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Continued fractions and irrationality exponents for modified Engel and Pierce series

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Abstract
An Engel series is a sum of reciprocals of a non-decreasing sequence (x_n) of positive integers, which is such that each term is divisible by the previous one, and a Pierce series is an alternating sum of the reciprocals of a sequence with the same property. Given an arbitrary rational number, we show that there is a family of Engel series which when added to it produces a transcendental number α whose continued fraction expansion is determined explicitly by the corresponding sequence (x_n), where the latter is generated by a certain nonlinear recurrence of second order. We also present an analogous result for a rational number with a Pierce series added to or subtracted from it. In both situations (a rational number combined with either an Engel or a Pierce series), the irrationality exponent is bounded below by $(3+\sqrt{5})/2$, and we further identify infinite families of transcendental numbers α whose irrationality exponent can be computed precisely. In addition, we construct the continued fraction expansion for an arbitrary rational number added to an Engel series with the stronger property that x_j^2 divides x_{j+1} for all j.

Keywords Continued fractions · Engel series · Pierce series · Irrationality degree

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1 Introduction

Given a sequence of positive integers (x_n), which is such that $x_n | x_{n+1}$ for all n, the sum of the reciprocals is the Engel series

$$\sum_{j=1}^{\infty} \frac{1}{x_j} = \sum_{j=1}^{\infty} \frac{1}{y_1 y_2 \cdots y_j}, \tag{1.1}$$

where $y_1 = x_1$ and $y_{n+1} = x_{n+1}/x_n$ for $n \geq 1$, and the alternating sum of the reciprocals is the Pierce series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{x_j} = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{y_1 y_2 \cdots y_j}. \tag{1.2}$$

(It should be assumed that (x_n) is eventually increasing, in the sense that for all n there is some $n' > n$ with $x_{n'} > x_n$, which guarantees the convergence of both sums 1.1 and 1.2.) Every positive real number admits both an Engel expansion, of the form (1.1), and a Pierce expansion (1.2) [5]. Although they are not quite so well known, Engel expansions and Pierce expansions are in many ways analogous to continued fraction expansions, both in the sense that they are determined recursively, and from a metrical point of view; for instance, see [6] for the case of Engel series.

In recent work [7], the first author presented a family of sequences (x_n) generated by a nonlinear recurrences of second order, of the form

$$x_{n+1}x_{n-1} = x_n^2 \Big(1 + x_n\, G(x_n)\Big), \qquad n \geq 2, \qquad G(x) \in \mathbb{Z}[x], \tag{1.3}$$

where the polynomial G takes positive values at positive arguments, such that the corresponding Engel series (1.1) yields a transcendental number whose continued fraction expansion is explicitly given in terms of the x_n. More recently [17], the second author proved that, when the sequence (x_n) is generated by a recurrence like (1.3), an analogous result holds for the associated Pierce series (1.2), although the structure of the corresponding continued fractions is different. In fact, in the latter work the polynomial $G(x_n)$ was replaced by an arbitrary sequence of positive integers, as it had already been noted in [9] that the recurrence (1.3) could be modified in this way and further allow the explicit continued fraction expansion to be determined for the sum of an arbitrary rational number $r = p/q$ and an Engel series, that is

$$\alpha = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j}, \qquad \text{with } x_1 = q. \tag{1.4}$$

In the next section we show that the initial conditions for the sequence (x_n) can be specified more generally than in [9], allowing dependence on a non-negative integer parameter m, and present analogous results for a family of transcendental numbers defined by a rational number with a Pierce series added to or subtracted from it, of the form

$$\alpha = \frac{p}{q} \pm \sum_{j=2}^{\infty} \frac{(-1)^j}{x_j}. \tag{1.5}$$

In Sect. 3 it is proved that, in both cases (1.4) and (1.5), α has irrationality exponent $\mu(\alpha) \geq (3+\sqrt{5})/2$, and if the nonlinear recurrence for (x_n) has a particular form then $\mu(\alpha)$ can be computed precisely. Explicit continued fractions for series of the form (1.4) with the stronger property that $x_j^2 | x_{j+1}$ for all j are constructed in the final section, generalizing the results in [8].

2 Explicit continued fractions

Before proceeding, we fix our notation for continued fractions and briefly mention some of their standard properties, which can be found in many books [3,5,10]. We denote a finite continued fraction by

$$[a_0; a_1, a_2, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}} = \frac{p_n}{q_n}, \tag{2.1}$$

where $a_0 \in \mathbb{Z}$, $a_j \in \mathbb{Z}_{>0}$ and p_n/q_n is in lowest terms with $q_n > 0$. Every $r \in \mathbb{Q}$ can be written as a finite continued fraction (2.1), although this representation is not unique (see 2.5 below). Each $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is given uniquely by an infinite continued fraction with convergents p_n/q_n of the form (2.1), that is (with $a_0 = \lfloor \alpha \rfloor$)

$$\alpha = [a_0; a_1, a_2, \ldots] = \lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n] = \lim_{n\to\infty} \frac{p_n}{q_n}. \tag{2.2}$$

The three-term recurrence relation satisfied by the numerators and denominators of the convergents is encoded in the matrix relation

$$\begin{pmatrix} p_{n+1} & p_n \\ q_{n+1} & q_n \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

valid for $n \geq -1$, with

$$\begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By taking the determinant of both sides of (2.3), one obtains the identity

$$p_j q_{j-1} - p_{j-1} q_j = (-1)^{j-1}, \qquad j \geq 1. \tag{2.4}$$

Note that any finite continued fraction can be rewritten as another one of different length, since one can always apply one of two operations, namely

$$[a_0; a_1, a_2, \ldots, a_j] = \begin{cases} [a_0; a_1, a_2, \ldots, a_j - 1, 1], & \text{if } j = 0 \text{ or } a_j > 1, \\ [a_0; a_1, a_2, \ldots, a_{j-1} + 1], & \text{if } a_j = 1, \end{cases} \tag{2.5}$$

and both operations change the parity of the length.

Henceforth we fix a rational number $r = p/q$ in lowest terms, with $q \geq 1$, and an integer parameter $m \in \mathbb{Z}_{\geq 0}$. Without loss of generality, because of (2.5), we may specify the continued fraction of r to be

$$\frac{p}{q} = [a_0; a_1, a_2, \ldots, a_{2k}], \tag{2.6}$$

with the index of the last partial quotient being even (and shifting r by an integer changes a_0 but otherwise makes no difference). Given $y_0 \in \mathbb{Z}_{>0}$, which in due course will be fixed differently according to the context, we define two sequences $(y_n)_{n\geq 0}$ and $(x_n)_{n\geq 1}$ via the recursion relations

$$y_j = y_{j-1}(1 + u_j x_j), \quad x_{j+1} = x_j y_j, \quad \text{for } j \geq 1, \text{ with } x_1 = q, \tag{2.7}$$

where $(u_n)_{n\geq 1}$ is an arbitrary sequence of positive integers. Note that the second relation guarantees the property $x_j | x_{j+1}$ required for an Engel series or a Pierce series. It is an immediate consequence of (2.7) that, given $x_1 = q$ and $x_2 = x_1 y_1 = q y_0 (1 + u_1 q)$, the subsequent terms of the sequence (x_n) are determined by the nonlinear recurrence

$$x_{n+1} x_{n-1} = x_n^2 (1 + u_n x_n), \qquad n \geq 2. \tag{2.8}$$

It may happen that the sequence (u_n) is defined entirely in terms of the sequence (x_n), by specifying a function G such that

$$u_n = G(x_n), \qquad G : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}, \tag{2.9}$$

which means that (2.8) becomes an autonomous recurrence of the form (1.3), although the function G need not necessarily be a polynomial. Yet in general the recurrence (2.8) is non-autonomous, whenever (u_n) is specified by

$$u_n = \widehat{G}(x_n, n), \qquad \widehat{G} : \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}, \tag{2.10}$$

where $\widehat{G}$ is a non-trivial function of its second argument.

The following theorem extends the results of [9] (corresponding to the case $m = 0$), being themselves extensions of [7] (corresponding to $p/q = 1$).

Theorem 2.1 *Let α be given by the series*

$$\alpha = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j}, \tag{2.11}$$

for (x_n) defined by (2.7) with

$$y_0 = mq + q_{2k-1} + 1, \tag{2.12}$$

where q_{2k-1} is the denominator of the $(2k-1)$th convergent of (2.6). Then the continued fraction expansion of α has the form (2.2), where the partial quotients a_j coincide with those of (2.6) for $0 \le j \le 2k$, while

$$a_{2k+1} = m + u_1 y_0, \quad \textit{and} \quad a_{2k+2j} = x_j, \; a_{2k+2j+1} = u_{j+1} y_j, \; \forall j \ge 1. \tag{2.13}$$

Proof The proof consists of showing that the partial sums of the series (2.11) coincide with the convergents with even index, that is

$$\frac{p}{q} + \sum_{j=2}^{N} \frac{1}{x_j} = \frac{p_{2k+2N-2}}{q_{2k+2N-2}}, \tag{2.14}$$

and then taking the limit $N \to \infty$. We omit the details, since the inductive proof of (2.14) is almost identical to that of Theorem 2.1 in [9], with part of the hypothesis being that the denominators of the convergents are

$$q_{2k+2N-3} = y_{N-1} - 1, \qquad q_{2k+2N-2} = x_N \tag{2.15}$$

for $N \ge 1$. The only difference is in verifying the base step: specifically, that, for $N = 1$, the three-term recurrence $q_{2k+2N-1} = a_{2k+2N-1} q_{2k+2N-2} + q_{2k+2N-3}$ gives the correct expression for q_{2k+1}. But when $N = 1$, by using (2.12) and (2.13), together with the first recursive relation in (2.7), the right-hand side becomes $(m + u_1 y_0)q + y_0 - mq - 1 = y_0(1 + u_1 x_1) - 1 = y_1 - 1$, which is the required formula for q_{2k+1}. $\square$

Continued fractions for some alternating series whose sum is a transcendental number were considered in [4], and the Pierce series in [17] provide other examples. As a first attempt at generalizing the latter results, we consider a rational number added to a Pierce series.

Theorem 2.2 *Suppose $x_1 = q > 1$, and let α be given by the series*

$$\alpha = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{(-1)^j}{x_j}, \tag{2.16}$$

for (x_n) defined by (2.7) with

$$y_0 = mq + q_{2k-1} - 1, \tag{2.17}$$

where q_{2k-1} is the denominator of the $(2k-1)$th convergent of (2.6), and the restriction $m \geq 1$ should be imposed if $q_{2k-1} = 1$. Then the continued fraction expansion of α has the form (2.2), where the partial quotients a_j coincide with those of (2.6) for $0 \leq j \leq 2k$, while

$$a_{2k+1} = m + u_1 y_0 - 1, \tag{2.18}$$

and

$$a_{2k+3j-1} = 1, \;\; a_{2k+3j} = x_j - 1, \;\; a_{2k+3j+1} = u_{j+1} y_j - 1 \;\; \forall j \geq 1. \tag{2.19}$$

Proof Let S_N denote the Nth partial sum of the series (2.16), that is

$$S_N = \frac{p}{q} + \sum_{j=2}^{N} \frac{(-1)^j}{x_j}.$$

We will show by induction that

$$S_N = [a_0; a_1, a_2, \ldots, a_{2k+3N-3}],$$

and then the result follows in the limit $N \to \infty$. As part of the inductive hypothesis, we also require the following expressions for the denominators of the convergents, for $N \geq 1$:

$$q_{2k+3N-3} = x_N, \; q_{2k+3N-2} = y_N - x_N + 1, \; q_{2k+3N-1} = y_N + 1. \tag{2.20}$$

For the base case $N = 1$ it is clear that $S_1 = p/q$ is given by (2.6), and $q_{2k} = q = x_1$, while by using the three-term recurrence it follows from (2.17), (2.18) and (2.7) that

$$\begin{aligned} q_{2k+1} = a_{2k+1} q_{2k} + q_{2k-1} &= (m + u_1 y_0 - 1)q + y_0 - mq + 1 \\ &= y_0(1 + u_1 x_1) - x_1 + 1 \\ &= y_1 - x_1 + 1, \end{aligned}$$

and $q_{2k+2} = a_{2k+2} q_{2k+1} + q_{2k} = y_1 - x_1 + 1 + x_1 = y_1 + 1$, which confirms (2.20) in this case. For the inductive step, the first expression in (2.20) is also verified with the three-term recurrence, as

$$\begin{aligned} q_{2k+3N} &= a_{2k+3N}\, q_{2k+3N-1} + q_{2k+3N-2} \\ &= (x_N - 1)(y_N + 1) + y_N - x_N + 1 \\ &= x_N\, y_N = x_{N+1}, \end{aligned}$$

using (2.7) once more, and the other two expressions for the denominators of the convergents are verified similarly. Now let $\mathbf{M}_{n+1}$ denote the matrix on the left-hand side of (2.3), and let

$$\mathbf{A}_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix};$$

then observe that

$$\begin{aligned} \mathbf{M}_{2k+3N} &= \mathbf{M}_{2k+3N-3}\, \mathbf{A}_{2k+3N-2}\, \mathbf{A}_{2k+3N-1}\, \mathbf{A}_{2k+3N} \\ &= \mathbf{M}_{2k+3N-3} \begin{pmatrix} u_N x_N y_{N-1} - 1 & u_N y_{N-1} \\ x_N & 1 \end{pmatrix}. \end{aligned}$$

The first column of the above identity yields the formula

$$p_{2k+3N} = (u_N x_N y_{N-1} - 1)\, p_{2k+3N-3} + x_N\, p_{2k+3N-4}, \tag{2.21}$$

and the analogous expression for q_{2k+3N}. Hence

$$\begin{aligned} p_{2k+3N} &= (y_N - y_{N-1} - 1)\, p_{2k+3N-3} + x_N\, p_{2k+3N-4} \\ &= (y_N - q_{2k+3N-4})\, p_{2k+3N-3} + q_{3k+3N-3}\, p_{2k+3N-4} \\ &= y_N\; p_{2k+3N-3} + (-1)^{N+1}, \end{aligned}$$

where we have used (2.20) and the first relation in (2.7), followed by (2.4). (Note that the calculation leading to the latter expression is slightly different in the base case $N = 1$, involving the use of (2.17) and (2.18), but the conclusion is the same.) Thus, using the first formula in (2.20), and the fact that $y_N / x_{N+1} = 1/x_N$, we have

$$\begin{aligned} p_{2k+3N}/q_{2k+3N} &= {x_{N+1}}^{-1} \left(y_N\; p_{2k+3N-3} + (-1)^{N+1} \right) \\ &= p_{2k+3N-3}/q_{2k+3N-3} + (-1)^{N+1}/x_{N+1}. \end{aligned}$$

So by the inductive hypothesis,

$$\frac{p_{2k+3N}}{q_{2k+3N}} = S_N + \frac{(-1)^{N+1}}{x_{N+1}} = S_{N+1},$$

as required. □

Remark 2.3 The assumption that $q > 1$ implies $k \geq 1$ in (2.6), and is made to ensure that $a_{2k+3} > 0$. However, when $q = 1$, the appearance of a zero in the continued fraction can be dealt with by applying the concatenation operation

$$[\ldots, A, 0, B, \ldots] \mapsto [\ldots, A + B, \ldots] \tag{2.22}$$

(see Proposition 3 in [16], for instance).

In the same spirit as [9], it is perhaps more natural to replace the first term in (1.2) with an arbitrary $r \in \mathbb{Q}$, resulting in a rational number minus a Pierce series. Such a modified Pierce series can be obtained immediately from (2.16), simply by regrouping the terms as

$$\left(\frac{p}{q}+\frac{1}{x_2}\right)-\frac{1}{x_2}+\frac{1}{x_3}-\cdots=\frac{p'}{q'}-\sum_{j=2}^{\infty}\frac{(-1)^j}{x'_j}, \tag{2.23}$$

where, since $q = x_1$ divides x_2, $p' = p(x_2/q)+1 \in \mathbb{Z}$, $q' = x_2$ and $x'_j = x_{j+1}$ for $j \geq 1$. In the proof of the preceding theorem, the initial term of the regrouped series appears with the continued fraction expansion $p'/q' = S_2 = [a_0; a_1, \ldots, a_{2k+3}]$, ending in a partial quotient with an odd index.

In order to formulate the most general result possible, we once again start with an arbitrary $r = p/q \in \mathbb{Q}$ and $m \in \mathbb{Z}_{\geq 0}$, but this time take the continued fraction expansion

$$\frac{p}{q} = [a_0; a_1, a_2, \ldots, a_{2k+1}], \tag{2.24}$$

ending in an odd index, which is always possible by (2.5). The following result includes the regrouped series (2.23) obtained from Theorem 2.2 as the special case $m = 0$.

Theorem 2.4 *Suppose $x_1 = q > 1$, and let α be given by the series*

$$\alpha = \frac{p}{q} - \sum_{j=2}^{\infty}\frac{(-1)^j}{x_j}, \tag{2.25}$$

for (x_n) defined by (2.7) with

$$y_0 = mq + q_{2k} - 1, \tag{2.26}$$

where q_{2k} is the denominator of the $2k$th convergent of (2.24), and the restriction $m \geq 1$ should be imposed if $q_{2k} = 1$. Then the continued fraction expansion of α has the form (2.2), where the partial quotients a_j coincide with those of (2.24) for $0 \leq j \leq 2k+1$, while

$$a_{2k+2} = m + u_1 y_0 - 1, \tag{2.27}$$

and

$$a_{2k+3j} = 1,\ a_{2k+3j+1} = x_j - 1,\ a_{2k+3j+2} = u_{j+1}y_j - 1\ \ \forall j \geq 1, \tag{2.28}$$

with the denominators of the convergents being given by

$$q_{2k+3j-2} = x_j,\ q_{2k+3j-1} = y_j - x_j + 1,\ q_{2k+3j} = y_j + 1\ \ \forall j \geq 1. \tag{2.29}$$

We omit the proof of the above, based on showing that the partial sums are given by

$$\frac{p}{q} - \sum_{j=2}^{N} \frac{1}{x_j} = \frac{p_{2k+3N-2}}{q_{2k+3N-2}},$$

since the steps are essentially the same as for Theorem 2.2, but the formulae (2.29) have been included for completeness.

Remark 2.5 Similarly to (2.23), the case $m = 0$ of Theorem 2.2 can be obtained from Theorem 2.4 by combining the first two terms of the series (2.25) into one. The series (2.11) can be reduced to the case $m = 0$ of Theorem 2.1 in the same way.

3 Irrationality exponents

The irrationality exponent $\mu(\alpha)$ of a real number α is defined to be the supremum of the set of real numbers μ such that there are infinitely many rational numbers P/Q satisfying the inequality $0 < |\alpha - P/Q| < 1/Q^{\mu}$. For an irrational number, $\mu(\alpha) \geq 2$, since the convergents of its continued fraction expansion (2.2) provide infinitely many P/Q with $|\alpha - P/Q| < 1/Q^2$. In fact, in the sense of Lebesgue measure, almost all real numbers have irrationality exponent equal to 2, while a famous theorem of Roth [11] says that every algebraic irrational number has $\mu(\alpha) = 2$. Large classes of transcendental numbers with $\mu(\alpha) = 2$ are presented in [2], but as we shall see, the transcendental numbers defined by the modified Engel and Pierce series above do not belong to these classes.

The transcendence of each of the numbers α defined in the previous section is essentially a consequence of the rapid growth of the associated sequence (x_n). According to the result of Lemma 2.2 in [9], $x_{n+1} > x_n^{5/2}$ for $n \geq 3$, which allows one to show that $\mu(\alpha) \geq 5/2$, but here we present a significant improvement on this result.

Lemma 3.1 *For all $\epsilon > 0$, there is some N such that*

$$x_{n+1} > x_n^{\mu^* - \epsilon}, \qquad \mu^* = \frac{3 + \sqrt{5}}{2}, \tag{3.1}$$

for all $n \geq N$.

Proof From (2.8) and the fact that $y_n = x_{n+1}/x_n > 1$ for $n \geq 1$, it is clear that $x_{n+1} > x_n^2$ for $n \geq 2$, which implies that $x_{n-1} < x_n^{1/2}$ for $n \geq 3$. Thus, using (2.8) once again it follows that $x_{n+1} > x_n^3/x_{n-1} > x_n^{5/2}$, which is the basic estimate given in [7,9]. To improve on this, we proceed by induction, assuming that

$$x_{n+1} > x_n^{\rho_k} \quad \text{for } n \geq k + 2,$$

and then from (2.8), the same argument as before (for $k = 0$) gives

$$x_{n+1} > x_n^3/x_{n-1} > x_n^{\rho_{k+1}} \quad \text{for } n \geq k+3, \quad \text{where } \rho_{k+1} = 3 - \rho_k^{-1}, \tag{3.2}$$

with $\rho_0 = 2$. The solution of the recurrence for ρ_k in (3.2) is obtained via $\rho_k = f_{k+1}/f_k$, which implies $f_{k+2} - 3f_{k+1} + f_k = 0$, so that the sequence (f_k) is just $1, 2, 5, 13, 34, \ldots$, i.e. a bisection of the Fibonacci numbers. Hence $\lim_{k\to\infty} \rho_k = (3+\sqrt{5})/2 = \mu^*$, and the result follows. □

Corollary 3.2 *All of the numbers α defined in Theorems 2.1, 2.2 and 2.4 are transcendental, with irrationality exponent $\mu(\alpha) \geq (3+\sqrt{5})/2$.*

Proof This follows from the same argument as used to prove Theorem 4 in [7], and Theorem 2.3 in [9], so we only sketch the details. For any $\epsilon > 0$, the terms of the sequence (x_n) satisfy the inequality (3.1) for sufficiently large n. Due to the fact that the partial sums of the series (2.11), (2.16) and (2.25) coincide with particular convergents of the continued fraction expansion of α, and the fact that, from (2.15), (2.20), and (2.29), the denominators of these convergents coincide with the terms of the sequence (x_n), a comparison with a geometric series shows that for all $\delta > 0$, the inequality

$$\left|\alpha - \frac{P}{Q}\right| < \frac{1}{Q^{\mu^*-\delta}}$$

holds for infinitely many rational approximations P/Q. □

The irrationality exponent can be computed explicitly in terms of the continued fraction expansion of α, using one of the formulae below:

$$\mu(\alpha) = 1 + \limsup_{n\to\infty} \frac{\log q_{n+1}}{\log q_n} = 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n} \tag{3.3}$$

(see [2], or Theorem 1 in [15], for instance). If the function $\widehat{G}$ in (2.10) is chosen suitably, then these limits can be evaluated precisely.

Theorem 3.3 *For some integer $d \geq 1$, let*

$$\lambda = \frac{d+2+\sqrt{d(d+4)}}{2}, \tag{3.4}$$

and let α be given by a modified Engel or Pierce series, defined according to one of Theorems 2.1, 2.2 or 2.4, with the sequence (u_n) in (2.7) being specified by a polynomial in x_n, namely

$$u_n = \sum_{j=0}^{d-1} v_n^{(j)}\, x_n^{d-1-j}, \tag{3.5}$$

where for each j, the coefficient $(v_n^{(j)})$ is an integer sequence, with $v_n^{(0)} \in \mathbb{Z}_{>0}$, $n \geq 1$, and as $n \to \infty$, for some $\nu < \lambda$,

$$\log v_n^{(0)} = O(\nu^n), \quad \textit{and} \quad v_n^{(j)} = O(v_n^{(0)}), \;\; j = 1, \ldots, d-1. \tag{3.6}$$

Then α has irrationality exponent $\mu(\alpha) = \lambda$.

Proof If u_n is given by (3.5), then setting $\Lambda_n = \log x_n$ and taking logarithms in (2.8) yields

$$\Lambda_{n+1} - (d+2)\,\Lambda_n + \Lambda_{n-1} = \Delta_n, \tag{3.7}$$

where, with $v_n^{(d)} = 1$,

$$\Delta_n = \log v_n^{(0)} + \log\left(1 + \sum_{j=1}^{d} \frac{v_n^{(j)}}{v_n^{(0)}}\, x_n^{-j}\right) = \log v_n^{(0)} + O(x_n^{-1}) = O(\nu^n)$$

as $n \to \infty$, by (3.6). By adapting the method of [1], the solution of (3.7) is found formally as

$$\Lambda_n = A\,\lambda^n + B\,\lambda^{-n} + \sum_{j=1}^{n-1}\left(\frac{\lambda^{n-j} - \lambda^{j-n}}{\lambda - \lambda^{-1}}\right)\Delta_j, \tag{3.8}$$

for constants A, B which can be fixed from the initial values $\Lambda_1 = \log q$, $\Lambda_2 = \log x_2$ (cf. Proposition 5 in [7]). Hence

$$\Lambda_n \sim C\,\lambda^n, \qquad C = A + \sum_{j=1}^{\infty} \frac{\lambda^{-j}\Delta_j}{\lambda - \lambda^{-1}}, \tag{3.9}$$

and $C > 0$ since $x_n \to \infty$ as $n \to \infty$. Then in the case of modified Engel series, from the formulae (2.15) there are two different cases for the first limit in (3.3): when n is even, the limit is

$$1 + \lim_{N\to\infty} \frac{\log(y_N - 1)}{\log x_N} = 1 + \lim_{N\to\infty} \frac{\Lambda_{N+1} - \Lambda_N}{\Lambda_N} = \lambda$$

from the asymptotic behaviour (3.9), while for odd n it is $1 + 1/(\lambda - 1) \leq \lambda$, where this inequality holds for all $\lambda \geq (3+\sqrt{5})/2$; so the limit superior is λ. For the modified Pierce series in Theorem 2.2, from (2.20) the corresponding limit varies with n mod 3, and gives λ, 2 or $1 + \lambda/(\lambda - 1) \leq \lambda$, so again the limit superior is λ, and the case of Theorem 2.4 is identical. □

Theorem 3.4 *For all $\nu \geq (3 + \sqrt{5})/2$ there are infinitely many α defined by modified Engel or Pierce series with irrationality exponent $\mu(\alpha) = \nu$.*

Proof If ν is one of the special values (3.4) then for any fixed $r = p/q$ and m there are uncountably many choices of the sequence (u_n) that take the form (3.5), satisfy (3.6) and produce α with distinct continued fraction expansions. So suppose that $d \geq 1$ (and hence λ) is fixed, and take any $\nu > \lambda$ and (u_n) of the same form as in (3.5), except that now

$$\log v_n^{(0)} \sim C' \, \nu^n, \quad \text{and} \ \ v_n^{(j)} = O(v_n^{(0)}), \ \ j = 1, \ldots, d-1 \tag{3.10}$$

for some $C' > 0$ (for instance, one could take $v_n^{(0)} = \lceil \exp(C'\nu^n) \rceil$). Then the formal expression (3.8) for the solution of (3.7) is still valid, but now from (3.10) the sum on the right is the dominant term, growing like ν^n as $n \to \infty$. Substituting this leading order asymptotic behaviour back into (3.7) yields

$$\Lambda_n \sim C \, \nu^n, \qquad C = \Big(\nu - (d+2) + \nu^{-1} \Big)^{-1} C' > 0,$$

and then the first limit in (3.3) is evaluated as before to yield $\mu(\alpha) = \nu$. $\square$

4 More explicit continued fractions

In [8] one of us obtained the explicit continued fraction expansion for an Engel series (1.1) with the stronger divisibility property

$$x_j^2 | x_{j+1}, \qquad \text{with} \ \ z_{j+1} = \frac{x_{j+1}}{x_j^2} \in \mathbb{Z}_{>0}, \quad j \geq 1 \tag{4.1}$$

being an arbitrary sequence of ratios. (In fact the condition $z_j \geq 2$ was imposed in [8], but it was explained how to deal with some $z_j = 1$ by applying the operation 2.22.) Series of this particular form include

$$\sum_{n=0}^{\infty} \frac{1}{u^{2^n}} \tag{4.2}$$

for integer $u \geq 2$, which for $u = 2$ is known as the Kempner number. All of the numbers (4.2) are transcendental, with irrationality exponent 2 [2]; their continued fraction expansions were found in recursive form in [12], with a non-recursive representation described in [13], and further generalizations with a similar recursive structure being given in [14] and later [16]. The continued fraction expansion of an Engel series (1.1) with the stronger divisibility property above has the same sort of recursive structure, defined by a particular subsequence of the convergents with finite continued fractions whose length approximately doubles at each step. Here we further generalize this result by considering an arbitrary $r = p/q \in \mathbb{Q}$ added to an Engel series of this type, of the form (1.4) with the property (4.1).

Given a finite continued fraction (2.1) for $n \geq 2$, written as $[a_0; \mathbf{a}]$, where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is the word defining the fractional part, it is convenient to define the following family of transformations:

$$\varphi_z : \quad [a_0; \mathbf{a}] \mapsto [a_0; \mathbf{a}, z-1, \hat{\mathbf{a}}], \tag{4.3}$$

where the word $\hat{\mathbf{a}}$ is given by

$$\hat{\mathbf{a}} = (1, a_n - 1, a_{n-1}, \ldots, a_2, a_1).$$

For each z, starting from a continued fraction whose final partial quotient has index n, φ_z produces a new continued fraction $\varphi_z([a_0; \mathbf{a}])$ whose final partial quotient has index $2n+2$. It is also helpful to define the one-parameter family of transformations

$$\varphi_z^* : \quad [a_0; \mathbf{a}] \mapsto [a_0; \mathbf{a}, z-1, \mathbf{a}^*], \tag{4.4}$$

where the word at the end is

$$\mathbf{a}^* = (a_{n-1}+1, a_{n-2}, \ldots, a_2, a_1).$$

For the latter family, $[a_0; \mathbf{a}]$ is sent to $\varphi_z^*([a_0; \mathbf{a}])$, whose final partial quotient has index $2n$. The transformation φ_z^* is just the result of applying φ_z to a continued fraction (2.1) for which $a_n = 1$, followed by using the concatenation operation (2.22) to remove the zero that appears.

Our interest in the above transformations is due to

Lemma 4.1

$$\frac{p_n}{q_n} + \frac{(-1)^n}{z q_n^2} = \varphi_z([a_0; \mathbf{a}]).$$

Proof This is a corollary of Proposition 2 in [16], and follows from the remarks made beneath the proof given there. The case of even $n = 2k$ also follows from the proof of Proposition 2.1 in [8]. □

We now consider a rational number $r = p/q$ added to an Engel series with the property (4.1). Without loss of generality, we exclude the case $r \in \mathbb{Z}$ (when $q = x_1 = 1$), since when $x_2 \geq 2$ one can instead take $p'/q' = p/q + 1/x_2$ as the initial rational term.

Theorem 4.2 *Let $p/q \in \mathbb{Q}\setminus\mathbb{Z}$ be a rational number in lowest terms, with continued fraction expansion*

$$\mathcal{C}_1 = [a_0; \mathbf{a}]$$

taken in the form (2.6), and let $(x_n)_{n\geq 1}$ be the integer sequence defined by

$$x_1 = q, \qquad x_{j+1} = x_j^2 z_{j+1}, \quad j \geq 1,$$

where $(z_n)_{n\geq 2}$ is an arbitrary sequence of positive integers. Define a sequence of finite continued fractions according to

$$\mathcal{C}_2 = \begin{cases} \varphi^*_{z_2}(\mathcal{C}_1), & \text{if } a_{2k} = 1, \\ \varphi_{z_2}(\mathcal{C}_1), & \text{otherwise}, \end{cases}$$

and, for all $j \geq 2$, with the restriction $z_j \geq 2$ imposed,

$$\mathcal{C}_{j+1} = \begin{cases} \varphi^*_{z_{j+1}}(\mathcal{C}_j), & \text{if } a_1 = 1, \\ \varphi_{z_{j+1}}(\mathcal{C}_j), & \text{otherwise}. \end{cases}$$

Then the modified Engel series

$$\alpha = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j} \tag{4.5}$$

has continued fraction expansion $\lim_{N\to\infty} \mathcal{C}_N$.

Proof We show by induction that the partial sum

$$S_N = \frac{p}{q} + \sum_{j=2}^{N} \frac{1}{x_j}$$

has continued fraction expansion $\mathcal{C}_N$, and then the result for α given by (4.5) follows in the limit $N \to \infty$. When $N = 1$ we have $S_1 = p/q = p_{n_1}/q_{n_1} = \mathcal{C}_1$, where $n_1 = 2k$ is the index of the final partial quotient of $\mathcal{C}_1$, chosen to be even as in (2.6). If $a_{n_1} = a_{2k} = 1$, then we apply the transformation (4.4) with $z = z_2$, and otherwise we apply (4.3). In either case we obtain a new continued fraction $\mathcal{C}_2$ whose final partial quotient has an even index, n_2 say, with $a_{n_2} = a_1$. At each subsequent step $N = j$, we have a continued fraction $\mathcal{C}_j$, with an even number n_j being the index of its final partial quotient, that is $a_{n_j} = a_1$, so that to obtain $\mathcal{C}_{j+1}$, if $a_1 = 1$ then we must apply (4.4) with $z = z_{j+1}$, or (4.3) otherwise. Then, from Lemma 4.1 we have

$$S_{j+1} = S_j + \frac{1}{x_{j+1}} = \frac{p_{n_j}}{q_{n_j}} + \frac{1}{z_{j+1}q_{n_j}^2} = \mathcal{C}_{j+1},$$

where we used the additional hypothesis that $q_{n_j} = x_j$. So the induction is almost complete, apart from the verification that $q_{n_{j+1}} = x_{j+1}$ is also a consequence of the above. However, $\mathcal{C}_{j+1}$ equals

$$\frac{p_{n_{j+1}}}{q_{n_{j+1}}} = \frac{p_{n_j}(x_{j+1}/q_{n_j}) + 1}{x_{j+1}} = \frac{p_{n_j}x_j z_{j+1} + 1}{x_{j+1}},$$

since $q_{n_j} = x_j$ and $x_{j+1} = x_j^2 z_{j+1}$. Now any prime P that divides x_{j+1} must divide x_j or z_{j+1}, so the numerator on the right-hand side above is congruent to 1 mod P. Thus the fraction on the right-hand side, with denominator x_{j+1}, is in lowest terms, and since the convergent $p_{n_{j+1}}/q_{n_{j+1}}$ is also in lowest terms this means that $q_{n_{j+1}} = x_{j+1}$ as required. □

Remark 4.3 The case where one or more of the $z_j = 1$ (as in the sum (4.2), for instance) can be dealt with by applying (2.22). For various examples of this, see [8].

Example 4.4 For a given denominator q it is sufficient to consider $0 < p/q < 1$, so picking $q = x_1 = 5$ and $z_j = (j+1)^2 + 1$ for $j \geq 2$, α is the sum

$$\frac{p}{5} + \frac{1}{5^2 \cdot 10} + \frac{1}{5^4 \cdot 10^2 \cdot 17} + \frac{1}{5^8 \cdot 10^4 \cdot 17^2 \cdot 26} + \frac{1}{5^{16} \cdot 10^8 \cdot 17^4 \cdot 26^2 \cdot 37} + \cdots,$$

and we list the continued fractions for $1 \leq p \leq 4$ in order:

$$\begin{gathered}
[0; 4, 1, 9, 5, 16, 1, 4, 9, 1, 4, 25, 1, 3, 1, 9, 4, 1, 16, 5, 9, 1, 4, 36, 1, 3, 1, 9, 5, 16, \ldots], \\
[0; 2, 2, 9, 1, 1, 2, 16, 1, 1, 1, 1, 9, 2, 2, 25, 1, 1, 2, 9, 1, 1, 1, 1, 16, 2, 1, 1, 9, 2, 2, \ldots], \\
[0; 1, 1, 1, 1, 9, 2, 1, 1, 16, 2, 2, 9, 1, 1, 1, 1, 25, 2, 1, 1, 9, 2, 2, 16, 1, 1, 2, 9, 1, 1, \ldots], \\
[0; 1, 4, 9, 1, 3, 1, 16, 4, 1, 9, 4, 1, 25, 5, 9, 1, 4, 16, 1, 3, 1, 9, 4, 1, 36, 5, 9, 1, 3, \ldots].
\end{gathered}$$

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