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Stress field and spin axis relaxation for inelastic triaxial ellipsoids

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ABSTRACT

A compact formula for the stress tensor inside a self-gravitating, triaxial ellipsoid in an arbitrary rotation state is given. It contains no singularity in the incompressible medium limit. The stress tensor and the quality factor model are used to derive a solution for the energy dissipation resulting in the damping (short-axis mode) or excitation (long axis) of wobbling. In the limit of an ellipsoid of revolution, we compare our solution with earlier ones and show that, with appropriate corrections, the differences in damping times estimates are much smaller than it has been claimed.

Key words: methods: analytical – celestial mechanics – minor planets, asteroids: general.

1 INTRODUCTION

Most asteroids rotate in the principal, shortest axis mode: their spin axes practically coincide with the directions of the maximum moment of inertia. Only 45 out of almost 5500 entries of the light curve data base (LCDB) (Warner, Harris & Pravec 2009, March 2012 version) refer to objects that are possible non-principal axis (NPA) rotators, also known as 'tumblers' or wobbling objects. With one exception of 253 Mathilde, tumblers are rather small, with estimated diameters below 20 km, but even in this size range they belong to a minority among about 2000 objects of this size with known rotation periods.

Attitude dynamics of asteroids is shaped mainly by gravitational torques (exerted either systematically by the Sun and giant planets or sporadically during encounters with other bodies), collisions, optical and thermal radiation recoil torques, i.e. the Yarkovsky–O'Keefe–Radzievskii–Paddack (YORP) effect, and – last but not least – by energy dissipation due to inelastic deformations. As far as NPA rotation is concerned, collisions and close approaches trigger tumbling (Scheeres et al. 2000; Paolicchi, Burns & Weidenschilling 2002). Small fragments created from collisions of larger objects are also expected to start their lives in an NPA rotation state. The YORP effect also excites wobbling (Rubincam 2000; Vokrouhlický et al. 2007; Breiter, Rożek & Vokrouhlický 2011), whereas – save for possible resonances – distant body gravitation torques are neutral in this respect. Thus, even accounting for observational selection effects mentioned by Pravec et al. (2005), the dissipative damping seems to override other effects in most of the cases.

The mechanism of wobble damping was first identified by Prendergast (1958). In NPA rotation, the centrifugal acceleration oscillates periodically, deforming each body fragment. The deformation is not perfectly elastic, so some fraction of fluctuating strain–stress energy is dissipated during each precession period and converted into heat. Draining the elastic energy affects the kinetic energy of rotation which also decreases. Thus, the rotation axis is driven towards the minimum energy state – rotation around the principal axis of maximum inertia. The angular momentum, however, is not affected by the energy dissipation, as far as we ignore thermal radiation and consider the body as an isolated system. Prendergast provided a general form of the energy dissipation rate equation for an oblate spheroid[1] based upon the solution of 3D elasticity equations and the assumption that a constant fraction of the oscillating part of elastic energy is dissipated at each precession period. The latter assumption defines the now commonly adopted 'Q-model'.

Burns & Safronov (1973) built upon the general idea of Prendergast using the combination of a spheroidal shape for rotation and a bent slender beam approximation for elastic energy. Their simple estimate of spin-axis alignment time is still in use – sometimes in the version provided by Harris (1994). However, some scepticism towards it has been brought by observations of asteroids that do rotate around the principal axis in spite of having Burns–Safronov damping time estimate longer than the age of the Solar system. Meanwhile, the problem migrated to geophysics (e.g. the Chandler wobble damping), rotation dynamics of comets and interstellar dust grain physics. The last branch, stemming from Purcell (1979), was finally brought back to the dynamics of comets and asteroids with the sequence of papers by Efroimsky and Lazarian (Lazarian & Efroimsky 1999; Efroimsky 2000, 2001, 2002; Efroimsky & Lazarian 2000). Their main point of novelty is the attempt to discuss a triaxial object, represented by a rectangular prism (brick), by solving the complete, quasi-static stress tensor equation (Efroimsky 2000). Later on, Molina, Moreno & Martínez-López (2003) issued

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[1] In this paper, we use the word 'spheroid' for an arbitrary ellipsoid of revolution.

the damping model for a spheroid using the same starting point as Prendergast (1958), i.e. solving equations for displacements. The work of Sharma, Burns & Hui (2005) not only provides the solution for a spheroid with two different ways of estimating the peak elastic energy required for the Q model, but it also offers a long discussion of shortcomings and problems related to earlier papers mentioned in this paragraph.

Trying to combine the YORP effect with a damping mechanism, we first intended to use the spheroid-based model of Sharma et al. (2005) for arbitrary shape asteroids. This approach, mentioned in Vokrouhlický et al. (2007), became less appealing after a closer inspection, because of a substantial difference in the dynamics of bodies with and without axial symmetry. On the other hand, the solution of Efroimsky (2000), albeit referring to a triaxial shape, exhibits a number of drawbacks.

(i) As a consequence of using a non-smooth, brick-shaped object, the solution of stress equations is inexact, with unknown error bounds.

(ii) Compatibility conditions are not fulfilled, i.e. there is no displacement field that might produce the strain tensor found by Efroimsky (Sharma et al. 2005).

(iii) Rotation dynamics is treated by approximate formulae valid only in the neighbourhood of the principal axis.

Later on, Efroimsky (2001) suggested the Fourier series involving the Jacobi nome as a remedy for the last item, but none of subsequent works has implemented this guideline. In these circumstances, we have decided to resume the problem at the point where Efroimsky has abandoned it, not only using the Fourier series to resolve the last problem, but also applying the triaxial ellipsoid shape which resolves the first two objections as well. From this point of view, this work combines a stress solution in the style of Sharma et al. (2005) with energy dissipation treatment in the spirit of Efroimsky (2001).

In Section 2, we first formulate the problem of determining the stress tensor and enumerate the assumptions, hoping to help a reader less familiar with elasticity problems. Basic facts are recalled according the textbooks of Landau & Lifshitz (1959), Saad (2005) and Wilmanski (2010). Two independent methods (displacement approach and stress approach) are used to derive and cross-check the final expressions of the stress tensor.

The Q model of energy dissipation is introduced in Section 3 and applied in Section 4 to derive an energy dissipation rate formula. Section 5 presents wobble damping time equations based upon the results of Section 4 and some exemplary results. In Section 6, we present the reduction to a specific case of a spheroid where a comparison of our solution with those reported in earlier works is possible. We use this opportunity to resolve controversies concerning drastically different energy dissipation rates in various models.

2 LINEAR ELASTIC MODEL OF ROTATING DEFORMABLE ELLIPSOID

2.1 Basic terms: strain, stress and body forces

In an arbitrary reference frame, we consider a body as a dense union of material points. Let the set of position vectors $\boldsymbol{r}$ define a reference state (configuration); if the points, for any reason, move with respect to the reference state, their position vectors will be incremented by displacement vectors $\boldsymbol{u}(\boldsymbol{r}, t)$, dependent on time t as well as on position $\boldsymbol{r}$, creating a new state with

$$\boldsymbol{r}'(t) = \boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r}, t). \tag{1}$$

The notion of displacements is too general, because it may include rigid body motion – translation and rotation of the entire body. The rigid body motions are discarded by the introduction of the strain $\mathbf{e}$ – the dimensionless, tensor quantity describing deformations of an infinitesimal volume element in terms of displacement gradient.

Assumption 1. The gradient of displacements is small and its square can be neglected.

Under the above assumption, the strain tensor $\mathbf{e}$ is symmetric by its definition

$$\mathbf{e} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right], \tag{2}$$

where the derivatives are taken with respect to the components of $\boldsymbol{r}$.

Two kinds of forces have to be considered in a continuous medium: volumetric forces and surface forces, known also as body forces and tractions, respectively. Body forces represent the 'external' force field. In our case, they include self-gravitation and forces of inertia. They are defined as a vector field and specified in terms of their volumetric density $\boldsymbol{b}(\boldsymbol{r}, t)$ – force divided by the mass of the volume element, so the total volumetric force $\boldsymbol{F}$ acting on a body with density $\rho(\boldsymbol{r})$ is the result of the volume integral

$$\boldsymbol{F} = \int_V \rho \boldsymbol{b}\, \mathrm{d}V. \tag{3}$$

Surface traction $\boldsymbol{t}^{\boldsymbol{n}}$ is a vector of the force acting on an infinitesimal oriented surface, divided by the area. It describes interactions between adjacent volume elements or forces applied directly on the boundary. The surface is defined by its unit normal vector $\boldsymbol{n}$ and may lie either on a boundary or inside the body. In order to describe traction at each possible direction of a plane passing through the point $\boldsymbol{r}$, each component vector of $\boldsymbol{t}^{\boldsymbol{n}}$ in a given basis is projected on each component vector of $\boldsymbol{n}$, creating the Cauchy stress tensor $\mathbf{T}$. Thus, traction on a surface defined by $\boldsymbol{n}$ can be obtained from $\mathbf{T}$ as (unless explicitly stated, repeated index summation assumed in all formulae)

$$t_i^{\boldsymbol{n}} = T_{ji}\, n_j. \tag{4}$$

The units of stress tensor components are those of force per area. A good illustration of the two forces of nature is a glass of water: body force density is constant throughout the volume (homogeneous gravitational field), whereas tractions define a hydrostatic pressure – vanishing on the top surface, reaching maximum at the bottom and depending, as a vector, on the direction of the surface element.

Assumption 2. The deformable body forms an isolated system without internal heat sources.

The consequences of this assumption are numerous. First of all, we can use the linear momentum conservation principle in the form of the Cauchy equation, linking stress tensor, body forces and the acceleration of mass particles in an inertial reference frame

$$\nabla \cdot \mathbf{T} + \rho \boldsymbol{b} = \rho\,(\ddot{\boldsymbol{r}} + \ddot{\boldsymbol{u}}). \tag{5}$$

If the reference configuration is not fixed in space and still we want it to define the reference frame for the Cauchy equation, then $\ddot{\boldsymbol{r}}$ should be transferred to the body forces $\boldsymbol{b}$ (subtracted) as the density of forces of inertia. In the case of rotation, some Corolis-type terms involving $\dot{\boldsymbol{u}}$ may also appear on the right-hand side (Tokis 1974).

Assumption 3. Quasistatic approximation.

Quasistatic approximation results from setting $\ddot{\boldsymbol{u}} = 0$ (and any Coriolis type $\dot{\boldsymbol{u}}$) on the right-hand side of (5). This simplification

was generally adopted since Prendergast (1958) and we proceed similarly in this work. Having included forces of inertia in $\boldsymbol{b}$, so that $\ddot{\boldsymbol{r}} = \mathbf{0}$, we solve a static equilibrium equation for $\mathbf{T}$:

$$\nabla \cdot \mathbf{T} = -\rho \boldsymbol{b}(\boldsymbol{r}, t), \tag{6}$$

although body forces can be time dependent. The validity of equation (6) can be justified if the solution of the original Cauchy equation is a sum of 'free' acoustic waves of high frequency and forced vibrations whose frequencies – presumably much lower – come from $\boldsymbol{b}$. Free vibrations can be neglected from two points of view: either we consider them to have zero amplitudes at some initial moment and then a slow, adiabatic forcing will not excite them considerably, or – looking forward to the introduction of some dissipation mechanism – the high-frequency terms will be quickly damped. In both cases, the stationary regime oscillations derived from (6) will have amplitudes that differ from the actual ones by a small quantity comparable with the ratio of free to forced vibration periods.

Assumption 4. Traction-free surface.

Boundary conditions for the stress tensor will be specified as homogeneous Neumann conditions on the surface of a deformable body

$$\mathbf{T}\,\boldsymbol{n} = \mathbf{0}. \tag{7}$$

The uniqueness of $\mathbf{T}$ as a solution of this boundary problem is guaranteed by general theorems (Saad 2005), but it is not seen immediately from three scalar equations of (7), even if we add an additional property: according to Assumption 2, the angular momentum is conserved, so the stress tensor should be symmetric, i.e. $\mathbf{T} = \mathbf{T}^{\mathrm{T}}$. At this point, Molina et al. (2003) felt free to postulate $\mathbf{T} = \mathbf{0}$, which was harshly criticized by Sharma et al. (2005).

Assumption 5. Hookean constitutive relations.

Let us assume that an asteroid is made of a linear, isotropic elastic material with adiabatic Lamé shear modulus μ and the Poisson ratio ν describing the compressibility (incompressible materials have the maximum possible $\nu = 0.5$). With these assumptions, Hooke's law – serving as a constitutive equation – states a linear relation between the strain and stress:

$$T_{ij} = 2\mu \left(\frac{\nu}{1-2\nu}\, \delta_{ij} \operatorname{tr} \mathbf{e} + e_{ij} \right), \tag{8}$$

or, conversely,

$$e_{ij} = \frac{1}{2\,\mu} \left(T_{ij} - \frac{\nu}{1+\nu} \delta_{ij} \operatorname{tr} \mathbf{T} \right), \tag{9}$$

where δ_{ij} is the Kronecker delta.

2.2 Stress approach versus displacement approach

Our first goal is to find the symmetric stress tensor $\mathbf{T}$ as a solution of equations (6) with boundary conditions (7). In the stress approach, the problem is solved directly, by assuming some ansatz on $\mathbf{T}$ as a function of $\boldsymbol{r}$. But if we accept Assumption 5, additional conditions have to be imposed. Strain is a mathematically meaningful quantity if there exists a displacement field that generates it through equation (2). Even without explicit knowledge of $\boldsymbol{u}$, this is guaranteed by Saint Venant's compatibility conditions (Saad 2005; Wilmanski 2010)

$$\nabla \times (\nabla \times \mathbf{e}) = \mathbf{0}, \tag{10}$$

providing six independent relations between e_{ij}. Through the constitutive relations (8), compatibility equations provide the identities that a meaningful stress tensor has to obey in addition to boundary conditions. In the following section, we show that (7) and (10) together admit a unique solution for $\mathbf{T}$. The stress approach was applied by Efroimsky (2000) to the problem of a rotating rectangular prism. Yet, the postulated form of $\mathbf{T}$ satisfied only the Cauchy equation (6); neither boundary conditions nor compatibility equations could be satisfied exactly (the latter were not tested at all) and the level of resulting error remains unknown (Sharma et al. 2005).

Another way of solving the Cauchy equation is the displacement approach. Using equations (2) and (8), we convert the first-order differential equation for stress tensor (6) into a second-degree equation for the displacement vector field $\boldsymbol{u}$, obtaining the quasistatic Lamé or Cauchy–Navier equation

$$\mu \left[\frac{\operatorname{tr}(\nabla \boldsymbol{u})}{1-2\nu} + \nabla^2 \boldsymbol{u} \right] = -\rho\, \boldsymbol{b}, \tag{11}$$

with boundary conditions

$$2\nu\, (\nabla \cdot \boldsymbol{u})\, \boldsymbol{n} + (1-2\nu) \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}} \right] \boldsymbol{n} = \mathbf{0}, \tag{12}$$

derived from (7). Equations (11) with only three Neumann boundary conditions (12) admit a solution $\boldsymbol{u}(\boldsymbol{r}, t)$ which is not unique and an arbitrary rigid motion may be added to displacements. But since the definition of $\mathbf{e}$ involves differentiation, the resulting strain and stress tensors are uniquely defined regardless of remaining arbitrary terms. Following Denisov & Novikov (1987), we will impose two special conditions: the volume integral of the displacements field should vanish

$$\int_V \boldsymbol{u}\, \mathrm{d}V = \mathbf{0}, \tag{13}$$

and the moment of displacements should also vanish, i.e.

$$\int_V \boldsymbol{r} \times \boldsymbol{u}\, \mathrm{d}V = \mathbf{0}. \tag{14}$$

These six conditions aim at suppressing rigid translation and rotation terms in displacements and allow a unique determination of $\boldsymbol{u}$, which is of minor interest for the stress tensor recovered through (2) and (8), but gives more insight into the question of the reference configuration choice and simplify energy and momentum balance discussion. Up to the ambiguity in the last two conditions, the displacement approach was taken by Chree (1895), Denisov & Novikov (1987), Molina et al. (2003) and Sharma et al. (2005).

We can also observe that restoring the term $\rho\ddot{\boldsymbol{u}}$ in equation (11), we obtain a quantitative measure of quasistatic approximation error. The homogeneous solution will involve a frequency close to

$$\omega_{\mathrm{f}} = \sqrt{\frac{\mu}{\rho a^2}}, \tag{15}$$

where a is the radius of an object. If body forces are periodic with frequency Ω (the precession frequency in our case) then, with μ of the order of 10 GPa, the ratio $\Omega/\omega_{\mathrm{f}}$ may be safely considered small.

2.3 Ellipsoid stress solution

2.3.1 Homogeneous ellipsoid body forces

Let the reference configuration be a homogeneous rigid ellipsoid with semiaxes $c \le b \le a$. Its shape will be described by two dimensionless parameters

$$h_1 = \frac{b}{a}, \qquad h_2 = \frac{c}{b}, \tag{16}$$

both taking values $0 < h_i \leq 1$. In the reference frame whose centre coincides with the centre of mass and the basis vectors $\boldsymbol{e}_i$ are directed along the principal axes, the parametric equation of the interior reads

$$\boldsymbol{r} = q\,a\,(\sin\vartheta\cos\phi\,\boldsymbol{e}_1 + h_1\sin\vartheta\sin\phi\,\boldsymbol{e}_2 + h_1\,h_2\,\cos\vartheta\,\boldsymbol{e}_3)\,, \tag{17}$$

where $0 \leq q < 1$, $0 \leq \vartheta \leq \pi$, $0 \leq \phi < 2\pi$. The boundary is specified by $q = 1$ and the unit normal vector on the boundary is

$$\boldsymbol{n} = \Phi(h_1, h_2, \vartheta, \phi)\Big(\sin\vartheta\cos\phi\,\boldsymbol{e}_1 + h_1^{-1}\sin\vartheta\sin\phi\,\boldsymbol{e}_2 + h_1^{-1}h_2^{-1}\cos\vartheta\,\boldsymbol{e}_3\Big), \tag{18}$$

where Φ is some nonzero function; its explicit definition is not required for the vanishing traction condition (7). According to the postulate (13), the centre of ellipsoid remains the centre of mass even in a deformed state.

Given an arbitrary function $F(\boldsymbol{r})$, the volume integration rule for an ellipsoid is

$$\int_V F\,\mathrm{d}V = a^3h_1^2h_2\int_0^1 q^2\,\mathrm{d}q\int_0^\pi \sin\vartheta\,\mathrm{d}\vartheta\int_0^{2\pi} F\,\mathrm{d}\phi, \tag{19}$$

where $F(\boldsymbol{r})$ should be expressed in terms of q, ϑ, ϕ according to equation (17).

The body forces acting on a freely rotating ellipsoid include the forces of inertia due to rotation, with the force density vector

$$\boldsymbol{b}_{\mathrm{in}} = \boldsymbol{r}\times\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\boldsymbol{r}\times\boldsymbol{\omega}), \tag{20}$$

where the time derivative of rotation vector $\boldsymbol{\omega}$ is given by Euler equations

$$\dot{\boldsymbol{\omega}} = -\frac{1-h_2^2}{1+h_2^2}\omega_2\omega_3\boldsymbol{e}_1 + \frac{1-h_1^2h_2^2}{1+h_1^2h_2^2}\omega_1\omega_3\boldsymbol{e}_2 - \frac{1-h_1^2}{1+h_1^2}\omega_1\omega_2\boldsymbol{e}_3, \tag{21}$$

and the moments of inertia I_i for an ellipsoid with mass m

$$\frac{I_1}{ma^2} = \frac{h_1^2\left(1+h_2^2\right)}{5}, \quad \frac{I_2}{ma^2} = \frac{1+h_1^2h_2^2}{5}, \quad \frac{I_3}{ma^2} = \frac{1+h_1^2}{5} \tag{22}$$

are substituted.

Gravitation inside the ellipsoid results in body forces density

$$\boldsymbol{b}_{\mathrm{gr}} = -\gamma_1 x\,\boldsymbol{e}_1 - \gamma_2 y\,\boldsymbol{e}_2 - \gamma_3 z\,\boldsymbol{e}_3, \tag{23}$$

with constants

$$\begin{aligned}
\gamma_1 &= \frac{\gamma m}{a^3}R_J\left(1, h_1^2, h_1^2h_2^2, 1\right),\\
\gamma_2 &= \frac{\gamma m}{a^3}R_J\left(1, h_1^2, h_1^2h_2^2, h_1^2\right),\\
\gamma_3 &= \frac{\gamma m}{a^3}R_J\left(1, h_1^2, h_1^2h_2^2, h_1^2h_2^2\right),
\end{aligned} \tag{24}$$

expressed in terms of gravitation constant γ and Carlson's elliptic integral

$$R_J(u,v,w,p) = \frac{3}{2}\int_0^\infty \frac{\mathrm{d}s}{(p+s)\sqrt{(u+s)(v+s)(w+s)}}. \tag{25}$$

The body forces are linear in coordinates of reference configuration, so we write them as

$$\boldsymbol{b} = \boldsymbol{b}_{\mathrm{in}} + \boldsymbol{b}_{\mathrm{gr}} = \mathbf{B}\boldsymbol{r}, \tag{26}$$

with coordinates-independent matrix $\mathbf{B}$ having elements

$$\begin{aligned}
B_{11} &= \omega_2^2 + \omega_3^2 - \gamma_1,\\
B_{22} &= \omega_3^2 + \omega_1^2 - \gamma_2,\\
B_{33} &= \omega_1^2 + \omega_2^2 - \gamma_3,\\
B_{12} &= -\frac{2\omega_1\omega_2}{1+h_1^2},\\
B_{13} &= -\frac{2\omega_1\omega_3}{1+h_1^2h_2^2},\\
B_{23} &= -\frac{2\omega_2\omega_3}{1+h_2^2},
\end{aligned} \tag{27}$$

and

$$B_{21} = h_1^2B_{12}, \quad B_{31} = h_1^2h_2^2B_{13}, \quad B_{32} = h_2^2B_{23}. \tag{28}$$

2.3.2 More assumptions

Assumption 6. Displacements and their partial derivatives are small quantities of the first order.

This stronger variant of Assumption 1 allows us to treat the problem in terms of a first-order approximation. Namely, considering boundary conditions we can impose them on the reference ellipsoid surface, neglecting the displacements that deform it. It also means we do not restrict reference ellipsoids to a figure of equilibrium-type solutions like e.g. Jacobi ellipsoids (Chandrasekhar 1969). In the same spirit, we ignore the variations of density due to strain and use a constant, mean ρ whenever it serves to define displacements or stress – either explicitly or indirectly (like in body forces $\boldsymbol{b}_{\mathrm{gr}}$).

It turns out that postulates (13) and (14) are inherently related to the choice of reference configuration that satisfies our assumption. In the first approximation, i.e. evaluating volume integrals over the homogeneous reference ellipsoid, we interpret (13) as a postulate that the centre of mass position is not altered by displacements. Similarly, equation (14) indirectly leads to the statement that displacement velocities do not contribute to the angular momentum of the system. Observing that $\dot{\boldsymbol{u}}$ will depend on $\dot{B}_{ij}$ in exactly the same form as $\boldsymbol{u}$ depends on B_{ij} (the Cauchy–Navier equation is linear and does not involve time derivative), we find

$$\int_V \rho(\boldsymbol{r}\times\dot{\boldsymbol{u}})\,\mathrm{d}V = \boldsymbol{0}, \tag{29}$$

as a consequence of (14), provided ρ is constant and the same bounding surface is used in both integrals, i.e. within the first-order approximation. In other words, the postulate (14) implies that we use Tisserand's mean axes (Munk & MacDonald 1960) as the reference frame and they approximately coincide with the principal axes of the reference ellipsoid. Such choice has a property of minimizing the displacements and their velocities.

Sharma et al. (2005) postulated a pre-stressed state of their reference spheroid and dropped the constant part of the stress due to self-gravitation on the onset of their derivation. They were not consequent in this point, because they did not do the same with a mean part of centrifugal stress. For typical asteroids, both effects may be comparable. They may even mutually cancel. Thus, we do not find the pre-stressed state assumption necessary for asteroids, although it is important for major objects, like the Earth, where it was originally introduced (Love 1934). On the other hand, its role would be to provide the rationale for the validity of the six assumptions we have already made.

2.3.3 Displacement approach

The problem of finding displacements and stress tensor for a freely rotating ellipsoid (without self-gravitation) was solved by Denisov & Novikov (1987). Their impressive solution, apparently worked out without the support of computer algebra, was based upon the displacement approach, which helped in establishing the influence of displacement field on the moments of inertia. Earlier results of Chree (1895) included the self-gravitation, but only the principal axis rotation was considered. We used the stress tensor of Denisov & Novikov (1987) as a test of our solution.

The main advantage of using an ellipsoid is the possibility of finding the displacement field that satisfies equations (11), (12) and (26) as a sum of two homogeneous polynomials of degrees 1 and 3, i.e. a sum of 13 monomials with dimensionless vector coefficients $\boldsymbol{f}$:

$$\begin{aligned}\boldsymbol{u} = {} & \boldsymbol{f}^{100}x + \boldsymbol{f}^{010}y + \boldsymbol{f}^{001}z + a^{-2}\left(\boldsymbol{f}^{003}z^3 + \boldsymbol{f}^{012}yz^2 + \boldsymbol{f}^{021}y^2z\right.\\ & + \boldsymbol{f}^{030}y^3 + \boldsymbol{f}^{102}xz^2 + \boldsymbol{f}^{111}xyz + \boldsymbol{f}^{120}xy^2 + \boldsymbol{f}^{201}x^2z\\ & \left. + \boldsymbol{f}^{210}x^2y + \boldsymbol{f}^{300}x^3\right), \end{aligned} \tag{30}$$

which involves 39 arbitrary constants: 9 for the linear part and 30 for the cubic terms. It is instructive to add B_{21}, B_{31} and B_{32} as additional, unspecified parameters, raising the number of unknowns to 42. Our choice for the form of $\boldsymbol{u}$ is straightforward, although not necessarily optimal; for example, Sharma et al. (2005) used the spherical harmonics basis and reported a smaller number of undetermined coefficients for a spheroid and in an unpublished solution for the ellipsoid (Sharma, private communication).

Using the 'brute force' attack, allowed by the use of an algebraic manipulator, we formulate 42 independent conditions involving linearly the coefficients of $\boldsymbol{u}$ and unspecified elements B_{ij} of the matrix $\mathbf{B}$ – the ones with $i \le j$ are parameters and those with $i > j$ are unknowns. First, we observe that the centre of mass condition (13) is satisfied identically by (30). Actually, it has been used implicitly to drop coordinate-independent and quadratic parts of $\boldsymbol{u}$.

(i) Equation (11) generates three equations linear in x, y and z. This amounts to nine equations linking each B_{ij} with five coefficients of the cubic part of $\boldsymbol{u}$.

(ii) Postulating equation (14), we obtain three equations – each linking two coefficients of the linear and six of the cubic part of $\boldsymbol{u}$.

(iii) The remaining 30 equations result from boundary conditions (12). They can be derived either directly, i.e. from the polynomial form of the ellipsoid surface and normal vector (Denisov & Novikov 1987) or by considering trigonometric polynomials of ϕ and ϑ resulting from the substitution of equations (17) and (18). In the latter case, we first equate to 0 the coefficients of $\sin 3\phi$, $\cos 3\phi$, $\sin 2\phi$ and $\cos 2\phi$ which have a common factor $(\sin\vartheta)^3$ or $(\cos\vartheta)^3$; this provides 12 conditions. Then, in the coefficients of $\sin\phi$ and $\cos\phi$, we separate the terms with $\sin\vartheta$ and $\sin 3\vartheta$, and set them to 0, obtaining another 12 conditions. Finally, in the part independent of ϕ we separate terms factored by $\cos\vartheta$ and $\cos 3\vartheta$ resulting in the last six conditions.

Solving the Cramer system of 42 linear equations, we obtain a unique solution for the 39 coefficients of $\boldsymbol{u}$ as linear combinations of $B_{i,j}$ with $i \le j$. For the remaining three unknowns, we recover equation (28) which becomes the condition of existence of the displacements solution (30) for an ellipsoid – slightly more general than the condition found by Denisov & Novikov (1987).

Unfortunately, the resulting expressions of the displacement field $\boldsymbol{u}$ are too long to be quoted. Each B_{ij} appearing in $\boldsymbol{f}$ has its own multiplier – a rational function of h_1^2, h_2^2 and ν. The only common factor of all $\boldsymbol{f}$ vectors is $\rho\, a^2\, \mu^{-1} = \omega_{\rm f}^{-2}$. This indicates a link between Assumptions 3 and 6.

2.3.4 Stress approach

The structure of the displacement field is inherited by the stress and strain tensors. Each T_{ij} and e_{ij} is a linear combination of B_{ij}, including zero- and second-degree monomials of x, y and z. We focus the discussion on the stress tensor, because $\mathbf{T}$ can be of interest in studying the breakup of spinning asteroids (e.g. Washabaugh & Scheeres 2002), whereas strain components e_{ij} in the elastic material are easily found from $\mathbf{T}$ using constitutive relations (9).

Introducing

$$\tilde{y} = \frac{y}{h_1}, \qquad \tilde{z} = \frac{z}{h_1 h_2}, \tag{31}$$

to benefit from some symmetries, we can write the general form of $\mathbf{T}$ for the elastic ellipsoid as

$$\begin{aligned}\mathbf{T} = \rho\left(a^2\mathbf{A} - x^2\mathbf{A}^{11} - \tilde{y}^2\mathbf{A}^{22} - \tilde{z}^2\mathbf{A}^{33}\right.\\ \left. - x\tilde{y}\mathbf{A}^{12} - \tilde{y}\tilde{z}\mathbf{A}^{23} - x\tilde{z}\mathbf{A}^{13}\right),\end{aligned} \tag{32}$$

where $\rho a^2\mathbf{A}$ represents the stress tensor at the origin $x = y = z = 0$, and all matrices are symmetric. Thus, the stress definition requires 42 matrix elements to be determined. Their expressions resulting from direct substitution of displacement solution are unwieldy, so we decided to apply the second possible way of determining $\mathbf{T}$ – the stress approach. The quasistatic equilibrium condition (6) does not involve $\mathbf{A}$ and leads to nine linear relations between $\mathbf{A}^{ij}$ and the body force matrix $\mathbf{B}$. Boundary conditions (7) generate 30 relations between the elements of $\mathbf{A}$ and $\mathbf{A}^{ij}$. However, the resulting set of 39 linear equations admits solutions only if conditions (28) are satisfied and then its rank drops to 36 allowing a unique solution for all $\mathbf{A}^{ij}$ as linear combinations of $\mathbf{A}$ and $\mathbf{B}$ elements.

The 'central stress' $\mathbf{A}$ results from compatibility equations (10), forming two independent subsystems of equations:

$$\begin{aligned}\frac{\partial^2 e_{11}}{\partial y\partial z} &= \frac{\partial^2 e_{12}}{\partial x\partial z} - \frac{\partial^2 e_{23}}{\partial x^2} + \frac{\partial^2 e_{13}}{\partial x\partial y},\\ \frac{\partial^2 e_{22}}{\partial x\partial z} &= \frac{\partial^2 e_{23}}{\partial x\partial y} - \frac{\partial^2 e_{13}}{\partial y^2} + \frac{\partial^2 e_{12}}{\partial y\partial z},\\ \frac{\partial^2 e_{33}}{\partial x\partial y} &= \frac{\partial^2 e_{13}}{\partial y\partial z} - \frac{\partial^2 e_{12}}{\partial z^2} + \frac{\partial^2 e_{23}}{\partial x\partial z}\end{aligned} \tag{33}$$

and

$$\begin{aligned}2\frac{\partial^2 e_{12}}{\partial x\partial y} &= \frac{\partial^2 e_{11}}{\partial y^2} + \frac{\partial^2 e_{22}}{\partial x^2},\\ 2\frac{\partial^2 e_{23}}{\partial y\partial z} &= \frac{\partial^2 e_{22}}{\partial z^2} + \frac{\partial^2 e_{33}}{\partial y^2},\\ 2\frac{\partial^2 e_{13}}{\partial x\partial z} &= \frac{\partial^2 e_{11}}{\partial z^2} + \frac{\partial^2 e_{33}}{\partial x^2}.\end{aligned} \tag{34}$$

Relating $\mathbf{e}$ with $\mathbf{T}$ by means of (9), substituting the general form of stress (32) and making use of $\mathbf{A}^{ij}$ expressed in terms of $\mathbf{A}$ and $\mathbf{B}$, we find that subsequent equations of (33) directly define A_{23}, A_{13} and A_{12}, respectively, in terms of B_{ij} with the same subscripts. On the other hand, equations (34) form a system of coupled linear equations for A_{ii} with right-hand sides depending on B_{ii}. Its solution

takes the form

$$\begin{pmatrix} 2A_{11} \\ 2h_1^{-2}A_{22} \\ 2h_{12}^{-2}A_{33} \end{pmatrix} = \left(\mathbf{1} + \mathbf{L}^{-1}\mathbf{R}\right) \begin{pmatrix} B_{11} \\ B_{22} \\ B_{33} \end{pmatrix}, \tag{35}$$

requiring the inverse of a 3×3 matrix $\mathbf{L}$. Although $\mathbf{L}$ is not complicated and invertible by elementary means, an explicit formula for $\mathbf{L}^{-1}$ is too long to be explicitly quoted. The detailed solution for $\mathbf{A}^{ij}$ and matrices $\mathbf{L}$, $\mathbf{R}$ are given in Appendix A.

We have verified that both methods (displacements approach and stress equations with compatibility conditions) lead to the same final results. Nevertheless, the reduction of $\mathbf{T}$ to the simple form given in Appendix A starting from the displacement solution would be a tedious exercise.

3 THE Q MODEL OF DISSIPATION

3.1 Energy of a deformable ellipsoid

For a rigid ellipsoid in free rotation with angular velocity $\boldsymbol{\omega}$, the only part of energy that matters is the kinetic energy

$$K_0 = \frac{1}{2}\int_V \rho\,(\boldsymbol{\omega}\times\boldsymbol{r})^2\,\mathrm{d}V = \frac{1}{2}\omega_i I_{ij}\omega_j \tag{36}$$

conserved during the motion. In the principal axes system, the tensor of inertia $\mathbf{I}$ is diagonal, i.e $I_{ij} = \delta_{ij}I_i$, given in equation (22). In the absence of external torques, the angular momentum

$$\boldsymbol{H} = \int_V \rho\,\boldsymbol{r}\times(\boldsymbol{\omega}\times\boldsymbol{r})\,\mathrm{d}V = I_{ij}\omega_j\boldsymbol{e}_i \tag{37}$$

is also conserved in an inertial frame, i.e. provided we account for the rotation of body-fixed basis vectors $\boldsymbol{e}_i$. Only the norm $H = ||\boldsymbol{H}||$ is constant in the body frame.

In a deformable body, the velocity of displacements $\dot{\boldsymbol{u}}$ adds up to the total kinetic energy K and angular momentum. Moreover, the results of volume integration are affected by the fluctuations of density and by the bounding surface deformations. However, under Assumption 2, as long as the deformations are elastic and the centre of mass is not altered by displacements, the angular momentum remains constant in the fixed frame. The total kinetic energy may vary, but the sum $K + U$, where U is the potential energy of deformation, remains constant. A rigorous discussion of energy exchange between K and U can be found in Munk & MacDonald (1960) or Lambeck (1980). For the present discussion, we approximate the kinetic energy as $K \approx K_0$ and consider potential energy $U \approx U_\mathrm{e}$, storing the work of both body forces and tractions, to be the elastic energy

$$U_\mathrm{e} = \int_V \epsilon\,\mathrm{d}V \geq 0, \tag{38}$$

with the energy density ϵ defined as

$$\epsilon = \frac{1}{2}e_{ij}T_{ij}, \tag{39}$$

which is evaluated, according to the first-order approximation, with the volume integral taken over the reference ellipsoid. Under Assumption 5, we can also express ϵ in terms of the stress tensor alone, obtaining (Efroimsky 2000)

$$\epsilon = \frac{1}{4\mu}\left(-\frac{\nu\,(\mathrm{tr}\,\mathbf{T})^2}{1+\nu} + T_{11}^2 + T_{22}^2 + T_{33}^2 + 2\left(T_{12}^2 + T_{23}^2 + T_{13}^2\right)\right). \tag{40}$$

Inelasticity disturbs the ideal picture of Hookean oscillations. Different rheological models have been constructed in hope to adjust constitutive relations between stress and strain to the reality of the material world. Extended constitutive relations are formulated either as differential equations, involving $\dot{\mathbf{T}}$ and/or $\dot{\mathbf{e}}$, or in terms of integrals of creep and relaxation functions. This leads to the occurrence of a time lag between forced oscillations of strain and stress. In this respect, the situation becomes similar to a periodically driven harmonic oscillator with damping: due to a lag between velocity and position, the time derivative of potential energy integrated over a forcing cycle does not vanish and generates a power deficit credited by the external forcing in order to sustain stationary oscillations. In the case of our study, the power supply comes from the kinetic energy K_0 and is not unlimited. Moreover, a coupling exists between the power supply and demand, because the amplitude of stationary vibrations depends on the excess of kinetic energy over the ground state of $\frac{1}{2}I_3\omega^2$.

3.2 Quality factor principle

Introducing the quality factor Q as an empirical parameter, one may, in principle, discuss the energy dissipation in vibrating inelastic materials without any explicit knowledge of their constitutive relations. In practice, however, the problem of unknown rheology leaks into the question of a proper definition of Q and of its dependence on driving frequency, temperature, etc. (Efroimsky & Williams 2009). In the limit, a perfect definition of Q is probably not easier than defining an adequate rheological model and solving the related inelastic oscillation problem.

O'Connell & Budiansky (1978) tried to put some order into a growing number of different Q definitions. They warned about 'confusion between some ill-defined Q of a process' and 'an intrinsic Q of the material'. Their generally acclaimed definition of quality factor as the ratio of real and imaginary parts of compliance leads 'for a large class of viscoelastic materials' to the common rephrasing as

$$Q = 2\pi\frac{(2E_\mathrm{av})}{\Delta E}, \tag{41}$$

where ΔE is the energy lost during one period of a harmonic (pure sine) loading, and E_av is the 'average stored energy' during the loading cycle (O'Connell & Budiansky 1978). The difficult point of this apparently simple definition is how to interpret E_av in some particular problem, even if we use the strain energy (38), (39) for this purpose.

Sharma et al. (2005) complained that 'definitions of measures of energy fluctuations corresponding to the type of loadings encountered with tumbling bodies are not readily available', meaning the presence of a constant term in body forces that are not purely periodic. Although their main solution followed the usual habit of dropping the constant part from strain and stress when plugging U_e into the definition (41), they express serious doubts and find 'no easily identifiable reason' for it, except that of comparison with earlier works. They proposed an alternative approach including the effect of the average $\langle\boldsymbol{b}\rangle$. When starting this work, we did share the doubts of Sharma et al. (2005) and were attracted by the alternative way of estimating the fluctuating energy amount. But, having rejected them at a later stage, we feel obliged to explain our point of view.

Consider the example given by Sharma et al. (2005): a scalar stress $T = a_0 + a_1\sin t$. There is no doubt that $\langle T^2\rangle$ with $a_0 \neq 0$ is different than for $a_0 = 0$. Moreover, with $a_0 \neq 0$, the plot of $U_\mathrm{e}(t)$ can be dominated by $\sin t$, whereas dropping the constant part we obtain only $\cos 2t$. However, a similar situation is met in a damped

harmonic oscillator driven by $a_0 + a_1 \cos t$: the potential energy of the stationary solution has the same dependence on a_0, but the power loss is completely independent of a_0. The statement that only a time variable part of stress may dissipate the energy can be found already in the paper of Prendergast (1958). There is no reason to doubt it. Thus, the only question is: does the dissipation by the periodic part of the stress depend on the constant part? We cannot rule out such possibility if relations between stress and strain are strongly nonlinear or the dissipation mechanism is complicated, but – on the other hand – it is quite unlikely that in these circumstances a simple term $a_0 a_1 \sin t$ will properly describe the dependence, as Sharma et al. (2005) suggest, mentioning a friction due to grain boundary or crack surface sliding. So, the alternative estimate of the stored energy, as proposed by Sharma, neither looks promising far from the almost-linear, weakly damped elasticity model, nor it behaves properly within this area – the latter readily seen if we consider the harmonic oscillator. Moreover, it does not account for the fact that a kth harmonic in a general harmonic load works k times during the fundamental period – the comment that applies both the main and alternative solution of Sharma et al. (2005). As a minor remark, we can add the contradiction between qualifying gravitation as an ignorable pre-stress and mean centrifugal force as a factor that contributes to the energy dissipation, present in Sharma et al. (2005). The alternative recipe for $E_{\rm av}$ has a mathematical meaning, but in our opinion, it will define an alternative 'Q of a process' which may be too far from the 'intrinsic Q' in terms of numerical values and dependence or independence on parameters.

We believe that the main line of reasoning and the warnings issued by O'Connell & Budiansky (1978) are sufficient to formulate the recipe for $E_{\rm av}$ leading to a reasonable, material-based quality factor. The rule seems to be: stay close to the property that, for sufficiently high Q values,

$$Q^{-1} \approx \tan \varphi, \tag{42}$$

where φ is the phase lag angle between the stress and the strain (the so-called loss angle). This rule validates the separation of contributions from subsequent harmonics of body forces $\boldsymbol{b}$, as well as the rejection of its constant part in the stress solution, i.e. the procedure of Efroimsky (2000). Note that for a single harmonic, taking twice the mean value of its square, we obtain its squared amplitude which explains why $2E_{\rm av}$ in equation (41) is often replaced by the 'peak energy' $E_{\rm p}$ (O'Connell & Budiansky 1978; Lambeck 1980).

Energy is dissipated by the work of body forces. The basic formula for the work rate $\dot{W}$ reads

$$\dot{W} = \int_V \dot{\epsilon}\, {\rm d}V = \int_V T_{ij} \dot{e}_{ij}\, {\rm d}V. \tag{43}$$

In the conservative, elastic case, when stress and strain are defined by equations (32) and (9), depending on time-periodic functions B_{ij}, the integral over the fundamental period of wobbling renders no work, because each term of the sum $T_{ij}\dot{e}_{ij}$ is a purely periodic function of time.

Assumption 7. Inelastic oscillations will be described by the following, heuristic approximation.

(i) Periodic terms of B_{ij} in the stress tensor $\mathbf{T}$ are taken directly from definitions (27) and (28).

(ii) Periodic terms of B_{ij} in the strain tensor $\mathbf{e}$, derived using equation (9), are modified by adding a phase lag φ to the argument of each harmonic, and dividing the amplitudes by $\cos \varphi$.

(iii) The phase lag is independent of the coordinates x, y, z, and related to the quality factor by equation (42).

(iv) The quality factor is independent of the frequency of forcing terms.

Let the fundamental frequency of wobbling be Ω, with an associated period

$$P = \frac{2\pi}{\Omega}, \tag{44}$$

and consider an exemplary product of periodic terms appearing in equation (43) under Assumption 7:

$$p_{\rm ex} = \Big(c_p \cos(p\Omega t) + s_p \sin(p\Omega t)\Big) \frac{\rm d}{{\rm d}t} \left(\frac{c_q}{\cos\varphi} \cos(q\Omega t - \varphi) + \frac{s_q}{\cos\varphi} \sin(q\Omega t - \varphi) \right). \tag{45}$$

Straightforward computation leads to the conclusion that the time integral over the period P vanishes for $p \neq q$, whereas $p = q \neq 0$ leads to

$$\int_0^P p_{\rm ex}\, {\rm d}t = p\pi \left(c_p^2 + s_p^2\right) \tan\varphi = p \frac{2\pi}{Q} \Big\langle \Big(c_p \cos(p\Omega t) + s_p \sin(p\Omega t)\Big)^2 \Big\rangle. \tag{46}$$

This example establishes a link between Assumption 7 and the operational rule of computing the energy loss due to the work per cycle P as the sum

$$\Delta E = -\int_0^P \dot{W}\, {\rm d}t = -\frac{2\pi}{Q} \sum_{p\geq 1} p \Big\langle 2\, U_p \Big\rangle, \tag{47}$$

in agreement with equation (41), where U_p is the part of elastic energy $U_{\rm e}$ from equation (38) involving only the pth harmonic in each B_{ij} term entering strain and stress.

A recent in-depth critical review of some issues concerning Assumption 7 can be found in (Efroimsky 2012).

4 ENERGY DISSIPATION RATE

4.1 Volume integration

The energy dissipation rate can be obtained from equation (47) as

$$\dot{E} = \frac{\Delta E}{P} = -\frac{2\Omega}{Q} \sum_{p\geq 1} p \Big\langle U_p \Big\rangle. \tag{48}$$

Having assumed the independence of φ and Q on coordinates, we can perform the volume integration required for U_p before the time average. This allows for a considerable economy of expressions; substituting the general form of $\mathbf{T}$ from equation (32) into equation (40), making use of the expressions for $\mathbf{A}$, $\mathbf{A}^{ij}$ from Appendix A, plugging in the definition of $\mathbf{B}$ in (27), (28) and integrating over the ellipsoid volume according to equation (19), we find

$$\begin{aligned} \Big\langle U_p \Big\rangle = \frac{a^4 \rho\, m}{\mu} \Big(&\alpha_{11} \Big\langle \big[\omega_1^2\big]_p^2 \Big\rangle + \alpha_{22} \Big\langle \big[\omega_2^2\big]_p^2 \Big\rangle + \alpha_{33} \Big\langle \big[\omega_3^2\big]_p^2 \Big\rangle \\ &+ \alpha_{12} \Big\langle \big[\omega_1^2\big]_p \big[\omega_2^2\big]_p \Big\rangle + \alpha_{13} \Big\langle \big[\omega_1^2\big]_p \big[\omega_3^2\big]_p \Big\rangle \\ &+ \alpha_{23} \Big\langle \big[\omega_2^2\big]_p \big[\omega_3^2\big]_p \Big\rangle + \beta_{12} \Big\langle \big[\omega_1\omega_2\big]_p^2 \Big\rangle \\ &+ \beta_{13} \Big\langle \big[\omega_1\omega_3\big]_p^2 \Big\rangle + \beta_{23} \Big\langle \big[\omega_2\omega_3\big]_p^2 \Big\rangle \Big), \end{aligned} \tag{49}$$

where m is the ellipsoid mass and α_{ij}, β_{ij} are dimensionless rational functions of h_1, h_2 and ν. For any function F represented as the Fourier series, the symbol $[F]_p$ designates its pth harmonic, i.e. a trigonometric monomial with argument $p\Omega t$.

Regretfully, the full form of α_{ij} and β_{ij} is too long to be explicitly quoted (although short enough to be efficiently programmed), but the expressions are available from the authors in an electronic form.

4.2 Fourier harmonics of angular velocity

Equation (49) requires the recall of basic facts about the free rotation of the rigid body (Whittaker 1952) for the specific case of a homogeneous ellipsoid. As usually, we have to distinguish the short-axis mode (SAM) of rotation, when the angular velocity vector $\boldsymbol{\omega}$ circulates around $\boldsymbol{e}_3$, and the long-axis mode (LAM), when $\boldsymbol{\omega}$ circulates around $\boldsymbol{e}_1$. Quantities referring to the SAM will have subscript $s = 3$, and those for the LAM, subscript $s = 1$ (in this paper, we use s exclusively for labelling the rotation mode).

Using two invariants of the free top, i.e. kinetic energy K_0 and angular momentum $\boldsymbol{H}$ (equations 36 and 37), with the diagonal matrix of inertia I given by equation (22), we define an auxiliary quantity $\mathcal{A}$ (Deprit & Elipe 1993; Breiter et al. 2011)

$$\mathcal{A} = \frac{2K_0}{H^2}, \tag{50}$$

such that $a_3 \le \mathcal{A} \le a_1$, where

$$a_i = I_i^{-1}. \tag{51}$$

A nominal angular velocity of rotation Ha_3 is often adopted as a scaling factor in the energy dissipation models. Its value would be equal to ω only in the principal axis rotation around $\boldsymbol{e}_3$. We prefer to use a quantity appropriate for the LAM as well, so let us define the nominal angular rates as

$$\tilde{\omega}_j = Ha_j, \tag{52}$$

such that each $\tilde{\omega}_j = \omega$ in the principal axis rotation around $\boldsymbol{e}_j$.

Let us define

$$n_3 = \sqrt{(a_1 - \mathcal{A})(a_2 - a_3)}, \qquad n_1 = \sqrt{(a_1 - a_2)(\mathcal{A} - a_3)}. \tag{53}$$

Then, the components of angular velocity ω_i in the body fixed frame are expressible in terms of the Jacobi elliptic functions with argument $\tau_s = Hn_s\, t$ and modulus k_s, where

$$k_3 = \frac{n_1}{n_3} = \frac{1}{k_1}. \tag{54}$$

Thus, in any rotation mode,

$$\begin{aligned}
\omega_1 &= \tilde{\omega}_1 \sqrt{\frac{\mathcal{A} - a_3}{a_1 - a_3}}\, F_s^{(1)},\\
\omega_2 &= \tilde{\omega}_2 \sqrt{\frac{\mathcal{A} - a_3}{a_2 - a_3}}\, F_s^{(2)},\\
\omega_3 &= \tilde{\omega}_3 \sqrt{\frac{a_1 - \mathcal{A}}{a_1 - a_3}}\, F_s^{(3)},
\end{aligned} \tag{55}$$

with specific functions

$$F_3^{(1)} = \pm\mathrm{cn}(\tau_3, k_3), \quad F_3^{(2)} = \mathrm{sn}(\tau_3, k_3), \quad F_3^{(3)} = \pm\mathrm{dn}(\tau_3, k_3) \tag{56}$$

and

$$F_1^{(1)} = \pm\mathrm{dn}(\tau_1, k_1), \quad F_1^{(2)} = \pm k_1\,\mathrm{sn}(\tau_1, k_1), \quad F_1^{(3)} = \mathrm{cn}(\tau_1, k_1) \tag{57}$$

for the SAM and LAM, respectively.

The fundamental frequency of wobbling, appearing in equation (44) as Ω, is

$$\Omega_s = \frac{\pi H n_s}{2\mathrm{K}_s}, \tag{58}$$

where K_s stands for the complete elliptic integral of the first kind:

$$\mathrm{K}_s = \mathrm{K}(k_s) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{1 - k_s^2 \sin^2 x}}. \tag{59}$$

Similarly, we will use $\mathrm{E}_s = \mathrm{E}(k_s)$ for the complete elliptic integral of the second kind.

In this work, we are interested only in the squares and products of angular velocity components. Given the Jacobi elliptic functions with argument τ_s and modulus k_s, we can expand their squares and products in the Fourier series of angle

$$\psi_s = \frac{\pi}{2\mathrm{K}_s}\,\tau_s = \Omega_s\, t. \tag{60}$$

From the expressions available in Byrd & Friedman (1954), we can easily derive the series

$$\mathrm{sn}^2(\tau_s, k_s) = X_0 - \sum_{p=1}^{\infty} X_{2p} \cos 2p\psi_s, \tag{61}$$

$$\mathrm{cn}^2(\tau_s, k_s) = 1 - X_0 + \sum_{p=1}^{\infty} X_{2p} \cos 2p\psi_s, \tag{62}$$

$$\mathrm{dn}^2(\tau_s, k_s) = 1 - k_s^2 X_0 + k_s^2 \sum_{p=1}^{\infty} X_{2p} \cos 2p\psi_s, \tag{63}$$

$$\mathrm{sn}(\tau_s, k_s)\,\mathrm{cn}(\tau_s, k_s) = \sum_{p=1}^{\infty} Y_{2p} \sin 2p\psi_s, \tag{64}$$

$$\mathrm{cn}(\tau_s, k_s)\,\mathrm{dn}(\tau_s, k_s) = k_s \sum_{p=1}^{\infty} X_{2p-1} \cos(2p-1)\psi_s, \tag{65}$$

$$\mathrm{sn}(\tau_s, k_s)\,\mathrm{dn}(\tau_s, k_s) = k_s \sum_{p=1}^{\infty} Y_{2p-1} \sin(2p-1)\psi_s, \tag{66}$$

with coefficients

$$X_0 = \frac{1}{k_s^2}\left(1 - \frac{\mathrm{E}_s}{\mathrm{K}_s}\right), \tag{67}$$

$$X_j = \left(\frac{\pi}{k_s\,\mathrm{K}_s}\right)^2 \frac{j\,\mathrm{q}_s^{j/2}}{1 - \mathrm{q}_s^j}, \tag{68}$$

$$Y_j = \frac{1 - \mathrm{q}_s^j}{1 + \mathrm{q}_s^j}\, X_j, \tag{69}$$

involving Jacobi's nome

$$\mathrm{q}_s = \exp\left(-\pi \mathrm{K}'_s/\mathrm{K}_s\right), \tag{70}$$

where $\mathrm{K}'_s = \mathrm{K}(k'_s)$ is the elliptic integral with complementary modulus

$$k'_s = \sqrt{1 - k_s^2}. \tag{71}$$

The quickly convergent series for the nome

$$\mathrm{q}_s = \zeta_s \left(1 + 2\zeta_s^4 + 15\,\zeta_s^8 + 150\,\zeta_s^{12} + 1707\,\zeta_s^{16} + \cdots\right), \tag{72}$$

with

$$\zeta_s = \frac{1 - \sqrt{k'_s}}{2\left(1 + \sqrt{k'_s}\right)} \tag{73}$$

can be used (Innes 1902; Byrd & Friedman 1954). Even close to the separatrix ($\mathcal{A} \approx a_2$), the first five terms provide a relative error

of 10^{-6} for $k_s = 0.999$. However, for the values of k_s close to 1, it is better to use

$$\zeta_s' = \frac{1-\sqrt{k_s}}{2\left(1+\sqrt{k_s}\right)}, \tag{74}$$

on the right-hand side of (72) instead of ζ, and obtaining the complementary nome q_s', which may serve to compute q_s through the relation

$$\ln \mathrm{q}_s \, \ln \mathrm{q}_s' = \pi^2. \tag{75}$$

An additional benefit of the nome is also a quickly convergent series for the elliptic integral

$$\mathrm{K}_s = \frac{\pi}{2}\left(1 + 2\sum_{p=1}^{\infty} \mathrm{q}_s^{p^2}\right)^2. \tag{76}$$

The mean values required in equation (49) follow directly from the presented Fourier series. For example, in the SAM

$$\left\langle \left[\omega_1^2\right]_{2p}^2 \right\rangle = \tilde{\omega}_1^4 \left(\frac{\mathcal{A}-a_3}{a_1-a_3}\right)^2 \left\langle \left(X_{2p}\cos 2p\psi_3\right)^2 \right\rangle = \frac{\tilde{\omega}_1^4}{2}\left(\frac{\mathcal{A}-a_3}{a_1-a_3}\right)^2 X_{2p}^2, \tag{77}$$

$$\left\langle \left[\omega_1^2\right]_{2p-1}^2 \right\rangle = 0, \tag{78}$$

and in the LAM

$$\left\langle \left[\omega_1^2\right]_{2p}^2 \right\rangle = \frac{\tilde{\omega}_1^4}{2}\left(\frac{\mathcal{A}-a_3}{a_1-a_3}\right)^2 k_1^4 X_{2p}^2, \qquad \left\langle \left[\omega_1^2\right]_{2p-1}^2 \right\rangle = 0, \tag{79}$$

with X_{2p} depending on q_3 and q_1, respectively. Note that some of the mean values can be negative, like

$$\left\langle \left[\omega_1^2\right]_{2p} \left[\omega_2^2\right]_{2p} \right\rangle = -\frac{\tilde{\omega}_1^2\tilde{\omega}_2^2}{2}\frac{(\mathcal{A}-a_3)^2}{(a_1-a_3)(a_2-a_3)} X_{2p}^2, \tag{80}$$

in the SAM. However, their associated α_{ij} are also negative, so there is no subtraction in equation (49).

4.3 Final expressions

Performing the necessary substitutions in equation (48), we find an expression for the energy loss rate

$$\dot{E}_s = -\frac{a^4 \rho\, m \tilde{\omega}_s^5}{\mu\, Q}\, \Psi_s(k_s, h_1, h_2, \nu) \tag{81}$$

with dimensionless

$$\Psi_3 = Z_3^5 \left(P_1(k_3)M_{13} + P_2(k_3)M_{23} + P_3(k_3)M_0 + P_4(k_3)M_{12}\right), \tag{82}$$

$$\Psi_1 = Z_1^5 \left(P_1(k_1)M_{13} + P_2(k_1)M_{12} + P_3(k_1)M_0 + P_4(k_1)M_{23}\right) \tag{83}$$

(note the swapped M_{23} and M_{12}), where

$$Z_s = \frac{\Omega_s}{\tilde{\omega}_s} = \frac{\pi n_s}{2 a_s \mathrm{K}_s}. \tag{84}$$

First, we recall that the leading factor depends on the semi-major axis of ellipsoid a, its mass m, density ρ, the fifth power of the nominal rotation rate $\tilde{\omega}_s$ (resulting from the division of angular momentum H by the related moment of inertia), Lamé shear modulus μ and quality factor Q. Functions $P_i(k_s)$ depend on the ratio of kinetic energy and angular momentum through an elliptic modulus k_s that enters Jacobi's nome q_s and have the form of infinite sums

$$P_1(k_s) = \sum_{p=1}^{\infty} \frac{(2p-1)^3 \mathrm{q}_s^{2p-1}}{\left(1-\mathrm{q}_s^{2p-1}\right)^2}, \tag{85}$$

$$P_2(k_s) = \sum_{p=1}^{\infty} \frac{(2p-1)^3 \mathrm{q}_s^{2p-1}}{\left(1+\mathrm{q}_s^{2p-1}\right)^2}, \tag{86}$$

$$P_3(k_s) = \sum_{p=1}^{\infty} \frac{(2p)^3 \mathrm{q}_s^{2p}}{\left(1-\mathrm{q}_s^{2p}\right)^2}, \tag{87}$$

$$P_4(k_s) = \sum_{p=1}^{\infty} \frac{(2p)^3 \mathrm{q}_s^{2p}}{\left(1+\mathrm{q}_s^{2p}\right)^2}, \tag{88}$$

although in practice only a few leading terms should be sufficient. Finally, M_{ij} and M_0 are dimensionless, positive coefficients depending only on the shape (through h_1, h_2) and on Poisson's ratio ν. In terms of the coefficients from equation (49), they are

$$M_{ij} = \frac{16\, a_i^2 a_j^2 \beta_{ij}}{d_{12}d_{13}d_{23}d_{ij}}, \tag{89}$$

$$M_0 = \frac{16}{d_{12}d_{13}d_{23}}\left(\frac{a_1^4 d_{23}\alpha_{11}}{d_{12}d_{13}} + \frac{a_2^4 d_{13}\alpha_{22}}{d_{12}d_{23}} + \frac{a_3^4 d_{12}\alpha_{33}}{d_{13}d_{23}} - \frac{a_1^2 a_2^2 \alpha_{12}}{d_{12}} + \frac{a_1^2 a_3^2 \alpha_{13}}{d_{13}} - \frac{a_2^2 a_3^2 \alpha_{23}}{d_{23}}\right). \tag{90}$$

where $d_{ij} = (a_i - a_j)$. Appendix B contains the full expressions of M_{ij} and M_0. For the reasons explained in the next section, we give them with the fixed Poisson's ratio $\nu = 0.25$.

4.4 Poisson's ratio

All recent models of spin-axis relaxation assume Poisson's ratio $\nu = 0.25$, i.e. equal Lamé constants $\lambda = \mu$. The authors justify it by the fact that this is approximately a typical value for most of cold solids (Efroimsky 2000). Earlier, Prendergast (1958) considered an incompressible object with $\nu = 0.5$. Only Molina et al. (2003) maintain the explicit dependence on ν in their final formulae for a spheroid.

The present model also maintains ν in the final expressions, so we are in a favorable situation to estimate the sensitivity of $\dot{E}_s$ on its value. Interestingly, in contrast to the results of Molina et al. (2003), the dependence of M_{ij} and M_0 on Poisson's occurs to be very weak. As a function of $0 \le \nu \le 0.5$, the values of M coefficients vary on the level of at most 10^{-2} (relatively), whereas the solution of Molina et al. (2003) exhibits the dependence on the level of 10^{-1}. This property came unexpected, because α_{ij} still contained a factor $(1-\nu^2)^{-1}$ that later vanished in M_0. In these circumstances, we fix the value of $\nu = 0.25$ as a physically realistic one, which considerably simplifies expressions, but the results will fairly well apply to an incompressible case with $\nu = 0.5$.

5 WOBBLE DAMPING TIME

Let us define a 'wobbling angle' θ_s as the maximum angle between the angular momentum vector $\boldsymbol{H}$ and a relevant axis (Oz in the SAM or Ox in the LAM) attained during the wobbling cycle of a rigid body, namely

$$\theta_s = \max\left(\arccos\left|\frac{\boldsymbol{H}\cdot\boldsymbol{e}_s}{H}\right|\right). \tag{91}$$

In the principal axis rotation $\theta_s = 0$, regardless of the prograde or retrograde case. The other limit, $\theta_s = 90^\circ$, refers to unstable rotation around an intermediate axis of inertia Oy or to the nonperiodic rotation on the separatrix, which falls beyond our model.

There is a direct relation between θ_s and the variable $\mathcal{A}$:

$$\mathcal{A}_s = a_2 - (a_2 - a_s)\cos^2\theta_s. \tag{92}$$

We have introduced the subscript s to $\mathcal{A}$ in order to facilitate the distinction of modes, although the primary definition (50) is universal. The modulus k_s is also related to the wobbling angle through

$$k_s = \frac{\sin\theta_s}{\sqrt{1+\kappa_s\cos^2\theta_s}}, \tag{93}$$

$$\kappa_3 = \frac{1}{\kappa_1} = \frac{a_2 - a_3}{a_1 - a_2} = h_1^4\,\frac{1-h_2^4}{1-h_1^4}. \tag{94}$$

Since the dissipation of energy does not affect angular momentum, and the energy is drained from the kinetic K_0, the differentiation of equation (50) leads to

$$\dot{\mathcal{A}}_s = \frac{2\,\dot{E}_s}{H^2} = -\frac{2a^4\rho\, m\tilde{\omega}_s^3}{\mu\, Q}\, a_s^2\Psi_s, \tag{95}$$

where $\dot{E}_s$ has been taken from equation (81).

On the other hand,

$$\dot{\mathcal{A}}_s = 2\,(a_2 - a_s)\sin\theta_s\cos\theta_s\dot{\theta}_s, \tag{96}$$

and we can equate the two relations, obtaining the differential equation

$$\frac{\mathrm{d}\theta_s}{\mathrm{d}t} = \frac{\dot{E}_s}{H^2\,(a_2 - a_s)\sin\theta_s\cos\theta_s}. \tag{97}$$

The resulting quadrature gives the time T_s required to change the wobbling angle from the initial θ_s^0 to the final θ_s':

$$T_s = \frac{(a_s - a_2)\,\mu Q}{a^4\rho m\tilde{\omega}_s^3 a_s^2}\int_{\theta_s^0}^{\theta_s'}\frac{\sin\theta_s\cos\theta_s}{\Psi_s}\,\mathrm{d}\theta_s, \tag{98}$$

where Ψ_s should be expressed in terms of the wobbling angle using equation (93).

In particular,

$$T_3 = -\frac{\mu Q}{a^2\rho\tilde{\omega}_3^3}\left[\frac{h_1^2\left(1+h_1^2\right)\left(1-h_2^2\right)}{5\,\left(1+h_1^2h_2^2\right)}\right]\int_{\theta_3^0}^{\theta_3'}\frac{\sin\theta_3\cos\theta_3}{\Psi_3}\,\mathrm{d}\theta_3, \tag{99}$$

implying $\theta_3' \le \theta_3^0$ and

$$T_1 = \frac{\mu Q}{a^2\rho\tilde{\omega}_1^3}\left[\frac{h_1^2\left(1-h_1^2\right)\left(1+h_2^2\right)}{5\left(1+h_1^2h_2^2\right)}\right]\int_{\theta_1^0}^{\theta_1'}\frac{\sin\theta_1\cos\theta_1}{\Psi_1}\,\mathrm{d}\theta_1, \tag{100}$$

with $\theta_1^0 \le \theta_1'$. For the reference with earlier works, we will use a shape parameter D_s (Sharma et al. 2005) defined as

$$D_s(h_1, h_2) = T_s\frac{a^2\rho\tilde{\omega}_s^3}{\mu Q} \tag{101}$$

for prescribed integration limits.

Of course, the notion of wobble damping time is properly related to T_3 or, if the evolution starts in the LAM and continue through the SAM, to the sum $T_1 + T_3$. In the LAM, energy dissipation excites wobbling, driving the angular momentum vector towards the separatrix. A term 'excitation time' seems more appropriate for T_1.

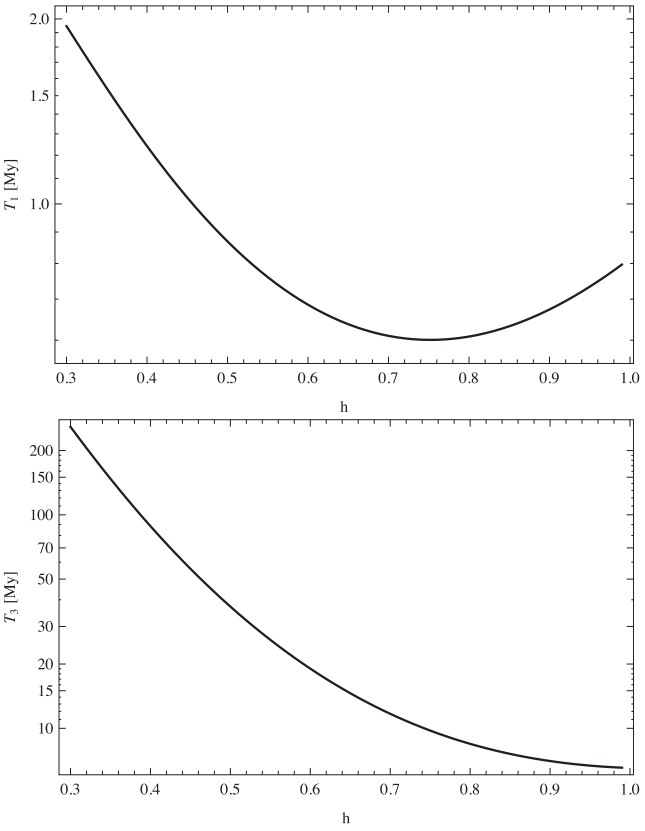

Figure 1. Logarithmic plots of wobble damping/excitation times for ellipsoids with $h_1 = h_2 = h$. Top: time spent in the LAM with θ_1 increasing from 5° to 85°; bottom: time spent in the SAM with θ_3 decreasing from 85° to 5°. Physical parameters – see the text.

As an illustrative example, we first plot damping/excitation times for a family of ellipsoids with $h_1 = h_2 = h$, assuming sample physical data $a = 1$ km, $\rho = 2000\,\mathrm{kg\,m^{-3}}$, $\mu = 10^9$ Pa and $Q = 100$, chosen more for an ease of scaling than for the reference to some specific case. We have assumed $\tilde{\omega}_3 = 2\pi/10^{\mathrm{h}}$ for T_3 and computed an equivalent

$$\tilde{\omega}_1 = \frac{1+h_1^2}{h_1^2\,\left(1+h_2^2\right)}\tilde{\omega}_3 \tag{102}$$

to be used in T_1.

Interested in a possibly complete history, we start the evolution at the LAM, with $\theta_1^0 = 5^\circ$ and integrate equation (100) up to $\theta_1' = 85^\circ$. The results are displayed in Fig. 1 (top). The dependence of T_1 on h is not monotonic: the shortest excitation time, 0.6 Myr, occurs at $h = 0.75$. Increasing asphericity, we reach $T_1 \approx 2$ Myr at $h = 0.3$. The other extreme would be $h = 1$, but we stop at $h = 0.99$, because the results would be meaningless for a sphere. After crossing the separatrix, the angular momentum vector is driven towards $\boldsymbol{e}_3$ and we computed the damping times T_3 take to evolve from $\theta_3^0 = 85^\circ$ to 5°. This time (Fig. 1, bottom), damping times are much longer than in the LAM for the same shape. T_3 is as high as 258 Myr at $h = 0.3$ and systematically decreases to 6.5 Myr at $h = 0.99$. Thus, the total damping time consists mostly in T_3 and – fixing the semi-axis a – we find that triaxiality inhibits the total damping process, save for a quick passage through the LAM, where we find a minimum at $h_1 = h_2 = 0.75$.

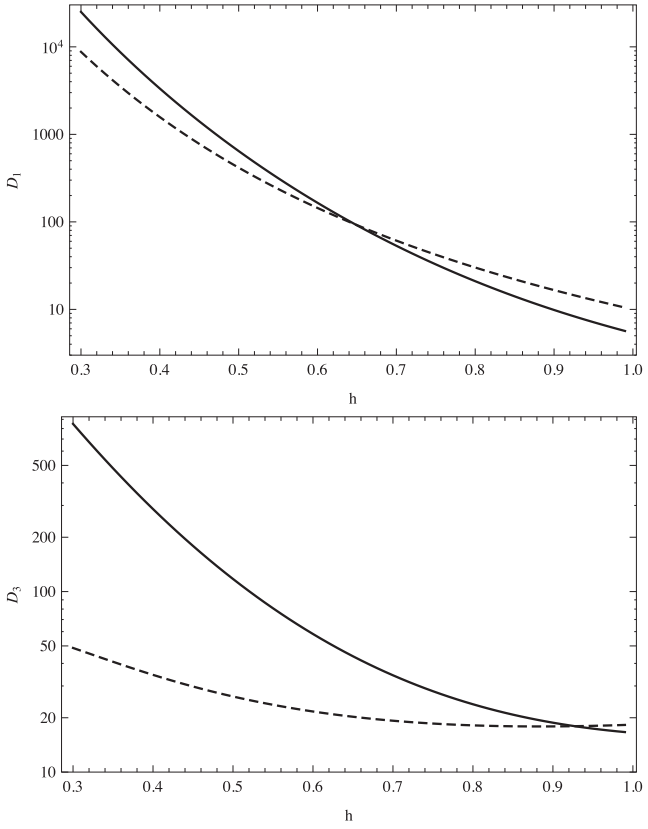

Figure 2. Logarithmic plots of shape functions D_s. Solid line: ellipsoids with $h_1 = h_2 = h$; dashed line: spheroids with $h_1 = h$ (LAM – top) or with $h_2 = h$ (SAM – bottom).

Integration limits in the above example have been wider than usually. For the reference with earlier works, we have also computed shape factors D_s for the limits $45^\circ \leq \theta_1 \leq 85^\circ$ and $45^\circ \geq \theta_3 \geq 5^\circ$ like Sharma et al. (2005). Fig. 2 confronts D_s for a family of $h_1 = h_2 = h$ ellipsoids with spheroids having an appropriate ratio h_1 (SAM) or h_2 (LAM) equal to 1, and the other one set as h. We note a systematic increase of D_s with increasing h for ellipsoids similar to T_3, but unlike T_1 from Fig. 1. For oblate spheroids there is a shallow minimum of D_3 close to $h \approx 0.9$. These discrepancies are not essential and result from an arbitrary choice of integration limits that cut off some more or less prominent (depending on the shape) parts of an integrand.

Integrated damping/excitation time is important for qualitative considerations, but if the joint action of inelastic dissipation and other torques is to be studied, the shape of $\Psi_s(\theta_s)$ becomes more interesting. Figs 3 and 4 demonstrate, how the ellipsoid's shapes affect $\Psi_1(\theta_1)$ and $\Psi_3(\theta_3)$. Each curve is normalized, i.e. Ψ_s values are divided by w – the mean value of Ψ_s with respect to θ_s on the interval $[0, \pi/2]$. The situation is a bit different in the LAM (Fig. 3) and SAM (Fig. 4). Departure from a spheroid (solid line) weakens the dissipation close to the principal axis and amplifies it for higher θ_3 in the SAM. In the LAM, this is not the rule, as seen for $h_1 = 0.3$ and $h_2 = 0.7$.

6 REDUCTION TO SPHEROID AND COMPARISON WITH OTHER WORKS

In the previous section, we have presented some results for spheroids. They could be computed by assuming h_1 or h_2 sufficiently close to 1, but in the strict limit the expressions involve singular factors. However, this singularity is only apparent. The point is that although $k_1 = q_1 = 0$ for $h_2 = 1$ and $k_3 = q_3 = 0$ for $h_1 = 1$ (hence all $P_k(k_s) = 0$), after we expand $P_k(k_s)$ in powers of k_s, substitute (93) and multiply by M_{ij}, some factors $(1 - h_1^2)$ or $(1 - h_2^2)$ cancel, leaving a well-defined limit for a spheroid. The resulting SAM expression for $h_1 = 1$, $h_2 = h$ is

$$\Psi_3(h, \theta_3) = \frac{8\left(1-h^2\right)\sin^2\theta_3\cos\theta_3}{35\left(1+h^2\right)^5}\left(2h^4C\cos^2\theta_3 + S\sin^2\theta_3\right), \tag{103}$$

where

$$C = \frac{26+35\,h^2}{13+20\,h^2}, \tag{104}$$

$$S = \frac{25+20\,h^2+16\,h^4}{15+10\,h^2+8\,h^4}, \tag{105}$$

and the LAM expression for $h_1 = h$, $h_2 = 1$ is simply

$$\Psi_1(h, \theta_1) = -h^4\,\Psi_3(h^{-1}, \theta_1). \tag{106}$$

The factor h^4 marks the difference between the LAM of the $b = c < a$ spheroid (present work) and a prolate $a = b < c$ spheroid used by other authors.

We have found it interesting to compare our solution with other published results. Remarkably, the latter can be reduced to the same

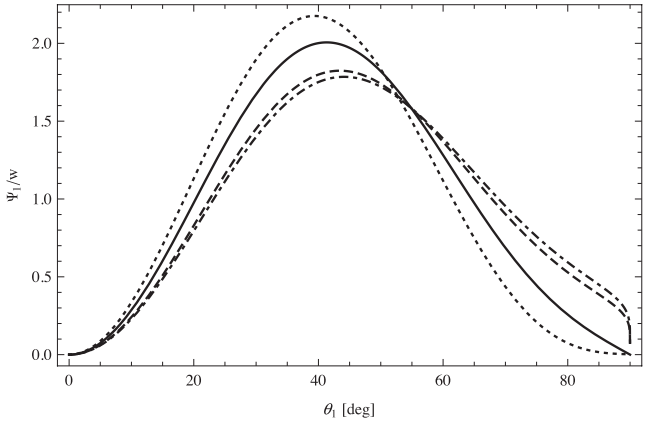

Figure 3. Normalized LAM energy dissipation rate functions $\Psi_1(\theta_1)/w$. Solid line: $(h_1, h_2, w) = (0.7, 1, 3000)$; dashed: (0.7, 0.7, 4000); dotted: $(0.3, 0.7, 9.6 \times 10^5)$; dot–dashed: (0.7, 0.3, 5200).

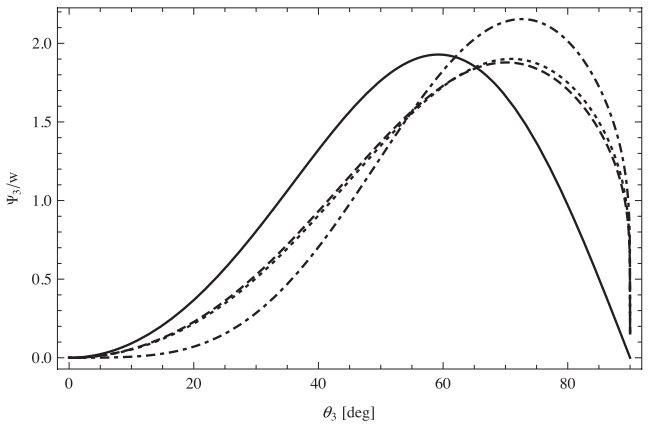

Figure 4. Normalized SAM energy dissipation rate functions $\Psi_s(\theta_s)/w$. Solid line: $(h_1, h_2, w) = (0.7, 1, 210)$; dashed: (0.7, 0.7, 530); dotted: $(0.3, 0.7, 1.6 \times 10^4)$; dot–dashed: (0.7, 0.3, 140).

general form (103) of Ψ_3, differing only with the particular expressions of the coefficients C and S.

Let us begin with Efroimsky & Lazarian (2000). Their solution for a rectangular prism with semi-edges $a = b$ and $c = ah$ has the form of equation (103) with

$$C = \frac{1323}{128}, \qquad S = \frac{105}{16}. \tag{107}$$

Obviously, the prism has a higher volume than an ellipsoid with the same a and h. In particular, the volume integral $\int x^4 \mathrm{d}V$ for a spheroid is smaller by a factor $\pi/14 \approx 0.224$. Thus, we propose to use

$$C = \frac{\pi}{14}\,\frac{1323}{128}, \qquad S = \frac{\pi}{14}\,\frac{105}{16}, \tag{108}$$

in equation (103) to make the comparison with a spheroid more even.

Molina et al. (2003) obtained for a spheroid

$$C = 1, \qquad S = \frac{2}{1+\nu}. \tag{109}$$

Since our solutions differ only by the choice of boundary conditions, we adopt for comparison the values from equation (109) with $\nu = 1/4$, i.e. $C = 1$ and $S = 8/5$.

The complete solution of Sharma et al. (2005) is known only indirectly through the coefficients provided by Sharma (private communication) and published in Vokrouhlický et al. (2007). In these circumstances, we have derived C and S using the stress tensor from Appendix D of Sharma et al. (2005) and following the recipe from their sections 5 and 6 (i.e. the mainline solution, not the alternative model). What we have obtained agrees with Vokrouhlický et al. (2007), and reads

$$C = \frac{1}{2}\left[\frac{26+35\,h^2}{13+20\,h^2}\right],$$
$$S = \frac{1}{4}\left[\frac{25+20\,h^2+16\,h^4}{15+10\,h^2+8\,h^4}\right], \tag{110}$$

where the factors in square brackets are the same as in our present solution (105). The difference in denominators 2 and 4 is easily understandable. First, the definition of Q used by Sharma et al. (2005) is based upon the mean value of energy; hence, their energy dissipation rate is twice as small as the one based upon our more common equation (41). On the other hand, Sharma et al. (2005) do not apply the multiplier p in the sum given by our formula (47), whereas S is directly related to the second harmonic of elastic energy. In these circumstances, we will use for comparison

$$C = \left[\frac{26+35\,h^2}{13+20\,h^2}\right], \quad S = \frac{1}{2}\left[\frac{25+20\,h^2+16\,h^4}{15+10\,h^2+8\,h^4}\right], \tag{111}$$

i.e. the same C as in our solution (105) and a half of our S.

Figs 5 and 6 present $\Psi_3(\theta_3)$ of the four solutions for oblate spheroids with $h = c/a = 0.9$ and 0.3. When the oblateness is moderate (Fig. 5), there is a reasonable proximity between the present model and a volume-scaled model of Efroimsky & Lazarian (2000), whereas the models of Molina et al. (2003) and Sharma et al. (2005) have maxima lower by 25 per cent than the present model and shifted with respect to each other by about 15°. Curiously, increasing the oblateness (Fig. 6), we find good agreement in the shape of the curves and differences of the maxima below 10 per cent, with the notable exception of Sharma et al. (2005) that dissipates energy twice as slow as the remaining models. A better agreement with Molina et al. (2003) for smaller h is understandable: the only difference with the present solution is due to different boundary

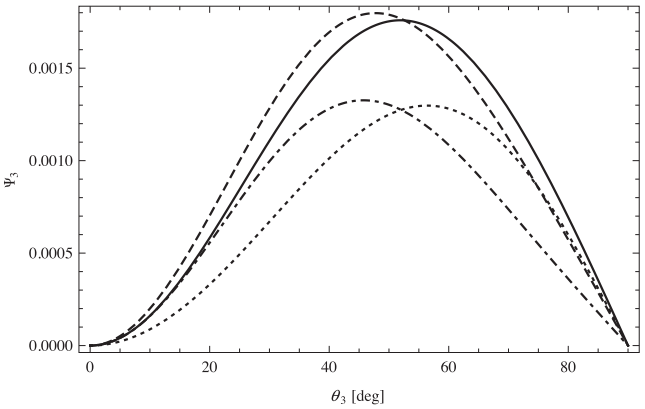

Figure 5. $\Psi_3(\theta_3)$ for a spheroid with $h = c/a = 0.9$. Solid line: present solution, dashed: volume-scaled Efroimsky & Lazarian (2000), dotted: Molina et al. (2003); dot–dashed: re-derived Sharma et al. (2005) for the equivalent Q definition.

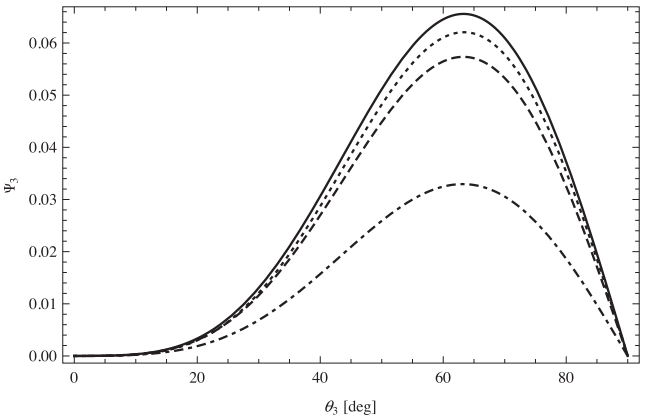

Figure 6. Same as Fig. 5 for $h = 0.3$.

conditions. They postulate a stress-free surface instead of the usual traction-free setup, and the deviatoric part of the stress tensor on the boundary in our solution decreases with h. Concerning the solution of Sharma et al. (2005), we find a systematic underestimation of Ψ_3 due to the missing multiplier of the second mode, with the ratio close to 2.

Let us recall the claims of Sharma et al. (2005) that their solution gives damping times longer than other ones by factor 10 or more, and that the difference is due to incomplete or incorrect solution of the elasticity problem in other papers. Why the differences in Figs. 5 and 6 are less drastic? At this point, we feel obliged to observe that for any h, numerical values of shape factors and damping times published and plotted in Sharma et al. (2005, Fig. 2) differ from the ones resulting from his energy dissipation formulae by a constant factor π. We have found no trace of this discrepancy in the published equations, so the difference, most likely, should be attributed to a purely computational error. Together with the incompatible definition of quality factor Q, it means that all damping times from Sharma et al. (2005) should be divided by 2π, so they are no longer to be considered unusually high. On the other hand, damping times shorter than Burns–Safronov estimates claimed by Efroimsky & Lazarian (2000) result partially (factor $14/\pi$) from

using an object (a prism) with a higher volume than any solid of revolution with the same ratio of axes.

7 CONCLUSIONS

Using the ensemble of standard assumptions, we have derived the stress tensor inside a freely rotating and self-gravitating ellipsoid. Writing about known solutions to this problem, Washabaugh & Scheeres (2002) put meaningful quotation marks around the word *available*. Our present solution, given in Appendix A has a form which is probably compact enough to suppress the marks, especially for the principal axis rotation. Interestingly, the presented form of $\mathbf{T}$ does not involve singularity at the incompressible limit $\nu = \frac{1}{2}$, where Washabaugh & Scheeres (2002) had to use $\nu = 0.499$.

The stress tensor has served us as the basis for the energy dissipation model built along the lines that Efroimsky (2002) proposed, but left unaccomplished. However, the use of an ellipsoid instead of an Efroimsky–Lazarian rectangular prism permitted us to avoid all objections related to partially satisfied boundary conditions and/or missing compatibility conditions. We have also found no reasons to impose superficial conditions of a stress-free boundary like Molina et al. (2003).

The solution hinges upon the use of the $P_k(\mathrm{q})$ series of the Jacobi nome q. Their convergence is very good and even taking $p \leq 4$ in equations (85)–(88) guarantees at least three significant digits in the area under $\Psi_s(\theta_s)$ for $0 < \theta_s < \pi/2$. Of course, the series are not legitimate exactly at $\mathrm{q}_s = 1$ (i.e. $\theta_s = 90^\circ$). An in-depth discussion of this limit was given by Efroimsky (2001). On the other hand, this state is not to be considered seriously, since any additional torque will trigger the emergence of a chaotic zone in the vicinity of separatrices.

In contrast to the results of Molina et al. (2003), we find that the role of compressibility in the energy dissipation process is marginal in the range of Poisson's ratio $0 \leqslant \nu \leqslant \frac{1}{4}$. Apparently, the stronger dependence obtained by Molina et al. (2003) for a spheroid resulted from too strong boundary conditions.

Investigating the spheroid as a particular case of our model, we have succeeded to resolve a major part of controversies concerning short damping times of Efroimsky & Lazarian (2000) and very long ones according to Sharma et al. (2005). In our opinion, the excess of energy dissipation rate over mainstream models is mostly due to a higher volume of the body shape assumed by Efroimsky & Lazarian (2000). The shape factors reported by Sharma et al. (2005) are overestimated mostly by an incompatible quality factor definition, a spurious factor π in their computations and a missing second-mode multiplier. We find the objections against boundary conditions used or compatibility condition violation, raised by Molina et al. (2003) or Sharma et al. (2005), formally justified, yet we note that they affect the accuracy of the damping/excitation times by at most 50 per cent.

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APPENDIX A: ELLIPSOID STRESS TENSOR

In order to shorten the expressions of the elements of $\mathbf{A}$ and $\mathbf{A}^{ij}$ in equation (32), we first introduce

$$B^{ijk} = \frac{1}{2}\left((-1)^i B_{11} + (-1)^j B_{22} + (-1)^k B_{33}\right) \tag{A1}$$

and

$$h_{12} = h_1 h_2. \tag{A2}$$

Then, we can explicitly define

$$A^{11}_{22} = h_1^2 A_{11} + 2A_{22} - h_2^{-2} A_{33} - h_1^2 B^{001}, \tag{A3}$$

$$A^{11}_{23} = 3A_{23} - h_{12}^2 B_{23}, \tag{A4}$$

$$A^{11}_{33} = h_{12}^2 A_{11} - h_2^2 A_{22} + 2A_{33} - h_{12}^2 B^{010}, \tag{A5}$$

$$A_{11}^{22} = 2A_{11} + h_1^{-2}A_{22} - h_{12}^{-2}A_{33} - B^{001}, \tag{A6}$$

$$A_{13}^{22} = 3A_{13} - h_{12}^2 B_{13}, \tag{A7}$$

$$A_{33}^{22} = -h_{12}^2 A_{11} + h_2^2 A_{22} + 2A_{33} - h_{12}^2 B^{100}, \tag{A8}$$

$$A_{11}^{33} = 2A_{11} - h_1^{-2}A_{22} + h_{12}^{-2}A_{33} - B^{010}, \tag{A9}$$

$$A_{12}^{33} = 3A_{12} - h_1^2 B_{12}, \tag{A10}$$

$$A_{22}^{33} = -h_1^2 A_{11} + 2A_{22} + h_2^{-2}A_{33} - h_1^2 B^{100}, \tag{A11}$$

$$A_{12}^{12} = -h_1\left(A_{11} + h_1^{-2}A_{22} - h_{12}^{-2}A_{33} - B^{001}\right), \tag{A12}$$

$$A_{13}^{12} = -h_1^{-1}\left(2A_{23} - h_{12}^2 B_{23}\right), \tag{A13}$$

$$A_{23}^{12} = -h_1\left(2A_{13} - h_{12}^2 B_{13}\right), \tag{A14}$$

$$A_{33}^{12} = 2h_{12}^2\left(2h_1^{-1}A_{12} - h_1 B_{12}\right), \tag{A15}$$

$$A_{11}^{23} = 2h_1^{-1}\left(2h_{12}^{-1}A_{23} - h_{12}B_{23}\right), \tag{A16}$$

$$A_{12}^{23} = -h_1\left(2h_{12}^{-1}A_{13} - h_{12}B_{13}\right), \tag{A17}$$

$$A_{13}^{23} = -h_{12}\left(2h_1^{-1}A_{12} - h_1 B_{12}\right), \tag{A18}$$

$$A_{23}^{23} = h_1 h_{12}\left(A_{11} - h_1^{-2}A_{22} - h_{12}^{-2}A_{33} + B^{100}\right), \tag{A19}$$

$$A_{12}^{13} = -2h_{12}^{-1}A_{23} + h_{12}B_{23}, \tag{A20}$$

$$A_{13}^{13} = -h_{12}\left(A_{11} - h_1^{-2}A_{22} + h_{12}^{-2}A_{33} - B^{010}\right), \tag{A21}$$

$$A_{22}^{13} = 2h_1^2\left(2h_{12}^{-1}A_{13} - h_{12}B_{13}\right), \tag{A22}$$

$$A_{23}^{13} = -h_{12}\left(2A_{12} - h_1^2 B_{12}\right). \tag{A23}$$

For the remaining 15 matrix elements that are not given above, we have (repeated index marks a pattern, no summation implied)

$$A_{jk}^{ii} = A_{jk}, \quad A_{jj}^{ii} = A_{jj}, \quad A_{kk}^{ij} = 0. \tag{A24}$$

The off-diagonal 'central stress' elements are given directly by

$$A_{12} = \left(1 - \frac{1+\nu}{2h_{12}^2h_2^2 + \left(3 + h_2^2 + h_{12}^2\right)(1+\nu)}\right)\frac{h_1^2 B_{12}}{2}, \tag{A25}$$

$$A_{13} = \left(1 - \frac{(1+\nu)h_2^2}{2h_1^2 + \left(1 + 3h_2^2 + h_{12}^2\right)(1+\nu)}\right)\frac{h_{12}^2 B_{13}}{2}, \tag{A26}$$

$$A_{23} = \left(1 - \frac{(1+\nu)h_1^2h_{12}^2}{2 + \left(h_1^2 + h_{12}^2 + 3h_1^2h_{12}^2\right)(1+\nu)}\right)\frac{h_{12}^2 B_{23}}{2}. \tag{A27}$$

The rest is obtained from equations (35) with

$$L_{11} = -2 - h_1^2 - h_1^4 - h_{12}^2\left(1 - h_1^2\right)\nu, \tag{A28}$$

$$L_{12} = -1 - h_1^2 - 2h_1^4 + h_{12}^2\left(1 - h_1^2\right)\nu, \tag{A29}$$

$$L_{13} = 1 + h_1^2 + h_1^4 + h_1^2\left(1 + 2h_2^2 + 2h_{12}^2\right)\nu, \tag{A30}$$

$$L_{21} = h_1^2\left(1 + h_2^2 + h_2^4\right) + \left(2 + 2h_2^2 + h_{12}^2\right)\nu, \tag{A31}$$

$$L_{22} = -h_1^2\left(2 + h_2^2 + h_2^4\right) - \left(1 - h_2^2\right)\nu, \tag{A32}$$

$$L_{23} = -h_1^2\left(1 + h_2^2 + 2h_2^4\right) + \left(1 - h_2^2\right)\nu, \tag{A33}$$

$$L_{31} = -2 - h_{12}^2 - h_{12}^4 - h_1^2\left(1 - h_{12}^2\right)\nu, \tag{A34}$$

$$L_{32} = 1 + h_{12}^2 + h_{12}^4 + h_1^2\left(2 + h_2^2 + 2h_{12}^2\right)\nu, \tag{A35}$$

$$L_{33} = -1 - h_{12}^2 - 2h_{12}^4 + h_1^2\left(1 - h_{12}^2\right)\nu \tag{A36}$$

and

$$\mathbf{R} = \begin{pmatrix} 1 - h_1^2\nu & h_1^2\left(h_1^2 - \nu\right) & -h_{12}^2\left(1 + h_1^2\right)\nu \\ -\left(1 + h_2^2\right)\nu & h_1^2\left(1 - h_2^2\nu\right) & h_{12}^2\left(h_2^2 - \nu\right) \\ 1 - h_{12}^2\nu & -h_1^2\left(1 + h_{12}^2\right)\nu & h_{12}^2\left(h_{12}^2 - \nu\right) \end{pmatrix}. \tag{A37}$$

APPENDIX B: COEFFICIENTS *M*

After setting Poisson's ratio $\nu = \frac{1}{4}$ and defining

$$N = \frac{32}{35}\left(\frac{h_{12}^2}{\left(1 - h_1^2\right)\left(1 - h_2^2\right)\left(1 - h_{12}^2\right)}\right)^2, \tag{B1}$$

where $h_{12} = h_1 h_2$, we obtain a compact form of three coefficients required in equations (82) and (83):

$$M_{13} = N\left(1 - h_1^4\right)\left(1 - h_2^4\right)\left(2 - \frac{5h_2^2}{5 + 8h_1^2 + 15h_2^2 + 5h_{12}^2}\right), \tag{B2}$$

$$M_{23} = N\left(1 - h_{12}^4\right)\left(1 - h_1^4\right)\left(2 - \frac{5h_1^4h_2^2}{8 + 5h_1^2 + 5h_{12}^2\left(1 + 3h_1^2\right)}\right), \tag{B3}$$

$$M_{12} = N\frac{\left(1 - h_{12}^4\right)\left(1 - h_2^4\right)}{h_2^4}\left(2 - \frac{5}{15 + 5h_2^2 + h_{12}^2\left(5 + 8h_2^2\right)}\right). \tag{B4}$$

The expression of the fourth one is more involved:

$$M_0 = \frac{N}{3h_2^4 N_9}\sum_{j=1}^{8} N_j\, h_2^{2j}. \tag{B5}$$

Using an auxiliary variable

$$\xi = \left(h_1 + h_1^{-1}\right)^2, \tag{B6}$$

we can compress N_j to read

$$N_0 = 225\left(\xi - 1\right), \tag{B7}$$

$$N_1 = 6\left(1 + h_1^2\right)\left(29\xi - 21\right), \tag{B8}$$

$$N_2 = h_1^2\left(31\xi^2 + 82\xi - 62\right), \tag{B9}$$

$$N_3 = h_1^2\left(1+h_1^2\right)\left(-92\xi^2+305\xi-216\right), \tag{B10}$$

$$N_4 = h_1^4\left(31\xi^3-341\xi^2+99\xi+295\right), \tag{B11}$$

$$N_5 = h_1^4\left(1+h_1^2\right)\left(174\xi^3-1012\xi^2+1185\xi-458\right), \tag{B12}$$

$$N_6 = h_1^6\left(225\xi^4-1404\xi^3+2412\xi^2-1409\xi-124\right), \tag{B13}$$

$$N_7 = h_1^6\left(1+h_1^2\right)\left(225\xi^3-1179\xi^2+1376\xi-368\right), \tag{B14}$$

$$N_8 = h_1^8\left(3\xi-4\right)\left(75\xi^2-292\xi+64\right), \tag{B15}$$

$$\begin{aligned} N_9 = {} & 48\xi-57+h_1^2h_2^4\left(48\xi^2-119\xi+100\right) \\ & +h_2^2\left(1+h_1^2\right)\left(32\xi-23+h_1^2h_2^4\left(39\xi-44\right)\right) \\ & +16h_1^4h_2^8(3\xi-4). \end{aligned} \tag{B16}$$

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