Representations of Quantum Nilpotent Algebras at Roots of Unity, and Their Completely Prime Quotients

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Abstract

This thesis studies algebras contained in a large class of iterated Ore extensions, as well as their quotient algebras by completely prime ideals, and develops methods for computing their *polynomial identity (PI)* degree and constructing irreducible representations of maximal dimension. This class contains quantum nilpotent algebras, including many examples of quantised coordinate rings and quantised enveloping algebras. When the deformation parameters are allowed to be roots of unity these algebras often become PI algebras. We focus our attention on such algebras in this work.

By extending Cauchon's deleting derivations algorithm in the generic setting [Cau03a] we are able, given a suitable PI algebra A and completely prime ideal $P \lhd A$, to construct a quantum affine space A' and completely prime ideal $Q \lhd A'$, such that the quotient algebras A/P and A'/Q share the same PI degree. This extends a result of Haynal [Hay08], where existence of Q was proved but no method of construction was provided. The PI degree of several small examples are then calculated.

For completely prime quotients of quantum matrices the PI degree is shown to be closely related to properties of *Cauchon-Le diagrams*. We prove that given any Cauchon-Le diagram, the invariant factors of its associated matrix are all powers of 2. Furthermore, we compute the *toric permutation* of Cauchon-Le diagrams corresponding to *quantum determinantal rings*, which then allows us to state an explicit formula for the PI degree of a quantum determinantal ring at a root of unity.

Finally, we show how certain irreducible representations of the quotient A'/Q may be passed through the deleting derivations algorithm to give an irreducible representation of A/P, and we construct an irreducible representation of a general quantum determinantal ring.

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Chapter 1

Introduction

The study of *quantum algebras*, or equivalently *quantum groups*, has been an active area of research since they first arose in the 1980s in the fields of theoretical physics and statistical mechanics. There is no axiomatic definition of a quantum algebra, instead, this class of algebras is made up of examples which share "quantum-like" properties. These are typically deformations of coordinate rings of algebraic groups or varieties, and deformations of enveloping algebras of semisimple Lie algebras, which are then labelled as *quantised* versions of the classical algebras.

Arguably the simplest example to work with, thanks to the results of Goodearl and Letzter [GL98], is a *quantum affine space*, denoted by $\mathscr{O}_\Lambda(\mathbb{K}^N)$, or by $\mathbb{K}_\Lambda[T_1,\dots,T_N]$ when the generators are known, for some multiplicatively antisymmetric matrix $\Lambda := (\lambda_{i,j})_{i,j}$ with nonzero entries in the base field, $\mathbb{K}^*$. This is a free algebra over $\mathbb{K}$ with generators $T_1,\dots,T_N$ subject to commutation relations determined by the matrix Λ:

$$T_iT_j = \lambda_{i,j}T_jT_i \quad \forall\, 1 \leq i,j \leq N.$$

This can be viewed as a deformation of the classical coordinate ring of affine N-space, the commutative polynomial ring $\mathscr{O}(\mathbb{K}^N) = \mathbb{K}[T_1,\dots,T_N]$. Properties of $\mathscr{O}_\Lambda(\mathbb{K}^N)$ can vary depending on the choice of entries in Λ. For example, take $N = 2$ to obtain the *quantum affine plane* $\mathbb{K}_\lambda[T_1,T_2]$ such that $T_1T_2 = \lambda T_2T_1$, and take $\mathbb{K}$ to be algebraically closed. If $\lambda \in \mathbb{K}^*$ is not a root of unity then all the prime ideals of $\mathbb{K}_\lambda[T_1,T_2]$ are completely prime, however, this is not the case if λ is a primitive ℓ^{th} root of unity since $\langle T_1^\ell - 1, T_2^\ell - 1\rangle$ provides a counterexample to this statement [BG02, II.6.8].

The choice of parameters in quantised coordinate rings and quantised enveloping algebras crudely splits their research into two main areas, each requiring different techniques to study them: the *generic* case, where suitable deformation parameters are taken to be non-roots of unity, and the *root of unity* case, where suitable deformation parameters are taken to be roots of unity. The generic case comprises algebras with much more rigid structures [BG02, II.1-II.9] than that of those in the root of unity case, whose algebras are typically finite modules over their centres and to which the theory of polynomial identities may be applied. It is this latter setting that we explore in thesis.

In the generic setting much progress has been made in uncovering the structures of the prime and primitive spectra of some quantum algebras. An important advancement is the *$\mathscr{H}$-stratification theory* of Goodearl and Letzter [GL98], which applies to certain algebras supporting a rational action of a torus, $\mathscr{H}$. It allows for the study of the prime and primitive spectra of the algebra through the study of the so-called *$\mathscr{H}$-primes*; prime ideals which are invariant under the torus action and which parametrise a finite partition of the prime spectrum. Another advancement is Cauchon's *deleting derivations algorithm* [Cau03a], which provides an embedding from the prime spectrum of a suitable algebra into the prime spectrum of a quantum affine space via a process which allows properties from the quantum affine space to be transferred back to the original algebra. This embedding respects certain rational torus actions that may be supported on the algebra, hence Cauchon's procedure is compatible with the $\mathscr{H}$-stratification theory.

Quantum algebras, specifically *iterated Ore extensions* (Section 2.3) that allow the application of both Cauchon's deleting derivations algorithm and Goodearl and Letzter's $\mathscr{H}$-stratification theory, are called *quantum nilpotent algebras* and originally appeared under the name Cauchon-Goodearl-Letzter extensions [LLR06]. The precise definition can be found in [GY15, Definition 2.3] and requires certain parameters to be generic. If, instead, we choose these parameters to be roots of unity then we obtain what we call *quantum nilpotent algebras at roots of unity* and these fall within the class of algebras studied in this thesis. Examples include quantum Schubert cell algebras, quantised Weyl algebras, quantised coordinate rings of affine, symplectic, and euclidean spaces, and quantum matrices.

Aim of Thesis

Many quantised coordinate rings and quantised enveloping algebras at roots of unity become *polynomial identity (PI) algebras*. When this is the case, the *PI degree* is a useful invariant for deducing various properties of the algebra (see Section 2.2 for definitions, basic results, and references to further literature on PI theory). For example, recently Brown and Yakimov [BY17] showed that, under some mild conditions on a prime PI algebra, knowing its Azumaya locus (an invariant which is linked to the PI degree of the algebra and the PI degree of quotients by maximal ideals) provides valuable information about its discriminant ideal (another invariant with applications in the study of automorphism groups of PI algebras ([CPWZ16] and [CPWZ15])). The PI degree also plays an important role in the representation theory of prime affine PI algebras, giving an upper bound on the dimension of irreducible representations (Theorem 2.17). The representations of an algebra shed light on its structure and, as such, they are worth investigating. For these reasons we require a method of calculating the PI degree of quantum algebras and their quotients.

This thesis focusses on extending Cauchon's deleting derivations algorithm in the generic setting [Cau03a] to include the root of unity case by utilising an adapted version, provided by Haynal [Hay08, Section 3], of the homomorphism which lies at the heart of the procedure. This allows, under certain conditions, for the computation of the PI degree of completely prime quotients of a large class of quantum algebras, including quantum nilpotent algebras. The algorithm associates to each suitable quantum PI algebra, A, a corresponding quantum affine space, A', with the same PI degree, and to each completely prime ideal, $P \lhd A$, it associates a completely prime ideal, $Q \lhd A'$, such that $\text{PI-deg}(A/P) = \text{PI-deg}(A'/Q)$. The algorithm also allows for an irreducible representation of A'/Q, satisfying certain conditions, to induce an irreducible representation of A/P of the same dimension. In the case where A/P is a *quantum determinantal ring* we calculate the PI degree explicitly, given a mild restriction on the deformation parameter q, and we construct an irreducible representation of A/P with dimension equal to the PI degree.

Deleting Derivations Algorithm and its Motivation

In the first part of this thesis (Chapters 3 and 4) we extend Cauchon's deleting derivations algorithm [Cau03a, Section 3] to make it applicable to all iterated Ore extensions of the form

$$A = \mathbb{K}[X_1][X_2;\sigma_2,\delta_2]\cdots[X_N;\sigma_N,\delta_N]$$

satisfying certain conditions (see Hypothesis 1) and whose maps (σ_i, δ_i) satisfy a q_i-skew relation, where the parameters q_i are allowed to be roots of unity. Cauchon requires these q_i to be non-roots of unity, however, we prove that many of the results in [Cau03a] still hold in this more general setting. In particular, we construct a corresponding quantum affine space $A' = \mathscr{O}_\Lambda(\mathbb{K}^N)$ such that certain localisations of A and A' are equal in $\text{Frac}(A)$, the total ring of fractions of A, and therefore

$$\text{Frac}(A) = \text{Frac}(A') \tag{1.1}$$

(Theorem 3.10). Furthermore, we produce an analogous canonical embedding to [Cau03a, Section 4] by defining an injective algebra homomorphism

$$\psi : \mathrm{C.Spec}(A) \longrightarrow \mathrm{C.Spec}(A'),$$

where $\mathrm{C.Spec}(R)$ denotes the completely prime spectrum of an algebra, R (Definition 4.7). From this we deduce that, given a completely prime ideal $P \in \mathrm{C.Spec}(A)$ and $Q := \psi(P) \in \mathrm{C.Spec}(A')$, certain localisations of A/P and A'/Q are equal in $\mathrm{Frac}(A/P)$, and hence

$$\mathrm{Frac}(A/P) = \mathrm{Frac}(A'/Q) \tag{1.2}$$

(Theorem 4.25). Similarly to [Cau03a, Proposition 4.4.1], we also recover a partition (Lemma 4.9) of the completely prime spectrum of A indexed by so-called *Cauchon diagrams*, $\mathscr{W}'$; a subset of the power set of $[\![1,N]\!]$.

The existence of Q and isomorphisms between the localisations in (1.1) and (1.2) are proved in [Hay08, Theorem 4.6 and Theorem 6.2] and, indeed, Haynal's modified deleting derivations homomorphism is an integral part of our algorithm. However, following Cauchon's construction, which we do closely, allows us to obtain, for each specific example, an explicit description of the generators of A' as elements in $\mathrm{Frac}(A)$, as well as of the ideal Q, in a way that the results of [Hay08] do not allow. The advantages of this procedure are outlined next as we discuss the motivations for extending Cauchon's algorithm.

Cauchon's deleting derivations algorithm has been applied by various authors to gain a better understanding of structure of quantum algebras in the generic case. Tauvel's height formula, for example, is given by

$$\mathrm{GKdim}(R/P) + \mathrm{ht}(P) = \mathrm{GKdim}(R)$$

and provides a useful connection between two invariants for a prime ideal P of certain algebras R; these are its height, $\mathrm{ht}(P)$, and the Gel'fand-Kirillov dimension of the quotient algebra, $\mathrm{GKdim}(R/P)$. Tauvel's height formula was shown to hold for (generic) quantum nilpotent algebras by Goodearl, Lenagan, and Launois [GLL18] and their proof relied on the use of Cauchon's deleting derivations algorithm.

Casteels also invoked said algorithm when he proved that the quantum minors in any torus-invariant prime ideal in the generic quantum matrices, $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$, form a Gröbner basis [Cas14]. This proves a conjecture of Goodearl and Lenagan [GL02], that all torus-invariant prime ideals of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$ for q a non-root of unity are generated by quantum minors, previously shown to hold for $\mathbb{K}$ of characteristic zero and q transcendental over $\mathbb{Q}$ by

Launois [Lau04b, Lau04a] and Goodearl, Launois, and Lenagan [GLL11], and independently by Yakimov [Yak10, Yak13].

Cauchon himself used the algorithm to give a positive answer to the *quantum Gel'fand-Kirillov conjecture* on prime quotients of a large class of quantum algebras in the generic setting [Cau03a, Théorème 6.1.1], covering examples such as quantum euclidean spaces, quantum symplectic spaces, quantum matrices, and quantum Weyl algebras, all at non-roots of unity. The general conjecture asserts that, if A is a $\mathbb{K}$-algebra which is either a quantised coordinate ring or a quantised enveloping algebra, and P is any prime ideal of A, then $\mathrm{Frac}(A/P)$ must be isomorphic to $\mathrm{Frac}(\mathscr{O}_\Lambda(K^t))$ for some t and some field extension K of $\mathbb{K}$ (see [BG02, I.2.11-14, II.10.4] for further discussion). Through developing a Poisson deleting derivations algorithm for Poisson $\mathbb{K}$-algebras, based on Cauchon's construction, Launois and Lecoutre [LL17] verify (under certain technical assumptions) that the Poisson version of the Gel'fand-Kirillov conjecture holds for a large class of quotients by Poisson prime torus-invariant ideals. A similar result had previously been proved by Goodearl and Launois for quotients by any Poisson prime ideals, when $\mathbb{K}$ has characteristic zero [GL07]. Extending the deleting derivations algorithm, as we do in this thesis, is therefore a step towards answering this conjecture for quantum nilpotent algebras at roots of unity.

These applications of Cauchon's algorithm rely on the explicit changes of indeterminates in the total ring of fractions to transfer certain, more easily obtainable information, from the quantum affine space back to information about the quantum algebra. One motivation for setting up an analogous deleting derivations algorithm in the root of unity case is, therefore, to obtain a similar tool which we might be able to wield to deduce analogous results to the aforementioned. These problems will not be covered any further in this thesis, except for the mention of Casteels' paper in Section 5.4 in our discussion of some open problems.

A second motivation, and the one that this thesis will focus on, comes from the utility of the PI degree in deducing various properties of PI algebras, as mentioned earlier in the introduction. Haynal proves *PI parity* between A and A' [Hay08, Corollary 4.7], that is, A is a PI algebra if and only if A' is a PI algebra and, in this case, both algebras share the same PI degree; a result which may be recovered from (1.1). Leroy and Matzcuk also recover this result in a slightly more general setting [LM11, Theorems 6 and 7]. Moreover, if A is a PI algebra then from (1.2) we deduce that $\text{PI-deg}(A/P) = \text{PI-deg}(A'/Q)$, thus recovering [Hay08, Corollary 6.4]. These equalities may simplify the computation of the PI degree of A and A/P thanks to a result by De Concini and Procesi [CP93, Proposition 7.1] which provides a method for computing the PI degree of a quantum affine space at a root of unity (we recap their result in Theorem 2.30). Hence, if A'/Q is a known quantum affine space then $\text{PI-deg}(A/P)$ is calculable using [CP93, Proposition 7.1].

Although Haynal's methods are successfully employed in [Hay08, Section 5] to compute the PI degree of some quantum algebras, A, they are not sufficient to deduce the PI degree of the completely prime quotients, A/P. The reason for this being that although A' is known, the completely prime ideal Q in A' corresponding to the completely prime ideal P in A cannot be computed using Haynal's methods alone. The process of deleting derivations presented in this thesis partially overcomes this issue by providing a method of constructing Q from P so that A'/Q can be calculated explicitly, given a specific example.

Applying the Deleting Derivations Algorithm

In Chapter 5 we work in the root of unity setting, where suitable parameters of A are given by powers of some primitive ℓ^{th} root of unity, $1 \neq q \in \mathbb{K}^*$. We bring together the technical results of Chapters 3 and 4, outlining how they can be applied in practice to compute the PI degree of A/P for specific examples. Of particular interest to us is when A'/Q becomes a quantum affine space, which happens, for example, when $Q = J_w = \langle T_i \mid i \in w \rangle$, where $T_1, \ldots, T_N$ generate A' and $w \subseteq [\![1, N]\!]$. In this case,

$$\text{PI-deg}(A/P) = \text{PI-deg}(\mathscr{O}_{q^M}(\mathbb{K}^n)), \tag{1.3}$$

for some additively skew-symmetric matrix $M \in M_n(\mathbb{K})$ (Remark 5.2), where, if we know to which subset w the ideal P corresponds, then we also know the matrix M. Applying the result of De Concini and Procesi we deduce (Lemma 5.7) the following formula for the PI degree of A/P which depends only on properties of the matrix M, namely the dimension of its kernel and its invariant factors, as well as the values of N and ℓ:

$$\text{PI-deg}(A/P) = \prod_{i=1}^{\frac{N-\dim(\ker(M))}{2}} \frac{\ell}{\gcd(h_i, \ell)}.$$

Using this formula we compute the PI degree of two examples of completely prime quotients A/P. The first example takes $A = U_q^+(\mathfrak{so}_5)$, the positive part of the quantised enveloping algebra of $\mathfrak{so}_5$, and computes the PI degree of a completely prime quotient $U_q^+(\mathfrak{so}_5)/\langle z' \rangle$ (Example 5.3.1). The second example (Example 5.3.2) computes the PI degree of the quantum Schubert variety $\mathscr{O}_q((G_{2,4}(\mathbb{K})_{\{1,3\}})$ by exploiting the link to a class of quotients of quantum matrices called *quantum determinantal rings*, discovered by Lenagan and Rigal in [LR08]. We also use this link to provide a formula for computing the PI degree of a general quantum Schubert variety, satisfying certain conditions (Corollary 5.13).

The method described in this chapter has some weaknesses in efficiency as it is not known what the set $\mathscr{W}'$ looks like for a general algebra A, or to which $Q \in \text{C.Spec}(A')$ the ideal P is sent. These open questions are discussed in Section 5.4. The formulae obtained in this chapter require the knowledge of properties of the matrix M from (1.3), which may be calculated for individual examples but for which we do not have a general closed form. The next chapter shows that for quantum determinantal rings we can improve on this to obtain an explicit formula for the PI degree, given that the deformation parameter is a primitive ℓ^{th} root of unity with ℓ odd.

Computing the PI degree using De Concini and Procesi's Result

Chapter 6 focusses on the application of the result of De Concini and Procesi in computing the PI degree of the quantum affine space on the right hand side of equation (1.3) in the case when $A = \mathscr{O}_q(M_{m,n}(\mathbb{K}))$, the single parameter quantum matrices at a root of unity q. It does this by using combinatorial arguments on certain types of diagrams to deduce the relevant properties of the matrix M.

In Theorem 4.37 we show that the set $\mathscr{W}'$ is in bijective correspondence with the set of $m \times n$ grids with each square coloured black or white adhering to a rule, which is stated in Section 4.5.1. These go by the name of *Cauchon-Le diagrams* (or sometimes *Cauchon diagrams* or *Le-diagrams* depending on the author), and were discovered independently by Postnikov [Pos06], in relation to the totally nonnegative Grassmannian, and by Cauchon [Cau03b]. For each $w \in \mathscr{W}'$ there exists a corresponding Cauchon-Le diagram C_w to which we can associate a matrix $M(C_w)$. If $\psi(P) = J_w$ then the matrix $M(C_w)$ replaces M in (1.3).

The main result in this chapter states that the invariant factors of the matrix $M(C_w)$ associated to a Cauchon-Le diagram C_w are all powers of 2 (Theorem 6.3). Furthermore, results of Bell, Casteels, and Launois [BCL12, Theorem 4.6 and Lemma 4.3] show that the dimension of $\ker(M(C_w))$ is contained within the properties of a combinatorial object associated to the diagram C_w called the *toric permutation*, which can be read off C_w using so-called *pipe dreams*. Combining these results we obtain a formula for the PI degree of A'/J_w using the corresponding Cauchon-Le diagram (Remark 6.4). We illustrate this with a small example. Work is still needed to trace the ideal J_w back through the deleting derivations algorithm in each example in order to find out to which $P \in \text{C.Spec}(A)$ it corresponds, so we don't get a closed formula for PI-deg(A/P) in general.

The situation improves when we take $A = \mathscr{O}_q(M_n(\mathbb{K}))$ and P to be the ideal generated by all $(t+1)\times(t+1)$ quantum minors of A. In this case, A/P becomes a quantum determinantal ring, denoted $R_t(M_n)$. Thanks to the results of Lenagan and Rigal [LR08, Lemma 4.4], the PI degree of $R_t(M_n)$ is equal to the PI degree of $\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})$ where the matrix $M = M(C)$ corresponds to an $n\times n$ Cauchon-Le diagram C consisting of an $(n-t)\times(n-t)$ block of black boxes in the top left corner. We calculate the toric permutation of C (Proposition 6.10) and deduce from this that $\dim(\ker(M(C))) = t$. This allows us to state the following explicit formula (Theorem 6.11), given that q is a primitive ℓ^{th} root of unity with ℓ odd:

$$\text{PI-deg}(R_t(M_n)) = \ell^{\frac{2nt-t^2-t}{2}}.$$

Chapter 6 ends with some open questions related to the work above.

Irreducible Representations

In Chapter 7 (written in collaboration with Samuel Lopes) we show that, in the root of unity setting, an irreducible representation of $A'/\psi(P)$ may be passed back through the deleting derivations algorithm to give an irreducible representation of A/P of the same dimension (Corollary 7.9). This is illustrated in Section 7.3.2 on the example $U_q^+(\mathfrak{so}_5)/\langle z'\rangle$ from Chapter 5. We specialise to the quantum determintantal ring $R_t(M_n)$ in Section 7.4, which we show sits between a quantum affine space and its corresponding quantum torus (Lemma 7.15). By first building an irreducible representation of this quantum affine space of dimension $\text{PI-deg}(R_t(M_n))$ (Proposition 7.7), we construct an irreducible representation of $R_t(M_n)$ of the same dimension in the case when the deformation parameter q is a primitive ℓ^{th} root of unity with ℓ odd (Proposition 7.16).

Notation

We adopt the following conventions and notation throughout this thesis:

- $\mathbb{N}$ denotes the set of non-negative integers (i.e. $0 \in \mathbb{N}$);
- $\mathbb{K}$ is an arbitrary field unless stated otherwise;
- $\mathbb{K}^* := \mathbb{K} \backslash \{0\}$;
- $[\![a,b]\!]$ denotes the set $\{a,\ldots,b\}$, for integers $a < b$;
- all algebras (often labelled A) are taken to be unital, associative $\mathbb{K}$-algebras unless stated otherwise;
- all ideals are two-sided ideals unless stated otherwise;
- $\bar{a} := a + I$ denotes the image of $a \in A$ in the quotient algebra A/I;
- P/I denotes the image of the ideal $P \lhd A$ in the quotient algebra A/I;
- $\mathrm{Spec}(A)$ denotes the set of all prime ideals of A and $\mathrm{C.Spec}(A)$ denotes the set of all completely prime ideals of A;
- $Z(A)$ denotes the centre of A;
- all homomorphisms and (skew) derivations are $\mathbb{K}$-linear;
- if we write $A = R[x;\sigma,\delta]$ it is understood that A is a left Ore extension over a $\mathbb{K}$-algebra R with $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$ and δ a σ-derivation on R;
- $\mathrm{Frac}(A)$ denotes the total ring of fractions of A, i.e. the (left and right) ring of fractions with respect to the set of all regular elements in A. By writing $\mathrm{Frac}(A)$ we're implying that such a construct exists for A;
- by calling a square matrix $M = (m_{i,j}) = (m_{i,j})_{i,j \in [\![1,n]\!]}$ *skew-symmetric* we mean that it is additively skew-symmetric, i.e. $m_{i,i} = 0$ and $m_{i,j} = -m_{j,i}$ for all $i,j \in [\![1,n]\!]$;
- $M_{m,n}(\mathbb{K})$ denotes the set of $m \times n$ matrices with entries in $\mathbb{K}$ and if $m = n$ we denote the resulting set simply as $M_n(\mathbb{K})$;
- if $I \subseteq [\![1,N]\!]$ then $|I|$ denotes the number of elements in the set I.

Chapter 2

Preliminaries

In this chapter we provide an overview of polynomial identity (PI) ring theory and introduce the specific algebras which will be studied in this thesis. We denote by $\mathbb{K}$ a field of arbitrary characteristic.

2.1 Rings of Fractions

The following definitions and results concerning rings of fractions of noetherian rings are well-known and can be found, for example, in [GW04, Chapters 6 and 10].

Definition 2.1. Given a multiplicative set $X \subseteq A$ in a ring A, X is called a *right Ore set* (or equivalently, X satisfies the *right Ore condition*) if, for all $a \in A$ and $x \in X$, there exists $b \in A$ and $y \in X$ such that $ay = xb$. A *left Ore set* is defined symmetrically and a *(two-sided) Ore set* is one which satisfies both the left and right Ore condition.

Given a right Ore set of regular elements $X \subset A$ in a ring A we call the localisation $AX^{-1} := \{ax^{-1} \mid a \in A, x \in X\}$ the *right ring of fractions for A with respect to X*. This is defined to be the overring $AX^{-1} \supseteq A$ such that every element of X is invertible in AX^{-1} and every element of AX^{-1} can be expressed in the form ax^{-1}, for some $a \in A$ and $x \in X$.

The *left ring of fractions $X^{-1}A$* is defined symmetrically and if X is both a right and left Ore set of regular elements we call $AX^{-1} = X^{-1}A$ a *(two-sided) ring of fractions for A with respect to X*.

A *right (left, two-sided) Ore domain* is any domain A in which the nonzero elements form a right (left, two-sided) Ore set.

Remark 2.2. If A is a noetherian ring then there exists a ring of fractions $AX^{-1} = X^{-1}A$ if and only if X is an Ore set [GW04, Theorem 10.3 and Proposition 10.7]. Furthermore, if A is a noetherian domain then it is an Ore domain [GW04, Corollary 6.7] where all nonzero elements are regular. It therefore has a *total ring of fractions* which is a division ring [GW04, Theorem 6.8] and we denote this by $\mathrm{Frac}(A)$ (this is the ring of fractions of A with respect to the set of all nonzero elements of A).

2.2 Polynomial Identity (PI) theory

2.2.1 Polynomial Identity Rings

Extensive background on PI theory can be found in books by Rowen ([Row88a] and [Row88b]) and Procesi [Pro73]. For a concise overview of more recent results relating specifically to the types of algebras discussed in this thesis, we turn to [BG02, Appendices I.13 and I.14].

Definition 2.3. An element $f = f(x_1,\dots,x_r)$ of the free algebra $\mathbb{Z}\langle x_1,\dots,x_r\rangle$ is called a *monic polynomial* if at least one monomial in f of highest degree has coefficient 1, where the degree of the monomial $x_1^{a_1}\cdots x_r^{a_r}$ is defined to be $a_1+\cdots+a_r$. A ring A is said to satisfy a monic polynomial f when $f(a_1,\dots,a_r)=0$, for all $a_1,\dots,a_r \in A$. In this case, A is said to be a *polynomial identity (PI) ring* (or, equivalently, A is PI). The *minimal degree* of a PI ring A is the least degree of all monic polynomial identities for A. We call an algebra a *PI algebra* if it satisfies some monic polynomial identity.

PI rings cover a large class of rings. We collect some examples and results below.

Example 2.4. Commutative rings satisfy the polynomial identity $x_1x_2 - x_2x_1$ and therefore have minimal degree 2.

Example 2.5. Nilpotent rings (rings in which there exists some $N \in \mathbb{N}_{>0}$ such that every product of N elements is 0 and there exists a nonzero product of $N-1$ elements) satisfy the polynomial identity x^N.

Example 2.6. Amitsur and Levitzki proved ([AL50, Theorem 1]) the celebrated result that the ring of $n\times n$ matrices $M_n(C)$ over a commutative ring C satisfies the so-called *standard identity* s_{2n} of degree $2n$ given by

$$s_{2n} := \sum_{\pi\in S_{2n}} (-1)^{\ell(\pi)} x_{\pi(1)}\cdots x_{\pi(2n)},$$

where S_{2n} is the symmetric group on $2n$ elements, and $\ell(\pi)$ gives the *length* of the permutation π (that is, the number of inversions). In fact, $M_n(C)$ satisfies no monic polynomial of degree less than $2n$, and thus it has minimal degree $2n$ (see, for example, [MR01, Proposition 13.3.2 and Theorem 13.3.3]).

Proposition 2.7 (Corollary 13.1.13, [MR01]). *Any ring which is finitely generated as a module over a commutative subring is a PI ring.*

Example 2.8. A *central simple algebra*, A, is defined as a simple $Z(A)$-algebra which is finite dimensional over $Z(A)$. Hence all central simple algebras are PI by Proposition 2.7.

Proposition 2.9 (I.13.2.4., [BG02]). *If A is a PI ring with minimal degree d, then any subring or factor ring of A is also PI with minimal degree at most d.*

Proposition 2.10 (Proposition 1.7.8, [Row88a]). *If A is a PI ring, and $S \neq \emptyset$ is a multiplicatively closed set of central elements of A, then the localisation AS^{-1} is also a PI ring. Suppose further that S contains no zero divisors then*

$$\textit{minimal degree of } A = \textit{minimal degree of } AS^{-1}.$$

2.2.2 PI Degree

In this section we define the PI degree of prime PI algebras. This definition will suffice because the algebras covered in this thesis are all prime.

As a consequence of the Artin-Wedderburn Theorem, any central simple algebra A is isomorphic to a matrix ring over a central simple division ring. Hence

$$\dim_{Z(A)}(A) = n^2 \tag{2.1}$$

for some $n \in \mathbb{N}_{>0}$. From this, we define the *PI degree* of A to be n.

We now recall one of the fundamental results from PI theory:

Theorem 2.11 (Posner's Theorem). *Let A be a prime PI ring with centre $Z(A)$ and minimal degree d. Let $S = Z(A)\backslash\{0\}$, $Q = AS^{-1}$ and $F = Z(A)S^{-1}$. Then Q is a central simple algebra with centre F and* $\dim_F(Q) = (d/2)^2$.

Note that the Q in Posner's theorem is PI and, since Q is a central simple algebra, we can state its PI degree to be $d/2$ by the discussion above. Furthermore, Proposition 2.10 tells us that Q has the same minimal degree as A, namely d. Recognising that the PI degree can be interpreted as some measure of how close to being commutative a PI algebra is and that this, in turn, is related to its minimal degree, the definition of PI degree given above can be extended to all prime PI rings in the following way:

Definition 2.12. The *PI degree* of a prime PI ring A with minimal degree d is

$$\text{PI-deg}(A) = d/2.$$

Remark 2.13. The reader is invited to refer to [BG02, I.13.3]) for a more technical motivation of this definition.

Remark 2.14. With this definition, Proposition 2.9 implies that PI-deg$(A/P) \leq$ PI-deg(A) for all prime PI rings A and prime ideals $P \in \mathrm{Spec}(A)$.

As a consequence of Posner's Theorem, every prime PI ring has a total ring of fractions $\mathrm{Frac}(A)$ obtained by inverting all nonzero central elements of A, and we obtain the following result [BG02, Corollary I.13.3]:

Corollary 2.15. *Let A be a prime PI ring. If B is a subring of* $\mathrm{Frac}(A)$*, with $A \subseteq B$, then B is also a prime PI ring and PI-deg$(B) =$ PI-deg(A).*

The following result (from [BG02, Section I.13.5.]) provides the important link between the PI degree of a prime affine PI algebra over an algebraically closed field and its irreducible representations, which will be utilised in Chapter 7. Before presenting this result, we give a definition:

Definition 2.16. An algebra is called *affine* if it is finitely generated as an algebra.

Theorem 2.17. *Let A be a prime affine PI algebra over the algebraically closed field $\mathbb{K}$, with PI-deg$(A) = n$.*

(1) If S is a primitive factor of A then

$$S \cong M_t(\mathbb{K})$$

for some integer t, where $t \leq n$.

(2) Let V be an irreducible A-module. Then V is a $\mathbb{K}$-vector space of dimension $t \in \mathbb{N}$, where $t \leq n$ and $A/\mathrm{ann}_A(V) \cong M_t(\mathbb{K})$.

Remark 2.18. The upper bounds in both parts of the theorem are, in fact, attained [BG02, Lemma III.1.2(2)].

2.3 Iterated Ore Extensions and Skew-Laurent Polynomial Rings

2.3.1 Ore Extensions

Let R be a $\mathbb{K}$-algebra and σ be a $\mathbb{K}$-algebra endomorphism on R. A *(left) σ-derivation* on R is defined to be a $\mathbb{K}$-linear additive map $\delta : R \to R$ such that, for all $r, s \in R$,

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s.$$

It may be easily verified that $\delta(1) = 0$, and that if $\sigma = \mathrm{Id}_R$ then δ becomes a derivation in the classical sense. In general, when we don't wish to specify σ, we often refer to the map δ defined above as a *skew derivation*.

We assume σ to be a $\mathbb{K}$-algebra automorphism on R for the rest of the thesis and denote this by $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$. From now on we will drop the "left" and simply call δ a σ-derivation whenever it satisfies the property above.

Given some $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$ and σ-derivation $\delta : R \to R$ we can form the *Ore extension of* R (or *skew-polynomial ring over* R), $A = R[x; \sigma, \delta]$, where

(1) A is a $\mathbb{K}$-algebra containing R as a subalgebra;

(2) $x \in A$;

(3) A is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(4) $xr = \sigma(r)x + \delta(r)$ for all $r \in R$.

When $\delta = 0$ we write $A = R[x; \sigma]$, and when $\sigma = \mathrm{Id}_R$ we write $A = R[x; \delta]$.

We give an example, taken from [GW04, Exercise 1L]:

Example 2.19. Consider the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$ with standard basis $\{e, f, h\}$, where $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. Then the enveloping algebra $U(\mathfrak{sl}_2(\mathbb{K}))$ is the $\mathbb{K}$-algebra generated by e, h, f subject to the following relations:

$$ef - fe = h, \qquad he - eh = 2e, \qquad hf - fh = -2f.$$

Let $R := \mathbb{K}[h]$ be the polynomial ring and let A be the subalgebra of $U(\mathfrak{sl}_2(\mathbb{K}))$ generated by e and h. Then A may be presented as an Ore extension over R, written

$$A = R[e; \sigma_1, 0] = R[e; \sigma_1],$$

where σ_1 is the $\mathbb{K}$-automorphism on R defined on h as $\sigma_1(h) = h-2$. It can be checked, using property (4) of Ore extensions given above, that $eh = \sigma_1(h)e + 0 = he - 2e$, as required.

Definition 2.20. $R[x;\sigma,\delta]$ is a *q-skew Ore extension* (or, equivalently, (σ,δ) is q-skew) if the automorphism and skew derivation satisfy the relation $\delta \circ \sigma = q\sigma \circ \delta$. Note that this is the opposite to the relation used in [Cau03a], however it matches [Hay08] and others. The derivation δ is *locally nilpotent* if, for every $r \in R$, there is an integer $n_r \geq 0$ such that $\delta^{n_r}(r) = 0$ and $\delta^m(r) \neq 0$ for any $m < n_r$. We define such an n_r as the *δ-nilpotence index of r*.

The existence of an Ore extension for any algebra R, $\mathbb{K}$-automorphism σ, and σ-derivation δ is given, for instance, in [GW04, Proposition 2.3]. Ore extensions satisfy a universal mapping property [GW04, Proposition 2.4] and are consequently unique up to isomorphism [GW04, Corollary 2.5]:

Proposition 2.21 (Universal Property of Ore Extensions)**.** *Let $A = R[x;\sigma,\delta]$ be an Ore extension. Suppose there exists an algebra B with an algebra homomorphism $\phi : R \to B$ and an element $y \in B$ such that $y\phi(r) = \phi \circ \sigma(r)y + \phi \circ \delta(r)$ for all $r \in R$. Then there exists a unique algebra homomorphism $\psi : A \to B$ such that $\psi(x) = y$ and $\psi|_R = \phi$.*

Corollary 2.22. *Let R be an algebra, $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$, and δ a σ-derivation on R. Set $A = R[x;\sigma,\delta]$ and $A' = R[x';\sigma,\delta]$. Then there exists a unique isomorphism $\phi : A \to A'$ such that $\phi(x) = x'$ and $\phi|_R = \mathrm{Id}_R$.*

The following results (see [BG02, Lemma I.1.12 and Theorem I.1.13]) will prove useful:

Theorem 2.23. *Let $A = R[x;\sigma,\delta]$. Then the following statements hold:*

(a) A is a domain if R is a domain.

(b) A is a prime ring if R is a prime ring.

(c) (Skew Hilbert Basis Theorem) *A is right (left) noetherian if R is right (left) noetherian.*

2.3.2 Iterated Ore Extensions

Iterated Ore extensions are constructed inductively: Starting with an Ore extension $A_1 := R[x_1;\sigma_1,\delta_1]$ we construct the algebra A_2 to be the Ore extension of A_1 by an automorphism, $\sigma_2 \in \mathrm{Aut}_{\mathbb{K}}(A_1)$, and a σ_2-derivation, $\delta_2 : A_1 \to A_1$. That is, $A_2 := A_1[x_2;\sigma_2,\delta_2] = R[x_1;\sigma_1,\delta_1][x_2;\sigma_2,\delta_2]$. We may continue this process to obtain the *iterated Ore extension*

$$A = A_n := R[x_1;\sigma_1,\delta_1]\cdots[x_n;\sigma_n,\delta_n],$$

where $\sigma_i \in \mathrm{Aut}_{\mathbb{K}}(A_{i-1})$ and δ_i is a σ_i-derivation on A_{i-1}, for all $1 \leq i \leq n$. The convention is to set $A_0 := R$.

Remark 2.24. Theorem 2.23 naturally extends to iterated Ore extensions.

We build upon Example 2.19 with another example taken from [GW04, Exercise 2S]:

Example 2.25. Let $A \subseteq U(\mathfrak{sl}_2(\mathbb{K}))$ be the subalgebra generated by e and h. Then $U(\mathfrak{sl}_2(\mathbb{K}))$ may be presented as an Ore extension over A or, equivalently, as an iterated Ore extension over $\mathbb{K}$:

$$U(\mathfrak{sl}_2(\mathbb{K})) = A[f;\sigma_2,\delta_2] = \mathbb{K}[h][e;\sigma_1][f;\sigma_2,\delta_2],$$

where σ_2 is the $\mathbb{K}$-automorphism on A with $\sigma_2(e) = e$ and $\sigma_2(h) = h+2$, δ_2 is the σ_2-derivation on A with $\delta_2(e) = -h$ and $\delta_2(h) = 0$, and σ_1 is the automorphism defined in Example 2.19.

2.3.3 Skew-Laurent Extensions

Given some $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$ we may write the *skew-Laurent extension of R* (or *skew-Laurent polynomial ring over R*) as $B = R[x^{\pm 1};\sigma]$, where

(1) B is a $\mathbb{K}$-algebra containing R as a subalgebra;

(2) $x \in B$ is invertible;

(3) B is a free left R-module with basis $\{1,x,x^{-1},x^2,x^{-2},\dots\}$;

(4) $xr = \sigma(r)x$ for all $r \in R$.

As with Ore extensions, it may be shown that the skew-Laurent extension for $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$ exists and is unique up to isomorphism, by a universal mapping property [GW04, Exercise 1N]. It may also be shown that a skew-Laurent extension is a localisation of an Ore extension; that is $B = AX^{-1}$, where $A = R[x;\sigma]$ and $X = \{x^i \mid i \in \mathbb{N}\}$ is a multiplicatively closed set in A.

Following an inductive process similar to the one used to get iterated Ore extensions, we may construct an *iterated skew-Laurent extension*

$$B = R[x_1^{\pm 1};\sigma_1]\cdots[x_n^{\pm 1};\sigma_n],$$

where $\sigma_i \in \mathrm{Aut}_{\mathbb{K}}(B_{i-1})$ for all $1 \leq i \leq n$.

As a consequence of the Skew Hilbert Basis Theorem, and by extending [GW04, Corollary 1.15] to iterated skew-Laurent extensions, we may state the following:

Corollary 2.26. *Let $B = R[x_1^{\pm 1};\sigma_1]\cdots[x_n^{\pm 1};\sigma_n]$ where $\sigma_i \in \mathrm{Aut}_{\mathbb{K}}(B_{i-1})$ for all $1 \leq i \leq n$. If R is right (left) noetherian then so is B.*

2.4 Algebras Used in this Thesis

The specific algebras of interest in this thesis arise either as deformations of classical coordinate rings or as deformations of enveloping algebras, which can be presented in terms of generators and relations. For example, given n indeterminates $\mathbf{x} := \{x_1,\ldots,x_n\}$ and t relations $R := \{r_i(\mathbf{x}) = 0 \mid i = 1,\ldots,t\}$, where $r_i \in \mathbb{K}\langle x_1,\ldots,x_n\rangle$, we write

$$\mathbb{K}\langle \mathbf{x} \mid r_1(\mathbf{x}) = 0,\ldots,r_t(\mathbf{x}) = 0\rangle$$

to mean the $\mathbb{K}$-algebra given by generators $\mathbf{x}$ satisfying the relations in R.

Example 2.27. The commutative polynomial ring in n indeterminates can be presented by generators $x_1,\ldots,x_n$ and relations $x_ix_j = x_jx_i$ for all $1 \leq i,\ j \leq n$, so

$$\mathbb{K}[x_1,\ldots,x_n] := \mathbb{K}\langle x_1,\ldots,x_n \mid x_ix_j = x_jx_i \quad \forall\, i,j \in [\![1,n]\!]\rangle.$$

Remark 2.28. In this thesis we will use the notation $\mathbb{K}\langle x_1,\ldots,x_n\rangle$ to denote **either** the free $\mathbb{K}$-algebra generated by $x_1,\ldots,x_n$ (where these generators satisfy no relations) **or** an algebra with these generators whose relations are known from the context (for example, a subalgebra of a given algebra, in which case the relations from the larger algebra carry over to the subalgebra).

2.4.1 Quantum Affine Spaces

Let N be a positive integer and $\Lambda := (\lambda_{i,j}) \in M_N(\mathbb{K}^*)$ be a multiplicatively antisymmetric matrix (that is, $\lambda_{i,j} = \lambda_{j,i}^{-1}$ and $\lambda_{i,i} = 1$ for all $1 \leq i,\ j \leq N$). Using Λ we can define relations on N indeterminates $T_1,\ldots,T_N$ by setting $T_iT_j = \lambda_{i,j}T_jT_i$ for all $i,j \in [\![1,N]\!]$. The $\mathbb{K}$-algebra presented by generators $T_1,\ldots,T_N$ with relations derived from Λ,

$$\mathbb{K}\langle T_1,\ldots,T_N \mid T_iT_j = \lambda_{i,j}T_jT_i \quad \forall\, i,j \in [\![1,N]\!]\rangle,$$

is called the *multiparameter quantum affine space* corresponding to Λ and is denoted by $\mathscr{O}_\Lambda(\mathbb{K}^N)$ or $\mathbb{K}_\Lambda[T_1,\ldots,T_N]$. In the special case where $N = 2$ we get the *quantum plane* which we denote by $\mathscr{O}_q(\mathbb{K}^2)$ or $\mathbb{K}_q[T_1,T_2]$ for some $q \in \mathbb{K}^*$, where $q = \lambda_{1,2}$ in the notation above.

If there exists some $q \in \mathbb{K}^*$ and an (additively) skew-symmetric matrix $M = (m_{i,j}) \in M_N(\mathbb{Z})$ such that $\lambda_{i,j} = q^{m_{i,j}}$ for all $1 \leq i, j \leq N$, then we call $\mathcal{O}_\Lambda(\mathbb{K}^N)$ a *uniparameter quantum affine space* corresponding to M with parameter q, and we denote it by $\mathcal{O}_{q^M}(\mathbb{K}^N)$ or $\mathbb{K}_{q^M}[T_1, \dots, T_N]$.

If all entries of Λ are roots of unity then there exists a root of unity $q \in \mathbb{K}^*$ such that $\lambda_{i,j} = q^{m_{i,j}}$ for some $m_{i,j} \in \mathbb{Z}$, for all $1 \leq i, j \leq N$. This may be seen by assuming that $\lambda_{i,j}$ is an $r_{i,j}^{\text{th}}$ root of unity, for all i, j, and setting $r := \text{lcm}\{r_{i,j} \mid i, j = 1, \dots, N\}$. There then exists an r^{th} root of unity, which we label q, and integers $m_{i,j}$ such that $q^{m_{i,j}} = \lambda_{i,j}$. Thus, $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathcal{O}_{q^M}(\mathbb{K}^N)$. We often refer to this as the *root of unity case*.

We can write the quantum affine space $\mathcal{O}_\Lambda(\mathbb{K}^N)$ as the iterated Ore extension

$$\mathbb{K}[T_1][T_2; \sigma_2] \cdots [T_N; \sigma_N],$$

where each σ_i is a $\mathbb{K}$-automorphism on the relevant algebra and $\sigma_i(T_j) = \lambda_{i,j} T_j$ for all $j < i$. Therefore $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is a noetherian domain by Theorem 2.23. It has a PBW $\mathbb{K}$-basis $\{T_1^{i_1} \cdots T_N^{i_N} \mid i_1, \dots, i_N \in \mathbb{N}\}$.

Remark 2.29 (PI setting)**.** In the root of unity case (where all $\lambda_{i,j} = q^{m_{i,j}}$ for a primitive ℓ^{th} root of unity, $q \in \mathbb{K}^*$), we can easily check that the quantum affine space $\mathcal{O}_{q^M}(\mathbb{K}^N)$ has $\mathbb{K}[T_1^\ell, \dots, T_N^\ell]$ as a central subalgebra. Hence $\mathcal{O}_{q^M}(\mathbb{K}^N)$ is a finitely generated module over its centre, with basis $\{T_1^{i_1} \cdots T_N^{i_N} \mid 0 \leq i_1, \dots, i_N < \ell\}$, and we can use Proposition 2.7 to conclude that it is a (prime) PI ring. This sufficient condition on the entries $\lambda_{i,j}$ for a quantum affine space to be PI is, in fact, necessary, as will be seen next.

The theorem that follows ([BG02, Proposition I.14.2] and [CP93, Proposition 7.1]) provides one of the key techniques for calculating the PI degree of the algebras we are interested in. It provides the motivation for the deleting derivations algorithm and, as such, it underpins this whole thesis.

Theorem 2.30 (De Concini and Procesi)**.** *Let $\Lambda = (\lambda_{i,j}) \in M_N(\mathbb{K})$ be a multiplicatively antisymmetric matrix.*

(i) The algebra $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is a PI ring if and only if all the $\lambda_{i,j}$ are roots of unity. In this case there exists a root of unity $q \in \mathbb{K}^*$ and integers $m_{i,j}$ such that $\lambda_{i,j} = q^{m_{i,j}}$ for all $i, j \in [\![1, N]\!]$.

(ii) Suppose $\lambda_{i,j} = q^{m_{i,j}}$ for all i, j, for some skew-symmetric matrix $M = (m_{i,j}) \in M_N(\mathbb{Z})$, and suppose $q \in \mathbb{K}^*$ is a primitive ℓ^{th} root of unity. Let h be the cardinality of the image

of the homomorphism

$$\mathbb{Z}^N \xrightarrow{M} \mathbb{Z}^N \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^N, \tag{2.2}$$

where π denotes the canonical epimorphism. Then PI-deg$(\mathscr{O}_\Lambda(\mathbb{K}^N)) = \sqrt{h}$.

2.4.2 Quantum Tori

Let N be a positive integer and $\Lambda = (\lambda_{i,j}) \in M_N(\mathbb{K}^*)$ be a multiplicatively antisymmetric matrix. The $\mathbb{K}$-algebra presented by generators $T_1, T_1^{-1}, T_2, T_2^{-1}, \dots, T_N, T_N^{-1}$ with relations coming from Λ,

$$\mathbb{K}\langle T_1, T_1^{-1}, T_2, T_2^{-1}, \dots, T_N, T_N^{-1} \mid T_iT_j = \lambda_{i,j}T_jT_i,\ \ T_iT_i^{-1} = T_i^{-1}T_i = 1 \quad \forall\, i,j \in [\![1,N]\!]\rangle,$$

is called the *multiparameter quantum torus* corresponding to Λ and is denoted by $\mathscr{O}_\Lambda((\mathbb{K}^*)^N)$ or $\mathbb{K}_\Lambda[T_1^{\pm1}, \dots, T_N^{\pm1}]$.

If there exists some $q \in \mathbb{K}^*$ and an (additively) skew-symmetric matrix $M = (m_{i,j}) \in M_N(\mathbb{Z})$, such that $\lambda_{i,j} = q^{m_{i,j}}$ for all $1 \le i,j \le N$, then we call $\mathscr{O}_\Lambda((\mathbb{K}^*)^N)$ a *uniparameter quantum torus* corresponding to M with parameter q and we denote it by $\mathscr{O}_{q^M}((\mathbb{K}^*)^N)$ or $\mathbb{K}_{q^M}[T_1^{\pm1}, \dots, T_N^{\pm1}]$.

If all entries in Λ are roots of unity, then there exists a root of unity $q \in \mathbb{K}^*$ such that $\lambda_{i,j} = q^{m_{i,j}}$, for some $m_{i,j} \in \mathbb{Z}$ and all $1 \le i,j \le N$. Thus, $\mathscr{O}_\Lambda((\mathbb{K}^*)^N) = \mathscr{O}_{q^M}((\mathbb{K}^*)^N)$. We often refer to this as the *root of unity case*.

We can write the quantum torus $\mathscr{O}_\Lambda((\mathbb{K}^*)^N)$ as the iterated skew-Laurent extension

$$\mathbb{K}[T_1^{\pm1}][T_2^{\pm1}; \sigma_2] \cdots [T_N^{\pm1}; \sigma_N]$$

where each σ_i is a $\mathbb{K}$-automorphism on the relevant algebra and $\sigma_i(T_j) = \lambda_{i,j}T_j$, for all $j < i$. Therefore $\mathscr{O}_\Lambda((\mathbb{K}^*)^N)$ is a noetherian domain by Theorem 2.23. It has a PBW $\mathbb{K}$-basis $\{T_1^{i_1} \cdots T_N^{i_N} \mid i_1, \dots, i_N \in \mathbb{Z}\}$.

Remark 2.31 (PI setting)**.** The quantum torus $\mathscr{O}_\Lambda((\mathbb{K}^*)^N)$ is the localisation of the quantum affine space $\mathscr{O}_\Lambda(\mathbb{K}^N)$ with respect to the multiplicatively closed set $S = \{T_1^{i_1}, \dots, T_N^{i_N} \mid i_1, \dots, i_N \in \mathbb{N}\}$, i.e.,

$$\mathscr{O}_\Lambda((\mathbb{K}^*)^N) = \mathscr{O}_\Lambda(\mathbb{K}^N)S^{-1}.$$

From this we arrive at the inclusions

$$\mathscr{O}_\Lambda(\mathbb{K}^N) \subset \mathscr{O}_\Lambda((\mathbb{K}^*)^N) = \mathscr{O}_\Lambda(\mathbb{K}^N)S^{-1} \subset \mathrm{Frac}(\mathscr{O}_\Lambda(\mathbb{K}^N)).$$

We saw in Theorem 2.30 that $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is a (prime) PI ring if and only if all the $\lambda_{i,j}$ are roots of unity. Using Corollary 2.15 with the inclusions above, we deduce that $\mathcal{O}_{q^M}((\mathbb{K}^*)^N)$ is a (prime) PI ring, with PI-deg$(\mathcal{O}_{q^M}((\mathbb{K}^*)^N)) =$ PI-deg$(\mathcal{O}_{q^M}(\mathbb{K}^N))$, if and only if all the $\lambda_{i,j}$ are roots of unity.

2.4.3 Quantum Matrices

The quantised coordinate ring of 2×2 matrices is now presented in terms of its generators and relations. The interested reader is referred to [BG02, Example I.1.6] for a detailed construction of this ring with justifications of the relations arising from the classical algebraic geometry setting.

The $\mathbb{K}$-algebra generated by a,b,c,d with relations

$$ab = qba, \quad ac = qca, \quad ad = da + (q-q^{-1})bc, \quad bc = cb, \quad bd = qdb, \quad cd = qdc, \tag{2.3}$$

for some $q \in \mathbb{K}^*$, is called the *single parameter quantised coordinate ring of* 2×2 *matrices* and is denoted by $\mathcal{O}_q(M_2(\mathbb{K}))$. Its generators can be thought of as lying in a 2×2 matrix, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

There is an analogue to the classical determinant in $\mathcal{O}_q(M_2(\mathbb{K}))$, called the *quantum determinant*, which is a central element and is defined as

$$D_q := ad - qbc.$$

Using the relations in the 2×2 case we can define the *single parameter quantised coordinate ring of* $m\times n$ *matrices* $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ for any $m,n \in \mathbb{N}_{>0}$ as follows: Arrange the mn generators $X_{i,j}$ of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ in a matrix,

$$\mathcal{X}_q = \begin{pmatrix} X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,n} \end{pmatrix},$$

which we call the *matrix of generators* for $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$. Then, for any 2×2 submatrix $\left(\begin{smallmatrix} X_{i,j} & X_{i,t} \\ X_{s,j} & X_{s,t} \end{smallmatrix}\right)$ of $\mathcal{X}_q$, we define the relations between the generators $X_{i,j}, X_{i,t}, X_{s,j}, X_{s,t}$ to be precisely those for a,b,c,d given in (2.3). That is, for $(1,1) \le (i,j) < (s,t) \le (m,n)$ (in lexicographic ordering), we have

$$X_{i,j}X_{s,t} = \begin{cases} X_{s,t}X_{i,j} & i < s,\ j > t; \\ qX_{s,t}X_{i,j} & (i = s,\ j < t) \text{ or } (i < s,\ j = t); \\ X_{s,t}X_{i,j} + (q - q^{-1})X_{i,t}X_{s,j} & i < s,\ j < t. \end{cases}$$

Remark 2.32. More often than not we use the terms *quantum $m \times n$ matrix algebra* or *$m \times n$ quantum matrices* to describe $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$, instead of the more formal *quantised coordinate ring of $m \times n$ matrices*.

We can also define the *multiparameter quantum $m \times n$ matrix algebra* (or *$m \times n$ multiparameter quantum matrices*), $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$, where $\lambda \in \mathbb{K}^*$ and $\mathbf{p} = (p_{i,j}) \in M_{m,n}(\mathbb{K}^*)$ is a multiplicatively antisymmetric matrix, as the $\mathbb{K}$-algebra with matrix of generators

$$\mathcal{X}_{\lambda,\mathbf{p}} = \begin{pmatrix} X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,n} \end{pmatrix}$$

and relations, for $(1,1) \leq (s,t) < (i,j) \leq (m,n)$,

$$X_{i,j}X_{s,t} = \begin{cases} p_{i,s}p_{t,j}X_{s,t}X_{i,j} + (\lambda - 1)p_{i,s}X_{s,j}X_{i,t} & i > s,\ j > t; \\ \lambda p_{i,s}p_{t,j}X_{s,t}X_{i,j} & i > s,\ j \leq t; \\ p_{t,j}X_{s,t}X_{i,j} & i = s,\ j > t. \end{cases}$$

Remark 2.33. We can recover the single parameter relations from those of the multiparameter case by setting $\lambda = q^{-2}$ and $p_{i,j} = q$ for all $i > j$.

Definition 2.34. We can extend the definition of the quantum determinant to arbitrary degree and use this to define quantum minors on $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ and $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$.

1. Single parameter case: The *single parameter quantum determinant D_q* of $\mathcal{O}_q(M_n(\mathbb{K}))$ can be expressed as

$$D_q = \sum_{\pi \in S_n} (-q)^{\ell(\pi)} X_{\pi(1),1}, \ldots X_{\pi(n),n},$$

where $\ell(\pi)$ gives the length of the permutation π.

2. Multiparameter case: The *multiparameter quantum determinant* $D_{\lambda,\mathbf{p}}$ of $\mathcal{O}_{\lambda,\mathbf{p}}(M_n(\mathbb{K}))$ can be expressed as

$$D_{\lambda,\mathbf{p}} = \sum_{\pi \in S_n} \left(\prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} (-p_{\pi(i),\pi(j)}) \right) X_{\pi(1),1}, \ldots X_{\pi(n),n}.$$

3. Quantum minors: Let $I \subseteq [\![1,m]\!]$ and $J \subseteq [\![1,n]\!]$, with $|I| = |J| = s$ for some $1 \le s \le \min(m,n)$. Order the elements in I and J so that $I = \{i_1 < \ldots < i_s\}$ and $J = \{j_1 < \ldots < j_s\}$ and let $\mathscr{X}_{I,J}$ denote the $s \times s$ submatrix of $\mathscr{X}_q$ (or $\mathscr{X}_{\lambda,\mathbf{p}}$) determined by taking the rows indexed by $\{i_1, \ldots, i_s\}$ and the columns indexed by $\{j_1, \ldots, j_s\}$. The *quantum minor* $[I|J] \in \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ (respectively $[I|J] \in \mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$) is defined to be the quantum determinant (respectively the multiparameter quantum determinant) of $\mathscr{X}_{I,J}$.

4. Index pairs: For I and J defined as above, we call the pair (I,J) an *index pair* and we denote the set of all index pairs by $\Delta_{m,n}$. Since there is a one-to-one correspondence between index pairs of $[\![1,m]\!] \times [\![1,n]\!]$ and quantum minors of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ we often identify these two sets and use $\Delta_{m,n}$ to denote the set of all quantum minors of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$.

Both the single parameter and multiparameter quantum matrices can be written as iterated Ore extensions with the generators appearing in lexicographic order. In the multiparameter case we can write

$$\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K})) = \mathbb{K}[X_{1,1}][X_{1,2}; \sigma_{1,2}, \delta_{1,2}] \cdots [X_{m,n}; \sigma_{m,n}, \delta_{m,n}],$$

where

$$\sigma_{i,j}(X_{s,t}) = \begin{cases} p_{i,s}p_{t,j}X_{s,t} & (i > s,\ j \neq t); \\ \lambda\, p_{i,s}p_{t,j}X_{s,t} & (i > s,\ j = t); \\ p_{t,j}X_{s,t} & (i = s,\ j > t), \end{cases}$$

$$\delta_{i,j}(X_{s,t}), = \begin{cases} (\lambda - 1)p_{i,s}X_{s,j}X_{i,t} & (i > s,\ j > t); \\ 0 & \text{otherwise}. \end{cases}$$

The single parameter case is recovered using Remark 2.33. $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ and $\mathcal{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$ are both noetherian domains by Theorem 2.23 and they both have a PBW $\mathbb{K}$-basis given by monomials in the generators, $X_{1,1}^{i_{1,1}} X_{1,2}^{i_{1,2}} \cdots X_{m,n}^{i_{m,n}}$, where $i_{1,1}, i_{1,2}, \ldots, i_{m,n} \in \mathbb{N}$.

Remark 2.35 (PI setting)**.** In the single parameter case when q is a primitive ℓ^{th} root of unity, we have that all ℓ^{th} powers of the generators are central elements. To see this, consider the 2×2 quantum matrices $\mathcal{O}_q(M_2(\mathbb{K}))$ generated by a,b,c,d (the calculations are then easily extended to $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$). It is straightforward to see that $b^\ell, c^\ell \in Z(\mathcal{O}_q(M_2(\mathbb{K})))$ and that a^ℓ, d^ℓ commute with b,c. It remains to check that $a^\ell d = da^\ell$ (and that $d^\ell a = ad^\ell$, which is shown in a similar way): It can be shown, via induction on $1\le k\le \ell$, that $a^k d = da^k + (q^{2k-1}-q^{-1})bca^{k-1}$. Therefore, when $k=\ell$, we get $q^{2\ell-1}-q^{-1} = q^{-1}-q^{-1} = 0$ and hence $a^\ell d = da^\ell$.

Define the *ℓ-centre* of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ to be the central subalgebra $Z_0 \subseteq Z(\mathcal{O}_q(M_{m,n}(\mathbb{K})))$, where $Z_0 := \mathbb{K}[X_{1,1}^\ell, \dots, X_{m,n}^\ell]$. Any monomial $X_{1,1}^{i_{1,1}} X_{1,2}^{i_{1,2}} \cdots X_{m,n}^{i_{m,n}} \in \mathcal{O}_q(M_{m,n}(\mathbb{K}))$ can be written as

$$X_{1,1}^{i_{1,1}} X_{1,2}^{i_{1,2}} \cdots X_{m,n}^{i_{m,n}} = (X_{1,1}^\ell)^{j_{1,1}} (X_{1,2}^\ell)^{j_{1,2}} \cdots (X_{m,n}^\ell)^{j_{m,n}} X_{1,1}^{k_{1,1}} X_{1,2}^{k_{1,2}} \cdots X_{m,n}^{k_{m,n}},$$

where $(X_{1,1}^\ell)^{j_{1,1}} (X_{1,2}^\ell)^{j_{1,2}} \cdots (X_{m,n}^\ell)^{j_{m,n}} \in Z_0$ and $0 \le k_{1,1}, \dots, k_{m,n} \le \ell-1$. Hence, $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is finite dimensional as a module over Z_0 and hence, by Proposition 2.7, it is a PI ring.

2.4.4 Quantum Determinantal Rings

For any $0 \le t < \min(m,n)$ let $I_t := \langle [I|J] \in \mathcal{O}_q(M_{m,n}(\mathbb{K})) \mid |I| = |J| = t+1 \rangle$ be the ideal in $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ generated by all $(t+1)\times(t+1)$ quantum minors. We call such an ideal a *quantum determinantal ideal* and we define the *quantum determinantal ring* as $R_t(M_{m,n}) := \mathcal{O}_q(M_{m,n}(\mathbb{K}))/I_t$. The algebra $R_t(M_{m,n})$ is noetherian (as the quotient ring of a noetherian ring) and also a domain, so that I_t is completely prime [GL00, Corollary 2.6]. Furthermore, if $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is PI then so too is $R_t(M_{m,n})$. Note that when $t=0$, the quantum determinantal ring $R_0(M_{m,n}(\mathbb{K}))$ is trivial, i.e. it becomes the field, $\mathbb{K}$.

Remark 2.36. From the geometric perspective, the set of $m\times n$ matrices of rank at most t, $V_t \subseteq M_{m,n}(\mathbb{K})$, is an irreducible variety with coordinate ring

$$\mathcal{O}(V_t) = \mathcal{O}(M_{m,n}(\mathbb{K}))/J_t,$$

where $J_t \lhd \mathcal{O}(M_{m,n}(\mathbb{K}))$ is the prime ideal generated by all $(t+1)\times(t+1)$ minors. Taking quantum analogues of the right hand side of this equality gives $\mathcal{O}_q(M_{m,n}(\mathbb{K}))/I_t$. Therefore the quantum determinantal ring $R_t(M_{m,n})$ can be thought of as the quantum analogue to $\mathcal{O}(V_t)$. (See [GL00] for more details on the motivation coming from determinantal varieties.)

Chapter 3

The Deleting Derivations Algorithm on Iterated Ore Extensions

3.1 Preliminary Work

In this chapter we lay out the main foundations of the deleting derivations algorithm. First we set up some definitions and notation of objects that will be used in the work that follows. For the following definition, see, for example, [Hay08, Definition 2.1]:

Definition 3.1. For an indeterminate t and integers $n \geq i \geq 0$, we define the following:

$$(i)_t := t^{i-1} + t^{i-2} + \cdots + t + 1, \tag{3.1}$$

$$(i)!_t := (i)_t(i-1)_t \cdots (1)_t, \quad \text{where } (0)!_t := 1, \tag{3.2}$$

$$\binom{n}{i}_t := \frac{(n)!_t}{(i)!_t(n-i)!_t}. \tag{3.3}$$

The expressions $\binom{n}{i}_t$ are called *t-binomial coefficients* and are polynomials in t over $\mathbb{Z}$ with similar properties to the regular binomial coefficients. We may evaluate the t-binomial coefficients at $t = q$, for some $q \in \mathbb{K}^*$, to give q-binomial coefficients.

The q-binomial coefficients defined above appear in the following *q-Leibniz rules*. These enable us to group together terms of the same degree in any q-skew Ore extension, $R[x;\sigma,\delta]$.

$$\delta^n(rs) = \sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i} \circ \delta^i(r)\delta^{n-i}(s) \text{ for all } r,s \in R \text{ and } n = 0,1,2,\ldots \tag{3.4}$$

$$x^n r = \sum_{i=0}^{n} \binom{n}{i}_q \sigma^{n-i} \circ \delta^i(r)x^{n-i} \text{ for all } r \in R \text{ and } n = 0,1,2,\ldots \tag{3.5}$$

In the case where q is a non-root of unity, Cauchon [Cau03a, Définition 2.1] defines a homomorphism for use in his algorithm which includes the multiplier $\frac{1}{(n)!_q}$ in a sum from $n = 1$ to ∞. If q were a root of unity with $q^\ell = 1$, then this map would be undefined for $n \geq \ell$ since $(\ell)!_q = 0$. In order to derive similar results to those of Cauchon for any $1 \neq q \in \mathbb{K}^*$, Haynal [Hay08, Section 2] defined a sequence of linear maps which satisfy properties similar to the q-Leibniz rules but do not involve q-binomial coefficients.

Definition 3.2. [Hay08, Definition 2.2] A *higher q-skew σ-derivation* (h.q-s.σ-d.) on a $\mathbb{K}$-algebra R is a sequence $d_0, d_1, d_2, \ldots$ of $\mathbb{K}$-linear operators on R such that the following three conditions are satisfied:

1. d_0 is the identity,
2. $d_n(rs) = \sum_{i=0}^{n} \sigma^{n-i} d_i(r) d_{n-i}(s)$ for all $r, s \in R$ and all n,
3. $d_n \circ \sigma = q^n \sigma \circ d_n$ for all n.

If a sequence of $\mathbb{K}$-linear maps satisfies the first two conditions then we refer to it as a *higher σ-derivation*. We often abbreviate the sequence $\{d_n\}_{n=0}^{\infty}$ as $\{d_n\}$ when it is obvious which subscript indexes the sequence. A h.q-s.σ-d. is *locally nilpotent* if, for all $r \in R$, there exists an integer $n_r \geq 0$ such that $d_n(r) = 0$ for all $n \geq n_r$, and $d_m(r) \neq 0$ for any $m < n_r$. In this case we call n_r the *d-nilpotence index of r*. A h.q-s.σ-d. is *iterative* if $d_n d_m = \binom{n+m}{m}_q d_{n+m}$ for all n, m. Note that this implies the d_n commute with each other. Finally, we say a q-skew σ-derivation δ *extends to a h.q-s.σ-d.* $\{d_n\}$ if such a sequence exists with $d_1 = \delta$.

We illustrate the definition given above with an example taken from [Hay08]:

Example 3.3. [Hay08, Section 2] The *quantised Weyl algebra* $A_1^{\varepsilon}(\mathbb{K})$ is defined as the $\mathbb{K}$-algebra generated by x and y subject to the relation $xy - \varepsilon yx = 1$, for some $\varepsilon \in \mathbb{K}^*$. It may be presented as an Ore extension $\mathbb{K}[y][x; \sigma', \delta']$ where $\sigma'(y) = \varepsilon y$, $\delta'(y) = 1$, and (σ', δ') is ε-skew. Taking $1 \neq \varepsilon$ to be a non-root of unity, we define the following $\mathbb{K}$-linear maps for all $n \geq 0$:

$$d'_n := \frac{(\delta')^n}{(n)!_\varepsilon}. \tag{3.6}$$

The sequence $\{d'_n\}$ then defines an iterative higher ε-skew σ'-derivation on $A_1^{\varepsilon}(\mathbb{K})$, as may be verified using the ε-Leibniz rules ((3.4) & (3.5), replacing q with ε), and by checking that

$$d'_n(y^i) = \begin{cases} \binom{i}{n}_\varepsilon y^{i-n} & n \leq i; \\ 0 & n > i. \end{cases}$$

In the example above, if $\varepsilon = q$, for some primitive ℓ^{th} root of unity $q \in \mathbb{K}^*$, then (3.6) is undefined for $n \geq \ell$. In [Hay08, Theorem 2.8] Haynal provides a result which gives a sufficient condition on a q-skew Ore extension $R[x;\sigma,\delta]$ for the σ-derivation, δ, to extend to a h.q-s.σ-d., $\{d_n\}$, for any $1 \neq q \in \mathbb{K}^*$. For algebras satisfying the required conditions, it is shown that $\{d_n\}$ comes from a higher ε-skew derivation $\{d'_n\}$ defined as in (3.6) on some $\mathbb{K}[\varepsilon^{\pm 1}]$-algebra satisfying certain properties. We will see in later chapters that the specific examples we are interested in for this thesis do indeed satisfy the conditions of the theorem, thus allowing us to define the higher q-skew derivations in this way.

In [Hay08, Section 3] these higher q-skew σ-derivations were used to define a *deleting derivations homomorphism* for any $1 \neq q \in \mathbb{K}^*$, which, when q is not a root of unity and under certain assumptions on $\{d_n\}$, returns the formula presented in [Cau03a, Section 2]. This algebra homomorphism is defined in the following proposition for use later on:

Proposition 3.4 ([Hay08], Proposition 3.4)**.** *For a $\mathbb{K}$-algebra R, let $A := R[x;\sigma,\delta]$, with $\sigma \in \mathrm{Aut}_{\mathbb{K}}(R)$ and δ an σ-derivation, and set $S := \{x^n \mid n \in \mathbb{N} \cup \{0\}\} \subset A$, the multiplicatively closed set. Suppose δ is locally nilpotent and extends to an iterative, locally nilpotent higher q-skew σ-derivation $\{d_n\}$ on R with $1 \neq q \in \mathbb{K}^*$. Then there is a unique injective algebra homomorphism,*

$$\begin{aligned} f : R[y;\sigma] &\longrightarrow AS^{-1} \\ y &\longmapsto x \\ r &\longmapsto \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} (q-1)^{-n} d_n \circ \sigma^{-n}(r) x^{-n} \end{aligned}$$

which we call the deleting derivation homomorphism. *The image of f is the subalgebra of AS^{-1} generated by x and $f(R)$, and is isomorphic (as an algebra) to $R[y;\sigma]$.*

Properties of the algebras which we consider in this thesis are listed in the hypothesis below for us to refer back to throughout this work. Note that these algebras are always noetherian domains, by Theorem 2.23.

Hypothesis 1.

H1.1: $A = \mathbb{K}[X_1][X_2;\sigma_2,\delta_2]\dots[X_N;\sigma_N,\delta_N]$ *is an iterated Ore extension, where $\mathbb{K}$ is a field, σ_i are $\mathbb{K}$-algebra automorphisms, and δ_i are σ_i-derivations.*

H1.2: $\sigma_i(X_l) = \lambda_{i,l} X_l$ *for all $l < i$ and $2 \leq i \leq N$, where $\lambda_{i,l} \in \mathbb{K}^*$.*

H1.3: $\Lambda := (\lambda_{i,j}) \in M_N(\mathbb{K}^*)$ *is a multiplicatively antisymmetric matrix. That is, $\lambda_{i,l} = \lambda_{l,i}^{-1}$ for all $1 \leq i,l \leq N$.*

H1.4: *For* $2 \le i \le N$ *there exists some* $1 \neq q_i \in \mathbb{K}^*$ *such that* $\delta_i \circ \sigma_i = q_i \sigma_i \circ \delta_i$, *i.e.* (σ_i, δ_i) *is* q_i*-skew.*

H1.5: $A_j := \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \ldots [X_j; \sigma_j, \delta_j]$, *so that* $A_0 := \mathbb{K}$ *and* $A_N := A$.

H1.6: *For all* $2 \le i \le N$ *each* δ_i *extends to a locally nilpotent, iterative h.*q_i*-s.*σ_i*-d.,* $\{d_{i,n}\}_{n=0}^{\infty}$, *on* A_{i-1}, *and* $\sigma_l \circ d_{i,n} = \lambda_{l,i}^n d_{i,n} \circ \sigma_l$ *on* A_{i-1} *for all* $n \ge 0$ *and* $i+1 \le l \le N$.

The following result is an inductive corollary to [Hay08, Lemma 4.1] which allows the reordering of extensions of A. This makes it possible to set up an algorithm to apply the deleting derivation homomorphism iteratively to successive subalgebras of $\mathrm{Frac}(A)$ by reordering the extensions of the subalgebra at each step so that those containing derivations come last.

Lemma 3.5. *Let*

$$A = A_{j-1}[X_j; \sigma_j, \delta_j][X_{j+1}; \sigma_{j+1}] \ldots [X_N; \sigma_N],$$
$$\hat{A} = A_{j-1}[X_j; \sigma_j, \delta_j][X_{j+1}^{\pm 1}; \sigma_{j+1}] \ldots [X_N^{\pm 1}; \sigma_N],$$

where the automorphisms and derivations satisfy properties H1.1-H1.5 and $\delta_j \neq 0$.

(I) Then

$$A = A_{j-1}[X_{j+1}; \sigma_{j+1}^*] \ldots [X_N; \sigma_N^*][X_j; \sigma_j', \delta_j'],$$
$$\hat{A} = A_{j-1}[X_{j+1}^{\pm 1}; \sigma_{j+1}^*] \ldots [X_N^{\pm 1}; \sigma_N^*][X_j; \sigma_j', \delta_j'],$$

where

(i) $\sigma_i^*|_{A_{j-1}} = \sigma_i|_{A_{j-1}}$ *for all* $j+1 \le i \le N$ *and* $\sigma_i^*(X_l) = \sigma_i(X_l) = \lambda_{i,l} X_l$ *for all* $j+1 \le l < i$;

(ii) $\sigma_j'|_{A_{j-1}} = \sigma_j$ *and* $\delta_j'|_{A_{j-1}} = \delta_j$;

(iii) $\sigma_j'(X_l) = \lambda_{j,l} X_l = \lambda_{l,j}^{-1} X_l$ *and* $\delta_j'(X_l) = 0$ *for all* $j+1 \le l \le N$.

(II) (σ_j', δ_j') *is* q_j*-skew.*

(III) Suppose that property H1.6 is also satisfied. Then δ_j' *extends to a locally nilpotent, iterative h.*q_j*-s.*σ_j'*-d.,* $\{d_{j,n}'\}_{n=0}^{\infty}$, *on* $A_{j-1}\langle X_{j+1}^{\pm 1}, X_{j+2}^{\pm 1}, \ldots, X_N^{\pm 1}\rangle$, *where the* $d_{j,n}'$ *coincide with the* $d_{j,n}$ *on* A_{j-1} *and, for all* $j+1 \le l \le N$ *and* $n \ge 1$, $d_{j,n}'(X_l) = 0$. *Moreover,* $\{d_{j,n}'\}_{n=0}^{\infty}$ *restricts to a h.*q_j*-s.*σ_j'*-d. on* $A_{j-1}\langle X_{j+1}, X_{j+2}, \ldots, X_N\rangle$ *which is also locally nilpotent and iterative.*

Proof. Fix $N \geq 3$. We prove the result by decreasing induction on $2 \leq j < N$, i.e. the index $2 \leq j < N$ such that $\delta_j \neq 0$ and $\delta_i = 0$ for all $j < i \leq N$. The initial case, where $j = N-1$ and $A = A_{N-2}[X_{N-1};\sigma_{N-1},\delta_{N-1}][X_N;\sigma_N]$, is proved in [Hay08, Lemma 4.1].

Suppose that each part of the lemma holds for some $3 \leq j+1 < N$ and consider the following algebras:

$$\begin{aligned} A &= A_{j-1}[X_j;\sigma_j,\delta_j][X_{j+1};\sigma_{j+1}]\dots[X_N;\sigma_N], \\ \hat{A} &= A_{j-1}[X_j;\sigma_j,\delta_j][X_{j+1}^{\pm1};\sigma_{j+1}]\dots[X_N^{\pm1};\sigma_N], \end{aligned}$$

where the automorphisms and derivations satisfy properties H1.1-H1.5 and $\delta_j \neq 0$.

Applying [Hay08, Lemma 4.1] to the subalgebras $A_{j-1}[X_j;\sigma_j,\delta_j][X_{j+1};\sigma_{j+1}] \subseteq A$ and $A_{j-1}[X_j;\sigma_j,\delta_j][X_{j+1}^{\pm1};\sigma_{j+1}] \subseteq \hat{A}$ we obtain

$$A_{j-1}[X_j;\sigma_j,\delta_j][X_{j+1};\sigma_{j+1}] = A_{j-1}[X_{j+1};\sigma_{j+1}^*][X_j;\bar{\sigma}_j,\bar{\delta}_j], \tag{3.7}$$
$$A_{j-1}[X_j;\sigma_j,\delta_j][X_{j+1}^{\pm1};\sigma_{j+1}] = A_{j-1}[X_{j+1}^{\pm1};\sigma_{j+1}^*][X_j;\bar{\sigma}_j,\bar{\delta}_j], \tag{3.8}$$

where

(a) $(\bar{\sigma}_j,\bar{\delta}_j)$ is q_j-skew;

(b) $\sigma_{j+1}^* = \sigma_{j+1}|_{A_{j-1}}$;

(c) $\bar{\sigma}_j|_{A_{j-1}} = \sigma_j$ and $\bar{\delta}_j|_{A_{j-1}} = \delta_j$;

(d) $\bar{\sigma}_j(X_{j+1}) = \lambda_{j+1,j}^{-1}X_{j+1} = \lambda_{j,j+1}X_{j+1}$ and $\bar{\delta}_j(X_{j+1}) = 0$.

Extending both sides of (3.7) by the Ore extensions $[X_{j+2};\sigma_{j+2}]\cdots[X_N;\sigma_N]$, and doing the same for (3.8) with the skew-Laurent extensions $[X_{j+2}^{\pm1};\sigma_{j+2}]\dots[X_N^{\pm1};\sigma_N]$, we obtain

$$\begin{aligned} A &= A_{j-1}[X_{j+1};\sigma_{j+1}^*][X_j;\bar{\sigma}_j,\bar{\delta}_j][X_{j+2};\sigma_{j+2}]\dots[X_N;\sigma_N] \\ &= B[X_j;\bar{\sigma}_j,\bar{\delta}_j][X_{j+2};\sigma_{j+2}]\dots[X_N;\sigma_N], \qquad (3.9) \\ \hat{A} &= A_{j-1}[X_{j+1}^{\pm1};\sigma_{j+1}^*][X_j;\bar{\sigma}_j,\bar{\delta}_j][X_{j+2}^{\pm1};\sigma_{j+2}]\dots[X_N^{\pm1};\sigma_N] \\ &= \hat{B}[X_j;\bar{\sigma}_j,\bar{\delta}_j][X_{j+2}^{\pm1};\sigma_{j+2}]\dots[X_N^{\pm1};\sigma_N], \qquad (3.10) \end{aligned}$$

where $B := A_{j-1}[X_{j+1};\sigma_{j+1}^*]$ and $\hat{B} := A_{j-1}[X_{j+1}^{\pm1};\sigma_{j+1}^*]$. The iterated Ore extension (3.9) and the iterated skew-Laurent extension (3.10) satisfy the conditions of the lemma and the inductive hypothesis, therefore we can apply inductive step to these algebras to obtain

$$\begin{aligned} A &= B[X_{j+2};\sigma_{j+2}^*]\dots[X_N;\sigma_N^*][X_j;\sigma_j',\delta_j'] \\ &= A_{j-1}[X_{j+1};\sigma_{j+1}^*][X_{j+2};\sigma_{j+2}^*]\dots[X_N;\sigma_N^*][X_j;\sigma_j',\delta_j'], \\ \hat{A} &= \hat{B}[X_{j+2}^{\pm 1};\sigma_{j+2}^*]\dots[X_N^{\pm 1};\sigma_N^*][X_j;\sigma_j',\delta_j'] \\ &= A_{j-1}[X_{j+1}^{\pm 1};\sigma_{j+1}^*][X_{j+2}^{\pm 1};\sigma_{j+2}^*]\dots[X_N^{\pm 1};\sigma_N^*][X_j;\sigma_j',\delta_j']. \end{aligned}$$

It is straightforward to verify that (I) and (II) hold for the algebras above by combining properties (a)-(d) with the properties coming from the inductive hypothesis. This proves (I) and (II) for all $2 \leq j < N$.

Suppose now that δ_j satisfies H1.6. From [Hay08, Lemma 4.1], $\bar{\delta}_j$ in (3.9) satisfies H1.6 so we can apply the inductive hypothesis to deduce that property (III) holds for δ_j'. This proves (III) for all $2 \leq j < N$.

□

3.2 The Deleting Derivations Algorithm

In the results that follow, we abuse notation slightly and use the same notation for maps defined on isomorphic algebras. We do this in the case where the action of the map on the generators of the algebra does not change, even though the algebras do. For example, suppose we have two isomorphic iterated Ore extensions

$$\mathbb{K}[X_1][X_2;\sigma_2,\delta_2] \cong \mathbb{K}[x_1][x_2;\bar{\sigma}_2,\bar{\delta}_2],$$

where $\sigma_2(X_1) = \lambda X_1$ and $\bar{\sigma}_2(x_1) = \lambda x_1$ for the same $\lambda \in \mathbb{K}^*$. Then, if we let $\iota : \mathbb{K}[X_1] \to \mathbb{K}[x_1]$ be the isomorphism sending X_1 to x_1, we see that $\sigma_2 = \iota^{-1} \circ \bar{\sigma}_2 \circ \iota$ and $\bar{\sigma}_2 = \iota \circ \sigma_2 \circ \iota^{-1}$. In this case, we simply denote $\bar{\sigma}_2$ by σ_2 (similarly for $\bar{\delta}_2$) and write $\mathbb{K}[x_1][x_2;\sigma_2,\delta_2]$. It is made explicit which algebra the map is defined on, if it is not already obvious from the context.

We may also abuse notation in a similar way for restrictions of maps to isomorphic subalgebras. Let R and S be algebras generated by $X_1,\dots,X_N$ and $x_1,\dots,x_N$ respectively and suppose $R \cong S$. Consider the following iterated Ore extensions where δ_1 and δ_2 are nonzero:

$$\begin{aligned} A &= R[Y_1;\sigma_1,\delta_1][Y_2;\sigma_2,\delta_2], \\ B &= S[y_1;\bar{\sigma}_1][y_2;\bar{\sigma}_2]. \end{aligned}$$

Then, if $\sigma_2(X_i) = \lambda_i X_i$ and $\bar{\sigma}_2(x_i) = \lambda_i x_i$ for all $i \in [\![1,N]\!]$ and some $\lambda_i \in \mathbb{K}^*$, we write $\bar{\sigma}_2|_S = \sigma_2|_R$.

We can now describe the deleting derivations algorithm for algebras A satisfying Hypothesis 1. It follows the method laid out in [Cau03a, Section 3.2] but utilises Haynal's homomorphism (Proposition 3.4).

For each $j \in [\![2,N+1]\!]$ we define a sequence $(X_1^{(j)},\dots,X_N^{(j)})$ of elements of $F := \mathrm{Frac}(A)$ and set $A^{(j)} := \mathbb{K}\langle X_1^{(j)},\dots,X_N^{(j)}\rangle$ to be the subalgebra of F generated by these elements. For $j = N+1$ we set $(X_1^{(N+1)},\dots,X_N^{(N+1)}) := (X_1,\dots,X_N)$ so that we we have that $A^{(N+1)} = A$. For ease of notation in the following proofs, for some fixed $j \in [\![2,N]\!]$ we set $(x_1,\dots,x_N) := (X_1^{(j+1)},\dots,X_N^{(j+1)})$ and assume that the algebra $A^{(j+1)}$ satisfies the following hypothesis:

Hypothesis 2.

H2.1: $A^{(j+1)} \cong \mathbb{K}[X_1]\dots[X_j;\sigma_j,\delta_j][X_{j+1};\sigma_{j+1}^{(j+1)}]\dots[X_N,\sigma_N^{(j+1)}]$ *by an isomorphism sending* $x_i \mapsto X_i$ *for all* $i \in \{1,\dots,N\}$.

H2.2: For each $i \in [\![j+1,N]\!]$, *the map* $\sigma_i^{(j+1)}$ *is an automorphism such that* $\sigma_i^{(j+1)}(X_l) = \lambda_{i,l}X_l$ *for all* $l \in [\![1,i-1]\!]$. *Furthermore, we have* $\sigma_i^{(j+1)} \circ d_{l,n} = \lambda_{i,l}^n d_{l,n} \circ \sigma_i^{(j+1)}$ *for all* $l \in [\![2,j]\!]$ *and* $n \geq 0$.

This allows us to write

$$A^{(j+1)} = \mathbb{K}\langle x_1,\dots,x_N\rangle \tag{3.11}$$

$$= \mathbb{K}[x_1]\dots[x_j;\sigma_j,\delta_j][x_{j+1};\sigma_{j+1}^{(j+1)}]\dots[x_N;\sigma_N^{(j+1)}] \tag{3.12}$$

where, for $i \in [\![2,j]\!]$, σ_i and δ_i satisfy Hypothesis 1 and δ_i extends to a locally nilpotent, iterative h.q_i-s.σ_i-d., $\{d_{i,n}\}_{n=0}^{\infty}$, on A_{i-1}. Note that the maps in (3.12) are not strictly the same as those defined in Hypothesis 2 because they are defined on different (albeit isomorphic) algebras. However, since the maps $\sigma_l^{(j+1)}$ act on the generators x_i in the same way as they act on the generators X_i, we perform the slight abuse of notation mentioned at the beginning of this section.

We define a new sequence of elements in F, $(y_1,\dots,y_N) := (X_1^{(j)},\dots,X_N^{(j)})$, in the following way:

$$y_l = \begin{cases} x_l & l \geq j; \\ \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j-1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n} & l < j. \end{cases} \tag{3.13}$$

Note that the sum stated above is finite, since the sequence $\{d_{j,n}\}_{n=0}^{\infty}$ is locally nilpotent: We see that $d_{j,n}$ commutes with σ_j^{-1} up to a factor of q_j, using property 3 of Definition 3.2:

$$\begin{aligned}
d_{j,n} \circ \sigma_j^{-1} &= (\sigma_j^{-1} \circ \sigma_j) \circ d_{j,n} \circ \sigma_j^{-1} \\
&= \sigma_j^{-1} \circ (\sigma_j \circ d_{j,n}) \circ \sigma_j^{-1} \\
&= \sigma_j^{-1} \circ (q_j^{-n} d_{j,n} \circ \sigma_j) \circ \sigma_j^{-1} \\
&= q_j^{-n} \sigma_j^{-1} \circ d_{j,n}.
\end{aligned}$$

Using the above observations, and denoting $n_l \in \mathbb{N}$ to be the d-nilpotence index of x_l, we can write the sum in (3.13) as:

$$\begin{aligned}
\sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n} = &\sum_{n=0}^{n_l - 1} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n} \\
&+ \sum_{n=n_l}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n},
\end{aligned}$$

where

$$\begin{aligned}
\sum_{n=n_l}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} d_{j,n} \circ \sigma_j^{-n}(x_l) x_j^{-n} &= \sum_{n=n_l}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} q_j^{-n^2} \sigma_j^{-n} \circ d_{j,n}(x_l) x_j^{-n} \\
&= \sum_{n=n_l}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} q_j^{-n^2} \sigma_j^{-n}(0) x_j^{-n} \\
&= 0.
\end{aligned}$$

We are therefore left with a finite sum from 0 to $n_l - 1$. With this we define $A^{(j)} := \mathbb{K}\langle y_1, \ldots, y_N \rangle$.

Theorem 3.6. *Let A be as in Hypothesis 1, with $A^{(j+1)}$ defined as above and satisfying Hypothesis 2. Then we have the following:*

(I) $A^{(j)} \cong \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}][X_j; \sigma_j^{(j)}] \cdots [X_N; \sigma_N^{(j)}]$ *by an isomorphism which sends y_l to X_l for all $1 \leq l \leq N$.*

(II) For all $j \leq i \leq N$, $\sigma_i^{(j)}$ are automorphisms satisfying

- *(i)* $\sigma_i^{(j)}(X_l) = \lambda_{i,l} X_i$ *for all $l \in [\![1, i-1]\!]$;*
- *(ii)* $\sigma_i^{(j)} \circ d_{l,n} = \lambda_{i,l}^n d_{l,n} \circ \sigma_i^{(j)}$ *for all $n \geq 0$ and all $l \in [\![2, j-1]\!]$.*

(III) Let $S_j := \{x_j^n \mid n \geq 0\} = \{y_j^n \mid n \geq 0\}$. This is a multiplicatively closed set of regular elements satisfying the two sided Ore-condition in $A^{(j+1)}$ and $A^{(j)}$ and, furthermore, $A^{(j)}S_j^{-1} = A^{(j+1)}S_j^{-1}$.

Proof. By Hypothesis 2, and the discussion thereafter, we can write $A^{(j+1)}$ as

$$A^{(j+1)} = \mathbb{K}[x_1]\dots[x_j;\sigma_j,\delta_j][x_{j+1};\sigma_{j+1}^{(j+1)}.]\dots[x_N;\sigma_N^{(j+1)}].$$

Defining

$$A_{j-1}^{(j+1)} := \mathbb{K}[x_1][x_2;\sigma_2,\delta_2]\dots[x_{j-1};\sigma_{j-1},\delta_{j-1}]$$

and applying Lemma 3.5 to $A^{(j+1)} = A_{j-1}^{(j+1)}[x_j;\sigma_j,\delta_j][x_{j+1};\sigma_{j+1}^{(j+1)}]\dots[x_N;\sigma_N^{(j+1)}]$ gives

$$A^{(j+1)} = A_{j-1}^{(j+1)}[x_{j+1};\sigma_{j+1}^{(j+1)*}]\dots[x_N;\sigma_N^{(j+1)*}][x_j;\sigma_j',\delta_j'], \tag{3.14}$$

where

(a) $\sigma_i^{(j+1)*}|_{A_{j-1}^{(j+1)}} = \sigma_i^{(j+1)}|_{A_{j-1}^{(j+1)}}$ for all $i \in [\![j+1,N]\!]$, and $\sigma_i^{(j+1)*}(x_l) = \sigma_i^{(j+1)}(x_l) = \lambda_{i,l}x_l$ for all $l \in [\![j+1,i-1]\!]$;

(b) $\sigma_j'|_{A_{j-1}^{(j+1)}} = \sigma_j$ and $\delta_j'|_{A_{j-1}^{(j+1)}} = \delta_j$;

(c) $\sigma_j'(x_l) = \lambda_{j,l}x_l = \lambda_{l,j}^{-1}x_l$ and $\delta_j'(x_l) = 0$ for all $l \in [\![j+1,N]\!]$.

In particular, δ_j' extends to a h.q_j-s.σ_j'-d., $\{d_{j,n}'\}_{n=0}^{\infty}$, on $A_{j-1}^{(j+1)}\langle x_{j+1}^{\pm 1},\dots,x_N^{\pm 1}\rangle$ where

$$d_{j,n}'|_{A_{j-1}^{(j+1)}} = d_{j,n} \quad (\forall\, n \geq 0), \tag{3.15}$$

$$d_{j,n}'(x_l) = 0 \qquad (\forall\, l \in [\![j+1,N]\!] \text{ and } n \geq 1). \tag{3.16}$$

Define

$$\widehat{A^{(j+1)}} := A_{j-1}^{(j+1)}[x_{j+1};\sigma_{j+1}^{(j+1)*}]\dots[x_N;\sigma_N^{(j+1)*}]$$

so that equation (3.14) becomes

$$A^{(j+1)} = \widehat{A^{(j+1)}}[x_j;\sigma_j',\delta_j'].$$

Applying [Hay08, Theorem 3.7] to $A^{(j+1)}$ yields the isomorphism,

$$\begin{aligned} f:\widehat{A^{(j+1)}}[x_j^{\pm1};\sigma_j'] &\longrightarrow \widehat{A^{(j+1)}}[x_j;\sigma_j',\delta_j']S_j^{-1} \\ \widehat{A^{(j+1)}} \ni a &\longmapsto f(a)=\sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}}(q_j-1)^{-n}d'_{j,n}\circ(\sigma_j')^{-n}(a)x_j^{-n} \\ x_j &\longmapsto x_j. \end{aligned}$$

Note that $f(x_l)=x_l$ for all $l\in[\![j+1,N]\!]$ since, by (3.16), $d'_{j,n}(x_l)=0$ for all $n\geq 1$. Therefore the only nonzero summands in $f(x_l)$ occur when $n=0$, in which case we get

$$\begin{aligned} f(x_l) &= \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}}(q_j-1)^{-n}d'_{j,n}\circ(\sigma_j')^{-n}(x_l)x_j^{-n} \\ &= q_j^{\frac{0(0+1)}{2}}(q_j-1)^{0}d'_{j,0}\circ(\sigma_j')^{0}(x_l)x_j^{0} \\ &= \mathrm{Id}(x_l) \\ &= x_l. \end{aligned}$$

If $l\in[\![1,j-1]\!]$, then x_l and $(\sigma_j')^{-n}(x_l)\in A_{j-1}^{(j+1)}$. Thus, by (3.15) and (b) above, we can replace σ_j' and $d'_{j,n}$ in $f(x_l)$ with σ_j and $d_{j,n}$ to obtain

$$f(x_l)=\sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}}(q_j-1)^{-n}d_{j,n}\circ\sigma_j^{-n}(x_l)x_j^{-n}=y_l,$$

as defined in (3.13). Therefore, for any $l\in[\![1,N]\!]$ we see that

$$f(x_l)=\begin{cases} x_l & l\geq j; \\ y_l & l<j, \end{cases}$$

hence the isomorphism f takes x_l to y_l for all $l\in[\![1,N]\!]$.

Using [Hay08, Theorem 3.7] we see that restricting f to $\widehat{A^{(j+1)}}[x_j;\sigma_j']$ gives the deleting derivation homomorphism as defined in Proposition 3.4. Therefore

$$\mathrm{Im}(f)\cong\widehat{A^{(j+1)}}[x_j;\sigma_j'], \tag{3.17}$$

where $\mathrm{Im}(f)=f(\widehat{A^{(j+1)}}[x_j;\sigma_j'])$ is the subalgebra of $\widehat{A^{(j+1)}}[x_j^{\pm1};\sigma_j',\delta_j']$ generated by x_j and $f(\widehat{A^{(j+1)}})$. Since $f(\widehat{A^{(j+1)}})$ is generated by $\mathbb{K}$ and y_l for all $l\neq j$, and since $x_j=y_j$, this simply tells us that

$$\mathrm{Im}(f) = \mathbb{K}\langle y_1, \dots, y_N \rangle = A^{(j)}.$$

Using (3.17) we see that

$$A^{(j)} = \mathrm{Im}(f) \cong \mathbb{K}[x_1] \dots [x_{j-1}; \sigma_{j-1}, \delta_{j-1}][x_{j+1}; \sigma_{j+1}^{(j+1)*}] \dots [x_N; \sigma_N^{(j+1)*}][x_j; \sigma_j'],$$

and therefore,

$$A^{(j)} = \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_{j+1}; \sigma_{j+1}^{(j+1)*}] \dots [y_N; \sigma_N^{(j+1)*}][y_j; \sigma_j']. \tag{3.18}$$

Finally we apply [Hay08, Proposition 3.6] to conclude that S_j is a multiplicatively closed set of regular elements in both $A^{(j+1)}$ and $A^{(j)}$, satisfying the two-sided Ore condition, and that

$$\begin{aligned} \mathrm{Im}(f) S_j^{-1} &= \widehat{A^{(j+1)}}[x_j; \sigma_j', \delta_j'] S_j^{-1}, \\ A^{(j)} S_j^{-1} &= A^{(j+1)} S_j^{-1}. \end{aligned}$$

Thus assertion (III) is proved.

The property $y_l y_j = \lambda_{l,j} y_j y_l$, along with the fact that $\lambda_{l,j} = \lambda_{j,l}^{-1}$, allows us to rearrange (3.18) to obtain

$$A^{(j)} = \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \sigma_j^{(j)}][y_{j+1}; \sigma_{j+1}^{(j)}] \dots [y_N; \sigma_N^{(j)}]. \tag{3.19}$$

Defining

$$A_{j-1}^{(j)} := \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}]$$

we see that $A_{j-1}^{(j)} \cong A_{j-1}$ and there is an isomorphism

$$\begin{aligned} A^{(j)} &= \mathbb{K}[y_1] \dots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \sigma_j^{(j)}][y_{j+1}; \sigma_{j+1}^{(j)}] \dots [y_N; \sigma_N^{(j)}] \\ &\cong \mathbb{K}[X_1] \dots [X_{j-1}; \sigma_{j-1}, \delta_{j-1}][X_j; \sigma_j^{(j)}][X_{j+1}; \sigma_{j+1}^{(j)}] \dots [X_N; \sigma_N^{(j)}] \end{aligned} \tag{3.20}$$

sending y_l to X_l for all $l \in [\![1, N]\!]$, where the maps (as defined on suitable subalgebras of $A^{(j)}$) are as follows:

(a′) $\sigma_j^{(j)} = \sigma_j'|_{A_{j-1}^{(j)}} = \sigma_j$;

(b′) $\sigma_i^{(j)}|_{A_{j-1}^{(j)}} = \sigma_i^{(j+1)*}|_{A_{j-1}^{(j)}} = \sigma_i^{(j+1)}|_{A_{j-1}^{(j)}}$ for all $i \in [\![j+1, N]\!]$;

(c′) $\sigma_i^{(j)}(y_l) = \lambda_{i,l} y_l$ for all $i \in [\![j+1, N]\!]$ and $l \in [\![1, i-1]\!]$.

Using the isomorphism in (3.20) along with the observations (a′)-(c′) above we can prove assertion (II) for all $i \in [\![j,N]\!]$: Observation (a′) proves both parts of assertion (II) when $i = j$, since σ_j satisfies assertion (II) by definition (see H.1.2 and H.1.6). When $i \in [\![j+1,N]\!]$, observation (b′) proves (II)(ii), since $\sigma_i^{(j+1)}$ satisfies H.2.2, and observation (c′) proves (II)(i). □

If A is an algebra satisfying Hypothesis 1 then Hypothesis 2 is satisfied for $j = N+1$. Theorem 3.6 then tells us that Hypothesis 2 is also satisfied for all $j \in [\![2,N+1]\!]$. We deduce the following:

Corollary 3.7. *The algebra $A' := A^{(2)}$ is a quantum affine space. More precisely, by setting $T_i := X_i^{(2)}$ for all $i \in [\![1,N]\!]$ and $\Lambda := (\lambda_{i,j}) \in M_N(\mathbb{K})$ to be the multiplicatively antisymmetric matrix, we obtain:*

$$A' = \mathbb{K}_\Lambda[T_1,\ldots,T_N].$$

Remark 3.8. For all $j \in [\![1,N]\!]$, we say that $A^{(j+1)}$ is the algebra obtained from A after $N-j$ steps of the deleting derivations algorithm.

3.3 Ring of Fractions

In order to be able to track the completely prime ideals along the deleting derivations algorithm we need the following two results regarding the total division ring of fractions of the algebras $A^{(j)}$ at each step of the algorithm. These results were discovered in [Cau03a, Subsection 3.3] in the generic setting and can be applied directly to our setting given the results proved above. They are included here, rewritten in the notation used thus far in this work, for completeness.

Let Σ be the multiplicatively closed set in A' generated by the elements $T_1,\ldots,T_N$. For $j \in [\![2,N]\!]$, define the sets Σ_j as:

$$\Sigma_2 := \Sigma,$$
$$\Sigma_{j+1} = A^{(j+1)} \cap \Sigma_j \quad \text{for } 2 \leq j \leq N.$$

Proposition 3.9. *For all $j \in [\![2,N+1]\!]$ the following are true:*

(i) Σ_j is a multiplicatively closed set of regular elements in $A^{(j)}$ containing $X_{j-1}^{(j)},\ldots,X_N^{(j)}$;

(ii) Σ_j satisfies the two-sided Ore condition in $A^{(j)}$;

(iii) The algebras $A^{(j)}\Sigma_j^{-1} \subset \mathrm{Frac}(A)$ are all equal.

Proof. We proceed by induction on j. When $j = 2$, assertions (i) and (ii) are trivially true from the definition of Σ and the fact that the generators (and monomials in these generators) of a quantum affine space are regular and normal.

Let $j \in [\![2,N]\!]$ and suppose assertions (i) and (ii) hold for j. We will show that assertions (i) and (ii) also hold for $j+1$ and that $A^{(j)}\Sigma_j^{-1} = A^{(j+1)}\Sigma_{j+1}^{-1}$.

Recall the notation from the previous section,

$$(x_1,\dots,x_N) := (X_1^{(j+1)},\dots,X_N^{(j+1)}) \quad \text{and} \quad (y_1,\dots,y_N) := (X_1^{(j)},\dots,X_N^{(j)}).$$

where $x_i = y_i$ for all $i \geq j$. By the induction hypothesis, Σ_j is a multiplicatively closed set of regular elements in $A^{(j)}$ containing $y_{j-1},\dots,y_N$. Therefore $\Sigma_{j+1} = A^{(j)} \cap \Sigma_j$ is a multiplicatively closed set of regular elements of $A^{(j+1)}$ containing $y_j = x_j,\ \dots,\ y_N = x_N$. This proves assertion (i).

Recall the set $S_j = \{x_j^n \mid n \in \mathbb{N}\} = \{y_j^n \mid n \in \mathbb{N}\} \subset \Sigma_j \cap \Sigma_{j+1}$ and use Theorem 3.6(III) to obtain the inclusions

$$A^{(j+1)} \subset A^{(j+1)}S_j^{-1} = A^{(j)}S_j^{-1} \subset A^{(j)}\Sigma_j^{-1}. \tag{3.21}$$

Since $\Sigma_{j+1} \subset \Sigma_j$ then Σ_{j+1} must be invertible in $A^{(j)}\Sigma_j^{-1}$. We use this to show that an element $a \in A^{(j)}\Sigma_j^{-1}$ can be rewritten as an element in $A^{(j+1)}\Sigma_{j+1}^{-1}$ in the following way: First, write $a = yu^{-1}$, where $y \in A^{(j)}$ and $u \in \Sigma_j$. Since

$$\Sigma_j \subset A^{(j)} \subset A^{(j)}S_j^{-1} = A^{(j+1)}S_j^{-1},$$

we can write $u = vs_1^{-1}$ and $y = xs_2^{-1}$, with $v,x \in A^{(j+1)}$ and $s_1,s_2 \in S_j$. Then a becomes

$$a = xs_2^{-1}(vs_1^{-1})^{-1} = xs_2^{-1}s_1v^{-1} = xs_1s_2^{-1}v^{-1} = xs_1(vs_2)^{-1}.$$

Observe that $vs_2 = (vs_1^{-1})s_1s_2 = us_1s_2 \in \Sigma_j$ since $u \in \Sigma_j$ and $s_1,s_2 \in S_j \subset \Sigma_j$. Also, $v \in A^{(j+1)}$ and $s_2 \in S_j \subset A^{(j+1)}$, therefore $vs_2 \in A^{(j+1)} \cap \Sigma_j = \Sigma_{j+1}$. Similarly, $xs_1 \in A^{(j+1)}$ so we can write

$$a = bc^{-1} \in A^{(j+1)}\Sigma_{j+1}^{-1},$$

where $b = xs_1 \in A^{(j+1)}$ and $c = vs_2 \in \Sigma_{j+1}$.

From the inductive hypothesis we know that $A^{(j)}\Sigma_j^{-1} = \Sigma_j^{-1}A^{(j)}$, so if $a \in A^{(j)}\Sigma_j^{-1}$ then it must also be true that $a \in \Sigma_j^{-1}A^{(j)}$. We also know from Theorem 3.6(III) that S_j is an Ore set in $A^{(j)}$ and $A^{(j+1)}$, thus $A^{(j)}S_j^{-1} = S_j^{-1}A^{(j)}$ and $A^{(j+1)}S_j^{-1} = S_j^{-1}A^{(j+1)}$. Using these results we can follow a similar method to before to rewrite $a \in A^{(j)}\Sigma_j^{-1} = \Sigma_j^{-1}A^{(j)}$ as

$$a = c'^{-1}b' \in \Sigma_{j+1}^{-1}A^{(j+1)},$$

with $c' \in \Sigma_{j+1}$ and $b' \in A^{(j+1)}$.

If we can prove that Σ_{j+1} is a two-sided Ore set in $A^{(j+1)}$ then the working above implies that $A^{(j)}\Sigma_j^{-1} \subseteq A^{(j+1)}\Sigma_{j+1}^{-1}$. Furthermore, $A^{(j+1)} \subset A^{(j)}\Sigma_j^{-1}$ and $\Sigma_{j+1} \subset \Sigma_j$ so we also have $A^{(j+1)}\Sigma_{j+1}^{-1} \subseteq A^{(j)}\Sigma_j^{-1}$. Hence assertion (iii) is true if we can prove that assertion (ii) holds.

From the inclusion $A^{(j+1)}\Sigma_{j+1}^{-1} \subseteq A^{(j)}\Sigma_j^{-1}$ we can write any $a = bc^{-1} \in A^{(j+1)}\Sigma_{j+1}^{-1}$ as $a \in A^{(j)}\Sigma_j^{-1}$ and, applying the above working, we see that there exist $c' \in \Sigma_{j+1}$ and $b' \in A^{(j+1)}$ such that $a = c'^{-1}b' \in \Sigma_{j+1}^{-1}A^{(j+1)}$. This verifies the two-sided Ore condition on $\Sigma_{j+1} \subset A^{(j+1)}$ necessary for proving assertion (ii) and, by the comment earlier, assertion (iii). □

From the above proposition it is clear that:

Theorem 3.10. *(i) There exists a multiplicatively closed set of regular elements $S \subseteq A$ such that $AS^{-1} = A'\Sigma^{-1} = \mathbb{K}_\Lambda[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$.*

(ii) $\mathrm{Frac}(A^{(j)}) = \mathrm{Frac}(A)$ *for all $j \in [\![2, N+1]\!]$ and, in particular,* $\mathrm{Frac}(A) = \mathrm{Frac}(A')$.

(iii) A is a PI algebra if and only if $\lambda_{i,j}$ are roots of unity for all $1 \leq i, j \leq N$ and, in this case, PI-deg(A) =PI-deg(A').

Proof. Take $S = \Sigma_{N+1}$ and apply Proposition 3.9(iii) to show that $AS^{-1} = A'\Sigma^{-1}$. The second statement follows from all the algebras $A^{(j)}$ having a common localisation (again, by Proposition 3.9(iii)). Finally, [Hay08, Corollary 4.7] shows that A is PI if and only if $\lambda_{i,j}$ are roots of unity for all $1 \leq i, j \leq N$ and, from Theorem 2.30, we see that under the same conditions A' is also PI. Therefore, since we have equality of their total rings of fractions, we conclude that we also have equality of their PI degrees. □

Chapter 4

Deleting Derivations Algorithm on Completely Prime Quotients

The aim of this chapter is to extend Theorem 3.10 to the quotient algebras A/P, for completely prime ideals $P \lhd A$. We first set up the canonical embedding, ψ, from the completely prime spectrum of A to the completely prime spectrum of A', which allows us to track properties of P through the deleting derivations algorithm. Using this, we prove that $\mathrm{Frac}(A/P) = \mathrm{Frac}(A'/\psi(P))$ and therefore that A/P and $A'/\psi(P)$ have the same PI degree provided they are both PI algebras (for example, if A is a PI algebra). Moreover, in this PI setting, when $\psi(P)$ is generated by a subset of the generators of A' then $A'/\psi(P)$ becomes a quantum affine space to which we can apply Theorem 2.30 and obtain its PI degree. Therefore, out of the completely prime ideals $Q \lhd A'$ we are interested in finding out which are in the image of the canonical embedding and, additionally, which of these are also generated by a subset of the generators of A'. To this end, we define criteria for $Q \lhd A'$ to lie in $\mathrm{Im}(\psi)$ and, in Section 4.5, we specialise to quantum matrices, where we give a combinatorial description of those $Q \in \mathrm{Im}(\psi)$ which are generated by a subset of the generators of A'.

Many of the results of this chapter are analogues to those found in the generic setting [Cau03a, Sections 4 and 5] and their proofs follow in almost the same way, thanks to the results of the previous chapter.

Let A satisfy Hypothesis 1 and retain the notation of Section 3.2. In particular, for some fixed $j \in [\![2, N]\!]$, we set $(x_1, \dots, x_N) := (X_1^{(j+1)}, \dots, X_N^{(j+1)})$ and $(y_1, \dots, y_N) := (X_1^{(j)}, \dots, X_N^{(j)})$, and define subalgebras of $F := \mathrm{Frac}(A)$ generated by these elements as $A^{(j+1)} := \mathbb{K}\langle x_1, \dots, x_N \rangle$ and $A^{(j)} := \mathbb{K}\langle y_1, \dots, y_N \rangle$. Recall, too, that $y_j = x_j$ and we have $A^{(j)} S_j^{-1} = A^{(j+1)} S_j^{-1}$ for the Ore set $S_j := \{x_j^n \mid n \in \mathbb{N}\} = \{y_j^n \mid n \in \mathbb{N}\}$.

For any ring R, let $\mathrm{Spec}(R)$ denote the set of prime ideals in R and $\mathrm{C.Spec}(R) \subseteq \mathrm{Spec}(R)$ denote the set of completely prime ideals in R.

4.1 The Canonical Embedding $\psi : \mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$

4.1.1 The Injection $\psi_j : \mathrm{C.Spec}(A^{(j+1)}) \to \mathrm{C.Spec}(A^{(j)})$

Recall the standard result that if X is a right Ore set in a right noetherian ring R, then extension and contraction provide inverse bijections between the set of prime ideals of RX^{-1} and the set of those prime ideals of R that are disjoint from X (see [GW04, Proposition 10.7 and Theorem 10.20]). That is: Let $I \subseteq R$ and $J \subseteq RX^{-1}$ be (two-sided) ideals and define the extension and contraction of these ideals (I^{e} and J^{c} respectively) to be the following:

$$I^{\mathrm{e}} := IX^{-1} = \{ix^{-1} \in RX^{-1} \mid i \in I, x \in X\} \subseteq RX^{-1},$$
$$J^{\mathrm{c}} := J \cap R = \{a \in R \mid a1^{-1} \in J\} \subseteq R.$$

Define a subset of $\mathrm{Spec}(R)$ to be $\mathscr{P}(R) := \{P' \in \mathrm{Spec}(R) \mid P' \cap X = \emptyset\}$. Then [GW04, Theorem 10.20] states that there is a bijection

$$\phi : \mathscr{P}(R) \longrightarrow \mathrm{Spec}(RX^{-1}),$$

with inverse, ϕ^{-1}, such that $\phi(P) = P^{\mathrm{e}}$ and $\phi^{-1}(Q) = Q^{\mathrm{c}}$ for all $P \in \mathscr{P}(R)$ and $Q \in \mathrm{Spec}(RX^{-1})$.

Lemma 4.1. *The inverse bijections ϕ and ϕ^{-1} defined above send completely prime ideals to completely prime ideals.*

Proof. Take $P \in \mathscr{P}(R)$ to be completely prime and consider $\phi(P) = P^{\mathrm{e}} = PX^{-1}$. Let $a, b \in RX^{-1}$ and assume $ab \in PX^{-1}$. Since $a, b \in RX^{-1}$ and $RX^{-1} = X^{-1}R$ then there exist $x, y \in X$ such that $xa, by \in R$ and, as PX^{-1} is a two-sided ideal, $x(ab)y \in PX^{-1}$. From this we see that $x(ab)y = (xa)(by) \in R$ and therefore that $(xa)(by) \in PX^{-1} \cap R = P$. Since P is completely prime then either $xa \in P$ or $by \in P$, whence we conclude that $a \in PX^{-1}$ or $b \in PX^{-1}$.

Now take $Q \in \mathrm{Spec}(RX^{-1})$ to be completely prime and consider $\phi^{-1}(Q) = Q^{\mathrm{c}}$. Let $a, b \in R$ and assume $ab \in Q^{\mathrm{c}}$. Note how this implies $ab \notin X$ so at least one of a and b is not in X. Certainly the natural map $\pi : R \longrightarrow RX^{-1}$ taking $r \in R$ to $r1^{-1}$ would take $ab \in Q^{\mathrm{c}}$ to $(ab)1^{-1} \in Q$. By the fact that Q is completely prime this implies $a1^{-1} \in Q$ or $b1^{-1} \in Q$ (since we cannot have $1^{-1} \in Q$ because this is an invertible element, which would mean that $Q = RX^{-1}$). But then $a \in Q^{\mathrm{c}}$ or $b \in Q^{\mathrm{c}}$ by the definition of Q^{c}. Therefore $Q^{\mathrm{c}} \subseteq R$ is completely prime for all $Q \subseteq RX^{-1}$ completely prime. $\square$

Lemma 4.2. *Let $P \in \mathrm{C.Spec}(A^{(j+1)})$. Then $x_j \notin P \iff P \cap S_j = \emptyset$.*

Proof. " $\Leftarrow$ " is immediate. To see " $\Rightarrow$ ", suppose $x_j \notin P$ and assume $P \cap S_j \neq \emptyset$. Then there exists some $n \in \mathbb{N}$ such that $x_j^n \in P \cap S_j$. Since P is completely prime, this implies $x_j \in P$, which gives a contradiction. Therefore our assumption that $P \cap S_j \neq \emptyset$ is false. $\square$

The observations above justify the following notation:

$$\begin{aligned}
\mathscr{P}_j^0(A^{(j)}) &:= \{P \in \mathrm{C.Spec}(A^{(j)}) \mid y_j \notin P\},\\
\mathscr{P}_j^1(A^{(j)}) &:= \{P \in \mathrm{C.Spec}(A^{(j)}) \mid y_j \in P\},\\
\mathscr{P}_j^0(A^{(j+1)}) &:= \{P \in \mathrm{C.Spec}(A^{(j+1)}) \mid x_j \notin P\},\\
\mathscr{P}_j^1(A^{(j+1)}) &:= \{P \in \mathrm{C.Spec}(A^{(j+1)}) \mid x_j \in P\}.
\end{aligned}$$

These sets allow us to define analogous maps to those defined in [Cau03a, Section 4.3], as we do in the following lemma. By the term *increasing* we mean that the homeomorphism is order-preserving with respect to inclusion of ideals. By *bi-increasing* we mean that both the homeomorphism and its inverse are increasing homeomorphisms.

Lemma 4.3. *There is an increasing homeomorphism $\psi_j^0 : \mathscr{P}_j^0(A^{(j+1)}) \to \mathscr{P}_j^0(A^{(j)})$ given by $\psi_j^0(P) := PS_j^{-1} \cap A^{(j)}$. Its inverse is defined by $(\psi_j^0)^{-1}(Q) := QS_j^{-1} \cap A^{(j+1)}$ and is also increasing. Hence ψ_j^0 is bi-increasing.*

Proof. We first endow the set $\mathrm{Spec}(A^{(j)}S_j^{-1})$ with the Zariski topology in the standard way and use this to induce the Zariski topology on the subset $\mathrm{C.Spec}(A^{(j)}S_j^{-1}) \subseteq \mathrm{Spec}(A^{(j)}S_j^{-1})$, defining the closed sets to be $V(J) := \{Q \in \mathrm{C.Spec}(A^{(j)}S_j^{-1}) \mid J \subseteq Q\}$ for all $J \lhd A^{(j)}S_j^{-1}$.

Using the inverse bijections in Lemma 4.1, which we label as ϕ_1 and ϕ_1^{-1} respectively, we endow $\mathscr{P}_j^0(A^{(j)})$ with the Zariski topology, taking the closed sets to be

$$V(I) := \{P \in \mathscr{P}_j^0(A^{(j)}) \mid I \subseteq P\},$$

for all $I \lhd A^{(j)}$. The bijective correspondence between $\mathscr{P}_j^0(A^{(j)})$ and $\mathrm{C.Spec}(A^{(j)}S_j^{-1})$ ensures a homeomorphism between these two topological spaces which is bi-increasing

In the same way as above, using inverse bijections ϕ_2 and ϕ_2^{-1} between $\mathscr{P}_j^0(A^{(j+1)})$ and $\mathrm{C.Spec}(A^{(j+1)}S_j^{-1})$, we may also induce the Zariski topology on $\mathscr{P}_j^0(A^{(j+1)})$ and obtain a bi-increasing homeomorphism between $\mathscr{P}_j^0(A^{(j+1)})$ and $\mathrm{C.Spec}(A^{(j+1)}S_j^{-1})$. Since $A^{(j)}S_j^{-1} = A^{(j+1)}S_j^{-1}$ then $\mathrm{C.Spec}(A^{(j)}S_j^{-1}) = \mathrm{C.Spec}(A^{(j+1)}S_j^{-1})$ as topological spaces. Therefore $\psi_j^0 := \phi_1^{-1} \circ \phi_2$ and $(\psi_j^0)^{-1} := \phi_2^{-1} \circ \phi_1$ are inverse bijections and give rise to a bi-increasing homeomorphism between $\mathscr{P}_j^0(A^{(j+1)})$ and $\mathscr{P}_j^0(A^{(j)})$. $\square$

Now we turn our attention to the sets $\mathscr{P}_j^1(A^{(j)})$ and $\mathscr{P}_j^1(A^{(j+1)})$.

Lemma 4.4. *There is a surjective algebra homomorphism $g_j : A^{(j)} \to A^{(j+1)}/\langle x_j\rangle$ which takes $y_i \mapsto x_i + \langle x_j\rangle$, for all $1 \le i \le N$, where $x_i + \langle x_j\rangle$ is the canonical image of x_i in $A^{(j+1)}/\langle x_j\rangle$.*

Proof. By Theorem 3.6 we have

$$A^{(j+1)} := \mathbb{K}\langle x_1,\dots,x_N\rangle \cong \mathbb{K}[X_1]\cdots[X_j;\sigma_j,\delta_j][X_{j+1};\sigma_{j+1}^{(j+1)}]\cdots[X_N;\sigma_N^{(j+1)}],$$
$$A^{(j)} := \mathbb{K}\langle y_1,\dots,y_N\rangle \cong \mathbb{K}[X_1]\cdots[X_{j-1};\sigma_{j-1},\delta_{j-1}][X_j;\sigma_j^{(j)}]\cdots[X_N;\sigma_N^{(j)}].$$

Restricting these algebras to $R := \mathbb{K}\langle x_1,\dots,x_{j-1}\rangle$ and $S := \mathbb{K}\langle y_1,\dots,y_{j-1}\rangle$ we see that there is an isomorphism $S \to R$ sending $y_i \mapsto x_i$ for all $i \in [\![1, j-1]\!]$. Since $R \subseteq A^{(j+1)}$ we can compose this isomorphism with the natural surjection $A^{(j+1)} \to A^{(j+1)}/\langle x_j\rangle$ to obtain the surjective algebra homomorphism $f : S \to A^{(j+1)}/\langle x_j\rangle$, which sends $y_i \mapsto x_i + \langle x_j\rangle$ for all $i \in [\![1, j-1]\!]$. Using the commutation rules for $A^{(j+1)}$, as stated in Hypothesis 2, we see that, for $i \in [\![j+1, N]\!]$ and $l \in [\![1, i-1]\!]$,

$$x_l x_i = \lambda_{l,i} x_i x_l \quad \Longrightarrow \quad (x_l + \langle x_j\rangle)(x_i + \langle x_j\rangle) = \lambda_{l,i}(x_i + \langle x_j\rangle)(x_l + \langle x_j\rangle),$$

and, since $x_j + \langle x_j\rangle = 0$, then $(x_j + \langle x_j\rangle)(x_i + \langle x_j\rangle) = \lambda_{j,i}(x_i + \langle x_j\rangle)(x_j + \langle x_j\rangle) = 0$ for all $i \in [\![1, j-1]\!]$. The relations on $x_i + \langle x_j\rangle \in A^{(j+1)}/\langle x_j\rangle$ therefore agree with those on $y_i \in S[y_j;\sigma_j^{(j)}]\cdots[y_N;\sigma_N^{(j)}]$, for all $i \in [\![1, N]\!]$. This, along with the surjective algebra homomorphism $S \to A^{(j+1)}/\langle x_j\rangle$ which we defined above, allows us to apply the universal property of Ore extensions (Proposition 2.21) to conclude that there exists a homomorphism,

$$\begin{aligned} g_j : A^{(j)} = S[y_j;\sigma_j^{(j)}]\cdots[y_N;\sigma_N^{(j)}] &\longrightarrow A^{(j+1)}/\langle x_j\rangle \\ y_i &\longmapsto x_i + \langle x_j\rangle, \end{aligned}$$

and that this homomorphism is surjective. $\square$

Lemma 4.5. *There is an increasing injective map $\psi_j^1 : \mathscr{P}_j^1(A^{(j+1)}) \to \mathscr{P}_j^1(A^{(j)})$ taking $P \mapsto \psi_j^1(P) := g_j^{-1}(P/\langle x_j\rangle)$, where $P/\langle x_j\rangle$ denotes the canonical image of P in $A^{(j+1)}/\langle x_j\rangle$, which induces a bi-increasing homeomorphism between $\mathscr{P}_j^1(A^{(j+1)})$ and the image $\psi_j^1(\mathscr{P}_j^1(A^{(j+1)}))$.*

Proof. By the First Isomorphism Theorem for algebras we can restrict the map g_j from Lemma 4.4 to the quotient algebra $A^{(j)}/\ker(g_j)$ to yield an isomorphism $g_j' : A^{(j)}/\ker(g_j) \xrightarrow{\sim} A^{(j+1)}/\langle x_j\rangle$. This map, along with the bijective correspondence between ideals in a quotient algebra R/I (for some algebra R and ideal $I \lhd R$) and ideals in R which contain I, induces the following bi-increasing homeomorphisms between sets endowed with the Zariski topology:

$$\begin{aligned}
f_1 : \mathscr{P}_j^1(A^{(j+1)}) &\longrightarrow \text{C.Spec}(A^{(j+1)}/\langle x_j \rangle),\\
f_2 : \text{C.Spec}(A^{(j+1)}/\langle x_j \rangle) &\longrightarrow \text{C.Spec}(A^{(j)}/\ker(g_j)),\\
f_3 : \text{C.Spec}(A^{(j)}/\ker(g_j)) &\longrightarrow \{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\}.
\end{aligned}$$

The composition of these maps gives a bi-increasing homeomorphism

$$\begin{aligned}
f_3 \circ f_2 \circ f_1 : \mathscr{P}_j^1(A^{(j+1)}) &\longrightarrow \{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\}\\
P &\longmapsto g_j^{-1}(P/\langle x_j \rangle).
\end{aligned}$$

Note that $g_j(y_j) = x_j + \langle x_j \rangle = 0$ so $\langle y_j \rangle \subseteq \ker(g_j)$, which leads to the inclusion

$$\{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\} \subseteq \mathscr{P}_j^1(A^{(j)}).$$

Therefore, from $f_3 \circ f_2 \circ f_1$, we can actually define an increasing injective map

$$\begin{aligned}
\psi_j^1 : \mathscr{P}_j^1(A^{(j+1)}) &\longrightarrow \mathscr{P}_j^1(A^{(j)})\\
P &\longmapsto g_j^{-1}(P/\langle x_j \rangle),
\end{aligned}$$

which induces a bi-increasing homeomorphism on its image, $\psi_j^1(\mathscr{P}_j^1(A^{(j+1)})) = \{Q \in \text{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\}$. $\square$

Using the two previous results we now define the map $\psi_j : \text{C.Spec}(A^{(j+1)}) \to \text{C.Spec}(A^{(j)})$ where, for $P \in \text{C.Spec}(A^{(j+1)})$, we set

$$\psi_j(P) := \begin{cases} \psi_j^0(P) & \text{if } P \in \mathscr{P}_j^0(A^{(j+1)});\\ \psi_j^1(P) & \text{if } P \in \mathscr{P}_j^1(A^{(j+1)}). \end{cases}$$

The next result follows immediately from our previous work.

Proposition 4.6. *For $j \in [\![2, N]\!]$ the map $\psi_j : \text{C.Spec}(A^{(j+1)}) \to \text{C.Spec}(A^{(j)})$ is injective. For $\varepsilon \in \{0,1\}$, ψ_j induces (by restriction) a bi-increasing homeomorphism $\mathscr{P}_j^\varepsilon(A^{(j+1)}) \to \psi_j(\mathscr{P}_j^\varepsilon(A^{(j+1)}))$ which is a closed subset of $\mathscr{P}_j^\varepsilon(A^{(j)})$.*

4.1.2 The Canonical Partition of $\text{C.Spec}(A)$

The maps defined in the previous section allow us to define a partition of the set $\text{C.Spec}(A)$ in a similar way to [Cau03a, Section 4.4].

Definition 4.7. Set $\psi := \psi_2 \circ \cdots \circ \psi_N$ to be the injective map $\psi : \mathrm{C.Spec}(A) \longrightarrow \mathrm{C.Spec}(A')$. We call ψ the *canonical embedding* of $\mathrm{C.Spec}(A)$ into $\mathrm{C.Spec}(A')$.

Let $\mathscr{W} := \mathbb{P}([\![1,N]\!])$ denote the power set of $[\![1,N]\!]$ and, for all $w \in \mathscr{W}$, set

$$\mathrm{C.Spec}_w(A') := \{Q \in \mathrm{C.Spec}(A') \mid Q \cap \{T_1, \dots T_N\} = \{T_i \mid i \in w\}\},$$

where $\{T_i\}_{i=1}^N$ are the generators of the quantum affine space A'.

Lemma 4.8. *The sets* $\{\mathrm{C.Spec}_w(A')\}_{w\in\mathscr{W}}$ *provide a partition of* $\mathrm{C.Spec}(A')$.

Proof. Let $Q \in \mathrm{C.Spec}(A')$. Then, for each T_i, either $T_i \in Q$ or $T_i \notin Q$. Therefore, for all $w \in \mathscr{W}$, either $\{T_i\}_{i\in w} \subseteq Q$ or $\{T_i\}_{i\in w} \nsubseteq Q$. This yields the following properties which make the sets $\{\mathrm{C.Spec}_w(A')\}_{w\in\mathscr{W}}$ into a partition of $\mathrm{C.Spec}(A')$:

- for $w_1, w_2 \in \mathscr{W}$, $\mathrm{C.Spec}_{w_1}(A') \cap \mathrm{C.Spec}_{w_2}(A') = \emptyset$;
- $\bigsqcup_{w\in\mathscr{W}} \mathrm{C.Spec}_w(A') = \mathrm{C.Spec}(A')$.

□

For each $w \in \mathscr{W}$ we define

$$\mathrm{C.Spec}_w(A) := \psi^{-1}(\mathrm{C.Spec}_w(A'))$$

and let $\mathscr{W}' \subseteq \mathscr{W}$ denote the set of all $w \in \mathscr{W}$ such that $\mathrm{C.Spec}_w(A) \neq \emptyset$. We immediately obtain a partition of $\mathrm{C.Spec}(A)$:

Lemma 4.9. *The set* $\mathrm{C.Spec}(A)$ *has a partition indexed by the family* $\mathscr{W}'$ *so that,*

$$\mathrm{C.Spec}(A) = \bigsqcup_{w\in\mathscr{W}'} \mathrm{C.Spec}_w(A), \qquad \textit{where } |\mathscr{W}'| \leq |\mathscr{W}| = 2^N.$$

Definition 4.10. We call the partition $\{\mathrm{C.Spec}_w(A)\}_{w\in\mathscr{W}'}$ the *canonical partition* of $\mathrm{C.Spec}(A)$, and we call each $w \in \mathscr{W}'$ a *Cauchon diagram of A*.

Note that $\mathscr{W}'$ depends on the expression of the algebra A as an iterated Ore extension, as we will illustrate in Example 4.38 once we've set up some necessary results.

4.1.3 Properties of the Canonical Embedding

In order to use the deleting derivations algorithm for the purpose of calculating the PI degree of quotient algebras, we need to be able to test whether a completely prime ideal of A' lies in the image of the canonical embedding. In this section we verify that ψ satisfies many of the same properties as the canonical embedding defined in the generic case and we give a membership criterion for $\mathrm{Im}(\psi)$.

The proofs of the topological properties of ψ follow in a similar manner to those in [Cau03a, Section 5.1], and the proofs concerning membership criteria for $\mathrm{Im}(\psi)$ closely mirror those in the Poisson setting, which can be found in [Lec14, Sections 5.4.3 and 5.4.4].

Lemma 4.11. *Fix some $j \in [\![2,N]\!]$ and let $Q \in \mathrm{C.Spec}(A^{(j)})$. Then,*

$$Q \in \mathrm{Im}(\psi_j) \iff [\textit{Either } x_j = y_j \notin Q \textit{ or } \ker(g_j) \subseteq Q].$$

Proof. Let $Q \in \mathrm{C.Spec}(A^{(j)})$. If $y_j \notin Q$ then $Q \in \mathscr{P}_j^0(A^{(j)})$, and in Lemma 4.3 we constructed a bijection between sets $\mathscr{P}_j^0(A^{(j+1)})$ and $\mathscr{P}_j^0(A^{(j)})$. Therefore $Q \in \mathrm{Im}(\psi_j^0) \iff y_j \notin Q$.

If $y_j \in Q$ then $Q \in \mathscr{P}_j^1(A^{(j)})$, and in Lemma 4.5 we constructed a bijection between sets $\mathscr{P}_j^1(A^{(j+1)})$ and $\{Q \in \mathrm{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq Q\}$. Therefore $Q \in \mathrm{Im}(\psi_j^1) \iff \ker(g_j) \subseteq Q$. □

We now define, inductively, injective maps $f_j : \mathrm{C.Spec}(A^{(j+1)}) \to \mathrm{C.Spec}(A')$, for all $j \in [\![1,N]\!]$. To do this, we set $f_1 := \mathrm{id}_{\mathrm{C.Spec}(A')}$ to be the identity map on $\mathrm{C.Spec}(A')$ and, for all $j \in [\![2,N]\!]$, we define $f_j := f_{j-1} \circ \psi_j$ so that $f_N = f_1 \circ \psi$. Using Lemma 4.11 we may deduce the following result:

Proposition 4.12. *Let $Q \in \mathrm{C.Spec}(A')$. The following are equivalent:*

- $Q \in \mathrm{Im}(\psi)$.
- *For all $j \in [\![2,N]\!]$ we have $Q \in \mathrm{Im}(f_{j-1})$ and either $X_j^{(j)} = X_j^{(j+1)} \notin f_{j-1}^{-1}(Q)$ or $\ker(g_j) \subseteq f_{j-1}^{-1}(Q)$.*

Proof. Let $Q \in \mathrm{C.Spec}(A')$. Suppose $Q \in \mathrm{Im}(\psi)$, then $Q = \psi(P)$ for some $P \in \mathrm{C.Spec}(A)$. Since $f_N = f_1 \circ \psi_2 \circ \cdots \circ \psi_N$ then $Q = f_{j-1}(P_j)$ for all $j \in [\![2,N]\!]$, where $P_j = \psi_j \circ \cdots \circ \psi_N(P)$. Hence $Q \in \mathrm{Im}(f_{j-1})$ for all $j \in [\![2,N]\!]$. From this we see that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\psi_j)$, for all $j \in [\![2,N]\!]$, and we may therefore apply Lemma 4.11 to $f_{j-1}^{-1}(Q)$ to conclude that either $X_j^{(j)} = X_j^{(j+1)} \notin f_{j-1}^{-1}(Q)$ or $\ker(g_j) \subseteq f_{j-1}^{-1}(Q)$.

Now suppose the second statement holds. By Lemma 4.11, $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\psi_j)$ for all $j \in [\![2,N]\!]$. Let $P' := f_{N-1}^{-1}(Q)$ so that $P' \in \mathrm{Im}(\psi_N)$. Then $P' = \psi_N(P)$ for some $P \in \mathrm{C.Spec}(A)$ and

$$Q = f_{N-1}(P') = f_{N-1}(\psi_N(P)) = \psi(P).$$

Therefore, $Q \in \mathrm{Im}(\psi)$. □

In order to utilise Proposition 4.12, we need to know the explicit form that each $\ker(g_j)$ takes, for all $j \in [\![2,N]\!]$. In Section 4.4 we provide a general form for $\ker(g_j)$ given any A satisfying Hypothesis 1, and in Section 4.5 we apply this result to the multiparameter quantum matrices to obtain a combinatorial description of the Cauchon diagrams, $\mathscr{W}'$.

We now state some properties of ψ which will be used to give a sufficient condition for a given completely prime ideal in A' to be in the image of the canonical embedding.

Lemma 4.13. *Fix some $j \in [\![2,N]\!]$ and let $i \in [\![j,N]\!]$ and $P \in \mathrm{C.Spec}(A^{(j+1)})$. Then,*

$$x_i \in P \iff y_i \in \psi_j(P).$$

Proof. When $i = j$ the result follows from the bijections constructed in Lemmas 4.3 and 4.5. Suppose $i > j$ and consider the two following cases:

Case 1: If $x_j \notin P$ then $\psi_j(P) = PS_j^{-1} \cap A^{(j)}$ and we see that

$$\begin{aligned} x_i \in P &\implies x_i \cdot 1^{-1} \in PS_j^{-1} &&\implies x_i = y_i \in PS_j^{-1} \cap A^{(j)} = \psi_j(P), \\ y_i \in \psi_j(P) &\implies y_i \cdot 1^{-1} \in \psi_j(P)S_j^{-1} &&\implies y_i = x_i \in \psi_j(P)S_j^{-1} \cap A^{(j+1)} = P. \end{aligned}$$

Case 2: If $x_j \in P$ then $\psi_j(P) = g_j^{-1}(P/\langle x_j \rangle)$. Therefore, since $y_i \in g_j^{-1}(g_j(y_i))$, we see that

$$\begin{aligned} x_i \in P \iff x_i + \langle x_j \rangle \in P/\langle x_j \rangle &\iff g_j(y_i) \in P/\langle x_j \rangle \\ &\iff g_j^{-1}(g_j(y_i)) \subseteq g_j^{-1}(P/\langle x_j \rangle) \\ &\iff y_i \in \psi_j(P). \end{aligned}$$

□

Corollary 4.14. *Let $i \in [\![1,N]\!]$ and $Q \in \mathrm{Im}(\psi)$. We have*

$$T_i := X_i^{(2)} \in Q \iff X_i^{(i+1)} \in f_i^{-1}(Q).$$

Proof. When $i = 1$, the result trivially holds since $f_1^{-1}(Q) = Q$. Suppose now that $i \in [\![2,N]\!]$. For each $j \in [\![2,i]\!]$ we may apply Lemma 4.13 to the ideal $P = f_j^{-1}(Q)$ with $\psi_j(P) = f_{j-1}^{-1}(P)$. This gives us

$$X_i^{(j+1)} \in f_j^{-1}(Q) \iff X_i^{(j)} \in \psi_j(f_j^{-1}(Q)) = f_{j-1}^{-1}(Q).$$

From this we easily conclude that

$$X_i^{(2)} \in Q \iff X_i^{(3)} \in f_2^{-1}(Q) \iff \cdots \iff X_i^{(i+1)} \in f_i^{-1}(Q).$$

$\square$

For the next two results we set up the following notation: Fixing some $w \in \mathscr{W}$ and $j \in [\![1,N]\!]$, we define

$$X_w := f_j^{-1}(\mathrm{C.Spec}_w(A')) \subseteq \mathrm{C.Spec}(A^{(j+1)}). \tag{4.1}$$

If $j = 1$ then $X_w = \mathrm{C.Spec}_w(A')$, and if $j \geq 2$ then we let $X_w = \psi_j^{-1}(Y_w)$, so that

$$Y_w := f_{j-1}^{-1}(\mathrm{C.Spec}_w(A')) \subseteq \mathrm{C.Spec}(A^{(j)}). \tag{4.2}$$

Lemma 4.15. *Let $j \in [\![1,N]\!]$ and $w \in \mathscr{W}$. For all $i \in [\![j,N]\!]$ we have*

(i) $i \notin w \implies [X_i^{(j+1)} \notin P,\ \ \forall P \in X_w] \implies X_w \subseteq \mathscr{P}_j^0(A^{(j+1)})$*;*

(ii) $i \in w \implies [X_i^{(j+1)} \in P,\ \ \forall P \in X_w] \implies X_w \subseteq \mathscr{P}_j^1(A^{(j+1)})$*.*

Proof. When $j = 1, X_i^{(j+1)} = T_i$ and the result follows from the definition of $X_w = \mathrm{C.Spec}_w(A')$.

Let $j \geq 2$ and assume the lemma is true for $j-1$. Suppose $i \notin w$ and consider $P \in X_w = f_j^{-1}(\mathrm{C.Spec}_w(A')) \subseteq \mathrm{C.Spec}(A^{(j+1)})$. If $X_i^{(j+1)} \in P$ then, since $i \geq j$, we may use Lemma 4.13 to deduce that $X_i^{(j)} \in \psi_j(P) \in Y_w$. However, this contradicts the inductive hypothesis, so we conclude that $X_i^{(j+1)} \notin P$ for all $P \in X_w$.

Similarly, if $i \in w$ then we deduce that $X_i^{(j+1)} \in P$ for all $P \in X_w$. $\square$

Lemma 4.16. *For each $j \in [\![1,N]\!]$, the set $f_j(X_w)$ is a closed subset of* $\mathrm{C.Spec}_w(A')$ *and f_j induces (by restriction) a bi-increasing homeomorphism from X_w to $f_j(X_w)$.*

Proof. When $j = 1$, the result is trivially true since f_1 is the identity map on $\mathrm{C.Spec}_w(A')$. Fix some $j \in [\![2,N]\!]$ and define sets X_w and Y_w as in (4.1) and (4.2). Assume the statements hold for $j-1$; that is, the subset $f_{j-1}(Y_w) \subseteq \mathrm{C.Spec}_w(A')$ is closed and f_{j-1} induces a bi-increasing homeomorphism from Y_w to $f_{j-1}(Y_w)$. Applying Lemma 4.15, for the fixed j and w of this proof and taking $i = j$, we see that

(i) $j \notin w \implies X_w \subseteq \mathscr{P}_j^0(A^{(j+1)})$;

(ii) $j \in w \implies X_w \subseteq \mathscr{P}_j^1(A^{(j+1)})$.

We therefore deduce, using Proposition 4.6, that $\psi_j^{\varepsilon}(X_w) \subseteq \mathscr{P}_j^{\varepsilon}(A^{(j)})$, where $\varepsilon = 0$ if $j \notin w$ and $\varepsilon = 1$ if $j \in w$. Since $\psi_j^{\varepsilon}(X_w) \subseteq Y_w$ for all $\varepsilon \in \{0,1\}$ then $\psi_j^{\varepsilon}(X_w) = Y_w \cap \psi_j^{\varepsilon}(\mathscr{P}_j^{\varepsilon}(A^{(j+1)}))$. Furthermore, $\psi_j^{\varepsilon}(\mathscr{P}_j^{\varepsilon}(A^{(j+1)})) \subseteq \mathscr{P}_j^{\varepsilon}(A^{(j)})$ is a closed subset, hence

$\psi_j^\varepsilon(X_w)$ is closed in Y_w. The result now follows from the induction hypothesis, as we may write the following inclusions

$$f_j(X_w) = f_{j-1}(\psi_j^\varepsilon(X_w)) \subseteq f_{j-1}(Y_w) \subseteq \mathrm{C.Spec}_w(A'),$$

where $f_{j-1}(Y_w)$ is a closed subset of $\mathrm{C.Spec}_w(A')$ (by the induction hypothesis) and $\psi_j^\varepsilon(X_w)$ is closed in Y_w. Since $f_{j-1}|_{Y_w} : Y_w \to f_{j-1}(Y_w)$ is a bi-increasing homeomorphism (again, by the induction hypothesis), which sends closed sets of Y_w to closed sets of $f_{j-1}(Y_w)$, we see that $f_j(X_w) = f_{j-1}(\psi_j^\varepsilon(X_w))$ is closed in $f_{j-1}(Y_w)$ and hence in $\mathrm{C.Spec}_w(A')$. Finally, $f_j|_{X_w} : X_w \to f_j(X_w)$ is the composition of the bi-increasing homeomorphisms ψ_j^ε and f_{j-1} (restricted to the appropriate subsets), and thus is also bi-increasing. □

When $j = N$, we have $f_j = f_N = \psi$ and $X_w = \mathrm{C.Spec}_w(A)$, whence we obtain the following result:

Theorem 4.17. *Let $\psi : \mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$ be the canonical embedding and let $w \in \mathscr{W}'$. Then $\psi(\mathrm{C.Spec}_w(A))$ is a (non-empty) closed subset of $\mathrm{C.Spec}_w(A')$ and the map ψ induces (by restriction) a bi-increasing homeomorphism from $\mathrm{C.Spec}_w(A)$ to $\psi(\mathrm{C.Spec}_w(A))$.*

Proposition 4.18. *Let $w \in \mathscr{W}'$, $P \in \mathrm{C.Spec}_w(A)$, and $Q \in \mathrm{C.Spec}_w(A')$. If $\psi(P) \subseteq Q$ then $Q \in \mathrm{Im}(\psi)$.*

Proof. We prove, by induction on $j \in [\![1,N]\!]$, that $Q \in \mathrm{Im}(f_j)$.

When $j = 1$, $f_1(Q) = Q$ and the result is holds trivially. Fix some $j \in [\![2,N]\!]$ and assume that $Q \in \mathrm{Im}(f_{j-1})$. We need to show that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\psi_j)$ so that $Q \in \mathrm{Im}(f_j)$ by the defining property $f_j = f_{j-1} \circ \psi_j$. Note that if $\psi(P) \subseteq Q$ then $f_{j-1}^{-1}(\psi(P)) \subseteq f_{j-1}^{-1}(Q)$. We investigate the following two cases:

Suppose $X_j^{(j+1)} \notin f_j^{-1}(\psi(P))$, then it follows by Corollary 4.14 that $T_j \notin \psi(P)$. Therefore $j \notin w$ and, by Lemma 4.15 (with $j-1$ in place of j and setting $i = j$), we see that $X_j^{(j+1)} \notin f_{j-1}^{-1}(Q)$. Finally, Lemma 4.11 allows us to conclude that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\psi_j)$.

Suppose $X_j^{(j+1)} \in f_j^{-1}(\psi(P))$, then $f_j^{-1}(\psi(P)) \in \mathscr{P}_j^1(A^{(j+1)})$. By Lemma 4.5, we obtain the bi-increasing homeomorphism ψ_j from $\mathscr{P}_j^1(A^{(j+1)})$ to $\{P' \in \mathrm{C.Spec}(A^{(j)}) \mid \ker(g_j) \subseteq P'\}$. Applying this map to $f_j^{-1}(\psi(P))$, we see that $\ker(g_j) \subseteq \psi_j(f_j^{-1}(\psi(P))) = f_{j-1}^{-1}(\psi(P))$. Finally, since $f_{j-1}^{-1}(\psi(P)) \subseteq f_{j-1}^{-1}(Q)$, the inclusion above gives $\ker(g_j) \subseteq f_{j-1}^{-1}(Q)$, and we may use Lemma 4.11 again to conclude that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\psi_j)$.

In both possible cases we conclude that $f_{j-1}^{-1}(Q) \in \mathrm{Im}(\psi_j)$ as desired and therefore, by induction, $Q \in \mathrm{Im}(f_j)$ for all $j \in [\![1,N]\!]$. □

4.2 Completely Prime Quotients of $A^{(j+1)}$ and $A^{(j)}$

In this section we follow closely the results and proofs found in [Cau03a, Section 5.3], showing that they also apply to algebras satisfying Hypothesis 1, and their completely prime quotients.

For some $j \in [\![2,N]\!]$, let $P \in \text{C.Spec}(A^{(j+1)})$ and $Q = \psi_j(P) \in \text{C.Spec}(A^{(j)})$ be its image under the canonical embedding. Set

$$B^{(j+1)} := A^{(j+1)}/P, \quad B^{(j)} := A^{(j)}/Q.$$

We denote by $(\bar{x}_1,\dots,\bar{x}_N) \in B^{(j+1)}$ and $(\bar{y}_1,\dots,\bar{y}_N) \in B^{(j)}$ the canonical images of $(x_1,\dots,x_N)$ in $B^{(j+1)}$ and $(y_1,\dots,y_N)$ in $B^{(j)}$, respectively.

Lemma 4.19. *Suppose $\bar{x}_j = 0$. Then there exists an algebra isomorphism $B^{(j)} \to B^{(j+1)}$ sending $\bar{y}_i \mapsto \bar{x}_i$ for all $i \in [\![1,N]\!]$.*

Proof. Since $x_j \in P$ then $Q = \psi_j^1(P) = g_j^{-1}(P/\langle x_j\rangle)$, where $P/\langle x_j\rangle$ is the canonical image of P in $A^{(j+1)}/\langle x_j\rangle$. Noting that $\langle x_j\rangle \subseteq P$, we can concatenate the surjective map g_j with the natural surjection $\pi : A^{(j+1)}/\langle x_j\rangle \to A^{(j+1)}/P$ to obtain the following surjective algebra homomorphism:

$$\begin{array}{ccccc} A^{(j)} & \xrightarrow{g_j} & A^{(j+1)}/\langle x_j\rangle & \xrightarrow{\pi} & A^{(j+1)}/P \\ y_i & \longmapsto & x_i + \langle x_j\rangle & \longmapsto & \bar{x}_i, \end{array}$$

Clearly $\ker(\pi) = P/\langle x_j\rangle$ and $\ker(\pi\circ g_j) = g_j^{-1}(P/\langle x_j\rangle) = Q$. We therefore get an algebra isomorphism,

$$\begin{array}{rcl} A^{(j)}/\ker(\pi\circ g_j) = A^{(j)}/Q & \longrightarrow & A^{(j+1)}/P \\ \bar{y}_i & \longmapsto & \bar{x}_i. \end{array}$$

□

Lemma 4.20. *Suppose $\bar{x}_j \neq 0$ and let $Z_j := \{\bar{x}_j^n \mid n \in \mathbb{N}\}$. Then the following hold:*

(i) Z_j is a multiplicative set of regular elements of $B^{(j+1)}$, which satisfies the two-sided Ore condition.

(ii) There exists an injective algebra homomorphism $\gamma : B^{(j)} \to B^{(j+1)}Z_j^{-1}$ defined on the generators of $B^{(j)}$ in the following way:

$$\gamma(\bar{y}_i) = \begin{cases} \bar{x}_i & \text{if } i \geq j; \\ \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j-1)^{-n}\lambda_{j,i}^{-n}\overline{d_{j,n}(x_i)}\bar{x}_j^{-n} & \text{if } i < j, \end{cases}$$

where $\overline{d_{j,n}(x_i)}$ denotes the canonical image of $d_{j,n}(x_i)$ in $B^{(j+1)}$.

(iii) If we identify $B^{(j)}$ with its image $\gamma(B^{(j)}) \subseteq B^{(j+1)}Z_j^{-1}$, then Z_j is also a multiplicative set of regular elements in $B^{(j)}$ satisfying the two-sided Ore condition. Furthermore, $B^{(j)}Z_j^{-1} = B^{(j+1)}Z_j^{-1}$.

Proof. Since $x_j \notin P$ then $Q = \psi_j^0(P) = PS_j^{-1} \cap A^{(j)}$, where $S_j = \{x_j^n \mid n \in \mathbb{N}\}$ is the multiplicatively closed set of regular elements in $A^{(j+1)}$ and $A^{(j)}$ satisfying the two-sided Ore condition, as shown in Theorem 3.6(III). We also see that $A^{(j+1)}S_j^{-1} = A^{(j)}S_j^{-1}$, as subalgebras of F, (again, by Theorem 3.6(III)) and we denote this algebra by Ω. It can be easily verified from the definition of ψ_j^0 that $PS_j^{-1} = QS_j^{-1} \lhd \Omega$, a completely prime ideal (by Lemma 4.1), which we denote by $\Theta \lhd \Omega$. It has the property that $\Theta \cap A^{(j+1)} = P$ and $\Theta \cap A^{(j)} = Q$.

We may define an algebra homomorphism

$$\begin{aligned} \gamma' : B^{(j+1)} &\longrightarrow \Omega/\Theta \\ a+P &\longmapsto a1^{-1} + \Theta, \end{aligned}$$

for all $a \in A^{(j+1)}$, hence all $a+P \in B^{(j+1)}$, and this homomorphism is injective. We identify $B^{(j+1)}$ with its image in Ω/Θ under the map γ' (that is, $b = a+P = a1^{-1} + \Theta$ for all $b = a+P \in B^{(j+1)}$). Note that, since $Z_j = \{\bar{x}_j^n \mid n \in \mathbb{N}\} = \{x_j^n + P \mid n \in \mathbb{N}\} = S_j + P$, its image under γ' becomes

$$\gamma'(Z_j) = Z_j 1^{-1} + \Theta = (S_j + P)1^{-1} + \Theta = S_j 1^{-1} + \Theta \subseteq B^{(j+1)} \subseteq \Omega/\Theta.$$

Identifying Z_j with its image $\gamma'(Z_j)$, we observe that all elements of the set Z_j are invertible in $B^{(j+1)}$. Since $\Omega = A^{(j+1)}S_j^{-1} = S_j^{-1}A^{(j+1)}$ and $PS_j^{-1} = S_j^{-1}P = \Theta$, we can write any element $b \in \Omega/\Theta$ as

$$b = a_1 s_1^{-1} + \Theta = s_2^{-1} a_2 + \Theta, \tag{4.3}$$

where $a_1, a_2 \in A^{(j+1)}$ and $s_1, s_2 \in S_j$. Let $b_1, b_2 \in B^{(j+1)}$ and $z_1, z_2 \in Z_j$ such that

$$b_1 = a_1 1^{-1} + \Theta, \quad b_2 = a_2 1^{-1} + \Theta, \quad z_1 = s_1 1^{-1} + \Theta, \quad z_2 = s_2 1^{-1} + \Theta.$$

Then, using (4.3), we see that, for all $b \in \Omega/\Theta$,

$$b = b_1 z_1^{-1} = z_2^{-1} b_2.$$

This shows that the set $Z_j \subset B^{(j+1)}$ satisfies the two-sided Ore condition, thus proving property (i) of the lemma. The working above also proves the equality $B^{(j+1)}Z_j^{-1} = \Omega/\Theta$, i.e. $(A^{(j+1)}/P)Z_j^{-1} = A^{(j+1)}S_j^{-1}/PS_j^{-1}$.

We now construct the homomorphism $\gamma\colon B^{(j)} \to B^{(j+1)}Z_j^{-1} = \Omega/\Theta$ defined in part (ii) of the theorem. Just as for $B^{(j+1)}$ we may use the fact that $\Theta \cap A^{(j)} = Q$ to define an injective homomorphism

$$\begin{aligned} \gamma\colon B^{(j)} &\longrightarrow \Omega/\Theta \\ a+Q &\longmapsto a1^{-1}+\Theta, \end{aligned}$$

for all $a \in A^{(j)}$. Recall,

$$y_i := \begin{cases} x_i & \text{if } i \geq j; \\ \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}}(q_j-1)^{-n}\lambda_{j,i}^{-n} d_{j,n}(x_i)x_j^{-n} & \text{if } i < j, \end{cases}$$

for all $y_i \in A^{(j)}$. Using $A^{(j+1)}S_j^{-1} = A^{(j)}S_j^{-1} = \Omega$ and $B^{(j+1)}Z_j^{-1} = \Omega/\Theta$, as was shown earlier in the proof, we observe that, for all $i \geq j$,

$$\gamma(\bar{y}_i) = \gamma(y_i+Q) = y_i1^{-1}+\Theta = x_i1^{-1}+\Theta = x_i+P = \bar{x}_i.$$

Similarly, for all $i < j$,

$$\begin{aligned} \gamma(\bar{y}_i) = \gamma(y_i+Q) = y_i1^{-1}+\Theta \\ &= \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}}(q_j-1)^{-n}\lambda_{j,i}^{-n} d_{j,n}(x_i)x_j^{-n}+\Theta \\ &= \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}}(q_j-1)^{-n}\lambda_{j,i}^{-n} \overline{d_{j,n}(x_i)}\bar{x}_j^{-n}, \end{aligned}$$

where $\overline{d_{j,n}(x_i)}$ is the image of $d_{j,n}(x_i)$ in $B^{(j+1)}$.

As before, identifying $B^{(j)}$ with its image in Ω/Θ allows us to show that Z_j is a multiplicative set of regular elements satisfying the two-sided Ore condition in $B^{(j)}$, and that $B^{(j)}Z_j^{-1} = \Omega/\Theta = B^{(j+1)}Z_j^{-1}$. □

Lemma 4.21. *$Frac(B^{(j+1)}) = Frac(B^{(j)})$.*

Proof. If $\bar{x}_j = 0$ then $B^{(j)} = B^{(j+1)}$ and the result is immediate. If $\bar{x}_j \neq 0$ then the result follows from the preceding lemma; specifically we see that $B^{(j)}Z_j^{-1} = B^{(j+1)}Z_j^{-1}$, with $B^{(j+1)}Z_j^{-1} \subseteq \mathrm{Frac}(B^{(j+1)})$ and $B^{(j)}Z_j^{-1} \subseteq \mathrm{Frac}(B^{(j)})$. □

4.3 Completely Prime Quotients of A and A'

We continue to extend Cauchon's algorithm to algebras satisfying Hypothesis 1 and in this section we present analogues to the results in [Cau03a, Section 5.4] using the same methods.

Let $P \in \text{C.Spec}(A)$ and $Q = \psi(P) \in \text{C.Spec}(A')$. In this section we prove a similar result to Lemma 4.21 for $\text{Frac}(A/P)$ and $\text{Frac}(A'/Q)$ by iterating the method in Section 4.2 and tracking the ideal P through this process. We begin be setting up some notation which allows us to do this:

- Let $B := A/P$ and $\bar{X}_1, \dots, \bar{X}_N \in B$ be the canonical images of $X_1, \dots, X_N \in A$.
- Let $B' := A'/Q$ and $t_1, \dots, t_N \in B'$ be the canonical images of $T_1, \dots, T_N \in A'$.
- For $j \in [\![2, N+1]\!]$, denote by $P_j := \psi_j \circ \cdots \circ \psi_N(P) \in \text{C.Spec}(A^{(j)})$ the image of P after $N - j + 1$ steps of the deleting derivations algorithm.
- For each $j \in [\![2, N+1]\!]$, define the algebra $B^{(j)} := A^{(j)}/P_j$ and denote by $\bar{X}_1^{(j)}, \dots, \bar{X}_N^{(j)}$ the canonical images of $X_1^{(j)}, \dots, X_N^{(j)}$ in $B^{(j)}$. Note that by taking $j = N+1$ we obtain $B^{(N+1)} = B$ with $(\bar{X}_1^{(N+1)}, \dots, \bar{X}_N^{(N+1)}) = (\bar{X}_1, \dots, \bar{X}_N)$, and by taking $j = 2$ we obtain $B^{(2)} = B'$ with $(\bar{X}_1^{(2)}, \dots, \bar{X}_N^{(2)}) = (t_1, \dots, t_N)$.

Proposition 4.22. *For each $j \in [\![2, N+1]\!]$, the following hold:*

(i) $B^{(j)}$ is a subalgebra of Frac(B) generated by $\bar{X}_1^{(j)}, \dots, \bar{X}_N^{(j)}$;

(ii) There is an algebra homomorphism,

$$\begin{aligned} f_j : \ A^{(j)} &\longrightarrow \mathit{Frac}(B) \\ X_i^{(j)} &\longmapsto \bar{X}_i^{(j)} \end{aligned}$$

with image $B^{(j)}$ and kernel P_j.

Proof.

(i) Applying Lemma 4.21 to each $j \in [\![2, N+1]\!]$, we see that $\text{Frac}(B^{(j)}) = \text{Frac}(B)$. Since $B^{(j)}$ is (trivially) a subalgebra of $\text{Frac}(B^{(j)})$, generated by $\bar{X}_1^{(j)}, \dots, \bar{X}_N^{(j)}$, then this observation leads us to deduce that $B^{(j)} \subseteq \text{Frac}(B)$.

(ii) Concatenating the natural embedding $B^{(j)} \hookrightarrow \text{Frac}(B)$ with the canonical surjection, $\pi_j : A^{(j)} \to A^{(j)}/P_j = B^{(j)}$, gives the homomorphism f_j described in the statement of the proposition. It is clear that π_j has kernel P_j and that the natural embedding $B^{(j)} \hookrightarrow \text{Frac}(B)$ has image $B^{(j)}$.

□

Lemmas 4.19 and 4.20 combine to give the following:

Proposition 4.23. *Let $j \in [\![2, N+1]\!]$.*

(i) If $\bar{X}_j^{(j+1)} = 0$ then $\bar{X}_i^{(j)} = \bar{X}_i^{(j+1)}$ for all $i \in [\![1, N]\!]$.

(ii) Suppose $\bar{X}_j^{(j+1)} \neq 0$ and set $(x_1, \dots, x_N) := (X_1^{(j+1)}, \dots, X_N^{(j+1)})$. Then the generators of $B^{(j)}$ are can be obtained as follows:

$$\bar{X}_i^{(j)} = \begin{cases} \bar{X}_i^{(j+1)} & \text{if } i \geq j; \\ \sum_{n=0}^{\infty} q_j^{\frac{n(n+1)}{2}} (q_j - 1)^{-n} \lambda_{j,i}^{-n} f_{j+1} \circ d_{j,n}(x_i) (\bar{X}_j^{(j+1)})^{-n} & \text{if } i < j, \end{cases}$$

where f_{j+1} is the map defined in Proposition 4.22.

(iii) Suppose $\bar{X}_j^{(j+1)} \neq 0$ and let $Z_j = \{(\bar{X}_j^{(j+1)})^n \mid n \in \mathbb{N}\} = \{(\bar{X}_j^{(j)})^n \mid n \in \mathbb{N}\}$ be a multiplicatively closed set of regular elements in $B^{(j)}$ and $B^{(j+1)}$. Then Z_j satisfies the two-sided Ore condition in both $B^{(j)}$ and $B^{(j+1)}$ and we have $B^{(j)} Z_j^{-1} = B^{(j+1)} Z_j^{-1}$.

Proof. Part (i) follows from the isomorphism given in Lemma 4.19 and the observation in Lemma 4.21. Part (ii) follows from Lemma 4.20 once one notes that $f_{j+1} \circ d_{j,n}(x_i) = \overline{d_{j,n}(x_i)} \cdot 1^{-1} \in \text{Frac}(B)$. Part (iii) also follows directly from Lemma 4.20. □

Let $w \in \mathscr{W}'$ with $P \in \text{C.Spec}_w(A)$ and $Q = \psi(P) \in \text{C.Spec}_w(A')$. By the definition of $\text{C.Spec}_w(A')$, we have that $T_i \in Q$ if and only if $i \in w$ or, equivalently, $t_i = 0$ if and only if $i \in w$. Let $i \in \bar{w} := \{1, \dots, N\} \backslash w$ so that $t_i \neq 0$. Then, since T_i is normal in A' and Q is completely prime, t_i is normal and regular in B', hence it is invertible in $\text{Frac}(B') = \text{Frac}(B)$.

We denote by Σ the multiplicatively closed set of regular elements in B' generated by all t_i such that $i \in \bar{w}$. From this set we define, recursively, the sets $\Sigma_j \subset B^{(j)}$ for $j \in [\![2, N+1]\!]$ in the following way:

$$\Sigma_2 := \Sigma, \quad \Sigma_{j+1} := B^{(j+1)} \cap \Sigma_j.$$

Proposition 4.24. *For each $j \in [\![2, N+1]\!]$ the following statements hold:*

(i) Σ_j is a multiplicatively closed set of regular elements in $B^{(j)}$ which contains, as a subset, $\{\bar{X}_i^{(j)} \mid i \in [\![j-1, N]\!] \text{ and } \bar{X}_i^{(j)} \neq 0\}$;

(ii) Σ_j satisfies the two-sided Ore condition in $B^{(j)}$;

(iii) The algebras $B^{(j)} \Sigma_j^{-1}$, when considered as subalgebras of Frac(B), are all equal.

Proof. When we take $j = 2$, properties (i) and (ii) are immediately satisfied by the discussion preceding this proposition.

Fix $j \in [\![2, N]\!]$ and assume properties (i) and (ii) are satisfied. We show that these properties are also satisfied when replacing j with $j+1$ and that $B^{(j)}\Sigma_j^{-1} = B^{(j+1)}\Sigma_{j+1}^{-1}$. We consider two cases: $\bar{X}_j^{(j+1)} = 0$ and $\bar{X}_j^{(j+1)} \neq 0$.

When $\bar{X}_j^{(j+1)} = 0$ we may apply Proposition 4.23(i) to obtain $\bar{X}_i^{(j)} = \bar{X}_i^{(j+1)}$ for all $i \in [\![1, N]\!]$. Therefore $B^{(j)} = B^{(j+1)}$, and properties (i) and (ii) follow immediately by inductive hypothesis and the fact that $\Sigma_{j+1} \subseteq \Sigma_j$.

Now suppose $\bar{X}_j^{(j+1)} \neq 0$. Applying Proposition 4.23(ii) to this case gives that $\bar{X}_i^{(j)} = \bar{X}_i^{(j+1)}$ for all $i \geq j$ and by the induction hypothesis we know that

$$\{\bar{X}_i^{(j)} \mid i \in [\![j-1, N]\!],\, \bar{X}_i^{(j)} \neq 0\} \;\subseteq\; \Sigma_j.$$

Therefore,

$$\begin{aligned}
\Sigma_{j+1} := B^{(j+1)} \cap \Sigma_j \;&\supseteq\; B^{(j+1)} \cap \{\bar{X}_i^{(j)} \mid i \in [\![j-1, N]\!],\, \bar{X}_i^{(j)} \neq 0\} \\
&=\; \{\bar{X}_i^{(j+1)} \mid i \in [\![j, N]\!],\, \bar{X}_i^{(j+1)} \neq 0\}.
\end{aligned}$$

The set Σ_{j+1} is multiplicatively closed by the induction hypothesis, which tells us that Σ_j is multiplicatively closed, and it contains regular elements because P_{j+1} is a completely prime ideal and hence $B^{(j+1)}$ is a domain in which all nonzero elements are regular. This proves part (i).

Note that $Z_j \subset \Sigma_{j+1}$ and $Z_j \subset \Sigma_j$. Applying Proposition 4.23(iii), we see that

$$B^{(j+1)} \subset B^{(j+1)} Z_j^{-1} = B^{(j)} Z_j^{-1} \subset B^{(j)} \Sigma_j^{-1}.$$

These inclusions mirror those found in (3.21), and we may apply the rest of the method used in the proof of Proposition 3.9 to conclude parts (ii) and (iii) of this proposition. □

An immediate consequence of this proposition is that we can now show equivalence of the total rings of fractions of A/P and $A'/\psi(P)$ in the case where $P \in \mathrm{C.Spec}_w(A)$, for some $w \in \mathscr{W}'$:

Theorem 4.25. *Let $w \in \mathscr{W}'$, $P \in \mathrm{C.Spec}_w(A)$, and $Q = \psi(P) \in \mathrm{C.Spec}_w(A')$. Let Σ be the multiplicatively closed set of elements in A'/Q which is generated by all the generators t_i such that $i \in \bar{w} = \{1, \ldots, N\} \backslash w$, where t_i is the canonical image of T_i in A'/Q. Then,*

(i) Σ is a multiplicatively closed set of regular, normal elements in A'/Q.

(ii) There exists a multiplicatively closed set of regular elements, Γ, in A/P which satisfies the two-sided Ore condition in A/P and is such that $(A/P)\Gamma^{-1} = (A'/Q)\Sigma^{-1}$.

(iii) $Frac(A/P) = Frac(A'/Q)$. Furthermore, A/P and A'/Q are PI algebras if all $\lambda_{i,j}$ are roots of unity and, in this case, PI-deg$(A/P) =$ PI-deg(A'/Q).

4.4 Kernel of g_j

Proposition 4.12 provides criteria for some $Q \in \mathrm{C.Spec}(A')$ to be in the image of the canonical embedding. Part of these criteria involves checking if $\ker(g_j) \subseteq f_{j-1}^{-1}(Q)$ for $j \in [\![2,N]\!]$. As such, in order to apply this result effectively it is necessary to know what $\ker(g_j)$ is for all $j \in [\![2,N]\!]$. In this section we compute the form that $\ker(g_j)$ takes for any algebra satisfying Hypothesis 1 and, in Section 4.5, we apply this result to multiparameter quantum matrices to obtain a combinatorial description of the set $\mathscr{W}'$.

Two-sided ideals in an algebra A will be denoted by $\langle \bullet \rangle \lhd A$ whereas left ideals will be denoted by $\langle \bullet \rangle^L \lhd_L A$. If $R \subseteq A$ is a subalgebra then we may need to specify where the ideal lies. Where clarity is necessary, we distinguish in which algebra the ideal lies by use of a subscript, i.e. $\langle \bullet \rangle_R^L \lhd_L R$ denotes the left ideal in R (similarly, $\langle \bullet \rangle_R \lhd R$ denotes the two-sided ideal in R).

Throughout this section we let $A = \mathbb{K}[X_1]\cdots[X_N;\sigma_N,\delta_N] = R[X_N;\sigma_N,\delta_N]$ be an iterated Ore extension satisfying Hypothesis 1 and we denote by $\langle X_N \rangle \lhd A$ the two-sided ideal in A generated by X_N.

Lemma 4.26. *For $A = \mathbb{K}[X_1]\cdots[X_N;\sigma_N,\delta_N] = R[X_N;\sigma_N,\delta_N]$ satisfying Hypothesis 1, we have*

$$\langle X_N \rangle = \sum_{t>0} R X_N^t + \langle \delta_N(R) \rangle_R^L.$$

Proof. Take $Y \in \langle X_N \rangle$ and write it as $Y = \sum_{t>0} Z_t X_N^t W_t$ for some $Z_t, W_t \in A$, where $Z_t = \sum_{k\geq 0} a_{k,t} X_N^k$ and $W_t = \sum_{i\geq 0} b_{i,t} X_N^i$, for some $a_{k,t}, b_{i,t} \in R$. That is,

$$Y = \sum_{\substack{t>0 \\ i,k\geq 0}} a_{k,t} X_N^{t+k} b_{i,t} X_N^i.$$

Applying the q-Leibniz rule (3.5) to $X_N^{t+k}b_{i,t}$, and splitting up the sums, gives,

$$\begin{aligned}
Y &= \sum_{\substack{t>0\\ i,k\geq 0}} a_{k,t}\left[\sum_{l=0}^{t+k}\binom{t+k}{l}_{q_N}\sigma_N^{t+k-l}\circ\delta_N^l(b_{i,t})X_N^{t+k-l+i}\right]\\
&= \sum_{\substack{t>0\\ i,k\geq 0}} a_{k,t}\sum_{l=0}^{t+k-1}\binom{t+k}{l}_{q_N}\sigma_N^{t+k-l}\circ\delta_N^l(b_{i,t})X_N^{t+k-l+i}\\
&\quad+\sum_{\substack{t,i>0\\ k\geq 0}} a_{k,t}\binom{t+k}{t+k}_{q_N}\delta_N^{t+k}(b_{i,t})X_N^i+\sum_{\substack{t>0\\ k\geq 0}} a_{k,t}\binom{t+k}{t+k}_{q_N}\delta_N^{t+k}(b_{0,t})\\
&= \sum_{t>0}\alpha_t X_N^t+\sum_{t>0}\beta_t\delta_N^t(b_t),
\end{aligned}$$

for some $b_t,\,\alpha_t,\,\beta_t\in R$. Thus $\langle X_N\rangle\subseteq\sum_{t>0}RX_N^t+\langle\delta_N(R)\rangle_R^L$, since $\delta_N^{t-1}(b_t)\in R$.

For the opposite inclusion we use the following identity arising from the definition of an Ore extension:

$$\delta_N(r)=X_Nr-\sigma_NX_N.$$

Hence $\delta_N(R)\subseteq\langle X_N\rangle$ and we deduce that $\sum_{t>0}RX_N^t+\langle\delta_N(R)\rangle_R^L\subseteq\langle X_N\rangle$. $\square$

Lemma 4.27. *For $A=\mathbb{K}[X_1]\cdots[X_N;\sigma_N,\delta_N]=R[X_N;\sigma_N,\delta_N]$ satisfying Hypothesis 1, we have*

$$\langle\delta_N(R)\rangle_R^L=\langle\delta_N^t(X_i)\mid i\in[\![1,N-1]\!],\,t>0\rangle_R.$$

Proof. Label the ideals in the statement of the lemma as $I:=\langle\delta_N(R)\rangle_R^L\lhd_L R$ and $J:=\langle\delta_N^t(X_i)\mid i\in[\![1,N-1]\!],\,t>0\rangle_R\lhd R$. Using properties of skew-derivations, we obtain

$$\delta_N(X_{i_1}\cdots X_{i_k})=\sum_{l=1}^{k}\sigma_N(X_{i_1}\cdots X_{i_{l-1}})\delta_N(X_{i_l})X_{i_{l+1}}\cdots X_{i_k}\in J$$

for all $k\in\mathbb{N}_{>0}$, where $i_1,\ldots,i_k\in[\![1,N-1]\!]$ are not necessarily distinct. Since any element of R can be written as a linear combination of some $X_{i_1}\cdots X_{i_k}$, we conclude that $\delta_N(R)\subseteq J$ and thus $I\subseteq J$.

We prove the opposite inclusion, $J\subseteq I$, through a series of claims:

Claim 1: For each $i\in[\![1,N-1]\!]$, $\delta_N^t(X_i)b\in I$ for any $b\in R$ and all $t>0$.

Let $i\in[\![1,N-1]\!]$ and $b\in R$. We prove by induction on $t>0$ that

$$\delta_N^t(X_ib)=\sum_{l=0}^{t}\alpha_l\delta_N^l(X_i)\delta_N^{t-l}(b), \tag{4.4}$$

for some $\alpha_l \in \mathbb{K}$ with $\alpha_t = 1$. The base case is shown by the identity

$$\delta_N(X_i b) = \sigma_N(X_i)\delta_N(b) + \delta_N(X_i)b.$$

Now assume that for some $t-1>1$ we have $\delta_N^{t-1}(X_i b) = \sum_{l=0}^{t-1} \alpha_l \delta_N^l(X_i)\delta_N^{t-1-l}(b)$, with $\alpha_{t-1}=1$. We use this, along with the identity $\sigma_N \circ \delta_N^l(X_i) = q_N^{-l}\lambda_{N,i}\delta_N^l(X_i)$ (a consequence of H.1.2 and H.1.4), to deduce the following:

$$\begin{aligned}
\delta_N^t(X_i b) &= \sum_{l=0}^{t-1} \alpha_l \delta_N \left(\delta_N^l(X_i)\delta_N^{t-1-l}(b)\right) \\
&= \sum_{l=0}^{t-1} \alpha_l \left(\sigma_N \circ \delta_N^l(X_i)\delta_N^{t-l}(b) + \delta_N^{l+1}(X_i)\delta_N^{t-1-l}(b)\right) \\
&= \sum_{l=0}^{t-1} \alpha_l q_N^l \lambda_{N,i}\delta_N^l(X_i)\delta_N^{t-l}(b) + \sum_{l=1}^{t} \alpha_{l-1}\delta_N^l(X_i)\delta_N^{t-l}(b) \\
&= \sum_{l=0}^{t} \alpha_l' \delta_N^l(X_i)\delta_N^{t-l}(b),
\end{aligned}$$

where $\alpha_l' \in \mathbb{K}$, for all $l \in [\![0,t]\!]$, and $\alpha_t' = \alpha_{t-1} = 1$, by the induction hypothesis. This proves the inductive step and therefore the equality in (4.4) holds for all $t>0$. To see more clearly that the claim now follows, we observe that $\delta_N^t(R) \subseteq I$, for all $t>0$, and thus rewrite (4.4) to obtain

$$\delta_N^t(X_i)b = \delta_N^t(X_i b) - \sum_{l=0}^{t-1} \alpha_l \delta_N^l(X_i)\delta_N^{t-l}(b) \ \in \ I.$$

Claim 2: $J \subseteq I$.

Take some $x \in J$ and write it as

$$x = \sum_{\underline{t} \in \mathbb{N}^{N-1}\setminus\{\underline{0}\}} a_{\underline{t}} \delta_N^{t_1}(X_1)\cdots\delta_N^{t_{N-1}}(X_{N-1})b_{\underline{t}}$$

for some $a_{\underline{t}}, b_{\underline{t}} \in R$. For each $\underline{t} \in \mathbb{N}^{N-1}\setminus\{\underline{0}\}$ there exists a largest number $k(\underline{t}) \in [\![1,N-1]\!]$ such that $t_{k(\underline{t})} \neq 0$ and $t_l = 0$ for all $l > k(\underline{t})$. Using this, we rewrite the equation above as

$$x = \sum_{\underline{t} \in \mathbb{N}^{N-1}\setminus\{\underline{0}\}} a_{\underline{t}} \delta_N^{t_1}(X_1)\cdots\delta_N^{t_{k(\underline{t})}}(X_{k(\underline{t})})b_{\underline{t}}.$$

By Claim 1 we know that $\delta_N^{t_{k(\underline{t})}}(X_{k(\underline{t})})b_{\underline{t}} \in I$ for each $\underline{t} \in \mathbb{N}^{N-1} \backslash \{0\}$, therefore $x \in I$. This completes the proof of this claim and thus proves the equality $I = J$.

□

Combining Lemmas 4.26 and 4.27 we obtain:

Corollary 4.28. *For $A = \mathbb{K}[X_1]\cdots[X_N;\sigma_N,\delta_N] = R[X_N;\sigma_N,\delta_N]$ satisfying Hypothesis 1, we have*

$$\langle X_N \rangle = \sum_{t>0} RX_N^t + \langle \delta_N^t(X_i) \mid i \in [\![2,N-1]\!], t > 0 \rangle_R.$$

As in Section 3.2, we denote by $A^{(N)}$ the subalgebra of $\mathrm{Frac}(A)$ obtained after one step of the deleting derivation algorithm. Setting $(Y_1,\ldots,Y_N) := (X_1^{(N)},\ldots,X_N^{(N)})$ allows us to write this algebra as

$$A^{(N)} = \mathbb{K}\langle Y_1,\ldots,Y_N \rangle = \mathbb{K}[Y_1]\cdots[Y_{N-1};\sigma_{N-1},\delta_{N-1}][Y_N;\sigma_N^{(N+1)}],$$

where $\sigma_N^{(N+1)}(Y_i) = \lambda_{N,i} \in \mathbb{K}^*$, for all $i \in [\![1,N-1]\!]$, and

$$Y_i := \begin{cases} X_i & i = N; \\ \sum_{n=0}^{\infty} q_N^{\frac{n(n+1)}{2}}(q_N-1)^{-n}\lambda_{N,i}^{-n} d_{N,n}(X_i) X_N^{-n} & i < N. \end{cases}$$

Recall from Section 4.1 the surjective algebra homomorphism $g_N : A^{(N)} \to A/\langle X_N \rangle$, which takes $Y_i \mapsto X_i + \langle X_N \rangle$, for all $i \in [\![1,N]\!]$. Define subalgebras

$$R := \mathbb{K}\langle X_1,\ldots,X_{N-1} \rangle \subseteq A, \quad S := \mathbb{K}\langle Y_1,\ldots,Y_{N-1} \rangle \subseteq A^{(N)}$$

and observe, as in the proof of Lemma 4.4, that there exists an isomorphism, $f : S \to R$, sending $Y_i \mapsto X_i$, for all $i \in [\![1,N-1]\!]$.

Using the isomorphism, f, we calculate expressions for the generators of the kernel of the map g_N. As a corollary to this result, we obtain expressions for the generators of $\ker(g_j)$ for all $j \in [\![2,N]\!]$.

Proposition 4.29. *Keep the notation above. Suppose that $A = \mathbb{K}[X_1]\cdots[X_N;\sigma_N,\delta_N] = R[X_N;\sigma_N,\delta_N]$ satisfies Hypothesis 1 and that f is the algebra isomorphism defined above. Then we have*

$$\ker(g_N) = \langle Y_N, f^{-1}(\delta_N^t(f(Y_i))) \mid i \in [\![1,N-1]\!], t > 0 \rangle \lhd A^{(N)}.$$

Proof. Consider the following commutative diagram:

$$
\begin{tikzcd}
S \arrow[r, hook, two heads, "f"] \arrow[d, hook, "\rho_1"] & R \arrow[d, hook, "\rho_2"] \arrow[dd, bend left, two heads, "F_2"] \\
A^{(N)} \arrow[d, two heads, "\pi_1"] & A \arrow[d, two heads, "\pi_2"] \\
A^{(N)}/\langle Y_N\rangle \arrow[uu, bend left, two heads, "F_1"] \arrow[uur, dashed, two heads, "G"] \arrow[r, dashed, two heads, "\iota"] & A/\langle X_N\rangle
\end{tikzcd}
\tag{4.5}
$$

where ρ_1, ρ_2 are the natural embeddings of subalgebras and π_1, π_2 are the natural surjections. Using the commutation rule $Y_N Y_i = \lambda_{N,i} Y_i Y_N$, for all $i \in [\![1, N-1]\!]$, it is straightforward to verify that $\pi_1 \circ \rho_1$ is an isomorphism taking Y_i to $Y_i + \langle Y_N\rangle$. We label its inverse $F_1 := (\pi_1 \circ \rho_1)^{-1}$. The concatenation of the algebra homomorphisms $\pi_2 \circ \rho_2$ is a surjection, taking X_i to $X_i + \langle X_N\rangle$ for all $i \in [\![1, N-1]\!]$. We denote this map by $F_2 := \pi_2 \circ \rho_2$. With these maps, we may define the following algebra homomorphisms:

$$
\begin{aligned}
G := f \circ F_1 : A^{(N)}/\langle Y_N\rangle &\longrightarrow R \\
Y_i + \langle Y_N\rangle &\longmapsto X_i, \qquad (4.6) \\
\iota := F_2 \circ G : A^{(N)}/\langle Y_N\rangle &\longrightarrow A/\langle X_N\rangle \\
Y_i + \langle Y_N\rangle &\longmapsto X_i + \langle X_N\rangle. \qquad (4.7)
\end{aligned}
$$

As the concatenation of two isomorphisms, G, too, is an isomorphism. As the concatenation of an isomorphism with a surjection, ι is a surjection.

We now consider the following diagram:

$$
\begin{tikzcd}
A^{(N)} \arrow[r, two heads, "\pi_1"] \arrow[rrd, two heads, "g_N"'] & A^{(N)}/\langle Y_N\rangle \arrow[r, hook, two heads, "G"] \arrow[rd, two heads, "\iota"] & R \arrow[d, two heads, "F_2"] \\
 & & A/\langle X_N\rangle
\end{tikzcd}
\tag{4.8}
$$

This diagram also commutes, since $g_N = \iota \circ \pi_1$. We see that

$$
\ker(g_N) = \ker(\iota \circ \pi_1) = \pi_1^{-1}(\ker(\iota)) = \pi_1^{-1}(\ker(F_2 \circ G)) = \pi_1^{-1}(G^{-1}(\ker(F_2))), \tag{4.9}
$$

where $\ker(F_2) = \rho_2^{-1}(\ker(\pi_2)) = \rho_2^{-1}(\langle X_N \rangle_A) = \langle X_N \rangle_A \cap R$. We now apply Corollary 4.28 to obtain

$$\begin{aligned}\ker(F_2) = \langle X_N \rangle_A \cap R &= \left(\sum_{t>0} R X_N^t + \langle \delta_N^t(X_i) \mid i \in [\![1, N-1]\!], t > 0 \rangle_R \right) \cap R \\ &= \langle \delta_N^t(X_i) \mid i \in [\![1, N-1]\!], t > 0 \rangle_R,\end{aligned}$$

since $X_N \notin R$. Substituting this into the expression (4.9) for $\ker(g_N)$, and defining the inverses $G^{-1} = F_1^{-1} \circ f^{-1}$ and $F_1^{-1} = \pi_1 \circ \rho_1$, we obtain

$$\begin{aligned}\ker(g_N) &= \pi_1^{-1}(G^{-1}(\ker(F_2))) \\ &= \pi_1^{-1}(G^{-1}\left(\langle \delta_N^t(X_i) \mid i \in [\![1, N-1]\!], t > 0 \rangle_R\right) \\ &= \pi_1^{-1}\left(F_1^{-1} \circ f^{-1}(\langle \delta_N^t(X_i) \mid i \in [\![1, N-1]\!], t > 0 \rangle_R)\right) \\ &= \pi_1^{-1} \circ F_1^{-1}\left(\langle f^{-1}(\delta_N^t(X_i)) \mid i \in [\![1, N-1]\!], t > 0 \rangle_S\right) \\ &= \pi_1^{-1}\left(\langle f^{-1}(\delta_N^t(X_i)) \mid i \in [\![1, N-1]\!], t > 0 \rangle_S / \langle Y_N \rangle\right) \\ &= \langle Y_N, f^{-1}(\delta_N^t(X_i)) \mid i \in [\![1, N-1]\!], t > 0 \rangle_{A^{(N)}} \\ &= \langle Y_N, f^{-1}(\delta_N^t(f(Y_i))) \mid i \in [\![1, N-1]\!], t > 0 \rangle_{A^{(N)}},\end{aligned}$$

as desired. $\square$

Fixing some $j \in [\![2, N]\!]$, we now consider the algebras $A^{(j+1)} := \mathbb{K}\langle X_1^{(j+1)}, \ldots, X_N^{(j+1)} \rangle$ and $A^{(j)} := \mathbb{K}\langle X_1^{(j)}, \ldots, X_N^{(j)} \rangle$, defined in Section 3.2. Define subalgebras of these,

$$\begin{aligned}R_j &:= \mathbb{K}\langle X_1^{(j+1)}, \ldots, X_{j-1}^{(j+1)}, X_{j+1}^{(j+1)}, \ldots, X_N^{(j+1)} \rangle \subseteq A^{(j+1)}, \\ S_j &:= \mathbb{K}\langle X_1^{(j)}, \ldots, X_{j-1}^{(j)}, X_{j+1}^{(j)}, \ldots, X_N^{(j)} \rangle \subseteq A^{(j)},\end{aligned}$$

and let $f_j : S_j \to R_j$ be the algebra homomorphism which takes $X_i^{(j)}$ to $X_i^{(j+1)}$, for all $i \in [\![1, N]\!] \setminus \{j\}$.

Corollary 4.30. *For each $j \in [\![2, N]\!]$ the algebra homomorphism f_j, defined above, is an isomorphism, and we have*

$$\ker(g_j) = \langle X_j^{(j)}, f_j^{-1}(\delta_j^t(f_j(X_i^{(j)}))) \mid i \in [\![1, j-1]\!], t > 0 \rangle \lhd A^{(j)}.$$

Proof. The case when $j = N$ is shown in Proposition 4.29.

Fix some $j \in [\![2,N-1]\!]$. Presenting $A^{(j+1)}$ and $A^{(j)}$ as iterated Ore extensions gives

$$A^{(j+1)} = \mathbb{K}[X_1^{(j+1)}]\cdots[X_j^{(j+1)};\sigma_j,\delta_j][X_{j+1}^{(j+1)};\sigma_{j+1}^{(j+1)}]\cdots[X_N^{(j+1)};\sigma_N^{(j+1)}],$$
$$A^{(j)} = \mathbb{K}[X_1^{(j)}]\cdots[X_{j-1}^{(j)};\sigma_{j-1},\delta_{j-1}][X_j^{(j)};\sigma_j^{(j)}]\cdots[X_N^{(j)};\sigma_N^{(j)}],$$

which may be rewritten, using Lemma 3.5, as

$$\begin{aligned}A^{(j+1)} &= \mathbb{K}[X_1^{(j+1)}]\cdots[X_{j-1}^{(j+1)};\sigma_{j-1},\delta_{j-1}][X_{j+1}^{(j+1)};\sigma_{j+1}^{(j+1)*}]\cdots[X_N^{(j+1)};\sigma_N^{(j+1)*}][X_j^{(j+1)};\sigma_j',\delta_j']\\ &= R_j[X_j^{(j+1)};\sigma_j',\delta_j'],\\ A^{(j)} &= \mathbb{K}[X_1^{(j)}]\cdots[X_{j-1}^{(j)};\sigma_{j-1},\delta_{j-1}][X_{j+1}^{(j)};\sigma_{j+1}^{(j)*}]\cdots[X_N^{(j)};\sigma_N^{(j)*}][X_j^{(j)};\sigma_j^{(j)\prime}]\\ &= S_j[X_j^{(j)};\sigma_j^{(j)\prime},]\end{aligned}$$

for $\mathbb{K}$-algebras R_j, S_j. The map $f_j : S_j \to R_j$ is seen to be an isomorphism by noting that we have $\mathbb{K}\langle X_1^{(j)},\ldots,X_{j-1}^{(j)}\rangle \cong \mathbb{K}\langle X_1^{(j+1)},\ldots,X_{j-1}^{(j+1)}\rangle$, from Lemma 4.4, to which we may adjoin the additional Ore extensions to both sides. An application of the universal property of Ore extensions then completes the proof that f_j is an isomorphism.

Since $A^{(j+1)} = R_j[X_j^{(j+1)};\sigma_j',\delta_j']$ is an iterated Ore extension satisfying Hypothesis 1, and f_j is an algebra isomorphism, we may apply Proposition 4.29 to see that

$$\ker(g_j) = \langle X_j^{(j)}, f_j^{-1}(\delta_j'^t(f_j(X_i^{(j)}))) \mid i \in [\![1,j-1]\!], t>0\rangle \lhd A^{(j)}.$$

Upon noting that $\delta_j'(X_i^{(j+1)}) = \delta_j(X_i^{(j+1)})$, for all $i \in [\![1,j-1]\!]$, the result of the corollary then follows.

□

4.5 Special Application: Quantum Matrices

In this section we provide a combinatorial description of the set of Cauchon diagrams $\mathscr{W}'$ (defined in Section 4.1.2) associated to $A = \mathscr{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$, the algebra of multiparameter quantum $m\times n$ matrices. We utilise Corollary 4.30 to calculate the kernel of the appropriately defined map g_j for quantum matrices, which allows for Propositions 4.34 and 4.36 to be proved in much the same way as in the analogous results for matrix Poisson varieties [Lec14, Propositions 7.3.5 and 7.3.7]. Theorem 4.37 then follows naturally.

4.5.1 Cauchon-Le diagrams

Before we see the deleting derivations algorithm applied to A, we first set up a correspondence linking the set $\mathscr{W} = \mathbb{P}(\{(1,1),(1,2),\ldots,(m,n)\})$ with $m \times n$ *diagrams* ($m \times n$ grids with each square coloured either black or white). As in Section 4.1.2, we let $\mathscr{W}' \subseteq \mathscr{W}$ denote the elements $w \in \mathscr{W}$ such that $\text{C.Spec}_w(A) \neq \emptyset$. We denote by G the set of all $m \times n$ diagrams. There is a bijection

$$\begin{aligned} \xi : \mathscr{W} &\longrightarrow G \\ w &\longmapsto C_w \end{aligned}$$

where C_w is the $m \times n$ diagram whose square in position (i,j) is coloured black if and only if $(i,j) \in w$. Figure 4.1 shows two examples of 3×4 diagrams, C_w and $C_{w'}$, corresponding to $w = \{(1,4),(2,2),(3,1),(3,2),(3,3)\}$ and $w' = \{(1,2),(1,4),(2,2),(3,1),(3,2),(3,3)\}$, respectively. For $C \in G$, we denote by w_C the pre-image of C under ξ, i.e. $w_C := \xi^{-1}(C)$. We investigate the effect this map has on the set of Cauchon diagrams $\mathscr{W}'$. To do this, we denote by $\mathscr{G} \subset G$ the subset of $m \times n$ diagram which satisfies the *Cauchon property*; that is, given a black square in the diagram, either all squares strictly above it are black or all squares strictly to the left are black. In Figure 4.1, the diagram $C_{w'}$ on the right satisfies the Cauchon property but the one on the left, C_w, does not.

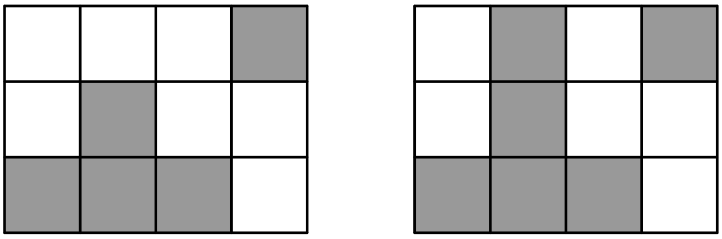

Fig. 4.1 Left diagram: $C_w \in G \backslash \mathscr{G}$. Right diagram: $C_{w'} \in \mathscr{G}$.

Definition 4.31. An $m \times n$ diagram satisfying the Cauchon property defined above is called a *Cauchon-Le diagram*.

4.5.2 Applying the Deleting Derivations Algorithm

In order to show the bijection $\mathscr{W}' \to \mathscr{G}$, we apply the deleting derivations algorithm to $A := \mathscr{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$ and use results in Section 4.1 to determine the set $\mathscr{W}'$ explicitly.

First, we set up some notation which will be used in the calculations to come. For some natural number n, set $[n] := \{1,\dots,n\}$. For $(u,v) \in ([m]\times[n])\setminus\{(1,1),(m,n)\}$ we denote by $(u,v)^-$ (respectively $(u,v)^+$) the largest (respectively smallest) element of $[m]\times[n]$ which is smaller (respectively larger) than (u,v) with respect to the lexicographic ordering. We set $(1,1)^+ := (1,2)$, $(m,n)^- := (m,n-1)$ and $(m,n)^+ := (m,n+1)$. We may sometimes also write $(u,v)^-$ for (u^-,v^-) (and similarly $(u,v)^+$ for (u^+,v^+)).

Recall the definition of the algebra of quantum $m\times n$ matrices, A, from Chapter 2 and its formulation as an iterated Ore extension (with generators in lexicographic order),

$$A := \mathbb{K}[X_{1,1}][X_{1,2};\sigma_{1,2},\delta_{1,2}]\cdots[X_{m,n};\sigma_{m,n},\delta_{m,n}],$$

with maps defined on $X_{s,t}$, for all $(1,1)\le(s,t)<(i,j)$, as

$$\sigma_{i,j}(X_{s,t}) = \begin{cases} p_{i,s}p_{t,j}X_{s,t} & (i>s,\ j\neq t);\\ \lambda p_{i,s}p_{t,j}X_{s,t} & (i>s,\ j=t);\\ p_{t,j}X_{s,t} & (i=s,\ j>t),\end{cases}$$

$$\delta_{i,j}(X_{s,t}) = \begin{cases} (\lambda-1)p_{i,s}X_{s,j}X_{i,t} & (i>s,\ j>t);\\ 0 & \text{otherwise,}\end{cases}$$

where $1\neq\lambda\in\mathbb{K}^*$ and $\mathbf{p}=(p_{i,j})\in M_{m,n}(\mathbb{K}^*)$ is multiplicatively antisymmetric. Define subalgebras, for all $(1,1)\le(i,j)\le(m,n)$, as

$$A_{(i,j)} := \mathbb{K}[X_{1,1}][X_{1,2};\sigma_{1,2},\delta_{1,2}]\cdots[X_{i,j};\sigma_{i,j},\delta_{i,j}].$$

Haynal shows [Hay08, Example 5.3] that each $(\sigma_{i,j},\delta_{i,j})$ is λ^{-1}-skew and $\delta_{i,j}^2=0$ for all (i,j), and therefore that A satisfies Hypothesis 1. Furthermore, she verifies that each $\delta_{i,j}$ extends to an iterative, locally nilpotent h.λ^{-1}-s.$\sigma_{i,j}$-d., $\{d_{(i,j),n}\}_n$, on the appropriate $\mathbb{K}$-subalgebra, $A_{(i,j)^-}\subseteq A$, by showing that the conditions of [Hay08, Theorem 2.8] hold for A. In particular, there exists a torsion-free $\mathbb{K}[\varepsilon^{\pm1}]$-algebra,

$$R = \mathbb{K}[\varepsilon^{\pm1}][X_{1,1}][X_{1,2};\sigma'_{1,2},\delta'_{1,2}]\cdots[X_{m,n};\sigma'_{m,n},\delta'_{m,n}],$$

such that $R/\langle\varepsilon-\lambda^{-1}\rangle\cong A$, with maps $\sigma'_{i,j}$ and $\delta'_{i,j}$ reducing to $\sigma_{i,j}$ and $\delta_{i,j}$ respectively. Furthermore, it is shown that $(\delta'_{i,j})^n(R_{(i,j)^-})\subseteq(n)!_\varepsilon R_{(i,j)^-}$, for all (i,j) and n, thus allowing a well-defined sequence of maps to be defined on the appropriate subalgebra $R_{(i,j)^-} \subseteq R$ as

$$d'_{(i,j),n} := \frac{(\delta'_{i,j})^n}{(n)!_\varepsilon}, \tag{4.10}$$

for all $n \geq 0$. The sequence $\{d'_{(i,j),n}\}_n$ forms a h.ε-s.$\sigma'_{i,j}$-d. on $R_{(i,j)^-}$ and induces a h.λ^{-1}-s.$\sigma_{i,j}$-d., $\{d_{(i,j),n}\}_n$, on $A_{(i,j)^-}$ via the quotient map.

We now take λ to be a primitive ℓ^{th} root of unity and all the entries $p_{i,j}$ to be roots of unity, so that A is a PI-algebra. Then, using (4.10) above, for $n < \ell$ we obtain

$$d_{(i,j),n} := \frac{\delta^n_{i,j}}{(n)!_{\lambda^{-1}}}. \tag{4.11}$$

In fact, we are able to deduce expressions for $d_{(i,j),n}(X_{s,t})$, for each $(1,1) \leq (s,t) < (i,j)$ and all $n \geq 0$, using the property $(\delta'_{i,j})^2 = \delta^2_{i,j} = 0$, for all $(1,1) \leq (i,j) \leq (m,n)$. More specifically, this property allows us to write, for all $(1,1) \leq (s,t) < (i,j)$ and $n \geq 2$,

$$d'_{(i,j),n}(X_{s,t}) = \frac{(\delta'_{i,j})^n}{(n)!_\varepsilon} = 0,$$

and from this we deduce that $d_{(i,j),n}(X_{s,t}) = 0$, for all $(1,1) \leq (s,t) < (i,j)$ and $n \geq 2$. Moreover,

$$\delta_{i,j}(X_{s,t}) = \begin{cases} (\lambda - 1)p_{i,s}X_{s,j}X_{i,t} & i > s \text{ and } j > t; \\ 0 & s = i \text{ or } t = j. \end{cases}$$

Thus, if $i > s$ and $j > t$, we obtain

$$d_{(i,j),n}(X_{s,t}) = \begin{cases} X_{s,t} & n = 0; \\ (\lambda - 1)p_{i,s}X_{s,j}X_{i,t} & n = 1; \\ 0 & n > 1, \end{cases} \tag{4.12}$$

and, if either $s = i$ or $t = j$, then $d_{(i,j),0}(X_{s,t}) = X_{s,t}$ and $d_{(i,j),n}(X_{s,t}) = 0$ for all $n > 0$.

Applying the deleting derivations algorithm to A gives, for each $(1,2) \leq (u,v) \leq (m,n)^+$, a subalgebra $A^{(u,v)}$ of $\mathrm{Frac}(A)$ generated by $X^{(u,v)}_{1,1}, \ldots, X^{(u,v)}_{m,n}$ which, when arranged in a matrix, gives the *matrix of generators for* $A^{(u,v)}$, $(X^{(u,v)}_{s,t})_{s,t} \in M_{m,n}(\mathrm{Frac}(A))$. We may write $A^{(u,v)}$ as an iterated Ore extension,

$$A^{(u,v)} = \mathbb{K}[X^{(u,v)}_{1,1}] \cdots [X^{(u,v)}_{u^-,v^-}; \sigma_{u^-,v^-}, \delta_{u^-,v^-}][X^{(u,v)}_{u,v}; \sigma^{(u,v)}_{u,v}] \cdots [X^{(u,v)}_{m,n}; \sigma^{(u,v)}_{m,n}],$$

for suitable maps $\sigma_{i,j}^{(u,v)}$, defined in Theorem 3.6. When the matrix $(X_{s,t}^{(u,v)^+})_{s,t}$ is known, we obtain the entries of $(X_{s,t}^{(u,v)})_{s,t}$ by setting

$$X_{s,t}^{(u,v)} := \begin{cases} X_{s,t}^{(u,v)^+} & (s,t) \geq (u,v); \\ X_{s,t}^{(u,v)^+} & (s,t) < (u,v) \text{ and } (s = u \text{ or } t = v); \\ X_{s,t}^{(u,v)^+} - p_{u,s} X_{s,v}^{(u,v)^+} X_{u,t}^{(u,v)^+} (X_{u,v}^{(u,v)^+})^{-1} & s < u \text{ and } t < v. \end{cases} \tag{4.13}$$

This is simply the result of applying one step of the deleting derivations algorithm to $A^{(u,v)^+}$. More precisely, we see immediately from (3.13) that if $(s,t) \geq (u,v)$ then $X_{s,t}^{(u,v)} = X_{s,t}^{(u,v)^+}$. When $(s,t) < (u,v)$ then, by (3.13), we write

$$X_{s,t}^{(u,v)} = \sum_{n=0}^{\infty} \lambda^{\frac{-n(n+1)}{2}} (\lambda^{-1} - 1)^{-n} d_{(u,v),n} \circ \sigma_{(u,v)}^{-n} (X_{s,t}^{(u,v)^+})(X_{u,v}^{(u,v)^+})^{-n}.$$

Using properties of $d_{(u,v),n}$ defined above we see that, if $s = u$ or $t = v$, then $d_{(u,v),1}(X_{s,t}^{(u,v)^+}) = \delta_{u,v}(X_{s,t}^{(u,v)^+}) = 0$ and the identity above reduces to $X_{s,t}^{(u,v)} = X_{s,t}^{(u,v)^+}$. If, however, $s < u$ and $t < v$ then, by (4.12), the identity becomes

$$\begin{aligned} X_{s,t}^{(u,v)} &= \sum_{n=0}^{\infty} \lambda^{\frac{-n(n+1)}{2}} (\lambda^{-1} - 1)^{-n} d_{(u,v),n} \circ \sigma_{(u,v)}^{-n} (X_{s,t}^{(u,v)^+})(X_{u,v}^{(u,v)^+})^{-n} \\ &= X_{s,t}^{(u,v)^+} - (\lambda - 1)^{-1} [(\lambda - 1) p_{u,s} X_{s,v}^{(u,v)^+} X_{u,t}^{(u,v)^+}] (X_{u,v}^{(u,v)^+})^{-1} \\ &= X_{s,t}^{(u,v)^+} - p_{u,s} X_{s,v}^{(u,v)^+} X_{u,t}^{(u,v)^+} (X_{u,v}^{(u,v)^+})^{-1}. \end{aligned}$$

This shows that $X_{s,t}^{(u,v)}$ is as defined in (4.13).

We set $A' := A^{(1,2)}$, with matrix of generators denoted by $(T_{i,j})_{i,j} := (X_{i,j}^{(1,2)})_{i,j}$, and we observe that A' is a quantum affine space in mn indeterminates. Finally, for each $w \in \mathscr{W}'$ we associate a completely prime ideal in A', which we define as

$$J_w := \langle T_{i,j} \mid (i,j) \in w \rangle \in \text{C.Spec}_w(A').$$

4.5.3 Combinatorial Description of Cauchon Diagrams, $\mathscr{W}'$

We now recall the maps defined in Section 4.1, updating the notation for $A = \mathscr{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$: Let $(1,2) \leq (u,v) \leq (m,p)$. By Proposition 4.6, there exists an injective map $\psi_{(u,v)}$:

$\mathrm{C.Spec}(A^{(u,v)^+}) \to \mathrm{C.Spec}(A^{(u,v)})$, defined on $P \in \mathrm{C.Spec}(A^{(u,v)^+})$ as

$$\psi_{(u,v)}(P) = \begin{cases} PS_{(u,v)}^{-1} \cap A^{(u,v)} & X_{u,v}^{(u,v)} \notin P; \\ g_{(u,v)}^{-1}(P/\langle X_{u,v}^{(u,v)^+}\rangle) & X_{u,v}^{(u,v)} \in P, \end{cases} \tag{4.14}$$

where $S_{(u,v)} \subset A^{(u,v)^+}$ is the multiplicatively closed set generated by $X_{u,v}^{(u,v)^+} = X_{u,v}^{(u,v)}$, and $g_{(u,v)}$ is the surjective algebra homomorphism

$$\begin{aligned} g_{(u,v)} : A^{(u,v)} &\longrightarrow A^{(u,v)^+}/\langle X_{u,v}^{(u,v)^+}\rangle \\ X_{i,j}^{(u,v)} &\longmapsto X_{i,j}^{(u,v)^+} + \langle X_{u,v}^{(u,v)^+}\rangle. \end{aligned}$$

The canonical embedding becomes $\psi : \mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$ to be $\psi := \psi_{(1,2)} \circ \cdots \circ \psi_{(m,n)}$. We also define injective maps $f_{(u,v)} : \mathrm{C.Spec}(A^{(u,v)^+}) \to \mathrm{C.Spec}(A')$, for all $(1,1) \le (u,v) \le (m,n)$, by setting $f_{(1,1)} := \mathrm{Id}_{\mathrm{C.Spec}(A')}$ and $f_{(u,v)} := f_{(u,v)^-} \circ \psi_{(u,v)}$. Then, for some $Q \in \mathrm{Im}(\psi)$ we set $P_{(u,v)} := f_{(u,v)^-}^{-1}(Q) \in \mathrm{C.Spec}(A^{(u,v)})$, for all $(1,2) \le (u,v) \le (m,n)^+$, and let $P := P_{(m,n)^+} \in \mathrm{C.Spec}(A)$.

To aid with notation in the next results, we provide the following definition:

Definition 4.32. Let R be a $\mathbb{K}$-algebra. The matrix $M = (m_{i,j}) \in M_{m,n}(R)$ is a *Cauchon matrix* if, for all $(i,j) \in [m] \times [n]$ we have

$$(m_{i,j} = 0) \iff (m_{k,j} = 0 \text{ for all } k \le i, \text{ or } m_{i,l} = 0 \text{ for all } l \le j).$$

Remark 4.33. Each Cauchon matrix M corresponds to a unique Cauchon-Le diagram $C_w \in \mathscr{G}$ via the rule that the square in position (i,j) is black if and only if $m_{i,j} = 0$.

Proposition 4.34. *Let ξ and $\mathscr{G}$ be as defined in Section 4.5.1. Then $\xi(\mathscr{W}') \subseteq \mathscr{G}$.*

Proof. Let $w \in \mathscr{W}'$ and $\xi(w) = C_w \in G$ be the associated diagram. By the definition of $\mathscr{W}'$, there must exist some ideal $P_w \in \mathrm{Im}(\psi) \cap \mathrm{C.Spec}_w(A')$, so that we have $T_{i,j} \in P_w$ if and only if $(i,j) \in w$, for each generator $T_{i,j} \in A'$. Consider the ring of matrices $M_{m,n}(A'/P_w)$, which contains all $m \times n$ matrices with entries of the form $a_{i,j} + P_w \in A'/P_w$, where $a_{i,j} \in A'$. Note that $T_{i,j} + P_w = 0$ if and only if $T_{i,j} \in P_w$, which is true if and only if $(i,j) \in w$. Hence,

$$C_w \in \mathscr{G} \iff (T_{i,j} + P_w)_{i,j} \text{ is a Cauchon matrix.}$$

For each $(1,1) \le (u,v) \le (m,n)^+$, we set ideals $P_{(u,v)} := f_{(u,v)^-}^{-1}(P_w) \in \mathrm{C.Spec}(A^{(u,v)})$, per the discussion at the beginning of this subsection with $Q = P_w$. We prove, by decreasing induction on $(1,2) \leq (u,v) \leq (m,n)^+$, that the matrix $(X_{i,j}^{(u,v)} + P_{(u,v)})_{i,j} \in M_{m,n}(A^{(u,v)}/P_{(u,v)})$ is a Cauchon matrix. The result will then follow from the case $(u,v) = (1,2)$.

First, suppose that $X_{i,j}^{(m,n)^+} \in P_{(m,n)^+}$. Then, for all $s < i$ and $t < j$,

$$X_{i,j}^{(m,n)^+} X_{s,t}^{(m,n)^+} - p_{i,s}p_{t,j} X_{s,t}^{(m,n)^+} X_{i,j}^{(m,n)^+} = (\lambda - 1) p_{i,s} X_{s,j}^{(m,n)^+} X_{i,t}^{(m,n)^+} \in P_{(m,n)^+}.$$

Since $P_{(m,n)^+} = \psi^{-1}(P_w)$ is a completely prime ideal, either $X_{s,j}^{(m,n)^+} \in P_{(m,n)^+}$ or $X_{i,t}^{(m,n)^+} \in P_{(m,n)^+}$. We may assume, without loss of generality, that $X_{i,t}^{(m,n)^+} \notin P_{(m,n)^+}$. Then, for all $r < i$,

$$X_{i,j}^{(m,n)^+} X_{r,t}^{(m,n)^+} - p_{i,r}p_{t,j} X_{r,t}^{(m,n)^+} X_{i,j}^{(m,n)^+} = (\lambda - 1) p_{i,r} X_{r,j}^{(m,n)^+} X_{i,t}^{(m,n)^+} \in P_{(u,v)^+}.$$

Using the completely prime property of $P_{(u,v)^+}$ again, along with the assumption $X_{i,t}^{(m,n)^+} \notin P_{(m,n)^+}$, we conclude that $X_{r,j}^{(m,n)^+} \in P_{(u,v)^+}$ for all $r < i$. Therefore $(X_{i,j}^{(m,n)^+} + P_{(m,n)^+})_{i,j}$ is a Cauchon matrix. This proves the base case. Note that if we were to assume $X_{s,j}^{(m,n)^+} \notin P_{(m,n)^+}$ instead then we would conclude that $X_{i,r}^{(m,n)^+} \in P_{(u,v)^+}$, for all $r < j$, and hence $(X_{i,j}^{(m,n)^+} + P_{(m,n)^+})_{i,j}$ is still a Cauchon matrix.

For the induction step, assume that $(X_{i,j}^{(u,v)^+} + P_{(u,v)^+})_{i,j} \in M_{m,n}(A^{(u,v)^+}/P_{(u,v)^+})$ is a Cauchon matrix, for some $(1,1) < (u,v) \leq (m,n)$, and consider the matrix $(X_{i,j}^{(u,v)} + P_{(u,v)})_{i,j} \in M_{m,n}(A^{(u,v)}/P_{(u,v)})$. Suppose $X_{i,j}^{(u,v)} \in P_{(u,v)}$, for some $(1,1) \leq (i,j) \leq (m,n)$. The proof now splits into three cases:

Case 1: If $(i,j) < (u,v)$, we observe that

$$X_{i,j}^{(u,v)} X_{s,t}^{(u,v)} - p_{i,s}p_{t,j} X_{s,t}^{(u,v)} X_{i,j}^{(u,v)} = (\lambda - 1) p_{i,s} X_{s,j}^{(u,v)} X_{i,t}^{(u,v)} \in P_{(u,v)},$$

for all $s < i$ and $t < j$. This allows us to conclude Case 1 using the same method as the base case.

Case 2: If $(i,j) = (u,v)$ then our supposition becomes $X_{u,v}^{(u,v)} \in P_{(u,v)}$ and, by Lemma 4.13, this implies that $X_{u,v}^{(u,v)^+} \in P_{(u,v)^+}$. We may apply Lemma 4.19 to obtain an isomorphism

$$\begin{aligned} A^{(u,v)}/P_{(u,v)} &\longrightarrow A^{(u,v)^+}/P_{(u,v)^+} \\ X_{s,t}^{(u,v)} + P_{(u,v)} &\longmapsto X_{s,t}^{(u,v)^+} + P_{(u,v)^+}. \end{aligned}$$

From this we see that $X_{s,t}^{(u,v)} \in P_{(u,v)}$ if and only if $X_{s,t}^{(u,v)^+} \in P_{(u,v)^+}$. The inductive hypothesis can now be applied, which states that either $X_{i,t}^{(u,v)^+} \in P_{(u,v)^+}$, for all $t \leq j$, or $X_{s,j}^{(u,v)^+} \in P_{(u,v)^+}$, for all $s \leq i$. This concludes the proof of this case.

Case 3: If $(i,j) > (u,v)$, we need to distinguish between two further cases.

Case 3.1: If $X_{u,v}^{(u,v)^+} \in P_{(u,v)^+}$ then we may apply Lemma 4.19 and proceed with the same method as used in Case 2.

Case 3.2: If $X_{u,v}^{(u,v)^+} \notin P_{(u,v)^+}$ then, since $(i,j) > (u,v)$, we have $X_{i,j}^{(u,v)^+} = X_{i,j}^{(u,v)}$. Furthermore, by Lemma 4.3, $\psi_{(u,v)}(P_{(u,v)^+}) = P_{(u,v)}S_{(u,v)}^{-1} \cap A^{(u,v)}$. Therefore,

$$\begin{aligned} X_{i,j}^{(u,v)^+} = X_{i,j}^{(u,v)} \in P_{(u,v)} \cap A^{(u,v)^+} &\subseteq P_{(u,v)}S_{(u,v)}^{-1} \cap A^{(u,v)^+} \\ &= P_{(u,v)^+}S_{(u,v)}^{-1} \cap A^{(u,v)^+} \\ &= P_{(u,v)^+}. \end{aligned}$$

The inductive hypothesis states that either $X_{s,j}^{(u,v)^+} \in P_{(u,v)^+}$, for all $s \leq i$, or $X_{i,t}^{(u,v)^+} \in P_{(u,v)^+}$, for all $t \leq j$. We use this to state the same result for $X_{s,j}^{(u,v)}, X_{i,t}^{(u,v)} \in P_{(u,v)}$. Note that, since $(i,j) > (u,v)$, we have $i \geq u$ but not necessarily $j \geq v$, which means that $(i,t) \geq (u,v)$ but we cannot say the same for (s,j). Thus these two cases are not symmetric and need to be checked separately.

Case 3.2.1: Suppose $X_{i,t}^{(u,v)^+} \in P_{(u,v)^+}$ for all $t \in [\![1,j]\!]$. Then, either $(i,t) \geq (u,v)$, or $(i,t) < (u,v)$ and $i = u$. In both cases we use (4.13) and the inductive hypothesis to deduce that

$$X_{i,t}^{(u,v)} = X_{i,t}^{(u,v)^+} \in P_{(u,v)^+} \cap A^{(u,v)} \subseteq P_{(u,v)}.$$

Case 3.2.2: Suppose $X_{s,j}^{(u,v)^+} \in P_{(u,v)^+}$ for all $s \in [\![1,i]\!]$. Then, by (4.13),

$$X_{s,j}^{(u,v)} = \begin{cases} X_{s,j}^{(u,v)^+} & \text{for } s \geq u \text{ or } j \geq v; \\ X_{s,j}^{(u,v)^+} - p_{u,s}X_{s,v}^{(u,v)^+}X_{u,j}^{(u,v)^+}(X_{u,v}^{(u,v)^+})^{-1} & \text{for } s < u \text{ and } j < v. \end{cases}$$

If $s \geq u$ or $j \geq u$ then $X_{s,j}^{(u,v)} \in P_{(u,v)}$ and we conclude as in Case 3.2.1. Otherwise, if $s < u$ and $j < v$ then $X_{s,j}^{(u,v)^+}$ and $X_{u,j}^{(u,v)^+}$ both belong to $P_{(u,v)^+}$ because $s,u \leq i$. Therefore, we conclude by observing that

$$X_{s,j}^{(u,v)} = X_{s,j}^{(u,v)^+} - p_{u,s}X_{s,v}^{(u,v)^+}X_{u,j}^{(u,v)^+}(X_{u,v}^{(u,v)^+})^{-1} \in P_{(u,v)^+}S_{(u,v)}^{-1} \cap A^{(u,v)} = P_{(u,v)}.$$

All three cases have been proved and thus the induction is complete. $\square$

In order to show the converse statement, $\xi^{-1}(\mathscr{G}) \subseteq \mathscr{W}'$, we need to know the explicit form for the kernel of $g_{(u,v)}$. This arises as a consequence of Corollary 4.30.

Lemma 4.35. *Let $A := \mathscr{O}_{\lambda,\mathbf{p}}(M_{m,n}(\mathbb{K}))$ and $(1,2) \leq (u,v) \leq (m,n)$. Then,*

$$\ker(g_{(u,v)}) = \langle X_{u,v}^{(u,v)}, X_{s,v}^{(u,v)} X_{u,t}^{(u,v)} \mid s \in [\![1,u-1]\!] \text{ and } t \in [\![1,v-1]\!] \rangle.$$

Proof. Using Lemma 3.5, we may rewrite the algebras $A^{(u,v)^+}$ and $A^{(u,v)}$ as

$$\begin{aligned}
A^{(u,v)^+} &= \mathbb{K}[X_{1,1}^{(u,v)^+}]\cdots[X_{u,v}^{(u,v)^+};\sigma_{(u,v)},\delta_{(u,v)}][X_{u^+,v^+}^{(u,v)^+};\sigma_{(u,v)^+}^{(u,v)^+}]\cdots[X_{m,n}^{(u,v)^+};\sigma_{(m,n)}^{(u,v)^+}] \\
&= R[X_{u,v}^{(u,v)^+};\sigma'_{(u,v)},\delta'_{(u,v)}], \\
A^{(u,v)} &= \mathbb{K}[X_{1,1}^{(u,v)}]\cdots[X_{u^-,v^-}^{(u,v)};\sigma_{(u,v)^-},\delta_{(u,v)^-}][X_{u,v}^{(u,v)};\sigma_{(u,v)}^{(u,v)}]\cdots[X_{m,n}^{(u,v)};\sigma_{(m,n)}^{(u,v)}] \\
&= S[X_{u,v}^{(u,v)};\sigma_{(u,v)}^{(u,v)'}],
\end{aligned}$$

where

$$\begin{aligned}
R &= \mathbb{K}\langle X_{1,1}^{(u,v)^+},\ldots,X_{u^-,v^-}^{(u,v)^+},X_{u^+,v^+}^{(u,v)^+},\ldots,X_{m,n}^{(u,v)^+}\rangle \subseteq A^{(u,v)^+}, \\
S &= \mathbb{K}\langle X_{1,1}^{(u,v)},\ldots,X_{u^-,v^-}^{(u,v)},X_{u^+,v^+}^{(u,v)},\ldots,X_{m,n}^{(u,v)}\rangle \subseteq A^{(u,v)},
\end{aligned}$$

and, for all $(s,t) < (u,v)$,

- $\sigma'_{(u,v)}(X_{s,t}^{(u,v)^+}) = \sigma_{(u,v)}(X_{s,t}^{(u,v)^+})$ and $\delta'_{(u,v)}(X_{s,t}^{(u,v)^+}) = \delta_{(u,v)}(X_{s,t}^{(u,v)^+})$;
- $\sigma_{(u,v)}^{(u,v)'}(X_{s,t}^{(u,v)}) = \sigma_{(u,v)}^{(u,v)}(X_{s,t}^{(u,v)})$;

and, for all $(s,t) > (u,v)$,

- $\sigma'_{(u,v)}(X_{s,t}^{(u,v)^+}) = \sigma_{(s,t)}^{-1}(X_{u,v}^{(u,v)^+})$ and $\delta'_{(u,v)}(X_{s,t}^{(u,v)^+}) = 0$;
- $\sigma_{(u,v)}^{(u,v)'}(X_{s,t}^{(u,v)}) = \sigma_{(s,t)}^{(u,v)^{-1}}(X_{u,v}^{(u,v)})$.

By applying Corollary 4.30 to this setting we obtain an algebra isomorphism $f : S \to R$ taking $X_{i,j}^{(u,v)} \mapsto X_{i,j}^{(u,v)^+}$, for all $(i,j) \in ([m]\times[n])\backslash\{(u,v)\}$, and

$$\ker(g_{(u,v)}) = \langle X_{u,v}^{(u,v)}, f^{-1}(\delta_{(u,v)}^t(f(X_{s,t}^{(u,v)}))) \mid (1,1) \leq (s,t) < (u,v),\ t > 0\rangle.$$

The observation $\delta_{(u,v)}^2 = 0$ allows us to simplify this to

$$\ker(g_{(u,v)}) = \langle X_{u,v}^{(u,v)}, f^{-1}(\delta_{(u,v)}(f(X_{s,t}^{(u,v)}))) \mid (1,1) \leq (s,t) < (u,v) \rangle.$$

The result then follows from this since, for all $(1,1) \leq (s,t) < (u,v)$, if $s \in [\![1, u-1]\!]$ and $t \in [\![1, v-1]\!]$ then

$$f^{-1}(\delta_{(u,v)}(f(X_{s,t}^{(u,v)}))) = f^{-1}((\lambda - 1)p_{i,s}X_{s,v}^{(u,v)^+}X_{u,t}^{(u,v)^+}) = (\lambda - 1)p_{i,s}X_{s,v}^{(u,v)}X_{u,t}^{(u,v)};$$

otherwise,

$$f^{-1}(\delta_{(u,v)}(f(X_{s,t}^{(u,v)}))) = f^{-1}(0) = 0.$$

$\square$

Proposition 4.36. *Let ξ and $\mathscr{G}$ be as defined in Section 4.5.1. Then $\xi^{-1}(\mathscr{G}) \subseteq \mathscr{W}'$.*

Proof. Let $C \in \mathscr{G}$ and define $w_C := \xi^{-1}(C)$. To show $w_C \in \mathscr{W}'$ it is sufficient to show that $J_{w_C} := \langle T_{i,j} \mid (i,j) \in w_C \rangle \in \mathrm{Im}(\psi)$ since then, by Proposition 4.18, any ideal $I \in \mathrm{C.Spec}(A')$ with $J_{w_C} \subseteq I$ will also be in the image of ψ. We prove, by increasing induction on $(1,1) \leq (u,v) \leq (m,n)$, that $J_{w_C} \in \mathrm{Im}(f_{(u,v)})$. The result then follows from the case $(u,v) = (m,n)$, since $f_{(m,n)} = \psi$.

When $(u,v) = (1,1)$, $f_{(1,1)}$ is simply the identity map $\mathrm{C.Spec}(A')$, so the result holds trivially. This proves the base case.

For the induction step, assume the result holds for $(u,v)^-$, where $(1,2) \leq (u,v) \leq (m,n)$; i.e. $J_{w_C} \in \mathrm{Im}(f_{(u,v)^-})$. Consider $P_{(u,v)} := f_{(u,v)^-}^{-1}(J_{w_C}) \in \mathrm{C.Spec}(A^{(u,v)})$. We will show that $P_{(u,v)} \in \mathrm{Im}(\psi_{(u,v)})$. We must consider two cases: $T_{u,v} \notin J_{w_C}$ and $T_{u,v} \in J_{w_C}$.

Case 1: If $T_{u,v} \notin J_{w_C}$ then $T_{u,v} = X_{u,v}^{(u,v)^+} = X_{u,v}^{(u,v)} \notin P_{(u,v)}$, and applying Proposition 4.12 allows us to deduce that $P_{(u,v)} \in \mathrm{Im}(\psi_{(u,v)})$.

Case 2: If $T_{u,v} \in J_{w_C}$ then $T_{u,v} = X_{u,v}^{(u,v)^+} = X_{u,v}^{(u,v)} \in P_{(u,v)}$. Again we apply Proposition 4.12, whence we see that $P_{(u,v)} \in \mathrm{Im}(\psi_{(u,v)})$ if and only if $\ker(g_{(u,v)}) \subseteq P_{(u,v)}$. We have an explicit form for $\ker(g_{(u,v)})$, given in Lemma 4.35, therefore we are done if we can show that either $X_{s,v}^{(u,v)} \in P_{(u,v)}$ for all $s \in [\![1,u]\!]$, or $X_{u,t}^{(u,v)} \in P_{(u,v)}$ for all $t \in [\![1,v]\!]$. Since $C \in \mathscr{G}$ then the matrix $(T_{i,j} + J_{w_C})_{i,j} \in M_{m,n}(A'/J_{w_C})$ is a Cauchon matrix. It follows from $T_{u,v} \in J_{w_C}$ that either $T_{s,v} \in J_{w_C}$ for all $s \in [\![1,u]\!]$, or $T_{u,t} \in J_{w_C}$ for all $t \in [\![1,v]\!]$. We treat these cases separately:

Case 2.1: Suppose $T_{u,t} \in J_{w_C}$ for all $t \in [\![1,v]\!]$. We prove by induction on $l \in [\![t,v]\!]$ that $X_{u,t}^{(u,l)} \in P_{(u,l)}$. If $l = t$ then, by Corollary 4.14, $X_{u,t}^{(u,t)^+} \in P_{(u,t)^+}$ and thus $X_{u,t}^{(u,t)} \in P_{(u,t)}$, by

Lemma 4.13. For the inductive step we assume that $X_{u,t}^{(u,l)} \in P_{(u,l)}$ for some $l \in [\![t, v-1]\!]$ and we show that $X_{u,t}^{(u,l)^+} \in P_{(u,l)^+}$. The proof splits into two further cases:

Case 2.1.1: If $X_{u,l}^{(u,l)} \notin P_{(u,l)}$ then, by (4.14), $P_{(u,l)^+} = P_{(u,l)} S_{(u,l)}^{-1} \cap A^{(u,l)^+}$. Since $X_{u,t}^{(u,l)}$ is in the same row as the pivot, $X_{u,l}^{(u,l)}$, then $X_{u,t}^{(u,l)} = X_{u,t}^{(u,l)^+} \in A^{(u,l)^+}$ and we obtain

$$X_{u,t}^{(u,l)} = X_{u,t}^{(u,l)^+} \in P_{(u,l)} \cdot 1^{-1} \cap A^{(u,l)^+} \subseteq P_{(u,l)^+}.$$

Case 2.1.2: If $X_{u,l}^{(u,l)} \in P_{(u,l)}$ then, by (4.14),

$$P_{(u,l)} = g_{(u,l)}^{-1}\left(P_{(u,l)^+}/\langle X_{u,l}^{(u,l)^+}\rangle\right).$$

By the induction hypothesis, we have $X_{u,t}^{(u,l)} \in P_{(u,l)}$ and, applying $g_{(u,l)}$ to both sides of the equation above, we see that

$$g_{(u,l)}(X_{u,t}^{(u,l)}) = X_{u,t}^{(u,l)^+} + \langle X_{u,l}^{(u,l)^+}\rangle \in P_{(u,l)^+}/\langle X_{u,l}^{(u,l)^+}\rangle.$$

Therefore, $X_{u,t}^{(u,l)^+} \in P_{(u,l)^+}$. This completes the proof for Case 2.1.

Case 2.2: Now suppose that $T_{s,v} \in J_{w_C}$ for all $s \in [\![1, u]\!]$. We induct on $(1,2) \leq (i,j) \leq (u,v)$ to show that $X_{s,v}^{(i,j)} \in P_{(i,j)}$. The base case, when $(i,j) = (1,2)$, is trivial since $P_{(1,2)} = J_{w_C}$. For the induction hypothesis, assume that $X_{s,v}^{(i,j)} \in P_{(i,j)}$ for some $(1,2) \leq (i,j) < (u,v)$. Again, we split the proof into two cases:

Case 2.2.1: If $X_{i,j}^{(i,j)} \notin P_{(i,j)}$ then $P_{(i,j)^+} = P_{(i,j)} S_{(i,j)}^{-1} \cap A^{(i,j)^+}$, and we have

$$X_{s,v}^{(i,j)} = \begin{cases} X_{s,v}^{(i,j)^+} & \text{for } s \geq i \text{ or } v \geq j; \\ X_{s,v}^{(i,j)^+} - p_{i,s} X_{s,j}^{(i,j)^+} X_{i,v}^{(i,j)^+} (X_{i,j}^{(i,j)^+})^{-1} & \text{for } s < i \text{ and } v < j. \end{cases}$$

If $s \geq i$ or $v \geq j$ we can conclude as in Case 2.1.1. Otherwise, if $s < i$ and $v < j$ then we rearrange the expression for $X_{s,v}^{(i,j)}$ to obtain

$$X_{s,v}^{(i,j)^+} = X_{s,v}^{(i,j)} + p_{i,s} X_{s,j}^{(i,j)^+} X_{i,v}^{(i,j)^+} (X_{i,j}^{(i,j)^+})^{-1}. \tag{4.15}$$

We now use the fact that generators $X_{a,b}^{(i,j)^+}$ remain unchanged under the deleting derivations algorithm if they lie in the same column or row as the pivot, $X_{i,j}^{(i,j)^+}$. Thus

$X_{s,j}^{(i,j)^+} = X_{s,j}^{(i,j)}$, $X_{i,v}^{(i,j)^+} = X_{i,v}^{(i,j)}$ and $X_{i,j}^{(i,j)^+} = X_{i,j}^{(i,j)}$. Substituting these into (4.15) gives

$$X_{s,v}^{(i,j)^+} = X_{s,v}^{(i,j)} + p_{i,s} X_{s,j}^{(i,j)} X_{i,v}^{(i,j)} (X_{i,j}^{(i,j)})^{-1}$$

and, upon noting that $X_{i,v}^{(i,j)} \in P_{(i,j)}$ (by the induction hypothesis), we deduce that

$$X_{s,v}^{(i,j)^+} = X_{s,v}^{(i,j)} + p_{i,s} X_{s,j}^{(i,j)} X_{i,v}^{(i,j)} (X_{i,j}^{(i,j)})^{-1} \in P_{(i,j)} S_{(i,j)}^{-1} \cap A^{(i,j)^+} = P_{(i,j)^+}.$$

Case 2.2.2: If $X_{i,j}^{(i,j)} \in P_{(i,j)}$ then we apply Lemma 4.19 to obtain an isomorphism $A^{(i,j)}/P_{(i,j)} \to A^{(i,j)^+}/P_{(i,j)^+}$, which sends $X_{a,b}^{(i,j)} + P_{(i,j)} \mapsto X_{a,b}^{(i,j)^+} + P_{(i,j)^+}$ for all $(1,1) \leq (a,b) \leq (m,n)$. Hence, if $X_{s,v}^{(i,j)} \in P_{(i,j)}$ for all $s \in [\![1,u]\!]$ then $X_{s,v}^{(i,j)^+} \in P_{(i,j)^+}$ for all $s \in [\![1,u]\!]$. This completes the induction for Case 2.2.

These two cases prove the main induction argument. $\square$

Propositions 4.34 and 4.36 combine to give the following result:

Theorem 4.37. *Let $w \in \mathscr{W}$. The following are equivalent:*

(i) $w \in \mathscr{W}'$;

(ii) $J_w \in \mathrm{Im}(\psi)$;

(iii) $C_w \in \mathscr{G}$;

(iv) $\psi^{-1}(\mathrm{C.Spec}_w(A')) \neq \emptyset$.

We are now able to present an example which illustrates how the presentation of the algebra as an iterated Ore extension affects the set of Cauchon diagrams $\mathscr{W}'$:

Example 4.38. Consider the single parameter quantum matrices $\mathscr{O}_q(M_2(\mathbb{K}))$ with matrix of generators $\begin{pmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{pmatrix}$, let $q \in \mathbb{K}^*$ be a primitive ℓ^{th} root of unity with $\ell > 2$, and let $\mathscr{W} := \mathbb{P}(\{(1,1),(1,2),(2,1),(2,2)\})$. We present this algebra as two different iterated Ore extensions, denoted A and B, by choosing two different orderings for the generators. We then show that the Cauchon diagrams of $\mathscr{O}_q(M_2(\mathbb{K}))$ with respect to these different orderings, denoted $\mathscr{W}_A'$ and $\mathscr{W}_B'$ respectively, are not equal.

- The standard ordering of generators in quantum matrices that is taken in this thesis is the lexicographic ordering, from which we obtain the following iterated Ore extension:

$$A := \mathbb{K}[X_{1,1}][X_{1,2};\sigma_{1,2}][X_{2,1};\sigma_{2,1}][X_{2,2};\sigma_{2,2},\delta_{2,2}],$$

where

$$\begin{aligned} \sigma_{2,2}(X_{2,1}) &= q^{-1}X_{2,1}, & \delta_{2,2}(X_{2,1}) &= 0, \\ \sigma_{2,2}(X_{1,2}) &= q^{-1}X_{1,2}, & \delta_{2,2}(X_{1,2}) &= 0, \\ \sigma_{2,2}(X_{1,1}) &= X_{1,1}, & \delta_{2,2}(X_{1,1}) &= (q^{-1}-q)X_{2,1}X_{1,2}. \end{aligned}$$

It can be verified that $(\sigma_{2,2}, \delta_{2,2})$ is q^2-skew. Then, by applying the deleting derivations algorithm once, we obtain the quantum affine space

$$A' = \mathbb{K}_{q^M}[T_{1,1}, T_{1,2}, T_{2,1}, T_{2,2}],$$

with $M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$ and generators $T_{2,2} := X_{2,2}$, $T_{2,1} := X_{2,1}$, $T_{1,2} := X_{1,2}$, and

$$\begin{aligned} T_{1,1} &:= X_{1,1} + q^2(q^2-1)^{-1}\delta_{2,2} \circ \sigma_{2,2}^{-1}(X_{1,1})X_{2,2}^{-1} \\ &= X_{1,1} - qX_{2,1}X_{1,2}X_{2,2}^{-1}. \end{aligned}$$

Note that $A^{(2,2)^+} = A$ and after one step of the algorithm we obtain $A^{(2,2)} = A'$. Let ψ_A : $\mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$ be the canonical embedding defined in Definition 4.7 and denote by $\mathscr{W}'_A \subseteq \mathscr{W}$ the set of all $w \in \mathscr{W}$ such that $\mathrm{C.Spec}_w(A) = \psi_A^{-1}(\mathrm{C.Spec}_w(A')) \neq \emptyset$. It can be shown (by applying Lemma 4.11 with the pivot generator $T_{2,2} = X_{2,2}$ in place of $y_j = x_j$) that the completely prime ideal $\langle T_{1,1} \rangle \in \mathrm{C.Spec}(A')$ lies in the image of ψ_A, since $T_{2,2} \notin \langle T_{1,1} \rangle$. Therefore $(1,1) \in \mathscr{W}'_A$.

- We may also present the generators in a reverse lexicographic order, giving the following iterated Ore extension:

$$B := \mathbb{K}[X_{2,2}][X_{2,1}; \bar{\sigma}_{2,1}][X_{1,2}; \bar{\sigma}_{1,2}][X_{1,1}; \bar{\sigma}_{1,1}, \bar{\delta}_{1,1}],$$

where

$$\begin{aligned} \bar{\sigma}_{1,1}(X_{1,2}) &= qX_{1,2}, & \bar{\delta}_{1,1}(X_{1,2}) &= 0, \\ \bar{\sigma}_{1,1}(X_{2,1}) &= qX_{2,1}, & \bar{\delta}_{1,1}(X_{2,1}) &= 0, \\ \bar{\sigma}_{1,1}(X_{2,2}) &= X_{2,2}, & \bar{\delta}_{1,1}(X_{2,2}) &= (q-q^{-1})X_{2,1}X_{1,2}. \end{aligned}$$

It can be verified that $(\bar{\sigma}_{1,1}, \bar{\delta}_{1,1})$ is q^{-2}-skew. Then, by applying the deleting derivations algorithm once, we obtain the quantum affine space

$$B' = \mathbb{K}_{q^{M'}}[T'_{2,2}, T'_{2,1}, T'_{1,2}, T'_{1,1}],$$

with $M' = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ and generators $T'_{1,1} := X_{1,1}$, $T'_{1,2} := X_{1,2}$, $T'_{2,1} := X_{2,1}$, and

$$\begin{aligned} T'_{2,2} &:= X_{2,2} + q^{-2}(q^{-2}-1)^{-1}\bar{\delta}_{1,1} \circ \bar{\sigma}_{1,1}^{-1}(X_{2,2})X_{1,1}^{-1} \\ &= X_{2,2} - q^{-1}X_{2,1}X_{1,2}X_{1,1}^{-1}. \end{aligned}$$

Care must be taken when applying the results from previous sections because the generators were always assumed to be arranged in lexicographic order, so that the subscripts would be increasing. This is not the case in this example. Instead we have $B^{(1,1)^-} = B$ and after one step of the algorithm we obtain $B^{(1,1)} = B'$. Let $\psi_B :$ C.Spec$(B) \to$ C.Spec(B') be the canonical embedding defined in Definition 4.7 and denote by $\mathscr{W}'_B \subseteq \mathscr{W}$ the set of all $w \in \mathscr{W}$ such that C.Spec$_w(B) = \psi_B^{-1}($C.Spec$_w(B')) \neq \emptyset$. We may now apply Lemma 4.11 to this case, this time using $T_{1,1} = X_{1,1}$ as the pivot generator, instead of $T_{2,2} = X_{2,2}$ as in the previous case. This changes the conclusion because $T'_{1,1} \in \langle T'_{1,1} \rangle$, hence we require $\ker(g_{(1,1)}) \subseteq \langle T'_{1,1} \rangle$ if $\langle T'_{1,1} \rangle$ is to be contained in the image of ψ_B. However, by applying Proposition 4.29 to this example it can be shown that $\ker(g_{(1,1)}) = \langle T'_{1,1}, T'_{1,2}T'_{2,1} \rangle$, and hence we see that $\ker(g_{(1,1)}) \nsubseteq \langle T'_{1,1} \rangle$. Therefore, $\langle T'_{1,1} \rangle \notin \mathrm{Im}(\psi_B)$ and $(1,1) \notin \mathscr{W}'_B$.

The above working shows that $(1,1) \in \mathscr{W}'_A$ but $(1,1) \notin \mathscr{W}'_B$, hence $\mathscr{W}'_A \neq \mathscr{W}'_B$.

Chapter 5

Method for Computing the PI Degree of A/P

Throughout this chapter we take A to be a $\mathbb{K}$-algebra satisfying Hypothesis 1, A' to be the quantum affine space obtained by deleting all derivations from A, $P \in \mathrm{C.Spec}(A)$, and $\psi : \mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$ to be the canonical embedding. We assume that each $\lambda_{i,j}$ (from H.1.3) is a root of unity so that $\lambda_{i,j} = q^{m_{i,j}}$ for some skew-symmetric matrix $M = (m_{i,j}) \in M_N(\mathbb{Z})$ and primitive ℓ^{th} root of unity, $1 \neq q \in \mathbb{K}^*$. In particular, A and A/P are PI algebras for all $P \in \mathrm{C.Spec}(A)$, by Theorem 3.10. Finally, set $\mathscr{W} := \mathbb{P}(\llbracket 1, N \rrbracket)$ and let $\mathscr{W}' \subseteq \mathscr{W}$ denote the set of Cauchon diagrams of A.

We set out the steps required to compute the PI degree of A/P. Provided the quotient $A'/\psi(P)$ is a quantum affine space, $\mathscr{O}_{q^{M'}}(\mathbb{K}^n)$, for some $n \in \mathbb{N}$ and $M' \in M_n(\mathbb{Z})$, the last step in the method is an application of the result by De Concini and Procesi (Theorem 2.30), which provides a method for calculating the PI degree of a quantum affine space using the matrix M'. As such, we first present a result which uses Theorem 2.30 to obtain a formula for the PI degree of $\mathscr{O}_{q^{M'}}(\mathbb{K}^n)$ in terms of the properties of M'.

To illustrate how the above method is applied in practice, we compute the PI degree on two small examples. The second of these examples introduces two new classes of algebras that are closely linked (in a way which will be explained): *quantum Schubert varieties* and *generalised quantum determinantal rings*. Finally, we see at the end of the chapter how this link may be exploited to provide a general strategy for computing the PI degree of specific examples of quantum Schubert varieties satisfying certain criteria.

5.1 General Method for Calculating PI-deg(A/P)

Given A and P as above, it was shown in Theorem 4.25 that PI-deg$(A/P) =$ PI-deg$(A'/\psi(P))$. To help in our discussion we single out a specific case of interest:

Definition 5.1. Let $w \in \mathscr{W}'$. We call $P \in \mathrm{C.Spec}_w(A)$ a *Cauchon ideal* if $\psi(P) = J_w = \langle T_i \in A' \mid i \in w \rangle \in \mathrm{C.Spec}_w(A')$.

Remark 5.2. With this definition we see that, if $P \in \mathrm{C.Spec}_w(A)$ is a Cauchon ideal, then

$$A'/\psi(P) = A'/J_w = \mathbb{K}_{q^M}[T_1, \ldots, T_N]/\langle T_i \mid i \in w \rangle = \mathbb{K}\langle t_i \mid i \notin w \rangle = \mathscr{O}_{q^{M'}}(\mathbb{K}^{N-|w|}),$$

where $M' \in M_{N-|w|}(\mathbb{Z})$ is the submatrix of M obtained by deleting the columns and rows indexed by $i \in w$, and $t_i := T_i + J_w$ for all $i \notin w$. This allows us to write

$$\text{PI-deg}(A/P) = \text{PI-deg}(A'/\psi(P)) = \text{PI-deg}(\mathscr{O}_{q^{M'}}(\mathbb{K}^{N-|w|})),$$

to which we can then apply De Concini and Procesi's result.

Knowing when $P \in \mathrm{C.Spec}(A)$ is a Cauchon ideal, and to which $w \in \mathscr{W}'$ it is associated to, is difficult in general. We discuss some open questions relating to these problems in Section 5.4. For specific examples these details can be worked out by hand, although this is often arduous for iterated Ore extensions with many derivations. The steps to follow in order to compute the PI degree of A/P are given below and, in the next section, we apply these steps to two specific examples.

1. Given an algebra A, verify that it satisfies Hypothesis 1 and that all $\lambda_{i,j}$ are roots of unity so that A is a PI algebra. Write $\lambda_{i,j} = q^{m_{i,j}}$ for some primitive ℓ^{th} root of unity, $1 \neq q \in \mathbb{K}^*$, and some $M = (m_{i,j}) \in M_N(\mathbb{Z})$.

2. Apply the deleting derivations algorithm (Theorem 3.6) to A, noting down the new indeterminates $(X_1^{(j)}, \ldots, X_N^{(j)})$ at each step, to obtain the quantum affine space A' after at most $N-1$ iterations.

3. For $P \in \mathrm{C.Spec}(A)$, apply Proposition 4.6, along with Lemmas 4.3 and 4.5, at each step of the algorithm to obtain $\psi(P) \in \mathrm{C.Spec}(A')$. Then PI-deg$(A/P) =$ PI-deg$(A'/\psi(P))$.

   (a) If $\psi(P) = J_w$, for some $w \in \mathscr{W}'$, then $A'/\psi(P)$ is the quantum affine space $\mathscr{O}_{q^{M'}}(\mathbb{K}^n)$, for some $n \leq N$. Proceed to Step 4.

(b) If $\psi(P) \neq J_w$, for all $w \in \mathscr{W}'$, then we do not have the techniques to compute its PI degree.

- Open question: When is P a Cauchon ideal? (See question 1 in Section 5.4.)

4. Apply Theorem 2.30 to $\mathscr{O}_{q^{M'}}(\mathbb{K}^n)$ and calculate the cardinality, h, of the image of the homomorphism $\pi \circ M' : \mathbb{Z}^n \to (\mathbb{Z}/\ell\mathbb{Z})^n$ to conclude that PI-deg$(A/P) =$ PI-deg$(A'/\psi(P)) = \sqrt{h}$.

Alternatively, Step 3 may be replaced with the following:

3*. For $w \in \mathscr{W}$, take the ideal $J_w \in$ C.Spec(A'), so that $A'/J_w = \mathscr{O}_{q^{M'}}(\mathbb{K}^n)$ for some $M' \in M_n(\mathbb{Z})$ and $n \leq N$. Verify whether J_w lies in the image of ψ by working backwards through the algorithm, applying Proposition 4.12 at each step to check that $J_w \in \text{Im}(f_j)$.

(a) If $J_w \in \text{Im}(\psi)$ then, by setting $P := \psi^{-1}(J_w) \in$ C.Spec(A), we have PI-deg$(A/P) =$ PI-deg$(A'/J_w) =$ PI-deg$(\mathscr{O}_{q^{M'}}(\mathbb{K}^n))$. Proceed to Step 4.

(b) If $J_w \notin \text{Im}(\psi)$ then there does not exist any $P \in$ C.Spec(A) such that $\psi(P) = J_w$.

- Open question: what is the set of Cauchon diagrams, $\mathscr{W}'$, for any algebra A? (See question 2 in Section 5.4.)

5.2 A Formula for PI-deg$(A/\psi^{-1}(J_w))$

Before presenting an application of the method described above, we give a result which simplifies the calculation of h in the statement of De Concini and Procesi's theorem (Theorem 2.30). Recall that their theorem stated the following: Given the quantum affine space $\mathscr{O}_{q^M}(\mathbb{K}^N)$, for some skew-symmetric matrix $M \in M_N(\mathbb{Z})$, the PI degree of $\mathscr{O}_{q^M}(\mathbb{K}^N)$ is given as $\sqrt{h}$, where h is the cardinality of the image of the following homomorphisms:

$$\mathbb{Z}^N \xrightarrow{M} \mathbb{Z}^N \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^N, \tag{5.1}$$

where π denotes the canonical epimorphism.

As will be seen in this section, the following definition [New72, Chapters II and IV] and lemma [Pan94, Corollary] simplifies the computation of h by allowing us to replace the matrix M in (5.1) with a simpler block diagonal matrix, S (see Remark 5.6).

Definition 5.3. Let M and S be two $N \times N$ integral matrices. We say that M is *equivalent* to S if there are two invertible matrices $U, V \in M_N(\mathbb{Z})$ (i.e. $\det(U) = \pm 1$, $\det(V) = \pm 1$) such that $M = USV$. We say that M is *congruent* to S (denoted $M \sim_C S$) if there is an invertible matrix $U \in M_N(\mathbb{Z})$ (i.e. $\det(U) = \pm 1$) such that $M = U^T SU$.

Lemma 5.4 (Panov). *Let $M, S \in M_N(\mathbb{Z})$ be two skew-symmetric matrices and let $\mathscr{O}_{q^M}((\mathbb{K}^*)^N)$ and $\mathscr{O}_{q^S}((\mathbb{K}^*)^N)$ be their respective associated quantum tori. Let $q \in \mathbb{K}^*$ be an element of the field. If M is congruent to S then $\mathscr{O}_{q^M}((\mathbb{K}^*)^N) \cong \mathscr{O}_{q^S}((\mathbb{K}^*)^N)$.*

Remark 5.5. Please note that in the interest of being consistent with Definition 5.3, the statement of Lemma 5.4 differs slightly from that of [Pan95, Lemma 2.4]. Panov stated the above result for equivalent matrices, however, his definition of equivalent matrices [Pan95, Definition 2.1] is precisely the definition of congruent matrices given in Definition 5.3, which appears to be more widely used in the literature.

Remark 5.6. It is a well known result that every skew-symmetric integral matrix $M \in M_N(\mathbb{Z})$ is congruent to its skew-normal form, which we denote by S or $\mathrm{Sk}(M)$ if it is not clear to which matrix M the skew-normal form S is congruent. This is a block diagonal matrix of the form

$$S = \begin{pmatrix} 0 & h_1 & & & & & & \\ -h_1 & 0 & & & & & & \\ & & 0 & h_2 & & & & \\ & & -h_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & h_s & \\ & & & & & -h_s & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ is a square matrix of zeros of dimension $\dim(\ker(M))$, so that $2s = N - \dim(\ker(M))$, and $h_i | h_{i+1} \in \mathbb{Z} \backslash \{0\}$ for all $i \in [\![1, s]\!]$. These nonzero $h_1, h_1, h_2, h_2, \ldots, h_s, h_s$ are called the *invariant factors* of M. As they always come in pairs, from now on we will avoid repetition and list the invariant factors simply as $h_1, h_2, \ldots, h_s$.

Panov's result is used in the following lemma, which states that the PI degree of the quantum affine space $\mathscr{O}_{q^M}(\mathbb{K}^N)$ is completely determined by properties of M, namely the dimension of its kernel along with its invariant factors and the value of ℓ.

Lemma 5.7. *Take $1 \neq q \in \mathbb{K}^*$, a primitive ℓ^{th} root of unity. Let $M \in M_N(\mathbb{Z})$ be a skew-symmetric integral matrix with invariant factors $h_1, \ldots, h_s$. Then the PI degree of $\mathscr{O}_{q^M}(\mathbb{K}^N)$ is given as*

$$\textit{PI-deg}(\mathscr{O}_{q^M}(\mathbb{K}^N)) = \prod_{i=1}^{\frac{N - \dim(\ker(M))}{2}} \frac{\ell}{\gcd(h_i, \ell)}.$$

Proof. The PI degree of this quantum affine space is $\sqrt{h}$, where h is the cardinality of the image of the homomorphism

$$\mathbb{Z}^N \xrightarrow{M} \mathbb{Z}^N \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^N,$$

and π denotes the canonical epimorphism. Let S be the skew-normal form of M. Then S is congruent to M and, by Lemma 5.4, the two quantum tori $\mathscr{O}_{q^M}((\mathbb{K}^*)^N)$ and $\mathscr{O}_{q^S}((\mathbb{K}^*)^N)$ are isomorphic. Therefore, since

$$\text{PI-deg}(\mathscr{O}_{q^M}(\mathbb{K}^N)) = \text{PI-deg}(\mathscr{O}_{q^M}((\mathbb{K}^*)^N) = \text{PI-deg}(\mathscr{O}_{q^S}((\mathbb{K}^*)^N) = \text{PI-deg}(\mathscr{O}_{q^S}(\mathbb{K}^N)),$$

it is enough to compute the cardinality, h, of the image of the homomorphism

$$\mathbb{Z}^N \xrightarrow{S} \mathbb{Z}^N \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^N. \tag{5.2}$$

Applying this map to a general element $\underline{z} = (z_1, \ldots, z_N)^T \in \mathbb{Z}^N$, we obtain the following:

$$(\pi \circ S)(\underline{z}) = (\overline{h_1 z_2}, \overline{-h_1 z_1}, \overline{h_2 z_4}, \overline{-h_2 z_3}, \ldots, \overline{h_s z_{2s}}, \overline{-h_s z_{2s-1}}, \overline{\mathbf{0}})^T,$$

where $\overline{h_i z_j}$ denotes the canonical image of $h_i z_j$ in $\mathbb{Z}/\ell\mathbb{Z}$. Since $\dim(\ker(M)) = \dim(\ker(S))$, the dimension of the zero matrix $\mathbf{0}$ in S is equal to $\dim(\ker(M))$ and hence $2s = N - \dim(\ker(M))$.

We now turn our attention to the entries of $(\pi \circ S)(\underline{z})$. Each nonzero entry is of the form $\pm\overline{h_i}\overline{z_j}$ for some $\overline{z_j} \in \mathbb{Z}/\ell\mathbb{Z}$. Consider then, for each invariant factor h_i, the map

$$\begin{aligned} f_i : \mathbb{Z} &\longrightarrow \mathbb{Z}/\ell\mathbb{Z} \\ z_j &\longmapsto \overline{h_i z_j}. \end{aligned}$$

The image of f_i is

$$\operatorname{Im}(f_i) = \{\overline{h}_i \overline{z}_j \mid \overline{z}_j \in \mathbb{Z}/\ell\mathbb{Z}\} \subseteq \mathbb{Z}/\ell\mathbb{Z}$$

and this forms an additive subgroup of $(\mathbb{Z}/\ell\mathbb{Z}, +)$. Furthermore, the image of f_i is the cyclic subgroup generated by $\overline{h}_i$ and denoted by $\langle \overline{h}_i \rangle$. Since $(\mathbb{Z}/\ell\mathbb{Z}, +)$ is the cyclic group generated by 1 containing ℓ elements then the order of $\overline{h}_i \in (\mathbb{Z}/\ell\mathbb{Z}, +)$ is $\ell / \gcd(\overline{h}_i, \ell)$. Hence

$$|\operatorname{Im}(f_i)| = |\langle \overline{h}_i \rangle| = \frac{\ell}{\gcd(\overline{h}_i, \ell)}.$$

The image of $(\pi \circ S)$ in $(\mathbb{Z}/\ell\mathbb{Z})^N$ consists of copies of the subgroup $\mathrm{Im}(f_i) \in \mathbb{Z}/\ell\mathbb{Z}$ in positions $2i$ and $2i-1$, for each $i \in [\![1,s]\!]$, and zeros in positions $2s+1,\ldots,N$. Therefore, for each $i \in [\![1,s]\!]$ the image of f_i describes two entries in the image of $(\pi \circ S)$. The cardinality of the whole image of $(\pi \circ S)$ is therefore given as

$$h = \prod_{i=1}^{s} |\mathrm{Im}(f_i)|^2 = \prod_{i=1}^{s} |\langle \overline{h}_i \rangle|^2 = \prod_{i=1}^{s} \left(\frac{\ell}{\gcd(\overline{h}_i, \ell)} \right)^2 .$$

As noted before, $2s = N - \dim(\ker(M))$, so the PI degree of $\mathscr{O}_{q^M}(\mathbb{K}^N)$ becomes

$$\sqrt{h} = \sqrt{\prod_{i=1}^{\frac{N-\dim(\ker(M))}{2}} \left(\frac{\ell}{\gcd(\overline{h}_i, \ell)} \right)^2} = \prod_{i=1}^{\frac{N-\dim(\ker(M))}{2}} \frac{\ell}{\gcd(\overline{h}_i, \ell)}. \tag{5.3}$$

□

5.3 Examples

We use the method described above to calculate examples of the PI degree of A/P for some Cauchon prime $P \in \mathrm{C.Spec}(A)$. Let $1 \neq q \in \mathbb{K}^*$ be a primitive ℓ^{th} root of unity for these examples, where we specify conditions on q for each example.

5.3.1 A Two Step Process: $U_q^+(\mathfrak{so}_5)/\langle z' \rangle$

In this example we take $\ell \notin \{2,4\}$ and compute the PI degree of a quotient of $U_q^+(\mathfrak{so}_5)$, the $\mathbb{C}$-algebra generated by two indeterminates E_1, E_2 subject to the the following relations:

$$E_1^3E_2 - (q^2+1+q^{-2})E_1^2E_2E_1 + (q^2+1+q^{-2})E_1E_2E_1^2 - E_2E_1^3 = 0$$
$$E_2^2E_1 - (q^2+q^{-2})E_2E_1E_2 + E_1E_2^2 = 0.$$

There is a PBW basis of $U_q^+(\mathfrak{so}_5)$ formed by monomials $E_1^{k_1}E_4^{k_4}E_3^{k_3}E_2^{k_2}$, where k_1,k_2,k_3,k_4 are nonnegative integers and E_3, E_4 are certain root vectors. This result can be found, for example, in [Lau06, Section 2.4]. The same paper also expresses this algebra as an iterated Ore extension over $\mathbb{C}$ generated by these four indeterminates, with the Ore extensions appearing in the following order: E_1, E_4, E_3, E_2. Because of this ordering, we relabel the indeterminates as

$$X_1 := E_1, \quad X_2 := E_4, \quad X_3 := E_3, \quad X_4 := E_2.$$

The relations between E_1, E_2, E_3, E_4 in [Lau06] then become

$$
\begin{aligned}
&X_2X_1 = q^{-2}X_1X_2, \\
&X_3X_1 = X_1X_3 - (q+q^{-1})X_2, \qquad X_3X_2 = q^{-2}X_2X_3, \\
&X_4X_1 = q^2X_1X_4 - q^2X_3, \qquad X_4X_2 = X_2X_4 - \frac{q^2-1}{q+q^{-1}}X_3^2, \qquad X_4X_3 = q^{-2}X_3X_4.
\end{aligned}
$$

This allows us to present $A := U_q^+(\mathfrak{so}_5)$ as the following iterated Ore extension:

$$U_q^+(\mathfrak{so}_5) = \mathbb{C}[X_1][X_2;\sigma_2][X_3;\sigma_3,\delta_3][X_4;\sigma_4,\delta_4]$$

where, using the notation in H1.5 to denote subalgebras $A_j := \mathbb{C}\langle X_1,\dots,X_j\rangle \subseteq A$, the automorphisms σ_i are defined on the generators as:

$$
\begin{aligned}
\sigma_2 : A_1 \longrightarrow A_1;\quad & X_1 \longmapsto q^{-2}X_1, \\
\sigma_3 : A_2 \longrightarrow A_2;\quad & X_1 \longmapsto X_1 \\
& X_2 \longmapsto q^{-2}X_2, \\
\sigma_4 : A_3 \longrightarrow A_3;\quad & X_1 \longmapsto q^2X_1 \\
& X_2 \longmapsto X_2 \\
& X_3 \longmapsto q^{-2}X_3;
\end{aligned}
\tag{5.4}
$$

and the σ_i-derivations δ_i are defined on the generators as:

$$
\begin{aligned}
\delta_3 : A_2 \longrightarrow A_2;\quad & X_1 \longmapsto -(q+q^{-1})X_2 \\
& X_2 \longmapsto 0, \\
\delta_4 : A_3 \longrightarrow A_3;\quad & X_1 \longmapsto -q^2X_3 \\
& X_2 \longmapsto -\frac{q^2-1}{q+q^{-1}}X_3^2 \\
& X_3 \longmapsto 0.
\end{aligned}
\tag{5.5}
$$

Verifying $U_q^+(\mathfrak{so}_5)$ satisfies Hypothesis 1:

We need to check that this presentation as an iterated Ore extension satisfies Hypothesis 1 in order to be able to apply the deleting derivations algorithm. Note, for example, that the field elements $\lambda_{i,j}$ in property H.1.3 are defined as $\lambda_{i,j} = q^{m_{i,j}}$, where the $m_{i,j}$ are the entries of the skew-symmetric matrix

$$M := \begin{pmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix} \in M_4(\mathbb{C}^*).$$

In fact, the only properties that need to be checked are H.1.4 and H.1.6 because the other properties follow immediately from the definitions above.

Routine computations show that (σ_3, δ_3) is q^2-skew and (σ_4, δ_4) is q^4-skew. So, since $\ell \notin \{2,4\}$, A satisfies property H.1.4 with $q_3 = q^2$ and $q_4 = q^4$. We define the following $\mathbb{C}[t^{\pm 1}]$-algebra:

$$R := \mathbb{C}[t^{\pm 1}][X_1][X_2;\bar{\sigma}_2][X_3;\bar{\sigma}_3,\bar{\delta}_3][X_4;\bar{\sigma}_4,\bar{\delta}_4],$$

where the $\mathbb{C}[t^{\pm 1}]$-automorphisms, $\bar{\sigma}_i$, and skew-derivations, $\bar{\delta}_i$, are defined analogously to (5.4) and (5.5) on the appropriate subalgebras of R, with t replacing q. We see immediately that $(\bar{\sigma}_3, \bar{\delta}_3)$ is t^2-skew and $(\bar{\sigma}_4, \bar{\delta}_4)$ is t^4-skew, and that

$$R/\langle t - q\rangle \cong A,$$

with $\bar{\sigma}_3$, $\bar{\sigma}_4$, $\bar{\delta}_3$, $\bar{\delta}_4$ reducing to σ_3, σ_4, δ_3, δ_4, respectively. For all $j \in [\![1,4]\!]$ denote by $R_j \subseteq R$ the subalgebra generated by $X_1, \ldots, X_j \in R$. To verify the first part of H.1.6 we need to show that $\bar{\delta}_3^n(R_2) \subseteq (n)!_{t^2} R_2$ and $\bar{\delta}_4^n(R_3) \subseteq (n)!_{t^4} R_3$ for all $n \geq 0$, since then the desired result follows from [Hay08, Theorem 2.8]. By [Hay08, Lemma 5.3] we note that we only need to show that this property holds on the generators, which we do by using (5.5) to see that $\bar{\delta}_3(X_2) = \bar{\delta}_4(X_3) = 0$, and

$$\begin{aligned} \bar{\delta}_4^n(X_1) &= -t^2 \bar{\delta}_4^{n-1}(X_3) = 0, \\ \bar{\delta}_4^n(X_2) &= -\frac{t^2-1}{t+t^{-1}} \bar{\delta}_4^{n-1}(X_3^2) = 0, \\ \bar{\delta}_3^n(X_1) &= -(t+t^{-1})\bar{\delta}_3^{n-1}(X_2) = 0, \end{aligned}$$

for all $n > 1$. Therefore, δ_3 extends to a h.q^2-s.σ_3-d., $\{d_{3,n}\}_{n=0}^{\infty}$, and δ_4 extends to a h.q^4-s.σ_4-d., $\{d_{4,n}\}_{n=0}^{\infty}$, both of which are locally nilpotent and iterative. Furthermore, $d_{3,n} = \frac{\delta_3^n}{(n)!_{q^2}}$ for all $n < \ell_3$, and $d_{4,n} = \frac{\delta_4^n}{(n)!_{q^4}}$ for all $n < \ell_4$, where q^2 is a primitive ℓ_3^{th} of unity and q^4 is a primitive ℓ_4^{th} root of unity. Note, in particular, that the results above allow us to write

$$d_{j,n}(X_l) = \begin{cases} X_l & n=0; \\ \delta_j(X_l) & n=1; \\ 0 & n>1, \end{cases} \tag{5.6}$$

for $j \in \{3,4\}$ and $l \in [\![1, j-1]\!]$.

Finally, we prove the remaining property of H.1.6, namely that $\sigma_4 \circ d_{3,n} = \lambda_{3,4}^n d_{3,n} \circ \sigma_4$ for all $n \geq 0$. This holds trivially when $n = 0$, as $d_{3,0} = \mathrm{Id}_{A_2}$. For $n > 0$ it is enough to prove $\sigma_4 \circ d_{3,1}(X_i) = \lambda_{4,3} d_{3,1} \circ \sigma_4(X_i)$, for $i = 1, 2$, as then the property holds on the whole of A_2, and for all n, by the following two observations: Firstly, if $\sigma_4 \circ d_{3,1}(a) = \lambda_{3,4} d_{3,1} \circ \sigma_4(a)$ and $\sigma_4 \circ d_{3,1}(b) = \lambda_{3,4} d_{3,1} \circ \sigma_4(b)$, for some $a, b \in A_2$, then

$$\begin{aligned}
\sigma_4 \circ d_{3,1}(ab) &= \sigma_4 \circ \delta_3(ab) \\
&= \sigma_4(\sigma_3(a)\delta_3(b) + \delta_3(a)b) \\
&= (\sigma_3 \circ \sigma_4)(a)(\sigma_4 \circ \delta_3)(b) + (\sigma_4 \circ \delta_3)(a)\sigma_4(b) \\
&= \lambda_{3,4}(\sigma_3 \circ \sigma_4)(a)(\delta_3 \circ \sigma_4)(b) + \lambda_{3,4}(\delta_3 \circ \sigma_4)(a)\sigma_4(b) \\
&= \lambda_{3,4}(\delta_3 \circ \sigma_4)(ab) \\
&= \lambda_{3,4} d_{3,1} \circ \sigma_4(ab).
\end{aligned}$$

Secondly, assuming the property $\sigma_4 \circ d_{3,n} = \lambda_{3,4}^n d_{3,n} \circ \sigma_4$ holds for some $n > 0$ then, by the definition of $d_{3,n}$, we may conclude that it holds for all $n > 0$, given that

$$\sigma_4 \circ \delta_3^{n+1} = (\sigma_4 \circ \delta_3^n) \circ \delta_3 = \lambda_{3,4}^n \delta_3^n \circ (\sigma_4 \circ \delta_3) = \lambda_{3,4}^n \lambda_{3,4} \delta_3^n \circ (\delta_3 \circ \sigma_4) = \lambda_{3,4}^{n+1} \delta_3^{n+1} \circ \sigma_4.$$

The following calculations therefore complete the proof that A satisfies H.1.6:

$$\begin{aligned}
\sigma_4 \circ d_{3,1}(X_i) = \sigma_4 \circ \delta_3(X_i) &= \begin{cases} -(q+q^{-1})X_2 & i=1; \\ 0 & i=2, \end{cases} \\
d_{3,1} \circ \sigma_4(X_i) = \delta_3(\lambda_{4,i} X_i) &= \begin{cases} -q^2(q+q^{-1})X_2 & i=1; \\ 0 & i=2, \end{cases}
\end{aligned}$$

and hence $\sigma_4 \circ d_{3,1}(X_i) = q^{-2} d_{3,1} \circ \sigma_4 = \lambda_{4,3} d_{3,1} \circ \sigma_4(X_i)$, as required. From this we conclude that A satisfies Hypothesis 1 and we may apply the techniques developed in Chapters 3 and 4.

Applying the deleting derivations algorithm:

Consider the ideal $\langle z' \rangle \lhd A$, generated by the central element (see [Lau06, Section 2.4])

$$z' := -(q^2 - q^{-2})(q + q^{-1})X_2X_4 + q^2(q^2-1)X_3^2.$$

We show that $\langle z' \rangle$ is a completely prime Cauchon ideal of A by tracking it through the deleting derivations algorithm to see that $\langle z' \rangle \in \mathrm{Im}(\psi)$, and then by noting that $\psi(\langle z' \rangle)$ is a completely prime ideal in A'.

First Step: Applying Theorem 3.6 to A results in the following subalgebra of $\mathrm{Frac}(A)$:

$$A^{(4)} = \mathbb{C}[X_1^{(4)}][X_2^{(4)};\sigma_2][X_3^{(4)};\sigma_3,\delta_3][X_4^{(4)};\sigma_4^{(4)}],$$

where $\sigma_4^{(4)}(X_l^{(4)}) = \lambda_{4,l}X_l^{(4)}$ for $l \in \{1,2,3\}$. The generators $X_l^{(4)} \in \mathrm{Frac}(A)$ are defined as in (3.13) and may be written in the following way, upon substituting in values for q_4 and $\sigma_4(X_l)$:

$$X_l^{(4)} := \begin{cases} X_l & l \geq 4; \\ \sum_{n=0}^{\infty} q^{\frac{4n(n+1)}{2}} (q^4-1)^{-n} \lambda_{4,l}^{-n} d_{4,n}(X_l) X_4^{-n} & l < 4. \end{cases}$$

The description of $d_{j,n}(X_l)$ in (5.6) reduces the sum above to just two terms, when $n = 0, 1$, and this gives the following:

$$\begin{aligned} X_4^{(4)} &:= X_4, \\ X_3^{(4)} &:= X_3 + q^4(q^4-1)^{-1}\lambda_{4,3}^{-1}\delta_4(X_3)X_4^{-1} \\ &= X_3, \\ X_2^{(4)} &:= X_2 + q^4(q^4-1)^{-1}\lambda_{4,2}^{-1}\delta_4(X_2)X_4^{-1} \\ &= X_2 - \frac{q^4}{(q^2+1)(q+q^{-1})}X_3^2X_4^{-1}, \\ X_1^{(4)} &:= X_1 + q^4(q^4-1)^{-1}\lambda_{4,1}^{-1}\delta_4(X_1)X_4^{-1} \\ &= X_1 - \frac{q^4}{q^4-1}X_3X_4^{-1}. \end{aligned}$$

Second Step: Applying Theorem 3.6 again, to $A^{(4)}$, results in the following algebra:

$$A^{(3)} = \mathbb{C}[X_1^{(3)}][X_2^{(3)};\sigma_2][X_3^{(3)};\sigma_3^{(3)}][X_4^{(3)};\sigma_4^{(3)}],$$

where $\sigma_4^{(3)}(X_l^{(3)}) = \lambda_{4,l}X_l^{(3)}$ for $l \in \{1,2,3\}$, and $\sigma_3^{(3)}(X_l^{(3)}) = \lambda_{3,l}X_l^{(3)}$ for $l \in \{1,2\}$. Performing similar calculations to before, we arrive at the following expressions for the generators of $A^{(3)}$:

$$\begin{aligned}
X_4^{(3)} &:= X_4^{(4)} = X_4,\\
X_3^{(3)} &:= X_3^{(4)} = X_3,\\
X_2^{(3)} &:= X_2^{(4)} + q^2(q^2-1)^{-1}\lambda_{3,2}^{-1}\delta_3(X_2^{(4)})(X_3^{(4)})^{-1}\\
&= X_2^{(4)}\\
&= X_2 - \frac{q^4}{(q^2+1)(q+q^{-1})}X_3^2X_4^{-1},\\
X_1^{(3)} &:= X_1^{(4)} + q^2(q^2-1)^{-1}\lambda_{3,1}^{-1}\delta_3(X_1^{(4)})(X_3^{(4)})^{-1}\\
&= X_1^{(4)} - q^2(q^2-1)^{-1}(q+q^{-1})X_2^{(4)}(X_3^{(4)})^{-1}\\
&= X_1 - \frac{q^2(q+q^{-1})}{q^2-1}X_2X_3^{-1}.
\end{aligned} \tag{5.7}$$

As there are no more derivations left in the presentation of $A^{(3)}$ as an iterated Ore extension, the algorithm stops here. We set $T_i := X_i^{(3)}$ for all $i \in [\![1,4]\!]$ and write $A^{(3)}$ as a uni-parameter quantum affine space,

$$A^{(3)} = A' = \mathbb{C}_{q^M}[T_1,T_2,T_3,T_4].$$

Finding $\psi(\langle z'\rangle) \in \mathrm{C.Spec}(A')$:

Recall (Definition 4.7) that $\psi := \psi_3 \circ \psi_4$, and set $S_3 := \{T_3^n \mid n \geq 0\}$ and $S_4 := \{(X_4^{(4)})^n \mid n \geq 0\}$ to be the multiplicatively closed Ore sets given in Theorem 3.6(III) (for $j=3$ and $j=4$, respectively). Note that in the equivalent subalgebras of F, i.e. $A'S_3^{-1} = A^{(4)}S_3^{-1} = A^{(4)}S_4^{-1} = AS_4^{-1}$ (Proposition 3.9(iii)), straightforward calculations confirm that

$$\hat{q}T_2T_4 = \hat{q}X_2^{(4)}X_4^{(4)} = z', \tag{5.8}$$

where $\hat{q} = -(q^2-q^{-2})(q+q^{-1}) \in \mathbb{C}^*$. Using this observation, we consider the ideal $\langle T_2\rangle \in \mathrm{C.Spec}(A')$ and track this back through the algorithm to show that $\psi^{-1}(\langle T_2\rangle) = \psi_4^{-2} \circ \psi_3^{-1}(\langle T_2\rangle) = \langle z'\rangle$, and hence that $\langle z'\rangle \in \mathrm{C.Spec}(A)$.

We first compute $\psi_3^{-1}(\langle T_2\rangle)$: Since $T_3 \notin \langle T_2\rangle$, then applying Proposition 4.6 and Lemma 4.3, and noting that $T_2 = X_2^{(4)}$ and $\langle T_2\rangle_{A'}S_3^{-1} = \langle T_2\rangle_{A'S_3^{-1}}$, we may write

$$\psi_3^{-1}(\langle T_2\rangle_{A'}) = \langle T_2\rangle_{A'S_3^{-1}} \cap A^{(4)} = \langle X_2^{(4)}\rangle_{A^{(4)}S_3^{-1}} \cap A^{(4)} = \langle X_2^{(4)}\rangle_{A^{(4)}}. \tag{5.9}$$

Next we compute $\psi_4^{-1}(\langle X_2^{(4)}\rangle)$: Since $X_4^{(4)} \notin \langle X_2^{(4)}\rangle$ then applying Proposition 4.6 and Lemma 4.3 gives

$$\psi_4^{-1}(\langle X_2^{(4)}\rangle) = \langle X_2^{(4)}\rangle_{A^{(4)}}S_4^{-1} \cap A = \langle X_2^{(4)}\rangle_{AS_4^{-1}} \cap A. \tag{5.10}$$

We now show that $\langle z'\rangle_{AS_4^{-1}} = \langle X_2^{(4)}\rangle_{AS_4^{-1}}$ using the observation in (5.8): Clearly, given any $a \in \langle z'\rangle_{AS_4^{-1}}$ there exists some $b \in AS_4^{-1}$ such that

$$a = z'b = \hat{q}X_2^{(4)}X_4^{(4)}b = X_2^{(4)}v \in \langle X_2^{(4)}\rangle_{AS_4^{-1}},$$

where $v = \hat{q}X_4^{(4)}b \in AS_4^{-1}$. Hence $\langle z'\rangle_{AS_4^{-1}} \subseteq \langle X_2^{(4)}\rangle_{AS_4^{-1}}$. Conversely, for any element $a' \in \langle X_2^{(4)}\rangle_{AS_4^{-1}}$ there exists some $b_1, b_2 \in AS_4^{-1}$ such that

$$a' = b_1X_2^{(4)}b_2 = b_1X_2^{(4)}X_4^{(4)}(X_4^{(4)})^{-1}b_2 = b_1\hat{q}^{-1}z'(X_4^{(4)})^{-1}b_2 = z'w \in \langle z'\rangle_{AS_4^{-1}},$$

where $w = \hat{q}^{-1}b_1(X_4^{(4)})^{-1}b_2 \in A^{(4)}S_4^{-1}$. Hence $\langle X_2^{(4)}\rangle_{AS_4^{-1}} \subseteq \langle z'\rangle_{AS_4^{-1}}$ and thus equality holds. The bijective correspondence between $\{I \in \text{C.Spec}(A) \mid I \cap S_4 = \emptyset\}$ and $\text{C.Spec}(AS_4^{-1})$ allows us to deduce that

$$\langle X_2^{(4)}\rangle_{AS_4^{-1}} \cap A = \langle z'\rangle_{AS_4^{-1}} \cap A = \langle z'\rangle_A.$$

Combining this with (5.9) and (5.10) we conclude that $\psi^{-1}(\langle T_2\rangle) = \langle z'\rangle$, hence $\langle z'\rangle \in \text{C.Spec}(A)$ and is a Cauchon ideal. In particular $\langle z'\rangle \in \text{C.Spec}_{\{2\}}(A)$ and therefore $\{2\} \in \mathscr{W}'$.

PI degree of $A'/\psi(\langle z'\rangle)$:

We are now in the position to apply Theorem 4.25 to this example, which implies that $A'/\langle T_2\rangle$ and $A/\langle z'\rangle$ are PI algebras with $\text{PI-deg}(A/\langle z'\rangle) = \text{PI-deg}(A'/\langle T_2\rangle)$. The quotient algebra $A'/\langle T_2\rangle$ simplifies to a quantum affine space of lower dimension,

$$A'/\langle T_2\rangle = \mathbb{C}_{q^{M'}}[t_1, t_3, t_4],$$

where $M' = \left(\begin{smallmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 2 & -2 & 0 \end{smallmatrix}\right)$ is obtained from M by deleting the second row and second column, and $t_i := T_i + \langle T_2 \rangle$, for all $i \in \{1,3,4\}$. It is easily verified that the skew normal form of M' is $S = \left(\begin{smallmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, hence M' has a kernel of dimension 1 and one pair of invariant factors, where $h_1 = 2$. Given that this is such a small example, we can apply Theorem 2.30 directly to $\mathbb{C}_{q^{M'}}[t_1,t_3,t_4]$ to calculate the cardinality h of the image of the homomorphism $\pi \circ M' : \mathbb{Z}^3 \to (\mathbb{Z}/\ell\mathbb{Z})^3$. We observe that

$$\pi \circ M' \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) = \begin{pmatrix} \overline{-2z} \\ \overline{2z} \\ \overline{2x-2y} \end{pmatrix}, \qquad \text{for all } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{Z}^3.$$

From this we see that we have 2 free entries in the image (in positions 1 and 3) and each of these entries has cardinality ℓ if ℓ is odd, and $\ell/2$ if ℓ is even. Therefore

$$h = \begin{cases} \ell^2 & \ell \text{ is odd;} \\ (\ell/2)^2 & \ell \text{ is even,} \end{cases}$$

and,

$$\text{PI-deg}(A/\langle z' \rangle) = \text{PI-deg}(A'/\langle T_2 \rangle) = \text{PI-deg}(\mathbb{C}_{q^{M'}}[T_1,T_3,T_4]) = \sqrt{h} = \begin{cases} \ell & \ell \text{ is odd;} \\ \ell/2 & \ell > 4 \text{ is even.} \end{cases}$$

We would recover the same result if, instead, we applied Lemma 5.7, using the properties of M' calculated above, to obtain

$$\text{PI-deg}(A/\langle z' \rangle) = \sum_{i=1}^{\frac{3-1}{2}} \frac{\ell}{\gcd(h_i,\ell)} = \frac{\ell}{\gcd(2,\ell)} = \begin{cases} \ell & \ell \text{ is odd;} \\ \ell/2 & \ell > 4 \text{ is even.} \end{cases}$$

5.3.2 A Quantum Schubert Variety: $\mathscr{O}_q(G_{2,4}(\mathbb{K}))_\gamma$ for $\gamma = \{1,3\}$

In this example we study the quantum analogue to the coordinate ring of a specific Schubert variety. *Quantum Schubert varieties* occur as a family of quotients of the so-called *quantum Grassmannian* and they are strongly linked to quantum determinantal rings. We exploit this strong link and apply the results in Section 4.5 to compute the PI degree of certain quantum Schubert varieties. The aforementioned link between quantum Schubert varieties and generalised quantum determinantal rings was shown by Lenagan and Rigal in [LR08].

Here, we present a summary of some of their results, before showing on a specific example how they may be used to compute the PI degree of a quantum Schubert variety.

The reader should beware the difference in notation: In their paper, Lenagan and Rigal define quantum determinantal rings for ideals I_t, where I_t is generated by all $t \times t$ quantum minors. In our definition (see Section 2.4.4), which agrees with [GL00], we define quantum determinantal rings for ideals I_t, where I_t is generated by all $(t+1) \times (t+1)$ quantum minors. When calling results from [LR08] we must therefore be careful to replace t with $t+1$ in order to match our definition.

The link to generalised quantum determinantal rings:

We begin by defining the objects mentioned above. For a full exposition of the following constructs, the reader is referred to [LR08].

Definition 5.8. Let $m,n \in \mathbb{N}_{>0}$ with $m \leq n$. The *quantum Grassmannian*, denoted by $\mathscr{O}_q(G_{m,n}(\mathbb{K}))$, is defined to be the subalgebra of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$ generated by the $m \times m$ quantum minors (called the *maximal quantum minors*). It is a deformation of the coordinate ring of the Grassmannian, $G_{m,n}(K)$, of m-dimensional subspaces of $\mathbb{K}^n$.

Just as index pairs $(I,J) \in \Delta_{m,n}$ correspond to quantum minors of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$, so too *index sets* $I = \{i_1 < \ldots < i_m\} \subseteq [\![1,n]\!]$ correspond to maximal minors $[[\![1,m]\!]|I]$ of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$, and hence to elements in $\mathscr{O}_q(G_{m,n}(\mathbb{K}))$. We denote by $\Pi_{m,n}$ the set of all index sets and we identify this with the set of all maximal minors of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$.

We equip the set $\Delta_{m,n}$ with a partial order $\leq_{\mathrm{st}}$ where, if $(I,J), (K,L) \in \Delta_{m,n}$ with $I = \{i_1 < \ldots < i_u\}, J = \{j_1 < \ldots < j_u\}, K = \{k_1 < \ldots < k_v\}, L = \{l_1 < \ldots < l_v\}$, then

$$(I,J) \leq_{\mathrm{st}} (K,L) \iff \begin{cases} u \geq v; \\ i_s \leq k_s & \text{for } s \in [\![1,v]\!]; \\ j_s \leq l_s & \text{for } s \in [\![1,v]\!]. \end{cases}$$

This partial order restricts to $\Pi_{m,n} \subset \Delta_{m,n}$ in the following way: For $I,J \in \Pi_{m,n}$, with $I = \{i_1 < \ldots < i_m\}$ and $J = \{j_1 < \ldots < j_m\}$, we have

$$I \leq_{\mathrm{st}} J \iff i_s \leq j_s \quad \text{for } s \in [\![1,m]\!].$$

We may now state the main object of interest for this example, as defined in [LR08, Definition 1.1]:

Definition 5.9. Let $\gamma \in \Pi_{m,n}$ and define the set $\Pi_{m,n}^{\gamma} = \{\alpha \in \Pi_{m,n} \mid \alpha \not\geq_{\text{st}} \gamma\}$. The *quantum Schubert variety associated to* γ is

$$\mathscr{O}_q(G_{m,n}(\mathbb{K}))_{\gamma} := \mathscr{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi_{m,n}^{\gamma} \rangle.$$

This is a deformation of the coordinate ring of a Schubert variety.

Lenagan and Rigal go on to define a class of quotients of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$, in a similar way [LR08, Definition 4.1]:

Definition 5.10. Let $\delta \in \Delta_{m,n}$ and set $\Delta_{m,n}^{\delta} = \{\alpha \in \Delta_{m,n} \mid \alpha \not\geq_{\text{st}} \delta\}$. The *generalised quantum determinantal ring associated to* δ is

$$\mathscr{O}_q(M_{m,n}(\mathbb{K}))_{\delta} := \mathscr{O}_q(M_{m,n}(\mathbb{K}))/\langle \Delta_{m,n}^{\delta} \rangle.$$

Remark 5.11. The quantum determinantal ring $R_t(M_{m,n}) := \mathscr{O}_q(M_{m,n}(\mathbb{K}))/I_t$ can be obtained from the definition above by taking $\delta = ([\![1,t-1]\!],[\![1,t-1]\!])$ (see [LR08, Remark 4.2(iv)]).

In [LR08, Proposition 4.3], an isomorphism is provided which links a localisation of a quantum Schubert variety with a skew-Laurent extension of a generalised quantum determinantal ring. This isomorphism is induced by the dehomogenisation map defined in [LR06, Section 3.5] and is constructed in the following way: For $\delta \in \Delta_{m,n}$, let $T = \{n+1,\ldots,n+m\}$ and $\gamma = \delta_{m,n}(\delta)$, where

$$\begin{aligned} \delta_{m,n} : \Delta_{m,n} &\longrightarrow \Pi_{m,m+n} \backslash \{T\} \\ (I,J) &\longmapsto K_{(I,J)} \end{aligned} \tag{5.11}$$

is an isomorphism of partially ordered sets and, for $I = \{i_1 < \ldots < i_t\}$ and $J = \{j_1 < \ldots < j_t\}$, we set $K_{(I,J)} := \{j_1,\ldots,j_t,n+1,\ldots,n+m\} \backslash \{n+m+1-i_1,\ldots,n+m+1-i_t\}$. Then there is an isomorphism,

$$D_{m,n}^{\delta} : \mathscr{O}_q(M_{m,n}(\mathbb{K}))_{\delta}[y,y^{-1};\phi] \longrightarrow \mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_{\gamma}[\overline{[T]}^{-1}], \tag{5.12}$$

which sends $\overline{[I|J]}$ to $\overline{[K_{(I,J)}]}\,\overline{[T]}^{-1}$ and y to $\overline{[T]}$ (where $\overline{[I|J]}$ denotes the canonical image of $[I|J]$ in the quotient algebra $\mathscr{O}_q(M_{m,n}(\mathbb{K}))_{\delta}$, and similarly for $\overline{[K_{(I,J)}]}$ in $\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_{\gamma}$). The automorphism ϕ of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))_{\delta}$ is defined as $\phi(\bar{X}_{i,j}) = q^{-1}\bar{X}_{i,j}$ for all generators $\bar{X}_{i,j} \in \mathscr{O}_q(M_{m,n}(\mathbb{K}))_{\delta}$.

We now proceed with our example, which is to compute the PI degree of $\mathscr{O}_q(G_{2,4}(\mathbb{K}))_{\{1,3\}}$. Let $m = n = 2$ and $\gamma = \{1,3\} \in \Pi_{2,4}$. From the maps defined above, we have $T = \{2,3\}$ and

$\delta = \delta_{2,2}^{-1}(\gamma) = (\{1\},\{1\}) \in \Delta_{2,2}$. Moreover,

$$\Delta_{2,2}^{\delta} = \{\alpha \in \Delta_{2,2} \mid \alpha \ngeq_{\mathrm{st}} (\{1\},\{1\})\} = (\{1,2\},\{1,2\}),$$
$$\Pi_{2,4}^{\gamma} = \{\alpha \in \Pi_{2,4} \mid \alpha \ngeq_{\mathrm{st}} (\{1,3\})\} = \{1,2\},$$

so that $\langle \Delta_{2,2}^{\delta} \rangle = \langle D_q \rangle$ and $\langle \Pi_{2,4}^{\gamma} \rangle = \langle [12|12] \rangle$. The isomorphism in (5.12) then becomes

$$D_{2,2}^{\delta} : \mathscr{O}_q(M_2(\mathbb{K}))/\langle D_q \rangle [y, y^{-1}; \phi] \longrightarrow \mathscr{O}_q(G_{2,4}(\mathbb{K}))_{\{1,3\}} [\overline{[1,2|2,3]}^{-1}]. \tag{5.13}$$

Since $1 \neq q \in \mathbb{K}^*$ is a primitive ℓ^{th} root of unity, both algebras in the map (5.13) are PI algebras. By the invariance of the PI degree under localisation (Corollary 2.15), and the isomorphism in (5.13), we see that

$$\text{PI-deg}(\mathscr{O}_q(M_2(\mathbb{K}))/\langle D_q \rangle [y; \phi]) = \text{PI-deg}(\mathscr{O}_q(G_{2,4}(\mathbb{K}))_{\gamma}). \tag{5.14}$$

We focus on computing the left hand side of this equality.

PI degree of $\mathscr{O}_q(M_2(\mathbb{K}))/\langle D_q \rangle [y; \phi]$:

Recalling the results from Section 4.5.1 (substituting $\lambda = q^{-2}$ and $p_{i,j} = q$, for all $i > j$, and relabelling the generators as $a := X_{1,1}$, $b := X_{1,2}$, $c := X_{2,1}$, $d := X_{2,2}$ for ease of reading), we may present $\mathscr{O}_q(M_2(\mathbb{K}))$ as the iterated Ore extension

$$\mathscr{O}_q(M_2(\mathbb{K})) = \mathbb{K}[a][b; \sigma_b][c; \sigma_c][d; \sigma_d, \delta],$$

where the actions of σ_b, σ_c, σ_d on the generators are determined by $q \in \mathbb{K}^*$ raised to the power of the entries of the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}.$$

For example, $\sigma_d(b) = q^{m_{4,2}} b = q^{-1} b$ and $\sigma_b(c) = q^{m_{2,3}} c = c$, etc. The σ_d-derivation is defined as

$$\begin{aligned} \delta : \mathbb{K}\langle a,b,c \rangle &\longrightarrow \mathbb{K}\langle a,b,c \rangle \\ a &\longmapsto (q^{-1} - q) bc \\ b, c &\longmapsto 0. \end{aligned}$$

We don't need the full force of the deleting derivations algorithm to obtain the results of Theorem 4.25 because this is such a small example. Instead we observe that, denoting the images of $a,\dots,d$ in $\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle$ as $\bar{a},\dots,\bar{d}$, we have $\bar{a} = q\bar{b}\bar{c}\bar{d}^{-1} \in \mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[\bar{d}^{-1}]$. Hence

$$\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[\bar{d}^{-1}] = \mathbb{K}_{q^{M'}}[\bar{b},\bar{c},\bar{d}^{\pm 1}], \tag{5.15}$$

where M' is the submatrix of M obtained by deleting the first row and first column.

Recall, from (5.14), that computing the PI degree of $\mathcal{O}_q(G_{2,4}(\mathbb{K}))_\gamma$ is equivalent to computing the PI degree of $\mathcal{O}_q(M_n(\mathbb{K}))/\langle D_q\rangle[y;\phi]$, where

$$\phi(\bar{a}) = q^{-1}\bar{a}, \quad \phi(\bar{b}) = q^{-1}\bar{b}, \quad \phi(\bar{c}) = q^{-1}\bar{c}, \quad \phi(\bar{d}) = q^{-1}\bar{d}.$$

Since ϕ is an automorphism, it extends to an automorphism ϕ' on $\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[\bar{d}^{-1}]$. We form Ore extensions over the algebras on either side of the equality in (5.15), using ϕ', which, as a consequence of the universal property of Ore extensions (Corollary 2.22), allows us to define an isomorphism

$$\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[\bar{d}^{-1}][y;\phi'] \longrightarrow \mathbb{K}_{q^{M'}}[\bar{b},\bar{c},\bar{d}^{\pm 1}][y';\phi']$$

taking y to y' and acting as the identity on $\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[\bar{d}^{-1}]$. From this, it is not hard to verify that

$$\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[y;\phi][\bar{d}^{-1}] \cong \mathbb{K}_{q^{\overline{M}}}[\bar{b},\bar{c},\bar{d},y][\bar{d}^{-1}],$$

where $\overline{M}$ is the (still skew-symmetric) matrix obtained from M' by attaching a column of 1's and a row of -1's as follows:

$$\overline{M} = \begin{pmatrix} M' & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

Therefore, by applying Corollary 2.15 and Lemma 5.7 we obtain

$$\text{PI-deg}(\mathcal{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[y;\phi]) = \text{PI-deg}(\mathbb{K}_{q^{\overline{M}}}[\bar{b},\bar{c},\bar{d},y]) = \prod_{i=1}^{\frac{4-\dim(\ker(\overline{M}))}{2}} \frac{\ell}{\gcd(h_i,\ell)},$$

where the h_i are the invariant factors of $\overline{M}$. The skew-normal form of $\overline{M}$ can be calculated easily (for example, using Maple), from which it may be verified that $\dim(\ker(\overline{M})) = 2$ and

$h_1 = 1$. This allows us to conclude this example by calculating the PI degree:

$$\text{PI-deg}(\mathscr{O}_q(G_{2,4}(\mathbb{K}))_\gamma) = \text{PI-deg}(\mathscr{O}_q(M_2(\mathbb{K}))/\langle D_q\rangle[y;\phi]) = \ell.$$

5.3.3 General Method for Quantum Schubert Varieties

In this section we present a more general method for calculating the PI degree of a quantum Schubert variety, which utilises the deleting derivations algorithm. We take $q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity, with $\ell > 2$, and m, n to be positive integers.

Recall the notation and constructs from the previous section, in particular the sets and maps which appear in (5.11) and (5.12). Let $\delta \in \Delta_{m,n}$ and $\gamma = \delta_{m,n}(\delta) \in \Pi_{m,m+n} \setminus \{T\}$, where $T = \{n+1, \dots, n+m\}$. Recall, too, the automorphism $\phi \in \text{Aut}_{\mathbb{K}}(\mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta$, defined as $\phi(\bar{X}_{i,j}) = q^{-1}\bar{X}_{i,j}$ for all generators $\bar{X}_{i,j} \in \mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta$.

Since q is a root of unity, the quotient algebra $\mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta$ is a PI algebra (as a finite module over a central subalgebra Z_0, as discussed in Section 2.4.3). It is clear, from the definition of ϕ, that y^ℓ is a central element and hence that the skew-Laurent extension $\mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta[y, y^{-1}; \phi]$ is finite dimensional over the central subalgebra $\mathbb{K}\langle Z_0, y^\ell\rangle$. Thus it is also a PI algebra. The isomorphism in (5.12) then allows us to deduce that

$$\text{PI-deg}(\mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta[y;\phi]) = \text{PI-deg}(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma). \tag{5.16}$$

We now make use of techniques from previous chapters to provide an expression for the left hand side of the equality above, thus giving an expression for the PI degree of the quantum Schubert variety on the right hand side of the equality.

Proposition 5.12. *Keep the notation and assumptions above. Let $A := \mathscr{O}_q(M_{m,n}(\mathbb{K}))$ and denote its set of Cauchon diagrams by $\mathscr{W}'$.*

(i) We have

$$\textit{PI-deg}(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma) = \textit{PI-deg}((\mathscr{O}_{q^M}(\mathbb{K}^{mn})/\psi(\langle\Delta^\delta_{m,n}\rangle))[y';\phi']),$$

where

- *$\mathscr{O}_{q^M}(\mathbb{K}^{mn})$ is the quantum affine space obtained by applying the deleting derivations algorithm to $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$;*
- *ψ is the canonical embedding $\psi : \text{C.Spec}(\mathscr{O}_q(M_{m,n}(\mathbb{K}))) \to \text{C.Spec}(\mathscr{O}_{q^M}(\mathbb{K}^{mn}))$;*
- *$\phi' \in \text{Aut}_{\mathbb{K}}(\mathscr{O}_{q^M}(\mathbb{K}^{mn})/\psi(\langle\Delta^\delta_{m,n}\rangle))$ is the automorphism defined as $\phi'(t_{r,s}) = q^{-1}t_{r,s}$ on the generators $t_{1,1}, \dots, t_{m,n}$ of $\mathscr{O}_{q^M}(\mathbb{K}^{mn})/\psi(\langle\Delta^\delta_{m,n}\rangle)$.*

(ii) If $\langle \Delta^{\delta}_{m,n} \rangle \in \mathrm{C.Spec}_w(\mathscr{O}_q(M_{m,n}(\mathbb{K})))$ is a Cauchon ideal, for some $w \in \mathscr{W}'$, then

$$PI\text{-}deg(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma) = PI\text{-}deg(\mathscr{O}_{q^{\overline{M}}}(\mathbb{K}^{k+1})),$$

where $k = mn - |w|$ and $\overline{M} \in M_{k+1}(\mathbb{Z})$ is obtained from M by first deleting the columns and rows indexed by w to obtain $M' \in M_k(\mathbb{Z})$, and then extending M' by a column of 1's and a row of -1's to obtain $\overline{M} = \begin{pmatrix} M' & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{pmatrix}$.

Proof. Applying the deleting derivations algorithm (Theorem 3.6) to $A = \mathscr{O}_q(M_{m,n}(\mathbb{K}))$ we obtain a quantum affine space

$$A' = \mathscr{O}_{q^M}(\mathbb{K}^{mn}) = \mathbb{K}_{q^M}[T_{1,1}, \dots, T_{m,n}],$$

for some $M \in M_{mn}(\mathbb{K})$, with $\text{PI-deg}(A) = \text{PI-deg}(A')$. Recall from [LR08, Proposition 4.3] that $\mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta = A/\langle \Delta^{\delta}_{m,n} \rangle$ is an integral domain, hence $\langle \Delta^{\delta}_{m,n} \rangle \in \mathrm{C.Spec}(A)$. Let $P := \langle \Delta^{\delta}_{m,n} \rangle$, so that $\mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta = A/P$, and denote its image under the canonical embedding by $Q = \psi(\langle \Delta^{\delta}_{m,n} \rangle) \in \mathrm{C.Spec}(A')$.

We adapt the notation in Section 4.3 to this example to make the proof more readable. Let $\bar{X}_{r,s} \in A/P$ be the canonical image of $X_{r,s} \in A$ and $t_{r,s} \in A'/Q$ be the canonical image of $T_{r,s} \in A'$, for all $(1,1) \le (r,s) \le (m,n)$. Using the maps defined in Proposition 4.6, we set $P_{u,v} := \psi_{(u,v)} \circ \cdots \circ \psi_{(m,n)}(P) \in \mathrm{C.Spec}(A^{(u,v)})$, for all $(1,2) \le (u,v) \le (m,n)^+$. Finally, for all $(1,2) \le (u,v) \le (m,n)^+$, we set $B^{(u,v)} := A^{(u,v)}/P_{u,v}$ and denote by $\bar{X}^{(u,v)}_{r,s} \in B^{(u,v)}$ the canonical image of $X^{(u,v)}_{r,s} \in A^{(u,v)}$, for all $(1,1) \le (r,s) \le (m,n)$. Note that $B^{(m,n)^+} = A/P$ with generators $\bar{X}^{(m,n)^+}_{r,s} = \bar{X}_{r,s}$, and $B^{(1,2)} = A'/Q$ with generators $\bar{X}^{(1,2)}_{r,s} = t_{r,s}$.

(i) Given the notation above, the statement that we wish to prove becomes

$$\text{PI-deg}(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma) = \text{PI-deg}((A'/Q)[y';\phi']), \tag{5.17}$$

where $\phi' \in \mathrm{Aut}_{\mathbb{K}}(A'/Q)$ is defined as $\phi'(t_{r,s}) = q^{-1}t_{r,s}$, for all $t_{r,s} \in A'/Q$. We apply Theorem 4.25 to $A/P = \mathscr{O}_q(M_{m,n}(\mathbb{K}))_\delta$ to see that there exists multiplicatively closed Ore sets, $\Gamma \subseteq A/P$ and $\Sigma \subseteq A'/Q$, such that

$$(A/P)\Gamma^{-1} = (A'/Q)\Sigma^{-1} \subseteq \mathrm{Frac}(A/P). \tag{5.18}$$

It is sufficient to show that

$$(A/P)[y;\phi]\Gamma^{-1} \cong (A'/Q)[y';\phi']\Sigma^{-1}, \tag{5.19}$$

as then, the equality in (5.17) will follow from Corollary 2.15 as well as the equality in (5.16), which, rewritten in the notation above, becomes

$$\text{PI-deg}((A/P)[y;\phi]) = \text{PI-deg}(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma).$$

We recall from the discussion at the beginning of this example that $\phi \in \mathrm{Aut}_{\mathbb{K}}(A/P)$ is defined as $\phi(\bar{X}_{i,j}) = q^{-1}\bar{X}_{i,j}$, for all generators $\bar{X}_{i,j} \in A/P$. This extends naturally to an automorphism $\bar{\phi}$ on $(A/P)\Gamma^{-1}$ and to an automorphism Φ on $\mathrm{Frac}(A/P)$. Restricting Φ to A/P and $(A/P)\Gamma^{-1}$ returns ϕ and $\bar{\phi}$, respectively. As a consequence of the universal property of Ore extensions (Corollary 2.22) and the equality in (5.18), we obtain the following isomorphism:

$$(A/P)\Gamma^{-1}[y;\bar{\phi}] \cong (A'/Q)\Sigma^{-1}[y';\bar{\phi}]. \tag{5.20}$$

We now show that $\bar{\phi}(t_{r,s}) = q^{-1}(t_{r,s})$, for all generators $t_{r,s} \in A'/Q$, by using an argument of decreasing induction on $(1,2) \le (u,v) \le (m,n)^+$ to show that $\Phi(\bar{X}_{r,s}^{(u,v)}) = q^{-1}\bar{X}_{r,s}^{(u,v)}$, for all $(1,1) \le (r,s) \le (m,n)$. Setting $(u,v) := (1,2)$ and noting that $\bar{\phi}(t_{r,s}) = \Phi(t_{r,s})$ will then prove the result.

Let $(u,v) = (m,n)^+$, then $\bar{X}_{r,s}^{(m,n)^+} = \bar{X}_{r,s}$ and $\Phi(\bar{X}_{r,s}) = \phi(\bar{X}_{r,s}) = q^{-1}\bar{X}_{r,s}$, by the definition of ϕ. This proves the base case.

Take some $(1,2) \le (u,v) \le (m,n)$ and assume that $\Phi(\bar{X}_{r,s}^{(u,v)^+}) = q^{-1}\bar{X}_{r,s}^{(u,v)^+}$, for all $(1,1) \le (r,s) \le (m,n)$. Using Proposition 4.23(ii) we may write

$$\bar{X}_{r,s}^{(u,v)} = \begin{cases} \bar{X}_{r,s}^{(u,v)^+} & (r,s) \ge (u,v); \\ \bar{X}_{r,s}^{(u,v)^+} & (r,s) < (u,v) \text{ and } (r = u \text{ or } s = v); \\ \bar{X}_{r,s}^{(u,v)^+} - q\bar{X}_{r,v}^{(u,v)^+}\bar{X}_{u,s}^{(u,v)^+}(\bar{X}_{u,v}^{(u,v)^+})^{-1} & r < u \text{ and } s < v. \end{cases}$$

From this we see that if $r \ge u$ or $s \ge v$, then

$$\Phi(\bar{X}_{r,s}^{(u,v)}) = \Phi(\bar{X}_{r,s}^{(u,v)^+}) = q^{-1}\bar{X}_{r,s}^{(u,v)^+} = q^{-1}\bar{X}_{r,s}^{(u,v)}.$$

Otherwise, if $r<u$ and $s<u$, then

$$\begin{aligned}\Phi(\bar{X}_{r,s}^{(u,v)}) &= \Phi(\bar{X}_{r,s}^{(u,v)^+}) - q\Phi(\bar{X}_{r,v}^{(u,v)^+})\Phi(\bar{X}_{u,s}^{(u,v)^+})\Phi((\bar{X}_{u,v}^{(u,v)^+})^{-1}) \\ &= q^{-1}\bar{X}_{r,s}^{(u,v)^+} - q(q^{-1}\bar{X}_{r,v}^{(u,v)^+}q^{-1}\bar{X}_{u,s}^{(u,v)^+}q(\bar{X}_{u,v}^{(u,v)^+})^{-1}) \\ &= q^{-1}(\bar{X}_{r,s}^{(u,v)^+} - q\bar{X}_{r,v}^{(u,v)^+}\bar{X}_{u,s}^{(u,v)^+}(\bar{X}_{u,v}^{(u,v)^+})^{-1}) \\ &= q^{-1}\bar{X}_{u,s}^{(u,v)}.\end{aligned}$$

This proves the inductive step and completes the proof that $\Phi(\bar{X}_{r,s}^{(u,v)}) = q^{-1}\bar{X}_{r,s}^{(u,v)}$ for all $(1,2)\le(u,v)\le(m,n)^+$ and all $(1,1)\le(r,s)\le(m,n)$. Therefore, taking $(u,v)=(1,2)$ we conclude that $\bar{\phi}(t_{r,s}) = q^{-1}(t_{r,s})$, for all $(1,1)\le(r,s)\le(m,n)$.

Using (5.20) we see that $\bar{\phi}$ restricts to both A/P and A'/Q. Restricting $\bar{\phi}$ to A/P returns the automorphism ϕ and, denoting the restriction of $\bar{\phi}$ to A'/Q as ϕ', we obtain

$$(A/P)[y;\phi]\Gamma^{-1} \cong (A'/Q)[y';\phi']\Sigma^{-1},$$

where $\phi'(t_{r,s}) = \bar{\phi}(t_{r,s}) = q^{-1}t_{r,s}$, for all $(1,1)\le(r,s)\le(m,n)$. This proves the isomorphism in (5.19), as required, and thus completes the proof of part (i).

(ii) Suppose $\langle\Delta_{m,n}^{\delta}\rangle \in \mathrm{C.Spec}_w(A)$ is a Cauchon ideal, for some $w\in\mathscr{W}'$. Using Remark 5.2 we write

$$A'/Q = \mathscr{O}_{q^M}(\mathbb{K}^{mn})/\psi(\langle\Delta_{m,n}^{\delta}\rangle) = \mathbb{K}_{q^M}[T_{1,1},\dots,T_{m,n}]/\langle T_{i,j}\mid(i,j)\in w\rangle = \mathscr{O}_{q^{M'}}(\mathbb{K}^k), \tag{5.21}$$

where $k=mn-|w|$, and $M'\in M_k(\mathbb{Z})$ is obtained from M by deleting the columns and rows indexed by w. Let $\phi'\in\mathrm{Aut}_{\mathbb{K}}(\mathscr{O}_{q^{M'}}(\mathbb{K}^k))$ be such that $\phi'(t_i)=q^{-1}t_i$ for all $i\in[\![1,k]\!]$, where $t_1,\dots,t_k$ are the generators of $\mathscr{O}_{q^{M'}}(\mathbb{K}^k)$. Note that this is the same map as defined in part (i), we have just relabelled the generators of $\mathscr{O}_{q^M}(\mathbb{K}^{mn})/\psi(\langle\Delta_{m,n}^{\delta}\rangle) = \mathscr{O}_{q^{M'}}(\mathbb{K}^k)$ from $t_{r,s}$ to t_i. Using ϕ' we form an Ore extension over $\mathscr{O}_{q^{M'}}(\mathbb{K}^k)$,

$$\mathscr{O}_{q^{M'}}(\mathbb{K}^k)[y';\phi'] = \mathbb{K}_{q^{M'}}[t_1,\dots,t_k][y';\phi'] = \mathbb{K}_{q^{\overline{M}}}[t_1,\dots,t_k,y'] = \mathscr{O}_{q^{\overline{M}}}(\mathbb{K}^{k+1}), \tag{5.22}$$

where $\overline{M} = \begin{pmatrix} M' & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{pmatrix} \in M_{k+1}(\mathbb{Z})$. Combining the result of part (i) with the equalities in (5.21) and (5.22), we may conclude:

$$\begin{aligned}\text{PI-deg}(\mathscr{O}_{q^{\overline{M}}}(\mathbb{K}^{k+1})) &= \text{PI-deg}(\mathscr{O}_{q^{M'}}(\mathbb{K}^k)[y';\phi']) \\ &= \text{PI-deg}((A'/Q)[y';\phi']) \\ &= \text{PI-deg}(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma).\end{aligned}$$

□

Combining this proposition with Lemma 5.7 gives the following:

Corollary 5.13. *Let $\delta \in \Delta_{m,n}$ and $\gamma \in \Pi_{m,m+n} \backslash \{T\}$ be as in Proposition 5.12. If $\langle \Delta_{m,n}^{\delta} \rangle \in$ C.Spec$_w(\mathscr{O}_q(M_{m,n}(\mathbb{K})))$ is a Cauchon ideal, for some $w \in \mathscr{W}'$, then*

$$\textit{PI-deg}(\mathscr{O}_q(G_{m,m+n}(\mathbb{K}))_\gamma) = \prod_{i=1}^{\frac{k+1-\dim(\ker(\overline{M}))}{2}} \frac{\ell}{\gcd(h_i,\ell)},$$

where $k = mn - |w|$, $\overline{M} \in M_{k+1}(\mathbb{Z})$ is the matrix defined in Proposition 5.12(ii), and $h_1,\ldots,h_s$ are the invariant factors of $\overline{M}$ (where $2s = k+1-\dim(\ker(\overline{M}))$).

The corollary above encourages us to investigate the properties of the matrix $\overline{M}$, the matrix extended from M'. In next chapter we present properties of M' in the special case of quantum determinantal rings, and we discuss the potential of using this information to gain information about the properties of $\overline{M}$.

5.4 Open Questions

We now address some of the questions and open problems arising from work in this chapter.

Steps 3(b) and 3*(b) in the general method for calculating PI-deg(A/P) motivate the following two questions:

1) When is $P \in$ C.Spec(A) a Cauchon ideal?

Analogues to this question have been answered in the generic case by Cauchon [Cau03a, Lemme 5.5.8], and in the Poisson setting by Lecoutre [Lec14, Lemma 5.5.10]. In both settings, this was achieved by considering a torus action on the starting algebra, with certain assumptions placed on the torus as well as, possibly, on the base field (see [Cau03a, Hypothèse 4.1.2 and Hypothèse 4.1.2] for these assumptions in the generic case, and [Lec14, Section 5.5.2] for these assumptions in the Poisson setting). In particular, the torus-invariant prime ideals (respectively, Poisson prime ideals) provide the answer to the analogous question in each setting.

Naturally, then, we investigated the effects of a rational torus action on the completely prime spectra of A and A' in the root of unity case. We can show that such an action commutes with the deleting derivation homomorphism and, therefore, extends to a rational torus action on A'. Furthermore, the torus-invariant completely prime ideals in A are mapped to torus-invariant completely prime ideals in A'. In order to conclude in a similar manner to Cauchon and Lecoutre we require the following statement to be true:

Let $T = (A'/J_w)S_w^{-1}$ be a quantum torus, for some $w \in \mathscr{W}$ where $w \neq [\![1,N]\!]$, and S_w the multiplicatively closed set generated by all $\bar{T}_i \in A'/J_w$ for $i \in [\![1,N]\!] \backslash w$. Assume T supports a rational torus action coming from a rational torus action, $\mathscr{H}$, on A. If $I \in \mathrm{C.Spec}(T)$ is an $\mathscr{H}$-invariant completely prime ideal, then $I = \{0\}$.

A positive result would imply that the only $\mathscr{H}$-invariant completely prime ideals of A' are the ideals J_w, for all $w \in \mathscr{W}$, and hence every $\mathscr{H}$-invariant ideal $P \in \mathrm{C.Spec}(A)$ is a Cauchon ideal. Unfortunately, the methods employed by Cauchon and Lecoutre to prove analogues to the statement above ([Cau03a, Lemme 5.5.3] and [Lec14, Proposition 3.4.3], respectively) break down in the root of unity case due to the existence of the so-called ℓ-centre - elements of the algebra A raised to the ℓ^{th} power which commute with everything and may form monomials which remain unchanged under the torus action.

We have found a counterexample to this statement, when $A = \mathscr{O}_q(M_2(\mathbb{K}))$, $w = \emptyset$ and $\ell = 2$, which suggests the need for at least some conditions on the primitive ℓ^{th} root of unity q; for example, that ℓ be greater than 2. Indeed, if $\ell > 2$ then, working in collaboration with Lewis Topley, we have been able to confirm that the statement above holds when $A = \mathscr{O}_q(M_2(\mathbb{K}))$ and $w = \emptyset$, using an algebraic geometric argument. That is, if $T = \mathbb{K}_{q^M}[a^{\pm1}, b^{\pm1}, c^{\pm1}, d^{\pm1}]$ with $M = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$, and the torus $\mathscr{H} = (\mathbb{K}^*)^3$ acts on T in the following way:

$$(\alpha, \beta, \gamma) \cdot (a, b, c, d) = (\alpha a, \beta b, \gamma c, \beta\gamma\alpha^{-1} d),$$

then T contains no non-trivial $\mathscr{H}$-invariant completely prime ideals. In and of itself, this result is not sufficient to conclude anything about which $\mathscr{H}$-invariant completely prime ideals in $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$ are Cauchon ideals, and attempts to induct on this result in order to prove it for quantum matrices of arbitrary dimension fail, again due to the existence of an ℓ-centre. For this reason we have omitted these results from the thesis. In light of the discussion above, it seems that different techniques may be required to investigate the validity of the statement for any T.

2) Given $J_w \in \mathrm{C.Spec}_w(A')$ for some $w \in \mathscr{W}'$, what is $\psi^{-1}(J_w) \in \mathrm{C.Spec}_w(A)$?

It is implied in the question that we know the full description of the set of Cauchon diagrams $\mathscr{W}'$ for A, although this in itself is still an open problem for general algebras. Since we have the description of $\mathscr{W}'$ for quantum matrices (Theorem 4.37), we focus on this question for $A = \mathscr{O}_q(M_{m,n}(\mathbb{K}))$.

Again, looking to the generic case for inspiration (that is when q is not a root of unity) we find a paper of Casteels [Cas14], in which he utilises Cauchon's deleting derivations algorithm to prove that all torus-invariant prime ideals of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$ are generated by quantum minors. We can show that, in the root of unity case, A still supports a rational torus action which transfers to a rational torus action on A', and that J_w and $\psi^{-1}(J_w)$ are invariant under these actions (see the discussion for Question 1 above). Therefore a natural starting point for resolving this open question in the root of unity setting is to see whether the method employed in [Cas14] still works when applying the deleting derivations algorithm introduced in Section 3.2 in place of Cauchon's algorithm.

Chapter 6

PI Degree of Quotients of Quantum Matrices

In this chapter we focus on the single parameter quantum matrices, $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$, where $q \in \mathbb{K}^*$ is some nonzero field element such that $q^2 \neq 1$, so that the algebra satisfies Hypothesis 1. Let $\mathscr{O}_{q^M}(\mathbb{K}^{mn})$, for $M \in M_{mn}(\mathbb{Z})$, be the quantum affine space obtained at the end of the deleting derivations algorithm applied to $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$ with the generators presented in lexicographic order. Let $\mathscr{W} := \mathbb{P}([\![1,m]\!] \times [\![1,n]\!])$ and $\mathscr{W}' \subseteq \mathscr{W}$ denote the set of Cauchon diagrams of $\mathscr{O}_q(M_{m,n}(\mathbb{K}))$. Let $P \in \mathrm{C.Spec}_w(\mathscr{O}_q(M_{m,n}(\mathbb{K})))$ be a Cauchon ideal, so that $\mathscr{O}_{q^M}(\mathbb{K}^{mn})/\psi(P) = \mathscr{O}_{q^{M'}}(\mathbb{K}^{mn-|w|})$, by Remark 5.2. When $q \in \mathbb{K}^*$ is a primitive ℓ^{th} root of unity (with $\ell > 2$) we have

$$\text{PI-deg}(\mathscr{O}_q(M_{m,n}(\mathbb{K}))/P) = \text{PI-deg}(\mathscr{O}_{q^{M'}}(\mathbb{K}^{mn-|w|})), \tag{6.1}$$

where $M' \in M_{mn-|w|}(\mathbb{Z})$ is the skew-symmetric matrix obtained from M by deleting the rows and columns indexed by w.

In Section 6.1 we associate to any $m \times n$ Cauchon-Le diagram C a matrix $M(C)$ and show how properties of $M(C)$, namely the dimension of its kernel and information about its invariant factors, may be obtained through properties of C. The main result of this chapter states that the invariant factors of the matrix corresponding to a Cauchon-Le diagram are all powers of 2. In Section 6.2 we specialise to quantum determinantal rings, where we give an explicit calculation of the PI degree of $R_t(M_n)$ when $q \in \mathbb{K}^*$ is a primitive ℓ^{th} root of unity and ℓ is odd.

6.1 Properties of M' Using Cauchon-Le Diagrams

Given an $m \times n$ Cauchon-Le diagram C we may compute its *toric permutation* τ, as defined in [BCL12, Section 4.1], by laying pipes over the squares such that we place a “cross" on each black square and a “hyperbola" on each white square. We label the sides of the diagram with the numbers $1,\dots,m+n$ such that each pair of opposite sides share the same labels in the same order. The permutation, τ, may then be read off this diagram by defining $\tau(i)$ to be the label (on the left or top side of C) reached by following the pipe starting at label i (on the right or bottom side of C).

For each $m \times n$ Cauchon-Le diagram, C, with N white squares, we may construct a skew-symmetric integral matrix, $M(C) \in M_N(\mathbb{Z})$, as follows: Label the white squares from 1 to N in such a way that the labels increase along the rows (from left to right) and down the columns (from top to bottom). Given such a labelling, we construct $M(C)$ according the rule

$$M(C)[i,j] = \begin{cases} 1 & \text{if square } i \text{ is stricly below or strictly to the right of square } j; \\ -1 & \text{if square } i \text{ is strictly above or strictly to the left of square } j; \\ 0 & \text{otherwise.} \end{cases}$$

Note that this construction of $M(C)$ is also valid if C is any $m \times n$ diagram, i.e. if C does not satisfy the Cauchon property stated in Section 4.5.1.

Recalling the notation from Section 4.5, Figure 6.1 shows the 3×5 labelled Cauchon-Le diagram $C = C_w$ associated to $w = \{(1,2),(1,4),(2,2),(3,1),(3,2),(3,3)\}$ on the left, and its pipe dream construction showing the toric permutation, $\tau = (17)(26384)$, on the right.

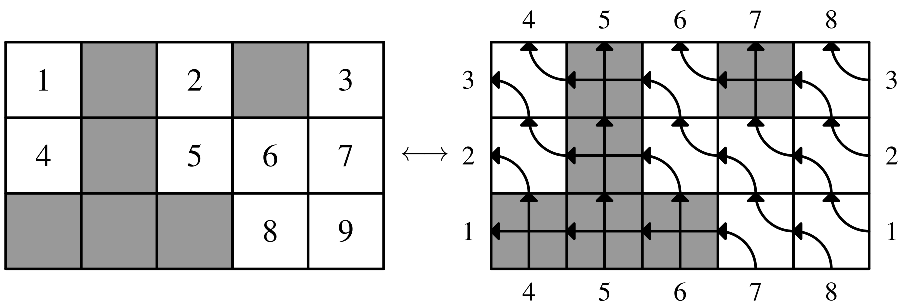

Fig. 6.1 A labelled Cauchon-Le diagram (left) with pipe dream construction (right) showing $\tau = (17)(26384)$

The matrix associated to the Cauchon-Le diagram C_w in Figure 6.1 is:

$$M(C_w) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}$$

Properties of a Cauchon-Le diagram provide information about its associated matrix. Indeed, combining [BCL12, Theorem 4.6 and Lemma 4.3] gives the following result:

Proposition 6.1 (Bell, Casteels, Launois; [BCL12]). *Let C be an $m \times n$ diagram and $M(C)$ be its associated matrix. Then the dimension of the kernel of $M(C)$ is the number of odd cycles in the disjoint cycle decomposition of its toric permutation τ.*

Remark 6.2. A cycle is odd if it can be written as an odd number of inversions. That is, odd cycles have even length. Given a diagram C with toric permutation τ, let $\mathrm{odd}(\tau)$ denote the number of odd cycles in the disjoint cycle decomposition of τ.

The next result provides information about the invariant factors of $M(C)$, which has powerful consequences for us when ℓ is odd.

Theorem 6.3. *Let C be an $m \times n$ Cauchon-Le diagram and $M(C)$ be its associated matrix. Then the invariant factors of $M(C)$ are all powers of 2.*

Proof. We prove that all invariant factors of $M(C)$ are powers of 2 using increasing induction on the dimension, d, of the kernel of $M(C)$.

If $d = 0$ then the matrix $M(C)$ is invertible and it follows, from [BLL10, Theorem 2.2], that the determinant of $M(C)$ is a power of 4. Let $\det(M(C)) = 4^a$ for some $a \in \mathbb{N}$. By Remark 5.2, $M(C)$ is congruent to the block diagonal matrix

$$\mathrm{Sk}(C) := \begin{pmatrix} 0 & h_1 & & & & & & \\ -h_1 & 0 & & & & & & \\ & & 0 & h_2 & & & & \\ & & -h_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & h_s \\ & & & & & & -h_s & 0 \end{pmatrix},$$

where $\mathrm{rank}(M(C)) = 2s$ and $h_i \mid h_{i+1} \in \mathbb{Z}\backslash\{0\}$, for $i \in [\![1, s-1]\!]$. The matrix $\mathrm{Sk}(C)$ is clearly equivalent to the Smith normal form of $M(C)$, which we denote by

$$\mathrm{Sm}(C) = \mathrm{diag}(h_1, h_1, h_2, h_2, \ldots, h_s, h_s).$$

We see that $\det(\mathrm{Sk}(C)) = \det(\mathrm{Sm}(C))$ since $\mathrm{Sm}(C)$ is obtained from $\mathrm{Sk}(C)$ by performing s distinct row swaps and multiplying s rows by -1, thus changing the determinant of $\mathrm{Sk}(C)$ by a factor of $(-1)^{2s} = 1$. The congruence relation then implies that

$$\det(M(C)) = \det(\mathrm{Sk}(C)) = \det(\mathrm{Sm}(C)).$$

Since the nonzero entries h_i in $\mathrm{Sm}(C)$ are the invariant factors of the matrix $M(C)$, we may use the observations above to deduce that

$$4^a = \det(M(C)) = \det(\mathrm{Sm}(C)) = \prod_{i=1}^{s} h_i^2 \quad \Longrightarrow \quad 2^a = \prod_{i=1}^{s} h_i.$$

Therefore, each h_i is a power of 2.

Now let $d > 0$ and assume the statement of the theorem holds for all Cauchon-Le diagrams whose associated matrix has kernel of dimension less than d. That is, for all Cauchon-Le diagrams C', with $\dim(\ker(M(C'))) < d$, the invariant factors of $M(C')$ are powers of 2. Consider a Cauchon-Le diagram C whose associated matrix $M(C)$ has kernel of dimension d, and let N be the number of white squares in C so that $M(C)$ is an $N \times N$ matrix. Then, by [BL10, Proposition 4.6], there exists a Cauchon-Le diagram C' obtained by adding exactly one black square to C, with the property

$$\dim(\ker(M(C'))) = d' = d - 1.$$

Label the white squares of C as $l_1 < \ldots < l_N$ and suppose we obtain C' by colouring the white square labelled l_i black, for some $i \in [\![1, N]\!]$. Then $M(C')$ is the $(N-1) \times (N-1)$ submatrix of $M(C)$ obtained by removing the row and column indexed by i. Noting that $d' = d - 1 < d$, we apply the induction hypothesis to $M(C')$ to see that all its invariant factors, denoted by $h'_1, \ldots, h'_{s'}$ for some $s' \in \mathbb{N}$, are powers of 2. Let $h_1, \ldots, h_s$ be the invariant factors of $M(C)$ and note that

$$s' = N - 1 - (d - 1) = N - d = s.$$

We may therefore write the Smith normal forms of $M(C)$ and $M(C')$, respectively, as

$$\mathrm{Sm}(C) = \mathrm{diag}(h_1, h_1, \ldots, h_s, h_s, 0, \ldots, 0), \tag{6.2}$$
$$\mathrm{Sm}(C') = \mathrm{diag}(h'_1, h'_1, \ldots, h'_s, h'_s, 0, \ldots, 0). \tag{6.3}$$

A well-known result for the Smith normal form (see [New72, Chapter II subsection 16]) applied to $M(C')$ allows us to write

$$D_{2i-1}(M(C')) = h'_i \cdot D_{2i-2}(M(C')), \tag{6.4}$$
$$D_{2i}(M(C')) = h'_i \cdot D_{2i-1}(M(C')), \tag{6.5}$$

where $D_j(M(C'))$ is the greatest common divisor of all $j \times j$ minors of the matrix $M(C')$, for all $j \in [\![1, N-1]\!]$, and $D_0(M(C')) := 1$. Clearly any minor of $M(C')$ of size j is also a minor of $M(C)$ of size j, so $D_j(M(C))$ divides $D_j(M(C'))$. Therefore, if $D_j(M(C'))$ is a power of 2, for all $j \in [\![1, 2s]\!]$, then $D_j(M(C))$ is also a power of 2, for all $j \in [\![1, 2s]\!]$. We now apply an induction argument on $i \in [\![1, s]\!]$ to show that $D_{2i-1}(M(C'))$ and $D_{2i}(M(C'))$ are powers of 2 for all i.

The base case, when $i = 1$, is easily seen to hold since h'_1 is a power of 2, by the inductive hypothesis on d, and (6.4) and (6.5) give

$$D_1(M(C')) = h'_1 \cdot D_0(M(C')) = h'_1 \cdot 1,$$
$$D_2(M(C')) = h'_1 \cdot D_1(M(C')) = h'_1 \cdot h'_1.$$

Assume now that $D_{2i-1}(M(C'))$ and $D_{2i}(M(C'))$ are powers of 2, for some $i \in [\![1, s-1]\!]$, and consider $D_{2i+1}(M(C'))$ and $D_{2i+2}(M(C'))$. Using (6.4) and (6.5), we write these as

$$D_{2i+1}(M(C')) = h'_{i+1} \cdot D_{2i}(M(C')),$$
$$D_{2i+2}(M(C')) = h'_{i+1} \cdot D_{2i+1}(M(C')),$$

where h'_{i+1} is a power of 2, by the induction hypothesis on d, and $D_{2i}(M(C'))$ is a power of 2, by the induction hypothesis on $i \in [\![1, s]\!]$. Hence, using the equations above, we may write

$$D_{2i+1}(M(C')) = 2^a \cdot 2^b = 2^{a+b},$$
$$D_{2i+2}(M(C')) = 2^a \cdot 2^{a+b} = 2^{2a+b},$$

for some $a, b \in \mathbb{N}$. This proves the inductive step for the induction on i and, hence, we conclude that $D_j(M(C))$ is a power of 2, for all $j \in [\![1, 2s]\!]$.

Finally, to conclude the main induction on d we apply the identity (6.5) to $M(C)$, and use the result of the previous induction argument, to show that, for any $i \in [\![1,s]\!]$,

$$h_i = \frac{D_{2i}(M(C))}{D_{2i-1}(M(C))} = \frac{2^e}{2^f} = 2^{e-f} \in \mathbb{Z},$$

for some $e, f \in \mathbb{N}$ such that $e \geq f$. This proves that all the invariant factors, h_i, of $M(C)$ are powers of 2, thus completing the main proof by induction. $\square$

Remark 6.4. Let $w \in \mathscr{W}'$ and take $P \in \mathrm{C.Spec}_w(\mathscr{O}_q(M_{mn}(\mathbb{K})))$ to be a Cauchon ideal, with $q \in \mathbb{K}^*$ a primitive ℓ^{th} root of unity and $\ell > 2$. It can be verified, through direct construction of $M(C_w)$, that the matrix M' in (6.1) is the one associated to the Cauchon-Le diagram C_w. Proposition 6.1 implies that $\dim(\ker(M')) = \mathrm{odd}(\tau)$, where τ is the toric permutation associated to C_w. Therefore, by Lemma 5.7, we may write

$$\text{PI-deg}(\mathscr{O}_{q^{M'}}(\mathbb{K}^{mn-|w|})) = \prod_{i=1}^{\frac{mn-|w|-\mathrm{odd}(\tau)}{2}} \frac{\ell}{\gcd(h_i, \ell)},$$

where the h_i are the invariant factors of the matrix $M' = M(C_w) \in M_{mn-|w|}(\mathbb{Z})$. These are all powers of 2, by Theorem 6.3, hence, if ℓ is odd then the PI degree becomes

$$\text{PI-deg}(\mathscr{O}_q(M_{m,n}(\mathbb{K})/P) = \ell^{\frac{mn-|w|-\mathrm{odd}(\tau)}{2}}.$$

We now illustrate these results with an example.

Example 6.5. Take $q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity, with $\ell > 2$, and let $A = \mathscr{O}_q(M_{3,5})(\mathbb{K})$. From Section 4.5 we see that A satisfies Hypothesis 1 and, applying the deleting derivations algorithm, we obtain the quantum affine space $A' = \mathbb{K}_{q^M}[T_{1,1}, \dots, T_{3,5}]$. Here, $M = M(D) \in M_{15}(\mathbb{K})$ is the skew-symmetric matrix corresponding to the 3×5 Cauchon-Le diagram D containing no black squares.

Taking $w = \{(1,2),(1,4),(2,2),(3,1),(3,2),(3,3)\}$ we construct the 3×5 diagram C_w, as seen in Figure 6.1, and confirm that it is a Cauchon-Le diagram. From Theorem 4.37 we deduce that $J_w \in \mathrm{C.Spec}_w(A') \cap \mathrm{Im}(\psi)$, where $J_w := \langle T_{i,j} \mid (i,j) \in w \rangle$ and $\psi : \mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$ is the canonical embedding. Thus

$$A'/J_w = \mathbb{K}_{q^{M'}}[T_{1,1}, T_{1,3}, T_{1,5}, T_{2,1}, T_{2,3}, T_{2,4}, T_{2,5}, T_{3,4}, T_{3,5}],$$

where $M' \subseteq M$ is the 9×9 submatrix obtained from M by deleting the rows and columns indexed by $(i,j)\in w$, that is, $M' = M(C_w)$. Therefore, by Theorem 4.25, we observe,

$$\text{PI-deg}(A/\psi^{-1}(J_w)) = \text{PI-deg}(A'/J_w) = \text{PI-deg}(\mathscr{O}_{q^{M(C_w)}}(\mathbb{K}^9)).$$

By Remark 6.4, we see that in order to compute $\text{PI-deg}(\mathscr{O}_q(M_{3,5}(\mathbb{K}))/\psi^{-1}(J_w))$ we require the values of the invariant factors of $M(C_w)$ along with $\text{odd}(\tau)$, where τ is the toric permutation of C_w. From Figure 6.1 we read off the toric permutation as $\tau = (17)(26384)$. This has 1 odd cycle, namely (17), so $\text{odd}(\tau) = 1$ and we are left with $(mn - |w| - \text{odd}(\tau))/2 = 4$ pairs of invariant factors of $M(C_w)$. Theorem 6.3 implies that the invariant factors h_i are all powers of 2, so we may write $h_i = 2^{a_i}$, for some $a_i \in \mathbb{Z}$ and all $i \in [\![1,4]\!]$. Therefore, by Remark 6.4,

$$\text{PI-deg}(A/\psi^{-1}(J_w)) = \prod_{i=1}^{4} \frac{\ell}{\gcd(2^{a_i},\ell)},$$

which simplifies to ℓ^4 when ℓ is odd.

In this small example we can actually calculate $M(C_w)$ explicitly and reduce it to its skew-normal form (using, for example, Maple) in order to find its invariant factors. Doing so confirms that $h_1 = h_2 = h_3 = 1$ and $h_4 = 2$. We may therefore state the result explicitly:

$$\text{PI-deg}(A/\psi^{-1}(J_w)) = \begin{cases} \ell^4 & \text{if } \ell \text{ is odd;} \\ \ell^3 m & \text{if } \ell = 2m. \end{cases}$$

It remains to trace J_w back through the deleting derivations algorithm, via the method given in Chapter 5, in order to calculate $P = \psi^{-1}(J_w)$. We know that such a completely prime ideal $P \in \text{C.Spec}_w(A)$ does exist, by Theorem 4.37.

6.2 Specialising to Quantum Determinantal Rings

In this section we provide a formula for the PI degree of the quantum determinantal ring $R_t(M_n) := \mathscr{O}_q(M_n(\mathbb{K}))/I_t$, for some $t \in [\![1, n-1]\!]$, in the case where q is a primitive ℓ^{th} root of unity with $\ell > 2$. To do this, we utilise an isomorphism defined in [LR08] to show that the PI degree of $R_t(M_n)$ is equal to the PI degree of $\mathscr{O}_{q^{M(C)}}(\mathbb{K}^{2nt-t^2})$, where C is the $n\times n$ Cauchon-Le diagram whose last t rows and t columns are white. We then calculate $\text{odd}(\tau)$, for the toric permutation τ associated to C, and conclude using Remark 6.4.

We take $1 \neq q \in \mathbb{K}^*$ to be any nonzero element of the field for the first subsection (this level of generality will be useful to us in the next chapter) however, we then specialise to q

being a root of unity so that we are in the PI setting. We will state explicitly whenever we require q to be a primitive ℓ^{th} root of unity.

6.2.1 PI Parity with a Quantum Affine Space

Let $1 \neq q \in \mathbb{K}^*$ be a nonzero field element. We begin by proving the following lemma, one isomorphism at a time:

Lemma 6.6. *Let $\delta := [1,\dots,t \mid 1,\dots,t]$ be a $t\times t$ quantum minor in $R = \mathscr{O}_q(M_n(\mathbb{K}))$, for some $q \in \mathbb{K}^*$. Let $\bar{\delta} \in R_t(M_n)$ be its canonical image and $\delta_t := [n+t-1,\dots,n, \mid n+t-1,\dots,n]$ be a $t\times t$ quantum minor in R^{op}. Then*

$$R_t(M_n)[\bar{\delta}^{-1}] \cong A_t[\delta^{-1}] \cong B_t[\delta_t^{-1}],$$

where A_t and B_t are the following subalgebras:

$$\begin{aligned} A_t &:= \langle X_{i,j} \in R \mid i \leq t \text{ or } j \leq t\rangle \subseteq R, \\ B_t &:= \langle X_{i,j} \in R^{\text{op}} \mid i \geq n-t+1 \text{ or } j \geq n-t+1\rangle \subseteq R^{\text{op}}. \end{aligned}$$

Remark 6.7. Lemma 6.6 implies that $\operatorname{Frac}(R_t(M_n)) \cong \operatorname{Frac}(B_t)$.

The first isomorphism in Lemma 6.6 is [LR08, Lemma 4.4], recalled below. Beware once again the difference in notation, as noted in the previous chapter: Where we use $t+1$ in defining quantum determinantal rings, Lenagan and Rigal use t. We have made the appropriate notation changes to the statement of this next lemma.

Lemma 6.8 (Lenagan & Rigal, 2008)**.** *Let $\delta := [1,\dots,t \mid 1,\dots,t]$ be a $t\times t$ quantum minor in R and $\bar{\delta} \in R_t(M_n)$ be its canonical image. Let $A_t := \langle X_{i,j} \in R \mid i \leq t \text{ or } j \leq t\rangle \subseteq R$ be a subalgebra. Then δ is normal in A_t and*

$$R_t(M_n)[\bar{\delta}^{-1}] \cong A_t[\delta^{-1}].$$

For the second isomorphism we use the map defined in [PW91, Proposition 3.7.1(3)]: Let $\mathbf{X} := \{X_{i,j} \mid 1 \leq i,j \leq n\}$, then the map $\mathbf{X} \to \mathbf{X}$ sending $X_{i,j}$ to $X_{n-j+1,n-i+1}$, for all $1 \leq i,j \leq n$, may be extended to an anti-automorphism

$$\begin{aligned} \rho_q : \quad R &\longrightarrow R \\ X_{i,j} &\longmapsto X_{n-j+1,n-i+1}. \end{aligned}$$

In particular, ρ_q defines an isomorphism between $(R,\cdot)$ and its opposite algebra $(R^{\mathrm{op}},*)$, where $\cdot$ denotes multiplication in R and $*$ denotes multiplication in R^{op}. Note, in particular, that $\rho_q^{-1}=\rho_q$. From [PW91, Lemma 4.3.1] we see that $\rho_q(\delta)=\delta_t$, where $\delta_t := [n-t+1,\dots,n \mid n-t+1,\dots,n]$ is the $t\times t$ quantum minor in R^{op}. The second isomorphism in Lemma 6.6 then follows by noting that ρ_q preserves the normality of δ, as is shown in the following lemma:

Lemma 6.9. *δ_t is normal in B_t and therefore $A_t[\delta^{-1}]\cong B_t[\delta_t^{-1}]$.*

Proof. To check that δ_t is normal in B_t we use the fact that δ is normal in A_t. First note that $\rho_q(A_t)=B_t$, where ρ_q is the isomorphism defined above. To see this, let $X_{i,j}\in A_t$ so that $i\le t$ or $j\le t$. Without loss of generality, assume $i\le t$ so that $n-i+1\ge n-t+1$. Then

$$\rho_q(X_{i,j})=X_{n-j+1,n-i+1}\in B_t$$

and hence $\rho_q(A_t)\subseteq B_t$. For the opposite inclusion, suppose $X_{i',j'}\in B_t$, for some i',j', and without loss of generality assume that $i'\ge n-t+1$. Then, since $n-i'+1\le t$, we see that

$$\rho_q(X_{i',j'})=X_{n-j'+1,n-i'+1}\in A_t.$$

It was noted earlier that $\rho_q^2=\mathrm{Id}_R$, hence, applying ρ_q to the equation above gives $X_{i',j'}\in \rho_q(A_t)$. From this we conclude that $B_t\subseteq\rho_q(A_t)$ and thus $B_t=\rho_q(A_t)$. Therefore, $A_t\cong B_t$.

Since δ is normal in A_t, we have that $\delta\cdot A_t=A_t\cdot\delta$. This implies that δ_t is normal in B_t, since

$$\delta_t*B_t=\rho_q(\delta)*\rho_q(A_t)=\rho_q(\delta\cdot A_t)=\rho_q(A_t\cdot\delta)=\rho_q(A_t)*\rho_q(\delta)=B_t*\delta_t.$$

Therefore, we conclude that

$$A_t[\delta^{-1}]\cong B_t[\delta_t^{-1}].$$

□

B_t may be expressed as an iterated Ore extension over $\mathbb{K}$ by taking the nonzero indeterminates $X_{i,j}\in B_t$, for $(i,j)\in\{(1,n-t+1),\dots,(n,n)\}$, and adjoining them in lexicographic order to the base field, $\mathbb{K}$. Let α_k be the $\mathbb{K}$-automorphism, and β_k be the α_k-derivation, appearing in the k^{th} Ore extension of B_t, and let $X_{(i,j)_k}$ and $X_{(i,j)_l}$ be the indeterminates in the k^{th} and l^{th} extensions of B_t respectively. That is,

$$X_{(i,j)_1}=X_{1,n-t+1},\quad\dots,\quad X_{(i,j)_t}=X_{1,n},\quad X_{(i,j)_{t+1}}=X_{2,n-t+1},\quad\dots,\quad X_{(i,j)_{2nt-t^2}}=X_{n,n}.$$

Then,

$$\begin{aligned} B &= \mathbb{K}[X_{1,n-t+1}][X_{1,n-t+2};\alpha_2,\delta_2]\cdots[X_{n,n};\alpha_{2nt-t^2},\delta_{2nt-t^2}] \\ &= \mathbb{K}[X_{(i,j)_1}][X_{(i,j)_2};\alpha_2,\delta_2]\cdots[X_{(i,j)_{2nt-t^2}};\alpha_{2nt-t^2},\delta_{2nt-t^2}] \end{aligned}$$

and, for all $1 \le l < k \le 2nt - t^2$, we have

$$\begin{aligned} X_{(i,j)_k} * X_{(i,j)_l} &= \alpha_k(X_{(i,j)_l}) * X_{(i,j)_k} + \beta_k(X_{(i,j)_l}) \\ &= q^{m_{k,l}} X_{(i,j)_l} * X_{(i,j)_k} + \beta_k(X_{(i,j)_l}), \end{aligned}$$

for some $m_{k,l} \in \mathbb{Z}$. The fact that $\alpha_k(X_{(i,j)_l}) = q^{m_{k,l}} X_{(i,j)_l}$ arises from the commutation rules of the single parameter quantum matrices, $R = \mathscr{O}_q(M_n(\mathbb{K}))$, and the α_k-derivations, β_k, may be worked out similarly. These $m_{k,l} \in \mathbb{Z}$ form a skew-symmetric matrix which we denote by $M \in M_{2nt-t^2}(\mathbb{Z})$.

It was shown in [Hay08, Example 5.4] that R satisfies the conditions of [Hay08, Theorem 4.6]. As B_t is isomorphic to a subalgebra of R, it too must satisfy these conditions because the relevant properties of the iterated Ore extension of R are preserved in B_t. We may therefore apply [Hay08, Corollary 4.7] to B_t to obtain the following:

1. $\mathrm{Frac}(B_t) \cong \mathrm{Frac}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2}))$.
2. If q is a root of unity then B_t is a PI algebra with $\text{PI-deg}(B_t) = \text{PI-deg}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2}))$.

6.2.2 The PI Setting

From now on we take $q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity, with $\ell > 2$, so that $R_t(M_n)$ and B_t are both PI algebras. We are interested in computing the PI degree of the quantum determinantal ring $R_t(M_n)$.

So far we have reduced the problem to finding the PI degree of a quantum affine space with associated matrix M. We apply De Concini and Procesi's result (Theorem 2.30) to this quantum affine space to obtain

$$\text{PI-deg}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})) = \sqrt{h},$$

where h is the cardinality of the image of the homomorphism

$$\mathbb{Z}^{2nt-t^2} \xrightarrow{M} \mathbb{Z}^{2nt-t^2} \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^{2nt-t^2}, \tag{6.6}$$

with π denoting the canonical epimorphism.

The cardinality of this map depends on the dimension of the kernel as well as the invariant factors of the matrix M (Lemma 5.6). Since these properties do not change upon multiplying M by $-\mathrm{Id}$, h is also the cardinality of the image of the homomorphism

$$\mathbb{Z}^{2nt-t^2} \xrightarrow{-M} \mathbb{Z}^{2nt-t^2} \xrightarrow{\pi} (\mathbb{Z}/\ell\mathbb{Z})^{2nt-t^2}. \tag{6.7}$$

That is,

$$\text{PI-deg}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})) = \text{PI-deg}(\mathscr{O}_{q^{-M}}(\mathbb{K}^{2nt-t^2})).$$

The advantage of considering the homomorphism in (6.7), instead of the one in (6.6), is that the matrix $-M$ corresponds to a Cauchon-Le diagram C (that is, there exists a Cauchon-Le diagram C such that $M(C) = -M$) whereas, for M there is no such Cauchon-Le diagram associated to it. In fact, one may verify that $-M$ is the matrix associated to the $n \times n$ Cauchon-Le diagram whose last t rows and t columns are white. This allows us to use the combinatorial techniques presented in this chapter to compute the dimension of the kernel and the invariant factors of $M(C)$. We do this next.

6.2.3 Calculating the Toric Permutation

Proposition 6.10. *Fix some $n \in \mathbb{N}_{>0}$ and $t \in [\![1, n-1]\!]$. Let C be the $n \times n$ Cauchon-Le diagram whose last t rows and t columns are white. Then the associated toric permutation, τ, is the product of t disjoint odd cycles $\tau = c_1 \cdots c_t$.*

In particular, write $n = ut + r$, for some $u \in \mathbb{N}$ and $r \in [\![0, t-1]\!]$. Then the cycles depend on $n-t$ and t, and are the following:

- $\underline{0 < n-t < t}$:

$$c_i := \begin{cases} (i, i+t, 2t+r+i, t+r+i), & 1 \le i \le r; \\ (i, t+r+i), & r < i \le t. \end{cases}$$

- $\underline{n-t = t}$:

$$c_i := (i, i+t, i+t+n, i+n), \quad 1 \le i \le t.$$

- $\underline{t < n-t}$:

$$c_i := \begin{cases} (i, i+t, \ldots, i+ut, i+ut+n, i+(u-1)t+n, \ldots, i+n), & 1 \le i \le r; \\ (i, i+t, \ldots, i+(u-1)t, i+(u-1)t+n, i+(u-2)t+n, \ldots, i+n), & r < i \le t. \end{cases}$$

Proof. We consider three cases: $t = n-t$, $t < n-t$, and $t > n-t$. For each of these cases, we provide the corresponding Cauchon-Le diagram with pipe dream construction (see Figures 6.2, 6.4, and 6.6), as defined in Section 6.2.3, and we read off the associated toric permutation, τ. Each diagram is drawn using the following conventions:

- The lines distinguishing the individual squares of the Cauchon-Le diagram are omitted to make the overall pattern of permutation paths clearer.
- The paths are coloured differently depending on which permutation rule they obey. These are given below:
  - A red path takes label i to $i+t$, for $i \in [\![1, n-t]\!]$;
  - A blue path takes label i to $i+n$, for $i \in [\![n-t+1, n]\!]$;
  - A pink path takes label i to $i-n$, for $i \in [\![n+1, n+t]\!]$;
  - A green path takes label i to $i-t$, for $i \in [\![n+t+1, 2n]\!]$.
- Only the first and last paths (with respect to source label i) of each permutation type are included on the Cauchon-Le diagram. It should be understood that all paths bounded by two paths of the same colour must also share that colour and, therefore, that specific permutation rule.

Our strategy is to consider only the permutation cycles starting at labels $i \in [\![1,t]\!]$. This will give t disjoint odd cycles $c_1, \ldots, c_t$ and we then show that each of the remaining labels $j \in [\![t+1, 2n]\!]$ appears in exactly one of these cycles. From this we will be able to conclude that $\tau = c_1 \cdots c_t$.

- $\underline{t < n-t}$: We start with the least straightforward case as once we've set up the notation and method of proof the other cases will follow easily. See Figure 6.2 for the Cauchon-Le diagram with pipe dreams corresponding to this case.

  Let $n = ut + r$ for some integers u, r, where $r < t$, and using this, rewrite the labels $i \in [\![1, 2n]\!]$ in terms of u, t, and r. This will allow us to track the permutations more easily, since τ either permutes a label i by $\pm t$ or $\pm n$ each time. We define the following sets:

$$\left.\begin{aligned} r_i &:= \{it+1, \ldots, it+r\} \\ r_i' &:= \{n+it+1, \ldots, n+it+r\} \end{aligned}\right\} \text{ for } i \in [\![0, u]\!]$$
$$\left.\begin{aligned} t_i &:= \{it+r+1, \ldots, (i+1)t\} \\ t_i' &:= \{n+it+r+1, \ldots, n+(i+1)t)\} \end{aligned}\right\} \text{ for } i \in [\![0, u-1]\!] \tag{6.8}$$

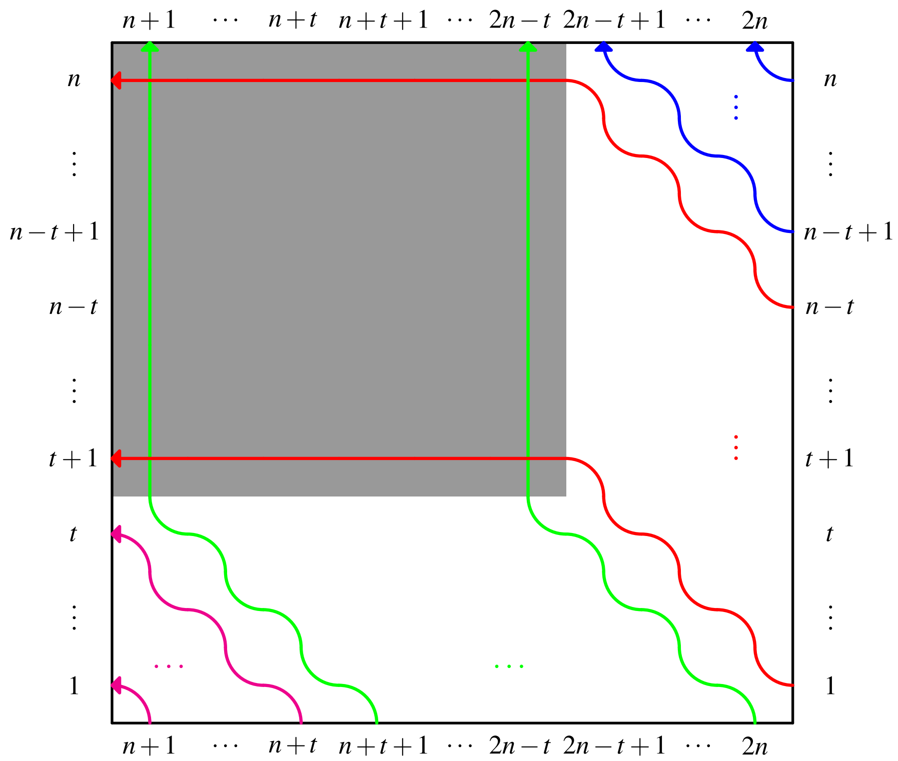

Fig. 6.2 Cauchon-Le diagram with pipe dreams for $t < n-t$

Note that $|r_i| = |r_i'| = r$ and $|t_i| = |t_i'| = t - r$. Note, also, the following important sets:

$$\begin{aligned} r_0 &= \{1, \ldots, r\}, \quad & t_0 &= \{r+1, \ldots, t\}, \\ t_{u-1} &= \{n-t+1, \ldots, ut\}, \quad & r_u &= \{ut+1, \ldots, n\}, \\ r_0' &= \{n+1, \ldots, n+r\}, \quad & t_0' &= \{n+r+1, \ldots, n+t\}, \\ r_u' &= \{n+ut+1, \ldots, 2n\}. \end{aligned}$$

We now redraw the Cauchon-Le diagram from Figure 6.2, labelling the sides r_i, r_i', t_i, t_i', and adding in grid lines to distinguish these sets of labels (See Figure 6.3).

We begin by considering the set r_0. Figure 6.3 shows the paths along which repeated applications of τ takes the set r_0. The colour of the path, again, denotes the permutation type that results in taking that path. The emboldened labels on the left and upper sides of the diagram show which sets r_0 is permuted through before returning to r_0.

We see from the diagram that the set r_0 is permuted along u red paths, 1 blue path, u green paths and 1 pink path. Applying the corresponding permutation type for each of

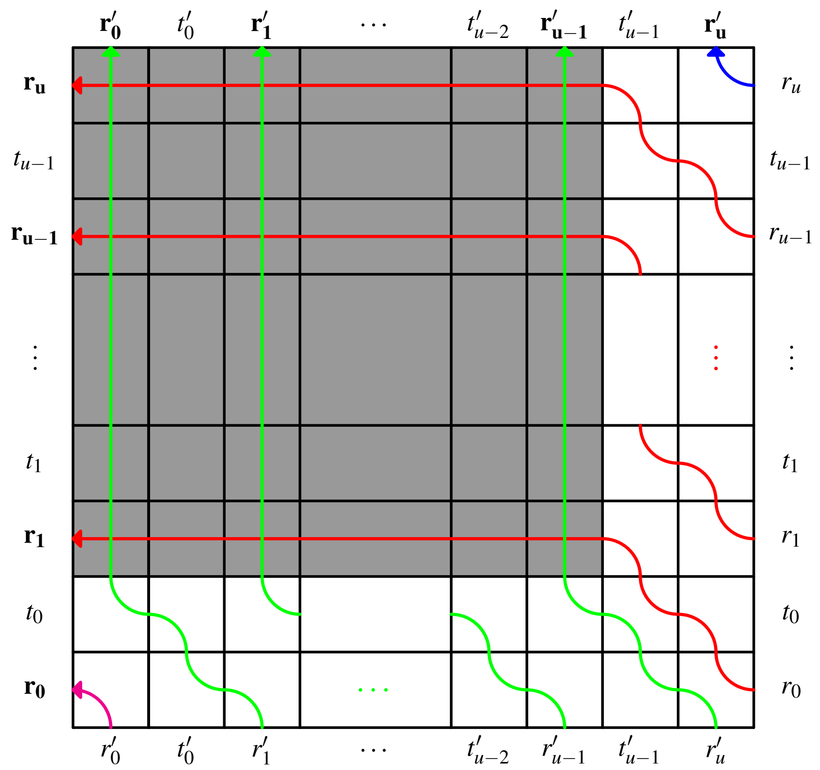

Fig. 6.3 Toric permutation applied to the set r_0 for $t < n-t$

these coloured paths gives us the following permutations:

$$c_i = (i, i+t, i+2t, \dots, i+ut, i+ut+n, i+(u-1)t+n, i+(u-2t)+n, \dots, i+n)$$

of length $2(u+1)$, for all $i \in r_0$.

Similarly, we see from Figure 6.3 that the set t_0 is permuted along $u-1$ red paths, 1 blue path, $u-1$ green paths and 1 pink path. Therefore, for any $i \in t_0$, we obtain the cyclic permutations

$$c_i = (i, i+t, i+2t, \dots, i+(u-1)t, i+(u-1)t+n, i+(u-2)t+n, \dots, i+n)$$

of length $2u$. Finally, it should be clear from the diagram that all other labels $j \in [\![t+1, 2n]\!]$ appear in exactly one of the cycles c_i, for some $i \in [\![1, t]\!]$. Therefore

$$c_i := \begin{cases} (i, i+t, \dots, i+ut, i+ut+n, i+(u-1)t+n, \dots, i+n), & 1 \le i \le r; \\ (i, i+t, \dots, i+(u-1)t, i+(u-1)t+n, i+(u-2)t+n, \dots, i+n), & r < i \le t. \end{cases}$$

- $\underline{0 < n-t < t}$:

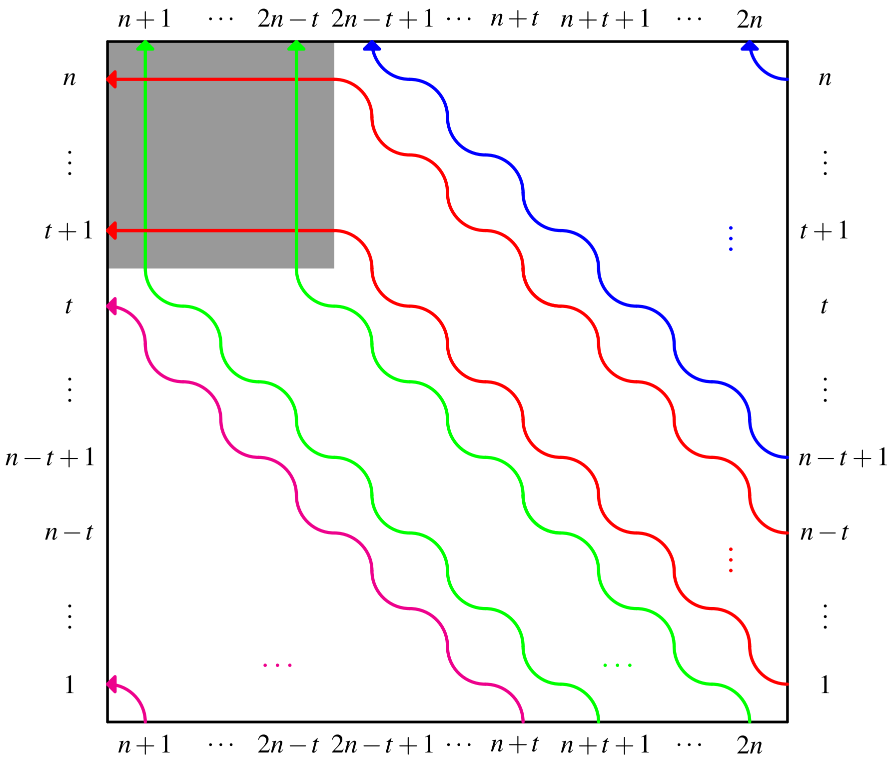

Fig. 6.4 Cauchon-Le diagram with pipe dreams for $0 < n-t < t$

Writing $n = ut + r$ we see that $0 < r < (2-u)t$ implying that $u = 1$ since, if $u = 0$ then $n - t = r - t < 0$, which is not possible. We may therefore write n as $n = t + r$. We obtain a similar Cauchon-Le diagram with pipe dreams to the previous case (Figure 6.4) and, again, we define sets r_i, t_i, r'_i, and t'_i as in (6.8). Since $u = 1$ in this case then there are only 6 sets to consider:

$$\begin{aligned} r_0 &= \{1,\dots,r\}, & r_1 &= \{t+1,\dots,t+r\}, & t'_0 &= \{t+2r+1,\dots,2t+r\}, \\ t_0 &= \{r+1,\dots,t\}, & r'_0 &= \{t+r+1,\dots,t+2r\}, & r'_0 &= \{2t+r+1,\dots,2t+2r\}. \end{aligned}$$

Proceeding as in the previous case, and using Figure 6.5, we conclude that we have the following t permutations of lengths 2 and 4:

$$c_i := \begin{cases} (i, i+t, 2t+r+i, t+r+i), & 1 \le i \le r; \\ (i, t+r+i), & r < i \le t. \end{cases}$$

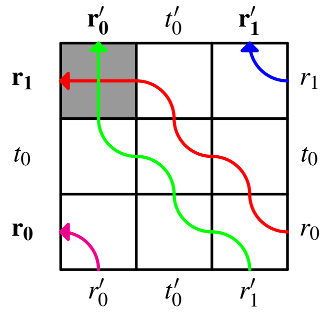

Fig. 6.5 Toric permutation applied to r_0 for $0 < n-t < t$

- $\underline{t = n-t}$: In this case the diagram is simplified even further since $n = 2t$ (see Figure

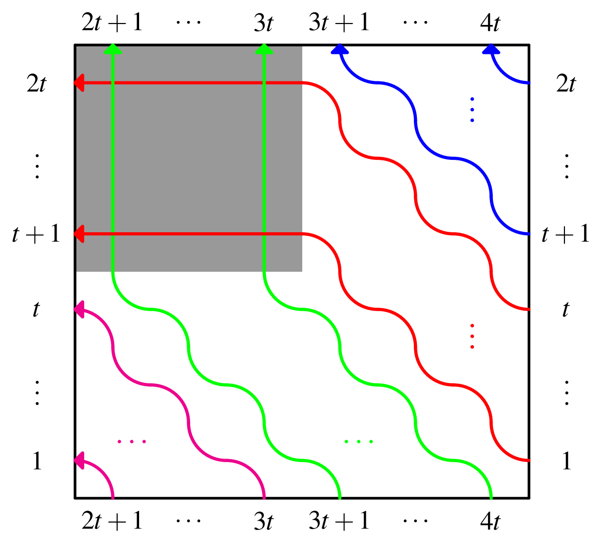

Fig. 6.6 Cauchon-Le diagram with pipe dreams for $t = n-t$

6.6) so we just need to consider $i \in [\![1,t]\!]$. We obtain 4 sets:

$$t_0 = \{1,\dots,t\}, \quad t_1 = \{t+1,\dots 2t\},$$
$$t'_0 = \{2t+1,\dots,3t\}, \quad t'_1 = \{3t+1,\dots 4t\},$$

and τ simply permutes these sets between one another.

From Figure 6.7 we see that there are t permutations of length 4, for $i \in [\![1,t]\!]$, given by

$$c_i := (i, i+t, i+t+n, i+n).$$

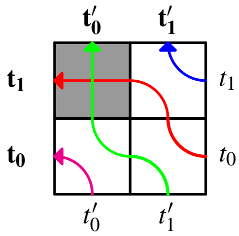

Fig. 6.7 Toric permutation applied to r_0 for $t = n - t$

$\square$

Theorem 6.11. *Let $n \in \mathbb{N}_{>0}$ and $t \in [\![1, n-1]\!]$ and take q to be a primitive ℓ^{th} root of unity with $\ell > 2$. Then*

$$PI\text{-}deg(R_t(M_n)) = \begin{cases} \ell^{\frac{2nt-t^2-t}{2}} & \ell \textit{ is odd;} \\ \prod_{i=1}^{\frac{2nt-t^2-t}{2}} \frac{\ell}{\gcd(h_i,\ell)} & \ell \textit{ is even,} \end{cases}$$

where $h_1, \ldots, h_{(2nt-t^2-t)/2} \in \mathbb{Z}$ are the invariant factors of the matrix $M(C)$ associated to the $n \times n$ Cauchon-Le diagram C whose last t rows and t columns are white.

Proof. Since isomorphic algebras have the same PI degree and the PI degree is invariant under localisation, Lemma 6.6 allows us to write

$$\text{PI-deg}(R_t(M_n)) = \text{PI-deg}(R_t(M_n)[\bar{\delta}^{-1}]) = \text{PI-deg}(B_t[\delta_t^{-1}]) = \text{PI-deg}(B_t).$$

By the second bullet point in Section 6.2.1, we obtain

$$\text{PI-deg}(R_t(M_n)) = \text{PI-deg}(B_t) = \text{PI-deg}(\mathcal{O}_{q^M}(\mathbb{K}^{2nt-t^2}))$$

where, by the discussion in Section 6.2.2, $M = M(C)$ for the Cauchon-Le diagram C consisting of white squares in the last t rows and t columns. Let τ be the toric permutation associated to C. By Proposition 6.10, the number of odd cycles in τ is t and, by Remark 6.4, we obtain

$$\text{PI-deg}(R_t(M_n)) = \text{PI-deg}(\mathcal{O}_{q^{M(C)}}(\mathbb{K}^{2nt-t^2})) = \prod_{i=1}^{\frac{2nt-t^2-t}{2}} \frac{\ell}{\gcd(h_i,\ell)}.$$

We now apply Theorem 6.3 to deduce that each h_i is a power of 2, hence, if ℓ is odd then the PI degree simplifies to $\text{PI-deg}(R_t(M_n)) = \ell^{\frac{2nt-t^2-t}{2}}$.

$\square$

6.3 Open Questions

There are a number of natural questions which arise from the work in this chapter. We address these in this section.

In the questions below take $q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity with $\ell > 2$ and let $M = M(C)$ be the matrix associated to some Cauchon-Le diagram C, and $\overline{M} = \left(\begin{smallmatrix} M & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{smallmatrix}\right)$.

Recall from Corollary 5.13 that knowing the properties of $\overline{M}$, for certain Cauchon-Le diagrams, would allow us to compute the PI degree of quantum Schubert varieties. This forms the basis of questions 2 and 3 below.

For some $n \geq 2$ and $1 \leq t \leq n$, denote by $\mathscr{M}_{n,t} := M(C) \in M_N(\mathbb{Z})$, the matrix associated to C, where C is the $n \times n$ Cauchon-Le diagram whose last t rows and t columns are white, and $N = 2nt - t^2$. Recall that the properties of the matrix $\mathscr{M}_{n,t}$ would allow us to compute the PI degree of quantum determinantal rings for any value of ℓ. That is,

$$\text{PI-deg}(R_t(M_n)) = \text{PI-deg}(\mathscr{O}_{q^{\mathscr{M}_{n,t}}}(\mathbb{K}^{2nt-t^2})).$$

This motivates the first question:

1) What are the values of the invariant factors of $\mathscr{M}_{n,t}$?

Fix $n \geq 2$ and let $1 \leq t \leq n$. By Theorem 6.11, $\dim(\ker(\mathscr{M}_{n,t})) = t$, leaving $s = \frac{2nt-t^2-t}{2}$ pairs of invariant factors. Denote these invariant factors by $h_i^{(t)} \in \mathbb{Z}$, for $i \in [\![1,s]\!]$. By Theorem 6.3 these invariant factors are all powers of 2. Knowing the exact values of each $h_i^{(t)}$ would allow us to state the PI degree of $R_t(M_n)$ for any $\ell > 2$.

It is possible to show that $h_i^{(t)} = 1$, for all $1 \leq t \leq n$ and $i \in [\![1,n-1]\!]$, using the following argument: When $t = 1$, $\mathscr{M}_{n,t}$ takes the form

$$\mathscr{M}_{n,1} = \begin{pmatrix} \mathscr{A}_{n-1} & \mathscr{B} \\ -\mathscr{B}^T & \mathscr{A}_n \end{pmatrix} \in M_{2n-1}(\mathbb{Z})$$

for matrices

$$\mathscr{A}_n = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & 0 & & 1 \\ \vdots & & \ddots & \vdots \\ -1 & -1 & \dots & 0 \end{pmatrix} \in M_n(\mathbb{Z}), \qquad \mathscr{B} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \end{pmatrix} \in M_{n-1,n}(\mathbb{Z}),$$

where $\mathbf{0} \in M_{n-1}(\mathbb{Z})$ is the zero square matrix, and $\mathbf{1} \in M_{n-1,1}(\mathbb{Z})$ is the $n-1$ dimensional vector with all entries equal to 1. It can be verified that the $(2n-2) \times (2n-2)$ minor of $\mathscr{M}_{n,1}$, obtained by removing the first row and first column, has determinant equal to 1 so that $D_{2n-2}(\mathscr{M}_{n,1}) = 1$. Recall that

$$h_i^{(1)} = \frac{D_{2i-1}(\mathscr{M}_{n,1})}{D_{2i-2}(\mathscr{M}_{n,1})} = \frac{D_{2i}(\mathscr{M}_{n,1})}{D_{2i-1}(\mathscr{M}_{n,1})}$$

so, in particular, $h_{n-1}^{(1)} = 1$. Using the property $h_i^{(1)} \mid h_j^{(1)}$ for all $j > i$ we deduce that $h_i^{(1)} = 1$ for all $i \in [\![1, n-1]\!]$.

The following inclusion of submatrices is immediately apparent:

$$\mathscr{M}_{n,1} \subseteq \mathscr{M}_{n,2} \subseteq \cdots \subseteq \mathscr{M}_{n,n}.$$

This implies that $h_i^{(j)} | h_i^{(l)}$, for all $j > l$ and $i \in [\![1, s]\!]$, and thus $h_i^{(t)} = 1$, for all $t \in [\![1, n]\!]$ and $i \in [\![1, n-1]\!]$. Furthermore, in [Hay08, Section 5.3] it was shown that $h_n^{(n)} = 2$. Therefore, $2 | h_n^{(t)}$ for all $t \in [\![1, n]\!]$ and this completes the proof that $\mathscr{M}_{n,t}$ has precisely $2(n-1)$ invariant factors equal to 1.

It is left to prove the exact values of $h_i^{(t)}$ for $i \in [\![n, s]\!]$ for all $t \in [\![1, n]\!]$. By extensive calculations on the invariant factors of $\mathscr{M}_{n,t}$ using Maple (for all $2 \leq n \leq 40$ and $1 \leq t \leq n$), we state the following conjecture:

Conjecture 6.12. *For $n \geq 2$ and all $t \in [\![1, n]\!]$ we have*

$$h_i^{(t)} = \begin{cases} 1 & \text{for } i \in [\![1, n-1]\!]; \\ 2 & \text{for } i \in [\![n, s]\!], \end{cases}$$

hence M has $2(n-1)$ invariant factors equal to 1, $2nt - t^2 - t - 2(n-1)$ invariant factors equal to 2, and a kernel of dimension t.

If the conjecture holds then it would allow us to state the PI degree of $R_t(M_n)$ for $\ell = 2m$, for some $m \in \mathbb{N}$. This would become PI-deg$(R_t(M_n)) = \ell^{n-1} m^{\frac{2nt - t^2 - t - 2n + 2}{2}}$.

2) Given any Cauchon-Le diagram C with corresponding matrix $M = M(C)$, are the invariant factors of the extended matrix $\overline{M}$ still all powers of 2?

For general Cauchon-Le diagrams this question doesn't necessarily have a positive answer, however the question is still open for Cauchon-Le diagrams with associated matrix $M(C) = \mathscr{M}_{n,t}$, as we discuss below.

Since M and $\overline{M}$ are skew symmetric, they have even rank (as one sees immediately by considering their skew-normal forms). Thus, upon extending M to obtain $\overline{M}$, as defined in the introduction to this section, the rank either remains the same or increases by 2. Let $h_1,\ldots,h_s \in \mathbb{Z}\backslash\{0\}$ be the invariant factors of M and let $\bar{S}$ denote the skew-normal form of $\overline{M}$, where

$$\bar{S} = \begin{pmatrix} 0 & \bar{h}_1 & & & & & & \\ -\bar{h}_1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & \bar{h}_s & & & \\ & & & -\bar{h}_s & 0 & & & \\ & & & & & 0 & \bar{h}_{s+1} & \\ & & & & & -\bar{h}_{s+1} & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix},$$

for $\bar{h}_1,\ldots,\bar{h}_s \in \mathbb{Z}\backslash\{0\}$, and $\bar{h}_{s+1} \in \mathbb{Z}$ (possibly zero). We may apply [Tho79, Theorem 3], with $C = \overline{M}$ and $A = M$, to obtain the following interlacing inequalities for the invariant factors:

$$\bar{h}_1|h_1|\bar{h}_2|h_2|\ldots|\bar{h}_s|h_s|\bar{h}_{s+1}|0. \tag{6.9}$$

Note that, in the notation of Thompson's paper, we have $h_{2i-1}(M) = h_{2i}(M) = h_i$ for all $i \in [\![1,s]\!]$, and $h_i(M) = 0$ for all $i > 2s$. Similarly, we have $h_{2i-1}(\overline{M}) = h_{2i}(\overline{M}) = \bar{h}_i$ for all $i \in [\![1,s+1]\!]$, and $h_i(\overline{M}) = 0$ for all $i > 2s+2$. Since each h_i is a power of 2, we deduce from (6.9) that $\bar{h}_1,\ldots,\bar{h}_s$ are also powers of 2. However, we are unable to make any such conclusion about $\bar{h}_{s+1}$.

If $\operatorname{rank}(\overline{M}) = \operatorname{rank}(M)$ then $\bar{h}_{s+1} = 0$ and we may conclude that all invariant factors of $\overline{M}$ are powers of 2. If $\operatorname{rank}(\overline{M}) = \operatorname{rank}(M)+2$, however, then $\bar{h}_{s+1} \neq 0$ and is not necessarily a power of 2.

Indeed, there are examples of Cauchon-Le diagrams C whose corresponding extended matrix $\overline{M}$ has an invariant factor which is not a power of 2: Let C be the 5×5 Cauchon-Le diagram in Figure 6.8. The invariant factors of its associated matrix, $M = M(C)$, consist of eight 1's and twelve 2's, however, its extended matrix $\overline{M(C)}$ has invariant factors consisting of ten 1's and two 6's.

We may restrict this question to Cauchon-Le diagrams C whose associated matrix is $M(C) = \mathscr{M}_{n,t}$. In this case, we have checked (using Maple) that, for $2 \leq n \leq 40$ and all $1 \leq t \leq n$, the invariant factors of $\bar{\mathscr{M}}_{n,t}$ are still all powers of 2. The question is therefore still open for matrices corresponding to quantum determinantal rings.

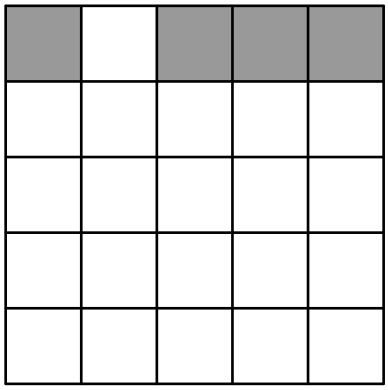

Fig. 6.8 Counterexample to question 2

3) Given $M = M(C)$ corresponding to a Cauchon-Le diagram C, and $\overline{M} := \left(\begin{smallmatrix} M & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{smallmatrix}\right)$, for what properties of C does rank$(M) =$ rank$(\overline{M})$?

In the discussion for question 2 we saw that if $\mathrm{rank}(\overline{M}) = \mathrm{rank}(M)$ then the invariant factors of $\overline{M}$ are all powers of 2. This is what motivates this question.

We focus on Cauchon-Le diagrams C with associated matrix $M(C) = \mathscr{M}_{n,t}$ and extended matrix $\bar{\mathscr{M}}_{n,t} = \left(\begin{smallmatrix} \mathscr{M}_{n,t} & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{smallmatrix}\right)$, for some $n \geq 2$ and $1 \leq t \leq n$. Computations in Maple (for $2 \leq n \leq 40$ and all $1 \leq t \leq n$) suggest that we have

$$\mathrm{rank}(\bar{\mathscr{M}}_{n,t}) = \begin{cases} \mathrm{rank}(\mathscr{M}_{n,t}) & n = 2m \text{ and } t|m; \\ \mathrm{rank}(\mathscr{M}_{n,t}) + 2 & \text{otherwise.} \end{cases}$$

Recall, from Sections 5.3.2 that $R_t(M_n) = \mathscr{O}_q(M_n(\mathbb{K}))_\delta$, for $\delta = ([\![1, t-1]\!], [\![1, t-1]\!])$. Thus, if our prediction above is proved to be true then we would be able to calculate the PI degree of $R_t(M_n)[y; \phi]$ explicitly in the case where $n = 2m$ and $t|m$, for some $m \in \mathbb{N}_{>0}$, when ℓ is even. By Corollary 5.13 this would then give us the the PI degree of the quantum Schubert variety $\mathscr{O}_q(G_{n,2n}(\mathbb{K})))_\gamma$, where $\gamma = \delta_{n,n}(\delta)$.

Chapter 7

Irreducible Representations

This chapter focuses on applying the techniques and results of previous chapters to construct irreducible representations of the algebras we have examined throughout this thesis. The results are joint work with Samuel Lopes.

Section 7.1 sets up the required standard background information for this chapter. In Section 7.2 we construct an irreducible representation of a uni-parameter quantum affine space at a root of unity, satisfying some mild conditions. From this representation we are able to induce irreducible representations of some completely prime quotient algebras A/P, where A satisfies Hypothesis 1, and of quantum determinantal rings at roots of unity.

Section 7.3 shows how to track certain irreducible representations of $A'/\psi(P)$ back through the deleting derivations algorithm to give an irreducible representation of A/P. If A is a PI algebra in the root of unity setting then this construction is always possible, and we give an explicit example of this in Section 7.3.2.

Quantum determinantal rings are a particularly nice class of algebras satisfying Hypothesis 1, as we have seen previously. In particular, we know the PI degree explicitly, provided that the deformation parameter q is a primitive ℓ^{th} root of unity and ℓ is odd. In Section 7.4 we show that quantum determinantal rings contain a particularly "nice" quantum affine space as a subalgebra, which we may use, along with the main result in Section 7.2, to construct an irreducible representation of the quantum determinantal ring at a root of unity. This section concludes with an example.

Aside from Section 7.1, throughout this chapter we will take $\mathbb{K}$ to be an algebraically closed field. This is required for several of the standard results we use, including the corollary to Schur's Lemma, the Jacobson Density Theorem, and Theorem 2.17, which bounds the dimension of irreducible representations of a PI algebra above by its PI degree. We take $1 \neq q \in \mathbb{K}^*$ to be a nonzero element of the field and specify when we require it to be a root of unity.

7.1 Representation Theory Background

For any field $\mathbb{K}$, let R be a $\mathbb{K}$-algebra and V be a $\mathbb{K}$-vector space. We denote by $\mathrm{End}_{\mathbb{K}}(V)$ the set of all $\mathbb{K}$-linear maps on V. This forms a ring which may also be viewed as a $\mathbb{K}$-algebra. A $\mathbb{K}$-algebra homomorphism

$$\begin{aligned} \varphi : \ R &\longrightarrow \mathrm{End}_{\mathbb{K}}(V) \\ r &\longmapsto \varphi_r \end{aligned}$$

defines a *representation of R on V*, which we denote by (φ, V). We will sometimes use the notation $\varphi(r)$ instead of φ_r, and write $\varphi(r) \cdot v$ to mean $\varphi_r(v)$, for all $v \in V$. The *dimension* of the representation is said to be the dimension of V as a vector space over the field, $\mathbb{K}$. A nonzero subspace $W \subseteq V$ gives rise to a *sub-representation* of V if $\varphi_r(W) \subseteq W$, for all $r \in R$. A representation (φ, V) of R is called *irreducible* if V is nonzero and has no sub-representations other than 0 and itself.

The algebra homomorphism φ is called the *structural homomorphism of V* as it makes V into a (left) *R-module*, with multiplication defined by $rv = \varphi_r(v)$, for all $r \in R$ and $v \in V$. In this chapter we assume all modules to be left modules. A *submodule* of an R-module, V, is a $\mathbb{K}$-subspace $W \subseteq V$ such that $RW \subseteq W$. We say that V is *simple* if it is nonzero and has no submodules other than 0 and itself. Therefore, simple modules correspond to irreducible representations. An R-module, V, is *semisimple* if it can be written as a direct sum of simple modules. In particular, every simple module is semisimple.

Given two R-modules, V and W, an *R-module homomorphism* $\phi : V \to W$ is a $\mathbb{K}$-linear map satisfying $\phi(rv) = r\phi(v)$ for all $r \in R$ and $v \in V$. We denote by $\mathrm{End}_R(V)$ the ring of R-module homomorphisms on V.

We recall two well-known results in representation theory. The first of these is Schur's Lemma (see, for example, [Lan02, Proposition 1.1]), which has important consequences when $\mathbb{K}$ is algebraically closed and V is finite-dimensional over $\mathbb{K}$. The second is the Jacobson Density Theorem (see, for example, [Lan02, Theorem 3.2]).

Proposition 7.1 (Schur's Lemma)**.** *Let R be $\mathbb{K}$-algebra and V a simple R-module. Then* $\mathrm{End}_R(V)$ *is a division $\mathbb{K}$-algebra. If V and W are simple R-modules, then every nonzero R-module homomorphism from V to W is an isomorphism.*

Corollary 7.2. *Let R be a $\mathbb{K}$-algebra, V be a simple R-module, and $\varphi : R \to \mathrm{End}_{\mathbb{K}}(V)$ be the structural homomorphism. Suppose $\mathbb{K}$ is algebraically closed and V has finite dimension over $\mathbb{K}$. Then*

(i) $\mathrm{End}_R(V) = \{\lambda \,\mathrm{Id}_V \mid \lambda \in \mathbb{K}\}$.

(ii) If $r \in Z(R)$ then $\varphi_r = \lambda\, \mathrm{Id}_V$, for some $\lambda \in \mathbb{K}$.

Proof. (i) Let $\varphi_r \in \mathrm{End}_R(V)$ be a nonzero R-linear endomorphism. The characteristic polynomial for φ_r is a polynomial over $\mathbb{K}$ and hence, since $\mathbb{K}$ is algebraically closed, all its roots lie in $\mathbb{K}$. Therefore, φ_r has an eigenvalue $\lambda \in \mathbb{K}$ and an eigenvector $v \in V$ such that $\varphi_r(v) = \lambda v$. That is, $v \in \ker(\varphi_r - \lambda\, \mathrm{Id}_V)$ and $\varphi_r - \lambda\, \mathrm{Id}_V \in \mathrm{End}_R(V)$ is not injective. This means that $\varphi_r - \lambda\, \mathrm{Id}_V$ is not invertible, thus, by Schur's Lemma, it must be 0, since $\mathrm{End}_R(V)$ is a division algebra. Hence $\varphi_r = \lambda\, \mathrm{Id}_V$.

(ii) If $r \in Z(R)$ then, for all $a \in R$ and $v \in V$, we have

$$\varphi_r(av) = \varphi_{ra}(v) = \varphi_{ar}(v) = arv = a\varphi_r(v).$$

Therefore $\varphi_r \in \mathrm{End}_R(V)$ and we may conclude using part (i).

□

Theorem 7.3 (Jacobson Density Theorem)**.** *Let V be semisimple over R and $R' := \mathrm{End}_R(V)$. Take some $f \in \mathrm{End}_{R'}(V)$ and $x_1, \ldots, x_n \in V$. Then there exists an element $r \in R$ such that $rx_i \in f(x_i)$ for all $i \in [\![1,n]\!]$. Furthermore, if V is finitely generated over R', the map $\rho : R \to \mathrm{End}_{R'}(V)$ taking $r \mapsto f_r$, where $f_r(v) = rv$ for $v \in V$, is surjective.*

When working over an algebraically closed field, the Jacobson Density Theorem can be used to show the following standard result:

Proposition 7.4. *Take $\mathbb{K}$ to be an algebraically closed field. Let R and S be $\mathbb{K}$-algebras, V a simple R-module and W a simple S-module. Then $V \otimes W$ is a simple module for the tensor product algebra $R \otimes S$.*

We end this introduction with a definition which provides a way to represent the tensor product of linear maps between vector spaces with fixed bases as matrices.

Definition 7.5. Let V_1, V_2, W_1, W_2 be $\mathbb{K}$-vector spaces and let $M = (m_{i,j}) \in M_{s,t}(\mathbb{Z})$ and $N = (n_{i,j}) \in M_{u,v}(\mathbb{Z})$ be matrices representing the $\mathbb{K}$-linear maps $V_1 \to W_1$ and $V_2 \to W_2$, respectively. Then the tensor product, also called the *Kronecker product*, $M \otimes N$, defined as

$$M \otimes N := \begin{pmatrix} m_{1,1}N & \cdots & m_{1,t}N \\ \vdots & \ddots & \vdots \\ m_{s,1}N & \cdots & m_{s,t}N \end{pmatrix} \in M_{su,tv}(\mathbb{Z}),$$

represents the tensor product of the two maps, $V_1 \otimes W_1 \to V_2 \otimes W_2$.

7.2 Irreducible Representations of Quantum Affine Spaces

In this section we take $\mathbb{K}$ to be algebraically closed and $1 \neq q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity.

We start by constructing an irreducible representation of a quantum affine space at a root of unity, with dimension equal to its PI degree. This will be called upon in later sections when we construct irreducible representations of quantum determinantal rings, and of algebras which have passed through the deleting derivations algorithm. To do this, we first construct an irreducible representation of a quantum affine plane at a root of unity which has maximal dimension. This result is not believed to be new, however, we present a proof of it here to convince the reader that we do indeed get an irreducible representation, as this is used heavily in the results that follow.

The notation of the lemma has been chosen to suit its application in the results to come, in which we will take tensor products of a family of quantum affine planes indexed by i.

Lemma 7.6. *Let $1 \neq q \in \mathbb{K}^*$ be a primitive ℓ^{th} root of unity, for some $\ell \in \mathbb{N}_{>1}$. Let $\mathbb{K}_{q^{h_i}}[x_i, y_i]$ denote the quantum affine plane with relations $x_iy_i = q^{h_i}y_ix_i$, for some $h_i \in \mathbb{Z}\backslash\{0\}$ satisfying $\gcd(h_i, \ell) = 1$. Let V_i be an ℓ-dimensional $\mathbb{K}$-vector space with basis $\{v_1^{(i)}, \dots, v_\ell^{(i)}\} \subseteq V_i$ and define the map*

$$\begin{aligned} \varphi_i : \mathbb{K}_{q^{h_i}}[x_i, y_i] &\longrightarrow \mathrm{End}_{\mathbb{K}}(V_i) \\ x_i &\longmapsto \varphi_i(x_i) \\ y_i &\longmapsto \varphi_i(y_i), \end{aligned} \tag{7.1}$$

where the endomorphisms $\varphi_i(x_i)$, $\varphi_i(y_i)$ act on the basis vectors $v_j^{(i)} \in \{v_1^{(i)}, \dots, v_\ell^{(i)}\}$ in the following way:

$$\varphi_i(x_i) \cdot v_j^{(i)} = \lambda_i q^{(j-1)h_i} v_j^{(i)}, \tag{7.2}$$

$$\varphi_i(y_i) \cdot v_j^{(i)} = \begin{cases} v_{j+1}^{(i)}, & \text{if } j \in [\![1, \ell-1]\!]; \\ v_1^{(i)}, & \text{if } j = \ell, \end{cases} \tag{7.3}$$

for some $\lambda_i \in K^*$. In particular, we can write $\varphi_i(x_i)$, $\varphi_i(y_i)$ as matrices in $M_\ell(\mathbb{K})$ with respect to this basis:

$$\varphi_i(x_i) = \begin{pmatrix} \lambda_i & & & & \\ & \lambda_i q^{h_i} & & & \\ & & \lambda_i q^{2h_i} & & \\ & & & \ddots & \\ & & & & \lambda_i q^{(\ell-1)h_i} \end{pmatrix}, \qquad \varphi_i(y_i) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & 0 \end{pmatrix}. \tag{7.4}$$

Then φ_i is a surjective algebra homomorphism which defines an irreducible representation of $\mathbb{K}_{q^{h_i}}[x_i,y_i]$ on V, of dimension ℓ, and satisfies the property $\varphi_i(x_i)^\ell = \lambda_i^\ell \mathrm{Id}_{V_i}$ and $\varphi_i(y_i)^\ell = \mathrm{Id}_{V_i}$.

Proof. Using (7.2) and (7.3) it may be verified that $\varphi_i(x_i)\varphi_i(y_i) = q^{h_i}\varphi_i(y_i)\varphi_i(x_i)$, so that the relations between generators $x_i, y_i \in \mathbb{K}_{q^{h_i}}[x_i,y_i]$ are preserved in the image of φ_i. By the universal property of algebras with generators and relations, φ_i then becomes an algebra homomorphism and thus defines a representation of $\mathbb{K}_{q^{h_i}}[x_i,y_i]$ on V_i. This gives V_i a $\mathbb{K}_{q^{h_i}}[x_i,y_i]$-module structure with multiplication defined as $rv = \varphi_i(r)\cdot v$, for all $r \in \mathbb{K}_{q^{h_i}}[x_i,y_i]$ and $v \in V$.

To show that (φ_i, V_i) defines an irreducible representation of the quantum affine space, we suppose that there exists a nonzero subspace $W \subseteq V_i$ which is a sub-representation of V_i and we argue that $W = V_i$. Assume (for contradiction) that $v_j^{(i)} \notin W$, for all $j \in [\![1,\ell]\!]$, and let $w := \sum_{j=1}^{\ell} \alpha_j v_j^{(i)} \in W$, with $\alpha_1,\dots,\alpha_\ell \in \mathbb{K}$, be a nonzero element of W which is minimal with respect to the number of nonzero summands. Note that the assumption $v_j^{(i)} \notin W$ implies that at least two coefficients in $\{\alpha_1,\dots,\alpha_\ell\}$ are nonzero. Since W is a sub-representation then $\varphi_i(y_i)\cdot w \in W$ and $\varphi_i(x_iy_i)\cdot w \in W$, where

$$\varphi_i(y_i)\cdot w = \sum_{j=1}^{\ell-1} \alpha_j v_{j+1}^{(i)} + \alpha_\ell v_1^{(i)},$$

$$\varphi_i(x_iy_i)\cdot w = \varphi_i(x_i)\cdot(\varphi_i(y_i)\cdot w) = \sum_{j=1}^{\ell-1} \lambda_i q^{jh_i}\alpha_j v_{j+1}^{(i)} + \lambda_i \alpha_\ell v_1^{(i)}.$$

We now use these elements of W to arrive at a contradiction. If $\alpha_\ell \neq 0$ then we consider $\varphi_i(y_i - \lambda_i^{-1}x_iy_i)\cdot w \in W$, where

$$\varphi_i(y_i - \lambda_i^{-1}x_iy_i)\cdot w = \sum_{j=1}^{\ell-1} \alpha_j (1-q^{jh_i}) v_{j+1}^{(i)}.$$

This is nonzero, since $\ell \nmid jh_i$ for all $j \in [\![1,\ell-1]\!]$, and it has fewer summands than w, thus contradicting the minimality of w. If, instead, $\alpha_\ell = 0$ and $\alpha_k \neq 0$, for some $k \in [\![1,\ell-1]\!]$, then we consider the element $\varphi_i(y_i - q^{-kh_i}\lambda_i^{-1}x_iy_i)\cdot w$ and argue in a similar way to show that this also contradicts the minimality of w. Therefore, our assumption is false and there must exist some $j \in [\![1,\ell]\!]$ such that $v_j^{(i)} \in W$. But then $W = V_i$, since $\varphi_i(y_i)$ permutes all the basis vectors, as can be seen in (7.3). We deduce from this that (φ_i, V_i) is irreducible.

To verify surjectivity of φ_i, note that V_i is a simple $\mathbb{K}_{q^{h_i}}[x_i,y_i]$-module, finite-dimensional over $\mathbb{K}$, and $\mathbb{K}$ is algebraically closed. Thus, by Schur's Lemma, $\mathrm{End}_{\mathbb{K}_{q^{h_i}}[x_i,y_i]}(V_i) \cong \mathbb{K}$ and we can apply the Jacobson Density Theorem to $R = \mathbb{K}_{q^{h_i}}[x_i, y_i]$, with $R' = \mathrm{End}_{\mathbb{K}_{q^{h_i}}[x_i,y_i]}(V_i)$, to see that the map $\rho : \mathbb{K}_{q^{h_i}}[x_i, y_i] \to \mathrm{End}_{\mathbb{K}}(V_i)$, sending $r \mapsto f_r$, is surjective. Recall that f_r was defined as $f_r(v) = rv$, for all $r \in \mathbb{K}_{q^{h_i}}[x_i, y_i]$ and $v \in V_i$, and this is precisely the definition of $\varphi_i(r)$ as stated earlier. Hence $\varphi_i = \rho$ is surjective.

Finally, it is easily verified that $\varphi_i(x_i)^\ell = \lambda_i^\ell \mathrm{Id}_{V_i}$ and $\varphi_i(y_i)^\ell = \mathrm{Id}_{V_i}$. This proves the final property in the statement. □

Consider the quantum affine space $\mathbb{K}_{q^M}[T_1, \ldots, T_N]$, where M has a kernel of dimension t, for some $t \in [\![0, N-1]\!]$. The skew normal form of M is

$$S = EME^T = \begin{pmatrix} 0 & h_1 & & & & & & \\ -h_1 & 0 & & & & & & \\ & & 0 & h_2 & & & & \\ & & -h_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & h_s & \\ & & & & & -h_s & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix} \in M_N(\mathbb{Z}),$$

where $E = (e_{i,j})_{i,j} \in M_N(\mathbb{Z})$ is invertible, $\mathbf{0}$ is the $t \times t$ zero matrix, the $h_i \in \mathbb{Z}\backslash\{0\}$ are the invariant factors of M, and $2s = N - t$. We define the quantum affine space associated to S as

$$D := \mathbb{K}_{q^S}[x_1, y_1, x_2, y_2, \ldots, x_s, y_s, z_1, \ldots, z_t].$$

If $\gcd(h_i, \ell) = 1$, for all $i \in [\![1, s]\!]$, then each subalgebra $\mathbb{K}_{q^{h_i}}[x_i, y_i] \subseteq D$ is a quantum affine plane satisfying the conditions of Lemma 7.6, hence it must have an irreducible representation (φ_i, V_i) of dimension ℓ. Let $V := V_1 \otimes \cdots \otimes V_s$ so that the dimension of V is ℓ^s. Using this notation we state the following proposition:

Proposition 7.7. *Let $1 \neq q \in \mathbb{K}^*$ be a primitive ℓ^{th} root of unity, for some $\ell \in \mathbb{N}_{>1}$, and $M \in M_N(\mathbb{Z})$ be a skew-symmetric matrix with invariant factors $h_1, \ldots, h_s \in \mathbb{Z}\backslash\{0\}$ satisfying $\gcd(h_i, \ell) = 1$, for all $i \in [\![1, s]\!]$. Then the following statements hold:*

(i) PI-deg$(\mathbb{K}_{q^M}[T_1, \ldots, T_N]) = \ell^s$.

(ii) There is an algebra homomorphism $\varphi : D \to \mathrm{End}_{\mathbb{K}}(V)$ which defines an irreducible representation of D on V of dimension ℓ^s. It is defined using the tensor product of the

maps φ_i found in Lemma 7.6 and it acts on the generators of D as follows:

$$\begin{aligned}
\varphi(x_i) &= \mathrm{Id}_{V_1} \otimes \cdots \otimes \mathrm{Id}_{V_{i-1}} \otimes \varphi_i(x_i) \otimes \mathrm{Id}_{V_{i+1}} \otimes \cdots \otimes \mathrm{Id}_{V_s}, \\
\varphi(y_i) &= \mathrm{Id}_{V_1} \otimes \cdots \otimes \mathrm{Id}_{V_{i-1}} \otimes \varphi_i(y_i) \otimes \mathrm{Id}_{V_{i+1}} \otimes \cdots \otimes \mathrm{Id}_{V_s}, \\
\varphi(z_j) &= \xi_j \, \mathrm{Id}_V,
\end{aligned}$$

where $\xi_j \in \mathbb{K}^*$ for all $j \in [\![1,t]\!]$, Id_{V_i} denotes the identity map on V_i for all $i \in [\![1,s]\!]$, and $\mathrm{Id}_V := \mathrm{Id}_{V_1} \otimes \cdots \otimes \mathrm{Id}_{V_s}$ denotes the identity map on V. Moreover, $\varphi(x_i)^{-1} = \lambda_i^{-\ell} \varphi(x_i)^{\ell-1}$ and $\varphi(y_i)^{-1} = \varphi(y_i)^{\ell-1}$, for all $i \in [\![1,s]\!]$.

(iii) (φ, V) induces an irreducible representation, (ϕ, V), of $\mathbb{K}_{q^M}[T_1, \ldots, T_N]$ where, for all $i \in [\![1,N]\!]$, we have

$$\phi(T_i) := \varphi(x_1)^{e'_{i,1}} \varphi(y_1)^{e'_{i,2}} \cdots \varphi(x_s)^{e'_{i,2s-1}} \varphi(y_s)^{e'_{i,2s}} \varphi(z_1)^{e'_{i,2s+1}} \cdots \varphi(z_t)^{e'_{i,2s+t}},$$

with $E^{-1} := (e'_{i,j})_{i,j} \in M_N(\mathbb{Z})$. Moreover, for all $i \in [\![1,2s+t]\!]$ there exists some $\nu_i \in \mathbb{K}^*$ such that $\phi(T_i)^{-1} = \nu_i^{-1} \phi(T_i)^{\ell-1}$.

Proof.

(i) This follows from a straightforward application of Lemma 5.7.

(ii) It is clear from the matrix S that that the elements $z_1, \ldots, z_t \in Z(D)$ generate a polynomial ring $\mathbb{K}[z_1, \ldots, z_t] \subseteq D$ and that, for each $i \in [\![1,s]\!]$, the pair x_i, y_i generates a quantum affine plane $\mathbb{K}_{q^{h_i}}[x_i, y_i] \subseteq D$ such that x_i, y_i commute with x_j, y_j, for all $j \in [\![1,s]\!] \setminus \{i\}$. The generators of D therefore share the same commutation relations as the generators of $\left(\bigotimes_{i=1}^s \mathbb{K}_{q^{h_i}}[x_i, y_i] \right) \otimes \mathbb{K}[z_1, \ldots, z_t]$. Using the universal property of Ore extensions we deduce that the following map extends to an algebra homomorphism, and that this is in fact an isomorphism:

$$\begin{aligned}
\iota : D &\longrightarrow \mathbb{K}_{q^{h_1}}[x_1, y_1] \otimes \ldots \otimes \mathbb{K}_{q^{h_s}}[x_s, y_s] \otimes \mathbb{K}[z_1, \ldots, z_t] \qquad (7.5) \\
x_i &\longmapsto 1_1 \otimes \cdots \otimes 1_{i-1} \otimes x_i \otimes 1_{i+1} \otimes \cdots \otimes 1_s \otimes 1 \\
y_i &\longmapsto 1_1 \otimes \cdots \otimes 1_{i-1} \otimes y_i \otimes 1_{i+1} \otimes \cdots \otimes 1_s \otimes 1 \\
z_j &\longmapsto 1_1 \otimes \cdots \otimes 1_s \otimes z_j,
\end{aligned}$$

where 1_i is the identity element in $\mathbb{K}_{q^{h_i}}[x_i, y_i]$, for all $i \in [\![1,s]\!]$.

That is, for a general element $d = \sum_{\underline{l}\in\mathbb{N}^{2s+t}} \alpha_{\underline{l}} x_1^{l_1} y_1^{l_2} \cdots x_s^{l_{2s-1}} y_s^{l_{2s}} z_1^{l_{2s+1}} \cdots z_t^{l_{2s+t}} \in D$,

$$\iota(d) = \sum_{\underline{l}\in\mathbb{N}^{2s+t}} \alpha_{\underline{l}} x_1^{l_1} y_1^{l_2} \otimes \cdots \otimes x_s^{l_{2s-1}} y_s^{l_{2s}} \otimes z_1^{l_{2s+1}} \cdots z_t^{l_{2s+t}}.$$

We can define an irreducible representation (φ', V) of the tensor product algebra on the right hand side of (7.5) by applying Proposition 7.4 and noting, by Schur's Lemma, that the image of $\mathbb{K}[z_1, \ldots, z_t]$ under φ' is contained in $\mathbb{K}$. Here, $\varphi' := \varphi_1 \otimes \cdots \varphi_s$ for the maps φ_i as defined in Lemma 7.6. We then pull this representation back to an irreducible representation (φ, V) of D via the isomorphism above by setting $\varphi := \varphi' \circ \iota$. This is defined on the generators of D as

$$\begin{aligned}
\varphi : D &\longrightarrow \operatorname{End}_{\mathbb{K}}(V) \\
x_i &\longmapsto \operatorname{Id}_{V_1} \otimes \cdots \otimes \operatorname{Id}_{V_{i-1}} \otimes \varphi_i(x_i) \otimes \operatorname{Id}_{V_{i+1}} \otimes \cdots \otimes \operatorname{Id}_{V_s} \\
y_i &\longmapsto \operatorname{Id}_{V_1} \otimes \cdots \otimes \operatorname{Id}_{V_{i-1}} \otimes \varphi_i(y_i) \otimes \operatorname{Id}_{V_{i+1}} \otimes \cdots \otimes \operatorname{Id}_{V_s} \\
z_j &\longmapsto \xi_j \operatorname{Id}_V,
\end{aligned} \tag{7.6}$$

where Id_{V_i} is the identity map on V_i and $\operatorname{Id}_V := \operatorname{Id}_{V_1} \otimes \cdots \otimes \operatorname{Id}_{V_s}$ is the identity map on V.

We may form a basis $\{\mu_{i_1,i_2,\ldots,i_s} \mid 1 \le i_1, i_2, \ldots, i_s \le \ell\}$ of V in terms of the basis of each V_i by setting:

$$\mu_{i_1,i_2,\ldots,i_s} := v_{i_1}^{(1)} \otimes v_{i_2}^{(2)} \otimes \cdots \otimes v_{i_s}^{(s)}.$$

Then, using (7.2) and (7.3), one may verify the following:

$$\begin{aligned}
\varphi(x_j)\cdot\mu_{i_1,\ldots,i_s} &= v_{i_1}^{(1)} \otimes \cdots \otimes v_{i_{j-1}}^{(j-1)} \otimes \varphi_j(x_j)\cdot v_{i_j}^{(j)} \otimes v_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes v_{i_s}^{(s)} \\
&= v_{i_1}^{(1)} \otimes \cdots \otimes v_{i_{j-1}}^{(j-1)} \otimes \lambda_j q^{(i_j-1)h_j} v_{i_j}^{(j)} \otimes v_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes v_{i_s}^{(s)} \\
&= \lambda_j q^{(i_j-1)h_j} \mu_{i_1,\ldots,i_s} \\
\varphi(y_j)\cdot\mu_{i_1,\ldots,i_s} &= v_{i_1}^{(1)} \otimes \cdots \otimes v_{i_{j-1}}^{(j-1)} \otimes \varphi_j(y_j)\cdot v_{i_j}^{(j)} \otimes v_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes v_{i_s}^{(s)} \\
&= \begin{cases} v_{i_1}^{(1)} \otimes \cdots \otimes v_{i_{j-1}}^{(j-1)} \otimes v_{1+i_j}^{(j)} \otimes v_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes v_{i_s}^{(s)}, & \text{if } i_j \ne \ell; \\ v_{i_1}^{(1)} \otimes \cdots \otimes v_{i_{j-1}}^{(j-1)} \otimes v_{1}^{(j)} \otimes v_{i_{j+1}}^{(j+1)} \otimes \cdots \otimes v_{i_s}^{(s)}, & \text{if } i_j = \ell, \end{cases} \\
&= \begin{cases} \mu_{i_1,\ldots,i_{j-1},1+i_j,i_{j+1},\ldots,i_s}, & \text{if } i_j \ne \ell; \\ \mu_{i_1,\ldots,i_{j-1},1,i_{j+1},\ldots,i_s}, & \text{if } i_j = \ell, \end{cases} \\
\varphi(z_j)\cdot\mu_{i_1,\ldots,i_s} &= \xi_j \mu_{i_1,\ldots,i_s}.
\end{aligned}$$

Fixing this basis for V allows us to represent the maps $\varphi(x_i)$, $\varphi(y_i)$, $\varphi(z_j) \in \mathrm{End}_{\mathbb{K}}(V)$ as matrices in $M_{\ell^s}(\mathbb{K})$ by first defining $\varphi_i(x_i)$, $\varphi_i(y_i)$ as the matrices in (7.4), and then taking the images in (7.6) to be the Kronecker product of these matrices.

(iii) The algebra homomorphism $\varphi : D \to \mathrm{End}_{\mathbb{K}}(V)$ preserves the relations

$$\varphi(x_i)\varphi(y_i) = q^{h_i}\varphi(y_i)\varphi(x_i),$$

for all $i \in [\![1,s]\!]$, and, as was shown in Lemma 7.6, $\varphi_i(x_i)^\ell = \lambda_i^\ell \mathrm{Id}_{V_i}$ and $\varphi_i(y_i)^\ell = \mathrm{Id}_{V_i}$. Hence we obtain

$$\left.\begin{aligned} \varphi(x_i)^{-1} &= \lambda_i^{-\ell}\varphi(x_i)^{\ell-1} = \varphi(\lambda_i^{-\ell}x_i^{\ell-1}) \\ \varphi(y_i)^{-1} &= \varphi(y_i)^{\ell-1} = \varphi(y_i^{\ell-1}) \\ \varphi(z_j)^{-1} &= \xi_j^{-1}\mathrm{Id}_V \end{aligned}\right\} \in \varphi(D). \tag{7.7}$$

These identities show that φ may be extended to define a representation of the quantum torus associated to D and hence, any negative powers of the generators x_i, y_i, z_j in the argument of φ may be replaced with positive powers, upon an appropriate multiplication by a scalar.

From the identity $E^{-1}S(E^{-1})^T = M$ we obtain the equality

$$m_{i,j} = \sum_{k=1}^{s} h_k(e'_{i,2k-1}e'_{j,2k} - e'_{i,2k}e'_{j,2k-1}). \tag{7.8}$$

Any algebra homomorphism $\phi : \mathbb{K}_{q^M}[T_1,\ldots,T_N] \to \mathrm{End}_{\mathbb{K}}(V)$ must preserve the commutation rules between the T_i. That is, for all $i, j \in [\![1,N]\!]$,

$$\phi(T_i)\phi(T_j) = q^{m_{i,j}}\phi(T_j)\phi(T_i) = q^{\sum_{k=1}^{s} h_k(e'_{i,2k-1}e'_{j,2k} - e'_{i,2k}e'_{j,2k-1})}\phi(T_j)\phi(T_i).$$

Let E_i^{-1} denote the i^{th} row of E^{-1}, for all $i \in [\![1,N]\!]$, and let $\underline{d} := x_1y_1\ldots x_sy_sz_1\ldots z_t \in D$ be the ordered monomial of all the generators of D. We define the following elements of the quantum torus $D\Sigma^{-1}$, where $\Sigma \subset D$ is the multiplicatively closed set generated by the generators of D:

$$\underline{d}^{E_i^{-1}} := x_1^{e'_{i,1}}y_1^{e'_{i,2}}\cdots x_s^{e'_{i,2s-1}}y_s^{e'_{i,2s}}z_1^{e'_{i,2s+1}}\cdots z_t^{e'_{i,2s+t}} \in D\Sigma^{-1}. \tag{7.9}$$

We may apply φ to $\underline{d}^{E_i^{-1}}$, using (7.7) to replace any negative powers $e'_{i,j}$ with positive powers $\ell - e'_{i,j}$ so that $\varphi(\underline{d}^{E_i^{-1}}) \in \varphi(D)$, and, using (7.8) and (7.9), it may be verified that

$$\varphi(\underline{d}^{E_i^{-1}})\varphi(\underline{d}^{E_j^{-1}}) = q^{m_{i,j}}\varphi(\underline{d}^{E_j^{-1}})\varphi(\underline{d}^{E_i^{-1}}),$$

for all $i, j \in [\![1,N]\!]$. Therefore, by the universal property of algebras with generators and relations, the following map extends to an algebra homomorphism and hence defines a representation of $\mathbb{K}_{q^M}[T_1,\dots,T_N]$ on V:

$$\begin{aligned} \phi : \mathbb{K}_{q^M}[T_1,\dots,T_N] &\longrightarrow \mathrm{End}_{\mathbb{K}}(V) \\ T_i &\longmapsto \varphi(\underline{d}^{E_i^{-1}}). \end{aligned}$$

Note that $\mathrm{Im}(\phi) \subseteq \mathrm{Im}(\varphi)$ and, for all $i \in [\![1,N]\!]$,

$$\phi(T_i)^\ell = \varphi(\underline{d}^{E_i^{-1}})^\ell = \varphi(x_1^{e'_{i,1}} y_1^{e'_{i,2}} \cdots x_s^{e'_{i,2s-1}} y_s^{e'_{i,2s}} z_1^{e'_{i,2s+1}} \cdots z_t^{e'_{i,2s+t}})^\ell = \nu_i \mathrm{Id}_V$$

for some $\nu_i \in \mathbb{K}^*$, by properties of φ. Thus $\phi(T_i)^{-1} = \nu_i^{-1}\phi(T_i)^{\ell-1}$, which allows us to deal with negative powers of T_i under ϕ in a similar manner to how we dealt with negative powers of x_i, y_i, z_j under φ.

We now wish to show that (ϕ, V) is irreducible. In a similar way to above, we may write each $\varphi(x_i)$, $\varphi(y_i)$, $\varphi(z_j)$ in terms of the $\phi(T_i)$. The identity $EME^T = S$ allows us to write the entries $s_{i,j} \in S$ in terms of the entries $m_{i,j} \in M$ and $e_{i,j} \in E$ as follows:

$$s_{i,j} = \sum_{l,k=1}^{N} m_{l,k} e_{i,l} e_{j,k}.$$

Denoting E_i to be the i^{th} row of E, for all $i \in [\![1,N]\!]$, and $\underline{T} := T_1 \cdots T_N$ to be the ordered monomial of the generators of $\mathbb{K}_{q^M}[T_1,\dots,T_N]$, we define the following elements of $\mathbb{K}_{q^M}[T_1^{\pm1},\dots,T_N^{\pm1}]$:

$$\underline{T}^{E_i} := T_1^{e_{i,1}} \cdots T_N^{e_{i,N}}. \tag{7.10}$$

Using this, one may verify, by explicit calculation on $\underline{T}^{E_i}\underline{T}^{E_j}$, that

$$\phi(\underline{T}^{E_i})\phi(\underline{T}^{E_j}) = q^{s_{i,j}}\phi(\underline{T}^{E_j})\phi(\underline{T}^{E_i}),$$

for all $i, j \in [\![1,N]\!]$. In particular, the elements

$$\phi(\underline{T}^{E_1}),\ \phi(\underline{T}^{E_2}),\ \dots,\ \phi(\underline{T}^{E_{2s-1}}),\ \phi(\underline{T}^{E_{2s}}),\ \phi(\underline{T}^{E_{2s+1}}),\ \dots,\ \phi(\underline{T}^{E_N})$$

share the same relations as the elements $x_1, y_1, \dots, x_s, y_s, z_1, \dots, z_t$. Therefore, again by the universal property of algebras with generators and relations, the following map extends to an algebra homomorphism:

$$\begin{aligned} \tau : D &\longrightarrow \mathrm{End}_{\mathbb{K}}(V) \\ x_i &\longmapsto \phi(\underline{T}^{E_{2i-1}}) \\ y_i &\longmapsto \phi(\underline{T}^{E_{2i}}) \\ z_j &\longmapsto \phi(\underline{T}^{E_{2s+j}}), \end{aligned}$$

with $\mathrm{Im}(\tau) \subseteq \mathrm{Im}(\phi)$. Since $E^{-1}E = \mathrm{Id}$, where Id is the identity matrix, we deduce that $\sum_{k=1}^{N} e'_{k,i} e_{j,k} = 1$ if $i = j$, and otherwise the sum is zero. Using this, along with the definition of ϕ, we obtain

$$\begin{aligned} \tau(x_i) = \phi(\underline{T}^{E_{2i-1}}) &= \phi(T_1^{e_{2i-1,1}} \cdots T_N^{e_{2i-1,N}}) \\ &= \varphi(\underline{d}^{E_1^{-1}})^{e_{2i-1,1}} \dots \varphi(\underline{d}^{E_N^{-1}})^{e_{2i-1,N}} \\ &= \varphi(x_1^{e'_{1,1}} y_1^{e'_{1,2}} \cdots z_t^{e'_{1,N}})^{e_{2i-1,1}} \cdots \varphi(x_1^{e'_{i,1}} y_1^{e'_{i,2}} \cdots z_t^{e'_{N,N}})^{e_{2i-1,N}} \\ &= \varphi(\nu x_1^{\sum_{k=1}^{N} e'_{k,1} e_{2i-1,k}} y_1^{\sum_{k=1}^{N} e'_{k,2} e_{2i-1,k}} \cdots z_t^{\sum_{k=1}^{N} e'_{k,N} e_{2i-1,k}}) \\ &= \varphi(\nu x_i), \end{aligned}$$

where $\nu \in \mathbb{K}^*$ is the scalar resulting from reordering the generators of D. Similarly we can show that $\tau(y_i) = \varphi(\nu' y_i)$ and $\tau(z_j) = \varphi(\nu'' z_j)$, for some $\nu', \nu'' \in \mathbb{K}^*$. From this we deduce that $\mathrm{Im}(\tau) = \mathrm{Im}(\varphi)$ and hence $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(\phi)$. It then follows that (ϕ, V) is an irreducible representation of $\mathbb{K}_{q^M}[T_1, \dots, T_N]$ because (φ, V) is irreducible.

□

7.3 Irreducible Representations and the Deleting Derivations Algorithm

Given a PI algebra A/P satisfying certain conditions, the deleting derivations algorithm provides a way to construct a quotient of a quantum affine space, $A'/\psi(P)$, such that $\text{PI-deg}(A/P) = \text{PI-deg}(A'/\psi(P))$. The importance of the PI degree comes from Theorem 2.17, which states that for a prime affine PI algebra, the PI degree gives an upper bound on the dimension of its irreducible representations. We show that we may pass certain irreducible representations of $A'/\psi(P)$ of dimension $d := \text{PI-deg}(A'/\psi(P)) = \text{PI-deg}(A/P)$

back through the deleting derivations algorithm to obtain an irreducible representation of A/P of degree d. In the case when $P \in \mathrm{C.Spec}(A)$ is a Cauchon ideal, the quotient $A'/\psi(P)$ is a quantum affine space at a root of unity and the result from the previous section may, in the case where its invariant factors are coprime to the order of the root of unity, provide us with an explicit irreducible representation of $A'/\psi(P)$ of dimension d.

This result is a specific case of a more general result, in which the algebra A/P doesn't need to be a PI algebra in order for an irreducible representation of A/P to be constructed from an irreducible representation of $A'/\psi(P)$. For this broader case, we simply require the irreducible representation of $A'/\psi(P)$ to satisfy a property on the generators. We prove this result first and the case where A/P is a PI algebra comes as a corollary. We end this section by providing an example of the theory.

In this section we continue to take $\mathbb{K}$ to be an algebraically closed field but we now let $1 \neq q \in \mathbb{K}^*$ be any element, unless stated otherwise.

7.3.1 The Construction

Recall the notation from Chapters 3 and 4. Specifically, let A be an algebra satisfying Hypothesis 1, A' be the quantum affine space obtained from applying Corollary 3.7 to A, and $\psi : \mathrm{C.Spec}(A) \to \mathrm{C.Spec}(A')$ be the canonical embedding.

Denote the power set $\mathscr{W} := \mathbb{P}(\llbracket 1,N \rrbracket)$ and let $\mathscr{W}' \subseteq \mathscr{W}$ be the set of Cauchon diagrams for A. Then, for some $w \in \mathscr{W}'$ and $P \in \mathrm{C.Spec}_w(A)$, set $P_j := \psi_j \circ \cdots \circ \psi_N(P) \in \mathrm{C.Spec}(A^{(j)})$, where we recall that $\psi := \psi_2 \circ \cdots \circ \psi_N$. For each $j \in \llbracket 2, N+1 \rrbracket$, let $B^{(j)} := A^{(j)}/P_j$ and, for all $i \in \llbracket 1,N \rrbracket$, denote by $\bar{X}_i^{(j)}$ the canonical image of $X_i^{(j)}$ in $B^{(j)}$, where $X_1^{(j)}, \dots, X_N^{(j)}$ generate $A^{(j)}$. In particular, for $j = 2$ and $j = N+1$, we set $B := A/P$ and $B' := A'/\psi(P)$ with generators $\bar{X}_i := X_i + P \in B$ and $t_i := T_i + \psi(P) \in B'$, respectively.

Define $\Sigma \subseteq B'$ to be the multiplicatively closed set generated by t_i for all $i \in \mathscr{W} \setminus \{w\}$ and let $\Sigma = \Sigma_2$. Finally, for all $j \in \llbracket 2,N \rrbracket$, set $\Sigma_{j+1} := B^{(j+1)} \cap \Sigma_j$.

Proposition
that (ϕ_j, V) is an irreducible representation of $B^{(j)}$ where, for each $e^{(j)} \in \Sigma_j$, there exists some $\xi \in \mathbb{K}^*$ and $\ell \in \mathbb{N}_{>1}$ such that $\phi_j(e^{(j)})^\ell = \xi \,\mathrm{Id}_V$.

Then, for any $k \in \llbracket 2, N+1 \rrbracket$ and $b^{(k)} \in B^{(k)}$, there exists some $b^{(j)} \in B^{(j)}$ and $e^{(j)} \in \Sigma_j$ such that

$$b^{(k)} = b^{(j)}(e^{(j)})^{-1} \in B^{(j)}\Sigma_j^{-1}.$$

If $k \geq j$ then ϕ_j induces an irreducible representation of $B^{(k)}$ on V, defined by the algebra homomorphism

$$\begin{aligned} \phi_k : B^{(k)} &\longrightarrow \mathrm{End}_{\mathbb{K}}(V) \\ b^{(k)} &\longmapsto \xi^{-1}\phi_j(b^{(j)})\phi_j(e^{(j)})^{\ell-1}. \end{aligned}$$

Moreover, for each $e^{(k)} \in \Sigma_k$, there exists some $\xi' \in \mathbb{K}^*$ and $\ell' \in \mathbb{N}_{>1}$ such that $\phi_k(e^{(k)})^{\ell'} = \xi' \,\mathrm{Id}_V$.

Proof. Let (ϕ_j, V) be an irreducible representation of $B^{(j)}$ satisfying the conditions of the proposition. We may rewrite the condition on ϕ_j as $\phi_j(e^{(j)})^{-1} = \xi^{-1}\phi_j(e^{(j)})^{\ell-1}$. This induces a representation of $B^{(j)}\Sigma_j^{-1}$,

$$\begin{aligned} \hat{\phi} :\ B^{(j)}\Sigma_j^{-1} &\longrightarrow \mathrm{End}_{\mathbb{K}}(V) \\ b^{(j)} &\longmapsto \phi_j(b^{(j)}) \\ (e^{(j)})^{-1} &\longmapsto \xi^{-1}\phi_j(e^{(j)})^{\ell-1}, \end{aligned}$$

from which we observe that $\hat{\phi}(e^{(j)})^{-1} = \xi^{-1}\hat{\phi}(e^{(j)})^{\ell-1}$, for all $e^{(j)} \in \Sigma_j$. The inclusion $B^{(j)} \subseteq B^{(j)}\Sigma_j^{-1}$ ensures that $(\hat{\phi}, V)$ is irreducible.

From Proposition 4.24(iii) we see that $B^{(j)}\Sigma_j^{-1} = B^{(k)}\Sigma_k^{-1}$, for all $k \in [\![2, N+1]\!]$, hence each element in $B^{(k)}\Sigma_k^{-1}$ can be written in terms of elements in $B^{(j)}\Sigma_j^{-1}$, and vice versa. This allows us to view $\hat{\phi}$ as an algebra homomorphism $\hat{\phi} : B^{(k)}\Sigma_k^{-1} \to \mathrm{End}_{\mathbb{K}}(V)$, hence $(\hat{\phi}, V)$ defines an irreducible representation of $B^{(k)}\Sigma_k^{-1}$.

Every element $b^{(k)} \in B^{(k)}$ may be written as an element $b^{(k)} \cdot 1^{-1} \in B^{(k)}\Sigma_k^{-1}$, which we write simply as $b^{(k)} \in B^{(k)}\Sigma_k^{-1}$. By the equality of localisations, $b^{(k)}$ may also be written as an element in $B^{(j)}\Sigma_j^{-1}$, that is $b^{(k)} = b^{(j)}(e^{(j)})^{-1} \in B^{(j)}\Sigma_j^{-1}$. Restricting $\hat{\phi}$ to $B^{(k)}$ gives the following algebra homomorphism:

$$\begin{aligned} \phi_k : B^{(k)} &\longrightarrow \mathrm{End}_{\mathbb{K}}(V) \\ b^{(k)} &\longmapsto \hat{\phi}\left(b^{(j)}(e^{(j)})^{-1}\right) = \xi^{-1}\phi_j(b^{(j)})\phi_j(e^{(j)})^{\ell-1}. \end{aligned}$$

This defines a (not necessarily irreducible) representation of $B^{(k)}$ on V. To show that this representation is irreducible when $k \geq j$ we note that, in this case, $\Sigma_k \subseteq \Sigma_j$. Therefore, for all $e^{(k)} \in \Sigma_k$, there exists some $\xi' \in \mathbb{K}^*$ and $\ell' \in \mathbb{N}$ such that $\hat{\phi}(e^{(k)})^{\ell'} = \xi' \,\mathrm{Id}_V$, hence

$$\phi_k(e^{(k)})^{\ell'} = \hat{\phi}(e^{(k)})^{\ell'} = \xi' \,\mathrm{Id}_V\,.$$

Using the identity $B^{(k)}\Sigma_k^{-1} = B^{(j)}\Sigma_j^{-1}$ we may write any element $b^{(j)} \in B^{(j)}$ as $b^{(j)} = b^{(k)}(e^{(k)})^{-1} \in B^{(k)}\Sigma_k^{-1}$. Thus

$$\phi_j(b^{(j)}) = \hat{\phi}(b^{(j)}) = \hat{\phi}\left(b^{(k)}(e^{(k)})^{-1}\right) = \xi'^{-1}\hat{\phi}(b^{(k)})\hat{\phi}(e^{(k)})^{\ell'-1} = \phi_k(\xi'^{-1}b^{(k)}(e^{(k)})^{\ell'-1}),$$

which shows that $\phi_j(B^{(j)}) \subseteq \phi_k(B^{(k)})$. Therefore, since (ϕ_j, V) is irreducible, so too is (ϕ_k, V). $\square$

Corollary 7.9. *Let A be a $\mathbb{K}$-algebra satisfying Hypothesis 1 and suppose that all the $\lambda_{i,j}$ in H1.2 are roots of unity. Write these as $\lambda_{i,j} = q^{m_{i,j}}$, for some skew-symmetric matrix $M = (m_{i,j})_{i,j} \in M_N(\mathbb{Z})$ and primitive ℓ^{th} root of unity $1 \neq q \in \mathbb{K}^*$, for $\ell \in \mathbb{N}_{>1}$.*

Then any irreducible representation (ϕ, V) of B' satisfies the conditions of Proposition 7.8 and therefore induces an irreducible representation (ϕ', V) of B.

Proof. A' is a prime affine PI algebra, by Theorem 3.10(iii). Since quotients of affine PI algebras are themselves affine PI algebras, and given that $\psi(P) \in \mathrm{C.Spec}(A')$, we therefore deduce that $A'/\psi(P)$ is a prime affine PI algebra. Hence, by Theorem 2.17, B' has an irreducible representation (ϕ, V) of dimension PI-deg(B'). Suppose $P \in \mathrm{C.Spec}_w(A)$, for some $w \in \mathscr{W}'$. Then the quotient algebra, $B' = A/\psi(P)$, is generated by all t_i such that $i \in \mathscr{W}\backslash\{w\}$, where $t_i := T_i + \psi(P)$. Hence $t_i t_j = q^{m_{i,j}} t_j t_i$, for all $i, j \in \mathscr{W}\backslash\{w\}$. We deduce from this that $t_i^{\ell} \in Z(B')$, for all $i \in \mathscr{W}\backslash\{w\}$, thus, by Schur's Lemma, $\phi(t_i)^{\ell} = \xi_i \mathrm{Id}_V$, for some $\xi_i \in \mathbb{K}^*$. Since Σ is defined to be the multiplicatively closed set generated by the set $\{t_i \mid i \in \mathscr{W}\backslash\{w\}\}$, then the conditions of Proposition 7.8 are satisfied. Applying this proposition to A with $j = 2$ and $k = N+1$ gives the desired result. $\square$

Remark 7.10. A special case of the corollary above arises when P is a Cauchon ideal, whence B' is a quantum affine space. Let $B' = \mathscr{O}_{q^M}(\mathbb{K}^n)$ and suppose the invariant factors of $M = (m_{i,j})$ satisfy the conditions of Proposition 7.7. We may then apply said proposition to obtain an irreducible representation (ϕ, V) of B' of dimension rank$(M)/2$. Then, by Corollary 7.9, we may apply Proposition 7.8 to obtain an irreducible representation of B on V.

Remark 7.11. If we let $P = \{0\} \lhd A$, so that $B^{(j)} = A^{(j)}$ for all $j \in [\![2, N+1]\!]$, then Proposition 7.8 and Corollary 7.9 give ways to induce an irreducible representation of A provided that there exists an irreducible representation (ϕ, V) of A' satisfying analogous conditions. In particular, if we define the sets Σ_j as in Section 3.3 (that is we take $\Sigma \subseteq A'$ to be the multiplicatively closed set generated by $T_1, \ldots, T_N$, we let $\Sigma_2 := \Sigma$, and we define $\Sigma_{j+1} := A^{(j+1)} \cap \Sigma_j$, for $j \in [\![2, N]\!]$) then the equality of localisations $A^{(j)}\Sigma_j^{-1} = A^{(k)}\Sigma_k^{-1}$, for all $j, k \in [\![1, N]\!]$, as stated in Proposition 3.9(iii), provides the condition required to transfer an irreducible representation of A' to an irreducible representation of A.

7.3.2 Example: $U_q^+(\mathfrak{so}_5)/\langle z'\rangle$

Let $\mathbb{K}=\mathbb{C}$ and $q\in\mathbb{C}^*$ be a primitive ℓ^{th} root of unity, with $\ell\notin\{2,4\}$. Recall Example 5.3.1, where we showed

$$\text{PI-deg}(U_q^+(\mathfrak{so}_5)/\langle z'\rangle)=\text{PI-deg}(\mathbb{C}_{q^{M'}}[t_1,t_3,t_4])=\begin{cases}\ell & \ell \text{ is odd};\\ \ell/2 & \ell>4 \text{ is even},\end{cases}$$

for $M'=\begin{pmatrix}0&0&-2\\0&0&2\\2&-2&0\end{pmatrix}$. In the notation of the previous section we write

$$B':=B^{(3)}=\mathbb{C}_{q^{M'}}[t_1,t_3,t_4],\quad B:=B^{(5)}=U_q^+(\mathfrak{so}_5)/\langle z'\rangle,\quad \bar{X}_i^{(5)}=\bar{X}_i,\quad \bar{X}_i^{(3)}=t_i,$$

for all $i\in[\![1,5]\!]$, and we denote by $\Sigma:=\Sigma_3\subseteq B'$ the multiplicatively closed set generated by $\{t_1,t_3,t_4\}$. Let ℓ be odd, then $\mathbb{C}_{q^{M'}}[t_1,t_3,t_4]$ satisfies the conditions of Proposition 7.7. The skew-normal form of M' is

$$S=\begin{pmatrix}0&2&0\\-2&0&0\\0&0&0\end{pmatrix}$$

and the quantum affine space associated to S is $D:=\mathbb{C}_{q^S}[x_1,y_1,z_1]$. By Proposition 7.7(ii) there is an ℓ-dimensional $\mathbb{C}$-vector space, V, and an algebra homomorphism $\varphi:D\to\text{End}_{\mathbb{C}}(V)$ whose image on the generators of D, upon fixing a basis $v_1,\dots,v_\ell\in V$, may be presented as matrices in the following way:

$$\varphi(x_1)=\begin{pmatrix}\lambda&&&&\\&\lambda q^2&&&\\&&\lambda q^4&&\\&&&\ddots&\\&&&&\lambda q^{(\ell-1)2}\end{pmatrix},\qquad \varphi(y_1)=\begin{pmatrix}0&0&\dots&0&1\\1&0&\dots&0&0\\0&1&\dots&0&0\\\vdots&&\ddots&\vdots&\vdots\\0&\dots&\dots&1&0\end{pmatrix},\qquad \varphi(z_1)=\xi\,\text{Id}_V\,.$$

That is, φ is the map φ_i defined in Lemma 7.6, with $\lambda_i=\lambda$. The pair (φ,V) defines an irreducible representation of D.

To define an irreducible representation of $\mathbb{C}_{q^{M'}}[t_1,t_3,t_4]$ we apply Proposition 7.7(iii) using $E^{-1}=\begin{pmatrix}1&0&1\\-1&0&0\\0&-1&0\end{pmatrix}$, where $E\in M_3(\mathbb{Z})$ is the invertible matrix satisfying $EME^T=S$. The resulting algebra homomorphism $\phi:\mathbb{C}_{q^{M'}}[t_1,t_3,t_4])\to\text{End}_{\mathbb{C}}(V)$, defined on t_1, t_3, t_4 as

$$\begin{aligned}\phi(t_1)&=\varphi(x_1^1y_1^0z_1^1)=\varphi(x_1)\varphi(z_1)=\xi\varphi(x_1),\\ \phi(t_3)&=\varphi(x_1^{-1}y_1^0z_1^0)=\varphi(x_1)^{-1}=\lambda^{-\ell}\varphi(x_1^{\ell-1}),\\ \phi(t_4)&=\varphi(x_1^0y_1^{-1}z_1^0)=\varphi(y_1)^{-1}=\varphi(y_1^{\ell-1}),\end{aligned}\tag{7.11}$$

then defines an irreducible representation of $\mathbb{C}_{q^{M'}}[t_1,t_3,t_4])$ on V. Using the definition of φ, we see that

$$\begin{aligned}
\phi(t_1)^\ell &= \xi^\ell \varphi(x_1)^\ell = \xi^\ell \mathrm{Id}_V,\\
\phi(t_3)^\ell &= \lambda^{-\ell^2}\varphi(x_1)^{\ell^2-\ell} = \lambda^{-\ell^2}\lambda^\ell \mathrm{Id}_V = \lambda^{-\ell^2+\ell}\mathrm{Id}_V\\
\phi(t_4)^\ell &= \varphi(y_1)^{\ell^2-\ell} = \mathrm{Id}_V,
\end{aligned}$$

hence the conditions of Proposition 7.8 are satisfied for $j=3$. Applying this proposition allows us to define an irreducible representation of $U_q^+(\mathfrak{so}_5)/\langle z'\rangle$ once we know how to write the generators $\bar{X}_1,\dots,\bar{X}_4 \in U_q^+(\mathfrak{so}_5)/\langle z'\rangle$ in terms of the generators $t_1,\dots,t_4 \in \mathbb{C}_{q^{M'}}[t_1,t_3,t_4]$.

In Example 5.3.1 we wrote the generators $T_1,\dots,T_4 \in \mathbb{C}_{q^M}[T_1,\dots,T_4]$ in terms of the generators $X_1,\dots,X_4 \in U_q^+(\mathfrak{so}_5)$ (see (5.7)). Taking the images of the generators T_i and X_i in the quotient algebras $\mathbb{C}_{q^{M'}}[t_1,t_3,t_4]$ and $U_q^+(\mathfrak{so}_5)/\langle z'\rangle$ respectively, gives the following:

$$t_4 := \bar{X}_4,\quad t_3 = \bar{X}_3,\quad t_2 = \bar{X}_2 - \frac{q^4}{(q^2+1)(q+q^{-1})}\bar{X}_3^2\bar{X}_4^{-1},\quad t_1 = \bar{X}_1 - \frac{q^2(q+q^{-1})}{q^2-1}\bar{X}_2\bar{X}_3^{-1}.$$

Rearranging these identities, and noting that $t_2=0$, allows us to write

$$\bar{X}_4 = t_4,\quad \bar{X}_3 = t_3,\quad \bar{X}_2 = \frac{q^4}{(q^2+1)(q+q^{-1})}t_3^2t_4^{-1},\quad \bar{X}_1 = t_1 + \frac{q^4}{q^4-1}t_3t_4^{-1}.$$

Applying Proposition 7.8, with $j=3$, $k=5$, and $\phi_j=\phi$, and using (7.11), we are able to deduce that there is an algebra homomorphism $\phi_5 : U_q^+(\mathfrak{so}_5)/\langle z'\rangle \to \mathrm{End}_{\mathbb{C}}(V)$ defined on its generators as

$$\begin{aligned}
\phi_5(\bar{X}_4) &= \phi(t_4) = \varphi(y_1)^{-1},\\
\phi_5(\bar{X}_3) &= \phi(t_3) = \varphi(x_1)^{-1},\\
\phi_5(\bar{X}_2) &= \phi\left(\frac{q^4}{(q^2+1)(q+q^{-1})}t_3^2t_4^{-1}\right)\\
&= \frac{q^4}{(q^2+1)(q+q^{-1})}\varphi(x_1)^{-2}\varphi(y_1),\\
\phi_5(\bar{X}_1) &= \phi\left(t_1 + \frac{q^4}{q^4-1}t_3t_4^{-1}\right)\\
&= \xi\varphi(x_1) + \frac{q^4}{q^4-1}\varphi(x_1)^{-1}\varphi(y_1),
\end{aligned}$$

which defines an irreducible representation (ϕ_5, V) of $U_q^+(\mathfrak{so}_5)/\langle z' \rangle$. Substituting in the matrices for $\varphi(x_1)$ and $\varphi(y_1)$ and taking $\ell = 5$ gives the following explicit form of this irreducible representation:

$$\phi_5(\bar{X}_4) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi_5(\bar{X}_3) = \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 & 0 \\ 0 & \lambda^{-1}q^3 & 0 & 0 & 0 \\ 0 & 0 & \lambda^{-1}q & 0 & 0 \\ 0 & 0 & 0 & \lambda^{-1}q^4 & 0 \\ 0 & 0 & 0 & 0 & \lambda^{-1}q^2 \end{pmatrix}$$

$$\phi_5(\bar{X}_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{(q^2+1)^2\lambda^2} \\ \frac{q}{(q^2+1)^2\lambda^2} & 0 & 0 & 0 & 0 \\ 0 & \frac{q^2}{(q^2+1)^2\lambda^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{q^3}{(q^2+1)^2\lambda^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{q^4}{(q^2+1)^2\lambda^2} & 0 \end{pmatrix}$$

$$\phi_5(\bar{X}_1) = \begin{pmatrix} \xi\lambda & 0 & 0 & 0 & \frac{q^4}{(q^4-1)\lambda} \\ \frac{q^2}{(q^4-1)\lambda} & \xi\lambda q^2 & 0 & 0 & 0 \\ 0 & \frac{1}{(q^4-1)\lambda} & \xi\lambda q^4 & 0 & 0 \\ 0 & 0 & \frac{q^3}{(q^4-1)\lambda} & \xi\lambda q & 0 \\ 0 & 0 & 0 & \frac{q}{(q^4-1)\lambda} & \xi\lambda q^3 \end{pmatrix}.$$

7.4 Irreducible Representations of Quantum Determinantal Rings

Recall the quantum determinantal ring $R_t(M_n)$ introduced in Chapter 5. We computed its PI degree to be PI-deg$(R_t(M_n)) = \ell^{\frac{2nt-t^2-t}{2}}$, in the case where the deformation parameter q is a primitive ℓ^{th} root of unity and ℓ is odd (Theorem 6.11). There must therefore exist an irreducible representation of $R_t(M_n)$ of this dimension, by Theorem 2.17. We will construct such a representation using the results of this chapter.

In order to apply the results in Section 7.3 to construct an irreducible representation of $R_t(M_n)$ we would need to know the explicit form of $\psi(I_t)$. As we do not yet have the techniques to compute this in general, we develop a different method to construct the irreducible representation. Our strategy is to first find a quantum affine space A which sits inside $R_t(M_n)$ as a subalgebra and whose associated quantum torus $A\Sigma^{-1}$ contains $R_t(M_n)$ as a subalgebra. We define an irreducible representation of A of the correct dimension, using Proposition 7.7, which we then push up to an irreducible representation of $A\Sigma^{-1}$. We then show that the restriction of the representation to $R_t(M_n)$ remains irreducible.

We end this chapter by providing an explicit computation of an irreducible representation of $R_2(M_3)$.

In this section we continue to take $\mathbb{K}$ to be algebraically closed and $1 \neq q \in \mathbb{K}^*$ to be any element, unless stated otherwise.

7.4.1 The Construction

First we introduce two dimensional invariants for algebras, which will be used to show algebraic independence of a set of elements. Definitions and basic results concerning the Gelfand-Kirillov dimension can be found in [KL00], whilst we use [Zha96] for definitions and results on the Gelfand-Kirillov transcendence degree.

Definition 7.12. For a $\mathbb{K}$-algebra A, the *Gelfand-Kirillov dimension of A* is defined to be

$$\text{GKdim}(A) = \sup_V \overline{\lim_{n\to\infty}} \log_n \dim(V^n),$$

and the *Gelfand-Kirillov transcendence degree of A* is defined to be

$$\text{Tdeg}(A) = \sup_V \inf_a \overline{\lim_{n\to\infty}} \log_n \dim((\mathbb{K}+aV)^n),$$

where V ranges over the finite-dimensional subspaces of A containing 1 (also known as *subframes*) of A, and a ranges over the regular elements of A.

Definition 7.13. An algebra A is called *Tdeg-stable* if the following hold:

(1) $\mathrm{Tdeg}(A) = \mathrm{GKdim}(A)$;

(2) $\mathrm{Tdeg}(S^{-1}A) = \mathrm{Tdeg}(A)$ for every Ore set S of regular elements.

Recall from Definition 2.34 that $\Delta_{n,n}$ denotes the set of all quantum minors of $\mathscr{O}_q(M_n(\mathbb{K}))$, or equivalently, the set of all index pairs $(I,J) \in [\![1,n]\!] \times [\![1,n]\!]$. We define the following elements, which form a subset of $\Delta_{n,n}$:

Definition 7.14. An index pair $(I,J) \in \Delta_{n,n}$ corresponds to a *final quantum minor* of $\mathscr{O}_q(M_n(\mathbb{K}))$ of size $s \in [\![1,n]\!]$ if $I = [\![i, i+(s-1)]\!]$ and $J = [\![j, j+(s-1)]\!]$, and either $i+(s-1) = n$ or $j+(s-1) = n$.

We denote the set of all final quantum minors of $\mathscr{O}_q(M_n(\mathbb{K}))$ by $\Omega_{n,n} \subseteq \Delta_{n,n}$, and the set of all final quantum minors of $\mathscr{O}_q(M_n(\mathbb{K}))$ of size less than or equal to t by $\Omega^t_{n,n} \subseteq \Omega_{n,n}$. Note that all the quantum minors in $\Omega^t_{n,n}$ "survive" in the quotient algebra $R_t(M_n)$ and generate a subalgebra, which we denote by $\mathbb{K}\langle\Omega^t_{n,n}\rangle \subseteq R_t(M_n)$. It may also be verified that $|\Omega^t_{n,n}| = 2nt - t^2$.

Lemma 7.15. *Take $1 \neq q \in \mathbb{K}^*$ to be a nonzero field element. Let $A = \mathbb{K}\langle\Omega^t_{n,n}\rangle$ and let $\Sigma \subset A$ be the multiplicative set generated by $\Omega^t_{n,n}$. Then,*

(i) The elements of $\Omega^t_{n,n}$ commute with each other up to powers of q;

(ii) $A \subseteq R_t(M_n) \subseteq A\Sigma^{-1}$;

(iii) A is the quantum affine space $\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})$ and $A\Sigma^{-1}$ is the quantum torus associated to A;

(iv) $M \in M_{2nt-t^2}(\mathbb{Z})$ has rank $2nt - t^2 - t$ and all its invariant factors are powers of 2.

Proof.

(i) This result follows from an analogous result for initial quantum minors of $\mathscr{O}_q(M_n(\mathbb{K}))$: A quantum minor $[I|J] \in \Omega_{n,n}$ is called *initial* if $I = \{i, i+1, \ldots, i+t\}$ and $J = \{j, j+1, \ldots, j+t\}$, for some $t \in [\![0,n]\!]$, and $1 \in I \cup J$. Noting that all pairs of initial quantum minors $[I|J]$, $[M|N]$ satisfying $i = m = 1$ are *weakly separated* in the sense of Leclerc and Zelevinsky [LZ98], we may apply their result [LZ98, Lemma 2.1] to see that the

$[I|J]$, $[M|N]$ above quasi-commute (that is, commute up to a power of q). Applying the transpose automorphism τ_q^1 of [PW91, Proposition 3.7.1(1)] reveals that any pair of initial quantum minors $[I'|J']$, $[M'|N']$ satisfying $j' = n' = 1$ also quasi-commute, as is shown in [Goo06, (4-13)]. Finally, [Goo06, Corollary 6.5] shows that pairs of initial quantum minors $[I|J]$, $[M'|N']$ satisfying $i = n' = 1$ quasi-commute and hence all initial quantum minors of $\mathcal{O}_q(M_n(\mathbb{K}))$ quasi-commute with each other. This result passes to the final quantum minors of $\mathcal{O}_q(M_n(\mathbb{K}))$, and hence to all elements of $\Omega^t_{n,n}$, by use of the anti-automorphism τ_q^2 of [PW91, Proposition 3.7.1(2)] and the property found in [PW91, Lemma 4.3.1].

(ii) $A \subseteq R_t(M_n)$ is immediate. We deduce from part (i) that the elements of $\Omega^t_{n,n}$ are normal in A. They are also regular, since $R_t(M_n)$ is a noetherian domain. Therefore, $\Sigma \subseteq A$ is an Ore set at which we can localise. We now show that each generator of $R_t(M_n)$ is contained in $A\Sigma^{-1}$:

Define sets $Q_1 := [\![n-t+1,n]\!] \times [\![1,n]\!]$, $Q_2 := [\![1,n]\!] \times [\![n-t+1,n]\!]$, and $Q := Q_1 \cup Q_2$. To prove $R_t(M_n) \subseteq A\Sigma^{-1}$ we first use decreasing induction on $(i,j) \in Q$ to show that $\bar{X}_{i,j} \in A\Sigma^{-1}$, where $\bar{X}_{i,j} \in R_t(M_n)$ is the canonical image of the generator $X_{i,j} \in R$. We then show that $\bar{X}_{i,j} \in A\Sigma^{-1}$, for all $(i,j) \in [\![1,n-t]\!] \times [\![1,n-t]\!]$. This will prove the result for all $(i,j) \in [\![1,n]\!] \times [\![1,n]\!]$ since $[\![1,n]\!] \times [\![1,n]\!] = Q \cup ([\![1,n-t]\!] \times [\![1,n-t]\!])$.

We start with the induction argument. Since $\bar{X}_{i,j} \in A$ for all $(i,j) \in (\{n\} \times [\![1,n]\!]) \cup ([\![1,n]\!] \times \{n\})$, by definition, the first non-trivial $(i,j) \in Q$ to prove in the induction is when $(i,j) = (n-1,n-1)$. We take this as our base case.

By the definition of quantum minors, we may write

$$[\{n-1,n\}|\{n-1,n\}] = \bar{X}_{n-1,n-1}\bar{X}_{n,n} - q\bar{X}_{n-1,n}\bar{X}_{n,n-1}.$$

Rearranging this we obtain

$$\bar{X}_{n-1,n-1} = ([\{n-1,n\}|\{n-1,n\}] + q\bar{X}_{n-1,n}\bar{X}_{n,n-1})\bar{X}_{n,n}^{-1} \in A\Sigma^{-1}.$$

Since $[\{n-1,n\}|\{n-1,n\}], \bar{X}_{n-1,n}, \bar{X}_{n,n-1}, \bar{X}_{n,n} \in A$, and $\bar{X}_{n,n}^{-1} \in A\Sigma^{-1}$, this proves the base case.

For ease of reading we now fix some notation: for some $(i,j) \in Q$, denote by $[I_{i,j}|J_{i,j}] \in \Delta_{n,n}$ the final quantum minor which has i as its first row index and j as its first column index; that is,

$$I_{i,j} := [\![i, i+\min\{n-i,n-j\}]\!], \;\; J_{i,j} := [\![j, j+\min\{n-i,n-j\}]\!].$$

Setting $s_{i,j} := \min\{n-i, n-j\}$ we see that $[I_{i,j}|J_{i,j}]$ has size $s_{i,j}+1$ and $s_{i,j}+1 \in [\![1,t]\!]$ because $(i,j) \in Q$. Therefore, $[I_{i,j}|J_{i,j}] \in \Omega^t_{n,n} \subseteq A$. One may also verify that $[I_{i,j}|J_{i,j}]$ is generated by sums of monomials in $\bar{X}_{l,k}$, where $(l,k) \geq (i,j)$ and $(l,k) \in Q$.

For the inductive step, fix some $(i,j) \in Q$, with $(i,j) < (n-1,n-1)$, and assume that $\bar{X}_{l,k} \in A\Sigma^{-1}$, for all $(i,j) < (l,k) \leq (n,n)$ and $(l,k) \in Q$. Consider $\bar{X}_{i,j}$. With the notation above we can rewrite $[I_{i,j}|J_{i,j}] \in A$, using the quantum Laplace relations from [NYM93, Proposition 1.1], as

$$[I_{i,j}|J_{i,j}] = \bar{X}_{i,j}[I_{i+1,j+1}|J_{i+1,j+1}] + \sum_{r=1}^{s_{i,j}} (-q)^r \bar{X}_{i,j+r}[I_{i,j}\backslash\{i\}|J_{i,j}\backslash\{j+r\}]. \tag{7.12}$$

Since $(i+1,j+1) \in Q$ then $[I_{i+1,j+1}|J_{i+1,j+1}] \in \Omega^t_{n,n}$ and is hence invertible in $A\Sigma^{-1}$. Using the inductive hypothesis we deduce that $[I_{i,j}\backslash\{i\}|J_{i,j}\backslash\{j+r\}] \in A\Sigma^{-1}$, for all $r \in [\![0, s_{i,j}]\!]$, as it is the sum of monomials in $\bar{X}_{l,k}$, where $(l,k) > (i,j)$ and $(l,k) \in Q$. Similarly, $\bar{X}_{i,j+r} \in A\Sigma^{-1}$, for all $r \in [\![1, s_{i,j}]\!]$, because $(i,j+r) \in Q$ and $(i,j+r) > (i,j)$. We may therefore rearrange (7.12) to obtain

$$\bar{X}_{i,j} = \left([I_{i,j}|J_{i,j}] - \sum_{r=1}^{s_{i,j}} (-q)^r \bar{X}_{i,j+r}[I_{i,j}\backslash\{i\}|J_{i,j}\backslash\{j+r\}] \right) [I_{i+1,j+1}|J_{i+1,j+1}]^{-1} \in A\Sigma^{-1}.$$

This proves the inductive step and we conclude that $\bar{X}_{i,j} \in A\Sigma^{-1}$, for all $(i,j) \in Q$.

For any $(i,j) \in [\![1, n-t]\!] \times [\![1, n-t]\!]$, define

$$\hat{I}_{i,j} := \{i, n-t+1, n-t+2, \dots, n\}, \quad \hat{J}_{i,j} := \{j, n-t+1, n-t+2, \dots, n\}$$

so that $|\hat{I}_{i,j}| = |\hat{J}_{i,j}| = t+1$. Then $[\hat{I}_{i,j}|\hat{J}_{i,j}] = 0$ in $R_t(M_n)$ and hence also in A. We may write this as

$$0 = \bar{X}_{i,j}[\hat{I}_{i,j}\backslash\{i\}|\hat{J}_{i,j}\backslash\{j\}] + \sum_{r=1}^{t} (-q)^r \bar{X}_{i,n-t+r}[\hat{I}_{i,j}\backslash\{i\}|\hat{J}_{i,j}\backslash\{n-t+r\}]. \tag{7.13}$$

Note that, for all $r \in [\![1,t]\!]$, the quantum minor $[\hat{I}_{i,j}\backslash\{i\}|\hat{J}_{i,j}\backslash\{n-t+r\}]$ is a sum of monomials in $\bar{X}_{l,k}$, where $l \in [\![n-t+1, n]\!]$ so that $(l,k) \in Q$. By the induction above we deduce that $\bar{X}_{l,k} \in A\Sigma^{-1}$ and hence $[\hat{I}_{i,j}\backslash\{i\}|\hat{J}_{i,j}\backslash\{n-t+r\}] \in A\Sigma^{-1}$, for all $r \in [\![1,t]\!]$. Furthermore, $[\hat{I}_{i,j}\backslash\{i\}|\hat{J}_{i,j}\backslash\{j\}] = [I_{n-t+1,n-t+1}, I_{n-t+1,n-t+1}] \in \Omega^t_{n,n}$ and is therefore invertible in $A\Sigma^{-1}$. Rearranging (7.13) we obtain

$$\bar{X}_{i,j} = \left(-\sum_{r=1}^{t} (-q)^r \bar{X}_{i,n-t+r} [\hat{I}_{i,j} \backslash \{i\} | \hat{J}_{i,j} \backslash \{n-t+r\}] \right) [\hat{I}_{i,j} \backslash \{i\} | \hat{J}_{i,j} \backslash \{j\}]^{-1} \in A\Sigma^{-1}.$$

Hence $\bar{X}_{i,j} \in A\Sigma^{-1}$, for all $(i,j) \in [\![1,n-t]\!] \times [\![1,n-t]\!]$, and, together with the induction proof above, this proves that $R_t(M_n) \subseteq A\Sigma^{-1}$.

(iii) It was shown in part (i) that elements of $\Omega^t_{n,n}$ commute with each other up to powers of q as determined by a skew-symmetric matrix, which we denote by $M \in M_{2nt-t^2}(\mathbb{Z})$. Hence, there is a surjective homomorphism

$$f : \mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2}) \longrightarrow A. \tag{7.14}$$

We will use the GK-dimension to show that this surjection is, in fact, an isomorphism.

Claim 1: $\mathrm{GKdim}(A) = \mathrm{GKdim}(R_t(M_n)) = 2nt - t^2$.

When q is not a root of unity then [GLL18, Corollary 4.8] applied to $R_t(M_n)$, with $J_w = I_t$ and $w = [\![1,n-t]\!] \times [\![1,n-t]\!]$, shows that $R_t(M_n)$ is Tdeg-stable in the generic case. When q is a root of unity then $R_t(M_n)$ is a PI algebra which is a noetherian domain. Thus [Zha96, Theorem 5.3] says that $R_t(M_n)$ is also Tdeg-stable in the root of unity case. Taking total rings of fractions of the algebras in part (ii) (possible since all algebras involved are noetherian domains), we obtain

$$\mathrm{Frac}(A) \subseteq \mathrm{Frac}(R_t(M_n)) \subseteq \mathrm{Frac}(A\Sigma^{-1}) = \mathrm{Frac}(A) \quad \Longrightarrow \quad \mathrm{Frac}(R_t(M_n)) = \mathrm{Frac}(A).$$

The algebras A and $R_t(M_n)$ therefore satisfy the conditions of [Zha96, Proposition 3.5(3)], which tells us that $\mathrm{GKdim}(A) \geq \mathrm{GKdim}(R_t(M_n))$. The equality of GK-dimension follows from the basic property that $A \subseteq R_t(M_n)$ implies $\mathrm{GKdim}(A) \leq \mathrm{GKdim}(R_t(M_n))$.

Finally, we see that $\mathrm{GKdim}(R_t(M_n)) = 2nt - t^2$, using [LR08, Remark 4.2 (iii) & (iv)], replacing the t with $t+1$ to make the result in the paper compatible with our definition of $R_t(M_n)$.

Claim 2: The map, f, in (7.14) is an isomorphism.

From (7.14) and the First Isomorphism Theorem, we have $\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})/\ker(f) \cong A$ and hence

$$\mathrm{GKdim}\left(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})/\ker(f) \right) = \mathrm{GKdim}(A) = 2nt - t^2.$$

As a consequence of Goldie's Theorem (see [GW04, Corollary 6.4]), every nonzero ideal of $\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})$ contains a regular element. Therefore, if $\ker(f)$ were not trivial then, by [KL00, Proposition 3.15],

$$\mathrm{GKdim}\left(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})/\ker(f)\right)+1 \leq \mathrm{GKdim}\left(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})\right). \tag{7.15}$$

However, $\mathrm{GKdim}\left(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})\right)=2nt-t^2$, by [LMO88, Lemma 2], so the inequality in (7.15) becomes $2nt-t^2+1 \leq 2nt-t^2$, which is clearly false. Therefore $\ker(f)$ must be trivial and $\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2}) \cong A$.

(iv) In Section 6.2.1 it was shown that $\mathrm{Frac}(R_t(M_n)) \cong \mathrm{Frac}(\mathscr{O}_{q^{M'}}(\mathbb{K}^{2nt-t^2}))$, and in Theorem 6.3 and Proposition 6.10 we deduced properties of $M' \in M_{2nt-t^2}(\mathbb{Z})$, namely that M' has rank $2nt-t^2-t$ and that all its invariant factors are powers of 2. Parts (ii) and (iii) of this lemma imply that $\mathrm{Frac}(R_t(M_n)) = \mathrm{Frac}(A) \cong \mathrm{Frac}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2}))$, therefore

$$\mathrm{Frac}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})) \cong \mathrm{Frac}(\mathscr{O}_{q^{M'}}(\mathbb{K}^{2nt-t^2})).$$

Taking q to be a non-root of unity allows us to apply [Pan95, Theorem 2.19] to this isomorphism, which states that

$$\mathrm{Frac}(\mathscr{O}_{q^M}(\mathbb{K}^{2nt-t^2})) \cong \mathrm{Frac}(\mathscr{O}_{q^{M'}}(\mathbb{K}^{2nt-t^2})) \quad \Longleftrightarrow \quad M \sim_C M',$$

where $\sim_C$ denotes the congruence relation of Definition 5.3. Therefore, M shares the same invariant factors and rank as M', thus proving part (iv) in the case when q is not a root of unity. Since the matrix M also defines the commutation relations on the quantum affine space A when q is a root of unity, this also proves part (iv) for any $1 \neq q \in \mathbb{K}^*$.

$\square$

Proposition 7.16. *Take $1 \neq q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity, with ℓ odd. Let $A := \mathbb{K}\langle \Omega^t_{n,n} \rangle \subseteq R_t(M_n)$ be the subalgebra generated by the final quantum minors of size less than or equal to t, denoted by $T_1, \ldots, T_{2s+t} \in R_t(M_n)$, where $2s = 2nt-t^2-t$. Let (ϕ, V) be the ℓ^s-dimensional irreducible representation of A defined in Proposition 7.7.*

Then, every element $r \in R_t(M_n)$ may be written as

$$r = \sum_{\underline{i} \in \mathbb{Z}^{2s+t}} \alpha_{\underline{i}} T_1^{i_1} \cdots T_{2s+t}^{i_{2s+t}} \in \mathbb{K}_{q^M}[T_1^{\pm 1}, \ldots, T_{2s+t}^{\pm 1}]$$

where $\alpha_{\underline{i}} \in \mathbb{K}$ and $i_j \in \mathbb{Z}$, for all $j \in [\![1, 2s+t]\!]$, and there is an algebra homomorphism

$$\begin{aligned} \rho : R_t(M_n) &\longrightarrow \operatorname{End}_{\mathbb{K}}(V) \\ r &\longmapsto \sum_{\underline{i} \in \mathbb{Z}^{2s+t}} \alpha_{\underline{i}} \phi(T_1)^{i_1} \cdots \phi(T_{2s+t})^{i_{2s+t}} \end{aligned}$$

which defines an irreducible representation of $R_t(M_n)$ of dimension ℓ^s.

Proof. By Lemma 7.15(iii), A is a quantum affine space so, denoting the final quantum minors in the set $\Omega^t_{n,n}$ by $T_1, \ldots, T_{2s+t}$, this allows us to write $A = \mathbb{K}_{q^M}[T_1, \ldots, T_{2s+t}]$, where $M \in M_{2nt-t^2}(\mathbb{Z})$ is a skew-symmetric matrix. We saw in Lemma 7.15(iv) that the dimension of the kernel of M is t and that all its invariant factors, $h_1, \ldots, h_s$, are powers of 2. Since ℓ is odd then $\gcd(h_i, \ell) = 1$, for all $i \in [\![1, s]\!]$, and we may apply Proposition 7.7 to obtain the irreducible representation (ϕ, V) of A of dimension ℓ^s defined therein.

Recall that $\Sigma \subseteq A$ is the Ore set generated by all T_i, and let $\pi : A \to A\Sigma^{-1}$ be the localisation map which defines $A\Sigma^{-1}$ as a right ring of fractions of A. Recall also, from Proposition 7.7, that $\phi(T_i) \subseteq \operatorname{End}_{\mathbb{K}}(V)$ is invertible for all $T_i \in \Sigma$. Then, by the universal property of localisations (see [GW04, Proposition 10.4]), there exists a unique ring homomorphism $\hat{\phi} : A\Sigma^{-1} \to \operatorname{End}_{\mathbb{K}}(V)$ such that $\phi = \hat{\phi} \circ \pi$. Since ϕ is a $\mathbb{K}$-algebra homomorphism then so too are $\hat{\phi}$ and π. We restrict $\hat{\phi}$ to $R_t(M_n)$ to obtain an algebra homomorphism

$$\begin{aligned} \rho : R_t(M_n) &\longrightarrow \operatorname{End}_{\mathbb{K}}(V) \\ r &\longmapsto \hat{\phi}(r \cdot 1^{-1}), \end{aligned}$$

which defines a representation (ρ, V) of $R_t(M_n)$. In particular, $\hat{\phi}(T_i^{-1}) = \hat{\phi}(T_i)^{-1} = \phi(T_i)^{-1}$, for all $T_i \in \Sigma$, because of the observation in Proposition 7.7(iii). Thus, writing $r \in R_t(M_n)$ in terms of $T_i^{\pm 1} \in A\Sigma^{-1}$, as is possible by the inclusions shown in Lemma 7.15(ii), we obtain

$$\rho(r) = \hat{\phi}(r \cdot 1^{-1}) = \hat{\phi}\left(\sum_{\underline{i} \in \mathbb{Z}^{2s+t}} \alpha_{\underline{i}} T_1^{i_1} \cdots T_{2s+t}^{i_{2s+t}} \right) = \sum_{\underline{i} \in \mathbb{Z}^{2s+t}} \alpha_{\underline{i}} \phi(T_1)^{i_1} \cdots \phi(T_{2s+t})^{i_{2s+t}},$$

where $i_j \in \mathbb{Z}$, for all $j \in [\![1, 2s+t]\!]$, and $\alpha_{\underline{i}} \in \mathbb{K}^*$.

To show irreducibility of (ρ, V) we use the fact that $A \subseteq R_t(M_n)$ to deduce $\operatorname{Im}(\phi) \subseteq \operatorname{Im}(\rho)$. Therefore, if (ρ, V) were reducible this would force (ϕ, V) to be reducible, thus contradicting Proposition 7.7. Hence (ρ, V) is an irreducible representation of $R_t(M_n)$ of dimension ℓ^s. $\square$

7.4.2 Example: $\mathscr{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle$

For this example we take $1 \neq q \in \mathbb{K}^*$ to be a primitive ℓ^{th} root of unity with ℓ odd, and we consider the quantum determinantal ring $R_2(M_3) = \mathscr{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle$, where $D_q \in \mathscr{O}_q(M_3(\mathbb{K}))$ is the 3×3 single parameter quantum determinant. We construct an explicit irreducible representation of $R_2(M_3)$ of dimension PI-deg$(R_2(M_3)) = \ell^3$ using the results of the previous section.

By Proposition 7.16 we see that we need to first construct an irreducible representation of the subalgebra $A \subseteq R_2(M_3)$ generated by the final quantum minors of size less than or equal to 2. There are eight such quantum minors:

$$T_1 := \bar{X}_{1,3}, \quad T_2 := \bar{X}_{2,3}, \quad T_3 := \bar{X}_{3,1}, \quad T_4 := \bar{X}_{3,2}, \quad T_5 := \bar{X}_{3,3},$$
$$T_6 := [12|23], \quad T_7 := [23|12], \quad T_8 := [23|23],$$

hence we set $A := \mathbb{K}\langle T_1, \ldots, T_8 \rangle \subseteq R_2(M_3)$. It can be verified that the commutation relations between the T_i correspond to the matrix

$$M := \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \end{pmatrix} \in M_8(\mathbb{Z}).$$

That is, $A = \mathbb{K}_{q^M}[T_1, \ldots, T_8]$. In order to construct an irreducible representation of A using Proposition 7.7 we need the skew-normal form, S, of M as well as the explicit matrix, E, such that $EME^T = S$. These are given below:

$$S = EME^T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -2 & 1 & -2 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad E^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

From S we observe that M has invariant factors $h_1 = 1, h_2 = 1, h_3 = 2$, so $\gcd(h_i, \ell) = \gcd(h_i, 3) = 1$ for all $i \in [\![1,3]\!]$, thus confirming that A satisfies the conditions of Proposition 7.7.

Applying Proposition 7.7(ii) to this example, we obtain an irreducible representation (φ, V) of $D := \mathbb{K}_{q^S}[x_1, y_1, x_2, y_2, x_3, y_3, z_1, z_2]$. Here, V is a ℓ^3-dimensional $\mathbb{K}$-vector space defined as $V = V_1 \otimes V_2 \otimes V_3$, for the three ℓ-dimensional $\mathbb{K}$-vector spaces V_1, V_2, V_3, and φ is the homomorphism from Proposition 7.7 defined using the maps φ_i found in Lemma 7.6. Hence,

$$\varphi(x_1^{i_1} y_1^{i_2} x_2^{i_3} y_2^{i_4} x_3^{i_5} y_3^{i_6} z_1^{i_7} z_2^{i_8}) = \xi_1^{i_7} \xi_2^{i_8} \varphi_1(x_1^{i_1} y_1^{i_2}) \otimes \varphi_2(x_2^{i_3} y_2^{i_4}) \otimes \varphi_3(x_3^{i_5} y_3^{i_6}),$$

for some $\xi_1, \xi_2 \in \mathbb{K}^*$ and $i_j \in \mathbb{N}$, for $j \in [\![1,8]\!]$. Recall from Proposition 7.7(ii) that $\varphi(x_i)^{-1} = \lambda_i^{-\ell} \varphi(x_i)^{\ell-1}$ and $\varphi(y_i)^{-1} = \varphi(y_i)^{\ell-1}$, for all $i \in [\![1,3]\!]$. This allows us to consider negative powers, i_j, of the generators when used as an argument in φ, as φ extends to the localisation $\mathbb{K}_{q^S}[x_1^{\pm 1}, y_1^{\pm 1}, x_2^{\pm 1}, y_2^{\pm 1}, x_3^{\pm 1}, y_3^{\pm 1}, z_1^{\pm 1}, z_2^{\pm 1}]$. From Proposition 7.7(iii) we then construct an irreducible representation (ϕ, V) of A using the rows of E^{-1}. Explicitly, we obtain

$$\begin{aligned}
\phi(T_1) &:= \varphi(y_1)\\
\phi(T_2) &:= \varphi(x_1)^{-1}\\
\phi(T_3) &:= \varphi(y_2)\\
\phi(T_4) &:= \varphi(x_2)^{-1}\\
\phi(T_5) &:= \varphi(x_1^{-1}y_1^{-1}x_2^{-1}y_2^{-1}x_3y_3)\\
\phi(T_6) &:= \xi_1\varphi(y_2)\\
\phi(T_7) &:= \xi_2\varphi(y_1)\\
\phi(T_8) &:= \varphi(x_1^{-1}x_2^{-1}x_3^{-1}).
\end{aligned}$$

Proposition 7.16 states that ϕ extends to an algebra homomorphism $\rho : R_2(M_3) \to \mathrm{End}_{\mathbb{K}}(V)$, making (ρ, V) into an irreducible representation of $R_2(M_3)$. To find out how ρ acts on the generators $\bar{X}_{i,j} \in R_2(M_3)$ we must write them as elements of $A\Sigma^{-1} = \mathbb{K}_{q^M}[T_1^{\pm 1}, \dots, T_8^{\pm 1}]$, where $\Sigma \subseteq A$ is the Ore set generated by T_i, for all $i \in [\![1, 8]\!]$. By the definition of $T_1, \dots, T_8$,

$$\bar{X}_{1,3} = T_1, \quad \bar{X}_{2,3} = T_2, \quad \bar{X}_{3,1} = T_3, \quad \bar{X}_{3,2} = T_4, \quad \bar{X}_{3,3} = T_5.$$

Hence, by Proposition 7.16,

$$\begin{aligned}
\rho(\bar{X}_{1,3}) &= \phi(T_1) = \varphi(y_1)\\
\rho(\bar{X}_{2,3}) &= \phi(T_2) = \varphi(x_1)^{-1}\\
\rho(\bar{X}_{3,1}) &= \phi(T_3) = \varphi(y_2)\\
\rho(\bar{X}_{3,2}) &= \phi(T_4) = \varphi(x_2)^{-1}\\
\rho(\bar{X}_{3,3}) &= \phi(T_5) = \varphi(x_1^{-1}y_1^{-1}x_2^{-1}y_2^{-1}x_3y_3).
\end{aligned}$$

Using the definition of quantum minors, we obtain expressions for $\bar{X}_{1,2}, \bar{X}_{2,1}, \bar{X}_{2,2}$ in terms of $\bar{X}_{1,3}^{\pm 1}, \bar{X}_{2,3}^{\pm 1}, \bar{X}_{3,1}^{\pm 1}, \bar{X}_{3,2}^{\pm 1}, \bar{X}_{3,3}^{\pm 1}, T_6^{\pm 1}, T_7^{\pm 1}, T_8^{\pm 1} \in A\Sigma^{-1}$:

$$\begin{aligned}
T_8 = [23|23] = \bar{X}_{2,2}\bar{X}_{3,3} - q\bar{X}_{2,3}\bar{X}_{3,2} \quad &\Longrightarrow \quad \bar{X}_{2,2} = (T_8 + q\bar{X}_{2,3}\bar{X}_{3,2})\bar{X}_{3,3}^{-1}\\
T_7 = [23|12] = \bar{X}_{2,1}\bar{X}_{3,2} - q\bar{X}_{2,2}\bar{X}_{3,1} \quad &\Longrightarrow \quad \bar{X}_{2,1} = (T_7 + q\bar{X}_{2,2}\bar{X}_{3,1})\bar{X}_{3,2}^{-1}\\
T_6 = [12|23] = \bar{X}_{1,2}\bar{X}_{2,3} - q\bar{X}_{1,3}\bar{X}_{2,2} \quad &\Longrightarrow \quad \bar{X}_{1,2} = (T_6 + q\bar{X}_{1,3}\bar{X}_{2,2})\bar{X}_{2,3}^{-1}.
\end{aligned}$$

This allows us to deduce the action of ρ on these generators, for example:

$$\begin{aligned}
\rho(\bar{X}_{2,2}) &= (\rho(T_8)+q\rho(\bar{X}_{2,3})\rho(\bar{X}_{3,2}))\rho(\bar{X}_{3,3})^{-1} \\
&= (\phi(T_8)+q\phi(T_2)\phi(T_4))\phi(T_5)^{-1} \\
&= \phi(T_8)\phi(T_5)^{-1}+q\phi(T_2)\phi(T_4)\phi(T_5)^{-1} \\
&= \varphi(x_1^{-1}x_2^{-1}x_3^{-1})\varphi(y_1x_1y_2x_2y_3^{-1}x_3^{-1})+q\varphi(x_1)^{-1}\varphi(x_2)^{-1}\varphi(y_1x_1y_2x_2y_3^{-1}x_3^{-1}) \\
&= \varphi(y_1y_2y_3^{-1}x_3^{-2}+x_1^{-1}y_1y_2y_3^{-1}x_3^{-1}).
\end{aligned}$$

Through similar calculations, we obtain

$$\begin{aligned}
\rho(\bar{X}_{2,1}) &= \varphi(\xi_2y_1x_2+qy_1y_2^2x_2y_3^{-1}x_3^{-2}+qx_1^{-1}y_1y_2^2x_2y_3^{-1}x_3^{-1}) \\
\rho(\bar{X}_{1,2}) &= \varphi(\xi x_1y_2+qy_1^2x_1y_2y_3^{-1}x_3^{-2}+y_1^2y_2y_3^{-1}x_3^{-1}).
\end{aligned}$$

For $\bar{X}_{1,1}$ we use the fact that the quantum determinant D_q is zero in $R_2(M_3)$. This gives

$$0=\overline{D_q}=\bar{X}_{1,1}[23|23]-q\bar{X}_{1,2}[23|13]+q^2\bar{X}_{1,3}[23|12]$$

which, upon rearranging, can be written in terms of elements of $A\Sigma^{-1}$:

$$\begin{aligned}
\bar{X}_{1,1} &= (q\bar{X}_{1,2}[23|13]-q^2\bar{X}_{1,3}[23|12])[23|23]^{-1} \\
&= (q\bar{X}_{1,2}(\bar{X}_{2,1}\bar{X}_{3,3}-q^2\bar{X}_{2,3}\bar{X}_{3,1})+q^2\bar{X}_{1,3}T_7)T_8^{-1}.
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
\rho(\bar{X}_{1,1}) = \varphi(q(&\xi_1x_1y_2+qy_1^2x_1y_2y_3^{-1}x_3^{-1}+y_1^2y_2y_3^{-1}x_3^{-1}) \\
&(\xi_2q^{-2}x_1^2x_2^3y_2^{-1}x_3^2y_3+q^{-7}x_1x_2^3y_2+q^{-2}x_1y_2x_2^3x_3-qy_2x_2x_3)-q^2\xi_2y_1^2x_1x_2x_3).
\end{aligned}$$

Using the definition of φ we write all $\rho(\bar{X}_{j,k})$ in terms of the maps φ_i, for $i,j,k\in[\![1,3]\!]$:

$$\begin{aligned}
\rho(\bar{X}_{1,1}) = q(&\xi_1\varphi_1(x_1)\otimes\varphi_2(y_2)\otimes\mathrm{Id}_{V_3}+q\varphi_1(y_1^2x_1)\otimes\varphi_2(y_2)\otimes\varphi_3(x_3^2y_3)^{-1} \\
&+\varphi_1(y_1^2)\otimes\varphi_2(y_2)\otimes\varphi_3(x_3y_3)^{-1})(\xi_2q^{-1}\varphi_1(x_1^2)\otimes\varphi_2(x_2)\varphi_2(y_2)^{-1}\otimes\varphi_3(x_3^2y_3) \\
&+q^{-7}\varphi_1(x_1^2)\otimes\varphi_2(x_2^3y_2)\otimes\mathrm{Id}_{V_3}+q^{-2}\varphi_1(x_1)\otimes\varphi_2(y_2x_2^3)\otimes\varphi_3(x_3) \\
&-q\,\mathrm{Id}_{V_1}\otimes\varphi_2(y_2x_2)\otimes\varphi_3(x_3))-q^2\xi_2\varphi_1(y_1^2x_1)\otimes\varphi_2(x_2)\otimes\varphi_3(x_3) \\
\rho(\bar{X}_{1,2}) = \ &\xi_1\varphi_1(x_1)\otimes\varphi_2(y_2)\otimes\mathrm{Id}_{V_3}+q\varphi_1(y_1^2x_1)\otimes\varphi_2(y_2)\otimes\varphi_3(x_3^2y_3)^{-1} \\
&+\varphi_1(y_1^2)\otimes\varphi_2(y_2)\otimes\varphi_3(x_3y_3)^{-1}
\end{aligned}$$

$$
\begin{aligned}
\rho(\bar{X}_{1,3}) &= \varphi_1(y_1) \otimes \mathrm{Id}_{V_2} \otimes \mathrm{Id}_{V_3} \\
\rho(\bar{X}_{2,1}) &= \xi_2 \varphi_1(y_1) \otimes \varphi_2(x_2) \otimes \mathrm{Id}_{V_3} + q\varphi_1(y_1) \otimes \varphi_2(y_2^2 x_2) \otimes \varphi_3(x_3^2 y_3)^{-1} \\
&\qquad + q\varphi_1(x_1)^{-1}\varphi_1(y_1) \otimes \varphi_2(y_2^2 x_2) \otimes \varphi_3(x_3 y_3)^{-1} \\
\rho(\bar{X}_{2,2}) &= \varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \varphi_3(x_3^2 y_3)^{-1} + \varphi_1(x_1)^{-1}\varphi_1(y_1) \otimes \varphi_2(y_2) \otimes \varphi_3(x_3 y_3)^{-1} \\
\rho(\bar{X}_{2,3}) &= \varphi_1(x_1)^{-1} \otimes \mathrm{Id}_{V_2} \otimes \mathrm{Id}_{V_3} \\
\rho(\bar{X}_{3,1}) &= \mathrm{Id}_{V_1} \otimes \varphi_2(y_2) \otimes \mathrm{Id}_{V_3} \\
\rho(\bar{X}_{3,2}) &= \mathrm{Id}_{V_1} \otimes \varphi_2(x_2)^{-1} \otimes \mathrm{Id}_{V_3} \\
\rho(\bar{X}_{3,3}) &= \varphi_1(y_1 x_1)^{-1} \otimes \varphi_2(y_2 x_2)^{-1} \otimes \varphi_3(x_3 y_3).
\end{aligned} \tag{7.16}
$$

Setting $\ell = 3$ and fixing the bases of V_1, V_2, V_3 allows us to present the images of x_i and y_i under φ_i as matrices in the following way:

$$
\varphi_1(x_1) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 q & 0 \\ 0 & 0 & \lambda_1 q^2 \end{pmatrix}, \quad \varphi_2(x_2) = \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 q & 0 \\ 0 & 0 & \lambda_2 q^2 \end{pmatrix}, \quad \varphi_3(x_3) = \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 q^2 & 0 \\ 0 & 0 & \lambda_3 q \end{pmatrix},
$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}^*$, and $\varphi_1(y_1) = \varphi_2(y_2) = \varphi_3(y_3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. The matrix representations of (7.16) can then be computed using the Kronecker product. For example,

$$
\rho(\bar{X}_{3,3}) = \begin{pmatrix} 0 & \lambda_1^{-1} & 0 \\ 0 & 0 & \lambda_1^{-1} q^2 \\ \lambda_1^{-1} q & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_2^{-1} q^2 \\ \lambda_2^{-1} q & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & \lambda_3 \\ \lambda_3 q^2 & 0 & 0 \\ 0 & \lambda_3 q & 0 \end{pmatrix}.
$$

As the matrices in this representation are of dimension $3^3 = 27$, it is impractical to write them explicitly here, however, they can be easily computed using, for example, Maple. The reader is invited to turn to Appendix A, where the Maple code required to generate the matrices in (7.16) for any choice of odd $\ell > 1$ is provided.

Appendix A

Irreducible representation of $\mathscr{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle$

This appendix contains Maple code which calculates the matrix representation of the irreducible representation of $\mathscr{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle$ constructed in Section 7.4.2, where $q \in \mathbb{K}^*$ is a primitive ℓ^{th} root of unity. The reader may choose ℓ to be any odd positive integer; the code will then return the matrices of dimension ℓ^3 which represent the generators $\bar{X}_{i,j} \in \mathscr{O}_q(M_3)/\langle D_q \rangle$ in the representation (ρ, V). The Maple code below relates to the notation in Section 7.4.2 in the following way:

$$\begin{gathered}
\mathtt{l} = \ell, \quad \mathtt{R} = \mathscr{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle, \quad \mathtt{A} = \mathbb{K}_{q^M}[T_1, \ldots, T_8], \\
\mathtt{X1} = \varphi_1(x_1), \quad \mathtt{X2} = \varphi_2(x_2), \quad \mathtt{X3} = \varphi_3(x_3), \\
\mathtt{Y1} = \varphi_1(y_1), \quad \mathtt{Y2} = \varphi_2(y_2), \quad \mathtt{Y3} = \varphi_3(y_3), \\
\mathtt{a} = \lambda_1, \quad \mathtt{b} = \lambda_2, \quad \mathtt{c} = \lambda_3, \quad \mathtt{d} = \xi_1, \quad \mathtt{e} = \xi_2, \\
\mathtt{T1} = \phi(T_1), \quad \mathtt{T2} = \phi(T_2), \quad \mathtt{T3} = \phi(T_3), \quad \mathtt{T4} = \phi(T_4), \\
\mathtt{T5} = \phi(T_5), \quad \mathtt{T6} = \phi(T_6), \quad \mathtt{T7} = \phi(T_7), \quad \mathtt{T8} = \phi(T_8), \\
\mathtt{X11} = \rho(\bar{X}_{1,1}), \quad \mathtt{X12} = \rho(\bar{X}_{1,2}), \quad \mathtt{X13} = \rho(\bar{X}_{1,3}), \quad \mathtt{X21} = \rho(\bar{X}_{2,1}), \quad \mathtt{X22} = \rho(\bar{X}_{2,2}), \\
\mathtt{X23} = \rho(\bar{X}_{2,3}), \quad \mathtt{X31} = \rho(\bar{X}_{3,1}), \quad \mathtt{X32} = \rho(\bar{X}_{3,2}), \quad \mathtt{X33} = \rho(\bar{X}_{3,3}).
\end{gathered}$$

The code is then as follows:

```
restart: with(LinearAlgebra):

# Set an odd value for l.
l:=3;
```

```
interface(rtablesize = l^3):

# Set the matrix M.
M:=Matrix(8,8,shape='antisymmetric'): M[1,2]:=1: M[1,5]:=1: M[1,8]:=1:
M[2,5]:=1: M[2,7]:=-1: M[3,4]:=1: M[3,5]:=1: M[3,8]:=1:
M[4,5]:=1: M[4,6]:=-1: M[5,6]:=-1: M[5,7]:=-1: M[6,8]:=1: M[7,8]:=1:
'M'=M;

# Set the matrix E and its inverse.
E:= Matrix(8,8): E[1,2]:=-1: E[2,1]:=1: E[3,4]:=-1:
E[4,3]:=1: E[5,2]:=1: E[5,4]:=1: E[5,8]:=-1: E[6,1]:=1:
E[6,2]:=-2: E[6,3]:=1: E[6,4]:=-2: E[6,5]:=1: E[6,8]:=1:
E[7,3]:=-1: E[7,6]:=1: E[8,1]:=-1: E[8,7]:=1:
'E'=E;
'E^{-1}'=MatrixInverse(E);

# Set S to be the skew-normal form of M.
S=E.M.Transpose(E);

# Set the matrices in the representation of the quantum affine planes.
Y1:=Matrix(l): Y2:=Matrix(l): Y3:=Matrix(l):
for i to l-1 do
    Y1[i+1,i]:=1: Y2[i+1,i]:= 1: Y3[i+1,i]:=1:
end do:
Y1[1,l]:=1: Y2[1,l]:=1: Y3[1,l]:=1:
'Y1'=Y1, 'Y2'=Y2, 'Y3'=Y3;

X1:=Matrix(l):
X2:=Matrix(l):
X3:=Matrix(l):
for i from 1 to l do
    X1[i,i]:=simplify(a*q^(i-1)):
    X2[i,i]:=simplify(b*q^(i-1)):
    X3[i,i]:=simplify(c*q^(2*(i-1))):
end do:
'X1'=X1;
```

```
'X2'=X2;
'X3'=X3;

# Compute matrices for the representation of A.
T1:=KroneckerProduct(KroneckerProduct(
    Y1,IdentityMatrix(l)),IdentityMatrix(l)):
T2:=KroneckerProduct(KroneckerProduct(
    X1^(-1),IdentityMatrix(l)),IdentityMatrix(l)):
T3:=KroneckerProduct(KroneckerProduct(
    IdentityMatrix(l),Y2),IdentityMatrix(l)):
T4:=KroneckerProduct(KroneckerProduct(
    IdentityMatrix(l),X2^(-1)),IdentityMatrix(l)):
T5:=KroneckerProduct(KroneckerProduct(
    X1^(-1).Y1^(-1), X2^(-1).Y2^(-1)), X3.Y3):
T6:=KroneckerProduct(KroneckerProduct(
    d*IdentityMatrix(l),Y2),IdentityMatrix(l)):
T7:=KroneckerProduct(KroneckerProduct(
    e*Y1,IdentityMatrix(l)),IdentityMatrix(l)):
T8:=KroneckerProduct(KroneckerProduct(
    X1^(-1), X2^(-1)), X3^(-1)):

# Compute matrices for the representation of R.
X13:=T1: X23:=T2: X31:=T3: X32:=T4: X33:=T5:
X22:=simplify((T8+q*X23.X32).X33^(-1)):
X21:=simplify((T7+q*X22.X31).X32^(-1)):
X12:=simplify((T6+q*X13.X22).X23^(-1)):
X11:=simplify((q*X12.(X21.X33-q*X23.X31)-q^2*X13.T7).T8^(-1)):

# Set q to be a primitive l-th root of unity in the complex numbers.
q:=exp(2*Pi*I/l);

# Output the matrices corresponding to the generators of R.
'X11'=simplify(X11);
'X12'=simplify(X12);
'X13'=simplify(X13);
'X21'=simplify(X21);
```

```
'X22'=simplify(X22);
'X23'=simplify(X23);
'X31'=simplify(X31);
'X32'=simplify(X32);
'X33'=simplify(X33);
```

In order to verify that the matrices calculated using the code above do indeed give a representation of $\mathcal{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle$, one must check that the output `X11`, `X12`, $\ldots$, `X33` share the same relations as $\bar{X}_{1,1}, \bar{X}_{1,2}, \ldots, \bar{X}_{3,3} \in \mathcal{O}_q(M_3(\mathbb{K}))/\langle D_q \rangle$. For example, for `X11` one should check that

```
simplify(X11.X12-q*X12.X11)
simplify(X11.X13-q*X13.X11}
simplify(X11.X21-q*X21.X11)
simplify(X11.X22-X22.X11-(q-q^(-1))*X12.X21)
simplify(X11.X23-X23.X11-(q-q^(-1))*X13.X21)
simplify(X11.X31-q*X31.X11)
simplify(X11.X32-X32.X11-(q-q^(-1))*X12.X31)
simplify(X11.X33-X33.X11-(q-q^(-1))*X13.X31)
```

are all zero.

References

[AL50] A. S. Amitsur and J. Levitzki. Minimal identities for algebras. *Proceedings of the American Mathematical Society*, 1(4):449–463, 1950.

[BCL12] J. Bell, K. Casteels, and S. Launois. Enumeration of $\mathscr{H}$-strata in quantum matrices with respect to dimension. *Journal of Combinatorial Theory. Series A*, 119(1):83–98, 2012.

[BG02] K. A. Brown and K. R. Goodearl. *Lectures on algebraic quantum groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002.

[BL10] J. Bell and S. Launois. On the dimension of H-strata in quantum algebras. *Algebra & Number Theory*, 4(2):175–200, 2010.

[BLL10] J. Bell, S. Launois, and J. Lutley. An automaton-theoretic approach to the representation theory of quantum algebras. *Advances in Mathematics*, 223(2):476–510, 2010.

[BY17] K. A. Brown and M. Yakimov. Azumaya loci and discriminant ideals of PI algebras. *arXiv:1702.04305*, 2017.

[Cas14] K. Casteels. Quantum matrices by paths. *Algebra & Number Theory*, 8(8):1857–1912, 2014.

[Cau03a] G. Cauchon. Effacement des dérivations et spectres premiers des algèbres quantiques. *Journal of Algebra*, 260(2):476–518, 2003.

[Cau03b] G. Cauchon. Spectre premier de $O_q(M_n(k))$: image canonique et séparation normale. *Journal of Algebra*, 260(2):519–569, 2003.

[CP93] C. De Concini and C. Procesi. Quantum groups. In *D-modules, representation theory, and quantum groups (Venice, 1992)*, volume 1565 of *Lecture Notes in Math.*, pages 31–140. Springer, Berlin, 1993.

[CPWZ15] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang. The discriminant controls automorphism groups of noncommutative algebras. *Advances in Mathematics*, 269:551–584, 2015.

[CPWZ16] S. Ceken, J. H. Palmieri, Y.-H. Wang, and J. J. Zhang. The discriminant criterion and automorphism groups of quantized algebras. *Advances in Mathematics*, 286:754–801, 2016.

[GL98] K. R. Goodearl and E. S. Letzter. Prime and primitive spectra of multiparameter quantum affine spaces. In *Trends in ring theory (Miskolc, 1996)*, volume 22 of *CMS Conf. Proc.*, pages 39–58. American Mathematical Society, Providence, RI, 1998.

[GL00] K. R. Goodearl and T. H. Lenagan. Quantum determinantal ideals. *Duke Mathematical Journal*, 103(1):165–190, 2000.

[GL02] K. R. Goodearl and T. H. Lenagan. Prime ideals invariant under winding automorphisms in quantum matrices. *International Journal of Mathematics*, 13(5):497–532, 2002.

[GL07] K. R. Goodearl and S. Launois. The Dixmier-Moeglin equivalence and a Gel'fand-Kirillov problem for Poisson polynomial algebras. *arXiv:0705.3486*, 2007.

[GLL11] K. R. Goodearl, S. Launois, and T. H. Lenagan. Torus-invariant prime ideals in quantum matrices, totally nonnegative cells and symplectic leaves. *Mathematische Zeitschrift*, 269(1-2):29–45, 2011.

[GLL18] K. R. Goodearl, S. Launois, and T. H. Lenagan. Tauvel's height formula for quantum nilpotent algebras. *arXiv:1805.06590*, 2018.

[Goo06] K. R. Goodearl. Commutation relations for arbitrary quantum minors. *Pacific Journal of Mathematics*, 228(1):63–102, 2006.

[GW04] K. R. Goodearl and R. B. Warfield, Jr. *An introduction to noncommutative Noetherian rings*, volume 61 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 2004.

[GY15] K. R. Goodearl and M. Yakimov. Unipotent and Nakayama automorphisms of quantum nilpotent algebras. In *Commutative algebra and noncommutative algebraic geometry. Vol. II*, volume 68 of *Math. Sci. Res. Inst. Publ.*, pages 181–212. Cambridge Univ. Press, New York, 2015.

[Hay08] H. Haynal. PI degree parity in q-skew polynomial rings. *Journal of Algebra*, 319(10):4199–4221, 2008.

[KL00] G. R. Krause and T. H. Lenagan. *Growth of algebras and Gelfand-Kirillov dimension*, volume 22 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2000.

[Lan02] S. Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[Lau04a] S. Launois. Generators for $\mathscr{H}$-invariant prime ideals in $O_q(\mathscr{M}_{m,p}(\mathbf{C}))$. *Proceedings of the Edinburgh Mathematical Society. Series II*, 47(1):163–190, 2004.

[Lau04b] S. Launois. Les idéaux premiers invariants de $\mathscr{O}_q(M_{m,p}(\mathbb{C}))$. *Journal of Algebra*, 272(1):191–246, 2004.

[Lau06] S. Launois. On the automorphism groups of q-enveloping algebras of nilpotent lie algebras. In *From Lie Algebras to Quantum Groups*, volume 28, pages 125–143, 2006.

[Lec14] C. Lecoutre. *Polynomial Poisson algebras: Gel'fand-Kirillov problem and Poisson spectra*. PhD thesis, University of Kent, December 2014.

[LL17] S. Launois and C. Lecoutre. Poisson deleting derivations algorithm and Poisson spectrum. *Communications in Algebra*, 45(3):1294–1313, 2017.

[LLR06] S. Launois, T. H. Lenagan, and L. Rigal. Quantum unique factorisation domains. *Journal of the London Mathematical Society. Second Series*, 74(2):321–340, 2006.

[LM11] A. Leroy and J. Matczuk. On q-skew iterated Ore extensions satisfying a polynomial identity. *Journal of Algebra and its Applications*, 10(4):771–781, 2011.

[LMO88] A. Leroy, J. Matczuk, and J. Okniński. On the Gel'fand-Kirillov dimension of normal localizations and twisted polynomial rings. In *Perspectives in ring theory (Antwerp, 1987)*, volume 233 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 205–214. Kluwer Acad. Publ., Dordrecht, 1988.

[LR06] T. H. Lenagan and L. Rigal. Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum Grassmannians. *Journal of Algebra*, 301(2):670–702, 2006.

[LR08] T. H. Lenagan and L. Rigal. Quantum analogues of Schubert varieties in the Grassmannian. *Glasgow Mathematical Journal*, 50(1):55–70, 2008.

[LZ98] B. Leclerc and A. Zelevinsky. Quasicommuting families of quantum Plücker coordinates. In *Kirillov's seminar on representation theory*, volume 181 of *Amer. Math. Soc. Transl. Ser. 2*, pages 85–108. Amer. Math. Soc., Providence, RI, 1998.

[MR01] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small.

[New72] M. Newman. *Integral matrices*. Academic Press, New York-London, 1972. Pure and Applied Mathematics, Vol. 45.

[NYM93] M. Noumi, H. Yamada, and K. Mimachi. Finite-dimensional representations of the quantum group $\mathrm{GL}_q(n;\mathbf{C})$ and the zonal spherical functions on $\mathrm{U}_q(n-1)\backslash\mathrm{U}_q(n)$. *Japanese Journal of Mathematics. New Series*, 19(1):31–80, 1993.

[Pan94] A. N. Panov. The skew field of rational functions on $\mathrm{GL}_q(n,K)$. *Functional Analysis and Its Applications*, 28(2):134–136, Apr 1994.

[Pan95] A. N. Panov. Skew fields of twisted rational functions and the skew field of rational functions on $\mathrm{GL}_q(n,K)$. *Algebra i Analiz*, 7(1):153–169, 1995.

[Pos06] A. Postnikov. Total positivity, Grassmannians, and networks. *arXiv:0609764*, 2006.

[Pro73] C. Procesi. *Rings with polynomial identities*. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, 17.

[PW91] B. Parshall and J. P. Wang. Quantum linear groups. *Memoirs of the American Mathematical Society*, 89(439):vi+157, 1991.

[Row88a] L. H. Rowen. *Ring theory. Vol. I*, volume 127 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[Row88b] L. H. Rowen. *Ring theory. Vol. II*, volume 128 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[Tho79] R. C. Thompson. Interlacing inequalities for invariant factors. *Linear Algebra and its Applications*, 24:1–31, 1979.

[Yak10] M. Yakimov. Invariant prime ideals in quantizations of nilpotent Lie algebras. *Proceedings of the London Mathematical Society. Third Series*, 101(2):454–476, 2010.

[Yak13] M. Yakimov. A proof of the Goodearl-Lenagan polynormality conjecture. *International Mathematics Research Notices. IMRN*, 2013(9):2097–2132, 2013.

[Zha96] J. J. Zhang. On Gel'fand-Kirillov transcendence degree. *Transactions of the American Mathematical Society*, 348(7):2867–2899, 1996.