple model structures on the same category. But different choices of the three classes of morphisms might produce different homotopy categories as we will see later.

Let Top denote the category of topological spaces and continuous maps.

Definition 2.1.11. A morphism $f : X \to Y$ of topological spaces is called a *Serre fibration* if, for each CW-complex A, it has the *right lifting property* (RLP) with respect to the inclusion

$$A \times \{0\} \to A \times [0,1].$$

In other words, a lift exists in any commutative diagram of the form

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ A \times [0,1] & \longrightarrow & Y, \end{array}$$

where A is a CW-complex. Equivalently, we can replace A by the n-disk D^n, i.e. f is a Serre fibration if and only if it has the RLP with respect to the inclusions $D^n \to D^n \times [0,1]$.

The following proposition from [DS95] makes the category Top into a model category.

Proposition 2.1.12. *[DS95, Proposition 8.3] We can define a model structure on the category of topological spaces* Top*, by defining a continuous map $f : X \to Y$ to be :*

- *a weak equivalence if it is a weak homotopy equivalence, i.e. for each base-point $x \in X$ the induced map $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ is a bijection of pointed sets for $i = 0$ and an isomorphism of groups for $i \geq 1$,*
- *a fibration if it is a Serre fibration,*
- *a cofibration if it has the LLP with respect to trivial fibrations, i.e. with respect to each morphism which is both a Serre fibration and a weak homotopy equivalence.*

Remark 2.1.13. In this model structure, every object is fibrant because the continuous map $X \to *$ to the point space is a Serre fibration, and the cofibrant objects are exactly the spaces which are retracts of generalised CW-complexes. Moreover, any space X has a CW-replacement X^{c}, in other words, cofibrant replacement in this model structure is given by CW-approximation.

The machinery of model categories is applicable beyond topology. Actually, many purely algebraic categories carry model category structures, for example the category Ch_R of non-negatively graded chain complexes over R, where R is an associative ring with unit. Recall that an object M of the category Ch_R is a collection of R-modules $\{M_n\}_{n\in\mathbb{N}}$, together with differentials

$$d_n : M_n \to M_{n-1}, \text{ such that } d_{n-1} \circ d_n = 0 \text{ for all } n > 1.$$

A morphism $f : M \to N$ of Ch_R is a collection of R-module morphisms

$$f_n : M_n \to N_n, \text{ such that } d_n \circ f_n = f_{n-1} \circ d_n.$$

Note that the category Ch_R has all small limits and colimits. The initial and terminal object is the chain complex 0, which is 0 in each degree.

Theorem 2.1.14. *[DS95, Theorem 7.2] Define a morphism $f : M \to N$ in the category Ch_R to be*

- *a weak equivalence if it induces isomorphisms on homology groups,*
- *a cofibration if for each $n \in \mathbb{N}$, the morphism $f_n : M_n \to N_n$ is a monomorphism with projective cokernel,*
- *a fibration if $M_n \to N_n$ is an epimorphism for each $n \in \mathbb{N}$.*

The above choice of classes of morphisms makes the category Ch_R into a model category. This is called the projective model structure on Ch_R.

Remark 2.1.15. With respect to the above model structure on the category Ch_R, the cofibrant objects are the chain complexes M such that each M_n is a projective R-module. Every chain complex is fibrant in this model structure. If we consider an R-module M as a chain complex concentrated in degree zero, a cofibrant replacement for M is simply a projective resolution.

The next example of a model category that will be relevant later on is the category of simplicial sets denoted sSets. Before talking about the model structure, we will briefly define the category of simplicial sets. The main references for the category of simplicial sets and the standard model structure on it are [GJ09] and [Hov99, Chapter 3].

Definition 2.1.16. The category of *finite ordinal numbers* $\mathbf{\Delta}$ has objects the ordered sets $[n] = \{0, \ldots, n\}$ for $n \geq 0$, and morphisms the order preserving maps. Moreover, the category $\mathbf{\Delta}$ is generated by the monomorphisms

$$d^i : [n-1] \to [n], \quad 0 \leq i \leq n$$
$$\{0, 1, \ldots, n-1\} \mapsto \{0, 1, \ldots, i-1, i+1, \ldots, n\}$$

and the epimorphisms

$$s^j : [n+1] \to [n], \quad 0 \leq j \leq n$$
$$\{0, 1, \ldots, n+1\} \mapsto \{0, 1, \ldots, j, j, \ldots, n\}.$$

In other words, all the morphisms in $\mathbf{\Delta}$ are compositions of the cofaces d^i and codegeneracies s^j. Plus, the *cosimplicial identities* are verified:

$$\begin{cases} d^j \circ d^i = d^i \circ d^{j-1} & \text{if} \quad i < j \\ s^j \circ d^i = d^i \circ s^{j-1} & \text{if} \quad i < j \\ s^j \circ d^j = \mathrm{Id} = s^j \circ d^{j+1} & \\ s^j \circ d^i = d^{i-1} \circ s^j & \text{if} \quad i > j+1 \\ s^j \circ s^i = s^i \circ s^{j+1} & \text{if} \quad i \leq j. \end{cases}$$

Definition 2.1.17. A *simplicial set* is a functor

$$X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Sets}$$
$$[n] \mapsto X_n,$$

where Sets is the category of sets, and X_n is called the set of n*-simplices* of X.

In other words, we can think of a simplicial set X as a collection of sets X_n

together with maps

$$d_i = X(d^i) : X_n \to X_{n-1},$$

$$s_j = X(s^j) : X_{n-1} \to X_n.$$

These are respectively, the face and degeneracy maps, and they satisfy the following *simplicial identities*:

$$\begin{cases} d_i \circ d_j = d_{j-1} \circ d_i & \text{if} \quad i < j \\ d_i \circ s_j = s_{j-1} \circ d_i & \text{if} \quad i < j \\ d_j \circ s_j = \mathrm{Id} = d_{j+1} \circ s_j & \\ d_i \circ s_j = s_{j+1} \circ d_i & \text{if} \quad i > j+1 \\ s_i \circ s_j = s_{j+1} \circ s_i & \text{if} \quad i \leq j. \end{cases}$$

Definition 2.1.18. A morphism $f : X \to Y$ between simplicial sets, called a *simplicial map*, is a collection of maps $f_n : X_n \to Y_n$ commuting with the face and degeneracy maps: $f_{n-1} \circ d_i = d_i \circ f_n$ and $f_{n+1} \circ s_i = s_i \circ f_n$.

The simplicial sets and simplicial maps form the category of simplicial sets denoted sSets.

A relatively simple example of a simplicial set is the standard n-simplex Δ^n of sSets, which is a combinatorial analogue to the standard topological n-simplex σ_n, i.e. the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$

$$\sigma_n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} t_i = 1, t_i \geq 0\}.$$

Example 2.1.19. (The standard n-simplices)
The functor from simplicial sets to sets sending $X \in s\mathrm{Sets}$ to its set of n-simplices X_n is representable. In fact, the Yoneda lemma gives the isomorphism

$$\mathrm{Hom}_{s\mathrm{Sets}}(\Delta^n, X) \cong X_n,$$

where Δ^n *is the standard* n*-simplex of* sSets *defined by*

$$\Delta^n = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]) : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Sets}.$$

Note that simplicial maps

$$\Delta^n \to \Delta^m,$$

correspond bijectively to morphisms

$$[n] \to [m], \text{ in } \mathbf{\Delta}.$$

Moreover, we have the boundary of the standard simplex

$$\partial\Delta^n := \bigcup_{0 \leq i \leq n} d^i \circ \Delta^{n-1} \subseteq \Delta^n,$$

and the horns Λ^n_k*, which are the union of all faces except the* k^{th} *one*

$$\Lambda^n_k := \bigcup_{i \neq k} d^i \circ \Delta^{n-1} \subseteq \Delta^n.$$

Another example of a simplicial set is the one constructed from a topological space X, called the singular set of X, and denoted $S(X)$.

Example 2.1.20. (The singular set)
We have the following functor

$$S(-) : \mathrm{Top} \to s\mathrm{Sets}$$
$$X \mapsto S(X)$$

defined by setting the singular n*-simplex*

$$S_n(X) = \mathrm{Hom}_{\mathrm{Top}}(\sigma_n, X),$$

where σ_n *is the standard topological* n*-simplex mentioned earlier.*

Definition 2.1.21. We can functorially associate a topological space to a simplicial set by the geometric realisation functor

$$\begin{aligned} |-| : s\mathrm{Sets} &\to \mathrm{Top} \\ X &\mapsto |X|. \end{aligned}$$

Here, $|X|$ is the topological space constructed in the following way. Consider X with the discrete topology and take

$$|X| = \bigcup_{n\geq 0} X_n \times |\Delta^n| / \sim,$$

where $|\Delta^n| = \sigma_n$, and the equivalence relation $\sim$ is generated by the relations

$$\begin{aligned} (d_i(x_n), v_{n-1}) &\sim (x_n, d^i(v_{n-1})), \quad x_n \in X_n \quad \text{and} \quad v_{n-1} \in |\Delta^{n-1}| \\ (s_i(x_n), v_{n+1}) &\sim (x_n, s^i(v_{n+1})), \quad x_n \in X_n \quad \text{and} \quad v_{n+1} \in |\Delta^{n+1}|. \end{aligned}$$

The maps d_i and s_i are, respectively, the faces and degeneracies of the simplicial set X. As for the maps d^i and s^i, notice that an order preserving map $[m] \to [n]$ induces a continuous map

$$|\Delta^m| \to |\Delta^n|.$$

Hence, by the maps d^i and s^i here, we mean the induced maps

$$\begin{aligned} d^i &: |\Delta^{n-1}| \to |\Delta^n| \\ s^j &: |\Delta^{n+1}| \to |\Delta^n|. \end{aligned}$$

Notation. Throughout this thesis, the top arrow in an adjoint functor pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ denotes the left adjoint, and the bottom arrow denotes the right adjoint.

The realisation functor and the singular set functor are part of an adjunction in the following sense.

Proposition 2.1.22. *[GJ09, Proposition 2.2] We have an adjunction*

$$|-|: s\mathrm{Sets} \rightleftarrows \mathrm{Top} : S(-).$$

In other words, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Top}}(|X|, Y) \cong \mathrm{Hom}_{s\mathrm{Sets}}(X, S(Y)).$$

The category of simplicial sets is another example of a model category, and this model structure will become useful when defining the model structure on spectra later.

Theorem 2.1.23. *[GJ09, Theorem 11.3] The category of simplicial sets is a model category, by defining a morphism of simplicial sets $f : X \to Y$ to be:*

- *a weak equivalence if its geometric realisation $|f| : |X| \to |Y|$ is a weak homotopy equivalence,*
- *a cofibration if for each n, the morphism $f_n : X_n \to Y_n$ is a monomorphism,*
- *a fibration if it is a Kan fibration, i.e. f has the left lifting property with respect to the inclusions of the horns $\Lambda^n_k \hookrightarrow \Delta^n$, $n \geq 1$, and $0 \leq k \leq n$.*

Remark 2.1.24. The initial object $\emptyset$ of the category sSets is the simplicial set which consists of the empty set in each degree. Since the morphisms from the empty set to all sets are monomorphisms, all the objects in sSets are cofibrant with respect to the above model structure.

The terminal object is the one consisting of the singleton set in each degree. The fibrant objects are what we call the *Kan complexes*. In other words, they are the simplicial sets Y such that the unique map $Y \to *$ is a Kan fibration, i.e. there exists a diagonal morphism making the diagram commute

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & *. \end{array}$$

Definition 2.1.25. Using the same concept of constructing a pointed topological space by choosing a basepoint, we also can apply this construction to simplicial sets. In that case, we obtain the category of *pointed simplicial sets* denoted $s\mathrm{Sets}_*$. An object of $s\mathrm{Sets}_*$ is a simplicial set X together with a distinguished 0-simplex $* \in X_0 = X([0])$. The morphisms in $s\mathrm{Sets}_*$ are the simplicial maps preserving the basepoints. Moreover, the category $s\mathrm{Sets}_*$ with the same weak equivalences, cofibrations and fibrations in $s\mathrm{Sets}$ form a model category.

In order to compare model categories, one studies functors between them that respect the model structure, so-called Quillen functors.

Definition 2.1.26. Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories.

(a) We call a functor $F : \mathcal{C} \to \mathcal{D}$ a *left Quillen functor* if F is a left adjoint and preserves cofibrations and trivial cofibrations.

(b) We call a functor $U : \mathcal{D} \to \mathcal{C}$ a *right Quillen functor* if U is a right adjoint and preserves fibrations and trivial fibrations.

(c) Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjunction. We call $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a *Quillen adjunction* if F is a left Quillen functor, or equivalently, if U is a right Quillen functor [Hov99, Lemma 1.3.4].

2.2 The homotopy category of a model category

In this section, $\mathcal{C}$ is some fixed model category, A and X are objects of $\mathcal{C}$. The homotopy category of $\mathcal{C}$ denoted $\mathrm{Ho}(\mathcal{C})$ will be a category with the same objects as $\mathcal{C}$, but the morphisms sets consist of equivalence classes of morphisms under a certain homotopy relation.

We will define the notion of left homotopy in terms of cylinder objects, and then a dual notion of right homotopy, defined in terms of path objects. It turns out that the two notions coincide if the source A is cofibrant and the target X is fibrant.

Definition 2.2.1. A *cylinder object* for A is an object $A \wedge I$ of $\mathcal{C}$ together with a diagram

$$A \amalg A \xrightarrow{i} A \wedge I \xrightarrow{\sim} A$$

which factors the folding map

$$\mathrm{Id}_A + \mathrm{Id}_A : A \amalg A \to A.$$

Note that the above folding map exists by the universal property of the coproduct. If the morphism

$$A \amalg A \xrightarrow{i} A \wedge I$$

is a cofibration, then $A \wedge I$ is called a *good cylinder object.* If in addition, the morphism

$$A \wedge I \to A$$

is a (necessarily trivial) fibration, then $A \wedge I$ is called a *very good cylinder object.* If $A \wedge I$ is a cylinder object for A, then we have two structure morphisms

$$i_1,\, i_2 : A \to A \wedge I$$

defined as the compositions $i_1 = i \circ i_{n_1}$ and $i_2 = i \circ i_{n_2}$, where i_{n_1} and i_{n_2} are the two canonical morphisms

$$i_{n_1},\ i_{n_2} : A \to A \amalg A.$$

Remark 2.2.2. By MC5(i), every object in a model category $\mathcal{C}$ has a very good cylinder object. Plus, we might have many cylinder objects $A \wedge I$, $A \wedge I', \ldots$ associated for an object A, since a cylinder object $A \wedge I$ is any object of $\mathcal{C}$ with the above formal property. However, all these cylinder objects are weakly equivalent.

Definition 2.2.3. Let $f, g : A \to X$ be two morphisms in a model category $\mathcal{C}$. By the universal property of the coproduct, there exists a morphism

$$f + g : A \amalg A \to X.$$

We say that f and g are *left homotopic*, denoted $f \sim^l g$, if the coproduct $f + g$ can be extended to $H : A \wedge I \to X$ as in the diagram

$$\begin{array}{ccc} A \amalg A & \xrightarrow{f+g} & X \\ {\scriptstyle i_0+i_1}\downarrow & \nearrow_{H} & \\ A \wedge I & & \end{array} .$$

Such a map H is said to be a *left homotopy* from f to g.

Example 2.2.4. *For a topological space $A \in$ Top, the product $A \times [0, 1]$ is a cylinder object for A with respect to the model structure in Theorem 2.1.12. The notion of left homotopy with respect to this cylinder object coincides with the usual notion of homotopy.*

We are interested in the case when the left homotopy relation forms an equivalence relation on the set $\mathrm{Hom}_{\mathcal{C}}(A, X)$.

Lemma 2.2.5. *[DS95, Lemma 4.7] If A is cofibrant, then $\sim^l$ is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(A, X)$.*

We can dualise the definition of cylinder objects and left homotopies to get similar notions of path objects and right homotopies.

Definition 2.2.6. A *path object* for X is an object X^I of $\mathcal{C}$ together with a diagram

$$X \xrightarrow{\sim} X^I \xrightarrow{p} X \times X$$

which factors the diagonal map

$$(\mathrm{Id}_X, \mathrm{Id}_X) : X \to X \times X.$$

A path object X^I is called a *good path object*, if $X^I \to X \times X$ is a fibration. If in addition, the morphism $X \to X^I$ is a (necessarily trivial) cofibration, then X^I is called a *very good path object*.

Example 2.2.7. *If* $\mathcal{C} = \mathrm{Top}$*, then one choice of path object for a space* X *is the mapping space* $\mathrm{Map}([0,1]\,, X)$*.*

Remark 2.2.8. In that sense, the notation X^I might suggest a space of paths in X. However that notion is not correct if we are working in a model category $\mathcal{C}$ that is not the category of topological spaces. In that case, a path object associated to an object A is any object of $\mathcal{C}$ with the above formal properties. Therefore, we might have several path objects associated to an object A, but they are all weakly equivalent.

Similarly, we define the right homotopy relation using path objects.

Definition 2.2.9. Two maps $f, g : A \to X$ are said to be *right homotopic* (written $f \sim^r g$) if there exists a path object X^I for X such that the product map $(f, g) : A \to X \times X$ lifts to a map $H : A \to X^I$ as in the diagram

$$\begin{array}{ccc} & & X^I \\ & \overset{H}{\nearrow} & \downarrow p \\ A & \xrightarrow[(f,g)]{} & X \times X \end{array} \quad .$$

Such a map H is said to be a *right homotopy* from f to g.

Lemma 2.2.10. *[DS95, Lemma 4.16] If* X *is fibrant, then* $\sim^r$ *is an equivalence relation on* $\mathrm{Hom}_{\mathcal{C}}(A, X)$*.*

Lemma 2.2.11. *[DS95, Lemma 4.21] If* A *is cofibrant and* X *is fibrant, then the left and right homotopy relations on* $\mathrm{Hom}_{\mathcal{C}}(A, X)$ *agree, and the identical right and left homotopy equivalence relations are denoted by the symbol "*$\sim$*". Two maps related by this relation are said to be* homotopic*, and the set of equivalence classes with respect to this relation is denoted* $\mathrm{Hom}_{\mathcal{C}}(A, X)/\sim$*.*

Actually, we have the following generalisation of *Whitehead's theorem*, which in the topological category says that a weak homotopy equivalence between CW-complexes is a homotopy equivalence.

Theorem 2.2.12. *[DS95, 4.24] Suppose A and X are objects in $\mathcal{C}$ which are both fibrant and cofibrant. Then $f : A \to X$ is a weak equivalence if and only if f is a homotopy equivalence, i.e. there exists a morphism $g : X \to A$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the respective identities.*

Definition 2.2.13. The *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$, and with

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X^{\mathrm{c,f}}, Y^{\mathrm{c,f}})/\sim .$$

Notation. For a given map $f : X \to Y$ in $\mathcal{C}$, denote by $[f]$ its homotopy class in the set of equivalence classes of $\mathrm{Hom}_{\mathcal{C}}(X^{\mathrm{c,f}}, Y^{\mathrm{c,f}})$ under the equivalence relation " $\sim$ " defined above.

Example 2.2.14. *Take the category of Ch_R of nonnegatively graded chain complexes over R, with the projective model structure defined in Theorem 2.1.14. In that case, the homotopy category $\mathrm{Ho}(\mathrm{Ch}_R)$ is equivalent to the category with objects the cofibrant chain complexes, i.e. projective chain complexes, and morphisms the ordinary chain homotopy classes of maps. Actually, $\mathrm{Ho}(\mathrm{Ch}_R)$ is equivalent to the derived category $\mathbf{D}(R)$, i.e. the localisation of Ch_R with respect to the quasi-isomorphisms.*

Example 2.2.15. *Another example is $\mathrm{Ho}(\mathrm{Top})$ the homotopy category of topological spaces with respect to the model structure in Proposition 2.1.12. We can see that $\mathrm{Ho}(\mathrm{Top})$ is equivalent to the usual homotopy category of CW-complexes, as shown in [DS95, Proposition 8.4].*

Remark 2.2.16. Since we did not assume the functoriality of the splittings in MC5(i) and MC5(ii), the assignments $X \mapsto X^{\mathrm{f}}$ and $X \mapsto X^{\mathrm{c}}$ need not to be functorial on the model level. However they are indeed functors on the homotopy level, and therefore the choices of X^{c} and X^{f} are unique up to homotopy equivalence. Note that our results in this thesis do not depend on the choice of cofibrant and fibrant replacements since any two choices will be weakly equivalent.

Definition 2.2.17. There is a functor between $\mathcal{C}$ and its homotopy category $\mathrm{Ho}(\mathcal{C})$ denoted

$$\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}(\mathcal{C}),$$

which is the identity map on objects, and sends a map $f : X \to Y$ in $\mathcal{C}$ to the map $[f] \in \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y)$.

The following proposition [DS95] is useful to detect the isomorphims in the homotopy category.

Proposition 2.2.18. *If f is a morphism of $\mathcal{C}$, then $[f]$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ if and only if f is a weak equivalence.*

If an adjoint pair of functors is a Quillen pair, it induces an adjoint pair of functors on the homotopy level

$$\mathrm{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathrm{R}G,$$

called the total derived adjunction. Since $\mathrm{L}F$ and $\mathrm{R}G$ are, respectively, the left and right derived functors of certain composites, we will first introduce the notion of left and right derived functors, denoted $\mathbb{L}F$ and $\mathbb{R}G$.

Definition 2.2.19. Suppose $\mathcal{C}$ is a model category, and $F : \mathcal{C} \to \mathcal{D}$ a functor. Consider pairs (G, s) where $G : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$ is a functor, and $s : G \circ \gamma_{\mathcal{C}} \to F$ is a natural transformation. Now, a *left derived functor* is a pair $(\mathbb{L}F, t)$ of this type that is universal from the left in the following sense. We have a functor

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathcal{D},$$

and a natural transformation

$$t : \mathbb{L}F \circ \gamma_{\mathcal{C}} \to F,$$

such that for any other pair (G, s) of that form, there exists a unique natural transformation $s' : G \to \mathbb{L}F$ making the below diagram commute

$$\begin{array}{ccc} & & F \\ & \overset{s}{\nearrow} & \uparrow t \\ G \circ \gamma & \xrightarrow[s' \circ \gamma]{} & (\mathbb{L}F) \circ \gamma\ . \end{array}$$

Similarly, a *right derived functor* for F is a pair $(\mathbb{R}F, t)$ that is universal from the right.

Remark 2.2.20. If a left (respectively right) derived functor for F exists, then it is unique up to canonical natural isomorphism.

Note that left and right derived functors might not exist, nevertheless the following proposition states when we can detect their existence.

Proposition 2.2.21. *[DS95, Proposition 9.3] Suppose $\mathcal{C}$ is a model category, and $F : \mathcal{C} \to \mathcal{D}$ is a functor taking weak equivalences between cofibrant (respectively fibrant) objects in $\mathcal{C}$ to isomorphims in $\mathcal{D}$. Then the left (respectively right) derived functor $(\mathbb{L}F, t)$ (respectively $(\mathbb{R}F, t)$) exists, and for each cofibrant (respectively fibrant) object X of $\mathcal{C}$ the morphism*

$$t_X : \mathbb{L}F(X) \to F(X)$$

$$(\text{respectively } t_X : F(X) \to \mathbb{R}F(X))$$

is an isomorphism.

Definition 2.2.22. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor between them. A *total left derived functor* $\mathrm{L}F$ for F is a left derived functor for the composite $\gamma_{\mathcal{D}} \circ F : \mathcal{C} \to \mathrm{Ho}(\mathcal{D})$. Similarly, a *total right derived functor* for F

is

$$\mathrm{R}F := \mathbb{R}(\gamma_{\mathcal{D}} \circ F) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D}).$$

Theorem 2.2.23. *[DS95, Theorem 9.7.(i)] Suppose $\mathcal{C}$ and $\mathcal{D}$ are model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen adjunction. Then the total derived functors* LF *and* RU *exist and are part of an adjunction*

$$\mathrm{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathrm{R}U$$

which we call the derived adjunction.

Note that, sometimes $\mathrm{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathrm{R}U$ is an adjoint equivalence of categories even when $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is not.

Definition 2.2.24. A Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is called a *Quillen equivalence* if and only if, for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, we have that a map

$$f : FX \to Y$$

is a weak equivalence in $\mathcal{D}$ if and only if its adjoint

$$f^\sharp : X \to UY$$

is a weak equivalence in $\mathcal{C}$.

Proposition 2.2.25. *[Hov99, Proposition 1.3.13] If $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen adjunction, then it is a Quillen equivalence if and only if the adjunction*

$$\mathrm{L}F : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathrm{R}U$$

is an adjoint equivalence of categories.

The following result proved in [Qui06], shows that the category of simplicial sets is a good model for topological spaces in combinatorial language. In other words, we have "well behaved" topological spaces that are Quillen equivalent to the original ones.

Theorem 2.2.26. *The adjoint pair of functors defined in Proposition 2.1.22*

$$|-| : s\mathrm{Sets} \rightleftarrows \mathrm{Top} : S(-)$$

is a Quillen equivalence.

Theorem 2.2.27. *[Hov99, Corollary 1.3.16] Suppose $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen adjunction. The following are equivalent:*

(a) *$F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen equivalence.*

(b) *F reflects weak equivalences between cofibrant objects, meaning that if f is a map between cofibrant objects such that its image $F(f)$ is a weak equivalence in $\mathcal{D}$, then f itself is a weak equivalence. Plus, for every fibrant Y, the map $F(UY)^{\mathrm{c}} \to Y$ is a weak equivalence.*

(c) *U reflects weak equivalences between fibrant objects and, for every cofibrant X, the map $X \to U(FX)^{\mathrm{f}}$ is a weak equivalence.*

As we have just seen, a Quillen equivalence induces an equivalence on the homotopy level, but its importance lies in the fact that it preserves higher order information which is not necessarily preserved by an equivalence of categories on the homotopy level. For such an example see, [Sch02].

Plus, note that not all equivalences between homotopy categories arise this way. There are examples of model categories $\mathcal{C}$ and $\mathcal{D}$ where $\mathrm{Ho}(\mathcal{C})$ is equivalent to $\mathrm{Ho}(\mathcal{D})$, but $\mathcal{C}$ and $\mathcal{D}$ are not Quillen equivalent. Examples of such cases are [DS09] and [Sch01, 2.1, 2.2].

2.3 Stable model categories

Stable model categories are much more interesting to study since they carry more structure in their homotopy categories. We will introduce them in this section and give some examples.

Definition 2.3.1. Let $\mathcal{C}$ be a pointed model category and, $X \in \mathcal{C}$. First construct X^{c}, a cofibrant replacement of X. Applying MC5(ii) to the folding map, we have the factorisation

$$X^{\mathrm{c}} \amalg X^{\mathrm{c}} \rightarrowtail X^{\mathrm{c}} \wedge I \xrightarrow{\sim} X^{\mathrm{c}},$$

where $X^{\mathrm{c}} \wedge I$ is a very good cylinder object of X^{c}. The *suspension* of X denoted ΣX is defined as the pushout diagram

$$\begin{array}{ccc} X^{\mathrm{c}} \amalg X^{\mathrm{c}} & \longrightarrow & X^{\mathrm{c}} \wedge I \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X. \end{array}$$

Dually, choosing a factorisation of the diagonal map by MC5(i), we have

$$X^{\mathrm{f}} \xrightarrow{\sim} \left(X^{\mathrm{f}}\right)^{I} \twoheadrightarrow (X^{\mathrm{f}}) \times (X^{\mathrm{f}}),$$

where $\left(X^{\mathrm{f}}\right)^{I}$ is a very good path object for a fibrant replacement of X. The *loop object* of X denoted ΩX is defined by the pullback diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & \left(X^{\mathrm{f}}\right)^{I} \\ \downarrow & & \downarrow \\ * & \longrightarrow & (X^{\mathrm{f}}) \times (X^{\mathrm{f}}). \end{array}$$

These constructions are not functorial or adjoint on $\mathcal{C}$, but they become functorial and adjoint in the homotopy category.

Theorem 2.3.2. *[Qui06, Section I.2] Let $\mathcal{C}$ be a pointed model category. We have well-defined functors on* $\mathrm{Ho}(\mathcal{C})$*, the suspension functor*

$$\Sigma : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}),$$

and the loop functor

$$\Omega : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{C}).$$

Furthermore, they form an adjunction

$$\Sigma : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \Omega.$$

Definition 2.3.3. A pointed model category $\mathcal{C}$ is called *stable* if Σ and Ω are inverse equivalences of homotopy categories.

Example 2.3.4. *We have the following non-stable model categories:*

- *The category of pointed topological spaces* Top_**, equipped with the model structure defined in Proposition 2.1.12, is not stable. In fact,* Σ *and* Ω *in this case are not inverse equivalences of homotopy categories. To see this, it is sufficient to consider a counterexample. We have that*

  $$\Sigma : \pi_2(S^1) \to \pi_3(S^2)$$

  is not an isomorphism since $\pi_2(S^1) \cong 0$ *and* $\pi_3(S^2) \cong \mathbb{Z}$*. Therefore* Σ *is not an equivalence of homotopy categories in that case.*

- *The category of pointed simplicial sets* $s\mathrm{Sets}_*$*, with the model structure defined in Theorem 2.1.23, is not stable. We can deduce that from the fact that for any* $X \in s\mathrm{Sets}_*$*, we have the following isomorphism between the suspension and the geometric realisation* $|-|$

  $$|\Sigma X| \cong \Sigma |X|.$$

  Since the category Top_* *is not stable, the category* $s\mathrm{Sets}_*$ *is not stable either.*

The following is an example of a stable model category

Example 2.3.5. *The category of non-negatively graded chain complexes* Ch_R*, is a stable model category with respect to the projective model structure in Theorem*

2.1.14. Take (M, ∂) a cofibrant chain complex in Ch_R, that is a degreewise projective chain complex. First, we know that in that case the coproduct $M \amalg M$ and the product $M \times M$ are the direct sum $M \oplus M$. A cylinder object $M \wedge I$ for M is the chain complex defined by

$$\begin{aligned} (M \wedge I)_n &= M_n \oplus M_{n-1} \oplus M_n, \text{with differentials} \\ \delta_n(a, b, c) &= (\partial_n a + b, -\partial_{n-1} b, \partial_n c - b). \end{aligned}$$

The chain maps $i_1, i_2 : M \to M \wedge I$ and $p : M \wedge I \to M$ are given by

$$\begin{aligned} i_1(m) &= (m, 0, 0), \\ i_2(m) &= (0, 0, m), \\ p(a, b, c) &= a + c. \end{aligned}$$

We can check that $p \circ i_1 = p \circ i_2 = \mathrm{Id}_M$, and that p is a quasi-isomorphism. Moreover, the map $i : M \oplus M \to M \wedge I$, defined by $i(a, b) = (a, 0, b)$, is a cofibration since M is projective. Now, the suspension of M, denoted by ΣM, is defined by the pushout diagram

$$\begin{array}{ccc} M_n \oplus M_n & \longrightarrow & M_n \oplus M_{n-1} \oplus M_n \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & (\Sigma M)_n \cong M_{n-1}. \end{array}$$

The differential of the suspension chain complex is $d_n = -\partial_{n-1}$. Analogously, we construct a path object for M denoted (M^I, σ), and that is the chain complex defined by

$$\begin{aligned} (X^I)_n &= X_n \oplus X_n \oplus X_{n+1}, \text{ with differentials} \\ \sigma_n(a, b, c) &= (\partial_n a, \partial_n b, -\partial_{n+1} c + a - b). \end{aligned}$$

We take the pullback of the resulting diagram, and we will have that the resulting *loop object is the chain complex* $(\Omega M, d')$ *defined as follows*

$$(\Omega M)_n = M_{n+1},$$
$$d'_n = -\partial_{n+1}.$$

We can conclude that the chain maps

$$M \to (\Omega \circ \Sigma)M, \text{ and}$$
$$(\Sigma \circ \Omega)M \to M$$

are quasi-isomorphisms since Σ *and* Ω *are degree shifts.*

Another important example of a stable model category is the category of spectra denoted Sp, to which we devote its own section in the next chapter.

Chapter 3

Stable homotopy theory

3.1 Spectra

Stable homotopy theory is concerned with the study of phenomena that remain unchanged after sufficiently many applications of the suspension functor. The motivating idea behind studying stable homotopy theory is the Freudenthal Suspension Theorem which implies that the sequence of homotopy classes of maps

$$[X, Y] \to [\Sigma X, \Sigma Y] \to \ldots \to [\Sigma^n X, \Sigma^n Y] \to \ldots$$

is eventually constant for finite-dimensional pointed CW-complexes X and Y. The goal is to form a "stable" category where the suspension operation is invertible, i.e. we should be able to desuspend any object in this category. To get this stability property, we have to pass from working with topological spaces to working with spectra

$$\text{Spaces} \xmapsto{\text{stabilisation}} \text{Spectra}.$$

Spectra are objects related to spaces and were developed around the 1960s by Lima [Lim59] and later generalised by Whitehead [Whi62]. The suspension is not invertible in the category of spectra, but becomes invertible when we pass to the associated homotopy category and obtain what is called the *stable homotopy category*. This stable homotopy category has many nice properties not found in the (unstable) homotopy category of spaces. Hence, studying this stable homotopy category has been an active field in algebraic topology, beginning with Boardman in his Ph.D thesis, and Adams [Ada95], and continuing to this day.

In this section, we will define the category of spectra denoted Sp, then briefly talk about the smash product $\wedge$, which plays a role in stable homotopy theory similar to that of the tensor product in algebra.

Definition 3.1.1. A *spectrum* X is a sequence of pointed simplicial sets $(X_0, X_1, \cdots)$ together with structure maps

$$\sigma_n^X : \Sigma X_n \to X_{n+1}, \text{ or equivalently}$$

$$\bar{\sigma}_n^X : X_n \to \Omega X_{n+1}.$$

A morphism $f : X \to Y$ of spectra is a collection of morphisms of pointed sets

$$f_n : X_n \to Y_n$$

that commute with the structure maps, that is,

$$f_{n+1} \circ \sigma_n^X = \sigma_n^Y \circ \Sigma f_n, \quad \text{for all } n \geq 0.$$

A spectrum X is called a *suspension spectrum* (respectively an Ω-*spectrum*) if σ_n^X (respectively $\bar{\sigma}_n^X$) is a weak homotopy equivalence for all n.

Remark 3.1.2. We can define a spectrum to be a sequence of CW-complexes, or pointed topological spaces. However, all these definitions will determine the same homotopy category, known as the stable homotopy category.

Definition 3.1.3. The *homotopy groups* of a spectrum X are the stable homotopy groups

$$\pi_n(X) = \operatorname*{colim}_i \pi_{n+i}(X_i),$$

where the homomorphisms of the direct system are

$$\pi_{n+i}(X_i) \xrightarrow{\Sigma} \pi_{n+i+1}(\Sigma X_i) \xrightarrow{(\sigma_i)_*} \pi_{n+i+1}(X_{i+1}).$$

Note that if X is an Ω-spectrum, then the homomorphism

$$\pi_{n+i}(X_i) \to \pi_{n+i+1}(X_{i+1})$$

is an isomorphism for $n+i \geq 1$, and we have $\pi_n(X) = \pi_{n+i}(X_i)$ for $n+i \geq 1$.

Example 3.1.4.

(a) Suspension spectrum

If X is a pointed space, we define its suspension spectrum denoted $\Sigma^\infty X$, by taking the n^{th}-term to be $(\Sigma^\infty X)_n = \Sigma^n X$ and the structure maps $\sigma_n = \mathrm{Id}$. The homotopy groups of the suspension spectrum $\Sigma^\infty X$ are the stable homotopy groups of the based space X

$$\pi_n(\Sigma^\infty X) = \operatorname*{colim}_i \pi_{n+i}(\Sigma^i X) = \pi_n^S(X).$$

(b) Sphere spectrum $\mathbb{S}^0$

The sphere spectrum is denoted $\mathbb{S}^0$ because of its special role. We take the n-sphere S^n to be the n^{th}- term and

$$\sigma_n : \Sigma S^n \cong S^{n+1} \to S^{n+1}$$

to be the canonical homeomorphisms. In that case, the homotopy groups of the sphere spectrum are the stable homotopy groups of spheres. Actually, by Freudenthal's theorem the sequence stabilises and the colimit is attained at a finite stage

$$\pi_n^{st}(\mathbb{S}^0) = \operatorname*{colim}_i \pi_{n+i}(S^i) = \pi_{n+i}(S^i), \ \textit{for } i \geqslant n+2.$$

(c) Eilenberg-MacLane spectra HG

Let G be an abelian group and $n \in \mathbb{N}$. An Eilenberg-MacLane space of type n, denoted $K(G,n)$, is a CW-complex with a single non-vanishing homotopy group G occuring in dimension n. Such spaces exist ([Gra75, Theorem 17.3]), and are uniquely determined up to weak homotopy equivalence. This sequence of spaces, as n varies, assembles to make a spectrum HG by taking $X_n = K(G,n)$. Since looping shifts homotopy, i.e.

$$\pi_n(\Omega X) \cong \pi_{n+1}(X),$$

we have a weak homotopy equivalence

$$\bar{\sigma}_n : K(G,n) \to \Omega K(G,n+1),$$

which is a homotopy equivalence by Whitehead's theorem. The map

$$\sigma_n : \Sigma K(G,n) \to K(G,n+1)$$

is the adjoint of the homotopy equivalence $\bar{\sigma}$. The homotopy groups of HG are concentrated in a single degree like the spaces from which it was built

$$\pi_n(HG) = \begin{cases} G & \text{if} \quad n = 0, \\ 0 & \text{if} \quad n \neq 0. \end{cases}$$

(d) Moore Spectrum $M(G)$

Let G be an abelian group. The Moore spectrum $M(G)$ has a Moore space $M(G,n)$ at the level n. A Moore space $M(G,n)$ is a CW-complex with one 0-cell and all other cells in dimensions n and $n+1$ such that

$$\begin{cases} \pi_n(M(G,n)) \cong G, & \\ \pi_k(M(G,n)) = 0 & \text{if} \quad k < n, \\ H_i(M(G,n),\mathbb{Z}) = 0 & \text{if} \quad i > n. \end{cases}$$

Given any abelian group G, there is a Moore spectrum $M(G)$ unique up to weak equivalence. The spectrum $M(G)$ has the following homology and homotopy groups

$$\begin{cases} \pi_0(M(G)) = H_0(M(G), \mathbb{Z}) = G, \\ \pi_i(M(G)) = 0 \quad \textit{for} \quad i < 0, \\ H_i(M(G), \mathbb{Z}) = 0 \quad \textit{for} \quad i \neq 0. \end{cases}$$

Note that the sphere spectrum is a Moore spectrum for $G = \mathbb{Z}$.

Remark 3.1.5. Since we defined a spectrum to be a sequence of simplicial sets, when we say a space in the above examples we mean the associated simplicial set to that space.

The following well-known theorem, see for example [Ada69] and [Ada95], allows us in certain cases to determine the homotopy groups of a certain Moore spectrum.

Theorem 3.1.6. (Universal coefficient theorem) *For a group G, we have a short exact sequence*

$$0 \to G \otimes_{\mathbb{Z}} \pi_n \mathbb{S}^0 \to \pi_n(M(G)) \to \mathrm{Tor}^{\mathbb{Z}}(G, \pi_{n-1}\mathbb{S}^0) \to 0.$$

We can use the above theorem to deduce the below result. Note that it is a well-known fact, but we choose to include the proof to show how the universal coefficient theorem can be used to deduce such a result.

Corollary 3.1.7. *For $G = \mathbb{Q}$, the rational numbers, there is an equivalence between the rational Moore spectrum and the Eilenberg-MacLane spectrum.*

$$M(\mathbb{Q}) \simeq H\mathbb{Q}.$$

Proof. If we take $G = \mathbb{Q}$ in Theorem 3.1.6, we end up with the following short exact sequence

$$0 \to \mathbb{Q} \otimes \pi_n \mathbb{S}^0 \to \pi_n(M(\mathbb{Q})) \to \mathrm{Tor}(\mathbb{Q}, \pi_{n-1}\mathbb{S}^0) \to 0.$$

However, we know that

$$\mathrm{Tor}(\mathbb{Q}, \pi_{n-1}\mathbb{S}^0) = 0, \forall n.$$

Hence, we have an isomorphism

$$\mathbb{Q} \otimes \pi_n \mathbb{S}^0 \to \pi_n(M(\mathbb{Q})).$$

Plus, by a Theorem of Serre, we know that

$$\begin{cases} \mathbb{Q} \otimes \pi_n \mathbb{S}^0 = 0, \text{ for } n \neq 0 \\ \mathbb{Q} \otimes \pi_0 \mathbb{S}^0 \cong \mathbb{Q}. \end{cases}$$

We conclude that the spectrum $M\mathbb{Q}$ is characterised by the homotopy groups

$$\pi_n M\mathbb{Q} = \begin{cases} \mathbb{Q}, \ n = 0 \\ 0, \ n \neq 0. \end{cases}$$

As seen in Example 3.1.4(c), that means that we have a weak equivalence between $M\mathbb{Q}$ and the rational Eilenberg-MacLane spectrum $H\mathbb{Q}$. □

Remark 3.1.8. For $G = \mathbb{Z}/p$, the mod-p Moore spectrum is defined as the cofibre of multiplication by p on the sphere spectrum, i.e. it is part of a distinguished triangle

$$\mathbb{S}^0 \xrightarrow{\cdot p} \mathbb{S}^0 \xrightarrow{\text{incl}} M(\mathbb{Z}/p) \xrightarrow{\text{pinch}} \Sigma\mathbb{S}^0.$$

We will learn about distinguished triangles in the next section. Here, incl is the inclusion of the bottom cell, and pinch is the map that "pinches" off the bottom cell so that only the top cell is left. Alternatively, $M(\mathbb{Z}/p)$ can be defined as $\Sigma^{-1}(\Sigma^\infty M(\mathbb{Z}/p, 1))$, the desuspension of the suspension spectrum of a mod-p Moore space $M(\mathbb{Z}/p, 1)$. The spectrum $\Sigma^\infty M(\mathbb{Z}/p, 1)$ is the suspension spectrum associated to the space $M(\mathbb{Z}/p, 1) = S^1 \cup_p D^2$, i.e. the space obtained from the circle S^1 by attaching a 2-disc along the degree p map

$$S^1 \xrightarrow{\cdot p} S^1$$
$$z \mapsto z^p.$$

Remark 3.1.9. The degree two map of the Moore spectrum, $2\mathrm{Id}_{M(\mathbb{Z}/2)}$, is non-zero in Ho(Sp) [Sch08, Proposition 4]. Furthermore, it factors as the composite

$$M(\mathbb{Z}/2) \xrightarrow{\text{pinch}} \mathbb{S}^1 \xrightarrow{\eta} \mathbb{S}^0 \xrightarrow{\text{incl}} M(\mathbb{Z}/2),$$

where $\eta : \mathbb{S}^1 \to \mathbb{S}^0$ is the well-known Hopf map. It is worth noticing that for p odd this is not true, and we have

$$p\,\mathrm{Id}_{M(\mathbb{Z}/p)} = 0 \text{ [Sch08, Proposition 5].}$$

Definition 3.1.10. Throughout this thesis, Sp denotes the model category of spectra with the stable Bousfield-Friedlander model structure [BF78]:
A map $f : X \to Y$ in Sp is:

- a weak equivalence if $f_* : \pi_* X \to \pi_* Y$ is an isomorphism,
- a cofibration if the induced map

$$\Sigma Y_n \cup_{\Sigma X_n} X_{n+1} \to Y_{n+1}$$

  is a cofibration of simplicial sets for all $n \geqslant 1$ and $X_0 \to Y_0$ is a cofibration of simplicial sets,
- a fibration if f has the right lifting property with respect to trivial cofibrations. Or equivalently, if f is a fibration in sSet$_*$ such that X and Y are fibrant in Sp.

Remark 3.1.11. (a) A spectrum X is fibrant in Sp with respect to this model structure if and only if X is an Ω-spectrum and each X_n is a Kan-complex.

(b) A spectrum X is cofibrant with respect to the model structure defined above if and only if each

$$\sigma_n^X : \Sigma X_n \to X_{n+1}$$

is an injection.

(c) The homotopy category of spectra denoted Ho(Sp) is called the *stable homotopy category* and is our centre of interest for the rest of this thesis.

Definition 3.1.12. Take X and Y two spectra. We construct the spectrum $X \wedge Y$ (we read it X *smash* Y) as a functor of two variables, with arguments and values in the stable homotopy category Ho(Sp)

$$- \wedge - : \mathrm{Ho(Sp)} \times \mathrm{Ho(Sp)} \to \mathrm{Ho(Sp)},$$

such that we have the following natural homotopy equivalences.

$$\begin{aligned}
&a = a(X, Y, Z) : (X \wedge Y) \wedge Z \to X \wedge (Y \wedge Z)\\
&\tau = \tau(X, Y) : X \wedge Y \to Y \wedge X\\
&l = l_X : \mathbb{S}^0 \wedge X \to X\\
&r = r_X : X \wedge \mathbb{S}^0 \to X\\
&s = s_{X,Y} : (\Sigma X) \wedge Y \to \Sigma(X \wedge Y).
\end{aligned}$$

Remark 3.1.13. In this thesis, we will not need to know more than the above properties. However, the construction of the smash product is somehow complicated and not as obvious as one would like. A good source to learn more about the construction of the smash product is [Ada95, chapter 4] and [Swi75, Theorem 13.40]. However, having a smash product defined on the category of spectra before passing to the homotopy level is possible, but with a different notion of spectra, for example symmetric spectra. For more details about that see [HSS00].

3.2 Tools from triangulated categories

Another important feature of the stable homotopy category Ho(Sp), as we will see later, is that it carries the structure of a triangulated category. Triangulated categories are a special class of categories which appear in many areas of mathematics, specifically stable homotopy theory. This kind of category first appeared implicitly in papers on stable homotopy theory in the work of Puppe, until Verdier axiomatised the properties of these categories in his Ph.D thesis [Ver96].

In this section, we define several tools that will play a relevant role in the upcoming chapters. We start by defining triangulated categories, then we talk about Toda brackets. Afterwards, we define cofiber and fiber sequences, and we close the subsection 3.2.3 by giving a precise definition of what we mean by *rigidity and exotic models* in stable homotopy theory. Lastly, we end this section by talking about compact generators.

3.2.1 Triangulated categories

Definition 3.2.1. A category $\mathcal{A}$ is called *additive* if the following conditions are satisfied.

(A1) Each morphism set $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is endowed with the structure of an abelian group, and the composition operation is bilinear.

(A2) We have a zero object $0_{\mathcal{A}}$, that is an object which is both the terminal and initial object of $\mathcal{A}$.

(A3) The category $\mathcal{A}$ has finite products which are isomorphic to finite coproducts.

Definition 3.2.2. A functor $F : \mathcal{A} \to \mathcal{A}'$ between additive categories is called *additive* if it is compatible with the additive structure. In other words, for any two objects X and Y in $\mathcal{A}$, the mapping induced by F

$$\mathrm{Hom}_{\mathcal{A}}(X,Y) \to \mathrm{Hom}_{\mathcal{A}'}(F(X),F(Y))$$

is a homomorphism of abelian groups.

Definition 3.2.3. Let $\mathcal{T}$ be an additive category equipped with an auto-equivalence

$$\Sigma : \mathcal{T} \to \mathcal{T}.$$

A *triangle* in $\mathcal{T}$ is a sequence (α,β,γ) of maps

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X,$$

and a morphism betweeen two triangles (α,β,γ) and (α',β',γ') is a triple (Φ_1,Φ_2,Φ_3) of maps in $\mathcal{T}$ making the following diagram commutative

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_3} & & \downarrow{\scriptstyle \Sigma\Phi_1} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'. \end{array}$$

The category $\mathcal{T}$ is *triangulated* if it is equipped with a class of distinguished triangles called *exact triangles*, satisfying the following axioms.

(TR0) The exact triangles are closed under isomorphisms.

(TR1) For all $X \in \mathcal{T}$, the triangle $0 \to X \xrightarrow{\mathrm{Id}_X} X \to 0$ is exact.

(TR2) Each map $\alpha : X \to Y$ fits into an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$.

(TR3) A triangle (α,β,γ) is exact if and only if $(\beta,\gamma,-\Sigma\alpha)$ is exact.

(TR4) Given two exact triangles (α,β,γ) and (α',β',γ'), each pair of maps Φ_1 and Φ_2 satisfying $\Phi_2 \circ \alpha = \alpha' \circ \Phi_1$, can be completed to a morphism of triangles

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ {\scriptstyle\Phi_1}\downarrow & & \downarrow{\scriptstyle\Phi_2} & & \vdots{\scriptstyle\Phi_3} & & \downarrow{\scriptstyle\Sigma\Phi_1} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X'. \end{array}$$

(TR5) (Octahedal axiom) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commute

$$\begin{array}{ccccccc} X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & \Sigma X \\ {\scriptstyle\mathrm{Id}_X}\downarrow & & \downarrow{\scriptstyle\beta_1} & & \vdots{\scriptstyle\delta_1} & & \downarrow{\scriptstyle\mathrm{Id}_{\Sigma X}} \\ X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & \Sigma X \\ & & \downarrow{\scriptstyle\beta_2} & & \vdots{\scriptstyle\delta_2} & & \\ & & W & \xrightarrow{\mathrm{Id}_W} & W & & \\ & & \downarrow{\scriptstyle\beta_3} & & \vdots{\scriptstyle\delta_3} & & \\ & & \Sigma Y & \xrightarrow{\Sigma\alpha_2} & \Sigma U. & & \end{array}$$

Lemma 3.2.4. *[Nee14, Proposition 1.1.20] Let (Φ_1, Φ_2, Φ_3) be a morphism between exact triangles. If two maps out of three are isomorphisms, then so is the third.*

Remark 3.2.5. We can see that by applying the previous proposition to $(\mathrm{Id}_X, \mathrm{Id}_Y, \Phi)$, the exact triangle in (TR2) is unique up to isomorphism. Note that this isomorphism is not unique because the completion of the diagram in (TR4) is not unique.

Definition 3.2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be triangulated categories. An *exact functor* $F : \mathcal{D} \to \mathcal{C}$ is an additive functor together with a natural equivalence $\Phi : F\Sigma \to \Sigma F$, with the property that for any exact triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

in $\mathcal{D}$, the following candidate triangle is exact in $\mathcal{C}$

$$F(X) \xrightarrow{F(\alpha)} F(Y) \xrightarrow{F(\beta)} F(Z) \xrightarrow{\Phi_X \circ F(\gamma)} \Sigma F(X).$$

In other words, an exact functor sends exact triangles to exact triangles.

Definition 3.2.7. A *triangulated subcategory* $\mathcal{T}'$ of $\mathcal{T}$ is a full additive subcategory with a triangulated structure, such that the inclusion functor $\mathcal{T}' \hookrightarrow \mathcal{T}$ is exact, and every object isomorphic to an object in $\mathcal{T}'$ is in $\mathcal{T}'$.

Remark 3.2.8. The above definition is equivalent to saying that the category $\mathcal{T}'$ is invariant under suspension, and if in a triangle two out of three objects are in $\mathcal{T}'$, then so is the third.

3.2.2 Toda brackets

Toda brackets are operations on homotopy classes of maps, named after Hiroshi Toda who defined them and used them to compute homotopy groups of spheres in [Tod62]. We will use Toda brackets mainly in Chapter 5 to complete short exact sequences.

We start by considering the following diagram in a triangulated category $\mathcal{T}$

$$\begin{array}{ccccccc} X_3 & \xrightarrow{\lambda_3} & X_2 & \xrightarrow{\lambda_2} & X_1 & \xrightarrow{\lambda_1} & X_0 \\ & & {\scriptstyle \iota}\downarrow & \overset{\beta}{\nearrow} & & & \\ & & C & & & \overset{\gamma}{\nearrow} & \\ & & {\scriptstyle \Pi}\downarrow & & & & \\ & & \Sigma X_3, & & & & \end{array}$$

in which λ_1, λ_2 and λ_3 are maps such that $\lambda_1 \circ \lambda_2 = 0 = \lambda_2 \circ \lambda_3$, and (λ_3, ι, Π) is an exact triangle in $\mathcal{T}$. If we apply the functor $\mathrm{Hom}_{\mathcal{T}}(-, X_1)$ to the exact triangle, we get an exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_1) \xrightarrow{\Pi^*} \mathrm{Hom}_{\mathcal{T}}(C, X_1) \xrightarrow{\iota^*} \mathrm{Hom}_{\mathcal{T}}(X_2, X_1) \xrightarrow{\lambda_3^*} \mathrm{Hom}_{\mathcal{T}}(X_3, X_1).$$

Hence the relation $\lambda_2 \circ \lambda_3 = 0$ tells us that $\lambda_2 \in \mathrm{Ker}\lambda_3^* = \mathrm{Im}\iota^*$, which implies the existence of a map $\beta : C \to X_1$ with $\beta \circ \iota = \lambda_2$. Note that this choice of β is not necessarily unique. Similarly, we apply $\mathrm{Hom}_{\mathcal{T}}(-, X_0)$ and obtain the exact sequence

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma X_2, X_0) \xrightarrow{(\Sigma\lambda_3)^*} \mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_0) \xrightarrow{\Pi^*} \mathrm{Hom}_{\mathcal{T}}(C, X_0) \xrightarrow{\iota^*} \mathrm{Hom}_{\mathcal{T}}(X_2, X_0).$$

Now, the relation $\lambda_1 \circ \lambda_2 = 0 = \lambda_1 \circ \beta \circ \iota$ implies that there exists a (non-unique) $\gamma \in \mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_0)$ such that $\gamma \circ \Pi = \lambda_1 \circ \beta$.

The construction of the element γ is not necessarily unique as there are choices involved in its construction. First, we can alter β by an element of $\Pi^*(\mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_1))$, and γ by an element of $(\Sigma\lambda_3)^*(\mathrm{Hom}_{\mathcal{T}}(\Sigma X_2, X_0))$ as we can read off from the exact sequences. Putting these choices together we see that γ is only defined modulo

$$(\Sigma\lambda_3)^*(\mathrm{Hom}_{\mathcal{T}}(\Sigma X_2, X_0)) + \lambda_{1*}(\mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_1)) \subseteq \mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_0).$$

Definition 3.2.9. Let

$$X_3 \xrightarrow{\lambda_3} X_2 \xrightarrow{\lambda_2} X_1 \xrightarrow{\lambda_1} X_0$$

be a sequence in a triangulated category $\mathcal{T}$ with $\lambda_1 \circ \lambda_2 = \lambda_2 \circ \lambda_3 = 0$. The *Toda bracket*

$$\langle \lambda_1, \lambda_2, \lambda_3 \rangle$$

is the set of all maps γ in $\mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_0)$ which can be constructed as above. Since there are choices involved in the construction of γ, the Toda bracket $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$ is an element of the quotient group $\mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_0)/R$, where

$$R = (\Sigma\lambda_3)^*(\mathrm{Hom}_{\mathcal{T}}(\Sigma X_2, X_0)) + \lambda_{1*}(\mathrm{Hom}_{\mathcal{T}}(\Sigma X_3, X_1)),$$

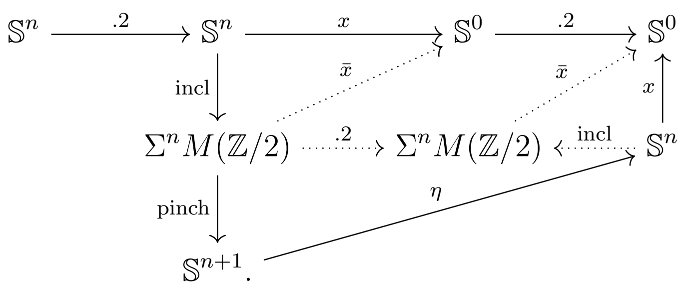
which we refer to as the *indeterminacy* of the Toda bracket involved.

Example 3.2.10. *If x is an element of $\pi_*^{st}(\mathbb{S}^0)$ such that $2x = 0$, then*

$$x \circ \eta \in \langle 2, x, 2 \rangle.$$

This can be seen from the commutative diagram below, and remember that

$$2\mathrm{Id}_{M(\mathbb{Z}/2)} = \mathrm{incl} \circ \eta \circ \mathrm{pinch}, \quad \textit{by Remark 3.1.9.}$$



The following theorem, as its name suggests, helps us to see what happens when we juggle around different elements in a Toda bracket.

Theorem 3.2.11. (Juggling Theorem)*[Rav03, Theorem A1.4.6] [Koc96, Proposition 5.7.4] Let*

$$X_3 \xrightarrow{\lambda_3} X_2 \xrightarrow{\lambda_2} X_1 \xrightarrow{\lambda_1} X_0 \xrightarrow{\alpha} Z$$

be composable morphisms in a triangulated category $\mathcal{T}$. Then the following inclusions hold if the involved Toda brackets are defined:

(a) $\langle \alpha \circ \lambda_1, \lambda_2, \lambda_3 \rangle \subseteq \langle \alpha, \lambda_1 \circ \lambda_2, \lambda_3 \rangle$

(b) $\alpha \circ \langle \lambda_1, \lambda_2, \lambda_3 \rangle \subseteq \langle \alpha \circ \lambda_1, \lambda_2, \lambda_3 \rangle$.

Remark 3.2.12. For a detailed list of relations between Toda brackets we suggest [Tod62, page 30]. We list some of these relations which we will need later on.

(a) If one of the elements λ_1, λ_2 or λ_3 is zero, then $\langle \lambda_1, \lambda_2, \lambda_3 \rangle = 0$.

(b) $\langle \lambda_1 + \lambda_1', \lambda_2, \lambda_3 \rangle \subseteq \langle \lambda_1, \lambda_2, \lambda_3 \rangle + \langle \lambda_1', \lambda_2, \lambda_3 \rangle$

(c) $\langle \lambda_1, \lambda_2, \lambda_3 \rangle + \langle \lambda_1, \lambda_2', \lambda_3 \rangle = \langle \lambda_1, \lambda_2 + \lambda_2', \lambda_3 \rangle$

(d) $\langle \lambda_1, \lambda_2, \lambda_3 + \lambda_3' \rangle \subseteq \langle \lambda_1, \lambda_2, \lambda_3 \rangle + \langle \lambda_1, \lambda_2, \lambda_3' \rangle$

3.2.3 Cofiber and fiber sequences

Before stating the result that gives a triangulated structure on the stable homotopy category, we briefly introduce fiber and cofiber sequences. For more details about their construction, we suggest [Hov99, Chapter 6] or [Qui06, Chapter I.3].

Definition 3.2.13. Let $\mathcal{C}$ be a pointed model category, and $f : X \to Y$ a morphism in our category. We define the *cofiber* of f as the pushout diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Cofib}(f). \end{array}$$

Dually, we define the *fiber* of f as the pullback diagram

$$\begin{array}{ccc} \mathrm{Fib}(f) & \longrightarrow & * \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Definition 3.2.14. Suppose $\mathcal{C}$ is a pointed model category. A *cofiber sequence* in $\mathrm{Ho}(\mathcal{C})$ is any diagram that is isomorphic to a diagram of the form

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

where f is a cofibration of cofibrant objects in $\mathcal{C}$ with cofiber C, together with a right coaction of ΣA on C given by [Hov99, Theorem 6.2.1] or [Qui06, Chapter I.3].

Remark 3.2.15. Suppose we have a cofiber sequence

$$X \to Y \to Z,$$

with a right coaction of ΣX on Z. Then we have a map

$$\partial : Z \to \Sigma X$$

called the *boundary map* of the cofiber sequence.

Definition 3.2.16. Dually, we can define a *fiber sequence* as a diagram

$$X \to Y \to Z$$

in $\mathrm{Ho}(\mathcal{C})$ together with a right action of ΩZ on X, which is isomorphic in $\mathrm{Ho}(\mathcal{C})$ to a diagram

$$E \xrightarrow{i} F \xrightarrow{p} G,$$

where p is a fibration between fibrant objects with fiber E. Furthermore, we have a *boundary map*

$$\partial : \Omega Z \to X$$

associated to the fiber sequence

$$X \to Y \to Z.$$

Proposition 3.2.17. *[Hov99, Proposition 6.3.4] Let $\mathcal{C}$ be a pointed model category, suppose that*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is a cofiber sequence in $\mathcal{C}$, then the cofiber sequence can be "extended" to the right. In other words, the sequence

$$Y \xrightarrow{g} Z \xrightarrow{\partial} \Sigma X$$

becomes a cofiber sequence. Dually, a fiber sequence $A \xrightarrow{h} B \xrightarrow{k} C$ *is "expandable" to the left, meaning that the sequence*

$$\Omega C \xrightarrow{\partial} A \xrightarrow{h} B$$

is a fiber sequence itself.

Proposition 3.2.18. *[Hov99, 7.1.6] The homotopy category* $\mathrm{Ho}(\mathcal{C})$ *of a stable model category* $\mathcal{C}$ *carries the structure of a triangulated category. The exact triangles are given by the fiber and cofiber sequences, since in this case they coincide up to sign.*

In particular, since the category of spectra with the model structure in Definition 3.1.10 is stable, its homotopy category $\mathrm{Ho}(\mathrm{Sp})$ is a triangulated category.

Notation. We denote the morphisms in a triangulated category $\mathcal{T}$ by $[A, B]^{\mathcal{T}}$. This is a group since triangulated categories are, in particular, additive. By $[A, B]^{\mathcal{T}}_n$ we mean $[\Sigma^n A, B]^{\mathcal{T}}$. If $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$ for some stable model category, we write $[A, B]^{\mathcal{C}}$ instead of $[A, B]^{\mathrm{Ho}(\mathcal{C})}$.

The following proposition is used as a basic tool for constructing long exact sequences out of exact triangles.

Theorem 3.2.19. *[Ada95, III Proposition 3.9.] Suppose we have an exact triangle in the stable homotopy category*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

then for each W *in* Sp *the sequence*

$$\ldots \to [X, W]^{\mathrm{Sp}}_{n+1} \to [Z, W]^{\mathrm{Sp}}_n \to [Y, W]^{\mathrm{Sp}}_n \to [X, W]^{\mathrm{Sp}}_n \to \ldots$$

$$(\textit{respectively}, \ \ldots \to [W, X]^{\mathrm{Sp}}_n \to [W, Y]^{\mathrm{Sp}}_n \to [W, Z]^{\mathrm{Sp}}_n \to [W, X]^{\mathrm{Sp}}_{n-1} \to \ldots)$$

is exact.

The next example will be useful later on, in the proof of Lemma 4.1.2, and in Section 5.2.12 where we are going to use it to compute certain homotopy groups.

Example 3.2.20. *We start first by remembering that a long exact sequence of modules over a ring R*

$$0 \to M_1 \xrightarrow{\Phi_1} M_2 \xrightarrow{\Phi_2} \dots \xrightarrow{\Phi_{n-1}} M_n \xrightarrow{\Phi_n} M_{n+1} \xrightarrow{\Phi_{n+1}} \dots$$

can be split into short exact sequences

$$0 \to \mathrm{Coker}(\Phi_{i-2}) \xrightarrow{\Phi_{i-1}} M_i \xrightarrow{\Phi_i} \mathrm{Im}\Phi_i \to 0.$$

By applying the above to the long exact homotopy sequence produced by Theorem 3.2.19 from the exact triangle

$$\mathbb{S}^0 \xrightarrow{p\cdot} \mathbb{S}^0 \xrightarrow{\mathrm{incl}} M \xrightarrow{\mathrm{pinch}} \mathbb{S}^1,$$

we get short exact sequences of the form

$$0 \to \big(\pi_{m+1}(\mathbb{S}^0)\big)/p \xrightarrow{\mathrm{incl}_*} \pi_{m+1}(M(\mathbb{Z}/p)) \xrightarrow{\mathrm{pinch}_*} \{\pi_m(\mathbb{S}^0)\}_p \to 0.$$

Here,

$$\{\pi_m(\mathbb{S}^0)\}_p = \{x \in \pi_m(\mathbb{S}^0) : px = 0\}$$

is the p-torsion of the group $\pi_m(\mathbb{S}^0)$.

After we have provided the necessary background, we can define *rigidity* and *exotic model.*

Definition 3.2.21. Let $\mathcal{C}$ be a fixed stable model category (for example $\mathcal{C} = \mathrm{Sp}$), and $\mathcal{D}$ any stable model category. Assuming that there is an equivalence of triangulated categories on their homotopy level

$$\Phi : \mathrm{Ho}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Ho}(\mathcal{D}),$$

are $\mathcal{C}$ and $\mathcal{D}$ Quillen equivalent?

- If the answer is affirmative, then we say that $\mathrm{Ho}(\mathcal{C})$ is *rigid*. For example, for $\mathcal{C} = \mathrm{Sp}$, Schwede showed that $\mathrm{Ho}(\mathrm{Sp})$ is rigid [Sch07a].
- If the answer is negative, and we have a counterexample where rigidity is not verified, then we say that $\mathcal{D}$ is an *exotic model* for $\mathcal{C}$.

3.2.4 Compact generators

Definition 3.2.22. Let $\mathcal{T}$ be a triangulated category with infinite coproducts, and $\mathcal{T}'$ a full triangulated subcategory of $\mathcal{T}$ with shift and triangles induced from $\mathcal{T}$. The subcategory $\mathcal{T}'$ is called *localising* if it is closed under coproducts in $\mathcal{T}$.

Definition 3.2.23. A set $\mathcal{G}$ of objects of a triangulated category $\mathcal{T}$ is called a set of *generators* if the only localising subcategory containing the objects of $\mathcal{G}$ is $\mathcal{T}$ itself.

Definition 3.2.24. We say that an object A of a triangulated category $\mathcal{T}$ is *compact* (also called small or finite) if the functor $[A, -]^{\mathcal{T}}$ from $\mathcal{T}$ to groups commutes with arbitrary coproducts, i.e. for any family of objects $\{A_i\}_{i\in\mathcal{I}}$ whose coproduct exists, the canonical map

$$\bigoplus_{i\in I}[A, A_i]^{\mathcal{T}} \to [A, \coprod A_i]^{\mathcal{T}}$$

is an isomorphism.

Note that objects of a stable model category are called "generators" or "compact" if they are so when considered as objects of the triangulated homotopy category.

The next theorem tells us what criterion should be satisfied by a set of compact objects, in order to become generators of a triangulated category.

Theorem 3.2.25. *[SS03, Lemma 2.2.1] Let $\mathcal{T}$ be a triangulated category with infinite coproducts, and $\mathcal{G}$ a set of compact objects. Then the following are equivalent.*

(i) The set $\mathcal{G}$ generates $\mathcal{T}$ in the sense of Definition 3.2.23.

(ii) The objects of $\mathcal{G}$ detect isomorphisms, meaning that a morphism $X \to Y$ in $\mathcal{T}$ is an isomorphism if and only if $[G, X]^{\mathcal{T}} \to [G, Y]^{\mathcal{T}}$ is an isomorphism for all $G \in \mathcal{G}$.

The previous theorem will form an important step in the proof of the main result in this thesis. Briefly speaking, if we want to prove that a criterion is true for all the objects in a certain triangulated category, then it is often sufficient to prove it true for a compact generator.

Remark 3.2.26. Note that in Theorem 3.2.25, the point (i) implies (ii) even without the hypotheses of compactness. In other words, the objects of a set of generators detect isomorphisms.

Example 3.2.27. *The sphere spectrum $\mathbb{S}^0$ is a compact generator of the stable homotopy category* Ho(Sp). *It "generates" the whole stable homotopy category under exact triangles and coproducts.*

In the next section, after introducing Bousfield localisation, we will give more examples that will be our center of interest for the rest of the thesis. For a list of interesting examples of compact generators see [SS03, Examples 2.3].

3.3 Bousfield localisation

A useful tool to create a new model category out of a given one is Bousfield localisation. This section starts by defining homology theories, and then constructing the function spectrum $F(X, Y)$. After that, we will talk about Bousfield localisation of spectra with respect to some homology theories, in particular with respect to Morava K-theory. We will end this section by discussing the "Periodicity Theorem" and the self-map on the Moore spectrum which will be needed later on.

Definition 3.3.1. A *generalised homology theory* E_* is a covariant functor from the category of pairs of pointed spaces to the category of graded abelian groups, such that the first three of the Eilenberg-Steenrod axioms are satisfied:

(*i*) (Homotopy axiom) Homotopic maps

$$f, g : (X, A) \to (Y, B)$$

induce the same homomorphism

$$E_*(f) = E_*(g) : E_*(X, A) \to E_*(Y, B).$$

(*ii*) (Exactness axiom) For each pair (X, A), there is a natural long exact sequence

$$\ldots \to E_n(A) \xrightarrow{E_n(i)} E_n(X) \xrightarrow{E_n(j)} E_n(X, A) \xrightarrow{\partial} E_{n-1}(A) \to \ldots,$$

where $E_n(i)$ is the homomorphism induced by the inclusion map

$$i : A \to X.$$

(*iii*) (Excision axiom) If we have the inclusions $C \subset A \subset X$, with the closure of C contained in the interior of A, then there is an isomorphism

$$E_*(X - C, A - C) \xrightarrow{\cong} E_*(X, A).$$

A *generalised cohomology theory* E^* is a contravariant functor with similar properties.

Actually, such theories can be constructed out of spectra. These constructions are due to G. W. Whitehead [Whi62].

Definition 3.3.2. Let E be an Ω-spectrum, and X a based space. The generalised cohomology theory E^* associated with E is defined by

$$E^n(X) = [\Sigma^\infty X, E]_{-n},$$

and the generalised homology theory E_* associated with E is defined by

$$E_n(X) = \pi_n(E \wedge X),$$

where $E \wedge X$ denotes the smash product of E with the suspension spectrum associated with X.

Notation. For a spectrum E, we denote $E_* := E_*(\mathbb{S}^0) \cong \pi_*(E)$. This is called the *coefficient ring* of the associated homology theory.

Example 3.3.3. *(i) Ordinary homology or cohomology with coefficients in an abelian group G can be constructed by taking E to be the Eilenberg-MacLane spectrum HG.*

$$E_*(X) = \pi_*(HG \wedge X) = H_*(X, G),$$
$$E^*(X) = [X, HG]_* = H^*(X, G).$$

(ii) If E is the sphere spectrum $\mathbb{S}^0$, then the resulting homology theory consists of the stable homotopy groups of X

$$E_n(X) = \pi_n^{st}(X),$$

and the coefficient ring in that case is the ring of stable homotopy groups of spheres.

(iii) Complex topological K-theory is usually defined in terms of stable equivalence classes of complex vector bundles, but the Bott periodicity theorem allows us to define complex K-theory in terms of spectra. In fact, complex K-theory is the cohomology theory associated with the periodic K-theory

spectrum

$$(KU)_n = \begin{cases} \mathbb{Z} \times BU & \textit{if } n \textit{ even,} \\ \Omega BU & \textit{if } n \textit{ odd.} \end{cases}$$

Here, U is the stable unitary group, and BU is its classifying space. The structure maps are the adjoints to

$$(\bar{\sigma} : KU_n \to \Omega KU_{n+1}) = \begin{cases} \textit{the Bott equivalence} \quad \mathbb{Z} \times BU \xrightarrow{\cong} \Omega^2 BU & \textit{if } n \textit{ even,} \\ \textit{the identification} \quad \Omega BU \xrightarrow{\cong} \Omega(\mathbb{Z} \times BU) & \textit{if } n \textit{ odd.} \end{cases}$$

Theories constructed from a spectrum satisfy the following axiom.

Definition 3.3.4 (Wedge Axiom)**.** Suppose that W is a (possibly infinite) wedge of spaces, $\bigvee_i X_i$, then

$$E^*(W) \cong \prod_\alpha E^*(X_\alpha), \text{ and}$$
$$E_*(W) \cong \bigoplus_\alpha E_*(X_\alpha).$$

Likewise, constructing a spectrum out of a generalised (co)homology theory is possible by the following theorem.

Theorem 3.3.5. (Brown Representability Theorem)*[Bro62] If h^* is a generalised cohomology theory satisfying the wedge axiom, then there is a spectrum E such that $h^* = E^*$.*

Remark 3.3.6. Note that with a finiteness assumption on the domain of the generalised homology theory, there is a homology version of the Brown representability theorem due to Adams [Ada71].

The complex bordism of a space X, denoted $\mathrm{MU}_*(\mathrm{X})$, can be interpreted geometrically as the group of bordism classes of manifolds over X with a complex linear structure. However, it can be represented by a spectrum MU, called the

Thom spectrum for the unitary group, and described in terms of Thom spaces:

$$\mathrm{MU}_{2n} = \mathrm{MU(n)},$$
$$\mathrm{MU}_{2n+1} = \Sigma\mathrm{MU(n)}.$$

Here, MU(n) is the Thom space of the universal n-dimensional complex vector bundle over the classifying space BU(n) of the unitary group U(n). Moreover, the complex bordism ring MU_* has many interesting algebraic properties, and is isomorphic to the following graded ring

$$\mathrm{MU}_* \cong \mathbb{Z}[x_1, x_2, \ldots], \;\; |x_i| = 2i.$$

Brown and Peterson showed that after localising MU at a prime p, the MU spectrum is homotopy equivalent to an infinite wedge of suspensions of “smaller” spectra named BP after them, with coefficient ring

$$\mathrm{BP}_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots], \;\; |v_i| = 2p^i - 2.$$

Here, $\mathbb{Z}_{(p)}$ is the ring of integers localised at p.

Many important homology theories are constructed from MU_* or BP_* via *Landweber's exact functor theorem* [Lan76]. Landweber's theorem states that given a BP_*-module M_*, and under certain conditions, the functor

$$BP_*(-) \otimes_{BP_*} M_* =: M_*(-)$$

is a generalised homology theory which can be represented by a spectrum M.

In this thesis we restrict ourselves to giving examples of such theories without going into details. Good references for such constructions are [Rav16] and [Rav03]. An important example of such homology theories satisfying Landweber's conditions and resulting in the construction of a generalised homology theory is the Johnson-Wilson theory.

Definition 3.3.7. For a fixed prime p and n a positive integer, the *Johnson-Wilson* spectrum denoted $E(n)$ is a spectrum with coefficients

$$E(n)_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}], \ |v_i| = 2p^i - 2.$$

The theory $E(0)_*$ agrees with singular homology with rational coefficients, while $E(1)_*$ is related to complex K-theory. By the *Adams splitting* [Ada69], the spectrum $E(1)$ is a summand of complex K-theory localised at p:

$$K_{(p)} \cong \bigvee_{i=0}^{p-2} \Sigma^{2i} E(1).$$

There is an analogue of the Landweber exact functor theorem on some spectrum $P(n)$ constructed from BP, and that enables us to construct the Morava K-theories $K(n)_*$. The existence of such theories was proved by Jack Morava in the early seventies in an unpublished work. The first published reference concerning the $K(n)$'s is an article by Johnson and Wilson [JW75]. In this thesis we will not talk about the construction of such theories, however a nice reference containing more details is [Wür91]. Instead, we will restrict ourselves to defining them and citing some of the many properties that made $K(n)$ play an important role in stable homotopy theory.

Definition 3.3.8. Let p be a fixed prime and $n \in \mathbb{N}$. The *Morava K-theory* $K(n)_*$ is a homology theory with coefficient ring

$$K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}], \ |v_n| = 2p^n - 2.$$

Similar to $E(n)_*$, we have by convention that $K(0) = E(0) = H\mathbb{Q} = M\mathbb{Q}$, and $K(1)$ is related to complex K-theory. More precisely, we have that $K(1)$ is a summand of mod-p complex K-theory [Rav16, Proposition 1.5.2(ii)]. In our case of interest in this thesis, $p = 2$, we have that mod-2 K-theory coincides with $K(1)$ since there is only one such summand. Since $K(n)_*$ is a *graded field* (i.e. each non-zero homogeneous element has a multiplicative inverse), every graded module over it is free. This leads to a special isomorphism called the *Künneth* isomorphism which provides us with a special tool to compute the homology of a smash product of spectra

$$K(n)_*(X \wedge Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y).$$

Actually, this *Künneth isomorphism* is only verified by Morava K-theory and ordinary homology with field coefficients.

Example 3.3.9. *Given spectra X and Y, the graded group $[W \wedge X, Y]_*$, regarded as a functor on the spectrum W, is a cohomology theory satisfying the wedge axiom 3.3.4. Therefore, by 3.3.5 there is a spectrum, denoted by $F(X,Y)$ and called the* function spectrum, *such that*

$$[W \wedge X, Y]_* \cong [W, F(X,Y)]_*.$$

When X is finite and $Y = \mathbb{S}^0$, then $F(X,Y)$ is the Spanier-Whitehead dual *of X, denoted DX, and discussed in more detail below.*

Theorem 3.3.10. *[Rav16, Theorem 5.2.1] Let X be a finite spectrum, that is any desuspension of the suspension spectrum of a finite CW-complex (i.e. its total number of cells is finite), or that is weakly equivalent to one of that form. Then, there is a unique finite spectrum DX (the Spanier-Whitehead dual of X) with the following properties.*

(i) For any spectrum Y, we have a natural isomorphism, in both X and Y, between the graded group $[X,Y]_*$ and $\pi_*(DX \wedge Y)$. In particular, we have $D\mathbb{S}^0 \simeq \mathbb{S}^0$.

(ii) This isomorphism is reflected in Morava K-theory, namely

$$\mathrm{Hom}(K(n)_*(X), K(n)_*(Y)) \cong K(n)_*(DX \wedge Y).$$

(iii) $DDX \simeq X$, and $[X,Y]_* \cong [DX, DY]_*$.

(*iv*) *For a homology theory* E_**, there is a natural isomorphism between* $E_k(X)$ *and* $E^{-k}(DX)$*.*

(*v*) *Spanier-Whitehead duality commutes with smash products, i.e. for finite spectra* X *and* Y*,*

$$D(X \wedge Y) = DX \wedge DY.$$

The following lemma is a known result, but we choose to include its proof because we could not find a reference.

Lemma 3.3.11. *For a fixed prime* p *and* $n \in \mathbb{N}$*, the mod-*p^n *Moore spectra are "self-dual" up to suspension. More precisely, the suspension* $\Sigma(DM(\mathbb{Z}/p^n))$ *is a mod-*p^n *Moore spectrum and we have the isomorphism*

$$j_n : M(\mathbb{Z}/p^n) \to \Sigma(DM(\mathbb{Z}/p^n))$$

in Ho(Sp)*.*

Proof. In this proof we will use the notation

$$M = M(\mathbb{Z}/p^n).$$

We have the exact triangle in Ho(Sp)

$$\mathbb{S}^0 \xrightarrow{.p^n} \mathbb{S}^0 \to M \to \mathbb{S}^1.$$

By applying Theorem 3.2.19, we have the commutative diagram, in which the upper and lower rows are exact

$$\begin{array}{ccccccccc}
\dots \to [\mathbb{S}^0, \mathbb{S}^0]_{n+1} & \longrightarrow & [\mathbb{S}^0, M]_{n+1} & \longrightarrow & [\mathbb{S}^0, \mathbb{S}^0]_n & \xrightarrow{(.p^n)_*} & [\mathbb{S}^0, \mathbb{S}^0]_n & \longrightarrow & \dots \\
\cong\downarrow & & \downarrow & & \downarrow\cong & & \downarrow\cong & & \\
\dots \to [\mathbb{S}^0, \mathbb{S}^0]_{n+1} & \longrightarrow & [M, \mathbb{S}^0]_n & \longrightarrow & [\mathbb{S}^0, \mathbb{S}^0]_n & \xrightarrow{(.p^n)^*} & [\mathbb{S}^0, \mathbb{S}^0]_n & \longrightarrow & \dots
\end{array}$$

By a Five Lemma argument, we conclude that

$$\left[\mathbb{S}^0, M\right]_{n+1} \cong \left[M, \mathbb{S}^0\right]_n, \text{ for all } n.$$

On the other hand, by Theorem 3.3.10(i), we have that

$$\left[M, \mathbb{S}^0\right]_n \cong \left[\mathbb{S}^0, DM\right]_n, \text{ for all } n.$$

which gives us the desired isomorphism in Ho(Sp)

$$\left[\mathbb{S}^0, M\right]_{n+1} \cong \left[\mathbb{S}^0, DM\right]_n \cong \left[\mathbb{S}^0, \Sigma DM\right]_{n+1}.$$ □

Now, let E_* be a generalised homology theory represented by a spectrum E.

Definition 3.3.12. A spectrum X is E_*-*acyclic* if

$$E_*(X) = \pi_*(E \wedge X) = 0.$$

Definition 3.3.13. A map $f : X \to Y$ is an E_*-*equivalence* if it induces an isomorphism on E_*-homology groups

$$E_*(f) : E_*(X) \to E_*(Y).$$

Definition 3.3.14. A spectrum X is E_*-*local* if for each E_*-acyclic spectrum A, we have

$$[A, X] = 0.$$

Equivalently, a spectrum X is E_*-*local* if each E_*-equivalence

$$f : A \to B$$

in Ho(Sp) induces a bijection

$$f^* : [B, X] \to [A, X].$$

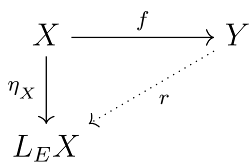
As we saw in Section 2.2, we can construct the homotopy category of a model category by inverting weak equivalences. This process is done by localising a model category by formally inverting a certain class of morphisms. In this thesis, we choose to talk about a specific localisation which is the Bousfield localisation of spectra with respect to some generalised homology theory. However, to read about localisation of a model category with respect to a certain class of morphisms, we refer the reader to [Hir09].

Mainly speaking, Bousfield localisation restricts attention to the part of the stable homotopy theory visible to a given homology theory E_*, which makes this tool very useful in studying the stable homotopy category. The main references for such constructions are [Bou79] and [Rav84]. This construction becomes particularly interesting when looking at some very special homology theories that give information about the structure of the p-local stable homotopy category for some prime p.

Definition 3.3.15. An *E_*-localisation functor* L_E is a covariant functor from Ho(Sp) to itself, along with a natural transformation η from the Id to L_E, such that:

(i) $\eta_X : X \to L_E X$ is an E_*-equivalence.

(ii) For any E_*-equivalence $f : X \to Y$, there is a unique $r : Y \to L_E X$ such that $rf = \eta_X$



Proposition 3.3.16. *[Rav84, Proposition 1.5] If the functor L_E exists, then it has the following properties.*

(i) It is unique.

(ii) It is idempotent, i.e. $L_E \circ L_E = L_E$.

(iii) For any map $g : X \to Y$, where Y is E_*-local, there is a unique map $\tilde{g} : L_E X \to Y$ such that $\tilde{g} \circ \eta_X = g$.

Theorem 3.3.17 (Localisation Theorem). *[Bou79] For every generalised homology theory E_*, there is a localisation functor*

$$L_E : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathrm{Sp}).$$

Theorem 3.3.18. *[Bou79, Theorem 1.1] Given E and A in $\mathrm{Ho}(\mathrm{Sp})$, there is a natural (in A) exact triangle*

$$C_E A \xrightarrow{\theta} A \xrightarrow{\eta_A} L_E A \to \Sigma(C_E A)$$

in $\mathrm{Ho}(\mathrm{Sp})$ such that $C_E A$ is E_*-acyclic and $L_E A$ is E_*-local.

We have a model structure on $L_E\mathrm{Sp}$ described in the following proposition. This construction can be seen as a special case of a more general result by Hirschhorn [Hir09]. More details about that can be found in [BR14b].

Proposition 3.3.19. *Let Sp be the category of spectra with the model structure of Definition 3.1.10, and E_* a generalised homology theory. Then there is a model category $L_E\mathrm{Sp}$ with the same objects as Sp and with the following model structure.*

- *The weak equivalences are the E_*-equivalences.*
- *The cofibrations are the cofibrations of Sp.*
- *The fibrations are the maps with the right lifting property with respect to cofibrations that are E_*-equivalences.*

Remark 3.3.20. As a consequence of the last proposition, we can see the model structure on the category $L_E\mathrm{Sp}$ as being formed out of the one on Sp such that:

- Every weak equivalence of Sp is a weak equivalence of $L_E\mathrm{Sp}$.
- The class of trivial fibrations of $L_E\mathrm{Sp}$ is equal to the class of trivial fibrations of Sp.

· Every fibration of $L_E\mathrm{Sp}$ is a fibration of Sp.

· Every trivial cofibration of Sp is a trivial cofibration of $L_E\mathrm{Sp}$.

Therefore, we can see that the adjunction

$$\mathrm{Id} : \mathrm{Sp} \rightleftarrows L_E\mathrm{Sp} : \mathrm{Id}$$

is a Quillen adjunction. Moreover, the categories Sp and $L_E\mathrm{Sp}$ have the same cofibrant objects, but the fibrant objects in $L_E\mathrm{Sp}$ are the ones which are fibrant in Sp and E_*-local. The set of homotopy classes of maps in $\mathrm{Ho}(L_E\mathrm{Sp})$ is denoted

$$[X, Y]^{L_E\mathrm{Sp}} = [L_E X, L_E Y]^{\mathrm{Sp}}.$$

As in Theorem 2.2.12, we have a special version of the Whitehead Theorem in the language of the E_*-local stable homotopy category, which enables us to get global data on spectra from local data.

Lemma 3.3.21. (The E_*-Whitehead Theorem)
If X, $Y \in \mathrm{Ho}(\mathrm{Sp})$ are E_*-local, and $f : X \to Y$ is an E_*-equivalence, then f is an isomorphism in $\mathrm{Ho}(\mathrm{Sp})$.

Lemma 3.3.22. *[Bou79, Lemma 1.4]Suppose we have an exact triangle in* $\mathrm{Ho}(\mathrm{Sp})$

$$W \to X \to Y \to \Sigma W.$$

If any two out of the three spectra W, X, Z are E_*-local, then so is the third.

Proposition 3.3.23. *[Bou79, Lemma 1.10] If we have an exact triangle in* $\mathrm{Ho}(\mathrm{Sp})$

$$W \to X \to Y \to \Sigma W,$$

then its E-localisation

$$L_E W \to L_E X \to L_E Y \to \Sigma(L_E W)$$

remains an exact triangle.

Remark 3.3.24. Bousfield localisation of a stable model category is not necessarily again stable. In [BR14b, Proposition 3.6], the authors construct a characterisation which allows us to verify when the stability of a model category is preserved by Bousfield localisation. In particular, this gives us that the E-local stable homotopy category $L_E\mathrm{Sp}$ is a stable model category itself.

The following lemma is a well-known fact, but we choose to include the proof since we could not find a reference for it.

Lemma 3.3.25. *Let X and Y be in* $\mathrm{Ho(Sp)}$, *then we have the isomorphism*

$$[X, L_EY]_n^{\mathrm{Sp}} \cong [L_EX, L_EY]_n^{\mathrm{Sp}},$$

for all $n \in \mathbb{N}$.

Proof. We have the following morphism of exact triangles

$$\begin{array}{ccccccc}
C_EX & \longrightarrow & X & \xrightarrow{\eta_X} & L_EX & \longrightarrow & \Sigma C_EX \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \eta_X^{-1}} & & \downarrow \\
* & \longrightarrow & X & \xrightarrow{\mathrm{Id}} & X & \longrightarrow & \Sigma * .
\end{array}$$

By Theorem 3.2.19, we have the commutative diagram, in which the upper and lower rows are exact

$$\begin{array}{ccccccc}
[*, L_EY]_{n+1} & \longrightarrow & [X, L_EY]_n & \xrightarrow{\mathrm{Id}^*} & [X, L_EY]_n & \longrightarrow & [*, L_EY]_n \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle (\eta_X^{-1})^*} & & \downarrow{\scriptstyle \mathrm{Id}^*} & & \downarrow{\scriptstyle \cong} \\
[C_EX, L_EY]_{n+1} & \longrightarrow & [L_EX, L_EY]_n & \xrightarrow{\eta_X^*} & [X, L_EY]_n & \longrightarrow & [C_EX, L_EY]_n .
\end{array}$$

We know that

$$[C_EX, L_EY] = 0,$$

because L_EY is E-local and C_EX is E-acyclic. Then by the five lemma, we conclude that

$$[X, L_E Y]_n^{\mathrm{Sp}} \cong [L_E X, L_E Y]_n^{\mathrm{Sp}}. \qquad \square$$

Remark 3.3.26. Note that in general,

$$[L_E X, L_E Y]_n^{\mathrm{Sp}} \cong [X, L_E Y]_n^{\mathrm{Sp}} \ncong [L_E X, Y]_n^{\mathrm{Sp}}.$$

Definition 3.3.27. Localisation at E is said to be *smashing* if for every spectrum X, the map

$$\mathrm{Id} \wedge \eta_{\mathbb{S}^0} : X \to X \wedge L_E \mathbb{S}^0$$

is an E-localisation.

Remark 3.3.28. Notice that in order to prove that a certain E-localisation is smashing, it is sufficient to check if the spectrum $X \wedge L_E \mathbb{S}^0$ is E_*-local because the map

$$Id \wedge \eta_{\mathbb{S}^0} : X \to X \wedge L_E \mathbb{S}^0$$

is always an E_*-equivalence.

Remark 3.3.29. The E-localisation functor L_E is triangulated and preserves generators. However, it does not preserve compactness in general, because the functor L_E does not commute with arbitrary coproducts. However, if the localisation is smashing, then the functor L_E commutes with arbitrary coproducts [Rav84, Proposition 1.27(d)], and the image of a compact generator is again a compact generator. Therefore, the spectrum $L_E \mathbb{S}^0$ is a generator in $\mathrm{Ho}(L_E \mathrm{Sp})$, but it is compact for a smashing localisation.

Example 3.3.30. *If $E = H\mathbb{Q} = M(\mathbb{Q})$, then*

$$L_{\mathbb{Q}} X := L_{H\mathbb{Q}} X \simeq X \wedge L_{\mathbb{Q}} \mathbb{S}^0 \simeq X \wedge H\mathbb{Q}$$

is smashing, and is called the rationalisation of X. In that case, the homotopy groups are

$$\pi_*(L_{\mathbb{Q}} X) \cong \pi_*(X) \otimes \mathbb{Q} \cong H_*(X) \otimes \mathbb{Q}.$$

Another smashing localisation that will be used later on is p-localisation. In other words, it is Bousfield localisation with respect to the p-local Moore spectrum $M(\mathbb{Z}_{(p)})$. Here, $\mathbb{Z}_{(p)}$ is the ring of integers localised at the prime p, that is, the subring of the rational numbers with denominator prime to p.

Proposition 3.3.31. *[Bou79, Proposition 2.4]For each $X \in \mathrm{Ho(Sp)}$, we have*

$$L_{M(\mathbb{Z}_{(p)})}X \simeq M(\mathbb{Z}_{(p)}) \wedge X, \text{ and}$$
$$\pi_*(L_{M(\mathbb{Z}_{(p)})}X) \cong \mathbb{Z}_{(p)} \otimes \pi_* X.$$

An interesting result arising from p-localisation is the *rigidity* of the 2-local stable homotopy category.

Theorem 3.3.32. *[Sch01] Let $\mathcal{C}$ be a stable model category. If we have an equivalence of triangulated categories between $\mathrm{Ho}(\mathcal{C})$ and the 2-local stable homotopy category*

$$\Phi : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathrm{Sp}_{(2)}),$$

then the category $\mathcal{C}$ and the 2-local spectra are Quillen equivalent.

Notation. Localisation of a spectrum X with respect to the p-local Moore spectrum will be denoted $X_{(p)}$.

Definition 3.3.33. The localisation of a spectrum X with respect to the mod-p Moore spectrum $M(\mathbb{Z}/p)$ is the *p-completion* of X denoted $X^\wedge_p$, i.e.

$$X^\wedge_p = L_{M(\mathbb{Z}/p)}X.$$

If a spectrum is $M(\mathbb{Z}/p)$-local, then we call it a *p-complete spectrum.*

Proposition 3.3.34. *[Bou79, Proposition 2.5]*

(a) *For each $X \in \mathrm{Ho(Sp)}$, we have*

$$X^\wedge_p = L_{M(\mathbb{Z}/p)}X \simeq F(\Omega M(\mathbb{Z}/p^\infty), X),$$

where $\mathbb{Z}/p^\infty$ *can be defined as the factor group* $\mathbb{Z}[1/p]/\mathbb{Z}$*, or as the colimit of the groups* $\mathbb{Z}/p^n$ *under multiplication by* p*. Additionally, there is a split short exact sequence*

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_* X) \to \pi_*(L_{M(\mathbb{Z}/p)}X) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{*-1}X) \to 0.$$

(b) *If the groups* $\pi_* X$ *are finitely generated, then*

$$\pi_*(L_{M(\mathbb{Z}/p)}X) \cong \mathbb{Z}_p^\wedge \otimes \pi_* X,$$

where $\mathbb{Z}_p^\wedge$ *denotes the* p*-adic integers, which can be defined as the limit of the group* $\mathbb{Z}/p^n$ *under multiplication by "*$.p$*".*

(c) *A spectrum* $X \in \mathrm{Ho(Sp)}$ *is* $M(\mathbb{Z}/p)$*-local (equivalently* p*-complete) if and only if the groups* $\pi_*(X)$ *are* Ext-p*-complete, i.e. the completion map*

$$\pi_*(X) \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_*(X))$$

is an isomorphism, and

$$\mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_*(X)) = 0.$$

Moreover, p-completion can be described as a homotopy limit as we will discuss in Section 3.4.

Notation. Localisation with respect to p-local complex K-theory $K_{(p)}$, where the coefficient ring is $K_{(p)_*} = \mathbb{Z}_p[v_1, v_1^{-1}]$, $|v_1| = 2p-2$ is denoted

$$L_{K_{(p)}} = L_{E(1)}.$$

The reason behind the above equality is the Adams splitting that we talked about earlier.

Remark 3.3.35. Localisation with respect to $E(n)$, $L_{E(n)}$, is also denoted L_n. By the notation L_0, we refer to the rationalisation, and we write

$$L_0 = L_{\mathbb{Q}} = L_{E(0)} = L_{K(0)} \text{ (see Definition 3.3.8).}$$

Furthermore, by [Rav84, Theorem 2.1]

$$L_{E(n)} = L_n = L_{K(0)\vee K(1)\vee \ldots \vee K(n)}.$$

This also illustrates that we can view localisation with respect to $K(n)$ as the "difference" between L_n and L_{n-1}. Thus in our case we have

$$L_1 = L_{E(1)} = L_{K_{(p)}} = L_{K(0)\vee K(1)}.$$

Remark 3.3.36. Another consequence of [Rav84, Theorem 2.1] to keep in mind is that if a spectrum is $K(1)$-local, it is also $E(1)$-local. But the converse is not true in general. As we will see in Section 4.1 some other condition needs to be added in order to achieve $K(1)$-locality out of $E(1)$-locality.

Another feature of the functors L_n is that, unlike the functors $L_{K(n)}$, they are *smashing*.

Theorem 3.3.37 (Smash product theorem)**.** *[Rav16, Theorem 7.5.6] For any spectrum X,*

$$L_n X \simeq X \wedge L_n \mathbb{S}^0.$$

The smashing property of the functor L_n not only makes some calculations easier, but also preserves the compactness of a generator as we saw in Remark 3.3.29. We conclude that in $\mathrm{Ho}(L_n\mathrm{Sp})$, the spectrum $L_n\mathbb{S}^0$ is a compact generator.

On the other hand, localisation with respect to the n^{th}-Morava K-theory is not smashing for $n > 0$. Although the $K(n)$-local sphere is still a generator, it is not a compact one. However, the following result provides us with a compact generator for $\mathrm{Ho}(L_{K(n)}\mathrm{Sp})$.

Lemma 3.3.38. *[HS99, Theorem 7.3] For a fixed prime p and $n > 0$, the spectrum $L_{K(n)}M(\mathbb{Z}/p)$ is a compact generator for the $K(n)$-local stable homotopy category* $\mathrm{Ho}(L_{K(n)}\mathrm{Sp})$.

Another feature of the $K(1)$-localisation that will become useful in the next chapter is the following result, which enables us to see the $K(1)$-localisation as the p-completion of the $E(1)$-localisation.

Lemma 3.3.39. *[Bou79, Proposition 2.11] [HS99, Propostion 7.10.(e)] For a fixed prime p and X any spectrum in* $\mathrm{Ho}(\mathrm{Sp})$*, we have*

$$L_{K(1)}X = L_{M(\mathbb{Z}/p)}L_1X = (L_1X)^{\wedge}_p.$$

Remark 3.3.40. The *rigidity* question of the E-local stable homotopy category is still wide open and very little is known. One of the open questions in this subject is whether we have rigidity for

$$E = K(1) \text{ at } p = 2.$$

In this thesis, we answer affirmatively this open question. As for the case where

$$E = E(1) \text{ and } p = 2,$$

we have the following result by [Roi07].

Theorem 3.3.41 ($E(1)$-Local Rigidity Theorem)**.** *Let $\mathcal{C}$ be a stable model category, $p = 2$, and let Φ be an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_1\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C}).$$

Then $L_1\mathrm{Sp}$ and $\mathcal{C}$ are Quillen equivalent.

However, if we take p to be odd, it has been shown by Franke [Fra96] that we lose rigidity of $\mathrm{Ho}(L_1\mathrm{Sp})$. Moreover, the author in [Roi07] established a criterion which enables us to check whether a stable model category $\mathcal{C}$ provides an exotic model for $L_1\mathrm{Sp}$ or not.

We will end this section by defining the "v_1-self map" that will be needed later on.

Definition 3.3.42. Let X be a spectrum, a map

$$f : \Sigma^d X \to X$$

is called a *self map* of X. We can iterate it up to suspension by considering the composites

$$\ldots \Sigma^{3d} X \xrightarrow{\Sigma^{2d} f} \Sigma^{2d} X \xrightarrow{\Sigma^{d} f} \Sigma^{d} X \xrightarrow{f} X.$$

For brevity, we denote these composite maps by

$$\begin{cases} f^2 = f \circ \Sigma^d f : \Sigma^{2d} X \to X, \\ f^3 = f^2 \circ \Sigma^{2d} f : \Sigma^{3d} X \to X, \\ \text{etc } \ldots \end{cases}$$

The self map is said to be *nilpotent* if some suspension of f, denoted f^t for some $t > 0$, is null-homotopic.

Definition 3.3.43. Let X be a p-local finite spectrum, and let $n \geq 1$. A v_n*-self map* is a map $f : \Sigma^k X \to X$ with the following properties.

(a) The map f is a $K(n)_*$-equivalence.

(b) For $m \neq n$, the induced map $K(m)_*(X) \to K(m)_*(Y)$ is nilpotent.

Definition 3.3.44. We say that a p-local finite spectrum X has *type* n if

$$K(n)_*(X) \neq 0, \text{ but } K(m)_*(X) = 0 \text{ for } m < n.$$

Example 3.3.45. *A spectrum X has type 0 if*

$$H_*(X, \mathbb{Q}) \ncong 0,$$

or equivalently if $H_i(X, \mathbb{Z})$ is not a torsion group for all i. The p-local sphere spectrum $\mathbb{S}^0_{(p)}$ is a type 0 spectrum which admits the multiplication by p map as a v_0-self map.

Example 3.3.46. *An example of a spectrum of type 1 is the mod-p Moore spectrum $M(\mathbb{Z}/p)$. Actually, since it has no rational homology, we have*

$$K(0)_*(M(\mathbb{Z}/p)) = H_*(M(\mathbb{Z}/p), \mathbb{Q}) = 0.$$

Moreover, if we look at the cofiber sequence

$$\mathbb{S}^0 \xrightarrow{\cdot p} \mathbb{S}^0 \to M(\mathbb{Z}/p),$$

the associated long exact sequence in $K(1)$-homology is of the form

$$\ldots \to K(1)_{i+1}M(\mathbb{Z}/p) \to K(1)_i\mathbb{S}^0 \xrightarrow{\cdot p} K(1)_i\mathbb{S}^0 \to K(1)_iM(\mathbb{Z}/p) \to \ldots$$

Since multiplication by p kills $K(1)_*(\mathbb{S}^0) \cong \mathbb{F}_p[v_1, v_1^{-1}]$, the maps

$$K(1)_*(\mathbb{S}^0) \to K(1)_*(M(\mathbb{Z}/p))$$

are injections, and $K(1)_*(M(\mathbb{Z}/p))$ is non-trivial.

Theorem 3.3.47 (Periodicity Theorem)**.** *[Rav16, Chapter 6] [HS98, §3] Let X be a finite p-local spectrum of type n. Then X admits a v_n-self map*

$$v_n^{p^i} : \Sigma^{p^i d} X \to X, \text{ for some } i \geq 0.$$

Where $d = 0$ if $n = 0$, and $d = 2p^n - 2$ if $n > 0$.

While Theorem 3.3.47 tells us that a v_n-self map exists for certain spectra, it does not address their periodicity. Finding the smallest integer such that the v_n-self map holds is not easy, and the list of known periodic self maps is very limited.

Example 3.3.48. *The earliest known periodic map was constructed on the mod-p Moore spectrum $M(\mathbb{Z}/p)$ by [Ada66], now known as the Adams map. For $p = 2$, it is denoted*

$$v_1^4 : \Sigma^8 M(\mathbb{Z}/2) \to M(\mathbb{Z}/2).$$

Note that there is no smaller degree v_1-self map that can be realised by $M(\mathbb{Z}/2)$. As for p odd, the period of the self map is $2p-2$ and we have a v_1^1-self map

$$v_1^1 : \Sigma^{2p-2} M(\mathbb{Z}/p) \to M(\mathbb{Z}/p).$$

Corollary 3.3.49. *The cofiber of a v_n-self map is of type $n+1$, hence it admits a v_{n+1}-self map.*

Example 3.3.50. *Let $V(1)$ denote the cofibre of the v_1-self map on the Moore spectrum. Starting with $p = 2$, the authors in [BHHM08] proved that $V(1)$, the cofibre of*

$$v_1^4 : \Sigma^8 M(\mathbb{Z}/2) \to M(\mathbb{Z}/2),$$

admits a minimal v_2-self map of the form

$$v_2^{32} : \Sigma^{192} V(1) \to V(1).$$

As for $p = 3$, by [BP04], the cofibre of

$$v_1^1 : \Sigma^{2p-2} M(\mathbb{Z}/p) \to M(\mathbb{Z}/p),$$

has a v_2^9-self map

$$v_2^9 : \Sigma^{144} V(1) \to V(1).$$

Lastly, for $p \geq 5$, the cofibre of

$$v_1^1 : \Sigma^{2p-2} M(\mathbb{Z}/p) \to M(\mathbb{Z}/p),$$

has a v_2^1-self map instead [Smi70].

3.4 Homotopy limits and colimits of spectra

The same way we have limits and colimits of abelian groups, we can mimic this construction in homotopy theory. Colimits and limits in a model category are not usually well-behaved with respect to homotopy equivalences, therefore they are not homotopy invariants. The homotopy (co)limit functor can be thought of as a correction to the (co)limit, modifying it so the result is homotopy invariant.

The main reference used in this section for the constructions of homotopy (co)limits in the stable homotopy category is [Rav16, A.5]. However, for a more detailed construction we refer the reader to [BK72, Chapter XI, XII]. Note that this is an old version of these constructions, a more modern construction of homotopy (co)limits in the language of model categories can be found in [Str11]. Nevertheless, we choose the old version because in the proof of our main theorem we need to see those homotopy (co)limits in the context displayed in this section.

We start this section by constructing the homotopy (co)limit in Ho(Sp). Afterwards, we will see how we can view localisation with respect to some homology theories as holim/hocolim. We will end this section by a special example of homotopy limits which are the homotopy pullback squares, and see how it relates to Bousfield localisation.

We begin by constructing the colimit (colim_i) of abelian groups since those will be needed in the definition of hocolim in the stable homotopy category. It is a well-known fact that colimits of abelian groups do exist and can be described as follows.

Proposition 3.4.1. *Let $(A_i, f_i)_{i\in\mathbb{N}}$ be a sequential direct system of abelian groups, where $f_i : A_i \to A_{i+1}$ is a homomorphism. Define the shift homomorphism*

$$s : \bigoplus_{i\in\mathbb{N}} A_i \to \bigoplus_{i\in\mathbb{N}} A_i, \text{ such that } s(a) = a - f_i(a) \text{ for } a \in A_i.$$

Then the cokernel of this shift map is the colimit (or direct limit) of the direct *system* $(A_i, f_i)_{i\in\mathbb{N}}$

$$\operatorname*{colim}_i A_i = \operatorname{coker}(s).$$

Proposition 3.4.2. *[Rav16, Section A.5] Let* $(X_i, f_i)_{i\in\mathbb{N}}$ *be a sequential direct system of spectra in* $\mathrm{Ho}(\mathrm{Sp})$*. By the axioms of the stable homotopy category [Mar83, Chapter 2 §1], the infinite coproduct or wedge* $\bigvee_{i\in\mathbb{N}} X_i$ *exists in* $\mathrm{Ho}(\mathrm{Sp})$ *and is defined by*

$$\left(\bigvee_{i\in\mathbb{N}} X_i\right)_n = \bigvee_{i\in\mathbb{N}} (X_i)_n .$$

It has the universal property that a collection of maps $g_i : X_i \to Y$ *leads to a unique map*

$$f : \bigvee_{i\in\mathbb{N}} X_i \to Y.$$

Moreover, we have

$$\pi_*(\bigvee_{i\in\mathbb{N}} X_i) = \bigoplus_{i\in\mathbb{N}} \pi_*(X_i).$$

Another property is that it distributes over smash product in the expected way.

Proposition 3.4.3. *[Swi75, Proposition 13.48] For any spectrum* $E \in \mathrm{Ho}(\mathrm{Sp})$*, we have a natural homotopy equivalence*

$$E \wedge (\bigvee_{i\in\mathbb{N}} X_i) \simeq \bigvee_{i\in\mathbb{N}} (E \wedge X_i).$$

Consequently, we have

$$E_*(\bigvee_{i\in\mathbb{N}} X_i) \cong \bigoplus_{i\in\mathbb{N}} E_*(X_i).$$

Within the same concept of defining the colimit of abelian groups as the cokernel of the shift map, we define the homotopy colimit as the cofibre of a shift map.

Proposition 3.4.4. *[Rav16, Section A.5]Let* $(X_i, f_i)_{i\in\mathbb{N}}$ *be a sequential direct system of spectra in* $\mathrm{Ho}(\mathrm{Sp})$*. Consider* $\bigvee_{i\in\mathbb{N}} X_i$*, the coproduct defined in Proposition 3.4.2.*

Then, there is a shift map

$$\sigma : \bigvee_{i\in\mathbb{N}} X_i \to \bigvee_{i\in\mathbb{N}} X_i$$

inducing the shift homomorphism of Proposition 3.4.1 in homology

$$s : \bigoplus_{i\in\mathbb{N}} E_*(X_i) \to \bigoplus_{i\in\mathbb{N}} E_*(X_i).$$

Hence, we can define the homotopy colimit of spectra as the cofibre of σ

$$\operatorname{hocolim}_i(X_i) = C_\sigma.$$

We end up with the following exact triangle in Ho(Sp)

$$\Omega C_\sigma \to \bigvee_{i\in\mathbb{N}} X_i \xrightarrow{\sigma} \bigvee_{i\in\mathbb{N}} X_i \to C_\sigma.$$

Remark 3.4.5. Since the wedge of spectra commutes with the smash product, the homotopy colimit commutes with it as well

$$\operatorname{hocolim}_i(X_i \wedge X) \simeq (\operatorname{hocolim}_i X_i) \wedge X.$$

The same applies for homology,

$$E_*(\operatorname{hocolim}_i X_i) = \operatorname{colim}_i E_*(X_i),$$

where colim_i is the colimit of abelian groups defined in Proposition 3.4.1.

Remark 3.4.6. The hocolim of E_*-local spectra is not necessarily E_*-local. See [Rav84, Example 1.9] for a counterexample.

Homotopy limits are defined in a similar way. First, we need to construct infinite products in the stable homotopy category. More information about the holim can be found in [Ada95, III].

Proposition 3.4.7. *[Rav16, Proposition A.4.3] Suppose we have a sequential inverse system of spectra* $(X_i, f_i)_{i\in\mathbb{N}}$ *in* Ho(Sp)*, where* $f_i : X_i \to X_{i-1}$ *is a morphism* *of spectra. There is a infinite product* $\prod_i X_i$ *satisfying*

$$[Y, \prod_i X_i]_* \cong \prod_i [Y, X_i]_*, \text{ for any } Y \in \mathrm{Ho}(\mathrm{Sp}).$$

Note that this product does not behave well with respect to the smash product, i.e.

$$\prod_i (X_i \wedge Y) \not\simeq (\prod_i X_i) \wedge Y$$

(this is analogous to the fact that infinite products of abelian groups do not commute with tensor products). Plus, it does not necessarily commute with Bousfield localisation L_E.

The same way as before, we construct a shift map on the product and define the holim_i as the fiber.

Proposition 3.4.8. *[Rav16, Section A.5] [Bou79, Lemma 1.8] Suppose we have a sequential inverse system of spectra* $(X_i, f_i)_{i\in\mathbb{N}}$ *in* $\mathrm{Ho}(\mathrm{Sp})$*. Let* $\prod_i X_i$ *be the infinite product defined in Proposition 3.4.7. By the universal property of the product* $\prod_i X_i$*, we can construct a shift map*

$$\prod_i X_i \xrightarrow{\sigma} \prod_i X_i$$

by describing the composite

$$\prod_i X_i \xrightarrow{\sigma} \prod_i X_i \xrightarrow{p_j} X_j$$

for each $j \in \mathbb{N}$*, where* p_j *is the evident projection. The map* σ *we want is given by*

$$p_j \circ \sigma = p_j - f_j \circ p_{j+1}.$$

The homotopy limit *of spectra*

$$\operatorname{holim}_i X_i$$

is the fibre F_σ *of the shift map defined above. Therefore, we have the exact triangle* *in Ho(Sp)*

$$F_\sigma \to \prod_i X_i \xrightarrow{\sigma} \prod_i X_i \to \Sigma F_\sigma.$$

Even though the holim does not commute with the smash product, it has a useful property and that is, unlike the hocolim, it preserves E-local spectra.

Proposition 3.4.9. *[Rav84, Proposition 1.7] The homotopy limit* holim *of E_*-local spectra is E_*-local.*

As mentioned earlier, some Bousfield localisations can be seen as homotopy colimits and limits.

Proposition 3.4.10. *[Bou79, Proposition 4.2] Let M denote the mod-2 Moore spectrum $M(\mathbb{Z}/2)$. The localisation of M with respect to $E(1)$ is the homotopy colimit of the sequence formed by the self-map on $M(\mathbb{Z}/2)$, i.e.*

$$\mathrm{hocolim}(M \xrightarrow{v_1^4} \Sigma^{-8} M \xrightarrow{v_1^4} \Sigma^{-16} M \xrightarrow{v_1^4} \dots) \simeq L_1 M.$$

The reference for the following proposition is [Bou79, Theorem 2.5]. We choose to include the proof since it will be needed later on to prove the corollary resulting from this proposition.

Proposition 3.4.11. *For a prime p, as we have seen in Proposition 3.3.34, the p-completion of a spectrum X is the function spectrum*

$$X_p^\wedge \simeq F(M(\mathbb{Z}/p^\infty), \Sigma X).$$

We can conclude that the p-completion $X_p^\wedge$ is the homotopy limit of

$$\dots \xrightarrow{F(\Psi_3, \Sigma X)} F(M(\mathbb{Z}/p^3), \Sigma X) \xrightarrow{F(\Psi_2, \Sigma X)} F(M(\mathbb{Z}/p^2), \Sigma X) \xrightarrow{F(\Psi_1, \Sigma X)} F(M(\mathbb{Z}/p), \Sigma X),$$

where $\Psi_n : M(\mathbb{Z}/p^n) \to M(\mathbb{Z}/p^{n+1})$ realizes the map $p\cdot : \mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$ on integral homology.

Proof. The Moore spectrum $M(\mathbb{Z}/p^\infty)$ is the homotopy colimit of the sequence

$$M(\mathbb{Z}/p) \xrightarrow{\psi_1} M(\mathbb{Z}/p^2) \xrightarrow{\psi_2} M(\mathbb{Z}/p^3) \xrightarrow{\psi_3} \dots$$

because the group $\mathbb{Z}/p^\infty$ is the colimit of the groups $\mathbb{Z}/p^n$ under multiplication by the $p\cdot$ maps. Since the exact functor $F(-, \Sigma X)$ is contravariant, it takes homotopy colimits to homotopy limits. Therefore, the spectrum

$$X_p^\wedge \simeq F(M(\mathbb{Z}/p^\infty), \Sigma X)$$

is the homotopy limit of

$$\dots \xrightarrow{F(\Psi_2, \Sigma X)} F(M(\mathbb{Z}/p^2), \Sigma X) \xrightarrow{F(\Psi_1, \Sigma X)} F(M(\mathbb{Z}/p), \Sigma X). \qquad \square$$

Note that the following corollary is stated as Remark 9.11 in [Sch07b].

Corollary 3.4.12. *As a consequence of Proposition 3.4.11, the p-completion of a spectrum X is*

$$X_p^\wedge \simeq \operatorname{holim}(\dots \to M(\mathbb{Z}/p^3) \wedge X \to M(\mathbb{Z}/p^2) \wedge X \to M(\mathbb{Z}/p) \wedge X).$$

Proof. The goal is to construct an isomorphism between the inverse systems of Proposition 3.4.11 and the corollary we are proving. First, we need to make precise the maps of the inverse system

$$\dots \to M(\mathbb{Z}/p^3) \wedge X \to M(\mathbb{Z}/p^2) \wedge X \to M(\mathbb{Z}/p) \wedge X.$$

As we have seen in Lemma 3.3.11, the suspension $\Sigma(DM(\mathbb{Z}/p^n))$ is a mod-p^n Moore spectrum. Moreover, we can choose an isomorphism

$$j_n : M(\mathbb{Z}/p^n) \to \Sigma(DM(\mathbb{Z}/p^n))$$

in Ho(Sp), such that the composition

$$\rho_n : M(\mathbb{Z}/p^{n+1}) \xrightarrow{j_{n+1}} \Sigma(DM(\mathbb{Z}/p^{n+1})) \xrightarrow{\Sigma D\psi_n} \Sigma(DM(\mathbb{Z}/p^n)) \xrightarrow{j_n^{-1}} M(\mathbb{Z}/p^n)$$

realises the reduction

$$\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$$

on homology. Now, we can construct the desired isomorphism between the inverse systems

$$(F(M(\mathbb{Z}/p^n), \Sigma X), F(\Psi_n, \Sigma X)) \text{ and } (M(\mathbb{Z}/p^n) \wedge X, \rho_n \wedge X).$$

For any spectrum X, the composite

$$M(\mathbb{Z}/p^n) \wedge X \xrightarrow{j_n \wedge \mathrm{Id}_X} \Sigma F(M(\mathbb{Z}/p^n), \mathbb{S}^0) \wedge X \to F(M(\mathbb{Z}/p^n), \Sigma X)$$

is an isomorphism, where the second isomorphism comes from the fact that

$$F(M(\mathbb{Z}/p^n), \Sigma X) \simeq \Sigma DM(\mathbb{Z}/p^n) \wedge X.$$

We have the commutative diagram in Ho(Sp)

$$\begin{array}{ccc} M(\mathbb{Z}/p^{n+1}) \wedge X & \xrightarrow{\rho_n \wedge X} & M(\mathbb{Z}/p^n) \wedge X \\ \simeq \downarrow & & \downarrow \simeq \\ F(M(\mathbb{Z}/p^{n+1}), \Sigma X) & \xrightarrow[F(\psi_n, \Sigma X)]{} & F(M(\mathbb{Z}/p^n), \Sigma X). \end{array}$$

Hence, the homotopy limits of the two systems are homotopy equivalent, and the p-completion $X_p^\wedge$ is also the homotopy limit of the sequence

$$\dots \xrightarrow{\rho_3 \wedge X} M(\mathbb{Z}/p^3) \wedge X \xrightarrow{\rho_2 \wedge X} M(\mathbb{Z}/p^2) \wedge X \xrightarrow{\rho_1 \wedge X} M(\mathbb{Z}/p) \wedge X. \qquad \square$$

In a model category, the homotopy pullback can be seen as a homotopy approximation of strict pullbacks. For a detailed construction of the homotopy pullback, see [DS95, Section 10].

Definition 3.4.13. Let A, B and C be objects in a model category $\mathcal{C}$. Consider a diagram

$$B \to A \leftarrow C.$$

Let

$$A \overset{\sim}{\rightarrowtail} A^{\mathrm{f}}$$

be a fibrant replacement of A. Applying MC5(ii), we factor the maps $B \to A^{\mathrm{f}}$ and $C \to A^{\mathrm{f}}$ into trivial cofibrations followed by fibrations

$$\begin{array}{ccccc} B & \longrightarrow & A & \longleftarrow & C \\ \sim\big\downarrow & & \sim\big\downarrow & & \sim\big\downarrow \\ B' & \longrightarrow & A^{\mathrm{f}} & \longleftarrow & C'. \end{array}$$

The (strict) pullback of

$$B' \to A^{\mathrm{f}} \leftarrow C'$$

is the *homotopy pullback* of

$$B \to A \leftarrow C.$$

The following result is well-known, but we choose to include a proof since we could not find a reference stating the result in the form we wanted.

Proposition 3.4.14. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array}$$

be a homotopy pullback square in Ho(Sp)*. Then for any X in* Ho(Sp)*, we have the following long exact sequence*

$$\ldots \to [X, A]_n^{\mathrm{Sp}} \to [X, B]_n^{\mathrm{Sp}} \oplus [X, C]_n^{\mathrm{Sp}} \to [X, D]_n^{\mathrm{Sp}} \to [X, A]_{n-1}^{\mathrm{Sp}} \to \ldots$$

Proof. By [Str11, Theorem 8.39], the homotopy pullback square gives us the following fiber sequence in Ho(Sp)

$$\Omega D \to A \to B \vee C.$$

The desired long exact sequence follows by Theorem 3.2.19. □

The following proposition is a widely known result. A nice proof can be found in [Bau11, Proposition 2.2].

Proposition 3.4.15. *Let E, F and X be spectra with $E_*(L_F X) = 0$. Then there is a homotopy pullback square*

$$\begin{array}{ccc} L_{E\vee F}X & \xrightarrow{r} & L_E X \\ {\scriptstyle r'}\downarrow & & \downarrow{\scriptstyle \eta_F} \\ L_F X & \xrightarrow[L_F(\eta_E)]{} & L_F L_E X. \end{array}$$

The map r in the diagram is the unique factorisation, as in Definition 3.3.15, of $\eta_E : X \to L_E X$ through $L_{E\vee F}X$, which exists because the map $X \to L_{E\vee F}X$ is an E_*-equivalence. The same holds for r', but in this case as the unique factorisation of $\eta_F : X \to L_F X$ since the map $X \to L_{E\vee F}X$ is an F_*-equivalence as well.

By applying the above proposition to

$$E = \bigvee_p M(\mathbb{Z}/p) \text{ and } F = M\mathbb{Q},$$

we get the *Sullivan arithmetic square*, which allows us to reconstruct a space if all its mod-p-localisations and its rationalisation are known.

Lemma 3.4.16. *[Bou79, Proposition 2.9] For any spectrum X, we have the following homotopy pullback square*

$$\begin{array}{ccc} X & \xrightarrow{\prod \eta_p} & \prod_p L_{M(\mathbb{Z}/p)}X \\ {\scriptstyle \eta_{\mathbb{Q}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathbb{Q}}} \\ L_{\mathbb{Q}}X & \xrightarrow[L_{\mathbb{Q}}(\prod_p \eta_p)]{} & L_{\mathbb{Q}}(\prod_p L_{M(\mathbb{Z}/p)}X). \end{array}$$

Another homotopy pullback square obtained from Proposition 3.4.15 is the *chromatic fracture square*, in that case we take

$$E = K(n) \text{ and } F = K(m), \text{ such that } m < n.$$

Lemma 3.4.17. *Let X be a spectrum. Then we have the following homotopy pullback square*

$$\begin{array}{ccc} L_{K(m)\vee K(n)}X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{K(m)}X & \longrightarrow & L_{K(m)}L_{K(n)}X, \end{array}$$

for $m < n$.

In particular, for $m = 0$ and $n = 1$, we have the following homotopy pullback square relating $E(1)$-localisation to $K(1)$-localisation

$$\begin{array}{ccc} L_1X & \longrightarrow & L_{K(1)}X \\ \downarrow & & \downarrow \\ L_0X & \longrightarrow & L_0L_{K(1)}X. \end{array}$$

Chapter 4

Proof of the $K(1)$ - local rigidity

This chapter is the heart of this thesis, in which we prove a new result of rigidity in stable homotopy theory.

$K(1)$-Local Rigidity Theorem. *Let $\mathcal{C}$ be a stable model category, $p = 2$, and let Φ be an exact equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C}).$$

Then the underlying model categories $L_{K(1)}\mathrm{Sp}$ and $\mathcal{C}$ are Quillen equivalent.

In order to establish the proof of our main theorem, we divide this chapter into three sections. The first section is the starting point, where we construct a new characterisation to detect $K(1)$-locality. In the second section, we construct the Quillen adjunction using the result of Section 4.1. Lastly, in Section 4.3 we finalise the proof of our main result, and the Quillen adjunction is proved to be a Quillen equivalence. In the process of proving our main result, the knowledge of certain homotopy groups in the $K(1)$-local setting is crucial. In order to make the task of reading this chapter easier and the proofs more comprehensible, we choose to write the details of all these computations separately in Chapter 5.

Notation. From now on, let $p = 2$, and let Sp denote the category of 2-local spectra. The mod-2 Moore spectrum $M(\mathbb{Z}/2)$ will be denoted by M.

4.1 From E(1)-locality to K(1)-locality

As we have seen in Example 3.3.48, the mod-2 Moore spectrum M has a periodic v_1^4-self map

$$v_1^4 : \Sigma^8 M \to M.$$

In [Bou79], a criterion involving this v_1^4-self map has been developed to show that a spectrum is $E(1)$-local.

Lemma 4.1.1. *[Bou79, §4] A spectrum X is $E(1)$-local if and only if v_1^4 induces an isomorphism*

$$(v_1^4)^* : [M, X]_n^{Sp} \to [M, X]_{n+8}^{Sp}, \text{ for all } n \in \mathbb{Z}.$$

In this section we extend this result to $K(1)$-locality by adding another condition. First, we need the following lemma.

Lemma 4.1.2. *For any spectrum $X \in \mathrm{Ho}(\mathrm{Sp})$, we have*

$$L_1(M \wedge X) \simeq L_{K(1)}(M \wedge X).$$

Proof. By Lemma 3.4.17, we have the following homotopy pullback square

$$\begin{array}{ccc} L_1 Y & \longrightarrow & L_{K(1)} Y \\ \downarrow & & \downarrow \\ L_0 Y & \longrightarrow & L_0 L_{K(1)} Y. \end{array}$$

Therefore, we have that if $L_0 Y \simeq *$ and $L_0 L_{K(1)} Y \simeq *$ then $L_1 Y \simeq L_{K(1)} Y$. This is the case for

$$Y = X \wedge M(\mathbb{Z}/2) =: X/2.$$

First, let us prove that $L_0(X \wedge M) \simeq *$. Similarly to Example 3.2.20, the long exact homotopy sequence of the exact triangle

$$X \xrightarrow{.2} X \xrightarrow{\text{incl}} X/2 \to \Sigma X$$

splits into short exact sequences of the form

$$0 \to (\pi_{m+1}X)/2 \to \pi_{m+1}(X/2) \to \{\pi_m X\}_2 \to 0.$$

As before, $\{\pi_m X\}_2$ denotes the 2-torsion of the group $\pi_m X$. Since tensoring with $\mathbb{Q}$ preserves exactness, we have

$$\pi_{m+1}(X/2) \otimes \mathbb{Q} \cong 0 \cong \pi_{m+1}(L_0(X/2)),$$

therefore $L_0(X/2) \simeq *$.

The same applies to $L_0 L_{K(1)}(X/2)$. We can see that by tensoring the following short exact sequence with $\mathbb{Q}$

$$0 \to (\pi_{m+1}L_{K(1)}X)/2 \to \pi_{m+1}L_{K(1)}(X/2) \to \{\pi_m L_{K(1)}X\}_2 \to 0,$$

we will have that

$$L_0 L_{K(1)}(X/2) \simeq *.$$

Hence, $L_1(M \wedge X) \simeq L_{K(1)}(M \wedge X)$ as desired. □

Remark 4.1.3. Even though the above lemma is written in the 2-local world, we can replace $p = 2$ in the proof by any prime p and the lemma will still be correct in the p-local setting.

Lemma 4.1.4. *A 2-complete spectrum X is $K(1)$-local if and only if v_1^4 induces an isomorphism*

$$(v_1^4)^* : [M, X]_n^{Sp} \to [M, X]_{n+8}^{Sp}$$

for all $n \in \mathbb{Z}$.

Proof. First, suppose that the spectrum X is $K(1)$-local. As we have seen in

Definition 3.3.43, the map v_1^4 induces a $K(1)_*$-isomorphism on M. Thus, its cofibre, denoted by $V(1)$, is $K(1)_*$-acyclic. The desired isomorphism is deduced from the long exact sequence

$$\ldots \to [V(1), X]_{n+1} \to [M, X]_n \xrightarrow{(v_1^4)^*} [\Sigma^8 M, X]_n \to [V(1), X]_n \to \ldots,$$

since by hypothesis $[V(1), X]_n = 0$ for all n.

To prove the other direction, we first note that the assumption is equivalent to

$$(v_1^4 \wedge X)^* : \pi_n(M \wedge X) \to \pi_{n+8}(M \wedge X)$$

being an isomorphism for all n because, as we proved in Lemma 3.3.11, the spectrum M is its own Spanier-Whitehead dual up to suspension

$$\Sigma(DM) \simeq M.$$

We conclude that

$$\text{hocolim}(M \wedge X \xrightarrow{v_1^4 \wedge X} \Sigma^{-8} M \wedge X \xrightarrow{v_1^4 \wedge X} \Sigma^{-16} M \wedge X \to \ldots) \simeq M \wedge X,$$

because all the arrows are weak equivalences. However, as we have seen in Proposition 3.4.10, we have that

$$\text{hocolim}(M \xrightarrow{v_1^4} \Sigma^{-8} M \xrightarrow{v_1^4} \Sigma^{-16} M \xrightarrow{v_1^4} \ldots) \simeq L_1 M.$$

We conclude that in our case,

$$M \wedge X \simeq (L_1 M) \wedge X$$

since unlike holim, hocolim commutes with the smash product "$\wedge$" as noted in Remark 3.4.5. However, by the Smash product theorem 3.3.37, L_1 is smashing, which tells us that

$$(L_1 M) \wedge X \simeq (L_1 \mathbb{S}^0 \wedge M) \wedge X \simeq L_1 \mathbb{S}^0 \wedge (M \wedge X) \simeq L_1(M \wedge X).$$

Consequently,

$$M \wedge X \simeq (L_1 M) \wedge X \simeq L_1(M \wedge X).$$

On the other hand, by Lemma 4.1.2

$$L_1(M \wedge X) \simeq L_{K(1)}(M \wedge X).$$

We conclude that $M \wedge X$ is $K(1)$-local.

By induction, we prove that $M(\mathbb{Z}/2^n) \wedge X$ is $K(1)$-local for all n. First, by applying the octahedral axiom (TR5 of Definition 3.2.3) to the diagram

$$\begin{array}{ccccccc}
\mathbb{S}^0 \wedge X & \xrightarrow{2\mathrm{Id}_{\mathbb{S}^0} \wedge X} & \mathbb{S}^0 \wedge X & \longrightarrow & M \wedge X & \longrightarrow & \mathbb{S}^1 \wedge X \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle 2^{n-1}\mathrm{Id}_{\mathbb{S}^0} \wedge X} & & \vdots & & \downarrow{\scriptstyle \mathrm{Id}} \\
\mathbb{S}^0 \wedge X & \xrightarrow{2^n\mathrm{Id}_{\mathbb{S}^0} \wedge X} & \mathbb{S}^0 \wedge X & \longrightarrow & M(\mathbb{Z}/2^n) \wedge X & \longrightarrow & \mathbb{S}^1 \wedge X \\
 & & \downarrow & & \vdots & & \\
 & & M(\mathbb{Z}/2^{n-1}) \wedge X & \xrightarrow{\mathrm{Id}} & M(\mathbb{Z}/2^{n-1}) \wedge X & & \\
 & & \downarrow & & \vdots & & \\
 & & \mathbb{S}^1 \wedge X & \longrightarrow & \Sigma M(\mathbb{Z}/2) \wedge X, & &
\end{array}$$

we obtain the following exact triangle in Ho(Sp)

$$M \wedge X \to M(\mathbb{Z}/2^n) \wedge X \to M(\mathbb{Z}/2^{n-1}) \wedge X \to \Sigma M(\mathbb{Z}/2) \wedge X.$$

If we suppose that $M(\mathbb{Z}/2^{n-1}) \wedge X$ is $K(1)$-local, then by a two out of three argument (Lemma 3.3.22) we deduce that $M(\mathbb{Z}/2^n) \wedge X$ is $K(1)$-local as well. By Corollary 3.4.12, the 2-completion of X denoted $X_2^\wedge$ is the homotopy limit (holim) of

$$\ldots \to M(\mathbb{Z}/2^n) \wedge X \to \ldots \to M(\mathbb{Z}/2^2) \wedge X \to M \wedge X.$$

As we have seen in Proposition 3.4.9, the homotopy limit of E_*-local spectra is E_*-local. Since every term of the above sequence is $K(1)$-local, the spectrum $X_2^\wedge$ is $K(1)$-local. As X is 2-complete (equivalently M-local), this must mean that X itself is $K(1)$-local. □

4.2 The Quillen functor pair

In order to obtain a Quillen equivalence between $L_{K(1)}\mathrm{Sp}$ and $\mathcal{C}$, we first need a Quillen adjunction between those categories. Forgetting the $K(1)$-local structure, Quillen adjunctions between spectra Sp and any stable model category have been studied first in [SS02] and were later generalised in [Len12].

Theorem 4.2.1. *[Len12, Section 6] Let $\mathcal{C}$ be a stable model category and $X \in \mathcal{C}$ a fibrant and cofibrant object. Then there is a Quillen adjunction*

$$X \wedge - : \mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

such that $X \wedge \mathbb{S}^0 \simeq X$.

Notation. The left derived functor of $X \wedge - : \mathrm{Sp} \to \mathcal{C}$ is denoted

$$X \wedge^L - : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C}),$$

and the right derived functor of $\mathrm{Hom}(X, -)$ is denoted

$$\mathrm{RHom}(X, -) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathrm{Sp}).$$

Looking at Sp and $L_{K(1)}\mathrm{Sp}$ as categories, they are the same, however, they have different model structures. Therefore, Theorem 4.2.1 gives us only an adjunction

$$X \wedge - : L_{K(1)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -).$$

We do not know if the above adjunction respects the model structures, and hence we cannot yet say that it is a *Quillen* adjunction.

Therefore, the goal now is to show that the Quillen adjunction of Theorem 4.2.1 also gets us a Quillen adjunction on $K(1)$-local spectra $L_{K(1)}\mathrm{Sp}$ as follows

$$\begin{array}{ccc} \mathrm{Sp} & \xrightarrow{X\wedge -} & \mathcal{C} \\ \downarrow{\scriptstyle \mathrm{Id}} & \nearrow{\scriptstyle X\wedge -} & \\ L_{K(1)}\mathrm{Sp}. & & \end{array}$$

To that end we use the following result by [BR11, Proposition 7.8].

Proposition 4.2.2. *As before, let $\mathcal{C}$ be a stable model category and $X \in \mathcal{C}$ a fibrant and cofibrant object. The Quillen adjunction*

$$X \wedge - : \mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

extends to a Quillen adjunction

$$X \wedge - : L_{K(1)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

if and only if the spectrum $\mathrm{RHom}(X, Y)$ *is $K(1)$-local for all $Y \in \mathcal{C}$.*

Important Notation. *For the rest of the thesis, let $\Phi : \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C})$ be an equivalence of triangulated categories, and X a fibrant-cofibrant replacement of $\Phi(L_{K(1)}\mathbb{S}^0)$. So from now on, unless stated, whenever we mention the spectrum X it means we are considering a precise choice of this spectrum which is a fibrant-cofibrant replacement of $\Phi(L_{K(1)}\mathbb{S}^0)$.*

In order to show that $\mathrm{RHom}(X, Y)$ is $K(1)$-local for all Y, we use Lemma 4.1.4. However, before being able to apply Lemma 4.1.4, we need to prove that $\mathrm{RHom}(X, Y)$ is 2-complete, to that end we use Proposition 3.3.34 (c).

Lemma 4.2.3. *The spectrum* $\mathrm{RHom}(X, Y)$ *is 2-complete for all* $Y \in \mathcal{C}$.

Proof. By Proposition 3.3.34 (c), in order to prove that the spectrum $\mathrm{RHom}(X, Y)$ is 2-complete for all $Y \in \mathcal{C}$, it is enough to show that the groups $\pi_*(\mathrm{RHom}(X, Y))$ are Ext-2-complete. However, the latter fact is the same as the following equivalent statements.

(i) The groups $[X \wedge^L \mathbb{S}^0, Y]_*^{\mathcal{C}}$ are Ext-2-complete, since we have an adjunction $(X \wedge^L -, \mathrm{RHom}(X, -))$.

(ii) The groups $[X, Y]_*^{\mathcal{C}}$ are Ext-2-complete since $X \wedge^L \mathbb{S}^0 \cong X$.

(iii) The groups $[\Phi^{-1}(X), \Phi^{-1}(Y)]_*^{L_{K(1)}\mathrm{Sp}}$ are Ext-2-complete (Φ is an equivalence of categories).

(iv) The groups $[\mathbb{S}^0, \Phi^{-1}(Y)]_*^{L_{K(1)}\mathrm{Sp}}$ are Ext-2-complete, because $\Phi^{-1}(X) \cong L_{K(1)}\mathbb{S}^0$.

(v) The groups $[\mathbb{S}^0, L_{K(1)}\Phi^{-1}(Y)]_*^{\mathrm{Sp}}$ are Ext-2-complete, as a consequence of the isomorphism $[A, B]^{L_{K(1)}\mathrm{Sp}} \cong [A, L_{K(1)}B]^{\mathrm{Sp}}$.

The last statement is the same as saying that the spectrum $L_{K(1)}\Phi^{-1}(Y)$ is 2-complete, which is indeed true, since by Lemma 3.3.39 we have

$$L_{K(1)}\Phi^{-1}(Y) = \left(L_1\Phi^{-1}(Y)\right)_2^{\wedge}. \qquad \square$$

Lemma 4.2.4. *The image of the mod-2 Moore spectrum* M *by the left derived functor*

$$X \wedge^L - : \mathrm{Ho}(\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C})$$

is

$$X \wedge^L M \cong \Phi(L_{K(1)}M) \cong \Phi(M).$$

Proof. Remember that the left derived functor $X \wedge^L -$ of the Quillen functor in Theorem 4.2.1 and the equivalence Φ are both exact. Therefore, from the isomorphisms

$$X \wedge^L \mathbb{S}^0 \cong X \cong \Phi(L_{K(1)}\mathbb{S}^0),$$

we can see that both $X \wedge^L M$ and $\Phi(L_{K(1)}M)$ are the cofibre of the multiplication by 2 on the object X. Thus, we deduce the first isomorphism

$$X \wedge^L M \cong \Phi(L_{K(1)}M).$$

As for the last isomorphism of the lemma, it is a consequence of the isomorphism

$$M \cong L_{K(1)}M$$

in the homotopy category $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$. □

Remark 4.2.5. We will need to know, on several occasions, what are the images of the maps incl and pinch under certain functors. So we choose to gather this information in this remark and use it when appropriate.

(a) First, we analyse the images of the maps incl and pinch by the functor $X \wedge^L -$. As we have seen in Lemma 4.2.4, we can choose an isomorphism in $\mathrm{Ho}(\mathcal{C})$

$$X \wedge^L M \cong \Phi(L_{K(1)}M)$$

completing the diagram

$$\begin{array}{ccccccc}
X \wedge^L \mathbb{S}^0 & \xrightarrow{2(X\wedge^L \mathrm{Id}_{\mathbb{S}^0})} & X \wedge^L \mathbb{S}^0 & \xrightarrow{X\wedge^L \mathrm{incl}} & X \wedge^L M & \xrightarrow{X\wedge^L \mathrm{pinch}} & X \wedge^L \mathbb{S}^1 \\
\cong\downarrow & & \downarrow\cong & & \vdots\cong & & \downarrow\cong \\
\Phi(L_{K(1)}\mathbb{S}^0) & \xrightarrow{2\Phi(\mathrm{Id}_{L_{K(1)}\mathbb{S}^0})} & \Phi(L_{K(1)}\mathbb{S}^0) & \xrightarrow{\Phi(\mathrm{incl})} & \Phi(L_{K(1)}M) & \xrightarrow{\Phi(\mathrm{pinch})} & \Phi(L_{K(1)}\mathbb{S}^1),
\end{array}$$

such that $X \wedge^L \mathrm{incl}$ corresponds to $\Phi(\mathrm{incl})$, and $X \wedge^L \mathrm{pinch}$ corresponds to $\Phi(\mathrm{pinch})$. Hence, we have the following equalities in $\mathrm{Ho}(\mathcal{C})$

$$X \wedge^L \mathrm{incl} = \Phi(\mathrm{incl}),$$

$$X \wedge^L \mathrm{pinch} = \Phi(\mathrm{pinch}).$$

(b) Now, we wonder what happens to the maps incl and pinch when we apply the homology theory $K(1)_*$. By applying the homology theory $K(1)_*$ to the exact triangle

$$\mathbb{S}^0 \xrightarrow{2\cdot} \mathbb{S}^0 \xrightarrow{\text{incl}} M \xrightarrow{\text{pinch}} \mathbb{S}^1,$$

we obtain the long exact sequence

$$\ldots \to K(1)_m(\mathbb{S}^0) \xrightarrow{K(1)_m(2)} K(1)_m\mathbb{S}^0 \xrightarrow{K(1)_m(\text{incl})} K(1)_m M \xrightarrow{K(1)_m(\text{pinch})} K(1)_{m-1}\mathbb{S}^0 \ldots$$

Remember that

$$K(1)_* := K(1)_*(\mathbb{S}^0) \cong \mathbb{F}_2[v_1, v_1^{-1}],\ |v_1| = 2.$$

This means that $K(1)_m(\mathbb{S}^0)$ is concentrated only in even degrees. Plus, for any m, notice that the maps $K(1)_m(2)$ are zero because multiplication by 2 annihilates $K(1)_*(\mathbb{S}^0)$. For m even, we have that $K(1)_{m-1}(\mathbb{S}^0) = 0$, and the long exact sequence is now of the form

$$\ldots \to K(1)_m(\mathbb{S}^0) \xrightarrow{0} K(1)_m(\mathbb{S}^0) \xrightarrow{K(1)_m(\text{incl})} K(1)_m(M) \xrightarrow{K(1)_m(\text{pinch})} 0 \to \ldots$$

We deduce that for m even, the map $K(1)_m(\text{incl})$ is an isomorphism, and the map $K(1)_m(\text{pinch})$ is zero. As for m odd, we would have that

$$K(1)_m(\mathbb{S}^0) = 0.$$

Hence, the exact sequence is now of the form

$$\ldots \to 0 \xrightarrow{K(1)_m(\text{incl})} K(1)_m(M) \xrightarrow{K(1)_m(\text{pinch})} K(1)_{m-1}(\mathbb{S}^0) \xrightarrow{0} \ldots$$

In conclusion, for m odd, we have that $K(1)_m(\text{pinch})$ is an isomorphism, and $K(1)_m(\text{incl}) = 0$. All the above data, tells us the below information about the values of $K(1)_m(M)$:

$$\begin{cases} K(1)_m(M) \cong K(1)_m(\mathbb{S}^0) \cong \mathbb{F}_2, \text{ for } m \text{ even} \\ K(1)_m(M) \cong K(1)_{m+1}(\mathbb{S}^0) \cong \mathbb{F}_2, \text{ for } m \text{ odd.} \end{cases}$$

Now, we can embark on the main goal of this section which is proving that $\mathrm{RHom}(X, Y)$ is $K(1)$-local for all Y, where $\mathrm{RHom}(X, -)$ is the right derived functor of the Quillen functor

$$\mathrm{Hom}(X, -) : \mathcal{C} \to \mathrm{Sp}.$$

To that end, we use Lemma 4.1.4, which tells us that a 2-complete spectrum Z is $K(1)$-local if and only if v_1^4 induces an isomorphism of its mod-2 homotopy groups $[M, Z]_*^{\mathrm{Sp}}$.

Lemma 4.2.6. *The map*

$$(v_1^4)^* : [M, \mathrm{RHom}(X, Y)]_n^{\mathrm{Sp}} \to [M, \mathrm{RHom}(X, Y)]_{n+8}^{\mathrm{Sp}}$$

is an isomorphism for all $n \in \mathbb{Z}$ and all $Y \in \mathcal{C}$.

Before we proceed to the proof of the above lemma, we need to compute the images of certain generators by the left derived functor $X \wedge^L -$ of the Quillen functor

$$X \wedge - : \mathrm{Sp} \to \mathcal{C} \text{ of Theorem 4.2.1.}$$

Namely, we will analyse the images of the Hopf elements $\eta \in \pi_1 L_{K(1)}\mathbb{S}^0$, $\nu \in \pi_3 L_{K(1)}\mathbb{S}^0$ and $\sigma \in \pi_7 L_{K(1)}\mathbb{S}^0$, plus the image of the element $\mu \in \pi_9 L_{K(1)}\mathbb{S}^0$. (We refer the reader to Chapter 5 for details about the generators of the stable homotopy groups of the $K(1)$-local sphere and their multiplicative relations.)

Lemma 4.2.7. *The functor $X \wedge^L -$ is taking the elements η, ν, σ, and μ in $\pi_* L_{K(1)}\mathbb{S}^0$ to the following values.*

- $X \wedge^L \eta = \begin{cases} \Phi(\eta), \ or \\ \Phi(\eta) + \Phi(y_1). \end{cases}$
- $X \wedge^L \nu = m\Phi(\nu)$ *for some odd* $m \in \mathbb{Z}$*,*
- $X \wedge^L \sigma = k\Phi(\sigma)$ *for some odd* $k \in \mathbb{Z}$*,*
- $X \wedge^L \mu = \begin{cases} \Phi(\mu), \ or \\ \Phi(\mu) + \Phi(\eta^2\sigma). \end{cases}$

Proof. $\boxed{X \wedge^L \eta}$

Remember that by Remark 3.1.9, the map $2\mathrm{Id}_M$ factors as

$$M \xrightarrow{\text{pinch}} \mathbb{S}^1 \xrightarrow{\eta} \mathbb{S}^0 \xrightarrow{\text{incl}} M.$$

If we $K(1)$-localise, as η survives the $K(1)$-localisation, we have that

$$2\mathrm{Id}_{L_{K(1)}M} \neq 0, \text{ and it factors as}$$

$$L_{K(1)}M \xrightarrow{\text{pinch}} L_{K(1)}\mathbb{S}^1 \xrightarrow{\eta} L_{K(1)}\mathbb{S}^0 \xrightarrow{\text{incl}} L_{K(1)}M.$$

To see the effect of the functor $X \wedge^L -$ on the element η, we first prove that $X \wedge^L 2\mathrm{Id}_M$ is different than zero. The functor $X \wedge^L -$ is additive, so we have the following in $\mathrm{Ho}(\mathcal{C})$:

$$\begin{aligned} X \wedge^L 2\mathrm{Id}_M &= 2\mathrm{Id}_{X\wedge^L M} \\ &= 2\mathrm{Id}_{\Phi(L_{K(1)}M)}, \text{ by Lemma 4.2.4} \\ &= 2\Phi(\mathrm{Id}_{L_{K(1)}M}) \\ &= \Phi(2\mathrm{Id}_{L_{K(1)}M}), \text{ because } \Phi \text{ is additive.} \end{aligned}$$

Since $2\mathrm{Id}_{L_{K(1)}M} \neq 0$, we have that

$$\Phi(2\mathrm{Id}_{L_{K(1)}M}) \neq 0$$

because Φ is an equivalence of categories, which means that we have a bijection between $[L_{K(1)}M, L_{K(1)}M]^{L_{K(1)}\mathrm{Sp}}$ and $[\Phi(L_{K(1)}M), \Phi(L_{K(1)}M)]^{\mathcal{C}}$. Consequently,

$$2\mathrm{Id}_{X\wedge^L M} \neq 0. \tag{1}$$

On the other hand, $2\mathrm{Id}_{X\wedge^L M}$ factors as

$$X \wedge^L M \xrightarrow{X\wedge^L \mathrm{pinch}} X \wedge^L \mathbb{S}^1 \xrightarrow{X\wedge^L \eta} X \wedge^L \mathbb{S}^0 \xrightarrow{X\wedge^L \mathrm{incl}} X \wedge^L M.$$

We already know by Remark 4.2.5 (a) that

$$X \wedge^L \mathrm{incl} = \Phi(\mathrm{incl})$$
$$X \wedge^L \mathrm{pinch} = \Phi(\mathrm{pinch}).$$

As for the element $X \wedge^L \eta$, we know so far that it cannot be zero because

$$2\mathrm{Id}_{X\wedge^L M} \neq 0, \text{ by } (1).$$

Plus,

$$\begin{aligned} X \wedge^L \eta \in [X \wedge^L \mathbb{S}^0, X \wedge^L \mathbb{S}^0]_1^{\mathcal{C}} &\cong [X, X]_1^{\mathcal{C}} \\ &\cong [\Phi(L_{K(1)}\mathbb{S}^1), \Phi(L_{K(1)}\mathbb{S}^0)]_0^{\mathcal{C}} \\ &\cong \mathbb{Z}/2\{\Phi(\eta), \Phi(y_1)\}, \text{ see Table 5.1.} \end{aligned}$$

Hence, we have three possibilities for $X \wedge^L \eta$:

$$\begin{cases} \Phi(\eta), \text{ or} \\ \Phi(y_1), \text{ or} \\ \Phi(\eta) + \Phi(y_1). \end{cases}$$

If we suppose that

$$X \wedge^L \eta = \Phi(y_1), \text{ then}$$

$$\begin{aligned}
2\mathrm{Id}_{X\wedge^L M} &= (X \wedge^L \mathrm{incl}) \circ (X \wedge^L \eta) \circ (X \wedge^L \mathrm{pinch}) \\
&= \Phi(\mathrm{incl}) \circ \Phi(y_1) \circ \Phi(\mathrm{pinch}) \\
&= \Phi(\mathrm{incl} \circ y_1 \circ \mathrm{pinch}) \\
&= 0, \text{ by Remark 5.2.2(a)},
\end{aligned}$$

and that is a contradiction to (1). So $X \wedge^L \eta$ cannot be equal to $\Phi(y_1)$, which leaves us with two possibilities for $X \wedge^L \eta$:

$$\begin{cases} \Phi(\eta), \text{ or} \\ \Phi(\eta) + \Phi(y_1). \end{cases} \tag{2}$$

$\boxed{X \wedge^L \nu}$

We start by remembering that we have the following relation

$$4\nu = \eta^3, \text{ (see Section 5.1 for all the relations in } \pi_* L_{K(1)}\mathbb{S}^0 \text{ used here).}$$

Therefore, to study $X \wedge^L \nu$, it makes sense to start by analysing the element

$$X \wedge^L \eta^3 = (X \wedge^L \eta)^3 = \begin{cases} \Phi(\eta)^3 = \Phi(\eta^3), \text{ or} \\ (\Phi(y_1) + \Phi(\eta))^3. \end{cases}$$

However, the second option, $(\Phi(y_1) + \Phi(\eta))^3$, is equal to $\Phi(\eta^3)$ since

$$\begin{aligned}
\eta y_1 &= 0, \text{ and} \\
y_1^2 &= 0.
\end{aligned}$$

Now, we know that

$$4(X \wedge^L \nu) = X \wedge^L \eta^3 = \Phi(\eta^3) = 4\Phi(\nu),$$

with $4(X \wedge^L \nu) \neq 0$ in

$$[X, X]_3^{\mathcal{C}} \cong [\Phi(L_{K(1)}\mathbb{S}^0), \Phi(L_{K(1)}\mathbb{S}^0)]_3^{\mathcal{C}} \cong \mathbb{Z}/8\{\Phi(\nu)\}.$$

All the above tells us that $X \wedge^L \nu$ has order 8 in the group $\mathbb{Z}/8\{\Phi(\nu)\}$ and is therefore a generator. In conclusion,

$$X \wedge^L \nu = m\Phi(\nu), \text{ for some odd integer } m \in \mathbb{Z}. \tag{3}$$

$\boxed{X \wedge^L \sigma}$

We start by looking at the Toda bracket relation

$$8\sigma = \big\langle \nu, 8, \nu \big\rangle \text{ ((5) in Section 5.1).}$$

Since the functor $X \wedge^L -$ is exact, the above gives us that

$$X \wedge^L 8\sigma \in \big\langle X \wedge^L \nu, X \wedge^L 8, X \wedge^L \nu \big\rangle.$$

The indeterminacy in this Toda bracket is zero, thus, equality holds. Therefore, we get

$$\begin{aligned} 8(X \wedge^L \sigma) = X \wedge^L 8\sigma &= \big\langle X \wedge^L \nu, X \wedge^L 8, X \wedge^L \nu \big\rangle \\ &= \big\langle m\Phi(\nu), \Phi(8), m\Phi(\nu) \big\rangle, \text{ by (3)} \\ &= m^2 \big\langle \Phi(\nu), \Phi(8), \Phi(\nu) \big\rangle \\ &= m^2 \Phi(8\sigma). \end{aligned}$$

Since Φ is an equivalence, we have that $\Phi(8\sigma) \neq 0$. Hence, the above tells us that

$$8(X \wedge^L \sigma) = m^2 8\Phi(\sigma) \neq 0 \text{ in } [X, X]_7^{\mathcal{C}} \cong \mathbb{Z}/16\{\Phi(\sigma)\}.$$

Consequently, $X \wedge^L \sigma$ has order 16 in this group, and

$$X \wedge^L \sigma = k\Phi(\sigma), \text{ for some odd integer } k \in \mathbb{Z}. \tag{4}$$

$\boxed{X \wedge^L \mu}$

We know that

$$\mu \in \langle \eta, 8\sigma, 2 \rangle, \text{ with indeterminacy} = \eta^2\sigma, \text{ by (6) in Section 5.1.}$$

It follows by (2) and (4) that

$$X \wedge^L \mu \in \langle X \wedge^L \eta, X \wedge^L 8\sigma, X \wedge^L 2 \rangle = \begin{cases} \langle \Phi(\eta), \Phi(8\sigma), 2 \rangle, \text{ or} \\ \langle \Phi(\eta) + \Phi(y_1), \Phi(8\sigma), 2 \rangle. \end{cases}$$

If we have that

$$\begin{aligned} X \wedge^L \mu \in \langle \Phi(\eta), \Phi(8\sigma), 2 \rangle &= \Phi(\langle \eta, 8\sigma, 2 \rangle) \\ &= \{\Phi(\mu), \Phi(\mu) + \Phi(\eta^2\sigma)\}, \end{aligned}$$

then

$$X \wedge^L \mu = \begin{cases} \Phi(\mu), \text{ or} \\ \Phi(\mu) + \Phi(\eta^2\sigma). \end{cases}$$

However, if

$$X \wedge^L \mu \in \langle \Phi(\eta) + \Phi(y_1), \Phi(8\sigma), 2 \rangle,$$

we shall prove that we will end up with the same possibilities as before. By Lemma 5.2.9, we know that

$$\begin{aligned} &\langle \eta + y_1, 8\sigma, 2 \rangle = \langle \eta, 8\sigma, 2 \rangle + \langle y_1, 8\sigma, 2 \rangle, \text{ and} \\ &\langle y_1, 8\sigma, 2 \rangle = \eta^2\sigma. \end{aligned}$$

Therefore,

$$X \wedge^L \mu \in \{\Phi(\mu), \Phi(\mu) + \Phi(\eta^2\sigma)\} + \Phi(\eta^2\sigma),$$

and this means that

$$X \wedge^L \mu \in \{\Phi(\mu), \Phi(\mu) + \Phi(\eta^2\sigma)\}, \text{ because } 2\Phi(\eta^2\sigma) = 0. \qquad \square$$

Now, we can move on to proving Lemma 4.2.6, namely that the mod-2 homotopy groups of $\mathrm{RHom}(X, Y)$ are v_1^4-periodic for all $Y \in \mathcal{C}$.

Proof. By adjunction, it suffices to prove that

$$(X \wedge^L v_1^4)^* : [X \wedge^L M, Y]_n^{\mathcal{C}} \to [X \wedge^L M, Y]_{n+8}^{\mathcal{C}}$$

is an isomorphism for all integers n. We know that

$$(v_1^4)^* : [M, \Phi^{-1}(Y)]_n^{L_{K(1)}\mathrm{Sp}} \to [M, \Phi^{-1}(Y)]_{n+8}^{L_{K(1)}\mathrm{Sp}}$$

is an isomorphism for all n. However, Lemma 4.2.4 tells us that

$$X \wedge^L M \cong \Phi(M),$$

and this means that

$$\Phi(v_1^4)^* : [X \wedge^L M, Y]_n^{\mathcal{C}} \to [X \wedge^L M, Y]_{n+8}^{\mathcal{C}}$$

is an isomorphism as well. Therefore, to show that $(X \wedge^L v_1^4)^*$ is an isomorphism, one compares the elements $(X \wedge^L v_1^4)$ and $\Phi(v_1^4)$ in the endomorphism ring

$$[X \wedge^L M, X \wedge^L M]_8^{\mathcal{C}} \cong [M, M]_8^{L_{K(1)}\mathrm{Sp}} \cong [M, L_{K(1)}M]_8^{\mathrm{Sp}}.$$

The calculation of the above endomorphism ring is done separately in Chapter 5.

By Computation 5.2.12 we know that

$$[M, L_{K(1)}M]_8^{\mathrm{Sp}} \cong \mathbb{Z}/4\{v_1^4\} \oplus \mathbb{Z}/2\{\widetilde{\eta\sigma} \circ \mathrm{pinch}, \mathrm{Id}_{L_{K(1)}M} \wedge \eta\sigma\}.$$

By Corollary 5.2.13,

$$2v_1^4 = \mathrm{incl} \circ \mu \circ \mathrm{pinch},$$

so

$$\begin{aligned} 2(X \wedge^L v_1^4) &= X \wedge^L (2v_1^4) \\ &= X \wedge^L (\mathrm{incl} \circ \mu \circ \mathrm{pinch}) \\ &= (X \wedge^L \mathrm{incl}) \circ (X \wedge^L \mu) \circ (X \wedge^L \mathrm{pinch}). \end{aligned}$$

First, by Remark 4.2.5(a) the images of the maps incl and pinch by the functor $X \wedge^L -$ are respectively $\Phi(\mathrm{incl})$ and $\Phi(\mathrm{pinch})$. On the other hand, by Lemma 4.2.7, $X \wedge^L \mu$ corresponds to either $\Phi(\mu)$ or $\Phi(\mu) + \Phi(\eta^2\sigma)$. Hence, we have two possibilities for $2(X \wedge^L v_1^4)$, either

- $$\begin{aligned} 2(X \wedge^L v_1^4) &= \Phi(\mathrm{incl}) \circ \Phi(\mu) \circ \Phi(\mathrm{pinch}) \\ &= \Phi(\mathrm{incl} \circ \mu \circ \mathrm{pinch}) \\ &= \Phi(2v_1^4) = 2\Phi(v_1^4) \end{aligned}$$

or

- $$\begin{aligned} 2(X \wedge^L v_1^4) &= \Phi(\mathrm{incl}) \circ (\Phi(\mu) + \Phi(\eta^2\sigma)) \circ \Phi(\mathrm{pinch}) \\ &= \Phi(\mathrm{incl} \circ \mu \circ \mathrm{pinch}) + \Phi(\mathrm{incl} \circ \eta^2\sigma \circ \mathrm{pinch}) \\ &= \Phi(\mathrm{incl} \circ \mu \circ \mathrm{pinch}) + 0, \text{ by Remark 5.2.2(b)} \\ &= \Phi(2v_1^4) = 2\Phi(v_1^4). \end{aligned}$$

This means that the elements $X \wedge^L v_1^4$ and $\Phi(v_1^4)$ are equal to each other up to an element in $[M, L_{K(1)}M]_8^{\mathrm{Sp}}$ of order 2, that is

$$X \wedge^L v_1^4 = \Phi(v_1^4) + \Phi(T), \text{ for some } T \in [M, L_{K(1)}M]_8^{\mathrm{Sp}}, \text{ such that } 2T = 0.$$

Showing that $X \wedge^L v_1^4$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ is now down to proving that all such $v_1^4 + T$ are isomorphisms in $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$. We know by Proposition 2.2.18 that $v_1^4 + T$ is an isomorphism in $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ if and only if it is a $K(1)_*$-equivalence. The map v_1^4 is indeed a $K(1)_*$-equivalence since it is a v_1-self map on the Moore spectrum. Now, to finish the proof, we need to check what $K(1)_*(T)$ is equal to for each $T \in [M, L_{K(1)}M]_8^{\mathrm{Sp}}$ with $2T = 0$. For that, it is enough to check what $K(1)_*$ is doing for

$$2v_1^4, \widetilde{\eta\sigma} \circ \mathrm{pinch} \text{ and } \mathrm{Id}_{L_{K(1)}M} \wedge \eta\sigma,$$

as each T in question is a sum of those. First, remember that $K(1)_*(v_1^4)$ is a morphism of $\mathbb{F}_2$-modules since $K(1)_*$ is a graded field. As a result, we have that

$$K(1)_*(2v_1^4) = 0.$$

Now, since $K(1)_m(\mathbb{S}^0)$ is concentrated only in even degrees, the induced map in $K(1)_*$-homology for the element $\eta \in \pi_1 L_{K(1)}\mathbb{S}^0$

$$K(1)_m(\eta) : K(1)_{m+1}(\mathbb{S}^0) \to K(1)_m(\mathbb{S}^0)$$

is zero whether m is odd or even. Consequently, we have

$$K(1)_*(\mathrm{Id}_{L_{K(1)}M} \wedge \eta\sigma) = 0, \text{ and}$$
$$K(1)_*(\eta\sigma \circ \mathrm{pinch}) = 0.$$

However, by Remark 5.2.1 we know that $K(1)_*(\eta\sigma \circ \mathrm{pinch})$ is also equal to

$$K(1)_*(\mathrm{pinch} \circ \widetilde{\eta\sigma} \circ \mathrm{pinch}) = K(1)_*(\mathrm{pinch}) \circ K(1)_*(\widetilde{\eta\sigma} \circ \mathrm{pinch}).$$

We already know that the above composition is equal to zero, and Remark 4.2.5(b) tells us that $K(1)_*(\mathrm{pinch})$ is either zero or an isomorphism, which means that in either case

$$K(1)_*(\widetilde{\eta\sigma} \circ \mathrm{pinch}) = 0.$$

Therefore, every $v_1^4 + T$ is a $K(1)_*$-isomorphism, which means that it is an isomorphism in $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ and its image

$$\Phi(v_1^4 + T) = X \wedge^L v_1^4$$

is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. In conclusion, by adjunction, the map in question

$$(v_1^4)^* : [M, \mathrm{RHom}(X,Y)]_n^{\mathrm{Sp}} \to [M, \mathrm{RHom}(X,Y)]_{n+8}^{\mathrm{Sp}}$$

is an isomorphism for all n and Y. □

Recall by Lemma 4.1.4, that a 2-complete spectrum is $K(1)$-local if and only if v_1^4 induces an isomorphism on its mod-2 homotopy groups. However, we know by Lemma 4.2.3 that $\mathrm{RHom}(X,Y)$ is 2-complete, and Lemma 4.2.6 tells us that v_1^4 induces an isomorphism on its mod-2 homotopy groups $[M, \mathrm{RHom}(X,Y)]_*^{\mathrm{Sp}}$. Consequently, by Proposition 4.2.2, we can construct the desired Quillen adjunction as we see below.

Corollary 4.2.8. *The spectrum* $\mathrm{RHom}(X,Y)$ *is* $K(1)$*-local for all* Y*. Thus,*

$$X \wedge - : L_{K(1)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

is a Quillen adjunction. □

Lemma 4.2.9. *The left derived functor*

$$X \wedge^L - : \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C})$$

of the Quillen functor in Corollary 4.2.8 is acting on the elements $y_0 \in \pi_0(L_{K(1)}\mathbb{S}^0)$ *and* $y_1 \in \pi_1(L_{K(1)}\mathbb{S}^0)$ *as follows.*

- $X \wedge^L y_0 = \Phi(y_0)$
- $X \wedge^L y_1 = \Phi(y_1)$

Proof. $\boxed{X \wedge^L y_0}$

The element y_0 is the only non-zero torsion element in

$$\pi_0(L_{K(1)}\mathbb{S}^0) \cong \mathbb{Z}_2^\wedge \oplus \mathbb{Z}/2,$$

and since the functor $X \wedge^L -$ is additive, the element $X \wedge^L y_0$ must be a torsion element as well. Hence, $X \wedge^L y_0$ is either equal to $\Phi(y_0)$ or zero. We have the following relation

$$\mu y_0 = \eta^2\sigma \text{ (Section 5.1)}.$$

We already know by Lemma 4.2.7 that

$$\begin{aligned} X \wedge^L \eta^2\sigma &= (X \wedge^L \eta^2) \circ (X \wedge^L \sigma) \\ &= \Phi(\eta^2) \circ \Phi(k\sigma) \text{ ,(for some odd } k \in \mathbb{Z}) \\ &= k\Phi(\eta^2\sigma) \neq 0. \end{aligned}$$

So,

$$X \wedge^L \mu y_0 = (X \wedge^L \mu) \circ (X \wedge^L y_0) \neq 0,$$

which tells us that $X \wedge^L y_0$ cannot be zero, and we are left with the other option of being equal to $\Phi(y_0)$.

$\boxed{X \wedge^L y_1}$

We have that $y_1 = \eta y_0$, which gives us by using the previous calculation and Lemma 4.2.7 the following.

$$\begin{aligned} X \wedge^L y_1 &= (X \wedge^L \eta) \circ (X \wedge^L y_0) \\ &= \begin{cases} \Phi(\eta) \circ \Phi(y_0), \text{ or} \\ \Phi(\eta) \circ \Phi(y_0) + \Phi(y_1) \circ \Phi(y_0) \end{cases} \\ &= \begin{cases} \Phi(y_1), \text{ or} \\ \Phi(y_1) + \Phi(\eta y_0^2). \end{cases} \end{aligned}$$

However, we know that

$$y_0^2 = 0,$$

and that tells us that we have in either case

$$X \wedge^L y_1 = \Phi(y_1). \qquad \square$$

4.3 The Quillen equivalence

As before, let $\Phi : \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C})$ be an equivalence of triangulated categories. In the previous section, we constructed the Quillen adjunction

$$X \wedge - : L_{K(1)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -),$$

where $X \simeq \Phi(L_{K(1)}\mathbb{S}^0)$. This Quillen adjunction induces the derived adjunction on the homotopy level (Theorem 2.2.23)

$$X \wedge^L - : \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \mathrm{RHom}(X, -).$$

Remember that

$$X \wedge^L \mathbb{S}^0 \cong X, \text{ and}$$
$$X \wedge^L M \cong \Phi(L_{K(1)}M).$$

Our goal now is to prove that the Quillen adjunction $(X \wedge -, \mathrm{Hom(X}, -))$ is a Quillen equivalence, or equivalently that the derived adjunction $(X \wedge^L -, \mathrm{RHom(X}, -))$ is an adjoint equivalence of categories (Proposition 2.2.25). To this end, we first start by looking at the homotopy type of the spectrum $\mathrm{RHom}(X, X \wedge^L M)$. Note that in the $E(1)$-local case in [Roi07], the author investigated the homotopy type of $\mathrm{RHom}(X, X \wedge^L \mathbb{S}^0)$. The reason behind it is that in $L_1\mathrm{Sp}$, the sphere spectrum is a compact generator, while in $L_{K(1)}\mathrm{Sp}$, the Moore spectrum M is a compact generator, and $\mathbb{S}^0$ is just a generator. Everything mentioned and the reason why we are looking at a compact generator will become apparent when we will be proving the equivalence.

As we have seen in Corollary 4.2.8, the spectrum $\mathrm{RHom}(X, X \wedge^L M)$ is $K(1)$-local. Therefore, by the Universal property of localisation (Proposition 3.3.16), the adjoint of the identity map factors over $L_{K(1)}M$

$$\begin{array}{ccc} M & \longrightarrow & \mathrm{RHom}(X, X \wedge^L M) \\ {\scriptstyle \eta_M}\downarrow & \overset{\exists! \lambda}{\nearrow} & \\ L_{K(1)}M & & \end{array}$$

Proposition 4.3.1. *The map*

$$\lambda : L_{K(1)}M \to \mathrm{RHom}(X, X \wedge^L M)$$

is a π_*-isomorphism.

Proof. Remember that any map that induces an isomorphism on the mod-2 homotopy groups of 2-complete spectra must be a weak equivalence. As the source and the target of the map λ are both 2-complete, it is enough to show that λ induces an isomorphism of mod-2 homotopy groups. In other words, we need to show that λ_* in the following commutative diagram is an isomorphism

$$\begin{array}{ccc} [M, L_{K(1)}M]_*^{\mathrm{Sp}} \cong [M, M]_*^{L_{K(1)}\mathrm{Sp}} & \xrightarrow{\lambda_*} & [M, \mathrm{RHom}(X, X \wedge^L M)]_*^{\mathrm{Sp}} \\ {\scriptstyle X\wedge^L -}\downarrow & \underset{\mathrm{adj}}{\overset{\cong}{\nearrow}} & \\ [X \wedge^L M, X \wedge^L M]_*^{\mathcal{C}}. & & \end{array}$$

The above diagram is commutative because, by definition of λ, for $\alpha \in [M, L_{K(1)}M]_*^{\mathrm{Sp}}$, the image of $X \wedge^L \alpha$ under the adjunction isomorphism is precisely $\lambda \circ \alpha$. All we need to show is that

$$X \wedge^L - : [M, M]_n^{L_{K(1)}\mathrm{Sp}} \longrightarrow [X \wedge^L M, X \wedge^L M]_n^{\mathcal{C}}$$

is an isomorphism for all n.

However, via the self map v_1^4, the endomorphisms of the Moore spectrum are periodic of period 8 in $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$

$$[M, M]_n^{L_{K(1)}\mathrm{Sp}} \cong [M, M]_{n+8}^{L_{K(1)}\mathrm{Sp}}.$$

Therefore, we only have to show that the desired isomorphism holds for $n = 1, \ldots, 8$. To that end, we first show that

$$X \wedge^L - : [\mathbb{S}^0, \mathbb{S}^0]_n^{L_{K(1)}\mathrm{Sp}} \longrightarrow [X, X]_n^{\mathcal{C}}$$

is an isomorphism for $n = 0, \ldots, 9$ by verifying that

$$\psi : [\mathbb{S}^0, \mathbb{S}^0]_n^{L_{K(1)}\mathrm{Sp}} \xrightarrow{X \wedge^L -} [X, X]_n^{\mathcal{C}} \xrightarrow{\Phi^{-1}} [\mathbb{S}^0, \mathbb{S}^0]_n^{L_{K(1)}\mathrm{Sp}}$$

is an isomorphism in that range. By Lemma 4.2.7, Lemma 4.2.9 and Table 5.1, we know that the above map ψ is acting on the generators of $\pi_*(L_{K(1)}\mathbb{S}^0)$ as follows.

x	$\psi(x)$
y_0	y_0
y_1	y_1
η	η or $\eta + y_1$
ν	$m\nu,\ m$ odd
σ	$k\sigma,\ k$ odd
μ	μ or $\mu + \eta^2\sigma$

Hence, the composition

$$\psi = \Phi^{-1} \circ (X \wedge^L -) : \pi_n(L_{K(1)}\mathbb{S}^0) \to \pi_n(L_{K(1)}\mathbb{S}^0)$$

is an isomorphism for $n = 0, \ldots, 9$. Since we already know that Φ^{-1} is an isomorphism, we conclude that

$$X \wedge^L - : [\mathbb{S}^0, \mathbb{S}^0]_n^{L_{K(1)}\mathrm{Sp}} \longrightarrow [X, X]_n^{\mathcal{C}}$$

is an isomorphism for $n = 0, \ldots, 9$. Now, the desired result will follow by using a five lemma argument. To be more specific, we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\pi_n L_{K(1)}\mathbb{S}^0)/2 & \xrightarrow{\text{incl}_*} & \pi_n(L_{K(1)}M) & \xrightarrow{\text{pinch}_*} & \{\pi_{n-1}L_{K(1)}\mathbb{S}^0\}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & [X,X]^{\mathcal{C}}_n/2 & \xrightarrow{\text{incl}_*} & [X, X \wedge^L M]^{\mathcal{C}}_n & \xrightarrow{\text{pinch}_*} & \{[X,X]^{\mathcal{C}}_{n-1}\}_2 & \longrightarrow & 0 \end{array}$$

where the two rows are short exact sequences, and the left-hand side, as well as the right-hand side arrows, are isomorphisms for $n = 0, \ldots, 9$. Therefore, we conclude that the middle vertical arrow is an isomorphism. Now, the statement that

$$X \wedge^L - : [M,M]^{L_{K(1)}\mathrm{Sp}}_n \longrightarrow [X \wedge^L M, X \wedge^L M]^{\mathcal{C}}_n$$

is an isomorphism for $n = 1, \ldots, 8$ is deduced from the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\pi_{n+1}(L_{K(1)}M))/2 & \xrightarrow{\text{pinch}^*} & [M,M]^{L_{K(1)}\mathrm{Sp}}_n & \xrightarrow{\text{incl}^*} & \{\pi_n L_{K(1)}M\}_2 & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 & \to & [X \wedge^L \mathbb{S}^0, X \wedge^L M]^{\mathcal{C}}_{n+1}/2 & \to & [X \wedge^L M, X \wedge^L M]^{\mathcal{C}}_n & \to & \{[X \wedge^L \mathbb{S}^0, X \wedge^L M]^{\mathcal{C}}_n\}_2 & \to & 0. \end{array}$$

Thus, we can conclude that $L_{K(1)}M$ and $\mathrm{Hom}(X, X \wedge^L M)$ are weakly equivalent in Sp. □

Now that we have all the necessary arguments, we can use the fact that M is a compact generator of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ (Lemma 3.3.38) to prove our main theorem.

Theorem 4.3.2. *The Quillen adjunction*

$$X \wedge - : L_{K(1)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

is a Quillen equivalence.

Proof. By [Hov99, 1.3.16], it is sufficient to show the following:

- $\mathrm{RHom}(X, -) : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ reflects isomorphisms.
- $A \to \mathrm{RHom}(X, X \wedge^L A)$ is an isomorphism for all $A \in \mathrm{Ho}(L_{K(1)}\mathrm{Sp})$.

Since Φ is an equivalence of triangulated categories, $\Phi(L_{K(1)}\mathbb{S}^0) = X$ is a generator for $\mathrm{Ho}(\mathcal{C})$, therefore as mentioned in Section 3.2 in Remark 3.2.26 it detects isomorphisms.

Let us first show the first point. For a morphism $f : Y \to Z$ in $\mathcal{C}$, let

$$\mathrm{RHom}(X, f) : \mathrm{RHom}(X, Y) \to \mathrm{RHom}(X, Z)$$

be an isomorphism in $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$, so

$$[\mathbb{S}^0, \mathrm{RHom}(X, Y)]_*^{L_{K(1)}(\mathrm{Sp})} \xrightarrow{\mathrm{RHom}(X,f)} [\mathbb{S}^0, \mathrm{RHom}(X, Z)]_*^{L_{K(1)}(\mathrm{Sp})}$$

is an isomorphism. By adjunction,

$$[X, Y]_*^{\mathcal{C}} \xrightarrow{f_*} [X, Z]_*^{\mathcal{C}}$$

is an isomorphism. Since X is a generator in $\mathrm{Ho}(\mathcal{C})$, we have that $f : Y \to Z$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ which proves the first point.

In order to prove the second point, we will use the Theorem 3.2.25 mentioned in Section 3.2 of this thesis. Consider the full subcategory $\mathcal{T}$ of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ containing those $A \in \mathrm{Ho}(L_{K(1)}\mathrm{Sp})$ such that

$$A \to \mathrm{RHom}(X, X \wedge^L A)$$

is an isomorphism. Our goal is to prove that $\mathcal{T} = \mathrm{Ho}(L_{K(1)}\mathrm{Sp})$. Since $\mathrm{RHom}(X, -)$ and $X \wedge^L -$ are exact functors, $\mathcal{T}$ is triangulated. By Proposition 4.3.1 it contains the Moore spectrum M, i.e. a compact generator of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$.

To be able to use Theorem 3.2.25, we still need to verify that this category $\mathcal{T}$ is also closed under coproducts. Now let A_i, $i \in \mathcal{I}$, be a family of objects in $\mathcal{T}$. We would like that $\coprod_i A_i \in \mathcal{T}$. As M reflects isomorphisms, this means that we need to show that

$$[M, \coprod_i A_i]_*^{L_{K(1)}\mathrm{Sp}} \to [M, \mathrm{RHom}(X, X \wedge^L (\coprod_i A_i))]_*^{L_{K(1)}\mathrm{Sp}}$$

is an isomorphism. On the other hand, by adjunction,

$$[M, \mathrm{RHom}(X, X \wedge^L (\coprod_i A_i))]_*^{L_{K(1)}\mathrm{Sp}} \cong [X \wedge^L M, X \wedge^L (\coprod_i A_i)]_*^{\mathcal{C}}.$$

Since $X \wedge^L -$ is a left adjoint, it commutes with coproducts, therefore

$$[X \wedge^L M, X \wedge^L (\coprod_i A_i)]_*^{\mathcal{C}} \cong [X \wedge^L M, \coprod_i (X \wedge^L A_i)]_*^{\mathcal{C}}.$$

Since Φ is an equivalence of triangulated categories, and $L_{K(1)}M$ is a compact generator of $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$, we have that $\Phi(L_{K(1)}M) = X \wedge^L M$ is a compact generator of $\mathrm{Ho}(\mathcal{C})$, and this means that

$$[X \wedge^L M, \coprod_i (X \wedge^L A_i)]_*^{\mathcal{C}} \cong \bigoplus_i [X \wedge^L M, X \wedge^L A_i]_*^{\mathcal{C}}.$$

Similarly, we know that

$$[M, \coprod_i A_i]_*^{L_{K(1)}\mathrm{Sp}} \cong \bigoplus_i [M, A_i]_*^{L_{K(1)}\mathrm{Sp}}.$$

As $A_i \in \mathcal{T}$, for all $i \in \mathcal{I}$,

$$[M, A_i]_*^{L_{K(1)}\mathrm{Sp}} \cong [M, \mathrm{RHom}(X, X \wedge^L A_i)]_*^{L_{K(1)}\mathrm{Sp}},$$

which is induced by

$$A_i \xrightarrow{\cong} \mathrm{RHom}(X, X \wedge^L A_i).$$

By naturality of those isomorphisms, we have that $\mathcal{T}$ is closed under coproducts, therefore $\mathcal{T} = \mathrm{Ho}(L_{K(1)}\mathrm{Sp})$, and our Quillen adjunction is indeed a Quillen equivalence. □

In conclusion, we proved that if we have by hypothesis an exact equivalence of triangulated categories for $p = 2$

$$\Phi : \mathrm{Ho}(L_{K(1)}\mathrm{Sp}) \to \mathrm{Ho}(\mathcal{C}),$$

then we have a Quillen equivalence

$$X \wedge - : L_{K(1)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -).$$

In other words, we showed that, 2-locally, the $K(1)$-local stable homotopy category is indeed *rigid*.

Chapter 5

Computations

In this chapter, we write details about computing certain homotopy groups in the $K(1)$-local setting. We start by talking about relations between generators of the homotopy groups $\pi_*(L_{K(1)}\mathbb{S}^0)$, using either equalities or Toda brackets. Afterwards, we compute homotopy groups and endomorphism rings of the $K(1)$-local Moore spectrum $M(\mathbb{Z}/2)$.

5.1 The groups $\pi_*(L_{K(1)}\mathbb{S}^0)$ and their generators

By [Bou79, Theorem 4.3] and Lemma 3.3.39, we know that the $K(1)$-local sphere is the fiber of $\Psi^3 - 1$ on $KO\mathbb{Z}_2$, where Ψ^3 is the Adams operation and $KO\mathbb{Z}_2$ is the 2-adic real K-theory spectrum. Therefore the long exact sequence produced by the fiber sequence

$$L_{K(1)}\mathbb{S}^0 \to KO\mathbb{Z}_2 \xrightarrow{\Psi^3-1} KO\mathbb{Z}_2$$

provides us with values of $\pi_n(L_{K(1)}\mathbb{S}^0)$ at $p = 2$. On the other hand, the long exact sequence provided by the homotopy pullback square

$$\begin{array}{ccc} L_1 Y & \longrightarrow & L_{K(1)}Y \\ \downarrow & & \downarrow \\ L_0 Y & \longrightarrow & L_0 L_{K(1)} Y \end{array}$$

tells us that 2-locally, we have

$$\pi_n(L_{K(1)}\mathbb{S}^0) \cong \pi_n(L_1\mathbb{S}^0), \text{ for } n \neq -2, \ -1, 0.$$

The final result from degree -2 until 9 reads as follows, see e.g. [Rav84, Theorem 8.15] or [Bou79, Corollary 4.5].

Table 5.1

n	$\pi_n(\mathbb{S}^0_{(2)})$	$\pi_n(L_1\mathbb{S}^0)$	$\pi_n(L_{K(1)}\mathbb{S}^0)$
-2	0	$\mathbb{Q}/\mathbb{Z}_{(2)} \cong \mathbb{Z}/2^\infty$	0
-1	0	0	$\mathbb{Z}_2^\wedge$
0	$\mathbb{Z}_{(2)}\{\iota\}$	$\mathbb{Z}_{(2)}\{\iota\} \oplus \mathbb{Z}/2\{y_0\}$	$\mathbb{Z}_2^\wedge\{\iota\} \oplus \mathbb{Z}/2\{y_0\}$
1	$\mathbb{Z}/2\{\eta\}$	$\mathbb{Z}/2\{\eta, y_1\}$	$\mathbb{Z}/2\{\eta, y_1\}$
2	$\mathbb{Z}/2\{\eta^2\}$	$\mathbb{Z}/2\{\eta^2\}$	$\mathbb{Z}/2\{\eta^2\}$
3	$\mathbb{Z}/8\{\nu\}$	$\mathbb{Z}/8\{\nu\}$	$\mathbb{Z}/8\{\nu\}$
4	0	0	0
5	0	0	0
6	$\mathbb{Z}/2\{\nu^2\}$	0	0
7	$\mathbb{Z}/16\{\sigma\}$	$\mathbb{Z}/16\{\sigma\}$	$\mathbb{Z}/16\{\sigma\}$
8	$\mathbb{Z}/2\{\eta\sigma, \varepsilon\}$	$\mathbb{Z}/2\{\eta\sigma\}$	$\mathbb{Z}/2\{\eta\sigma\}$
9	$\mathbb{Z}/2\{\eta^2\sigma, \eta\varepsilon, \mu\}$	$\mathbb{Z}/2\{\eta^2\sigma, \mu\}$	$\mathbb{Z}/2\{\eta^2\sigma, \mu\}$

The element y_0 is the unique element of order 2 of $\pi_0(L_{K(1)}\mathbb{S}^0)$, and $y_1 = \eta y_0$ is a generator of the second summand in $\pi_1(L_{K(1)}\mathbb{S}^0)$. As for the other elements of $\pi_n(L_{K(1)}\mathbb{S}^0)$, we give them the names of their (not necessarily unique) preimages in $\pi_n(\mathbb{S}^0)$. Moreover, we have the following relations, [Rav84, Theorem 8.15(d)]

$$4\nu = \eta^3, \ \eta y_1 = 0, \ y_0^2 = 0, \ y_1^2 = 0, \ \sigma y_1 = 0 \ \text{ and } \ \mu y_0 = \eta^2\sigma.$$

Furthermore, by [Tod62, Lemma 5.13, Lemma 10.5, Tables in Chapter XI], we have the following relations on Toda brackets

$$8\sigma = \langle \nu, 8, \nu \rangle, \tag{5}$$

$$\mu \in \langle \eta, 8\sigma, 2 \rangle, \text{ with indeterminacy} = \eta^2\sigma. \tag{6}$$

5.2 The endomorphism ring of $L_{K(1)}M(\mathbb{Z}/2)$

In this section, we compute the group $[M(\mathbb{Z}/2), L_{K(1)}M(\mathbb{Z}/2)]_8^{\mathrm{Sp}}$ which we used to prove Lemma 4.2.6.

Notation. As in the previous chapter, the mod-2 Moore spectrum $M(\mathbb{Z}/2)$ will be denoted by M.

We will first outline the method that we are going to adapt in order to compute the homotopy groups $\pi_m(L_{K(1)}M)$ for any m, and then use it to compute the groups $\pi_8(L_{K(1)}M)$ and $\pi_9(L_{K(1)}M)$. Afterwards, we will see how we are going to use $\pi_8(L_{K(1)}M)$ and $\pi_9(L_{K(1)}M)$ to find what $[M, L_{K(1)}M]_8^{\mathrm{Sp}}$ is equal to.

General Strategy. *Analogously to Example 3.2.20, the long exact homotopy sequence of the exact triangle*

$$L_{K(1)}\mathbb{S}^0 \xrightarrow{2\cdot} L_{K(1)}\mathbb{S}^0 \xrightarrow{\text{incl}} L_{K(1)}M \xrightarrow{\text{pinch}} L_{K(1)}\mathbb{S}^1, \qquad (\blacktriangle)$$

provides us with short exact sequences of the form

$$0 \to \big(\pi_{m+1}(L_{K(1)}\mathbb{S}^0)\big)/2 \xrightarrow{\text{incl}_*} \pi_{m+1}(L_{K(1)}M) \xrightarrow{\text{pinch}_*} \{\pi_m(L_{K(1)}\mathbb{S}^0)\}_2 \to 0.$$

As before, we have

$$\{\pi_m(\mathbb{S}^0)\}_2 = \{x \in \pi_m(\mathbb{S}^0) : 2x = 0\}.$$

Since Section 5.1 provided us with the groups $\pi_m(L_{K(1)}\mathbb{S}^0)$ in a certain range, finding the homotopy groups of $L_{K(1)}M$ and the group $[M, L_{K(1)}M]_8^{\mathrm{Sp}}$ is now a game of completing short exact sequences in which Toda brackets are the main players.

Remark 5.2.1. Since pinch_* is surjective, let $\tilde{x}$ denote the preimage of $x \in \{\pi_m(\mathbb{S}^0)\}_2$ such that

$$\text{pinch} \circ \tilde{x} = x.$$

We have

$$\begin{aligned}\text{pinch}_*(2\tilde{x}) &= 2\,\text{pinch} \circ \tilde{x} \\ &= 2x = 0.\end{aligned}$$

Therefore, $2\tilde{x} \in \text{Im}(\text{incl}_*)$, and has a unique preimage in $\big(\pi_{m+1}(L_{K(1)}\mathbb{S}^0)\big)/2$ under the map incl_*. However, we know that

$$\begin{aligned}2\tilde{x} &= \text{incl} \circ \eta \circ \text{pinch} \circ \tilde{x}, \text{ by Remark 3.1.9} \\ &= \text{incl} \circ \eta x \\ &= \text{incl}_*(\eta x),\end{aligned}$$

which tells us that this preimage is indeed ηx.

Notation. For x an element in $\{\pi_m(L_{K(1)}\mathbb{S}^0)\}_2$, we will denote its preimage under the map pinch_* by $\tilde{x}$. Note that this preimage need not to be unique, however our computations do not depend on the choice of such an $\tilde{x}$ unless stated.

Remark 5.2.2. Before starting the computations, it is worth noting that we have the following equalities.

$$\begin{aligned}(a)\ \text{incl} \circ y_1 \circ \text{pinch} &= \text{incl} \circ \eta y_0 \circ \text{pinch} \\ &= 2\widetilde{y_0} \circ \text{pinch}, \text{ by Remark 5.2.1} \\ &= \widetilde{y_0} \circ (2\,\text{pinch}), \text{ since all the functors involved are additive} \\ &= 0, \text{ because } 2\,\text{pinch} = 0. \\ (b)\ \text{incl} \circ \eta^2\sigma \circ \text{pinch} &= \text{incl}_*(\eta(\eta\sigma)) \circ \text{pinch} \\ &= 2\widetilde{\eta\sigma} \circ \text{pinch}, \text{ by Remark 5.2.1} \\ &= \widetilde{\eta\sigma}(2\,\text{pinch}) \\ &= 0.\end{aligned}$$

Computation 5.2.3. $\pi_0 L_{K(1)}M \cong \mathbb{Z}/2\{\text{incl}, \text{incl} \circ y_0\}$

Proof. As mentioned in the beginning of this section, we have the following short exact sequence

$$0 \to \big(\pi_0(L_{K(1)}\mathbb{S}^0)\big)/2 \xrightarrow{\text{incl}_*} \pi_0(L_{K(1)}M) \xrightarrow{\text{pinch}_*} \{\pi_{-1}(L_{K(1)}\mathbb{S}^0)\}_2 \to 0.$$

Since we have that

$$\{\pi_{-1}(L_{K(1)}\mathbb{S}^0)\}_2 = \{\mathbb{Z}_2^\wedge\}_2 = 0, \text{ and}$$
$$\mathbb{Z}_2^\wedge/2 \cong \mathbb{Z}/2,$$

the above short exact sequence gives us an isomorphism

$$\mathbb{Z}/2\{\iota, y_0\} \xrightarrow{\text{incl}_*} \pi_0 L_{K(1)}M. \qquad \square$$

Computation 5.2.4. $\pi_1 L_{K(1)}M \cong \mathbb{Z}/4\{\widetilde{y_0}\} \oplus \mathbb{Z}/2\{\text{incl} \circ \eta\}$

Proof. By using the same strategy as before and the calculations of Section 5.1, we end up having the following short exact sequence

$$0 \to \mathbb{Z}/2\{\eta, y_1\} \xrightarrow{\text{incl}_*} \pi_1 L_{K(1)}M \xrightarrow{\text{pinch}_*} \mathbb{Z}/2\{y_0\} \to 0.$$

We have two choices for $\pi_1 L_{K(1)}M$: either $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We will prove that the second choice cannot work. Since pinch_* is surjective, there exists an element $\widetilde{y_0} \in \pi_1 L_{K(1)}M$ such that $\text{pinch}_*(\widetilde{y_0}) = y_0$. By Remark 5.2.1, the equality $y_1 = \eta y_0$, and the fact that incl_* is injective, we have that

$$\text{incl}_*(y_1) = \text{incl}_*(\eta y_0) = 2\widetilde{y_0} \neq 0.$$

Hence, we conclude that

$$\pi_1 L_{K(1)}M \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2,$$

with $\widetilde{y_0}$, the preimage of y_0 by pinch$_*$, generating $\mathbb{Z}/4$, and incl$_*(\eta)$ generating the second summand. □

Computation 5.2.5. $\pi_8 L_{K(1)}M \cong \mathbb{Z}/2\{\mathrm{incl}\circ\eta\sigma, \widetilde{8\sigma}\}$.

Proof. We have the short exact sequence

$$0 \to \mathbb{Z}/2\{\eta\sigma\} \xrightarrow{\mathrm{incl}_*} \pi_8 L_{K(1)}M \xrightarrow{\mathrm{pinch}_*} \mathbb{Z}/2\{8\sigma\} \to 0,$$

because

$$\{\pi_7 L_{K(1)}\mathbb{S}^0\}_2 = \{\mathbb{Z}/16\{\sigma\}\}_2 = \mathbb{Z}/2\{8\sigma\}.$$

Hence,

$$\pi_8 L_{K(1)}M \cong \mathbb{Z}/4 \text{ or } \mathbb{Z}/2\oplus\mathbb{Z}/2.$$

The option of $\pi_8 L_{K(1)}M$ being isomorphic to $\mathbb{Z}/4$ with $\widetilde{8\sigma}$, the preimage of 8σ by pinch$_*$, as generator is not valid because

$$\begin{aligned} 2\,\widetilde{8\sigma} &= \mathrm{incl}_*(\eta(8\sigma)), \text{ by Remark 5.2.1} \\ &= \mathrm{incl}_*(8\eta\sigma) \\ &= 0, \text{ since } 8\eta\sigma = 0. \end{aligned}$$

As a consequence,

$$\pi_8 L_{K(1)}M \cong \mathbb{Z}/2\oplus\mathbb{Z}/2,$$

with $\widetilde{8\sigma}$ generating one summand, and incl$_*(\eta\sigma)$ generating the second one. □

To specify the element $\widetilde{8\sigma}$ in the above computation, and to be able to compute $[M, L_{K(1)}M]_8^{\mathrm{Sp}}$, we need the following equality.

Lemma 5.2.6. $8\sigma = \mathrm{pinch}\circ v_1^4\circ\mathrm{incl}$ *in* $\mathrm{Ho}(L_{K(1)}\mathrm{Sp})$.

Proof. The element pinch $\circ\, v_1^4 \circ$ incl lies in

$$\pi_7 L_{K(1)} \cong \mathbb{Z}/16\{\sigma\}.$$

However, we know that

$$2\,\mathrm{pinch} \circ v_1^4 \circ \mathrm{incl} = 0, \text{ because } 2\,\mathrm{pinch} = 0.$$

Therefore, the element $\mathrm{pinch} \circ v_1^4 \circ \mathrm{incl}$ can be either 0 or 8σ. Now, in order to prove the desired equality, what is left to do is to prove that the element in question cannot be equal to zero. Assume that

$$\mathrm{pinch} \circ v_1^4 \circ \mathrm{incl} = 0.$$

By looking at the long exact homotopy sequence below produced from the exact triangle (▲)

$$\ldots \to [\mathbb{S}^8, L_{K(1)}\mathbb{S}^0]^{\mathrm{Sp}} \xrightarrow{\mathrm{incl}_*} [\mathbb{S}^8, L_{K(1)}M]^{\mathrm{Sp}} \xrightarrow{\mathrm{pinch}_*} [\mathbb{S}^8, L_{K(1)}\mathbb{S}^1]^{\mathrm{Sp}} \to \ldots,$$

we see that our assumption gives us that

$$v_1^4 \circ \mathrm{incl} \in \mathrm{Ker}(\mathrm{pinch}_*) = \mathrm{Im}(\mathrm{incl}_*).$$

Hence, there exists an element $\varphi \in [\mathbb{S}^8, L_{K(1)}\mathbb{S}^0]^{\mathrm{Sp}}$ such that

$$\mathrm{incl} \circ \varphi = v_1^4 \circ \mathrm{incl}.$$

This element φ lies in

$$\pi_8(L_{K(1)}\mathbb{S}^0) \cong \mathbb{Z}/2\{\eta\sigma\},$$

therefore

$$\varphi = \eta a, \text{ for either } a = 0 \text{ or } a = \sigma.$$

By illustrating all the above factorisations in a commutative diagram and then applying the m^{th} $K(1)$-homology to it, we end up with the following commutative diagram

$$\begin{array}{ccccc} K(1)_m(L_{K(1)}\mathbb{S}^8) & \xrightarrow{K(1)_m(\mathrm{incl})} & K(1)_m(\Sigma^8 L_{K(1)}M) & \xrightarrow{K(1)_m(v_1^4)} & K(1)_m(L_{K(1)}M) \\ {\scriptstyle K(1)_m(\eta)}\downarrow & & & & \uparrow{\scriptstyle K(1)_m(incl)} \\ K(1)_m(L_{K(1)}\mathbb{S}^7) & & \xrightarrow{\qquad K(1)_m(a) \qquad} & & K(1)_m(L_{K(1)}\mathbb{S}^0). \end{array}$$

By Remark 4.2.5(b), we know that for m even, the map $K(1)_m(\mathrm{incl})$ is an isomorphism. Additionally, we know that the v_1-self map v_1^4 is a $K(1)_*$-isomorphism. Thus, the upper row in the above commutative diagram is an isomorphism for even m. However, the map $K(1)_m(\eta)$ lowers the degree by 1, so it must be zero, which leads us to a contradiction. This means that there is no such φ verifying the equality

$$\mathrm{incl} \circ \varphi = v_1^4 \circ \mathrm{incl},$$

and this means that the composition pinch $\circ\, v_1^4 \circ$ incl is indeed non-zero, which leave us with the only option of being equal to 8σ. □

Corollary 5.2.7. $\pi_8 L_{K(1)}M \cong \mathbb{Z}/2\{\mathrm{incl} \circ \eta\sigma, v_1^4 \circ \mathrm{incl}\}$

Proof. The element $\widetilde{8\sigma}$ in Computation 5.2.5 is a preimage of 8σ by pinch_* of order two, i.e. it is an element in $\pi_8 L_{K(1)}M$ such that

$$2\,\widetilde{8\sigma} = 0, \text{ and}$$
$$\mathrm{pinch} \circ \widetilde{8\sigma} = 8\sigma.$$

By Lemma 5.2.6, we have that

$$\mathrm{pinch} \circ \widetilde{8\sigma} = 8\sigma = \mathrm{pinch} \circ v_1^4 \circ \mathrm{incl}.$$

Hence, we can choose $\widetilde{8\sigma}$ to be equal to $v_1^4 \circ$ incl since it is a preimage of 8σ by pinch_* and is of order two. □

Computation 5.2.8. $\pi_9 L_{K(1)}M \cong \mathbb{Z}/4\{\widetilde{\eta\sigma}\} \oplus \mathbb{Z}/2\{\mathrm{incl} \circ \mu\}$

Proof. We start by looking at the information provided by the short exact sequence

$$0 \to \mathbb{Z}/2\{\eta^2\sigma, \mu\} \xrightarrow{\text{incl}_*} \pi_9 L_{K(1)}M \xrightarrow{\text{pinch}_*} \mathbb{Z}/2\{\text{incl} \circ \eta\sigma, v_1^4 \circ \text{incl}\} \to 0.$$

The situation is similar to Computation 5.2.4, and the maps are acting on generators as follows

$$0 \longrightarrow \mathbb{Z}/2\{\eta^2\sigma, \mu\} \longrightarrow \pi_9 L_{K(1)}M \longrightarrow \mathbb{Z}/2\{\eta\sigma\} \longrightarrow 0$$

$$\widetilde{\eta\sigma} \xmapsto{\text{pinch}_*} \eta\sigma$$

$$\eta^2\sigma \xmapsto{\text{incl}_*} 2\widetilde{\eta\sigma} \xmapsto{\text{pinch}_*} 0$$

$$\mu \xmapsto{\text{incl}_*} \text{incl} \circ \mu.$$

By Remark 5.2.1,

$$2\widetilde{\eta\sigma} = \text{incl} \circ \eta^2\sigma \neq 0, \text{ because incl}_* \text{ is injective.}$$

Hence, the element $\widetilde{\eta\sigma}$ is generating the $\mathbb{Z}/4$ summand, and the other summand is generated by the element $\text{incl} \circ \mu$. □

Before computing the groups $[M, L_{K(1)}M]_8^{\mathrm{Sp}}$, we need a couple of lemmas regarding some Toda bracket equalities.

Lemma 5.2.9. *We have the following equalities of Toda brackets.*

(a) $\langle y_1, 8\sigma, 2\rangle = \eta^2\sigma$.

(b) $\langle \eta + y_1, 8\sigma, 2\rangle = \langle \eta, 8\sigma, 2\rangle + \langle y_1, 8\sigma, 2\rangle$.

(c) $\langle \text{incl} \circ \eta, 8\sigma, 2\rangle = \text{incl} \circ \langle \eta, 8\sigma, 2\rangle$.

Proof. (a) We start by computing the indeterminacy of the Toda bracket involved. By going back to the Definition 3.2.9 of Toda brackets, we know that the indeterminacy here is

$$\begin{aligned}(\Sigma 2)^*([\mathbb{S}^9, L_{K(1)}\mathbb{S}^0]) + (y_1)_*([\mathbb{S}^9, L_{K(1)}\mathbb{S}^1]) &\cong \mathbb{Z}/2\{y_1\eta\sigma\} \text{ (since } \pi_9(L_{K(1)}\mathbb{S}^0) \cong \mathbb{Z}/2) \\ &= 0 \text{ (because } y_1\eta = 0).\end{aligned}$$

By Theorem 3.2.11(b) and the fact that $y_1 = y_0\eta$, we see that the bracket $\langle y_1, 8\sigma, 2\rangle$ contains

$$\begin{aligned} y_0 \circ \langle \eta, 8\sigma, 2\rangle &= \{y_0\mu, y_0\mu + y_0\eta^2\sigma\}, \text{ by (6) in Section 5.1} \\ &= \{\eta^2\sigma, \eta^2\sigma + y_0(y_0\mu)\}, \text{ because } \eta^2\sigma = \mu y_0 \\ &= \{\eta^2\sigma\}, \text{ because } y_0^2 = 0.\end{aligned}$$

This tells us that

$$\langle y_1, 8\sigma, 2\rangle = \eta^2\sigma.$$

(b) We first notice that by Remark 3.2.12 (b), we have

$$\langle \eta + y_1, 8\sigma, 2\rangle \subseteq \langle \eta, 8\sigma, 2\rangle + \langle y_1, 8\sigma, 2\rangle. \tag{7}$$

To show the other inclusion, we need first to determine the indeterminacy of the Toda bracket $\langle \eta + y_1, 8\sigma, 2\rangle$. As before, the indeterminacy now is

$(\Sigma 2)^*([\mathbb{S}^9, L_{K(1)}\mathbb{S}^0]) + (\eta + y_1)_*([\mathbb{S}^9, L_{K(1)}\mathbb{S}^1])$, and that consists of the elements

$$\{0, \eta^2\sigma, y_1\eta\sigma, \eta^2\sigma + y_1\eta\sigma\} = \{0, \eta^2\sigma\}, \text{ because } y_1\eta = 0.$$

Hence, the bracket on the left hand side in (7) has indeterminacy $\eta^2\sigma$, which is the same as the indeterminacy of the sum of brackets on the right hand side.

Plus, by part (a) of this lemma, and (6) we have

$$\begin{aligned}\langle \eta, 8\sigma, 2\rangle + \langle y_1, 8\sigma, 2\rangle &= \{\mu, \mu + \eta^2\sigma\} + \{\eta^2\sigma\}\\ &= \{\mu + \eta^2\sigma, \mu + 2\,\eta^2\sigma\}\\ &= \{\mu, \mu + \eta^2\sigma\},\ \text{because } 2\,\eta^2\sigma = 0.\end{aligned}$$

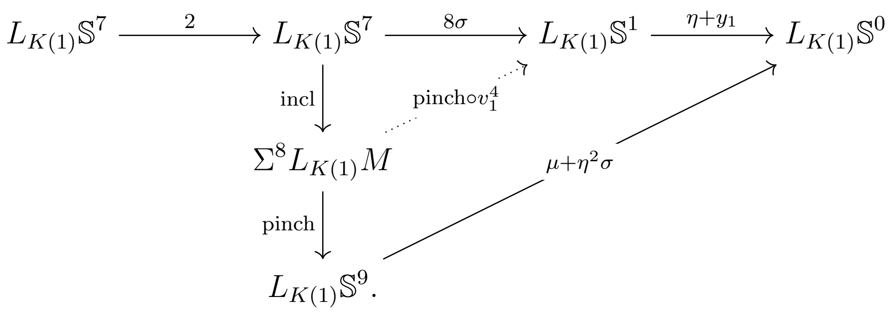
Therefore, in order to show the inclusion

$$\{\mu, \mu + \eta^2\sigma\} \subseteq \langle \eta + y_1, 8\sigma, 2\rangle,$$

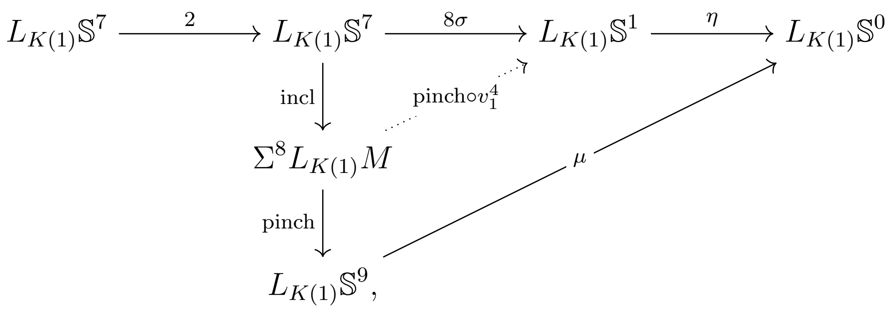
it is sufficient to prove that

$$\mu + \eta^2\sigma \in \langle \eta + y_1, 8\sigma, 2\rangle.$$

To prove that $\mu + \eta^2\sigma$ belongs to the Toda bracket $\langle \eta + y_1, 8\sigma, 2\rangle$, it is sufficient to verify that the diagram below commutes



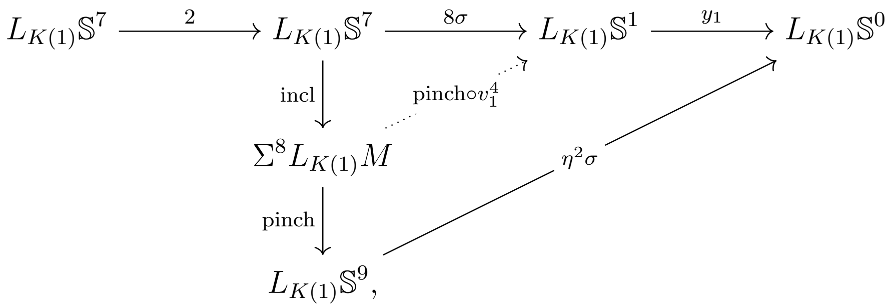
On the other hand, since the element μ is in $\langle \eta, 8\sigma, 2\rangle$, it verifies the commutativity of the diagram



i.e. we have the equality

$$\mu \circ \text{pinch} = \eta \circ \text{pinch} \circ v_1^4. \tag{8}$$

Plus, part (a) of this lemma ensures the commutativity of the diagram



which gives us the equality

$$\eta^2\sigma \circ \text{pinch} = y_1 \circ \text{pinch} \circ v_1^4. \tag{9}$$

All the above tells us that

$$\begin{aligned}(\mu + \eta^2\sigma) \circ \text{pinch} &= \mu \circ \text{pinch} + \eta^2\sigma \circ \text{pinch} \\ &= \eta \circ \text{pinch} \circ v_1^4 + y_1 \circ \text{pinch} \circ v_1^4, \text{ by (8) and (9).}\end{aligned}$$

Consequently, $\mu + \eta^2\sigma \in \langle \eta + y_1, 8\sigma, 2 \rangle$. However, remember that the indeterminacy of $\langle \eta + y_1, 8\sigma, 2 \rangle$ is $\eta^2\sigma$, which tells us that

$$\mu + \eta^2\sigma + \eta^2\sigma = \mu \in \langle \eta + y_1, 8\sigma, 2 \rangle.$$

Therefore, equality must hold in (7).

(c) By the Juggling Theorem 3.2.11 (b), we have

$$\text{incl} \circ \langle \eta, 8\sigma, 2 \rangle \subseteq \langle \text{incl} \circ \eta, 8\sigma, 2 \rangle.$$

Our goal is to establish equality, and hence show that

$$\left\langle \mathrm{incl} \circ \eta, 8\sigma, 2 \right\rangle \subseteq \mathrm{incl} \circ \left\langle \eta, 8\sigma, 2 \right\rangle. \tag{10}$$

We know by (6) that

$$\begin{aligned} \mathrm{incl} \circ \left\langle \eta, 8\sigma, 2 \right\rangle &= \{\mathrm{incl} \circ \mu, \mathrm{incl} \circ \mu + \mathrm{incl} \circ \eta^2\sigma\} \\ &= \{\mathrm{incl} \circ \mu, \mathrm{incl} \circ \mu + 2\widetilde{\eta\sigma}\}. \end{aligned}$$

As for the bracket $\left\langle \mathrm{incl} \circ \eta, 8\sigma, 2 \right\rangle$, its indeterminacy is

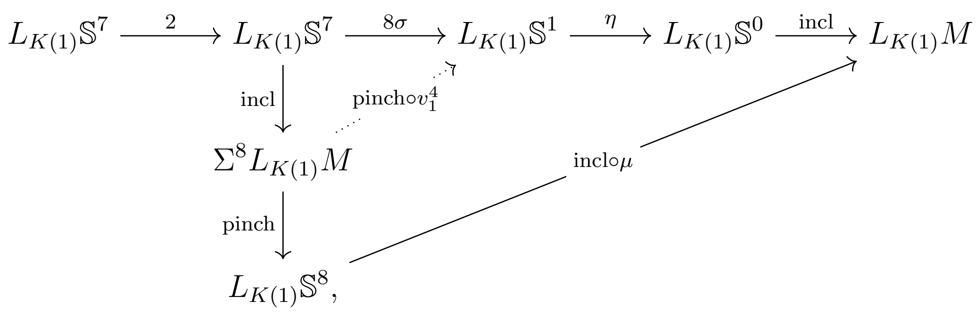
$$(\Sigma 2)^*([\mathbb{S}^9, L_{K(1)}M]) + (\mathrm{incl} \circ \eta)_*([\mathbb{S}^9, L_{K(1)}\mathbb{S}^1]) \text{ consisting of the elements}$$

$$\{0, 2\widetilde{\eta\sigma}, \mathrm{incl} \circ \eta^2\sigma, 2\widetilde{\eta\sigma} + \mathrm{incl} \circ \eta^2\sigma\} = \{0, 2\widetilde{\eta\sigma}, 4\widetilde{\eta\sigma}\}, \text{ because } \mathrm{incl} \circ \eta^2\sigma = 2\widetilde{\eta\sigma}.$$

However, the only non-trivial element in the above indeterminacy is $2\widetilde{\eta\sigma}$ since $4\widetilde{\eta\sigma} = 0$. We conclude that the indeterminacies of the brackets in question are equal. Hence, to finish our proof, it is enough to show that

$$\mathrm{incl} \circ \mu \in \left\langle \mathrm{incl} \circ \eta, 8\sigma, 2 \right\rangle.$$

To that end, it is enough to check the commutativity of the diagram



i.e. we need to check that

$$\mathrm{incl} \circ \mu \circ \mathrm{pinch} = \mathrm{incl} \circ \eta \circ \mathrm{pinch} \circ v_1^4. \tag{11}$$

However, since $\mu \in \langle \eta, 8\sigma, 2\rangle$, we have

$$\mu \circ \text{pinch} = \eta \circ \text{pinch} \circ v_1^4,$$

which means that the equality in (11) is true. Consequently, the equality in (10) holds because we have the same indeterminacy on both sides. □

Lemma 5.2.10. $\langle 2\text{Id}_{L_{K(1)}M}, v_1^4 \circ \text{incl}, 2\rangle = \text{incl} \circ \langle \eta, 8\sigma, 2\rangle$

Proof. First, we have the following inclusion

$$\begin{aligned}\langle 2\text{Id}_{L_{K(1)}M}, v_1^4 \circ \text{incl}, 2\rangle &= \langle \text{incl} \circ \eta \circ \text{pinch}, v_1^4 \circ \text{incl}, 2\rangle, \text{ by Remark 3.1.9} \\ &\subseteq \langle \text{incl} \circ \eta, \text{pinch} \circ v_1^4 \circ \text{incl}, 2\rangle, \text{by Theorem 3.2.11(a)}.\end{aligned}$$

On the other hand, we know that

$$\begin{aligned}\langle \text{incl} \circ \eta, \text{pinch} \circ v_1^4 \circ \text{incl}, 2\rangle &= \langle \text{incl} \circ \eta, 8\sigma, 2\rangle, \text{ by Lemma 5.2.6} \\ &= \text{incl} \circ \langle \eta, 8\sigma, 2\rangle, \text{by Lemma 5.2.9 (c)}. \\ &= \{\text{incl} \circ \mu, \text{incl} \circ \mu + 2\widetilde{\eta\sigma}\}, \text{ by (6)}.\end{aligned}$$

Hence, we have that

$$\langle 2\text{Id}_{L_{K(1)}M}, v_1^4 \circ \text{incl}, 2\rangle \subseteq \{\text{incl} \circ \mu, \text{incl} \circ \mu + 2\widetilde{\eta\sigma}\}. \tag{12}$$

The above inclusion tells us that the bracket $\langle 2\text{Id}_{L_{K(1)}M}, v_1^4 \circ \text{incl}, 2\rangle$ contains either one element or two elements, and if the latter is true we will have equality between the brackets involved. To that end, we study the indeterminacy of the bracket $\langle 2\text{Id}_{L_{K(1)}M}, v_1^4 \circ \text{incl}, 2\rangle$. The indeterminacy here is equal to

$$(\Sigma 2)^*([\mathbb{S}^9, L_{K(1)}M]) + (2\text{Id}_{L_{K(1)}M})_*([\mathbb{S}^9, L_{K(1)}M]) \cong \mathbb{Z}/2\{2\widetilde{\eta\sigma}\}.$$

Consequently, we have the desired equality

$$\langle 2\mathrm{Id}_{L_{K(1)}M}, v_1^4 \circ \mathrm{incl}, 2 \rangle = \mathrm{incl} \circ \langle \eta, 8\sigma, 2 \rangle. \qquad \square$$

Lemma 5.2.11. $\langle 2\mathrm{Id}_{L_{K(1)}M}, \mathrm{incl} \circ \eta\sigma, 2 \rangle = 0,$ *with indeterminacy*

$$2\pi_9 L_{K(1)}M \cong \mathbb{Z}/2\{2\widetilde{\eta\sigma}\}.$$

Proof. The indeterminacy of the Toda bracket in question is

$$(\Sigma 2)^*([\mathbb{S}^9, L_{K(1)}M]) + (2\mathrm{Id}_{L_{K(1)}M})_*([\mathbb{S}^9, L_{K(1)}M]) = 2\pi_9 L_{K(1)}M \cong \mathbb{Z}/2\{2\widetilde{\eta\sigma}\}.$$

On the other hand,

$$\begin{aligned}\langle 2\mathrm{Id}_{L_{K(1)}M}, \mathrm{incl} \circ \eta\sigma, 2 \rangle &= \langle \mathrm{incl} \circ \eta \circ \mathrm{pinch}, \mathrm{incl} \circ \eta\sigma, 2 \rangle, \text{ by Remark 3.1.9} \\ &\subseteq \langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ \mathrm{incl} \circ \eta\sigma, 2 \rangle, \text{ by the Juggling Theorem 3.2.11 (a).}\end{aligned}$$

However, the bracket $\langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ \mathrm{incl} \circ \eta\sigma, 2 \rangle$ is equal to zero because

$$\mathrm{pinch} \circ \mathrm{incl} = 0.$$

Plus, the indeterminacy of $\langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ \mathrm{incl} \circ \eta\sigma, 2 \rangle$ is

$$(\Sigma 2)^*([\mathbb{S}^9, L_{K(1)}M]) + (\mathrm{incl} \circ \eta)_*([\mathbb{S}^9, \mathbb{S}^1]),$$

consisting of the elements

$$\{0, 2\widetilde{\eta\sigma}, \mathrm{incl} \circ \eta^2\sigma, 2\widetilde{\eta\sigma} + \mathrm{incl} \circ \eta^2\sigma\}$$

in which the only non-trivial element is $2\widetilde{\eta\sigma}$ because $\mathrm{incl} \circ \eta^2\sigma = 2\widetilde{\eta\sigma}$, and $\widetilde{\eta\sigma}$ is of order 4. Hence, the two brackets on both sides of the inclusion have the same indeterminacy. The above tells us that the bracket $\langle 2\mathrm{Id}_{L_{K(1)}M}, \mathrm{incl} \circ \eta\sigma, 2 \rangle$ have at least one element, and is included in the bracket $\langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ \mathrm{incl} \circ \eta\sigma, 2 \rangle$ which consists of exactly one element. We conclude that the two brackets must be equal, and the bracket $\langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ \mathrm{incl} \circ \eta\sigma, 2 \rangle$ is equal to zero, with indeterminacy $\mathbb{Z}/2\{2\widetilde{\eta\sigma}\}$. □

Computation 5.2.12.

$$[M, L_{K(1)}M]_8^{\mathrm{Sp}} \cong \mathbb{Z}/4\{v_1^4\} \oplus \mathbb{Z}/2\{\widetilde{\eta\sigma} \circ \mathrm{pinch}, \mathrm{Id}_{L_{K(1)}M} \wedge \eta\sigma\}$$

Proof. Using the same method to produce short exact sequences as in previous calculations, we have the short exact sequence

$$0 \to \big(\pi_9 L_{K(1)}M\big)/2 \xrightarrow{\mathrm{pinch}^*} [M, L_{K(1)}M]_8^{\mathrm{Sp}} \xrightarrow{\mathrm{incl}^*} \{\pi_8 L_{K(1)}M\}_2 \to 0.$$

Note that this short exact sequence comes from the long exact sequence obtained when we apply the contravariant functor $[-, L_{K(1)}M]$ to the exact triangle (▲). By Corollary 5.2.7 and Computation 5.2.8, the above equals

$$0 \to \mathbb{Z}/2\{\widetilde{\eta\sigma}, \mathrm{incl} \circ \mu\} \xrightarrow{\mathrm{pinch}^*} [M, L_{K(1)}M]_8^{\mathrm{Sp}} \xrightarrow{\mathrm{incl}^*} \mathbb{Z}/2\{\mathrm{incl} \circ \eta\sigma, v_1^4 \circ \mathrm{incl}\} \to 0.$$

For $x \in \{\pi_8 L_{K(1)}M\}_2$, we denote the preimage of x under incl^* by $\bar{x} \in [M, L_{K(1)}M]_8^{\mathrm{Sp}}$, i.e.

$$\mathrm{incl}^*(\bar{x}) = \bar{x} \circ \mathrm{incl} = x.$$

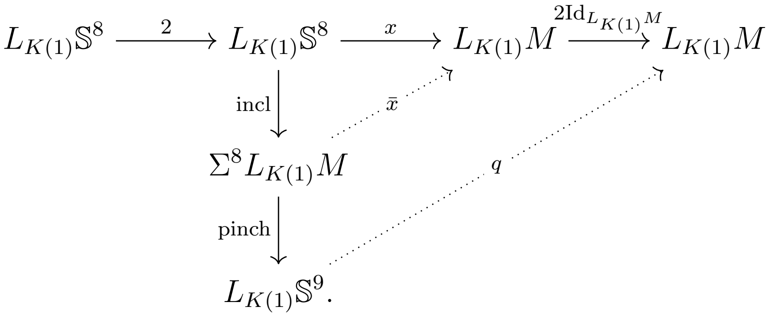
Since $2x = 0$, we have that

$$2x = 2(\bar{x} \circ \mathrm{incl}) = \mathrm{incl}^*(2\bar{x}) = 0,$$

and the element $2\bar{x} \in \mathrm{Im}(\mathrm{pinch}^*)$. Hence, there exists a unique $q \in \big(\pi_9 L_{K(1)}M\big)/2$ such that

$$\mathrm{pinch}^*(q) = q \circ \mathrm{pinch} = 2\bar{x}.$$

Now, for a fixed $x \in \{\pi_8 L_{K(1)}M\}_2$ if q is not equal to zero, then $2\bar{x}$ is not equal to zero as well, because pinch^* is injective. If the latter is true, then $\bar{x}$ generates a $\mathbb{Z}/4$-summand. This is where the Toda brackets come in handy. The element q lies in the Toda bracket $\langle 2\text{Id}_{L_{K(1)}M}, x, 2\rangle$. In other words, for a fixed x, q depends on x, and satisfies the commutativity of the diagram below



So to determine which $x \in \{\pi_8 L_{K(1)}M\}_2$ is giving us $2\bar{x} = 0$, we will make use of the previous Toda bracket calculations. For $x = v_1^4 \circ \text{incl}$, we have

$$q \in \langle 2\text{Id}_{L_{K(1)}M}, v_1^4 \circ \text{incl}, 2\rangle = \text{incl} \circ \langle \eta, 8\sigma, 2\rangle, \text{ by Lemma 5.2.10.}$$

However, we know that

$$\mu \in \langle \eta, 8\sigma, 2\rangle, \text{ with indeterminacy} = \eta^2\sigma \text{ (by (6)).}$$

This means that

$$q = \text{incl} \circ \mu, \text{ or}$$
$$q = \text{incl} \circ \mu + \text{incl} \circ \eta^2\sigma.$$

By Remark 5.2.2

$$\text{pinch}^*(\text{incl} \circ \eta^2\sigma) = \text{incl} \circ \eta^2\sigma \circ \text{pinch} = 0,$$

which gives us that in either case, for $x = v_1^4 \circ \mathrm{incl}$, we have

$$\mathrm{incl} \circ \mu \circ \mathrm{pinch} = 2\bar{x} = 2v_1^4 \neq 0, \text{ because pinch}^* \text{ is injective.} \qquad (\bigstar)$$

Now, for $x = \mathrm{incl} \circ \eta\sigma$, we have by Lemma 5.2.11 that

$$q \in \left\langle 2\mathrm{Id}_{L_{K(1)}M}, \mathrm{incl} \circ \eta\sigma, 2 \right\rangle = 2\pi_9 L_{K(1)}M.$$

However, the element q lies in $\big(\pi_9 L_{K(1)}M\big)/2$, and the above Toda bracket is telling us that it lies at the same time in $2\pi_9 L_{K(1)}M$. Hence, for $x = \mathrm{incl} \circ \eta\sigma$, the element q must be equal to zero. We deduce that we have just one $\mathbb{Z}/4$-summand, and two other $\mathbb{Z}/2$-summands. So the short exact sequence we are looking at now is of the form

$$0 \to \mathbb{Z}/4\{\widetilde{\eta\sigma}, \mathrm{incl}{\circ}\mu\} \xrightarrow{\mathrm{pinch}^*} \mathbb{Z}/4\{v_1^4\} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xrightarrow{\mathrm{incl}^*} \mathbb{Z}/2\{\mathrm{incl}{\circ}\eta\sigma, v_1^4{\circ}\mathrm{incl}\} \to 0.$$

One of the two $\mathbb{Z}/2$ summands in the middle is generated by

$$\mathrm{pinch}^*(\widetilde{\eta\sigma}) = \widetilde{\eta\sigma} \circ \mathrm{pinch}.$$

As for the other $\mathbb{Z}/2$ summand, it is generated by a preimage of $\mathrm{incl} \circ \eta\sigma$ under the map incl^*. Hence, any element $P \in [M, L_{K(1)}M]_8$ with

$$\mathrm{incl}^*(P) = P \circ \mathrm{incl} = \mathrm{incl} \circ \eta\sigma$$

can be taken to be a generator of the other $\mathbb{Z}/2$ summand. The choice

$$P = \mathrm{Id}_{L_{K(1)}M} \wedge \eta\sigma : L_{K(1)}M \wedge L_{K(1)}\mathbb{S}^8 \simeq \Sigma^8 L_{K(1)}M \to L_{K(1)}M \wedge L_{K(1)}\mathbb{S}^0 \simeq L_{K(1)}M$$

can be taken as a generator because it verifies our condition, which can be seen by the commutativity of the diagram

$$\begin{array}{ccccccc}
L_{K(1)}\mathbb{S}^8 & \xrightarrow{2.} & L_{K(1)}\mathbb{S}^8 & \xrightarrow{\text{incl}} & \Sigma^8 L_{K(1)}M \simeq L_{K(1)}M \wedge \mathbb{S}^8 & \xrightarrow{\text{pinch}} & L_{K(1)}\mathbb{S}^9 \\
\downarrow{\scriptstyle \eta\sigma} & & \downarrow{\scriptstyle \eta\sigma} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \eta\sigma} \\
L_{K(1)}\mathbb{S}^0 & \xrightarrow{2.} & L_{K(1)}\mathbb{S}^0 & \xrightarrow{\text{incl}} & L_{K(1)}M \simeq L_{K(1)}M \wedge L_{K(1)}\mathbb{S}^0 & \xrightarrow{\text{pinch}} & L_{K(1)}\mathbb{S}^1.
\end{array}$$

□

The equation (★) in the previous proof gives us the following equality.

Corollary 5.2.13. $2v_1^4 = \text{incl} \circ \mu \circ \text{pinch} \neq 0$.

Chapter 6

Future work

A natural question that arises after proving rigidity of the $K(1)$-local stable homotopy category is what happens if we go up one chromatic level and try to investigate, 2-locally, the rigidity of $\mathrm{Ho}(L_{K(2)}\mathrm{Sp})$?

In this chapter, I will outline what difficulties might arise at $n = 2$ and will explain some ideas in order to overcome these.

The first step in constructing the desired Quillen equivalence between $L_{K(2)}\mathrm{Sp}$ and a stable model category $\mathcal{C}$ is to build a Quillen functor from $L_{K(2)}\mathrm{Sp}$ to $\mathcal{C}$. As we have seen in Proposition 4.2.2, we have a Quillen adjunction

$$X \wedge - : L_{K(2)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

if and only if the spectrum $\mathrm{RHom}(X, Y)$ is $K(2)$-local for all $Y \in \mathcal{C}$. This significantly complicates the case we are trying to study. To be more specific, in the $K(1)$-local case, as we have proved in Lemma 4.1.4, a 2-complete spectrum A is $K(1)$-local if and only if the v_1^4-self map induces an isomorphism on its mod-2 homotopy groups $[M(\mathbb{Z}/2), A]_*^{\mathrm{Sp}}$. However, in the $K(2)$-local case, we cannot prove that a spectrum is $K(2)$-local, or $E(2)$-local to start with, by just testing it against a v_2-self map. This goes back to the fact that the Telescope conjecture is still open at $n = 2$. Therefore, a starting point is to try to adjust our localisation so it is enough to test against a v_2-self map. Hence, for many reasons listed below, a better idea is to consider the $K(2)$-*finite localisation*, $L^f_{K(2)}\mathrm{Sp}$, in the sense of

[Mil92] at $p = 2$. We start by defining what we mean by *finite localisation.*

Let E_* be a generalised homology theory represented by a spectrum E. The E-finite localisation with respect to some homology theory E_* can be characterised in exactly the same terms as the Bousfield localisation in Section 3.3, but with the addition of a finiteness assumption.

Definition 6.1.

- A spectrum W is *finitely* E_*-local if and only if $[A, W] = 0$ for every *finite* E_*-acyclic spectrum A.
- A spectrum Z is *finitely* E_*-acyclic if and only if $[Z, W] = 0$ for every *finitely* E_*-local spectrum W.
- A map $f : X \to Y$ is a *finite* E_*-equivalence if and only if its cofiber is *finitely* E_*-acyclic.

Theorem 6.2. [Mil92, Theorem 4] For any spectrum X, there is a finite E_*-equivalence from X to a finitely E_*-local spectrum $L_E^f X$. It is denoted by

$$\eta_X^f : X \to L_E^f X,$$

and called the finite E-localisation. The map η_X^f is *initial* among maps from X to finitely E_*-local spectra, and *terminal* among finite E_*-equivalences out of X.

Moreover, we can relate this finite localisation to Bousfield localisation of Section 3.3 as follows.

Theorem 6.3. [Mil92, Corollary 11] Finite E-localisation is Bousfield localisation with respect to the spectrum $L_E^f \mathbb{S}^0$. In other words, for any spectrum X, we have

$$L_E^f X = L_{L_E^f \mathbb{S}^0} X.$$

Remark 6.4. Notice that any E_*-local spectrum is in particular finitely E_*-local. Since the map η_X^f is initial among maps from X to finitely E_*-local spectra, we get a unique factorisation from $L_E^f X$ to $L_E X$

$$\begin{array}{ccc} X & \xrightarrow{\eta_X^f} & L_E^f X \\ {\scriptstyle \eta_X}\downarrow & \swarrow{\scriptstyle r} & \\ L_E X. & & \end{array}$$

The Telescope conjecture suggests that for $E = E(n)$, the Johnson-Wilson theory, the map

$$L_n^f X \to L_n X$$

is an equivalence for all X. It has been shown by Mahowald [Mah81] (for $p = 2$) and Miller [Mil81] (for $p > 2$) that the Telescope conjecture is true for $n = 1$. As for other values of n and p, the Telescope conjecture remains open.

By [Rav84, Theorem 2.11], $K(n)$-locality coincides with $E(n)$-locality for *finite* spectra. Hence, $K(n)$-finite localisation is the same as $E(n)$-finite localisation, and we denote it by

$$L_n^f := L_{E(n)}^f = L_{K(n)}^f.$$

Definition 6.5. As seen in Theorem 3.3.47, by the periodicity theorem, any finite spectrum X of type n (Definition 3.3.44) admits a v_n-self map. The *telescope* of X, denoted $\mathrm{Tel}(X)$, is the homotopy colimit of the sequence formed by the v_n-self map.

In some literature, the $E(n)$-finite localisation is called the "telescopic localisation" because the finite localisation of a type n spectrum X is the telescope of this spectrum.

Proposition 6.6. [Mil92, Proposition 14] If X is a spectrum of type n, then

$$L_n^f X = \mathrm{Tel}(X).$$

This $K(n)$-finite localisation is a nice localisation, and it will give us the main ingredients to start studying rigidity. To begin with, finite localisation is smashing [Mil92, Proposition 9], which is a desirable criterion when studying rigidity, because it provides us with the sphere spectrum as a compact generator, and that makes some calculations easier. Adding to that, as the next proposition suggests, this localisation provides us with a criterion of when a spectrum is finitely $K(2)$-local.

Proposition 6.7. [Mil92, Proposition 15] Let T be a finite p-local spectrum of type n, and denote its v_n-self map by

$$v_n^{p^i} : \Sigma^{p^i(2p^n-2)}T \to T, \text{ for some } i \geqslant 0.$$

Then a spectrum A is finitely $K(n)$-local (or finitely $E(n)$-local) if and only if $v_n^{p^i}$ induces an isomorphism

$$(v_n^{p^i})^* : [T, A]_k \to [T, A]_{k+p^i(2p^n-2)}, \text{ for all } k \in \mathbb{Z}.$$

Remark 6.8. As mentioned before, the Telescope conjecture is true for $n = 1$. Hence, by Proposition 6.6, we have that

$$L_1 X = L_1^f X = \mathrm{Tel}(X), \text{ for any spectrum } X.$$

Therefore, showing that a spectrum is $E(1)$-local is equivalent to showing that it is finitely $E(1)$-local. Consequently, by Proposition 6.7, to show that a spectrum is $E(1)$-local it is now enough to test it against a v_1-self map. However, for $n = 2$, the Telescope conjecture is still open, and hence we cannot prove that a spectrum is $E(2)$-local by only testing it against a v_2-self map. To overcome this obstacle, and study rigidity on a new chromatic level, we consider $K(2)$-*finite* localisation instead of $K(2)$-localisation since this *finite* localisation will provide us with the missing tool of verifying locality by using a v_2-self map.

The new rigidity theorem we are planning to prove using the tools mentioned earlier is thus:

$K(2)$-Finitely Local Rigidity Conjecture. Let $\mathcal{C}$ be a stable model category, $p = 2$, and let Φ be an equivalence of triangulated categories

$$\Phi \ : \ \mathrm{Ho}(L^f_{K(2)}\mathrm{Sp}) \ \rightarrow \ \mathrm{Ho}(\mathcal{C}).$$

Then the underlying model categories $L^f_{K(2)}\mathrm{Sp}$ and $\mathcal{C}$ are Quillen equivalent.

Plan of the proof. The starting point is to construct a Quillen adjunction

$$X \wedge - : L^f_{K(2)}\mathrm{Sp} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -).$$

As we have seen in Proposition 4.2.2, the above Quillen adjunction exists if and only if the spectrum $\mathrm{RHom}(X, Y)$ is *finitely* $K(2)$-local for all $Y \in \mathcal{C}$. Hence, our goal now is to prove that the spectrum $\mathrm{RHom}(X, Y)$ is finitely $K(2)$-local, and to that end we use Proposition 6.7. In conclusion, the aim now is to prove that

$$(v_2^{2^i})^* : [T, \mathrm{RHom}(X, Y)]_n \rightarrow [T, \mathrm{RHom}(X, Y)]_{n+2^i 6}$$

is an isomorphism for some type 2-spectrum T.

Since we can test against any spectrum of type 2, we have the advantage of choosing a suitable one that will make our calculations achievable. If we want to choose a spectrum related to the $K(1)$-local case, we might start considering $V(1)$, the cofibre of

$$v_1^4 : \Sigma^8 M(\mathbb{Z}/2) \rightarrow M(\mathbb{Z}/2).$$

The spectrum $V(1)$ is a type 2 spectrum and possesses a v_2^{32}-self map [BHHM08]

$$v_2^{32} : \Sigma^{192} V(1) \rightarrow V(1).$$

Although this is a well constructed spectrum, the period 192 of the self map will make our computations almost impossible. However, the good news is that in [BE16] a type 2 spectrum called Z with certain properties has been constructed, and it has been shown that it possesses a v_2^1-self map

$$v_2^1 : \Sigma^6 Z \to Z.$$

This makes the spectrum Z an ideal candidate to test finite locality against. To be more accurate, we want to prove that v_2^1 induces an isomorphism

$$(v_2^1)^* : [Z, \mathrm{RHom}(X,Y)]_n \to [Z, \mathrm{RHom}(X,Y)]_{n+6}, \text{ for all } n \in \mathbb{Z}.$$

To that end, similarly to Chapter 5, we will need to do some calculations in the $K(2)$-finite local setting. More precisely, we need to calculate the homotopy groups

$$[Z, L_2^f Z]_6^{\mathrm{Sp}} = [Z, Z]_6^{L_2^f \mathrm{Sp}}.$$

Therefore, a first step towards solving rigidity in this finite setting is to calculate the homotopy groups

$$\pi_n(L_2^f Z) \text{ for } 0 \leqslant n \leqslant 7.$$

For this calculation, we can use a special type of spectral sequence, the "localised Adams spectral sequence". This localised Adams spectral sequence was constructed by Mahowald and Sadofsky in [MS95]. It converges to $\pi_*(L_2^f Z)$ with the E_2-term

$$E_2^{s,t} \cong v_2^{-1} Ext_{\mathcal{A}}^{s,t}(H^*(Z), \mathbb{Z}/2).$$

Using this spectral sequence in order to calculate the desired homotopy groups is not an easy task. Even if we can calculate the E_2- terms, the biggest challenge is determining the differentials, and any progress in this direction will be of significant importance. This will not only help answering the rigidity question at the chromatic level 2 but will also contribute to other areas of chromatic homotopy theory, especially given that nothing is known about the homotopy groups of this recently constructed spectrum Z in the world of finite localisation.

Bibliography

[Ada66] J.F. Adams. On the groups J(X) IV. *Topology*, 5(1):21–71, 1966.

[Ada69] J.F. Adams. Lectures on generalised cohomology. In *Category theory, homology theory and their applications III*, pages 1–138. Springer, 1969.

[Ada71] J.F. Adams. A variant of E.H. Brown's representability theorem. *Topology*, 10(3):185–198, 1971.

[Ada95] J.F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, 1995.

[Bau11] T. Bauer. Bousfield localization and the hasse square. *Available at* http://math.mit.edu/conferences/talbot/2007/tmfproc/Chapter09/bauer.pdf, 2011.

[BE16] P. Bhattacharya and P. Egger. A class of 2-local finite spectra which admit a v_2^1-self-map. *arXiv preprint arXiv:1608.06250*, 2016.

[BF78] A.K. Bousfield and E.M. Friedlander. Homotopy theory of γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory II*, pages 80–130. Springer, 1978.

[BHHM08] M. Behrens, M. Hill, M.J. Hopkins, and M. Mahowald. On the existence of a v_2^{32}-self map on $M(1,4)$ at the prime 2. *Homology, Homotopy and Applications*, 10(3):45–84, 2008.

[BK72] A.K. Bousfield and D.M. Kan. *Homotopy limits, completions and localizations*, volume 304. Springer Science & Business Media, 1972.

[Bou79] A.K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.

[BP04] M. Behrens and S. Pemmaraju. On the existence of the self map v_9^2 on the Smith-Toda complex V(1) at the prime 3. Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory. *Contemporary Mathematics*, 346:9–49, 2004.

[BR11] D. Barnes and C. Roitzheim. Local framings. *New York Journal of Mathematics*, 17:513–552, 2011.

[BR14a] D. Barnes and C. Roitzheim. Rational equivariant rigidity. *Contemporary Mathematics*, 617(1):13–30, 2014.

[BR14b] D. Barnes and C. Roitzheim. Stable left and right Bousfield localisations. *Glasgow Mathematical Journal*, 56(1):13–42, 2014.

[Bro62] E.H. Brown. Cohomology theories. *Annals of Mathematics*, pages 467–484, 1962.

[DS95] W.G. Dwyer and J. Spalinski. Homotopy theories and model categories. in. *Handbook of algebraic topology*, 73:73–126, 1995.

[DS09] D. Dugger and B. Shipley. A curious example of triangulated-equivalent model categories which are not Quillen equivalent. *Algebraic & Geometric Topology*, 9(1):135–166, 2009.

[Fra96] J. Franke. Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence. *K-theory Preprint Archives*, 139, 1996.

[GJ09] P.G. Goerss and J.F. Jardine. *Simplicial homotopy theory.* Springer Science & Business Media, 2009.

[Gra75] B. Gray. *Homotopy theory: an introduction to algebraic topology*, volume 64. Academic Press, 1975.

[Hir09] P.S. Hirschhorn. *Model categories and their localizations*. Number 99. American Mathematical Society, 2009.

[Hov99] M. Hovey. Model categories, volume 63 of mathematical surveys and monographs. *American Mathematical Society, Providence, RI*, 109:109–112, 1999.

[HS98] M.J. Hopkins and J.H. Smith. Nilpotence and stable homotopy theory II. *Annals of mathematics*, 148(1):1–49, 1998.

[HS99] M. Hovey and N.P. Strickland. *Morava K-theories and localisation*, volume 666. American Mathematical Society, 1999.

[HSS00] M. Hovey, B. Shipley, and J. Smith. Symmetric spectra. *Journal of the American Mathematical Society*, 13(1):149–208, 2000.

[JW75] D.C. Johnson and W.S. Wilson. BP operations and Morava's extraordinary K-theories. *Mathematische Zeitschrift*, 144(1):55–75, 1975.

[Koc96] S.O. Kochman. *Bordism, Stable Homotopy, and Adams Spectral Sequences*, volume 7. American Mathematical Society, 1996.

[Lan76] P.S. Landweber. Homological properties of comodules over MU*(MU) and BP*(BP). *American Journal of Mathematics*, pages 591–610, 1976.

[Len12] F. Lenhardt. Stable frames in model categories. *Journal of Pure and Applied Algebra*, 216(5):1080–1091, 2012.

[Lim59] E.L. Lima. *The Spanier-Whitehead duality in new homotopy categories*. 1959.

[Mac13] S. MacLane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[Mah81] M. Mahowald. bo-resolutions. *Pacific Journal of Mathematics*, 92(2):365–383, 1981.

[Mar83] H.R. Margolis. Spectra and the Steenrod algebra, volume 29 of north-holland mathematical library, 1983.

[Mil81] H.R. Miller. On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space. *Journal of Pure and Applied Algebra*, 20(3):287–312, 1981.

[Mil92] H. Miller. Finite localizations. *Boletin de la Sociedad Matematica Mexicana (2)*, 37(1-2):383–389, 1992.

[MS95] M. Mahowald and H. Sadofsky. v_n- telescopes and the Adams spectral sequence. *Duke Mathematical Journal*, 78(1):101–129, 1995.

[Nee14] A. Neeman. *Triangulated Categories.(AM-148)*, volume 148. Princeton University Press, 2014.

[Pat16] I. Patchkoria. Rigidity in equivariant stable homotopy theory. *Algebraic & Geometric Topology*, 16(4):2159–2227, 2016.

[Pat17a] I. Patchkoria. The derived category of complex periodic K-theory localized at an odd prime. *Advances in Mathematics*, 309:392–435, 2017.

[Pat17b] I. Patchkoria. On exotic equivalences and a theorem of Franke. *Bulletin of the London Mathematical Society*, 49(6):1085–1099, 2017.

[PR17] I. Patchkoria and C. Roitzheim. Rigidity and exotic models for v_1-local G-equivariant stable homotopy theory. *arXiv preprint arXiv:1705.03855*, 2017.

[Qui06] D.G. Quillen. *Homotopical algebra*, volume 43. Springer, 2006.

[Rav84] D.C. Ravenel. Localization with respect to certain periodic homology theories. *American Journal of Mathematics*, 106(2):351–414, 1984.

[Rav03] D.C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. American Mathematical Society, 2003.

[Rav16] D.C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128. Princeton University Press, 2016.

[Roi07] C. Roitzheim. Rigidity and exotic models for the K-local stable homotopy category. *Geometry & Topology*, 11(4):1855–1886, 2007.

[Sch01] S. Schwede. The stable homotopy category has a unique model at the prime 2. *Advances in Mathematics*, 164(1):24–40, 2001.

[Sch02] M. Schlichting. A note on K-theory and triangulated categories. *Inventiones mathematicae*, 150(1):111–116, 2002.

[Sch07a] S. Schwede. The stable homotopy category is rigid. *Annals of Mathematics*, pages 837–863, 2007.

[Sch07b] S. Schwede. An untitled book project about symmetric spectra. *preprint*, 3(10), 2007. http://www.math.uni-bonn.de/~schwede/SymSpec-v3.pdf.

[Sch08] S. Schwede. Algebraic versus topological triangulated categories. *arXiv preprint arXiv:0807.2592*, 2008.

[Smi70] L. Smith. On realizing complex bordism modules: Applications to the stable homotopy of spheres. *American Journal of Mathematics*, 92(4):793–856, 1970.

[SS02] S. Schwede and B. Shipley. A uniqueness theorem for stable homotopy theory. *Mathematische Zeitschrift*, 239(4):803–828, 2002.

[SS03] S. Schwede and B. Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.

[Str11] J. Strom. *Modern classical homotopy theory*, volume 127. American Mathematical Society, 2011.

[Swi75] R.M. Switzer. *Algebraic topology–homotopy and homology*, volume 212. Springer, 1975.

[Tod62] H. Toda. *Composition methods in homotopy groups of spheres.* Number 49. Princeton University Press, 1962.

[Ver96] J.L. Verdier. *Des catégories dérivées des catégories abéliennes.* Société Mathématique de France, 1996.

[Whi62] G.W. Whitehead. Generalized homology theories. *Transactions of the American Mathematical Society*, 102(2):227–283, 1962.

[Wür91] U. Würgler. Morava K-theories : A survey. In *Algebraic topology Poznań 1989*, pages 111–138. Springer, 1991.