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Multiperiod portfolio optimization for asset–liability management with quadratic transaction costs

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Abstract: This paper investigates the multiperiod asset-liability management problem with quadratic transaction costs. Under the mean-variance criteria, we construct tractability models with/without the riskless asset and obtain the pre-commitment and time-consistent investment strategies through the application of embedding scheme and backward induction approach, respectively. In addition, some conclusions in the existing literatures can be regarded as the degenerated cases under our setting. Finally, the numerical simulations are given to show the difference of frontiers derived by different strategies. Also, some interesting findings on the impact of quadratic transaction cost parameters on efficient frontiers are discussed.

Keywords: Asset-liability management; Multiperiod portfolio optimization; Quadratic transaction costs; Pre-commitment strategies; Time-consistent strategies

1 Introduction

Asset-liability management (ALM) is a general risk management problem for financial services companies, such as pension funds and insurance companies. Typically, ALM involves the management of assets in such a way as to earn adequate returns while maintaining a comfortable surplus of assets over existing and future liabilities. Since the seminal work of Sharpe (1990), many attempts have been made to solve the ALM problem, among which the mean-variance criteria presented by Markowitz (1952) is of great importance. Actually, the mean-variance asset-liability management (MVALM) problem is a portfolio optimization problem, so as to realize the trade-off between the expectation of the terminal surplus maximization and minimum risk measured by the variance of the terminal surplus. Keel and Müller (1995) studied the asset-liability management problem in a single period setting, and validated the significant effect of liability on efficient frontier. However, the multiperiod MVALM problem faces with the difficulties in solving the analytical solution

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due to non-separability of variance. In this sense, the multiperiod mean-variance ALM problem cannot be directly solved by the dynamic programming approach.

Up to now, there are two mainstream approaches are applied to deal with this time-inconsistent problem. One is the embedding method initiated by Li and Ng (2000) and Zhou and Li (2000) in multiperiod and continuous-time portfolio optimization respectively, and the corresponding optimal investment strategy is called the pre-commitment strategy. Leippold et al. (2004) firstly applied the embedding method in multiperiod asset-liability management to acquire the closed form of efficient frontier. Subsequently, Xie et al. (2008) modeled the uncontrollable liability under the framework of Zhou and Li (2000) in continuous-time setting. Chang (2015) considered the issue of asset-liability management and derived effective strategy through the dynamic programming and Lagrangian duality theory. And he further validated the impact of liability on investment strategies. More studies can be found in Chen et al. (2008), Chen and Yang (2011) and Bensoussan et al. (2013). The other is the game approach, which is firstly developed by Bjork and Murgoci (2010). In this case, this optimization problem is treated as a non-cooperative game, in which the strategies at different period are determined by different players aiming at optimizing their own target functions. Nash equilibrium of these strategies was then utilized to define as the time-consistent strategy for the agent of the original problem. Wei et al. (2013) provided the first study in the time-consistent solution of the mean-variance asset-liability management. And the time-consistent strategy is derived in continuous-time setting. For more researches regarding the time-consistent strategy of asset-liability management, readers may refer to Li et al. (2012), Chen et al. (2013) and Long and Zeng (2016). Besides, some scholars have applied genetic algorithms to portfolio optimization for numerical solutions, such as Guo (2016), Li (2015), Yu et al. (2012, 2009, 2008).

However, these studies do not take into account market frictions, such as transaction costs. It is generated by investors to aggressively adjust their portfolio for the goal of the maximum profit and risk minimization. For the institutional investors engaged in bulk trading, transaction costs are particularly high. Thus, how to effectively allocate financial assets in the presence of transaction costs is a key problem to be solved. Further, Arnott (1990) found that ignorance of transaction costs would lead to invalid portfolios through empirical study.

Yoshimoto (1996) once again proved this judgment. Nevertheless, portfolio optimization with transaction costs has been an insurmountable problem. In order to obtain analytical solutions, Fu et al. (2015) represented a two-stage portfolio including a risky asset and a riskless asset, and deduced an analytical expression of the investment strategy when considering the proportional transaction cost. However, this approach is limited to one or two investment stages, and also the investor can only invest one riskless asset and one risky asset. To deal with this dilemma, Gârleanu and Pedersen (2013) promoted the optimal feedback solution for dynamic portfolios with a quadratic transaction cost which was followed by some researchers, such as Boyd et al. (2014), DeMiguel et al. (2015) and Zhang et al. (2017). As far as we know, there is very little research focus on ALM problem with transaction costs. Papi et al. (2006) considered the proportional transaction cost in ALM problem and proposed an approximation method based on the classical dynamic programming algorithm. Though the method reduces the computational and storage requirement of algorithm, it fails to acquire the analytical solutions.

Motivated by the difficulties for multiperiod asset-liability management problem with transaction costs, we provide the tractability framework to obtain the analytic solutions, which considers the quadratic transaction costs adopted by Gârleanu and Pedersen (2013). Since investors tend to pursue the goal of maximizing ultimate surplus, not the wealth of a particular period. We take the ultimate surplus of investment as the optimization target and cover the wealth accumulation of investment process, which is different from Gârleanu and Pedersen (2013). We then derive the pre-commitment and time-consistent investment strategies by applying the embedding scheme and backward induction approach, respectively. Also, we obtain the analytical expressions for the optimal investment strategies, the corresponding expectation and variance of surplus and the expected transaction costs. What is more, we study two cases, namely, the market containing a riskless asset and the investment without riskless assets. Finally, some numerical simulations are presented to compare the frontiers from different strategies and further verify the formulations derived in this paper.

The rest of this paper is organized as follows. In Section 2, we formulate the multiperiod MVALM problem containing a riskless asset with quadratic transaction costs. The pre-commitment strategy and time-consistent strategy are solved in Section 3. In Section 4, we consider the portfolio without riskless assets and derive the pre-commitment and time-consistent strategies. In Section 5, some numerical simulations are presented to show our findings for different strategies. Section 6 concludes this whole paper.

2 Problem formulation

Consider a capital market with $n+1$ assets and an investment process for T periods. Here, asset 0 is a riskless asset with a constant return rate r_t^0 while asset i is a risky asset with a random return rate e_t^i at period t for $i=1,2,...,n$ and $t=0,1,...,T-1$. It is assumed that the vector $e_t = [e_t^1, e_t^2, \cdots, e_t^n]'$, are statistically independent and return e_t has a known mean $E(e_t) = [E(e_t^1), E(e_t^2), \cdots, E(e_t^n)]'$ and a known covariance matrix

$$\Omega_t = \begin{bmatrix} Cov(e_t^1, e_t^1) & Cov(e_t^1, e_t^2) & \cdots & Cov(e_t^1, e_t^n) \\ Cov(e_t^2, e_t^1) & Cov(e_t^2, e_t^2) & \cdots & Cov(e_t^2, e_t^n) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(e_t^n, e_t^1) & Cov(e_t^n, e_t^2) & \cdots & Cov(e_t^n, e_t^n) \end{bmatrix}, \tag{2.1}$$

which is supposed to be positive definite. The investor allocates his initial wealth W_0 among all the securities in the market at initial time, along with the accumulation of wealth, and then adjust the amount of investment for each asset at period t. In order to better describe the investment process, we define v_t^i, $i=1,2,...,n$, as the investment amount on risky asset i which is allowed short selling and $\Delta v_t = \{\Delta v_t^1, \Delta v_t^2, ..., \Delta v_t^n\}$ as the adjustment amount on risky asset i at period t. Therefore, the investment amount on riskless asset is $v_t^0 = W_{t+1} - I'v_t$, and the adjustment amount on riskless asset Δv_t^0 is equal to $-1'\Delta v_t$ based on the self-financing assumption.

In addition, suppose that the investor has an exogenous liability. The initial liability is L_0. Let q_t be the return of liability at the t-th investment period, where $(e_t', q_t)'$ is statistically independent. We diagonalize the co-variance vector about the liability and risky assets, denoted as $\Omega_t^0 = diag(Cov(q_t, e_t^1), \ldots, Cov(q_t, e_t^n))$. Therefore, we have

$$L_{t+1} = q_t . L_t, \text{ for } t = 0,1,...,T-1 \tag{2.2}$$

and the surplus at period t can be expressed as $S_t = W_t - L_t$.

We follow the quadratic transaction costs adopted in Gârleanu and Pedersen (2013). Under this setting, the transaction cost (TC) associated with trading volumes Δv_t is given by

$$TC_T = \sum_{t=0}^{T-1} C_t = \sum_{t=0}^{T-1} \Delta v_t' \Lambda \Delta v_t . \tag{2.3}$$

where Λ is a symmetric positive-definite matrix measuring the level of total trading costs.

Note that the transaction cost C_t depicts the expense arising from changes on investment amounts at period t rather than trading shares shown in Gârleanu and Pedersen (2013). Trading volume Δv_t moves the average price by $\Lambda \Delta v_t$, and it leads to a total transaction costs according to T period, which can be denote as TC_T. More importantly, we assume that the transaction cost is paid beyond the amount of investment wealth W_t, that is, the transaction cost is independent of W_t. Obviously, the transaction cost is regarded as an undesirable payment, and it should be minimized in the objective function.

Let $\Delta v(t) = \{\Delta v_t, \Delta v_{t+1}, \ldots, \Delta v_{T-1}\}$ be the strategy at period t, and then the multiperiod asset-liability management problem with quadratic transaction costs can be expressed as:

$$F_t(J_t, \Delta v(t)) = \max\ E(S_T^{\Delta v(t)} | J_t) - \omega Var\,(S_T^{\Delta v(t)} | J_t) - \beta E(TC_T^{\Delta v(t)} | J_t), \tag{2.4}$$

where $S_T^{\Delta v(t)}$ and $TC_T^{\Delta v(t)}$ denote the terminal surplus and the total transaction cost corresponding to the investment strategy $\Delta v(t)$, respectively. In addition, J_t denotes the information at period, and $\omega \in (0,+\infty)$ is risk-aversion coefficient, $\beta \in (0,+\infty)$ is cost-aversion coefficient.

The difficulty in solving the problem $F_t(J_t, \Delta v(t))$ is cause by the non-separability of variance. That is, it does not satisfy the Bellman optimality principle. Therefore it can not be directly solved by dynamic programming approach. In the following, we will adopt embedding scheme and backward induction to solve this problem. According to the idea of Li and Ng (2000), for the pre-commitment strategy, we embed it into a separable auxiliary problem which can be solved by dynamic programming. Then the solution of the original problem can be obtained by the following theorem.

Theorem 2.1. If $\Delta v^*(t)=\{\Delta v_t^*, \Delta v_{t+1}^*, ..., \Delta v_{T-1}^*\}$ is the optimal strategy for the auxiliary problem

$$\widetilde{F}_t(J_t, \Delta v(t)) = \max \lambda E(S_T^{\Delta v(t)} | J_t) - \omega E((S_T^{\Delta v(t)})^2 | J_t) - \beta E(TC_T^{\Delta v(t)} | J_t), \tag{2.5}$$

then $\Delta v^*(t)$ is also the optimal strategy for the problem $F_t(J_t, \Delta v(t))$ for $\lambda = 1 + 2\omega E_t(S_T^*)$. $E_t(S_T^*)$ denotes the expectation of the investors' final surplus when he invests according to the optimal strategy $\Delta v^*(t)$. The proof of theorem 2.1 is detailed in Appendix A.

According to Theorem 2.1, the pre-commitment strategy can be obtained by the following steps:

1) We first construct the auxiliary problem

$$\max \lambda E(S_T^{\Delta v(t)} | J_t) - \omega E((S_T^{\Delta v(t)})^2 | J_t) - \beta E(TC_T^{\Delta v(t)} | J_t), \tag{2.6}$$

which is a separable structure in the sense of dynamic programming.

2) Through the idea of dynamic programming, we obtain the solution $\Delta \hat{v}(t)$ of the auxiliary problem, and $\Delta \hat{v}(t)$ is a function of λ.

3) By iterating each period of the strategy $\Delta \hat{v}(t)$ with the state transition equation of surplus, it is easy to find the expected final surplus $E_t(S_T)$, which is a function of λ. Then, by using equation $\lambda = 1 + 2\omega E_t(S_T)$ and the expression of $E_t(S_T)$ for λ, the pre-commitment strategy $\Delta v^*(t)$ for the problem $F_t(J_t, \Delta v(t))$ is solved.

Also, the problem $F_t(J_t, \Delta v(t))$ can be solved by the time-consistent strategy. Bjork and Murgoci (2010), from a mathematical point of view, proves the application of Nash equilibrium strategy to solving time-inconsistent problems. Then Wu (2013) investigates the time-consistent Nash equilibrium strategies for a multiperiod mean-variance portfolio selection problem. Mathematically, the time-consistent strategy can be defined as follows.

Definition 2.1. Let $\Delta\tilde{v}$ be a fixed control law. For an arbitrary point τ ($\tau = 0,1,...,T-1$), one selects an arbitrary control value Δv_τ and define the strategy $\Delta\bar{v}(\tau) = \{\Delta v_\tau, \Delta\tilde{v}_{\tau+1},...,\Delta\tilde{v}_{T-1}\}$. Then $\Delta\tilde{v}$ is call as the time-consistent strategy if for all $\tau < T$, it satisfies

$$\max_{\Delta v(\tau)} F_\tau(J_\tau;\Delta\bar{v}(\tau)) = F_\tau(J_\tau;\Delta\tilde{v}(\tau)) \tag{2.7}$$

Let $\Delta\tilde{v}(t)$ be the time-consistent strategy at period t, Definition 2.1 makes it possible to solve the problem by the following procedures:

1)
$$\begin{aligned}\Delta\tilde{v}(T-1) = \Delta\tilde{v}_{T-1} = \arg\max_{\Delta v_{T-1}}[&E(S_T^{\Delta v(T-1)}|J_{T-1}) \\ &-\omega Var(S_T^{\Delta v(T-1)}|J_{T-1}) - \beta E(TC_T^{\Delta v(T-1)}|J_{T-1})]\end{aligned} \tag{2.8}$$

2) Given that the decision maker $T-1$ will use $\Delta\tilde{v}_{T-1}$, $\Delta\tilde{v}_{T-2}$ is the optimal strategy by optimizing objective function $F_{T-2}(J_{T-2};(\Delta v_{T\text{-}2},\Delta\tilde{v}_{T\text{-}1}))$;

3) Generally, given that the forthcoming decision makers $t+1,...,T-1$ choose the strategy $\Delta\tilde{v}(t+1) = (\Delta\tilde{v}_{t+1},...,\Delta\tilde{v}_{T-1})$, $\Delta\tilde{v}_t$ is obtained by letting decision maker t choose Δv_t to maximize F_t. That is

$$\Delta\tilde{v}_t = \arg\max_{\Delta v_t} F_t(J_t;(\Delta v_t,\Delta\tilde{v}_{t+1},...,\Delta\tilde{v}_{T-1})) \tag{2.9}$$

For a mean-variance investor, the pre-commitment as well as time-consistent strategies are available. We will show them in the following sections.

3 Analytical solutions of multiperiod MVALM problem with a riskless asset

In this section, we consider the market with a riskless asset and derive the analytical solutions which contain pre-commitment strategy and time-consistent strategy. The corresponding investment strategies, the expectation and variance of surplus and the expected transaction costs are showed in this section.

To sum up, the formulation for the market with a riskless asset can be expressed by the following model:

$$
(P(\omega,\beta))\begin{cases} \max\limits_{\Delta v(t)} E_t(S_T)-\omega Var_t(S_T)-\beta E_t(TC_T) \\ s.t.\begin{cases} v_{t+1}=G_t(v_t+\Delta v_t), \quad t=0,1,\ldots,T-1 \\ v_{t+1}^0=r_t^0(v_t^0-I'\Delta v_t), \quad t=0,1,\ldots,T-1 \\ W_{t+1}=I'v_{t+1}+v_{t+1}^0, \quad t=0,1,\ldots,T-1 \\ L_{t+1}=q_tL_t, \quad t=0,1,\ldots,T-1 \\ S_t=W_t-L_t, \quad t=0,1,\ldots,T \\ TC_T=\sum_{t=0}^{T-1}\Delta v'_t\,\Lambda\Delta v_t, \quad t=0,1,\ldots,T-1 \end{cases} \end{cases} \tag{3.1}
$$

where $\omega \in (0,+\infty)$ is the risk-aversion coefficient, $\beta \in (0,+\infty)$ is the cost-aversion coefficient. For a specific investor, ω and β are constant.

3.1 Pre-commitment strategy for problem $(P(\omega,\beta))$

As the non-separability of variance in problem $(P(\omega,\beta))$, the objective function does not meet the requirement of dynamic programming approach. Thus, according to the Theorem 2.1, we first construct the auxiliary problem $(A(\lambda,\omega,\beta))$ and solve the problem $(P(\omega,\beta))$ based on solutions of problem $(A(\lambda,\omega,\beta))$.

$$
(A(\lambda,\omega,\beta))\begin{cases} \max\limits_{\Delta v(t)} \lambda\, E_t(S_T)-\omega E_t(S_T^2)-\beta E_t(TC_T) \\ s.t.\begin{cases} v_{t+1}=G_t(v_t+\Delta v_t), \quad t=0,1,\ldots,T-1 \\ v_{t+1}^0=r_t^0(v_t^0-I'\Delta v_t), \quad t=0,1,\ldots,T-1 \\ W_{t+1}=I'v_t+v_t^0, \quad t=0,1,\ldots,T-1 \\ L_{t+1}=q_tL_t, \quad t=0,1,\ldots,T-1 \\ S_t=W_t-L_t, \quad t=0,1,\ldots,T \\ TC_T=\sum_{t=0}^{T-1}\Delta v'_t\,\Lambda\Delta v_t, \quad t=0,1,\ldots,T-1 \end{cases} \end{cases} \tag{3.2}
$$

Obviously, $(A(\lambda,\omega,\beta))$ is a separable structure in the sense of dynamic programming. According to Theorem 2.1, we can obtain the optimal asset allocation and the optimal value of objective function by solving the analytical solution of auxiliary problem $(A(\lambda,\omega,\beta))$. For convenience, we list the notations of this section as following.

Define:

$$\widetilde{M}_T = \omega II', \widetilde{N}_T = \widetilde{O}_T = \omega, \widetilde{P}_T = \widetilde{Q}_T = 2\omega I, \widetilde{R}_T = 2\omega, \widetilde{X}_T = \lambda I, \widetilde{Y}_T = \widetilde{Z}_T = \lambda, \widetilde{K}_T = 0$$

$$\pi_t = E(G_t \widetilde{M}_{t+1} G_t), \quad \chi_t = \widetilde{N}_{t+1}(r_t^0)^2, \quad \tau_t = r_t^0 E(G_t)\widetilde{P}_{t+1}, \ t = 0,1,2,...,T-1$$

$$\kappa_t = 2[\pi_t + \beta\Lambda + \chi_t II' - \tau_t I' - I\tau'_t], \ t = 0,1,2,...,T-1$$

$$\psi_t = E(q_t G_t)\widetilde{Q}_{t+1}, \upsilon_t = \widetilde{R}_{t+1} r_t^0 E(q_t), \varpi_t = \widetilde{Y}_{t+1} r_t^0, \ t = 0,1,2,...,T-1$$

$$\widetilde{A}_t = \kappa_t^{-1}[E(G_t)x_{t+1} - \varpi_t I], \ t = 0,1,2,...,T-1$$

$$\widetilde{B}_t = \kappa_t^{-1}[\psi_t - \upsilon_t I], \ t = 0,1,2,...,T-1$$

$$\widetilde{C}_t = \kappa_t^{-1}[2\chi_t I - \tau_t], \ t = 0,1,2,...,T-1$$

$$\widetilde{D}_t = \kappa_t^{-1}[-2\pi_t + I\tau_t'], \ t = 0,1,2,...,T-1$$

$$\widetilde{M}_t = \widetilde{M}_t^1 + \beta\widetilde{M}_t^2, \quad \widetilde{N}_t = \widetilde{N}_t^1 + \beta\widetilde{N}_t^2, \quad \widetilde{O}_t = \widetilde{O}_t^1 + \beta\widetilde{O}_t^2, \quad t = 0,1,2,...,T-1$$

$$\widetilde{P}_t = \widetilde{P}_t^1 + \beta\widetilde{P}_t^2, \quad \widetilde{Q}_t = \widetilde{Q}_t^1 + \beta\widetilde{Q}_t^2, \quad \widetilde{R}_t = \widetilde{R}_t^1 + \beta\widetilde{R}_t^2, \ t = 0,1,2,...,T-1$$

$$\widetilde{X}_t = \widetilde{X}_t^1 + \beta\widetilde{X}_t^2 + \lambda\widetilde{X}_t^3, \quad \widetilde{x}_t = \widetilde{X}_t / \lambda, \ t = 0,1,2,...,T-1$$

$$\widetilde{Y}_t = \widetilde{Y}_t^1 + \beta\widetilde{Y}_t^2 + \lambda\widetilde{Y}_t^3, \quad \widetilde{y}_t = \widetilde{Y}_t / \lambda, \ t = 0,1,2,...,T-1$$

$$\widetilde{Z}_t = \widetilde{Z}_t^1 + \beta\widetilde{Z}_t^2 + \lambda\widetilde{Z}_t^3, \ t = 0,1,2,...,T-1$$

$$\widetilde{K}_t = \widetilde{K}_t^1 + \beta\widetilde{K}_t^2 + \lambda\widetilde{K}_t^3, \ t = 0,1,2,...,T-1$$

$$\widetilde{X}_T^3 = \lambda I, \widetilde{Y}_T^3 = \widetilde{Z}_T^3 = \lambda, \widetilde{K}_T^3 = 0$$

$$\nu_t^3 = E(G_t)\widetilde{X}_{t+1}^3, \ \varpi_t^3 = \widetilde{Y}_{t+1}^3 r_t^0, \ t = 0,1,2,...,T-1.$$

$$\widetilde{X}_t^3 = (\varepsilon + \widetilde{D}_t)'\nu_t^3 - \varpi_t^3 \widetilde{D}'_t I, \ t = 0,1,2,...,T-1.$$

$$\widetilde{Y}_t^3 = \nu_t^{3}{}' \widetilde{C}_t + \varpi_t^3(1 - I'\widetilde{C}_t), \ t = 0,1,2,...,T-1.$$

$$\widetilde{Z}_t^3 = -\nu_t^{3}{}' \widetilde{B}_t + \varpi_t^3 I' \widetilde{B}_t + \widetilde{Z}_{t+1}^3 E(q_t), \ t = 0,1,2,...,T-1.$$

$$\widetilde{K}_t^3 = \widetilde{K}_{t+1}^3 + \lambda\nu_t^{3}{}' \widetilde{A}_t - \lambda\varpi_t^3 I' \widetilde{A}_t, \ t = 0,1,2,...,T-1.$$

$$\widetilde{M}_T^1 = \omega II', \widetilde{N}_T^1 = \widetilde{O}_T^1 = \omega, \widetilde{P}_T^1 = \widetilde{Q}_T^1 = 2\omega I, \widetilde{R}_T^1 = 2\omega, \widetilde{X}_T^1 = \widetilde{Y}_T^1 = \widetilde{Z}_T^1 = \widetilde{K}_T^1 = 0$$

$$\widetilde{Q}_{T-1}^2 = \widetilde{D}'_{T-1}\Lambda\widetilde{B}_{T-1}, \widetilde{R}_{T-1}^2 = \widetilde{B}'_{T-1}\Lambda\widetilde{C}_{T-1}, \widetilde{X}_{T-1}^2 = \widetilde{D}'_{T-1}\Lambda\widetilde{A}_{T-1}, \widetilde{Y}_{T-1}^2 = \widetilde{A}'_{T-1}\Lambda\widetilde{C}_{T-1}$$

$\widetilde{Z}_{T-1}^2 = \widetilde{A}'_{T-1}\Lambda\widetilde{B}_{T-1}, \widetilde{K}_{T-1}^2 = \widetilde{A}'_{T-1}\Lambda\widetilde{A}_{T-1}$

The following notations are defined for $t = 0,1,2,...,T-i$ and $i = 1,2$:

$\pi_t^i = E(G_t\widetilde{M}_{t+1}^i G_t) + (i-1)\Lambda, \chi_t^i = \widetilde{N}_{t+1}^i (r_t^0)^2, \tau_t^i = r_t^0 E(G_t)\widetilde{P}_{t+1}^i$

$\psi_t^i = E(q_t G_t)\widetilde{Q}_{t+1}^i, \upsilon_t^i = \widetilde{R}_{t+1}^i r_t^0 E(q_t)$

$\widetilde{M}_t^i = (\varepsilon + \widetilde{D}_t)'\pi_t^i(\varepsilon + \widetilde{D}_t) + \chi_t^i\widetilde{D}'_t II'\widetilde{D}_t - (\varepsilon + \widetilde{D}_t)'\tau_t^i I'\widetilde{D}_t$

$\widetilde{N}_t^i = \widetilde{C}_t'\pi_t^i\widetilde{C}_t + \chi_t^i(1 - I'\widetilde{C}_t)^2 + \tau_t^i{}'\widetilde{C}_t(1 - I'\widetilde{C}_t)$

$\widetilde{O}_t^i = \widetilde{B}_t'\pi_t^i\widetilde{B}_t + \chi_t^i\widetilde{B}'_t II'\widetilde{B}_t + \widetilde{O}_{t+1}^i E(q_t^2) - \tau_t^i{}'\widetilde{B}_t I'\widetilde{B}_t - \psi_t^i{}'\widetilde{B}_t + \upsilon_t^i I'\widetilde{B}_t$

$\widetilde{P}_t^i = 2\widetilde{D}'_t\pi_t^i\widetilde{C}_t + 2\pi_t^i\widetilde{C}_t - 2\chi_t^i(1 - I'\widetilde{C}_t)\widetilde{D}'_t I + [(1 - I'\widetilde{C}_t)(\varepsilon + \widetilde{D}_t)' - \widetilde{D}'_t I\widetilde{C}'_t]\tau_t^i$

$\widetilde{Q}_t^i = -2\widetilde{D}'_t\pi_t^i\widetilde{B}_t - 2\pi_t^i\widetilde{B}_t - 2\chi_t^i\widetilde{D}'_t II'\widetilde{B}_t + [\widetilde{B}'_t I(\varepsilon + \widetilde{D}_t)' + \widetilde{D}'_t I\widetilde{B}_t]\tau_t^i + (\varepsilon + \widetilde{D}_t)'\psi_t^i - \upsilon_t^i\widetilde{D}'_t I$

$\widetilde{R}_t^i = 2[\widetilde{N}_{t+1}^i r_t^0\widetilde{B}'_t I(1 - I'\widetilde{C}_t) - \widetilde{B}_t'\pi_t^i\widetilde{C}_t] - \tau_t^i{}'[\widetilde{B}_t(1 - I'\widetilde{C}_t) - \widetilde{C}_t I'\widetilde{B}_t] + \psi_t^i{}'\widetilde{C}_t + \upsilon_t^i(1 - I'\widetilde{C}_t)$

$\widetilde{X}_t^i = \lambda\{2[-\widetilde{D}'_t\pi_t^i\widetilde{A}_t - \pi_t^i\widetilde{A}_t - \chi_t^i\widetilde{D}_t II'\widetilde{A}_t] + [\widetilde{A}'_t I(\varepsilon + \widetilde{D}_t)' + \widetilde{D}'_t I\widetilde{A}'_t]\tau_t^i\}$

$\widetilde{Y}_t^i = \lambda\{-2\widetilde{A}'_t\pi_t^i\widetilde{C}_t + 2\chi_t^i\widetilde{A}'_t I(1 - I'\widetilde{C}_t) - \tau_t^i{}'[\widetilde{A}_t(1 - I'\widetilde{C}_t) - \widetilde{C}_t I'\widetilde{A}_t]\}$

$\widetilde{Z}_t^i = \lambda\{2\widetilde{A}'_t(\pi_t^i + \chi_t^i II')\widetilde{B}_t - \tau_t^i{}'[\widetilde{A}_t I'\widetilde{B}_t + \widetilde{B}_t I'\widetilde{A}_t] - \psi_t^i{}'\widetilde{A}_t + \upsilon_t^i I'\widetilde{A}_t\}$

$\widetilde{K}_t^i = \widetilde{K}_{t+1}^i - \lambda^2[\widetilde{A}'_t(\pi_t^i - \chi_t II' + \tau_t^i{}'I')\widetilde{A}_t]$.

where I is the n-dimensional column vector of element 1, and ε is a unit matrix.

By using the procedure on the pre-commitment strategy in Section 2, the corresponding investment strategy for problem $(A(\lambda,\omega,\beta))$ can be given in the following Theorem 3.1.

Theorem 3.1. The optimal strategy Δv_t and optimal value function $f_t(v_t, v_t^0, L_t)$ at the period t for problem $(A(\lambda,\omega,\beta))$, respectively, is

$$\Delta v_t = \lambda\widetilde{A}_t + \widetilde{B}_t L_t + \widetilde{C}_t v_t^0 + \widetilde{D}_t v_t, \quad t = 0,1,2,...,T-1 \tag{3.3}$$

$$f_t(v_t, v_t^0, L_t) = -v'_t\widetilde{M}_{t+1}v_t - \widetilde{N}_t(v_t^0)^2 - \widetilde{O}_t L_t^2 - \widetilde{P}'_t v_t v_t^0 + \widetilde{Q}'_t v_t L_t + \widetilde{R}_t L_t v_t^0$$
$$+ \widetilde{X}'_t v_t + \widetilde{Y}_t v_t^0 - \widetilde{Z}_t L_t + \widetilde{K}_t, \; t = 0,1,2,...,T-1. \tag{3.4}$$

Proof. See Appendix B.

And then, in accordance with Theorem 2.1, we can obtain

$$\tilde{\lambda}^* = \frac{1 + 2\omega(\tilde{X}_0^3 + \tilde{Y}_0^3 - \tilde{Z}_0^3)}{1 - 2\omega \tilde{K}_0^3}. \tag{3.5}$$

Theorem 3.2. The optimal investment strategy of problem $(P(\omega, \beta))$, the corresponding expectation and variance of surplus and expected transaction cost for $t = 0,1,2,...,T-1$ is, respectively, as follows

$$\Delta v_t^* = \lambda^* \tilde{A}_t + \tilde{B}_t L_t + \tilde{C}_t v_t^0 + \tilde{D}_t v_t, \tag{3.6}$$

$$E_t(S_T{}^*) = \lambda^* \tilde{K}_t^3 + \tilde{X}_t^3 + \tilde{Y}_t^3 - \tilde{Z}_t^3 \tag{3.7}$$

$$\begin{aligned} Var_t(S_T^*) = {} & v'_t \tilde{M}_t^1 v_t + \tilde{N}_t^1 (v_t^0)^2 + \tilde{O}_t^1 L_t^2 + \tilde{P}_t^{1}{}' v_t v_t^0 - \tilde{Q}_t^{1}{}' v_t L_t - \tilde{R}_t^1 L_t v_t^0 \\ & - \lambda^* \tilde{x}_t'^1 v_t - \lambda^* \tilde{y}_t^1 v_t^0 + \lambda^* \tilde{z}_t^1 L_t - \tilde{K}_t^1 - [E_t(S_T^*)]^2 \end{aligned} \tag{3.8}$$

$$\begin{aligned} E_t(TC_T{}^*) = {} & v'_t \tilde{M}_t^2 v_t + \tilde{N}_t^2 (v_t^0)^2 + \tilde{O}_t^2 L_t^2 + \tilde{P}_t^{2}{}' v_t v_t^0 - \tilde{Q}_t^{2}{}' v_t L_t \\ & - \tilde{R}_t^2 L_t v_t^0 - \tilde{X}_t^{2}{}' v_t - \tilde{Y}_t^2 v_t^0 + \tilde{Z}_t^2 L_t - \tilde{K}_t^2. \end{aligned} \tag{3.9}$$

Remark 3.1. When the investor has no liability, that is $L_t = 0$, then the pre-commitment strategy reduces to

$$\Delta v_t^* = \lambda^* \tilde{A}_t + \tilde{C}_t v_t^0 + \tilde{D}_t v_t \tag{3.10}$$

and the expectation and variance of surplus and the expected transaction costs for $t = 0,1,2,...,T$, respectively, is：

$$E_t(S_T{}^*) = \lambda^* \tilde{K}_t^3 + \tilde{X}_t^3 + \tilde{Y}_t^3 \tag{3.11}$$

$$Var_t(S_T^*) = v'_t \tilde{M}_t^1 v_t + \tilde{N}_t^1 (v_t^0)^2 + \tilde{P}_t^{1}{}' v_t v_t^0 - \lambda^* \tilde{x}_t^{1}{}' v_t - \lambda^* \tilde{y}_t^1 v_t^0 - \tilde{K}_t^1 - [E_t(S_T^*)]^2 \tag{3.12}$$

$$E_t(TC_T{}^*) = v'_t \tilde{M}_t^2 v_t + \tilde{N}_t^2 (v_t^0)^2 + \tilde{P}_t^{2}{}' v_t v_t^0 - \tilde{X}_t^{2}{}' v_t - \tilde{Y}_t^2 v_t^0 - \tilde{K}_t^2. \tag{3.13}$$

Remark 3.2. When ignoring the transaction cost, that is $C_t = 0, t = 0,1,...,T-1$, the pre-commitment strategy can be acquired by setting $\beta = 0$ in equations (3.10). In addition, if the liability $L_t = 0$ ($t = 0,1,...,T-1$) at the same time, the pre-commitment optimal strategy and the frontier is equivalent to those in Li and Ng (2000).

In summary, Theorem 3.1 generally includes a portfolio optimization strategy and the corresponding frontier that does not contain transaction costs or liabilities, or both.

3.2 Time-consistent strategy for problem $(P(\omega,\beta))$

Here, we show the time-consistent strategy for multiperiod MVALM problem with quadratic transaction cost. The backwards induction is applied to solve the time-consistent strategy containing a riskless asset.

For $t=0,1,...,T-1$ and $\alpha=0$ if $t=T-1$, $\alpha=1$ if $t<T-1$, we define:

$$\tilde{x}_T=1,\tilde{y}_T=1,\tilde{z}_T=1$$

$$\tilde{\Omega}_t=\alpha E(G_t\tilde{M}_{t+1}G_t)+\tilde{x}_{t+1}\tilde{x}'_{t+1}\,\Omega_t$$

$$\tilde{\Omega}_t^0=\begin{cases}2\Omega_{T-1}^0 I,\ t=T-1\\ -E(G_tq_t)'\tilde{o}_{t+1}-\Omega_t^0{}'\tilde{x}_{t+1}\tilde{y}_{t+1}\ ,t=0,1,...,T-2\end{cases}$$

$$\tilde{\Lambda}_t=\Lambda+\alpha E(G_t\tilde{d}_{t+1}G_t)$$

$$\tilde{\Theta}_t=[2\omega\tilde{\Omega}_t+2\beta\tilde{\Lambda}_t]^{-1}$$

$$\tilde{a}_t=\tilde{\Theta}_t(2\omega\tilde{\Omega}_t+2\alpha\beta E(G_t\tilde{d}_{t+}G_t))$$

$$\tilde{c}_t=\tilde{\Theta}_t[E(G_t)\tilde{x}_{t+1}-\tilde{z}_{t+1}r_t^0 I-\alpha E(G_t)(\omega\tilde{p}_{t+1}-\beta\tilde{k}_{t+1})]$$

$$\tilde{b}_t=\tilde{\Theta}_t(\omega\tilde{\Omega}_t^0-\alpha\beta E(G_tq_t)\tilde{j}_{t+1}$$

$$\tilde{x}_t=(\varepsilon-\tilde{a}_t)'E(G_t)\tilde{x}_{t+1}+\tilde{z}_{t+1}r_t^0 I'\tilde{a}_t$$

$$\tilde{y}_t=\tilde{x}'_{t+1}\,E(G_t)\tilde{b}_t+\tilde{y}_{t+1}E(q_t)-\tilde{z}_{t+1}r_t^0 I'\tilde{b}_t$$

$$\tilde{z}_t=\tilde{z}_{t+1}r_t^0$$

$$\tilde{\xi}_t=\tilde{\xi}_{t+1}+[\tilde{x}_{t+1}E(G_t)-\tilde{z}_{t+1}r_t^0 I']\tilde{c}_t$$

$$\tilde{m}_t=(\varepsilon-a_t)'\tilde{\Omega}_t(\varepsilon-a_t)$$

$$\tilde{n}_t=\tilde{b}'_t\,\tilde{\Omega}_t\tilde{b}_t+\sigma_t-\tilde{\Omega}'^0_t\,\tilde{b}_t+\alpha\tilde{n}_{t+1}E(q_t^2)$$

$$\tilde{o}_t=(\varepsilon-\tilde{a}_t)'(2\tilde{\Omega}'_t\,\tilde{b}_t-\tilde{\Omega}_t^0)$$

$$\tilde{p}_t=(\varepsilon-\tilde{a}_{T-1})'[2\tilde{\Omega}'_t\,\tilde{c}_t+\alpha E(G_t)\tilde{p}_{t+1}]$$

$$\tilde{\varsigma}_t=2\tilde{b}'_t\,\tilde{\Omega}_t\tilde{c}_t-\tilde{\Omega}'^0_t\,\tilde{c}_t+\alpha[E(q_t)\tilde{\varsigma}_{t+1}+\tilde{p}_{t+1}'\,E(G_t)\tilde{b}_t]$$

$$\tilde{\zeta}_{\mathrm{t}}=\tilde{c}'_t\,\hat{\Omega}_t\tilde{c}_t+\alpha[\tilde{p}'_{t+1}\,E(G_t)\tilde{c}_t+\tilde{\zeta}_{\mathrm{t+1}}]$$

$$\widetilde{d}_t = \widetilde{a}'_t \Lambda \widetilde{a}_t + \alpha(\varepsilon - \widetilde{a}_t)' E(G_t \widetilde{d}_{t+1} G_t)(\varepsilon - \widetilde{a}_t)$$

$$\widetilde{h}_t = \widetilde{b}'_t \Lambda \widetilde{b}_t + \alpha[\widetilde{h}_{t+1} E(q_t^2) + \widetilde{j}'_{t+1} E(G_t q_t) \widetilde{b}_t]$$

$$\widetilde{j}_t = -2\widetilde{a}'_t \Lambda \widetilde{b}_t + \alpha[(\varepsilon - \widetilde{a}_t)'(2E(G_t \widetilde{d}_{t+1} G_t)\widetilde{b}_t + E(G_t q_t)\widetilde{j}_{t+1})]$$

$$\widetilde{k}_t = -2a'_t \Lambda c_t + \alpha[(\varepsilon - \widetilde{a}_t)' 2E(G_t \widetilde{d}_{t+1} G_t)\widetilde{c}_t + E(G_t q_t)\widetilde{k}_{t+1}]$$

$$\widetilde{u}_t = 2\widetilde{b}'_t \widetilde{\Lambda}_t \widetilde{c}_t + \alpha[\widetilde{j}'_{t+1} E(G_t q_t)\widetilde{c}_t + \widetilde{k}'_{t+1} E(G_t)\widetilde{b}_t + \widetilde{u}_{t+1} E(q_t)]$$

$$\widetilde{\iota}_t = \widetilde{c}'_t \widetilde{\Lambda}_t \widetilde{c}_t + \alpha[\widetilde{\iota}_{t+1} + \widetilde{k}'_{t+1} E(G_t)\widetilde{c}_t].$$

By applying Bellman's principle of optimality, the time-consistent investment strategy of problem $(P(\omega,\beta))$ is given in the following theorem.

Theorem 3.2. The time-consistent investment strategy of problem $(P(\omega,\beta))$ for $t = 0,1,...,T-1$ is given by

$$\Delta v_t = -\widetilde{a}_t v_t + \widetilde{b}_t L_t + \widetilde{c}_t, \tag{3.14}$$

and the expectation of surplus is

$$E_t(S_T) = \widetilde{x}'_t v_t + \widetilde{y}_t L_t + \widetilde{z}_t v_t^0 + \widetilde{\xi}_t \tag{3.15}$$

the variance of surplus is

$$Var_t(S_T) = v'_t \widetilde{m}_t v_t + \widetilde{n}_t L_t^2 + \widetilde{o}'_t v_t L_t + \widetilde{p}'_t v_t + \widetilde{\varsigma}_t L_t + \widetilde{\zeta}_t \tag{3.16}$$

the expected transaction costs is

$$E_t(TC_T) = v'_t \widetilde{d}_t v_t + \widetilde{h}_t L_t^2 + \widetilde{j}'_t v_t L_t + \widetilde{k}'_t v_t + \widetilde{u}_t L_t + \widetilde{\iota}_t. \tag{3.17}$$

Proof. See Appendix C.

Remark 3.3. If the investor have no liability, that is $L_t = 0$ for $t = 0,1,...,T-1$, then the time-consistent strategy reduces to

$$\Delta v_t = -\widetilde{a}_t v_t + \widetilde{c}_t \tag{3.18}$$

and the expectation and variance of surplus and the expected transaction costs, respectively, is:

$$E_t(S_T) = \widetilde{x}'_t v_t + \widetilde{z}_t v_t^0 + \widetilde{\xi}_t \tag{3.19}$$

$$Var_t(S_T) = v'_t \widetilde{m}_t v_t + \widetilde{p}'_t v_t + \widetilde{\zeta}_t \tag{3.20}$$

$$E_t(TC_T) = v'_t \widetilde{d}_t v_t + \widetilde{k}'_t v_t + \widetilde{\iota}_t. \tag{3.21}$$

Furthermore, the Theorem 3.2 still generalizes the situation without transaction cost when $\beta = 0$, and the situation without liability and transaction cost.

Remark 3.4. When ignoring the transaction cost, that is $C_t = 0, t = 0,1,...,T-1$, then the time-consistent strategy can be acquired by setting $\beta = 0$ in the equations (3.18). And the expectation and variance of surplus and the expected transaction costs at period t can be obtained in the same way.

Similarly, Theorem 3.2 generally includes a portfolio optimization strategy and the corresponding frontier that does not contain transaction costs or liabilities, or both.

4 Analytical optimal solutions of multiperiod MVALM problem without riskless assets

To our best knowledge, most existing literatures about portfolio selection only concern the market with a riskless asset and risky assets. However, Yao et al. (2014) pointed out that, in some real investments, the riskless asset does not exist due to the stochastic nature of real interest rates and the inflation risk. In addition, Viceira (2012) held that the expected return on riskless asset is time-varying especially in multiperiod investment. Ma et al. (2013), Gülpınar et al. (2016) and Chiu et al. (2017) also studied the market without riskless assets. Therefore, it is necessary to take an economy with only risky assets into account for the multiperiod asset allocation.

Here, we consider a market consisting of only n risky assets presented in Section 2. In this setting, this portfolio optimization problem can be written as follows:

$$(\hat{P}(\omega,\beta))\begin{cases} \max\limits_{\Delta v(t)} E_t(S_T) - \omega Var_t(S_T) - \beta E_t(TC_T) \\ s.t. \begin{cases} W_{t+1} = e_t'(v_t + \Delta v_t), \quad t = 0,1,...,T-1 \\ v_{t+1} = G_t(v_t + \Delta v_t), \quad t = 0,1,...,T-1 \\ I'\Delta v_t = 0, \quad t = 0,1,...,T-1 \\ L_{t+1} = q_t L_t, \quad t = 0,1,...,T-1 \\ S_t = W_t - L_t, \quad t = 0,1,...,T \\ TC_T = \sum\limits_{t=0}^{T-1} \Delta v'_t \Lambda \Delta v_t, \quad t = 0,1,...,T-1 \end{cases} \end{cases} \tag{4.1}$$

Obviously, the solving of multiperiod portfolio model without riskless assets is similar to that of the model with a riskless asset. Thus, we omit the proving process and only show the results in this section.

4.1 Pre-commitment strategy for problem $(\hat{P}(\omega,\beta))$

From a mathematical point of view, the nature of the problem $(\hat{P}(\omega,\beta))$ is similar to $(P(\omega,\beta))$. And the difference between problem $(\hat{P}(\omega,\beta))$ and problem $(P(\omega,\beta))$ is only the wealth equation. Therefore, it essentially has the same non-separable structure in the sense dynamic programming. Thus, we solve it by the Theorem 2.1. Similarly, we first construct auxiliary problem $(\hat{A}(\lambda,\omega,\beta))$ and then solve problem $(\hat{P}(\omega,\beta))$ through the relationship between them. The auxiliary problem is showed below:

$$(\hat{A}(\lambda,\omega,\beta))\begin{cases} \max\limits_{\Delta v(t)} \lambda E_t(S_T)-\omega E_t(S_T^2)-\beta E_t(TC_T) \\ s.t.\ \begin{cases} W_{t+1}=e_t'(v_t+\Delta v_t), \quad t=0,1,\ldots,T-1 \\ v_{t+1}=G_t(v_t+\Delta v_t), \quad t=0,1,\ldots,T-1 \\ I'\Delta v_t=0, \quad t=0,1,\ldots,T-1 \\ L_{t+1}=q_tL_t, \quad t=0,1,\ldots,T-1 \\ S_t=W_t-L_t, \quad t=0,1,\ldots,T \\ TC_T=\sum\limits_{t=0}^{T-1}\Delta v'_t\Lambda\Delta v_t, \quad t=0,1,\ldots,T-1 \end{cases} \end{cases} \tag{4.2}$$

The analytical solution and the optimal value of objective function to problem $(\hat{A}(\lambda,\omega,\beta))$ are derived by dynamic programming approach.

Define:

$$M_T=\omega II', \quad N_T=\lambda I, \quad P_T=2\omega I, \quad Q_T=-\omega, \quad R_T=-\lambda$$

$$E(G_tM_{t+1}G_t)+\beta\Lambda=\theta_t, E(G_tM_{t+1}G_t)=\phi_t, t=0,1,2,\ldots,T-1$$

$$A_t=\frac{1}{2}\theta_t^{-1}(\varepsilon-\frac{II'\theta_t^{-1}}{I'\theta_t^{-1}I})E(G_t)n_{t+1}, \ t=0,1,2,\ldots,T-1$$

$$B_t=\frac{1}{2}\theta_t^{-1}(\varepsilon-\frac{II'\theta_t^{-1}}{I'\theta_t^{-1}I})E(q_tG_t)P_{t+1}, \ t=0,1,2,\ldots,T-1$$

$$C_t=\theta_t^{-1}(\varepsilon-\frac{II'\theta_t^{-1}}{I'\theta_t^{-1}I})\phi_t, \ t=0,1,2,\ldots,T-1$$

$$M_t=M_t^1+\beta M_t^2, \quad N_t=N_t^1+\beta N_t^2+\lambda N_t^3, \quad n_t=N_t/\lambda, \ t=0,1,2,\ldots,T-1$$

$$P_t=P_t^1+\beta P_t^2, Q_t=Q_t^1+\beta Q_t^2, \ t=0,1,2,\ldots,T-1$$

$R_t = R_t^1 + \beta R_t^2 + \lambda R_t^3$, $\quad O_t = O_t^1 + \beta O_t^2 + \lambda O_t^3$, $t = 0,1,2,...,T-1$

$N_T^3 = I, R_T^3 = -1, O_T^3 = 0$

$N_t^3 = (\varepsilon - C_t)E(G_t)N_{t+1}^3$, $t = 0,1,2,...,T-1$

$R_{t+1}^3 = N_{t+1}^3 E(G_t)B_t + R_{t+1}^3 E(q_t)$, $t = 0,1,2,...,T-1$

$O_t^3 = O_{t+1}^3 + N_{t+1}^3 E(G_t)A_t, t = 0,1,2,...,T-1$.

$M_T^1 = \omega II', \quad N_T^1 = 0, \quad P_T^1 = 2\omega I, \quad R_T^1 = 0, \quad Q_T^1 = -\omega, \quad O_T^1 = 0$

$M_{T-1}^2 = C'_{T-1}\Lambda C_{T-1}, \quad N_{T-1}^2 = 2C'_{T-1}\Lambda A_{T-1}, \quad P_{T-1}^2 = 2C'_{T-1}\Lambda B_{T-1}$

$Q_{T-1}^2 = -B'_{T-1}\Lambda B_{T-1}, \quad R_{T-1}^2 = -2A'_{T-1}\Lambda B_{T-1}, \quad O_{T-1}^2 = -A'_{T-1}\Lambda A_{T-1}$.

The following notations are defined for $t = 0,1,2,...,T-i$ and $i = 1,2$:

$M_t^i = (\varepsilon - C_t)'\phi_t^i(\varepsilon - C_t) + (i-1)C_t'\Lambda C_t$, $\phi_t^i = E(G_t M_{t+1}^i G_t)$

$N_t^i = 2\lambda[(C'_t - \varepsilon)\phi_t^i A_t + (i-1)C'_t \Lambda A_t]$

$P_t^i = 2(C_t - \varepsilon)'\phi_t^i B_t + (\varepsilon - C_t)' E(q_t G_t) P_{t+1}^i + 2(i-1)C_t'\Lambda B_t$

$Q_t^i = -B_t'[\phi_t^i + (i-1)\Lambda]B_t + P_{t+1}^i E(q_t G_t)B_t + Q_{t+1}^i E(q_t^2)$

$R_t^i = -2\lambda A_t'[\phi_t^i + (i-1)\Lambda]B_t + \lambda P_{t+1}^i E(q_t G_t)A_t$

$O_t^i = O_{t+1}^i - \lambda^2 A_t'[\phi_t^i + (i-1)\Lambda]A_t$.

where I is the n-dimensional column vector of element 1, and ε is a unit matrix.

Theorem 4.1. The optimal strategy Δv_t of problem $(\hat{P}(\omega,\beta))$ for $t = 0,1,...,T-1$ is specified by:

$$\Delta v^*{}_t = \lambda^* A_t + B_t L_t - C_t v_t. \tag{4.3}$$

And the expectation and variance of surplus and the expected transaction costs, respectively, is:

$$E_t(S_T{}^*) = \lambda^* O_t^3 + N_t^3 + R_t^3 \tag{4.4}$$

$$Var_t(S_T^*) = v'_t M_t^1 v_t - N'^1_t v_t - P'^1_t L_t v_t - Q_t^1 L_t^2 - R_t^1 L_t - O_t^1 - [E_t(S_T^*)]^2 \tag{4.5}$$

$$E_t(TC_T^*) = v'_t M_t^2 v_t - N_t'^2 v_t - P_t'^2 L_t v_t - Q_t^2 L_t^2 - R_t^2 L_t - O_t^2 , \tag{4.6}$$

where $\lambda^* = \dfrac{1+2\omega(N_0^3 + R_0^3)}{1-2\omega O_0^3}$ and $t = 0,1,2,...,T$.

Remark 4.1. When the investor has no liability, that is $L_t = 0$ for $t = 0,1,...,T-1$, the above optimal strategy remains valid. Under this situation, the optimal strategy Δv_t is specified by

$$\Delta v^*{}_t = \lambda^* A_t - C_t v_t , \quad t = 0,1,2,...,T-1 . \tag{4.7}$$

And the expressions of expectation and variance of surplus and the expected transaction costs for $t = 0,1,...,T$, respectively, is

$$E_t(S_T^*) = \lambda^* O_t^3 + N_t^3 \tag{4.8}$$

$$Var_t(S_T^*) = v'_t M_t^1 v_t - N'^1_t v_t - O_t^1 - [E_t(S_T^*)]^2 \tag{4.9}$$

$$E_t(TC_T^*) = v'_t M_t^2 v_t - N'^2_t v_t - O_t^2 . \tag{4.10}$$

This implies that the Theorem 4.1 can generalize the situation without liability.

Remark 4.2. When ignoring the transaction cost, that is $C_t = 0, t = 0,1,...,T-1$, the pre-commitment strategy can be acquired by setting $\beta = 0$ in equations (4.7). If the liability $L_t = 0$ ($t = 0,1,...,T-1$) and cost-aversion coefficient $\beta = 0$ that ignores the transaction cost, the pre-commitment optimal strategy is equivalent to that in Li and Ng (2000). Therefore, Theorem 4.1 generally includes three situations just like Theorem 3.1.

4.2 Time-consistent strategy for problem $(\hat{P}(\omega,\beta))$

It is not difficult to find the problem $(\hat{P}(\omega,\beta))$ can be solved by the time consistent strategy. By using backwards induction, we derive the time-consistency strategy for multiperiod MVALM problem without riskless assets by using the procedures presented in Section 2.

Define:

For $\alpha = 0$ if $t = T-1$ and $\alpha = 1$ if $t < T-1$, we define:

$$x_T = 1\,,\, y_T = 1\,,\, z_T = 1$$

$$\hat{\Omega}_t = \alpha E(G_t M_{t+1} G_t) + x_{t+1} x_{t+1}{}' \Omega_t$$

$$\hat{\Omega}_t^0 = \begin{cases} 2\Omega_{T-1}^0 I,\, t = T-1 \\ -E(G_t q_t) o_{t+1} - \Omega_t^0 x_{t+1} y_{t+1},\, t = 0,1,\ldots,T-2 \end{cases}$$

$$\hat{\Lambda}_t = \Lambda + \alpha \mathrm{E}(\mathrm{G}_t \mathrm{M}_{t+1} \mathrm{G}_t)$$

$$\Psi_t = \begin{cases} E(e_{T-1}),\ \ t = T-1 \\ E(G_t)(x_{t+1} - \omega p_{t+1}),\, t = 0,1,\ldots,T-2 \end{cases}$$

$$\Theta_t = [2\omega\hat{\Omega}_t + 2\beta\hat{\Lambda}_t]^{-1} \left\{ \varepsilon - \frac{II'[2\omega\hat{\Omega}_t + 2\beta\hat{\Lambda}_t]^{-1}}{I'[2\omega\hat{\Omega}_t + 2\beta\hat{\Lambda}_t]^{-1} I} \right\}$$

$$a_t = 2\omega\Theta_t\hat{\Omega}_t,\ \ b_t = \omega\Theta_t\hat{\Omega}_t^0,\ \ c_t = \Theta_t\Psi_t$$

$$x_t = \begin{cases} (\varepsilon - A_{T-1})' E(e_{T-1}), t = T-1 \\ (\varepsilon - A_t)' E(G_t) x_{t+1}\ , t = 0,1,\ldots,T-2 \end{cases}$$

$$y_t = x'_{t+1} E(G_t) b_t + y_{t+1} E(q_t)$$

$$z_t = z_{t+1} + x'_{t+1} E(G_t) c_t$$

$$m_t = (\varepsilon - a_t)' \hat{\Omega}_t (\varepsilon - a_t)$$

$$n_t = b'_t \hat{\Omega}_t b_t + \sigma_t - \hat{\Omega}'^0_t b_t + \alpha n_{t+1} E(q_t^2)$$

$$o_t = (\varepsilon - a_t)'(2\hat{\Omega}'_t b_t - \hat{\Omega}_t^0)$$

$$p_t = (\varepsilon - a_{T-1})'[2\hat{\Omega}'_t c_t + \alpha E(G_t) p_{t+1}]$$

$$\varsigma_t = 2b'_t \hat{\Omega}_t c_t - \hat{\Omega}'^0_t c_t + \alpha[E(q_t)\varsigma_{t+1} + p_{t+1}' E(G_t) b_t]$$

$$\zeta_\mathrm{t} = \zeta_{\mathrm{t}+1} + c'_t \hat{\Omega}_t c_t + \alpha p'_{t+1} E(G_t) c_t$$

$$d_t = a'_t \Lambda a_t + \alpha(\varepsilon - a_t)' E(G_t d_{t+1} G_t)(\varepsilon - a_t)$$

$$h_t = b'_t \Lambda b_t + \alpha[h_{t+1} E(q_t^2) + j'_{t+1} E(G_t q_t) b_t]$$

$$j_t = -2a'_t \Lambda b_t + \alpha[(\varepsilon - a_t)'(2E(G_t d_{t+1} G_t) b_t + E(G_t q_t) j_{t+1})]$$

$$k_t = -2a'_t \Lambda c_t + \alpha[(\varepsilon - a_t)'(2E(G_t d_{t+1} G_t) c_t + E(G_t q_t) k_{t+1})]$$

$$u_t = 2b'_t \Lambda_t c_t + \alpha[j'_{t+1} E(G_t q_t) c_t + k'_{t+1} E(G_t) b_t + u_{t+1} E(q_t)]$$

$$\iota_t = \iota_{t+1} + c'_t \hat{\Lambda}_t c_t + \alpha[k'_{t+1} E(G_t) c_t].$$

Subsequently, by applying the procedures of time consistent strategy, we have the following conclusions.

Theorem 4.2. For $t = 0,1,2,...,T-1$, the time-consistent investment strategy of problem $(\hat{P}(\omega,\beta))$ is given by

$$\Delta v_t = -a_t v_t + b_t L_t + c_t, \tag{4.11}$$

the corresponding expectation and variance of surplus and the expected transaction costs, respectively, is:

$$E_t(S_T) = x'_t v_t + y_t L_t + z_t \tag{4.12}$$

$$Var_t(S_T) = v'_t m_t v_t + n_t L_t^2 + o'_t v_t L_t + p'_t v_t + \varsigma_t L_t + \zeta_t \tag{4.13}$$

$$E_t(TC_T) = v'_t d_t v_t + h_t L_t^2 + j'_t v_t L_t + k'_t v_t + u_t L_t + \iota_t. \tag{4.14}$$

Remark 4.3. Similarly, if the investor does not have any liability, that is $L_t = 0$ for $t = 0,1,...,T-1$, then the time-consistent strategy reduces to

$$\Delta v_t = -a_t v_t + c_t \tag{4.15}$$

and the expectation and variance of surplus and the expected transaction costs, respectively, is:

$$E_t(S_T) = x'_t v_t + z_t \tag{4.16}$$

$$Var_t(S_T) = v'_t m_t v_t + p'_t v_t + \zeta_t \tag{4.17}$$

$$E_t(TC_T) = v'_t d_t v_t + k'_t v_t + \iota_t \tag{4.18}$$

Thus the same to Theorem 3.2, the Theorem 4.2 generalizes three situations as well.

5 Numerical simulations

In this section, some numerical simulations are given, which provide twofold contributions. Firstly, we compare the results of application of quadratic transaction costs and no costs in Example 5.1. Further, in the present of quadratic cost, we compare the frontiers under different strategies and different settings in Example 5.2, including strategies with/without riskless assets. Secondly, to disclose the impact of quadratic transaction costs on frontiers, we discuss the transaction cost parameters β and Λ in

Example 5.3 and 5.4, respectively.

Consider a riskless asset with a constant return rate of 1.04 and three risky assets whose corresponding expected return vector and the covariance matrix are given as $E(e_t) = [1.162, 1.246, 1.228]$ and

$$\Omega_t = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix} \text{ for } t = 0,1,\ldots,T-1$$

respectively. The expected return of liability is 1.136 and the corresponding variance is 0.01. $\Omega_t^0 = diag([0.0006, 0.0149, 0.0050])$ is the diagonal matrix of covariance vectors of liability and risky assets.

The investor has 1 unit of wealth and 0.3 unit of liability at the beginning of the planning horizon, and conducts a multiperiod investment process with $T = 4$. The parameter ω ranges from 0.4 to 1.2 and β varies from different examples. Due to the importance of the investor's terminal net surplus, all simulated results are demonstrated by the frontiers that take $E(S_T - TC_T)$ as ordinate and $Var(S_T - TC_T)$ as abscissa, referred to as M-V frontier. According to the conclusions shown in previous sections, we can find that $Var(S_T - TC_T)$ is equivalent to $Var(S_T)$.

Example 5.1 Comparison of strategies with/without cost

Although the empirical evidence shows that the transaction cost affects the strategy, it fails to quantify the extent of the change intuitively. Thus, we compare the strategies of considering the transaction cost and that of ignoring it.

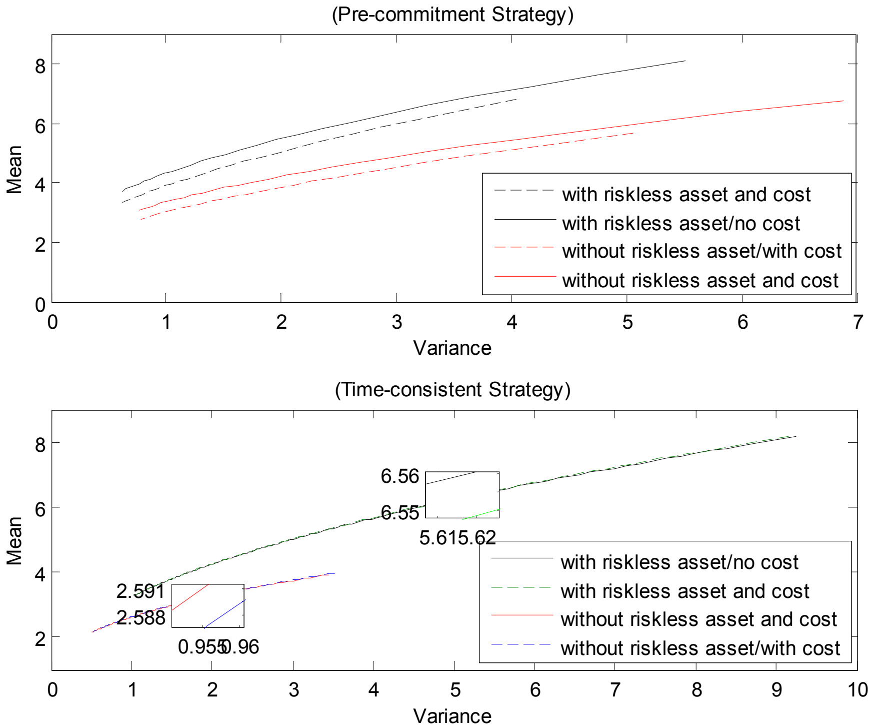

Fig. 5.1 The M-V frontiers under different strategies with/without cost

From Fig.5.1, we can know that the existence of transaction costs does affect the pre-commitment strategies greatly. And the time-consistent strategies have been affected to some extent, but not seriously. Ignoring transaction costs can lead to invalid pre-commitment strategies. At the same time, the investors who has high transaction cost aversion could be inclined to consider time-consistent strategies.

Example 5.2 Comparison of the frontiers under different strategies

In order to better understand the difference among different investment strategies, we will discuss the frontiers under the following two situations:

(a) β =0.5, Λ =0.001* ε , where ε is a unit matrix;

(b) β =10, Λ =0.001* ε , where ε is a unit matrix.

When other parameters remain unchanged, different cost aversion coefficients will produce different frontiers. The detailed simulation results are shown in Fig 5.2.

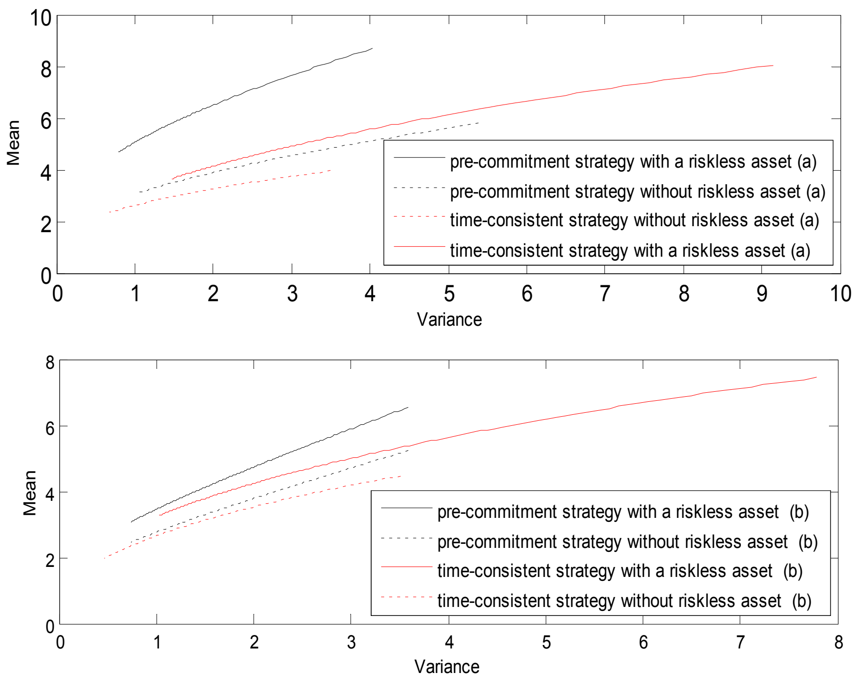

Fig. 5.2 The M-V frontiers under different strategies

From Fig.5.2, we can draw two conclusions. One is, for the given risk level, the expected net surplus of pre-commitment strategy is better than that of time-consistent strategy no matter that there is a riskless asset or not in the asset pool. In other words, we can obtain higher income by following the pre-commitment strategy. This can be explained by that the pre-commitment strategy is the global optimal investment strategy for the initial period, while the time-consistent strategy only considers local incentives and ignores global objectives. The existence of the quadratic transaction cost does not affect the superiority of the pre-commitment strategy. The other interesting conclusion is that when the value of β is particularly large, the gap between the frontier of pre-commitment strategy and that of time-consistent strategy have been reduced. Comparatively speaking, the cost constraint is more punitive to the pre-commitment strategy. If the investor adopts the pre-commitment strategy without considering the transaction cost, then it will lead to ineffective investment strategy, especially for the individual investor with higher cost aversion.

Example 5.3 Impact of cost-aversion coefficient on different frontiers

To explore the impact of cost-aversion coefficient on frontiers, we set β is, in turn, equivalent to 0,0.8,1.6 and 2.4. Fig. 5.3 shows the sensitivity of the corresponding frontier under different strategies to the cost coefficients.

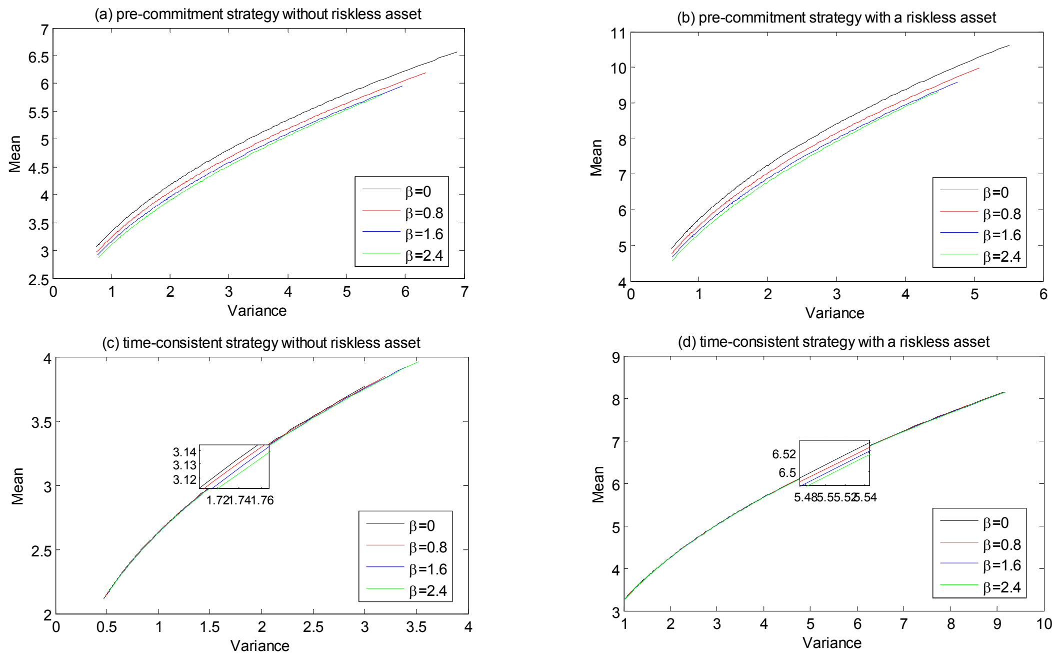

Fig. 5.3 The efficient frontiers of strategies under different cost-aversion coefficient

As shown in Fig. 5.3, we can find that the frontiers move downward with the increase of β for all strategies. The existence of transaction costs has a significant effect on investment strategies and the efficient frontier. Comparatively speaking, with the increase of β, it causes a smaller change on the frontier of time-consistent strategy. This indicates that the frontier of time-consistent strategy is less sensitive to cost-aversion coefficient than that of pre-commitment strategy, regardless of whether there exist riskless assets in the asset pool. More importantly, no matter how large the cost aversion coefficient is, the produced cost is relatively small for the time-consistent strategy.

Example 5.4 Impact of parameter Λ on different frontiers

The positive definite matrix Λ in the quadratic transaction cost function can be diagonalized into a matrix consisting of eigenvalues, which dominate the corresponding unit cost of risky assets. In this example, we will discuss the impact of these eigenvalues on different frontiers.

Here, we set the matrix Λ as Table 5.1 and β equals to 0.5. Fig. 5.4 shows the frontiers of different strategies when Λ takes different value.

Table 5.1 The parameter-set.

Λ^1	Λ^2	Λ^3	Λ^4
$0.001*\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$0.001*\begin{bmatrix}3&0&0\\0&1&0\\0&0&1\end{bmatrix}$	$0.001*\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$	$0.001*\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}$

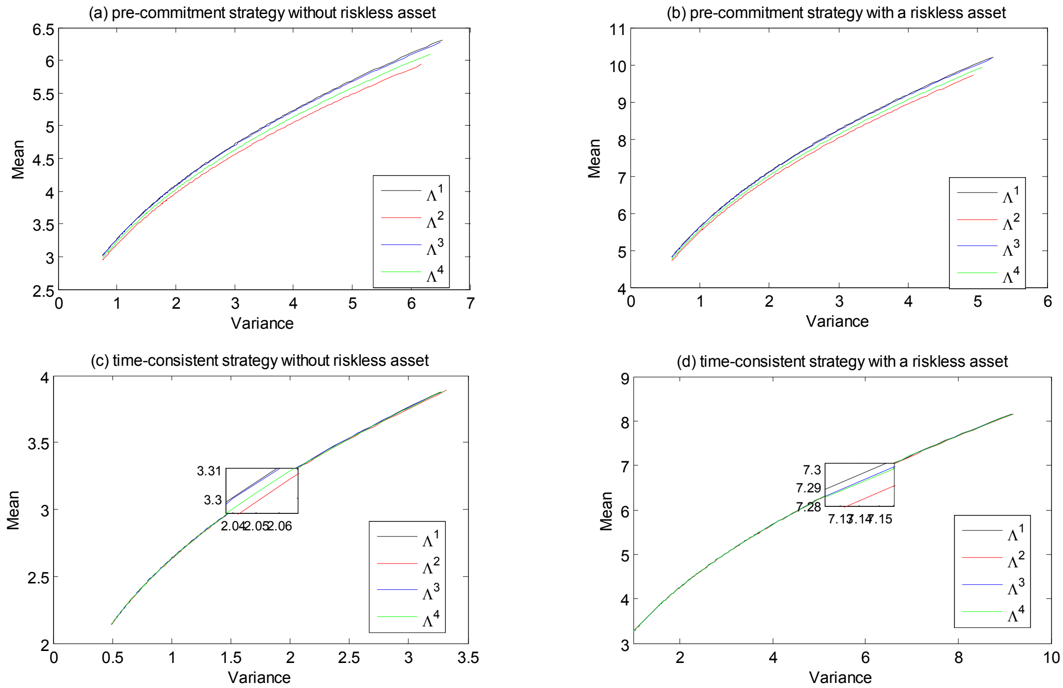

Fig. 5.4 The efficient frontiers of strategies under different parameter Λ

It is easy to find that both the pre-commitment and time-consistent strategies follow the same law. That is, the frontiers drop in the same order with the change of parameter Λ, and they are all below the frontier where Λ remains unchanged. The changes of the elements on the parameter Λ also affect the frontiers, whether it is a pre-commitment strategy or a time-consistent strategy. More importantly, the increase of unit cost has less impact on the frontier of time-consistent strategy. For the change of parameter, the time consistent strategy might be more stable which is coincident with the conclusion of Example 5.3. Comparatively speaking, for the instability of the market environment and the aversion of the investors to the cost, the time consistent strategy might be the better choice in the complex market.

6. Conclusion

This paper provides the highly tractable multiperiod asset-liability management frameworks for the study of optimal trading strategies in presence of quadratic transaction costs. For different investment setting (with/without riskless assets), the pre-commitment and time-consistent investment strategies are derived by applying the embedding scheme and backward induction approach, respectively. The derived strategies cover several optimal strategies in existing literatures. This provides investors with a sensible investment strategy when transaction costs or liabilities are not considered, or neither is considered. Finally, some numerical simulations are carried out. The results indicate that the transaction costs play an important role in investment markets. Furthermore, when considering transaction costs, the time-consistent strategy is more robust than the pre-commitment strategy in asset-liability management.

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Appendix A

Proof of Theorem 2.1

Define $\prod_F(\omega,\beta)$ to be the set of the optional solution ψ^* of problem $F_t(J_t,\Delta v(t))$ for given ω and β. Similarly, for problem $\widetilde{F}_t(J_t,\Delta v(t))$, $\prod_{\widetilde{F}}(\lambda,\omega,\beta)$ is the set of the optional solution for given λ, ω and β. And denote

$$d(\pi,\omega)=1+2\omega E_t(S_T)\big|_\pi . \tag{1}$$

We firstly proof that for any $\psi^*\in\prod_F(\omega,\beta),\psi^*\in\prod_{\widetilde{F}}(d(\pi^*,\omega),\omega,\beta)$.

Assume that $\psi^*\notin\prod_{\widetilde{F}}(d(\psi^*,\omega),\omega,\beta)$, then exist ψ to make

$$(d(\psi^*,\omega),-\omega,-\beta)\begin{pmatrix}E_t(S_T)\\ E_t(S_T^2)\\ E_t(TC_T)\end{pmatrix}\Bigg|_{\psi} > (d(\psi^*,\omega),-\omega,-\beta)\begin{pmatrix}E_t(S_T)\\ E_t(S_T^2)\\ E_t(TC_T)\end{pmatrix}\Bigg|_{\psi^*} \tag{2}$$

Consider the function

$$\begin{aligned}U[E_t(S_T),E_t(S_T^2),E_t(TC_T)]&=E_t(S_T)-\omega Var_t(S_T)-\beta E_t(TC_T)\\ &=E_t(S_T)-\omega[E_t(S_T^2)-E_t^{\,2}(S_T)]-\beta E_t(TC_T)\end{aligned} \tag{3}$$

Obviously, this equation is a convex function about $E_t(S_T)$, $E_t(S_T^2)$ and $E_t(TC_T)$, so we can get

$$\begin{aligned}&U[E_t(S_T),E_t(S_T^2),E_t(TC_T)]\big|_{\psi}-U[E_t(S_T),E_t(S_T^2),E_t(TC_T)]\big|_{\psi^*}\\ &\geq\left(\frac{\partial U}{\partial E_t(S_T)}\bigg|_{\psi^*},\frac{\partial U}{\partial E_t(S_T^2)}\bigg|_{\psi^*},\frac{\partial U}{\partial E_t(\mathrm{T}C_T)}\bigg|_{\psi^*}\right)\left[\begin{pmatrix}E_t(S_T)\\ E_t(S_T^2)\\ E_t(\mathrm{T}C_T)\end{pmatrix}\Bigg|_{\psi}-\begin{pmatrix}E_t(S_T)\\ E_t(S_T^2)\\ E_t(\mathrm{T}C_T)\end{pmatrix}\Bigg|_{\psi^*}\right] \qquad (4)\\ &=(d(\pi^*,\omega),-\omega,-\beta)\left[\begin{pmatrix}E_t(S_T)\\ E_t(S_T^2)\\ E_t(TC_T)\end{pmatrix}\Bigg|_{\psi}-\begin{pmatrix}E_t(S_T)\\ E_t(S_T^2)\\ E_t(TC_T)\end{pmatrix}\Bigg|_{\psi^*}\right]\end{aligned}$$

Combine (2) and (3), we can obtain

$$U[E_t(S_T),E_t(S_T^2),E_t(TC_T)]\Big|_{\psi} > U[E_t(S_T),E_t(S_T^2),E_t(TC_T)]\Big|_{\psi^*}. \tag{5}$$

Apparently, it is conflict with the assumption $\psi^* \in \prod_F(\omega,\beta)$, so that is proved.

Then, for given (ω,β), the optimal solution of problem $(\widetilde{F}(\lambda,\omega,\beta))$ can be expressed by parameter λ, and then the corresponding wealth and transaction costs can be expressed at the same way, marked $S_T(\lambda,\omega)$ and $TC_T(\lambda,\beta)$, respectively. Because of $\prod_F(\omega,\beta) \subset \bigcup_{\lambda} \prod_{\widetilde{F}}(\lambda,\omega,\beta)$, the problem $F(\omega,\beta)$ can be degenerated into equivalent problem as follows:

$$\max_{\lambda} U(E_t[S_T(\lambda,\omega)], E_t[S_T^2(\lambda,\omega), E_t[TC_T(\lambda,\beta)])$$
$$= \max_{\lambda} E_t[S_T(\lambda,\omega)] - \omega\{E_t[S_T^2(\lambda,\omega) - E_t^{\,2}[S_T(\lambda,\omega)]\} - \beta E_t[TC_T(\lambda,\beta)]. \tag{6}$$

On the other hand, due to $\pi^* \in \prod_{\widetilde{F}}(\lambda^*,\omega,\beta)$,according to the discussion of Reidand Citron, there is

$$\lambda^* \frac{\partial E_t[S_T(\lambda^*,\omega)]}{\partial\lambda} - \omega\frac{\partial E_t[S_T^2(\lambda^*,\omega)]}{\partial\lambda} - \beta\frac{\partial E_t(TC_T)}{\partial\lambda} = 0 \tag{7}$$

And because of $\lambda^* = 1 + 2\omega E_t[S_T(\lambda^*,\omega)] = 1 + 2\omega E_t(S_T)\big|_{\psi^*}$,

then the first order necessary condition of optimal solution about λ^* is $\left.\frac{\partial U}{\partial\lambda}\right|_{\lambda^*} = 0$, namely

$$\frac{\partial E_t[S_T(\lambda^*,\omega)]}{\partial\lambda}(1 + 2\omega E_t[S_T(\lambda^*,\omega)]) - \omega\frac{\partial E_t[S_T^2(\lambda^*,\omega)]}{\partial\lambda} - \beta\frac{\partial E_t(TC_T)}{\partial\lambda} = 0 \tag{8}$$

So ψ^* is also the optimal control of the problem $F(\omega,\beta)$ for $d(\pi,\omega) = 1 + 2\omega E_t(S_T)\big|_{\psi^*}$.

Q.E.D

Appendix B

Proof of Theorem 3.1 We adopt the dynamic programming of reverse solving method to solve the problem $(A(\lambda,\omega,\beta))$ beginning from period $T-1$.

Denoting $f_T(v_T, v_T^0, L_T)$ as the optimal function of problem $(A(\lambda,\omega,\beta))$ at the period T, then we have

$$f_T(v_T, v_T^0, L_T) = -v'_T \widetilde{M}_T v_T - \widetilde{N}_T (v_T^0)^2 - \widetilde{O}_T L_T^2 - \widetilde{P}'_T v_T v_T^0 + \widetilde{Q}'_T v_T L_T + \widetilde{R}_T L_T v_T^0 + \widetilde{X}'_T v_T + \widetilde{Y}_T v_T^0 - \widetilde{Z}_T L_T \tag{A.1}$$

So $f_T(v_T, v_T^0, L_T)$ also meets Theorem 3.1.

When $t = T-1$, applying dynamic programming principles gives rise to

$$\begin{aligned} & f_{T-1}(v_{T-1}, v_{T-1}^0, L_{T-1}) \\ &= \max_{\Delta v_{T-1}} E(f_T(v_T, v_T^0, L_T) - \beta \Delta v'_{T-1} \Lambda \Delta v_{T-1}) \\ &= \max_{\Delta v_{T-1}} [-(v_{T-1} + \Delta v_{T-1})' E(G_{T-1} \widetilde{M}_T G_{T-1})(v_{T-1} + \Delta v_{T-1}) + \widetilde{N}_T (r_{T-1}^0)^2 (v_{T-1}^0 - I' \Delta v_{T-1})^2 \\ &\quad - \widetilde{O}_T E(q_{T-1}^2) L_{T-1}^2) - \widetilde{P}'_T r_{T-1}^0 E(G_{T-1})(v_{T-1} + \Delta v_{T-1})(v_{T-1}^0 - I' \Delta v_{T-1}) \\ &\quad + \widetilde{Q}'_T E(q_{T-1} G_{T-1})(v_{T-1} + \Delta v_{T-1}) L_{T-1} + \widetilde{R}_T r_{T-1}^0 E(q_{T-1})(v_{T-1}^0 - I' \Delta v_{T-1}) L_{T-1} \\ &\quad + \widetilde{X}'_T E(G_{T-1})(v_{T-1} + \Delta v_{T-1}) + \widetilde{Y}_T r_{T-1}^0 (v_{T-1}^0 - I' \Delta v_{T-1}) \\ &\quad - \widetilde{Z}_T E(q_{T-1}) L_{T-1} - \beta \Delta v'_{T-1} \Lambda \Delta v_{T-1}] \end{aligned} \tag{A.2}$$

Applying the first order condition about Δv_{T-1} yields the following optimal strategy

$$\Delta v^*{}_{T-1} = \lambda \widetilde{A}_{T-1} + \widetilde{B}_{T-1} L_{T-1} + \widetilde{C}_{T-1} v_{T-1}^0 + \widetilde{D}_{T-1} v_{T-1}. \tag{A.3}$$

Substituting (A.3) into (A.2) and simplifying the resulting equation yield

$$\begin{aligned} f_{T-1}(v_{T-1}, v_{T-1}^0, L_{T-1}) &= -v'_{T-1} \widetilde{M}_{T-1} v_{T-1} - \widetilde{N}_{T-1} (v_{T-1}^0)^2 - \widetilde{O}_{T-1} L_{T-1}^2 \\ &\quad - \widetilde{P}'_{T-1} v_{T-1} v_{T-1}^0 + \widetilde{Q}'_{T-1} v_{T-1} L_{T-1} + \widetilde{R}_{T-1} L_{T-1} v_{T-1}^0 \\ &\quad + \widetilde{X}'_{T-1} v_{T-1} + \widetilde{Y}_{T-1} v_{T-1}^0 - \widetilde{Z}_{T-1} L_{T-1} + \widetilde{K}_{T-1}. \end{aligned} \tag{A.4}$$

Next, for every $t = 0,1,2,...,T-2$, by using mathematical induction we can suppose

$$\begin{aligned} f_{t+1}(v_{t+1}, v_{t+1}^0, L_{t+1}) &= -v'_{t+1} \widetilde{M}_{t+1} v_{t+1} - \widetilde{N}_{t+1} (v_{t+1}^0)^2 - \widetilde{O}_{t+1} L_{t+1}^2 - \widetilde{P}'_{t+1} v_{t+1} v_{t+1}^0 \\ &\quad + \widetilde{Q}'_{t+1} v_{t+1} L_{t+1} + \widetilde{R}_{t+1} L_{t+1} v_{t+1}^0 + \widetilde{X}'_{t+1} v_{t+1} + \widetilde{Y}_{t+1} v_{t+1}^0 \\ &\quad - \widetilde{Z}_{t+1} L_{t+1} + \widetilde{K}_t. \end{aligned} \tag{A.5}$$

According to the state transition equations, there is

$$\begin{aligned} & f_t(v_t, v_t^0, L_t) \\ &= \max_{\Delta v_t} E(f_{t+1}(v_{t+1}, v_{t+1}^0, L_{t+1}) - \beta \Delta v'_t \Lambda \Delta v_t) \\ &= \max_{\Delta v_t} [-(v_t + \Delta v_t)' E(G_t \widetilde{M}_{t+1} G_t)(v_t + \Delta v_t) + \widetilde{N}_{t+1} (r_t^0)^2 (v_t^0 - I' \Delta v_t)^2 \end{aligned}$$

$$
\begin{aligned}
&-\widetilde{O}_{t+1}E(q_t^2)L_t^2)-\widetilde{P}'_{t+1}r_t^0E(G_t)(v_t+\Delta v_t)(v_t^0-I'\Delta v_t)\\
&+\widetilde{Q}'_{t+1}E(q_tG_t)(v_t+\Delta v_t)L_t+\widetilde{R}_{t+1}r_t^0E(q_t)(v_t^0-I'\Delta v_t)L_t\\
&+\widetilde{X}'_{t+1}E(G_t)(v_t+\Delta v_t)+\widetilde{Y}_{t+1}r_t^0(v_t^0-I'\Delta v_t)-\widetilde{Z}_{t+1}E(q_t)L_t-\beta\Delta v'_t\Lambda\Delta v_t]
\end{aligned}
\quad .\text{(A.6)}
$$

Applying the first order condition about Δv_t yields the following optimal strategy

$$
\Delta v_t^* = \lambda\widetilde{A}_t+\widetilde{B}_tL_t+\widetilde{C}_tv_t^0+\widetilde{D}_tv_t\,. \tag{A.7}
$$

Substituting (A.7) into (A.6) and simplifying the resulting equation yields

$$
\begin{aligned}
f_t(v_t,v_t^0,L_t)=-v'_t\widetilde{M}_tv_t-\widetilde{N}_t(v_t^0)^2-\widetilde{O}_tL_t^2-\widetilde{P}'_tv_tv_t^0+\widetilde{Q}'_tv_tL_t\\
+\widetilde{R}_tL_tv_t^0+\widetilde{X}'_tv_t+\widetilde{Y}_tv_t^0-\widetilde{Z}_tL_t+\widetilde{K}_t.
\end{aligned}
\tag{A.8}
$$

According to above elaborating, we can obtain the optimal strategy and value function for $t=0,1,2,...,T-1$, as follows:

$$
\Delta v_t = \lambda\widetilde{A}_t+\widetilde{B}_tL_t+\widetilde{C}_tv_t^0+\widetilde{D}_tv_t \tag{A.9}
$$

$$
\begin{aligned}
f_t(v_t,v_t^0,L_t)=-v'_t\widetilde{M}_tv_t-\widetilde{N}_t(v_t^0)^2-\widetilde{O}_tL_t^2-\widetilde{P}'_tv_tv_t^0+\widetilde{Q}'_tv_tL_t\\
+\widetilde{R}_tL_tv_t^0+\widetilde{X}'_tv_t+\widetilde{Y}_tv_t^0-\widetilde{Z}_tL_t+\widetilde{K}_t.
\end{aligned}
\tag{A.10}
$$

Q.E.D

Appendix C

Proof of Theorem 3.2 According to the procedures for the solution of time consistent strategy, the proof process is as follows.

Denoting $g_t(v_t,v_t^0,L_t)$ as the optimal function of problem $(P(\omega,\beta))$ at the period t.

When $t=T-1$, there is the objective function

$$
\begin{aligned}
\mathrm{g}_{T-1}(\Delta v_{T-1})=\max_{\Delta v_{T-1}}\{&E(e'_{T-1})(v_{T-1}+\Delta v_{T-1})+r_{T-1}^0(v_{T-1}^0-I'\Delta v_{T-1})-E(q_{T-1})L_{T-1}\\
&-\omega[(v_{T-1}+\Delta v_{T-1})'\widetilde{\Omega}_{T-1}(v_{T-1}+\Delta v_{T-1})-\widetilde{\Omega}'^0_{T-1}(v_{T-1}+\Delta v_{T-1})L_{T-1}\\
&+L_{T-1}^2\sigma_{T-1}]-\beta\Delta v'_{T-1}\Lambda\Delta v_{T-1}-\mu I'\Delta v_{T-1}\}
\end{aligned}
\tag{B.1}
$$

Applying the first order condition about $\Delta v_{T\text{-}1}$ yields the following optimal strategy

$$
\Delta v_{T-1}=-\widetilde{a}_{T-1}v_{T-1}+\widetilde{b}_{T-1}L_{T-1}+\widetilde{c}_{T-1}\,. \tag{B.2}
$$

And the condition expectation and variance of final wealth and condition expected cost, respectively, is

$$
\begin{aligned}
E_{T-1}(S_T) &= E(e'_{T-1})(v_{T-1} + \Delta v_{T-1}) - E(q_{T-1})L_{T-1} \\
&= \tilde{x}'_{T-1} v_{T-1} + \tilde{y}_{T-1} L_{T-1} + \tilde{z}_{T-1} v^0_{T-1} + \tilde{\xi}
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
Var_{T-1}(v_T) &= (v_{T-1} + \Delta v_{T-1})'\tilde{\Omega}_{T-1}(v_{T-1} + \Delta v_{T-1}) + \sigma_{T-1} L^2_{T-1} - \tilde{\Omega}'^0_{T-1}(v_{T-1} + \Delta v_{T-1})L_{T-1} \\
&= v'_{T-1}\tilde{m}_{T-1} v_{T-1} + \tilde{n}_{T-1} L^2_{T-1} + \tilde{o}'_{T-1} v_{T-1} L_{T-1} + \tilde{p}'_{T-1} v_{T-1} + \tilde{\varsigma}_{T-1} L_{T-1} + \tilde{\zeta}_{T-1}
\end{aligned} \tag{B.4}
$$

$$
E_{T-1}(TC_T) = v'_{T-1}\tilde{d}_{T-1} v_{T-1} + \tilde{h}_{T-1} L^2_{T-1} - \tilde{j}'_{T-1} v_{T-1} L_{T-1} - \tilde{k}'_{T-1} v_{T-1} + \tilde{u}_{T-1} L_{T-1} + \tilde{\iota}_{T-1}. \tag{B.5}
$$

It is easy to find that Theorem 3.2 holds for period $T-1$.

Assume that Theorem 3.2 also holds for $t+1$, then for t, we can obtain that

$$
\begin{aligned}
g_t(\Delta v_t) &= \max_{\Delta v_t}\{E_t(S_T) - \omega Var_t(S_T) - \beta(E_{t+1}(TC_T) + \Delta v'_t \Lambda \Delta v_t) - \mu I'\Delta v_t\} \\
&= \max_{\Delta v_t}\{E_t[E_{t+1}(S_T)] - \omega\{E_t[Var_{t+1}(S_T)] + Var_t[E_{t+1}(S_T)]\} \\
&\qquad - \beta(E_t(TC_{t+1}) + \Delta v'_t \Lambda \Delta v_t) - \mu I'\Delta v_t\} \\
&= \max_{\Delta v_t}\{\tilde{x}'_{t+1} E(G_t)(v_t + \Delta v_t) + \tilde{y}_{t+1} E(q_t) L_t + \tilde{z}_{t+1} r^0_t (v^0_t - I'\Delta v_t) \\
&\quad + \tilde{\xi}_{t+1} - \mu I'\Delta v_t - \omega\{(v_t + \Delta v_t)' E(G_t \tilde{m}_{t+1} G_t)(v_t + \Delta v_t) \\
&\quad + \tilde{n}_{t+1} E(q^2_t) L^2_t + \tilde{o}'_{t+1} E(G_t q_t)(v_t + \Delta v_t) L_t + E(q_t)\tilde{\varsigma}_{t+1} L_t \\
&\quad + \tilde{p}_{t+1}' E(G_t)(v_t + \Delta v_t) + \tilde{\zeta}_{t+1} + (v_t + \Delta v_t)'\tilde{x}_t \tilde{x}'_t \Omega_t (v_t + \Delta v_t) \\
&\quad + \sigma^q_t \tilde{y}^2_t L^2_t + \tilde{x}'_t \tilde{y}_t \Omega^0_t (v_t + \Delta v_t) L_t\} - \beta((v_t + \Delta v_t)' E(G_t \tilde{d}_{t+1} G_t)(v_t + \Delta v_t) \\
&\quad + \tilde{h}_{t+1} E(q^2_t) L^2_t + \tilde{j}'_{t+1} E(G_t q_t)(v_t + \Delta v_t) L_t + \tilde{k}_{t+1}' E(G_t)(v_t + \Delta v_t) \\
&\quad + E(q_t)\tilde{u}_{t+1} L_t + \tilde{\iota}_{t+1} + \Delta v'_t \Lambda \Delta v_t)\}
\end{aligned} \tag{B.6}
$$

Applying the first order condition about Δv_t yields the following optimal strategy

$$
\Delta v_t = -\tilde{a}_t v_t + \tilde{b}_t L_t + \tilde{c}_t. \tag{B.7}
$$

And the condition expectation and variance of final wealth and condition expected cost for period t, respectively, is

$$
E_t(S_T) = \tilde{x}'_t v_t + \tilde{y}_t L_t + \tilde{z}_t v^0_t + \tilde{\xi}_t \tag{B.8}
$$

$$
Var_t(S_T) = v'_t \tilde{m}_t v_t + \tilde{n}_t L^2_t + \tilde{o}'_t v_t L_t + \tilde{p}'_t v_t + \tilde{\varsigma}_t L_t + \tilde{\zeta}_t \tag{B.9}
$$

$$
E_t(TC_T) = v'_t \tilde{d}_t v_t + \tilde{h}_t L^2_t - \tilde{j}'_t v_t L_t - \tilde{k}'_t v_t + \tilde{u}_t L_t + \tilde{\iota}_t. \tag{B.10}
$$

It is easy to find that Theorem 3.2 also holds at period t for $t = 0,1,\dots,T-1$. By mathematical induction, we complete the proof of Theorem 3.2.

Q.E.D