THE MASSIVE ODE/IM CORRESPONDENCE FOR SIMPLY-LACED LIE ALGEBRAS

A THESIS SUBMITTED TO

THE UNIVERSITY OF KENT AT CANTERBURY

IN THE SUBJECT OF MATHEMATICS

FOR THE DEGREE

OF DOCTOR OF PHILOSOPHY BY RESEARCH

By

Neal Carr

March 2019

The massive ODE/IM correspondence for simply-laced Lie algebras

Neal Carr

Abstract

The ODE/IM correspondence is a connection between the properties of particular differential equations (ODEs) and certain quantum integrable models in two dimensions (IMs). In its original form, the ODE/IM correspondence originally connected the spectral determinants of a set of second-order ODEs and the ground-state eigenvalues of **Q**-operators defined in a conformal field theory. The spectral determinants for these ODEs and the **Q**-operator eigenvalues were found to satisfy the same functional relations.

In this thesis, we are concerned with two generalisations of this correspondence. The first of these is the extension of the correspondence to encompass the excited states of the conformal field theory. The corresponding ODEs are defined by a set of parameters z_i which are constrained by a set of algebraic locus equations. Studying the space of solutions of these equations, we find an apparent discrepancy between the number of solutions of the locus equations and the number of states in a particular level subspace of the conformal field theory, which is not explained by the occurrence of singular vectors in the conformal field theory. This discrepancy is resolved by considering a more general set of locus equations defined using a result due to Duistermaat on the single-valuedness of solutions of second-order ODEs of the correct form.

The second generalisation of the correspondence of interest is the connection between linear systems of differential equations constructed as Lax pairs from the affine Toda field theory equation of motion (for a given affine Lie algebra $\hat{\mathfrak{g}}$), and the ground-state eigenvalues of $\mathbf{Q}$-operators associated with a massive integrable model with symmetry generated by the Lie algebra $\hat{\mathfrak{g}}$. We consider the cases where $\mathfrak{g}$ is a simply-laced Lie algebra, deriving asymptotics of the solutions of the associated linear systems, and from these we construct Q-functions, which encode various properties of the massive IM in the functional relations they satisfy and their asymptotic expansions. In the case of $A_r^{(1)}$, we also derive T-functions that satisfy additional sets of functional relations which arise in the IMs.

Declaration

The work in this thesis is based on research completed at the School of Mathematics, Statistics and Actuarial Science at the University of Kent. This thesis, nor any part of it, has been submitted elsewhere for any other degree or qualification. Sections 2.4, 4.3, 5, 6 are believed to be original unless otherwise stated. All other sections contain necessary background information, for which no originality is claimed.

Acknowledgments

Firstly, I would like to thank my supervisor, Dr Clare Dunning, for her guidance, support and patience over the last four years. Thank you for giving me the opportunity to study at Kent, and for always patiently answering the many questions that arose during my time here. I also thank Dr Steffen Krusch for being my second supervisor.

I gratefully acknowledge the School of Mathematics, Statistics, and Actuarial Science at the University of Kent and the Engineering and Physical Sciences Research Council (award reference 1511097), whose financial support made this research possible.

Many thanks are also due to the many people who I have depended on over the last four years for their kindness and support. A particular debt of thanks is owed to Claire Carter, who has done so much to ensure my welfare and was always available for advice and kind words. I am also pleased to have met so many other fantastic people while studying here, especially Acki, Katherine, Neil, Reuben, Clare, Christoph, Jenny, Ellen, Clelia, Alfredo, John, Chris and Mark. I would also like to particularly thank Acki for being such a brilliant flatmate and putting up with me over the last two years.

Lastly, I would like to thank my family, Mum, Dad, Thomas and William, for all the love and support they have given me over the years.

Contents

List of Figures

Chapter 1

Introduction

The ODE/IM correspondence [23, 37, 1] is an intriguing connection between two seemingly disparate areas of mathematical physics: the study of the spectral properties of particular differential equations (ODEs), and certain quantum integrable models in two dimensions (IMs). This connection first manifested in the form of identical functional relations occurring in the study of particular ODEs and IMs. We will now introduce these two halves of the ODE/IM correspondence, before elaborating on the precise connection between them.

1.1 ODEs and eigenvalue problems

1.1.1 Beginnings: the anharmonic oscillator

The story of the ODE/IM correspondence begins with the study of the spectral properties of the anharmonic oscillator, with dynamics determined by the

Schrödinger equation with potential x^{2M}:

$$-\frac{d^2\psi(x)}{dx^2} + |x|^{2M}\psi(x) = E\psi(x), \tag{1.1.1}$$

where $M > 1$ is a positive integer or half-integer. When (1.1.1) is considered on the real line, a set of normalisable solutions $\{\psi_k(x)\}_{k=1}^{\infty}$ exists with associated discrete eigenvalues E_k. These eigenvalues E_k can be encoded into a spectral determinant $D(E)$, an entire function in the parameter E with the zeroes $D(E_k) = 0$. The spectral determinant $D(E)$ admits an infinite product expansion

$$D(E) = D(0)\prod_{k=1}^{\infty}\left(1 - \frac{E}{E_k}\right), \tag{1.1.2}$$

which, due to the invariance of (1.1.1) under the parity symmetry $x \to -x$, further factorises into a product of two spectral determinants $D(E) = D_+(E)D_-(E)$, where

$$D^+(E) = D^+(0)\prod_{k \text{ even}}\left(1 - \frac{E}{E_k}\right), \quad D^-(E) = D^-(0)\prod_{k \text{ odd}}\left(1 - \frac{E}{E_k}\right). \tag{1.1.3}$$

As a consequence of the parity symmetry, the solutions $\psi_k(x)$ with k even (odd respectively) are even (odd) functions. The spectral determinants $D^+(E)$, $D^-(E)$ satisfy a particular functional relation [55, 56]:

$$e^{\frac{i\pi}{2(M+1)}}D^+(e^{\frac{-i\pi}{M+1}}E)D^-(e^{\frac{i\pi}{M+1}}E) - e^{\frac{-i\pi}{2(M+1)}}D^+(e^{\frac{i\pi}{M+1}}E)D^-(e^{\frac{-i\pi}{M+1}}E) = 2i, \tag{1.1.4}$$

The first manifestation of the ODE/IM correspondence was the observation by Dorey and Tateo in [25] that the functional relation (1.1.4) matched a functional relation satisfied by the rescaled eigenvalues of **Q**-operators which arise in the study of a particular class of IMs [7], conformal field theories. (We will discuss these models further in section 1.2). These IMs are defined for $M > 0$, and this fact along with numerical investigations and the study of the solvable cases $M = 1/2, 1$ (the Airy equation and the harmonic oscillator respectively) led the authors of [25] to conjecture the extension of the ODE/IM correspondence to eigenvalue problems of the form (1.1.1) with arbitrary $M > 0$.

1.1.2 Anharmonic oscillator with angular momentum term

Bazhanov, Lukyanov and Zamolodchikov [9] then extended the correspondence by adding an angular momentum term to the anharmonic oscillator (1.1.1)

$$-\frac{d^2\psi(x)}{dx^2} + \left(x^{2M} + \frac{l(l+1)}{x^2}\right)\psi(x) = E\psi(x). \tag{1.1.5}$$

The equation (1.1.5) is considered on the positive real axis, and is subject to boundary conditions at $x = 0$; the solution $\psi(x)$ is constrained to satisfy $\psi(x) \sim x^{l+1}$ or $\psi(x) \sim x^{-l}$ in the neighbourhood of $x = 0$. The corresponding IM is a natural generalisation of the IM related to the anharmonic oscillator (1.1.1). The equation (1.1.5) is the prototype of all the other differential equations we will consider in this thesis, so we now take the time to consider this equation more carefully, defining its spectral determinants and functional relations satisfied by them. In the rest of this section we follow closely the notation in section 5 of the review paper [23], which itself is derived from the original papers [26, 9].

To define eigenvalue problems associated with (1.1.5) we stipulate boundary conditions that solutions must satisfy at the regular singular point $x = 0$ and at the irregular singular point at $x = \infty$. We will require solutions of (1.1.5) to decay as $x \to \infty$ along the positive real axis. Using the WKB approximation [13] to analyse equation (1.1.5) in the large-x limit, we define a solution $y(x, E, l)$ of (1.1.5) which decays as $x \to \infty$ along the positive real axis, with asymptotic expansion in that limit given by

$$y(x,E,l) \sim \frac{x^{-M/2}}{\sqrt{2i}} \exp\left(-\frac{x^{M+1}}{M+1}\right) \quad \text{as } x \to \infty. \tag{1.1.6}$$

The choice of normalisation in (1.1.6) simplifies the form of the spectral determinants we will construct in this section.

In the neighbourhood of the regular singular point at $x=0$, the behaviour of any solution of (1.1.5) is a linear combination of x^{l+1} and x^{-l}. Following [26], we choose a solution $\psi^+(x,E,l)$ to satisfy the $x \to 0$ asymptotic

$$\psi^+(x,E,l) \sim x^{l+1}, \quad \text{as } x \to 0. \tag{1.1.7}$$

Due to the remaining linearly independent asymptotic solution x^{-l}, $\psi^+(x,E,l)$ is only uniquely defined for $l > -1/2$. We extend the definition of $\psi^+(x,E,l)$ to all l by exploiting the symmetry of (1.1.5) under the mapping $l \to -1-l$, and define

$$\psi^-(x,E,l) = \psi^+(x,E,-1-l) \sim x^{-l}, \quad \text{as } x \to 0. \tag{1.1.8}$$

The solutions $\psi^\pm(x,E,l)$ then form a basis of solutions of (1.1.5) in the small-x limit for generic l. The basis also respects the symmetry $l \to -1-l$ of (1.1.5).

The two solutions $\psi^\pm(x,E,l)$ define two separate eigenvalue problems; we consider solutions $\psi(x,E,l)$ of (1.1.5) with associated eigenvalues $E_k^\pm$ which satisfy

$$\psi(x,E_k^\pm,l) \sim \psi^\pm(x,E_k^\pm,l) \quad \text{as } x \to 0, \tag{1.1.9}$$

$$\psi(x,E_k^\pm,l) \sim y(x,E_k^\pm,l) \quad \text{as } x \to +\infty. \tag{1.1.10}$$

To define spectral determinants $D_{\mp}(E,l)$ associated with these eigenvalue problems, we first define the Wronskian of two functions of x

$$W[f,g] = f(x)g'(x) - f'(x)g(x), \tag{1.1.11}$$

which allows us to define a notion of linear independence for solutions of (1.1.5). Specifically, two solutions $f(x)$ and $g(x)$ of (1.1.5) are linearly independent if and only if their Wronskian is non-zero. If their Wronskian is zero, $f(x)$ and $g(x)$ are effectively the same solution of (1.1.5), up to an overall normalisation constant. The Wronskians

$$D_{\mp}(E,l) = W[y,\psi^{\pm}](E,l), \tag{1.1.12}$$

are therefore zero at the values of E where $y(x,E,l)$ and $\psi^{\pm}(x,E,l)$ are proportional to one another. At these values of E, there exists a global solution with the required asymptotic behaviours, which is precisely the requirement of the eigenvalue problems (1.1.9)-(1.1.10). The functions $D_{\mp}(E,l)$ are therefore spectral determinants. We also note the identification $D_+(E,-l-1) = D_-(E,l)$ follows from the definitions of the asymptotics (1.1.7)-(1.1.8).

1.1.3 Functional relations

In order to construct functional relations that the spectral determinants $D_{\mp}(E)$ satisfy, we first note the invariance of equation (1.1.5) under the transformation

$$x \to \omega^{-k}x, \quad E \to \omega^{2k}E, \quad k \in \mathbb{Z},\ \omega = e^{\frac{2\pi i}{2M+2}}. \tag{1.1.13}$$

Given a solution $\chi(x,E,l)$ of (1.1.5), we define a set of rotated functions

$$\chi_k(x,E,l) = \omega^{k/2}\chi(\omega^{-k}x,\omega^{2k}E,l), \tag{1.1.14}$$

which due to the invariance of (1.1.5) under the transformation (1.1.13), are all solutions of (1.1.5). It is also convenient to define Stokes sectors $\mathcal{S}_k$ in the complex x-plane

$$\mathcal{S}_k = \left|\arg(x) - \frac{2\pi k}{2M+2}\right| < \frac{\pi}{2M+2}, \tag{1.1.15}$$

and the rotations of the large-x asymptotic solution $y(x,E,l)$ by

$$y_k(x,E,l) = \omega^{k/2}y(\omega^{-k}x,\omega^{2k}E,l), \tag{1.1.16}$$

where $k \in \mathbb{Z}$. The solutions $y_k(x,E,l)$ are the most rapidly decaying solutions of (1.1.5) as $|x| \to \infty$ on the Stokes sector $\mathcal{S}_k$. We also introduce rotations of the small-x asymptotic solutions $\psi^{\pm}(x,E)$ by

$$\psi_k^{\pm}(x,E,l) = \omega^{k/2}\psi^{\pm}(\omega^{-k}x,\omega^{2k}E,l). \tag{1.1.17}$$

We then compute the Wronskians

$$\begin{gathered} W[\psi_k^+,\psi_p^-] = -(2l+1)\omega^{(k-p)(l+1/2)}, \\ W[\psi_k^+,\psi_p^+] = W[\psi_k^-,\psi_p^-] = 0, \quad k,p \in \mathbb{Z}, \end{gathered} \tag{1.1.18}$$

and see that for generic $l > -1/2$, the solutions $\{\psi_k^+,\psi_k^-\}$ are linearly independent solutions and thus form a basis for the solution space of (1.1.5). (The papers [23, 26] briefly discusses the isolated values of l where this assumption breaks down; from here on we assume we choose a value of l where this does not happen.) The linear independence of ψ_k^+,ψ_k^- implies that we may write a solution $y_k(x,E,l)$

as a linear combination of ψ_k^+ and ψ_k^-

$$y_k(x,E,l) = B_-(E,l)\,\psi_k^-(x,E,l) + B_+(E,l)\,\psi_k^+(x,E,l), \tag{1.1.19}$$

where $B_-(E,l)$ and $B_+(E,l)$ are independent of x. By taking Wronskians of (1.1.19) with respect to ψ_k^+ and ψ_k^- respectively and using (1.1.16), (1.1.17) and (1.1.12) we find

$$B_\pm(E,l) = \mp\frac{D_\pm(\omega^{2k}E,l)}{2l+1}. \tag{1.1.20}$$

Using (1.1.18) and the definitions of the spectral determinants $D_\pm(E,l)$, we find

$$(2l+1)y_k(x,E,l) = D_-(\omega^{2k}E,l)\psi_k^-(x,E,l) - D_+(\omega^{2k}E,l)\psi_k^+(x,E,l). \tag{1.1.21}$$

To find a functional relation involving only $D_\pm(E,l)$, we consider (1.1.21) at $k = -1$ and $k = 0$ and compute $W[y_{-1}, y_0]$ to find:

$$\begin{aligned}(2l+1)^2 W[y_{-1},y_0] = &-D_-(\omega^{-2}E,l)D_+(E,l)W[\psi_{-1}^-,\psi_0^+] \\ &- D_+(\omega^{-2}E,l)D_-(E,l)W[\psi_{-1}^+,\psi_0^-].\end{aligned} \tag{1.1.22}$$

From the large-x asymptotic expressions for y_0 (1.1.6) and y_{-1} (computed by acting on (1.1.6) with the transformation (1.1.13)), we find $W[y_{-1}, y_0] = 1$. Substituting this result into (1.1.22), shifting $E \to \omega E$ and simplifying using (1.1.18), we are left with the functional relation

$$\omega^{-(l+1/2)}D_+(\omega^{-1}E,l)D_-(\omega E,l) - \omega^{l+1/2}D_+(\omega E,l)D_-(\omega^{-1}E,l) = 2l+1. \tag{1.1.23}$$

When $l = 0$, this functional reproduces (1.1.4) associated with the equation (1.1.1) studied by Dorey and Tateo, up to the disparity between the constants on the right-hand sides of (1.1.4) and (1.1.23). This difference arises from the choice of normalisation in the large-x asymptotics (1.1.6). The functional relation (1.1.23) also occurs in the related IM, and is called the *quantum Wronskian* in the IM literature [7].

Other sets of functional relations occur in the associated IM, and these may also be constructed using solutions of the differential equation (1.1.5). A particularly important specimen of functional relations are the so-called *TQ-relations*, constructed in [26]. The construction begins with the expansion of the rotated solution $y_{-1}(x, E, l)$ in the basis $\{y_0, y_1\}$:

$$y_{-1}(x, E, l) = C(E, l)y_0(x, E, l) + \widetilde{C}(E, l)y_1(x, E, l). \tag{1.1.24}$$

(Any pair of rotated solutions $\{y_{n-1}, y_n\}$ form a basis of solutions of (1.1.5) as $W[y_{n-1}, y_n] = 1$.) Taking Wronskians of (1.1.24) with respect to y_0 and y_1 we find

$$C(E, l)y_0(x, E, l) = y_{-1}(x, E, l) + y_1(x, E, l). \tag{1.1.25}$$

We follow [23] and take Wronskians of (1.1.25) with respect to $\psi^{\pm}$. We use the result (5.12) in [23]

$$W[y_k, \psi^{\pm}] = \omega^{\pm(l+1/2)k} W[y, \psi^{\pm}](\omega^{2k}E, l) = \omega^{\pm(l+1/2)k} D_{\mp}(\omega^{2k}E, l), \tag{1.1.26}$$

to find the so-called *TQ-relations*

$$C(E, l)D_{\mp}(E, l) = \omega^{\mp(l+1/2)} D_{\mp}(\omega^{-2}E, l) + \omega^{\pm(l+1/2)} D_{\mp}(\omega^{2}E, l). \tag{1.1.27}$$

The functions $C(E, l)$ and $D(E, l)$ correspond to the ground-state eigenvalues of **T**- and **Q**-operators respectively in a conformal field theory, which is the origin of the name TQ-relation. The precise nature of this correspondence will be given in section 1.2, where we discuss the related conformal field theory and the origin of the **T**- and **Q**-operators.

The last class of functional relations we will encounter in this thesis are *fusion relations*. For the differential equation (1.1.5) these are constructed by expanding y_{-1} in the basis $\{y_{n-1}, y_n\}$:

$$y_{-1}(x, E, l) = C_0^{(n)}(E, l) y_{n-1}(x, E, l) + \tilde{C}_0^{(n)}(E, l) y_n(x, E, l) \tag{1.1.28}$$

The authors of [23] define

$$C^{(n)}(E, l) = C_0^{(n)}(\omega^{1-n} E, l) \tag{1.1.29}$$

and show that they satisfy the fusion relations

$$C^{(n-1)}(\omega^{-1} E) C^{(n-1)}(\omega E) = 1 + C^{(n)}(E) C^{(n-2)}(E). \tag{1.1.30}$$

Besides the functional relations we have exhibited here, analogues of other objects from conformal field theory may also be constructed from the spectral determinants $D_\pm(E)$; in the following chapters we will encounter *Bethe ansatz equations* satisfied by the zeroes of generalisations of the spectral determinants $D_\pm(E)$. These Bethe ansatz equations, along with the asymptotic behaviour of the spectral determinants, also determine non-linear integral equations which encode thermodynamic properties of the associated integrable models. In the next section, we will elucidate these links more precisely, giving the precise correspondence between the spectral determinants discussed in this section and the eigenvalues of the **T**- and **Q**-operators associated with a particular family of conformal field theories.

1.2 Integrable models (IMs)

We now introduce the other half of the ODE/IM correspondence, which is composed of various integrable quantum field theories. What does it mean for a quantum field theory to be integrable? One of the characteristics of an integrable field theory is the existence of infinitely many commuting local integrals of motion in the theory. This is a direct generalisation of the notion of integrability in a classical mechanical system. Such a system with n degrees of freedom is integrable if there exists n integrals of motion; that is, n functions of the positions and velocities of particles in the system that are constants throughout the motion of the system. These n functions must also pairwise commute with respect to the Poisson bracket. In the context of field theory, however, the number of degrees of freedom is infinite, and so the process of ensuring that all such integrals of motion are accounted for is somewhat more involved. Nevertheless, this general notion of integrability will be the definition we will adhere to in this thesis. Other possible definitions of quantum integrability are discussed in [14].

1.2.1 Baxter's T and Q functions in conformal field theory

A prolific source of integrable field theories as defined above is the family of two-dimensional conformal field theories [17]. These are two-dimensional field theories in Euclidean spacetime, parametrised by independent light-cone coordinates $z, \bar{z}$, that are invariant under holomorphic/anti-holomorphic transformations of the coordinates $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$. In two dimensions, this symmetry group is infinite-dimensional, generated by the family of transformations $z \to z^p$, $\bar{z} \to \bar{z}^q$ for $p, q \in \mathbb{Z}$. This large symmetry group constrains the class of possible field theories with this symmetry enormously, and allows for the complete construction of the possible states and operators in the theory.

Conformal field theories arise in the physical description of integrable lattice models at critical points, where the physical system undergoes a phase transition. The prototypical example of such a lattice model is the two-dimensional Ising model, described at its critical point by one of a particular family of conformal field theories called minimal models [12]. A slight generalisation of this model, the six-vertex ice-type model, defined on an N-by-N' lattice (see [23] and [4] for more details) is most relevant to our current discussion. The partition function of this model can be written [44, 52] in terms of a transfer matrix $\mathbb{T}$, with the eigenvalues of this transfer matrix determining the thermodynamic properties of the system at the critical point. The eigenvalues are calculated using the Bethe ansatz technique; a possible candidate for an eigenvector of $\mathbb{T}$ dependent on some parameters ν_i is constructed, with the result that it is an eigenvector of $\mathbb{T}$ if and only if the parameters ν_i satisfy Bethe ansatz equations. Once the eigenvalues of the transfer matrix are found (usually in the limit $N, N' \to \infty$), physical information about the model can be extracted from them, and the model is considered solved.

In his treatment of the six-vertex model, Baxter introduced an additional matrix $\mathbb{Q}$ and found it, along with the transfer matrix $\mathbb{T}$ satisfied a matrix equation that is the integrable lattice model analogue to the TQ-relation. Bazhanov, Lukaynov and Zamolodchikov [6, 7, 8] subsequently demonstrated how to generalise these $\mathbb{T}$ and $\mathbb{Q}$ matrices to operators in a conformal field theory, with central charge

$$c = 1 - 6(\beta - \beta^{-1})^2, \quad 0 < \beta < 1, \tag{1.2.1}$$

and with an additional free 'vacuum parameter' p. The space of states of the conformal field theory is inhabited by representations $\mathcal{V}_\Delta$ of the Virasoro algebra, generated by a highest weight state $|\Delta\rangle$, where the highest weight Δ is given by

$$\Delta = \left(\frac{p}{\beta}\right)^2 + \frac{c-1}{24} \tag{1.2.2}$$

The states in $\mathcal{V}_\Delta$ are generated by acting on $|\Delta\rangle$ with operators L_n, with $n \leq 0$. The operators L_n satisfy the commutation relations of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{1.2.3}$$

where $[L_m, L_n]$ is the Lie bracket of the Virasoro algebra.

The authors of [6, 7, 8] define a transfer matrix operator $\mathbf{T}(s,p) : \mathcal{V}_\Delta \to \mathcal{V}_\Delta$, and a pair of other operators $\mathbf{Q}_\pm(s,p) : \mathcal{V}_\Delta \to \mathcal{V}_\Delta$, which were found to satisfy the TQ-relations

$$\mathbf{T}(s,p)\mathbf{Q}_\pm(s,p) = \mathbf{Q}_\pm(q^2 s, p) + \mathbf{Q}_\pm(q^{-2}s, p), \tag{1.2.4}$$

where $q = e^{i\pi\beta^2}$. The vacuum state $|\Delta\rangle$ is an eigenstate of the $\mathbf{T}$- and $\mathbf{Q}$-operators, and we define the corresponding ground state eigenvalues in the same way as the review paper [23]:

$$T(s,p) = \langle\Delta| \, \mathbf{T}(s,p) \, |\Delta\rangle \,, \tag{1.2.5}$$

$$Q_\pm(s,p) = \langle\Delta| \, s^{\mp\mathbf{P}/\beta^2} \mathbf{Q}_\pm(s,p) \, |\Delta\rangle \,, \tag{1.2.6}$$

where the operator $\mathbf{P}$ satisfies $\mathbf{P}\,|\Delta\rangle = p\,|\Delta\rangle$. Applying both sides of the operator TQ-relation to the vacuum state $|\Delta\rangle$ we find the TQ-relations as satisfied by the ground state eigenvalues of $T(s,p)$ and $Q_\pm(s,p)$

$$T(s,p)Q_\pm(s,p) = e^{\mp 2\pi i p}Q_\pm(q^{-2}s,p) + e^{\pm 2\pi i p}Q_\pm(q^2 s,p). \tag{1.2.7}$$

This matches with the TQ-relations (1.1.27) we found earlier satisfied by spectral determinants of (1.1.5). Specifically, setting

$$\beta^2 = \frac{1}{M+1}, \quad p = \frac{2l+1}{4M+4}, \tag{1.2.8}$$

and associating the functions $T, Q_\pm$ with $C, D_\mp$ respectively identifies these two TQ-relations derived in the context of ordinary differential equations and integrable field theory. To make this identification exact, the analytical properties of C, D must match those of T and Q. In [26], it was shown that $C(E,l)$ and $D_-(E,l) = D_+(E,-1-l)$ satisfy the following:

1. $C(E,l)$ and $D(E,l)$ are entire functions of E,

2. The zeroes of $D_-(E,l)$ are all real and, if $l > -1/2$, they are all positive,

3. The zeroes of $C(E,l)$ are all real, and, if $-1-M/2 < l < M/2$, they are all negative,

4. If $M > 1$, the large-E asymptotics of $D(E,l)$ are given by

$$D_-(E,l) \sim \exp\left(\frac{a_0}{2}(-E)^{\frac{M+1}{2M}}\right), \text{ as } |E| \to \infty, \quad |\arg(-E)| < \pi, \tag{1.2.9}$$

   where

$$a_0 = -B\left(\frac{M+1}{2M} + \frac{1}{2}, -\frac{M+1}{2M}\right), \quad \text{where } B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{1.2.10}$$

5.

$$D_-(0,l) = \frac{\Gamma(1+\frac{2l+1}{2M+2})}{\sqrt{2\pi i}}(2M+2)^{\frac{2l+1}{2M+2}+\frac{1}{2}}, \tag{1.2.11}$$

6. $D_\pm(E,l)$ can be written as a well-defined product over its zeroes $E_k^\pm$:

$$D_\pm(E,l) = D_\pm(0,l)\prod_{k=1}^{\infty}\left(1-\frac{E}{E_k^\pm}\right). \tag{1.2.12}$$

The analogous properties satisfied by $T(s,p)$ and $Q_+(s,p)$ given in [7] where $0 < \beta^2 < 1/2$ are

1. $T(s,p)$ and $Q_+(s,p)$ are entire functions of s,
2. The zeroes of $Q_+(s,p)$ are all real, and if $2p > \beta^2$, they are all strictly positive,
3. The zeroes of $T(s,p)$ are all real, and if $|p| < 1/4$, they are all negative,
4. The large-s asymptotics of $Q_\pm(s,p)$ are given by

$$Q_+(s,p) \sim \exp\left(\frac{a_0}{\beta^2}(-2)^{\frac{1}{2(1-\beta^2)}}\Gamma(1-\beta^2)^{\frac{1}{(1-\beta^2)}}\right), \tag{1.2.13}$$

5. $Q_+(0,p) = 1$,
6. $Q_\pm(s,p)$ can be written as a well-defined product over its zeroes s_k^+:

$$Q_\pm(s,p) = \prod_{k=1}^{\infty}\left(1-\frac{s}{s_k^\pm}\right). \tag{1.2.14}$$

With these properties satisfied by C, D_- and T, Q_+, the identification of the T and $Q_\pm$ functions with the C and $D_\mp$ functions is precisely

$$Q_\pm(s,p) = \frac{D_\mp(\frac{s}{v},\frac{2p}{\beta^2}-\frac{1}{2})}{D_\mp(0,\frac{2p}{\beta^2}-\frac{1}{2})}, \tag{1.2.15}$$

$$T(s,p) = C\left(\frac{s}{v},\frac{2p}{\beta^2}-\frac{1}{2}\right), \tag{1.2.16}$$

where $M = \beta^{-2} - 1$, and

$$v = (2M+2)^{-\frac{2M}{M+1}} \Gamma\left(\frac{M}{M+1}\right)^{-2}. \tag{1.2.17}$$

The spectral determinants $D_\pm(E,l)$ exhibit other features that natively occur in the study of integrable field theories. By setting $E = \omega E_k^\pm$ and $E = \omega^{-1} E_k^\pm$ in (1.1.23) and dividing the resulting expression, we see that the zeroes of $E_k^\pm$ of $D_\pm(E,l)$ satisfy Bethe ansatz equations

$$\omega^{\pm(2l+1)} \frac{D_\pm(\omega^2 E_k^\pm, l)}{D_\pm(\omega^2 E_k^\pm, l)} = -1, \tag{1.2.18}$$

which may be expanded using the product expansion (1.2.12) to yield an infinite set of equations satisfied by the zeroes $E_k^\pm$

$$\omega^{\pm(2l+1)} \prod_{j=1}^{\infty} \frac{E_k^\pm - \omega^2 E_j^\pm}{E_k^\pm - \omega^{-2} E_j^\pm} = -1. \tag{1.2.19}$$

Bethe ansatz equations of this type, along with the properties satsfied by the zeroes of $D_\pm(E)$ and the asymptotics of $D_\pm(E)$, may be encoded into non-linear integral equations [16]. The asymptotic expansion of $D_\pm(E)$ (1.2.9) picks out a particular solution of the BAEs, corresponding to the ground state $|\Delta\rangle$ of the conformal field theory. The non-linear integral equation can be solved numerically for $\log D_\pm(E)$, and hence the spectrum of the eigenvalue problems associated with (1.1.5) may be found numerically. Using the non-linear integral equation, $\log D_\pm(E)$ may also be expanded [24] as an asymptotic power series in $E^{\frac{M+1}{2M}}$ and E^{M+1}, and the coefficients in this expansion are the ground-state eigenvalues of the integrals of motion of the corresponding integrable field theory.

We have seen above how the authors of [9] and [26], building on [25], demonstrated the ODE/IM correspondence between eigenvalue problems associated with the anharmonic oscillator with an angular momentum term (1.1.5) and the ground-state eigenvalues of **Q**-operators associated with conformal field theory. The scope of the ODE/IM correspondence has since been expanded to encompass links between more eigenvalue problems and other integrable field theories. In the next section, we briefly survey some of these generalisations, introducing the two major generalisations that will concern us for the rest of this thesis.

1.3 Generalisations of the ODE/IM correspondence

Since the early papers [25, 9, 42], there have been large generalisations to the ODE/IM correspondence, matching ever larger classes of eigenvalue problems to other quantum integrable models. The example of the ODE/IM correspondence we have studied in sections 1.1 and 1.2 is related to the Lie algebra $A_1 = \mathfrak{su}(2)$. It is natural, then, to consider examples of the ODE/IM correspondence connected with more elaborate Lie algebras. In [54, 22], the eigenvalue problem (1.1.5) was considered with the x^{2M} term replaced with $x^{2M} + \alpha x^{M-1}$, where α is a constant. Functional relations are constructed in a similar manner to the A_1 case considered in sections 1.1 and 1.2. The algebra related to this class of examples of the ODE/IM correspondence is the Lie superalgebra $\mathfrak{sl}(2|1)$.

The ODE/IM correspondence has also been extended beyond second-order ordinary differential equations; in [27] a third-order differential equation was found to be related to an integrable field theory related to the affine Lie algebra $A_2^{(2)}$. This work was then extended to differential equations related to the Lie algebra $A_r^{(1)}$ in [53, 21]. The spectral determinants of these differential equations were found to satisfy functional relations related to an integrable field theory associated with the Lie algebra A_r. Moreover, from these functional relations, the authors of [21] derived A_r Bethe ansatz equations and a set of related non-linear integral equations, which matched non-linear integral equations derived in [58].

A natural generalisation, after considering the ODE/IM correspondence related to the Lie algebra $A_r = \mathfrak{su}(r+1)$, is to bring the other classical families of simple Lie algebras $B_r = \mathfrak{so}(2r+1)$, $C_r = \mathfrak{sp}(2r)$ and $D_r = \mathfrak{so}(2r)$ into the fold. In [18, 19], Bethe ansatz equations for the classical Lie algebras were derived from specially constructed pseudo-differential equations; these are equations which incorporate an inverse derivative operator $\left(\frac{d}{dx}\right)^{-1}$. Additionally, in [47, 48], the ODE/IM correspondence was considered for arbitrary simple Lie algebras $\mathfrak{g}$ by studying a set of linear systems constructed from representations of Lie algebra generators of the Langlands dual algebra $\mathfrak{g}^\vee$. The authors of [47, 48] demonstrate the solutions of these linear systems satisfy the Ψ-system, from which they derive quantum Wronskians and Bethe ansatz equations associated with the simple Lie algebra $\mathfrak{g}$. These results were written in the language of affine opers in [34], and the quantum Wronskians were rederived in that paper as a consequence of relations between elements of representations of subalgebras of quantum affine algebras $U_q(\hat{\mathfrak{g}})$, which contain the previously mentioned $\mathbf{Q}$-operators and their generalisations to general simple Lie algebras.

There are two other generalisations of the ODE/IM correspondence that are particularly relevant to this thesis. The prototypical example of the ODE/IM correspondence we have encountered in sections 1.1 and 1.2 related the spectral determinants constructed from a second-order differential operator to the ground state eigenvalues of the $\mathbf{Q}$-operators. For each vacuum state $|\Delta\rangle$ there exists an infinite family of excited states, constructed by acting on the vacuum state with generators of the Virasoro algebra (1.2.3). Each of these excited states were naturally expected to correspond to a particular member of a family of unique second-order ODEs. This family of ODEs, first studied in [10], depend on a set of parameters $\{z_i\}_{i=1}^{L}$ and are generalisations of the ODE (1.1.5):

$$-\frac{d^2\psi}{dx^2} + \left(\frac{l(l+1)}{x^2} + x^{2M} - 2\frac{d^2}{dx^2} \sum_{k=1}^{L} \log\bigl(x^{2M+2} - z_k\bigr) \right) \psi = E\psi. \tag{1.3.1}$$

In order for the spectral determinants associated with (1.3.1) (with the same boundary conditions as (1.1.5)) to match the properties of the excited state eigenvalues of the **Q**-operators, the solutions of (1.3.1) must be single-valued at all points of the complex x-plane except for $x = 0$ and $x = \infty$. This requirement [29] leads to the algebraic locus equations [10, 33]:

$$\sum_{j\neq k} \frac{z_k(z_k^2 + (1+2M)(3+M)z_k z_j + M(1+2M)z_j^2)}{(z_k - z_j)^3} - \frac{M z_k}{4(1+M)} + \Delta = 0, \quad z_k \text{ distinct},\ k = 1, \dots, L. \tag{1.3.2}$$

The ODEs (1.3.1) were denoted as 'monstrous' by the authors of [10] because of their apparent lack of utility in ODE theory. However, equations of the form (1.3.1) for $M = 1$ were studied in [32], where the zeroes of Wronskians of Hermite polynomials related to the equations (1.3.1) were found to form patterns in the complex-x plane corresponding to certain partitions of integers. In chapter 2 we will study the locus equations (1.3.2) and show how the presence of singular vectors in the conformal field theory are telegraphed by the loss of one or more solutions of the algebraic locus equations (1.3.2). We also solve a puzzle that occurs at certain values of M and l, namely the loss of solutions of (1.3.2) but without the presence of these singular vectors. This puzzle is resolved by a slight generalisation of the assumptions used to derive the locus equations.

The second important generalisation was the more recent extension of the ODE/IM correspondence to massive integrable field theory. All the examples of the ODE/IM correspondence we have seen so far were associated with massless integrable field theory; namely, various conformal field theories. The first indication of an extension to massive integrable field theory was given in [11], where the authors suggested the study of certain partial differential equations in order to extend the ODE/IM correspondence to massive integrable field theory. This goal was first realised in [45], with the ODE side of the correspondence replaced with a classical partial differential equation (in the case of $A_1^{(1)}$, the massive sinh-Gordon equation) expressible in terms of a Lax pair of linear equations. It is these linear equations and the properties of their spectral determinants that contain information on the corresponding massive integrable field theory. We will review the $A_1^{(1)}$ case of the massive ODE/IM correspondence in chapter 3, following the calculations in [45]. We will begin with the massive sinh-Gordon equation, define the related Q-functions, and the functional relations and Bethe ansatz equations that they satisfy. We will also see the $\mathrm{Re}\,\theta \to \pm\infty$ asymptotics of Q will also contain the ground state eigenvalues of the integrals of motion of the related massive integrable field theory.

The remainder of the thesis will consist of generalising the procedure given in chapter 3 to systems of classical PDEs with more involved Lie algebra structure. This was partly performed in [2, 37, 38], where the authors determined Bethe ansatz equations satisfied by Q-functions in the conformal limit. The relevant non-linear integral equations for the $A_r^{(1)}$ case were also given in [41]. We will generalise the analysis in these papers, following [45] to derive integrals of motion for the integrable field theories associated with the simply-laced Lie algebras.

We begin with a brief overview of the relevant theory of Lie algebras in chapter 4, which will serve to fix the notation we will use throughout the thesis. This chapter will also demonstrate methods of converting systems of differential equations to pseudo-differential equations present in the literature [18, 19, 1], and will contain a generalisation of the WKB approxmiation [13] to systems of differential equations. Having established all the relevant prerequisites, chapter 5 will extend the massive ODE/IM correspondence to the $A_r^{(1)}$ case, building on results in [2, 37, 38]. We consider the remaining simply-laced Lie algebras, namely the family $D_r^{(1)}$ and the exceptional Lie algebras $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ in chapter 6. Finally, in chapter 7 we close with some concluding remarks and an outlook for future research.

Chapter 2

Excited states of conformal field theory and the Schrödinger equation

2.1 Introduction

The prototypical example of the ODE/IM correspondence (1.1.5) we considered in the introduction was a connection between eigenvalue problems defined by a second-order Schrödinger-type differential equation and the ground state eigenvalues of **Q**-operators associated with a particular class of integrable models, conformal field theories. Such theories are also inhabited by excited states, which are themselves eigenstates of the **Q**-operators. A natural generalisation of the example of the ODE/IM correspondence in the introduction would be to find ODEs that correspond to these excited states.

A family of ODEs (1.3.1) which corresponded to the excited states were found in [10]. The authors of [10] constructed a set of differential equations dependent on a family of parameters $\{z_i\}_{i=1}^{L}$, which are constrained by a set of algebraic locus equations (1.3.2). Each solution of the locus equations was conjectured in [10] to correspond to a particular state in the conformal field theory, although the exact number of solutions of (1.3.2) for all values of the parameters M and l is not known definitively. Numerical investigations have so far corroborated the conjecture of [10], and the case when $M = 1$ has been explored in detail in our paper to appear that will also include work in this chapter.

In this chapter, we begin in section 2.2 by introducing information about the conformal field theories of interest and the spaces of states that define them. We then introduce the relevant ODEs in section 2.3, whose potentials are constrained by conditions on the asymptotics and the requirement of single-valuedness of the solutions of the ODEs. These constraints imply a set of algebraic locus equations that determine the possible ODEs. Lastly, in section 2.4 we consider the solutions of the locus equations more closely, solving an apparent discrepancy between the number of states at certain levels in the conformal field theory and the corresponding admissible ODEs. A more general form of the locus equations than that given in [10] will rectify this mismatch.

2.2 Conformal field theory

In this section we will briefly introduce the relevant concepts relating to conformal field theory (CFT). For a more complete introduction to the subject we refer to the standard text [17]. For our purposes, a CFT is a two-dimensional quantum field theory, with a Hilbert space of states

$$\mathcal{H} = \bigoplus_{\Delta} \mathcal{V}_{\Delta}, \tag{2.2.1}$$

(here we have omitted the anti-holomorphic space of states $\bar{\mathcal{H}}$, populated by subspaces $\mathcal{V}_{\bar{\Delta}}$; the full Hilbert space is then $\mathcal{H} \otimes \bar{\mathcal{H}}$) where the subspaces $\mathcal{V}_\Delta$ are representations of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \tag{2.2.2}$$

generated by a highest weight state $|\Delta\rangle$. The constant parameter c is the central charge of the CFT. A representation (or Verma module [17]) $\mathcal{V}_\Delta$ of the algebra (2.2.2) is generated by a highest weight state $|\Delta\rangle$, defined by

$$L_0 |\Delta\rangle = \Delta |\Delta\rangle\,, \quad L_n |\Delta\rangle = 0 \quad \text{for } n > 1. \tag{2.2.3}$$

The remaining states in $\mathcal{V}_\Delta$ are generated by the repeated action of the raising operators L_{-n}. Using the commutation relations (2.2.2) a general state in $\mathcal{V}_\Delta$ $L_{-k_1}L_{-k_2}\dots L_{-k_m}|\Delta\rangle$, (with $k_1, k_2, \dots, k_m > 0$) is also an eigenstate of L_0:

$$L_0 \left(L_{-k_1}L_{-k_2}\dots L_{-k_m}|\Delta\rangle\right) = (\Delta + k_1 + \dots + k_m)L_{-k_1}L_{-k_2}\dots L_{-k_m}|\Delta\rangle\,. \tag{2.2.4}$$

The representation $\mathcal{V}_\Delta$ then decomposes into a direct sum of subspaces $\mathcal{V}_\Delta^{(L)}$,

$$\mathcal{V}_\Delta = \bigoplus_{L=0}^{\infty} \mathcal{V}_\Delta^{(L)}, \quad L_0\mathcal{V}_\Delta^{(L)} = (\Delta + L)\mathcal{V}_\Delta^{(L)}. \tag{2.2.5}$$

where $L \in \mathbb{Z}_{\geq 0}$ is the level of the subspace $\mathcal{V}_\Delta^{(L)}$. The subspaces $\mathcal{V}_\Delta^{(L)}$ are spanned by $p(L)$ linearly independent states, where $p(L)$ is the number of partitions of the integer L. Labelling the states in $\mathcal{V}_\Delta^{(L)}$ by $\{|1\rangle, |2\rangle, \dots, |p(L)\rangle\}$, we define the Kacs determinant

$$\det\Big(\{\langle i|j\rangle\}_{i,j=1,\dots,p(L)}\Big), \tag{2.2.6}$$

where, if $|i\rangle = L_{-i_1} \dots L_{-i_m} |\Delta\rangle$ and $|j\rangle = L_{-j_1} \dots L_{-j_n} |\Delta\rangle$, then

$$\langle i|j\rangle = \langle\Delta| L_{i_m} \dots L_{i_1} L_{-j_1} \dots L_{-j_n} |\Delta\rangle . \tag{2.2.7}$$

Using the Virasoro algebra (2.2.2) and the properties of the highest weight state (2.2.3), the Kacs determinant for each level subspace $\mathcal{V}_{\Delta}^{(L)}$ may be found as a function of Δ and c. Zeroes of the Kacs determinant indicate the presence of singular vectors $|i\rangle$, which are orthogonal to all other states in $\mathcal{V}_{\Delta}$ and satisfy $\langle i|i\rangle = 0$. These singular vectors are the highest weight states of a sub-representation of the Virasoro algebra, indicating the representation $\mathcal{V}_{\Delta}$ becomes reducible at these points. The singular vectors should also arise naturally in the related set of differential equations, and the authors of [10] gave some evidence that this was indeed the case.

The $\mathbf{Q}_{\pm}$-operators were constructed in [7, 8, 10] as a CFT analogue to Baxter's Q matrices used in the description of the statistical mechanics of six and eight-vertex ice-type models [4]. These $\mathbf{Q}_{\pm}$-operators respect the decomposition of the representation $\mathcal{V}_{\Delta}$ (2.2.5)

$$\mathbf{Q}_{\pm} : \mathcal{V}_{\Delta}^{(L)} \to \mathcal{V}_{\Delta}^{(L)}. \tag{2.2.8}$$

The highest weight state $|\Delta\rangle$ is an eigenvector of the $\mathbf{Q}$-operators

$$\mathbf{Q}_{\pm}(s) |\Delta\rangle = Q_{\pm}^{(vac)}(s) |\Delta\rangle , \tag{2.2.9}$$

where s is a complex parameter. It is the vacuum eigenvalues $Q_{\pm}^{(vac)}(s)$ that correspond to spectral determinants $D_{\pm}(E)$ associated with two eigenvalue problems concerning the Schrödinger equation

$$-\frac{d^2\psi(x)}{dx^2} + \left(x^{2M} + \frac{l(l+1)}{x^2}\right)\psi(x) = E\psi(x), \quad M > 1,\ l > -1/2, \tag{2.2.10}$$

on the positive real axis. The equation (2.2.10) has two solutions in the $|x| \to 0$ limit

$$\chi_+(x,E,l) \sim x^{l+1}, \quad \chi_-(x,E,l) \sim x^{-l} \quad |x| \to 0, \tag{2.2.11}$$

and a unique decaying solution on the positive real axis as $|x| \to \infty$:

$$y(x) \sim x^{-M/2} \exp\left(-\frac{x^{M+1}}{M+1}\right). \tag{2.2.12}$$

We define two eigenvalue problems by searching for eigenvalues $E = E_k^{\mp}$ that produce solutions $\psi(x, E_k^{\pm}, l)$ satisfying

$$\psi(x, E_k^{\pm}, l) \sim \chi_\pm(x) \quad \text{as } |x| \to 0, \tag{2.2.13}$$

$$\psi(x, E_k^{\pm}, l) \sim y(x) \quad \text{as } |x| \to \infty. \tag{2.2.14}$$

These eigenvalues $E_k^{\pm}$ then define the spectral determinants

$$D_\mp(E) = D_\mp(0)\prod_{k=1}^{\infty}\left(1 - \frac{E}{E_k^{\mp}}\right). \tag{2.2.15}$$

The key result of the example of the ODE/IM correspondence we considered in the introduction was the relation between the spectral determinants $D_\mp(E)$ and the vacuum eigenvalues of the **Q**-operators in the following way [10]:

$$Q_\pm^{(vac)}(s) = (-s)^{\pm\frac{2l+1}{4}} D_\mp(\nu s), \tag{2.2.16}$$

where the constant ν is given by

$$\nu = \left(\frac{2\sqrt{\pi}\Gamma(\frac{3}{2} + \frac{1}{2M})}{\Gamma(1 + \frac{1}{2M})} \right). \tag{2.2.17}$$

The constants M and l are related to the central charge c (defined in (1.2.1)) and the highest weight Δ (1.2.2) in the following way

$$c = 1 - \frac{6M^2}{M+1}, \quad \Delta = \frac{(2l+1)^2 - 4M^2}{16(M+1)}. \tag{2.2.18}$$

The result of [10] was to extend the correspondence between the vacuum eigenvalues of the $\mathbf{Q}$-operators and the spectral determinants of the Schrödinger equation (2.2.10) to excited eigenvalues of the $\mathbf{Q}$-operators, corresponding to eigenstates in $\mathcal{V}_\Delta^{(L)}$ with $L > 0$. The corresponding differential equations are of the form

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \tag{2.2.19}$$

where the so-called monstrous [10] potentials $V(x)$ are given by

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 2\frac{d^2}{dx^2}\sum_{k=1}^{L} \log\bigl(x^{2M+2} - z_k\bigr), \tag{2.2.20}$$

and the constants $\{z_k\}_{k=1}^L$ (with $z_j \neq z_k$) satisfy the algebraic locus equations (1.3.2).

In the next section we will derive the monstrous potentials (2.2.20) and the locus equations (1.3.2) constraining the parameters $\{z_k\}_{k=1}^L$, from constraints on the asymptotic and single-valuedness properties of the potentials. A similar calculation is performed in sections 4.4-4.6 of [31]: in that paper the authors work with $\hat{\mathfrak{sl}}_2$-opers which are equivalent to second-order Schrödinger operators.

2.3 Algebraic locus equations

We begin with the general Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \tag{2.3.1}$$

and we note [10] that eigenvalue problems of this form (with the boundary conditions (2.2.13) and (2.2.14)) correspond to eigenvalues of the **Q**-operators if and only if the potential $V(x)$ satisfies the following properties:

1. $$V(\omega x) = \omega^{-2}V(x), \quad \text{where } \omega = e^{i\pi/(M+1)},$$

   (this symmetry ensures that if $\chi(x, E, l)$ is a solution of (2.3.1), rotated functions of the form (1.1.14) are also solutions of (2.3.1)),

2. $$V(x) \sim \frac{l(l+1)}{x^2} \quad \text{as } |x| \to 0,$$

3. $$V(x) \sim x^{2M} \quad \text{as } |x| \to \infty,$$

4. For any value of E all solutions $\psi(x, E, l)$ of (2.3.1) are single-valued except at $x = 0$ and $x = \infty$. By this, we mean for any solution $\psi(x, E, l)$ and any $x' \in \mathbb{C} \setminus \{0\}$, $\psi(x, E, l)$ has a convergent Laurent series in some sufficiently small punctured neighbourhood of x'.

With these conditions, the spectral determinants associated with (2.3.1) satisfy the same analytic properties and functional relations as the corresponding eigenvalues of $\mathbf{Q}_{\pm}(s)$. To implement these conditions, we rewrite $V(x)$ as

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} + v(x). \tag{2.3.2}$$

Property 1 implies

$$v(x) = x^{-2} F(x^{2M+2}), \tag{2.3.3}$$

where F is a rational function of x^{2M+2}. Properties 2 and 3 constrain the function F further, mandating that

$$F(0) = 0, \quad |F(\infty)| < \infty. \tag{2.3.4}$$

These constraints on F along with Liouville's theorem imply that there exist poles at finite values of x. Following appendix B of [10], consider the Laurent expansion of $V(x)$ about a given pole $x = x_{k,p}$

$$V(x) = \sum_{m=-\infty}^{\infty} (x - x_{k,p})^m V_m, \tag{2.3.5}$$

where we will see the double index $x_{k,p}$ is a convenient labelling for the poles of $V(x)$. The Laurent expansion (2.3.5) is constrained by Property 4 above; to ensure the single-valuedness of the solution $\psi(x, E, l)$ we invoke a result due to Duistermaat and Grünbaum (Proposition 3.3 in [29]), which states $\psi(x, E, l)$ is single-valued about $x = x_{k,p}$ if and only if the coefficients of the Laurent expansion of $V(x)$ satisfy the following conditions:

$$V_n = 0, \qquad \text{where } n < -2, \tag{2.3.6}$$

$$V_{-2} = \nu_{k,p}(\nu_{k,p} + 1), \qquad \text{where } \nu_{k,p} \in \mathbb{Z}_{\geq 0}, \tag{2.3.7}$$

$$V_{2k-1} = 0, \qquad \text{where } k = 0, 1, \ldots, \nu_{k,p}. \tag{2.3.8}$$

For now, we consider the simplest non-trivial case considered in [10], where all the poles have $\nu_{k,p} = 1$. We will see that there are particular values of L, l and M where this assumption breaks down, but for generic values of L, l and M the following computation of the locus equations (1.3.2) will be valid. We will discuss the cases where the locus equations break down in section 2.4.

The boundedness of F in the limit $|x| \to \infty$ implies that the potential $V(x)$ may be written as a sum over Laurent expansions about its poles $x = x_{k,p}$:

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} + \sum_{k,p} \frac{2}{(x - x_{k,p})^2} \tag{2.3.9}$$

where the constraint $V_{-1} = 0$ in (2.3.8) implies the poles at $x = x_{k,p}$ are double poles. The symmetry constraint imposed by Property 1 also constrains the poles to be $(2M+2)^{\text{th}}$ roots of some constants z_k, so that

$$x_{k,p} = z_k^{1/(2M+2)} e^{2\pi i p/(2M+2)}, \quad p = 0, 1, \ldots, 2M+1. \tag{2.3.10}$$

This pattern for the roots is only valid for rational M. The final locus equations are valid for all M by continuity from rational M. The sum in (2.3.9) then takes the form

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} + \sum_{k=1}^{L} \sum_{p=0}^{2M+1} \frac{2}{(x - x_{k,p})^2}, \tag{2.3.11}$$

which we rewrite as a sum of second derivatives of logarithms

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 2\frac{d^2}{dx^2} \sum_{k=1}^{L} \sum_{p=0}^{2M+1} \log(x - x_{k,p}), \tag{2.3.12}$$

which simplifies using (2.3.10)

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 2\frac{d^2}{dx^2}\sum_{k=1}^{L}\log\bigl(x^{2M+2} - z_k\bigr), \tag{2.3.13}$$

which matches the form of the monstrous potential given in [10]. With this general form of the potential, we now enforce the additional constraints on the Laurent expansion of (2.3.13) at its poles, given by (2.3.8). Specifically, for the case $\nu_{k,p} = 1$ we consider here, we require the component V_1 in the Laurent expansion (2.3.5) about each of the poles of $V(x)$ to be zero.

Without loss of generality, let us consider the Laurent expansion of $V(x)$ about a pole $x = w$, where $w^{2M+2} = z_k$. To aid in the calculation of the coefficient V_1 of $(x - w)$ in this Laurent expansion, we rewrite $V(x)$ in a more convenient form, separating the contributions from the roots of z_k from the other roots

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 2\frac{d^2}{dx^2}\sum_{q=0}^{2M+1}\log\Bigl(x - we^{\frac{2\pi iq}{2M+2}}\Bigr) - 2\frac{d^2}{dx^2}\sum_{j\neq k}\log\bigl(x^{2M+2} - z_j\bigr) \tag{2.3.14}$$

The term proportional to $(x - w)$ in the Laurent expansion of $V(x)$ is given by

$$\begin{aligned} -\frac{2l(l+1)}{w^3} + 2Mw^{2M-1} - 2\sum_{q=1}^{2M+1}\frac{d^3}{dx^3}\log\Bigl(x - we^{\frac{2\pi iq}{2M+2}}\Bigr)\Bigg|_{x=w} \\ - 2\sum_{j\neq k}\frac{d^3}{dx^3}\log\bigl(x^{2M+2} - z_j\bigr)\Bigg|_{x=w}. \end{aligned} \tag{2.3.15}$$

We set (2.3.15) equal to zero and evaluate the derivatives

$$\begin{aligned} &-\frac{2l(l+1)}{w^3} + 2Mw^{2M-1} - \frac{4}{w^3}\sum_{q=1}^{2M+1}\frac{1}{(1 - e^{\frac{2\pi iq}{2M+2}})^3} \\ &-\frac{8(1+M)}{w^3}\sum_{j\neq k}\frac{a(z_j, z_k, M)}{(z_k - z_j)^3} = 0, \end{aligned} \tag{2.3.16}$$

where $a(z_j, z_k, M)$ is the polynomial

$$a(z_j, z_k, M) = (2+2M)^2 z_k^3 - 3(1+2M)(1+M)z_k(z_k - z_j) \\ + M(1+2M)z_k(z_k - z_j)^2. \tag{2.3.17}$$

The sum of roots of unity in (2.3.16) is given by [36]:

$$\sum_{q=1}^{2M+1} \frac{1}{(1 - e^{\frac{2\pi i q}{2M+2}})^3} = \frac{1 - 4M^2}{8}. \tag{2.3.18}$$

After algebraic manipulation, (2.3.16) then simplifies to the locus equations

$$\sum_{j \neq k} \frac{z_k(z_k^2 + (1+2M)(3+M)z_k z_j + M(1+2M)z_j^2)}{(z_k - z_j)^3} - \frac{M z_k}{4(1+M)} + \Delta = 0. \tag{2.3.19}$$

The solutions $(z_1, z_2, \ldots, z_L)$ of the locus equations (1.3.2) up to permutations of z_k define monstrous potentials (2.2.20) which themselves define eigenvalue problems with their associated Schrödinger equations. For a given level L and for generic l and M, there should then be $p(L)$ solutions of the locus equations, corresponding to the $p(L)$ states in the subspace $\mathcal{V}_\Delta^{(L)}$. For certain values of M and l, the Kacs determinant (2.2.6) will be zero, indicating the presence of a singular vector in the space $\mathcal{V}_\Delta^{(L)}$. As an example, we compute the Kacs determinant of $\mathcal{V}_\Delta^{(2)} = \{L_{-2}\left|\Delta\right\rangle, L_{-1}^2\left|\Delta\right\rangle\}$ using the Virasoro commutation relations (2.2.2):

$$\begin{vmatrix} \langle\Delta| L_2 L_{-2} |\Delta\rangle & \langle\Delta| L_1^2 L_{-2} |\Delta\rangle \\ \langle\Delta| L_2 L_{-1}^2 |\Delta\rangle & \langle\Delta| L_1^2 L_{-1}^2 |\Delta\rangle \end{vmatrix} = \begin{vmatrix} 4\Delta + c/2 & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{vmatrix} \tag{2.3.20}$$

$$= 2\Delta(16\Delta^2 + 2(4+c)\Delta + c - 18).$$

We set this Kacs determinant equal to zero and see that, for

$$2\Delta(16\Delta^2 + 2(4+c)\Delta + c - 18) = 0, \quad c = 1 - \frac{6M^2}{M+1}, \tag{2.3.21}$$

$$\implies \Delta = 0, \frac{1-2M}{4(1+M)}, \text{ or } \frac{1+3M}{4}. \tag{2.3.22}$$

These roots match the expressions in [17] for the roots of the Kac determinant for the level subspace $\mathcal{V}_\Delta^{(L)}$,

$$\Delta_{r,s}\left(\frac{1}{M}\right) = \frac{(rM + (r-s))^2 - M^2}{4(M+1)}, \quad r, s \geq 1, \ rs \leq L. \tag{2.3.23}$$

We find

$$\Delta_{1,1}\left(\frac{1}{M}\right) = 0, \quad \Delta_{1,2}\left(\frac{1}{M}\right) = \frac{1-2M}{4(1+M)}, \quad \Delta_{2,1}\left(\frac{1}{M}\right) = \frac{1+3M}{4}. \tag{2.3.24}$$

If Δ matches one of these roots, a singular vector arises in the associated CFT. At these roots, one of the solutions (z_1, z_2) of the associated locus equations (2.3.19) disappears as one or both of the z_i goes to zero. The number of solutions of the locus equations should then match the number of non-singular vectors in $\mathcal{V}_L^\Delta$.

For $L \geq 3$, however, there exist points in the (l, M) parameter space where the number of solutions of the locus equations (1.3.2) reduces, and yet the Kacs determinant is non-zero, indicating the absence of any singular vectors. Numerical investigation of the locus equations uncovered this peculiar behaviour at the point $L = 3$, $l = 3/4$, $M = 1$. As $l \to 3/4$, one of the solutions (z_1, z_2, z_3) converges on the point $(-15/16, -15/16, -15/16)$, with the solution disappearing entirely at the point $l = 3/4$. The locus equations cannot describe this solution, as they were derived with the assumption that the constants z_j were pairwise distinct, which obviously fails to be true at this so called 'triple point'. In the next section, we will investigate the nature of these triple points and present a method of locating them.

2.4 Solutions of the locus equations

The algebraic locus equations in the form (1.3.2) cease to have validity at points in the (l, M) parameter space where any of the solutions z_j fail to be distinct. How then do we handle solutions like the triple point at $L = 3$, $l = 3/4$, $M = 1$? This problem is resolved by considering the result (2.3.7)-(2.3.8) due to Duistermaat and Grünbaum again; recall that we chose the integers $\nu_{k,p} = 1$, following the authors of [10]. We may, in principle, relax this condition, although (2.3.8) implies we must now ensure the cubic and other terms in the Laurent expansion must be zero as well. For general $\nu_{k,p}$ we then have a set of locus equations, which must be simultaneously solved to locate the points where the solutions z_j coalesce for a given l and M.

To demonstrate this, we consider the potential

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 6\frac{d^2}{dx^2}\log\bigl(x^{2M+2} - z_1\bigr). \tag{2.4.1}$$

This potential has poles at $x = x_{1,p} = z_1^{1/(2M+2)} e^{2\pi i p/(2M+2)}$, and the Laurent expansion of $V(x)$ at each of these poles has dominant behaviour

$$V(x) \sim \frac{6}{(x - x_{1,p})^2} + \dots \quad \text{as } x \to x_{1,p}, \tag{2.4.2}$$

i.e. we have set $\nu_{1,p} = 2$ in (2.3.7). This choice of $\nu_{1,p}$ means that about any pole $x = w$, the terms proportional to both $(x - w)$ and $(x - w)^3$ must be zero. We therefore expand $V(x)$ about an arbitrary pole $x = w$, with $w^{2M+2} = z_1$, and consider the linear and cubic terms of the Laurent expansion.

Setting the linear term of the Laurent expansion of $V(x)$ equal to zero yields

$$-2l(l+1) + 2Mz_1 - 12 \sum_{q=1}^{2M+1} \frac{1}{(1-e^{\frac{2\pi i q}{2M+2}})} = 0. \tag{2.4.3}$$

We recall the sum of roots of unity in (2.4.3) was evaluated in (2.3.18), so that

$$-2l(l+1) + 2Mz_1 - \frac{3(1-4M^2)}{2} = 0, \tag{2.4.4}$$

is the constraint on z_1, l, M from the constraint $V_1 = 0$ in (2.3.8).

The cubic term set equal to zero yields

$$-24l(l+1) + 2M(2M-1)(2M-2)z_1 - 144 \sum_{q=1}^{2M+1} \frac{1}{(1-e^{\frac{2\pi i q}{2M+2}})^5} = 0, \tag{2.4.5}$$

with the sum of roots of unity in this expression given by

$$\sum_{q=1}^{2M+1} \frac{1}{(1-e^{\frac{2\pi i q}{2M+2}})^5} = \frac{(2M-3)(2M+1)(4M^2+20M-3)}{288}. \tag{2.4.6}$$

The cubic term then yields an additional constraint on l, M and z_1 which must be satisfied to allow the presence of a triple point

$$\begin{aligned} &-48l(l+1) + 4M(2M-1)(2M-2)z_1 \\ &\qquad - (2M-3)(2M+1)(4M^2+20M-3) = 0. \end{aligned} \tag{2.4.7}$$

The presence of two constraints on (l, M, z) indicates that triple points will only occur at certain values of l and M. As a check on our calculation, we substitute $l = 3/4$ and $M = 1$ into the equations (2.4.4) and (2.4.7). The second of these reduces to zero; the first of these yields a linear equation for z_1

$$\frac{15}{8} + 2z_1 = 0 \tag{2.4.8}$$

which proves the existence of a triple point at $z_1 = -15/16$ for $l = 3/4$ and $M = 1$. We have therefore demonstrated that if the potential (2.3.13) can be rewritten in the form (2.4.1) (i.e. if $z_1 = z_2 = z_3$) the locus equations (2.3.19) are no longer sufficient to determine the values of the z_i. In the case of these triple points, the cubic term of the Laurent expansion potential $V(x)$ about $x = w$, with $w^{2M+2} = z_1$ must be set equal to zero, satisfying Duistermaat's condition for single-valuedness (2.3.8).

One may of course consider more general potentials of the form

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - \sum_{k=1}^{L} \nu_k(\nu_k + 1)\frac{d^2}{dx^2}\log\bigl(x^{2M+2} - z_k\bigr), \tag{2.4.9}$$

where ν_k is an integer ≥ 1. For $\nu_k \geq 1$, higher-order terms of odd power in the Laurent expansion of the potential must be zero, as decreed by (2.3.8). Considering more general potentials of the form (2.4.9) allows the general analysis of points where the solutions of the original algebraic locus equations (1.3.2) coincide.

As a final note, we have yet to find an example of a 'sextuple point', or other more complicated examples. In principle, points where $\nu(\nu+1)/2$ solutions (where $\nu = 0, 1, 2, \dots$) of the original locus equations (1.3.2) coalesce are possible. However, we have already seen the presence of triple points constrains the allowed values of l and M by imposing an additional equation l, M and the z_k must satisfy. Higher order points such as the sextuple point can only occur at specific values of l, M and z_k.

We illustrate a sextuple point by considering the potential

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 12\frac{d^2}{dx^2}\log\left(x^{2M+2} - z_1\right) \tag{2.4.10}$$

where $\nu_{1,p} = 3$ for $p = 0, \ldots, 2M-1$. For solutions of the associated Schrödinger eigenvalue problem to be single-valued about a pole $x = w$ ($w^{2M+2} = z_1$), we require the linear, cubic, and quintic terms of the Laurent expansion of (2.4.10) about $x = w$ to vanish. This leads to three equations in l, M and z_1:

$$2M(6M + z_1) - 2l(l+1) - 3 = 0, \tag{2.4.11}$$

$$\begin{aligned} 9 + 24l(l+1) + 8M(-6 + M(-13 + 2M(4 + M))) \quad & (2.4.12) \\ -4(M-1)M(2M-1)z_1 = 0, \quad & \end{aligned}$$

$$\begin{aligned} -135 - 1440l(l+1) + 16M(144 + 120M - 428M^2 - 27M^3 + 48M^4 \quad & (2.4.13) \\ +8M^5 + (-2+M)(-1+M)(-3+2M)(-1+2M)z_1) = 0. \quad & \end{aligned}$$

Exploring the solution space of these coupled polynomial equations, we have found only two solutions that satisfy both $l \geq -1/2$ and $M > 0$. They are

$$l = 0.214905263947..., M = 0.185911063538..., z_1 = 8.35728635815... \tag{2.4.14}$$

$$l = 14.56857388290..., M = 3.263779478909..., z_1 = 50.3705538334... \tag{2.4.15}$$

(2.4.11) and the reality of l and M (due to the inequalities on l and M) enforces the reality of z_1. Numerical investigation of the original locus equations (1.3.2) indicates as predicted the coalescence of the six points z_i in one of the eleven solutions of (1.3.2) near these points.

Points where $\nu_k = 4$ (where 10 solutions of the locus equations coincide) would require the coefficients of $(x-w)$, $(x-w)^3$, $(x-w)^5$ and $(x-w)^7$ in the Laurent expansion of $V(x)$ about $x = w$. This will induce an overdetermined system of

$L+3$ equations in $L+2$ parameters l, M and $z_1, \ldots, z_L$. Numerical investigation of the locus equations associated with the potential

$$V(x) = \frac{l(l+1)}{x^2} + x^{2M} - 20\frac{d^2}{dx^2}\log\bigl(x^{2M+2} - z_1\bigr), \tag{2.4.16}$$

revealed no solutions l, M, z_1 which satisfied the four equations in the space $l \geq -1/2$, $M > 0$.

2.5 Conclusions

In this section we have studied the algebraic locus equations given in [10]. We have given a derivation of these equations, and solved an apparent mismatch between the number of states in a given level L of the state space of a conformal field theory at certain values of the central charge c and highest weight Δ and the number of solutions of the locus equations (2.3.19). This problem was resolved by considering higher-order terms in the Laurent expansion of the potential $V(x)$, setting them equal to zero as mandated by Duistermaat's conditions (2.3.8). This generates a set of generalised locus equations, and the solutions of the locus equations in this more general setting then account for all the states in the conformal field theory.

In principle, the study of excited eigenstates of the **Q**-operators defined on $A_1^{(1)}$ conformal field theories should extend to all the other field theories we have considered in this thesis. Particularly, the excited eigenstates conformal field theories with $A_r^{(1)}$ Lie algebra symmetry should be straightforward to match with more exotic differential operators, perhaps depending on sets of parameters $z_{k_1}^{(1)}, z_{k_2}^{(2)}, \ldots z_{k_r}^{(r)}$, with $1 \leq k_i \leq p(L_i)$ and with $\{L_1, \ldots, L_r\}$ being a set of r integers. To perform this generalisation, however, we first require a result similar to Duistermaat and Grünbaum's result in [29], guaranteeing single-valuedness of solutions of differential operators of the form

$$-\frac{d^{r+1}\psi}{dx^{r+1}} + a_1(x)\frac{d^{r-1}\psi}{dx^{r-1}} + \cdots + +a_{n-2}(x)\frac{d\psi}{dx} + V(x)\psi(x) = E\psi(x) \qquad (2.5.1)$$

about poles in the coefficients $a_1(x), \ldots, a_{n-2}(x), V(x)$. With this result and a suitable generalisation of the asymptotic and symmetry properties given in section [10], it will be possible to derive a class of suitable monstrous (even more so) potentials and to relate their spectral determinants to eigenvalues of the associated $\mathbf{Q}_i$-operators in the $A_r^{(1)}$ conformal field theory.

Chapter 3

The massive ODE/IM correspondence

3.1 Introduction

The examples of the ODE/IM correspondence we have seen thus far have related the spectral determinants of second-order Schrödinger-type differential operators to the eigenvalues of **Q**-operators that appear in certain conformal field theories. We now consider another major generalisation of the ODE/IM correspondence, first indicated in [11] and applied by Lukyanov and Zamolodchikov in [45], which extends the ODE/IM correspondence to massive integrable field theories. The story starts with classical partial differential equations (PDEs), with a Lax pair representation defining associated systems of differential equations. Q-functions are then defined for these systems, and it is these that contain information on the ground state eigenvalues of the **Q**-operators in the massive integrable field theory.

In later chapters, we will explore this massive ODE/IM correspondence related to classical PDEs related to the simply-laced Lie algebras. We first describe the smallest non-trivial simple Lie algebra $A_1^{(1)}$, as in [45], but introducing notation that will generalise easily to the cases involving larger Lie algebras. In section 3.2 we define the relevant PDE, the modified sinh-Gordon equation, and its Lax pair representation, a pair of systems of differential equations. The solutions of these systems define Q-functions, which are the objects of study in section 3.4. The Q-functions satisfy functional relations and Bethe ansatz equations related to the integrable field theory, and from these, we define a non-linear integral equation and use this equation in section 3.5 to derive expressions for the integrals of motion of the associated massive integrable field theory.

The above procedure, which will provide the framework for our study of more general Lie algebras, is summarised in Figure 1. We will not consider the T-functions for the $A_1^{(1)}$ case in this chapter; this topic will be covered, along with the T-functions for the $A_r^{(1)}$ case, in chapter 5. The Ψ-system is also unnecessary for the $A_1^{(1)}$ case, as the quantum Wronskian is sufficient in this case to derive the $A_1^{(1)}$ Bethe ansatz equations.

3.2 The modified sinh-Gordon equation

3.2.1 The Lax pair representation

Lukyanov and Zamolodchikov [45] began with the modified sinh-Gordon equation, given by

$$\beta\partial_z\partial_{\bar{z}}\phi - m^2e^{2\beta\phi} + p(z)p(\bar{z})m^2e^{-2\beta\phi} = 0, \tag{3.2.1}$$

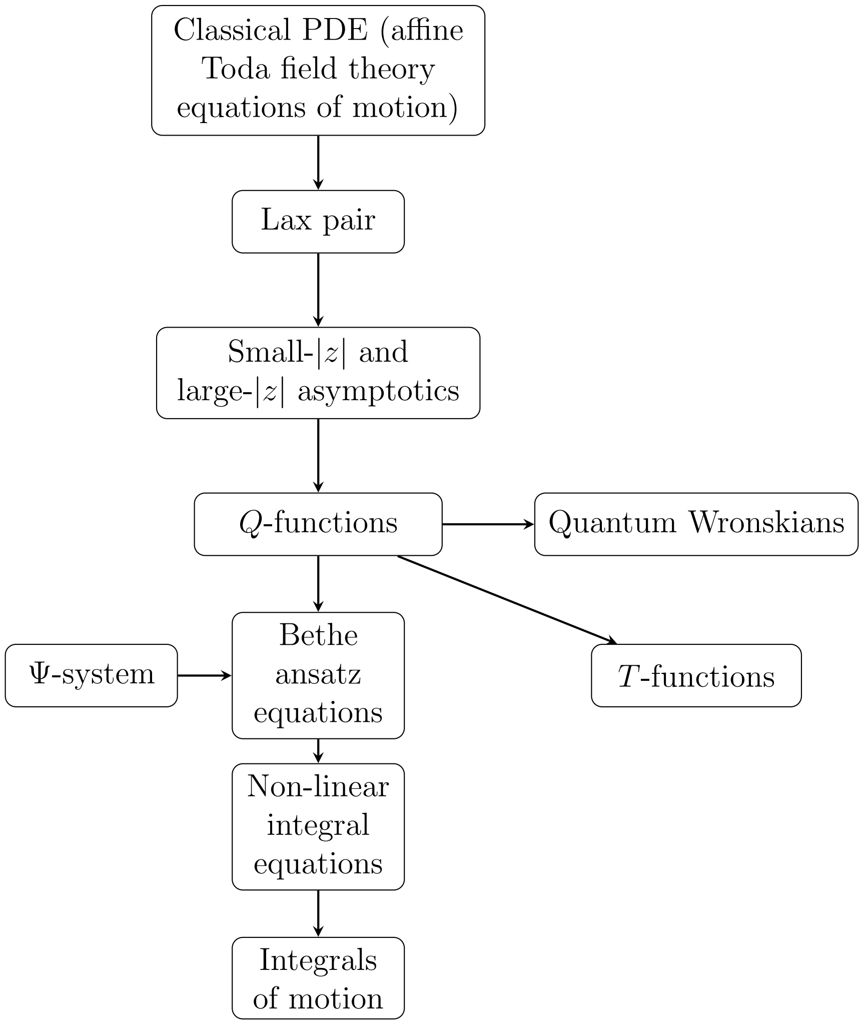


Figure 1: Diagram outlining the procedure that will be followed for the study of the massive ODE/IM correspondence for the simply-laced Lie algebras.

where $\phi(z,\bar{z})$ is a scalar field in the independent complex coordinates z and $\bar{z}$, β is a dimensionless coupling constant, m is a mass parameter, and

$$p(z) = z^{2M} - s^{2M}, \tag{3.2.2}$$

where $M > 0$, and $s > 0$. The constant β can be removed by rescaling $\phi \to \phi/\beta$, but we will retain it to match notation which matches that found in [37, 38] which generalises more readily to larger Lie algebras. We are also solely concerned with real solutions to (3.2.1). We will therefore treat z and $\bar{z}$ as independent complex variables, but we will only consider the solutions of (3.2.1) on the subset of $\mathbb{C}^2$ where $\bar{z} = z^*$.

The result of [45] was to connect the modified sinh-Gordon equation (3.2.1) to the quantum sine- and sinh-Gordon massive integrable field theories. They began this process by recasting (3.2.1) in the form of a Lax pair. Following [45], we define the generators $\{H, E_\pm\}$ of the Lie algebra $A_1 = \mathfrak{su}(2)$, and the commutation relations that define that algebra

$$[H, E_\pm] = \pm 2E_\pm, \quad [E_+, E_-] = H. \tag{3.2.3}$$

We then define the Lax pair

$$(\partial_z + A)\Psi = 0, \tag{3.2.4}$$

$$(\partial_{\bar{z}} + \bar{A})\Psi = 0, \tag{3.2.5}$$

where A and $\bar{A}$ are given by

$$A = \frac{\beta}{2}\partial_z\phi\, H + me^{\theta}e^{\beta\phi}\, E_+ + me^{\theta}p(z)e^{-\beta\phi}\, E_-, \tag{3.2.6}$$

$$\bar{A} = -\frac{\beta}{2}\partial_{\bar{z}}\phi\, H + me^{-\theta}e^{\beta\phi}\, E_- + me^{-\theta}p(\bar{z})e^{-\beta\phi}\, E_+. \tag{3.2.7}$$

The modified sinh-Gordon equation (3.2.1) may then be recovered from the compatibility condition

$$\partial_z \bar{A} - \partial_{\bar{z}} A + [A, \bar{A}] = 0, \tag{3.2.8}$$

using the commutation relations (3.2.3).

In this chapter, we work with the fundamental representation of A_1

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.2.9}$$

(In [45], $H = \sigma_3$, $E_\pm = \sigma_\pm$.) In the representation (3.2.9), the Lax pair (3.2.4)-(3.2.5) form two two-dimensional systems of differential equations. The solutions Ψ of these systems of equations in the $|z| \to 0$ and $|z| \to \infty$ limits will allow us to define Q-functions which will encode information on the related massive integrable field theory.

The presence of the exponential terms in the matrices A and $\bar{A}$ make a consideration of the asymptotics of the Lax pair equations (3.2.4)-(3.2.5) more complicated, and make a connection to the eigenvalue problem (1.1.5) discussed in the Introduction more opaque. To remedy this, we define, for an arbitrary 2-by-2 matrix $U(z, \bar{z})$, a gauge transformation

$$\begin{aligned} &A \to UAU^{-1} + U\partial_z U^{-1}, \\ &\bar{A} \to U\bar{A}U^{-1} + U\partial_{\bar{z}} U^{-1}, \quad \Psi \to U\Psi. \end{aligned} \tag{3.2.10}$$

Using $\partial(UU^{-1}) = U\partial U^{-1} + \partial U U^{-1} = 0$, (where $\partial = \partial_z$ or $\partial_{\bar{z}}$) it is straightforward to show that the Lax pair equations (3.2.4)-(3.2.5) and the compatibility condition (3.2.8) are invariant under the gauge transformation (3.2.10). By an astute choice of gauge it is then possible to remove the exponential terms from A or $\bar{A}$, although this is not possible for both simultaneously.

We demonstrate the utility of this gauge transformation by setting $U = e^{-\beta\phi H/2}$, where the exponential of an operator X is defined in the standard way as a power series

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}. \tag{3.2.11}$$

Gauge transforming A using this matrix U has the effect of removing the inconvenient exponential terms from A:

$$A \to \widetilde{A} = \beta\partial_z\phi\, H + me^{\theta}E_+ + me^{\theta}p(z)E_- \tag{3.2.12}$$

where the derivation of (3.2.12) uses the identity

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots. \tag{3.2.13}$$

Under this gauge transformation, $\bar{A}$ becomes

$$\bar{A} \to \widetilde{\bar{A}} = me^{-\theta}e^{2\beta\phi}E_- + me^{-\theta}p(\bar{z})e^{-2\beta\phi}E_+, \tag{3.2.14}$$

retaining the exponential terms. If we wish to consider the linear system $(\partial_{\bar{z}} + \bar{A})\Psi = 0$ with the exponential terms removed, we must perform another gauge transformation on the original Lax pair (3.2.4)-(3.2.5) with $U = e^{\beta\phi H/2}$. We will mostly work with the choice of gauge defined by $U = e^{-\beta\phi H/2}$, removing the exponential terms from the holomorphic equation (3.2.4). This choice of gauge does not affect the final outcome of our calculations; it is merely helpful to consider the asymptotics of a simpler form of one of the Lax pair equations and then undo the gauge transformation to find the asymptotic solutions of the original equations.

It is also useful to introduce the Symanzik rotation [37] Ω_k for $k \in \mathbb{Z}$:

$$\Omega_k : z \to z e^{\pi i k/M}, \ \bar{z} \to \bar{z} e^{-\pi i k/M}, \ \theta \to \theta - \frac{\pi i k}{M}, \ s \to s e^{\pi i k/M}. \tag{3.2.15}$$

Under a Symanzik rotation, the matrices A and $\bar{A}$ are rotated in the complex plane

$$A \to e^{-\pi i k/M} A, \quad \bar{A} \to e^{\pi i k/M} \bar{A}. \tag{3.2.16}$$

The derivative operators ∂_z and $\partial_{\bar{z}}$ have the same respective behaviours under a Symanzik rotation and so the linear systems (3.2.4)-(3.2.5) are invariant under Symanzik rotation. Any Symanzik rotation $\Omega_k[\Psi]$ of the linear systems is also a solution of the linear systems. We will often exploit this property, defining solutions of the linear systems (3.2.4)-(3.2.5) that respect the Symanzik rotation.

3.2.2 Solutions of the modified sinh-Gordon equation

The gauge-transformed linear system $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$, written out using the fundamental representation (3.2.9), is given by

$$\begin{pmatrix} \partial_z + \beta \partial_z \phi & m e^{\theta} \\ m e^{\theta} p(z) & \partial_z - \beta \partial_z \phi \end{pmatrix} \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} = 0, \tag{3.2.17}$$

where $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2)^T$. To analyse the asymptotics of the solutions of the system of equations (3.2.17) we must first define a particular solution $\phi(z, \bar{z})$ of the modified sinh-Gordon equation (3.2.1). Following [37] we choose a solution $\phi(z, \bar{z})$ which satisfies the following conditions:

- $\phi(z, \bar{z})$ should be real and finite everywhere, except at $|z| = 0$.

- Periodicity

$$\phi(ze^{\pi ik/M}, \bar{z}e^{-\pi ik/M}) = \phi(z, \bar{z}), \quad k \in \mathbb{Z}, \tag{3.2.18}$$

- Large-$|z|$ asymptotics:

$$\phi(z, \bar{z}) = \frac{M}{2\beta} \log(z\bar{z}) + o(1) \quad \text{as } |z| \to \infty, \tag{3.2.19}$$

- Small-$|z|$ asymptotics (where $g \in \mathbb{R}$):

$$\phi(z, \bar{z}) = g \log z\bar{z} + O(1) \quad \text{as } |z| \to 0. \tag{3.2.20}$$

The constant g is not entirely free; it is constrained by the requirement that $g \log z\bar{z}$ is the dominant behaviour for ϕ in the small-$|z|$ limit. To see this, substitute the ansatz

$$\phi(z, \bar{z}) = g \log z\bar{z} + f(z, \bar{z}) \tag{3.2.21}$$

into the modified sinh-Gordon equation (3.2.1). The result is an equation for $f(z, \bar{z})$

$$z\bar{z}\partial_z\partial_{\bar{z}} f = \frac{2m^2}{\beta}(z\bar{z})^{1+2\beta g} e^{2\beta f} - \frac{2m^2}{\beta}(z^{2M} - s^{2M})(\bar{z}^{2M} - s^{2M})(z\bar{z})^{1-2\beta g} e^{-2\beta f}. \tag{3.2.22}$$

$f(z, \bar{z})$ is then expanded as a power series in powers of $(z\bar{z})^{1\pm 2\beta g}$, z^{2M}, $\bar{z}^{2M}$:

$$f(z, \bar{z}) = \sum_{a_0, a_1, b, c = 0}^{\infty} F(a_0, a_1, b, c)(z\bar{z})^{a_0(1-2\beta g) + a_1(1+2\beta g)} z^{2bM} \bar{z}^{2cM}, \tag{3.2.23}$$

where $F(a_0, a_1, b, c)$ are constants fixed by the substitution of (3.2.23) into the modified sinh-Gordon equation (3.2.1). Our desired solution ϕ must satisfy (3.2.20) in the small-$|z|$ limit, implying that $f(z, \bar{z}) = O(1)$ in that limit and that all powers of z and $\bar{z}$ in (3.2.23) must be positive. This leads to the constraints on g

$$1 - 2\beta g > 0, \quad 1 + 2\beta g > 0 \implies |\beta g| < 1/2. \tag{3.2.24}$$

Setting $\beta g = l$, this constraint matches the constraint $|l| < 1/2$ in [45].

With the solution of the modified sinh-Gordon equation (3.2.1) fixed, we now consider the asymptotic solutions of the Lax pair in the small-$|z|$ and large-$|z|$ limits. These will allow us to define the Q-functions which contain information on the quantum sine-Gordon massive integrable field theory.

3.3 Asymptotics of the linear systems

We first consider the gauge-transformed linear system $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$, as given by equation (3.2.17). Having chosen a solution $\phi(z, \bar{z})$ of the modified sinh-Gordon equation, we now consider the linear system (3.2.17) in the small-$|z|$ and large-$|z|$ limits.

3.3.1 Small-$|z|$ asymptotics of the linear systems

Substituting the small-$|z|$ behaviour of ϕ into the linear system (3.2.17), we find

$$\begin{pmatrix} \partial_z + \frac{\beta g}{z} & me^{\theta} \\ me^{\theta} p(z) & \partial_z - \frac{\beta g}{z} \end{pmatrix} \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} = 0, \tag{3.3.1}$$

a system of equations solely in z. We then take the $|z| \to 0$ limit; in this limit, the off-diagonal terms become irrelevant, and the system becomes

$$\begin{pmatrix} \partial_z + \frac{\beta g}{z} & 0 \\ 0 & \partial_z - \frac{\beta g}{z} \end{pmatrix} \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} = 0. \tag{3.3.2}$$

This system is a decoupled pair of equations, and has the pair of solutions

$$c_0 \begin{pmatrix} z^{-\beta g} \\ 0 \end{pmatrix}, \quad c_1 \begin{pmatrix} 0 \\ z^{\beta g} \end{pmatrix}, \tag{3.3.3}$$

where c_0 and c_1 are arbitrary constants. We then define two solutions $\widetilde{\Xi}_0$ and $\widetilde{\Xi}_1$ of $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$, defined by their asymptotics in the small-$|z|$ limit

$$\widetilde{\Xi}_0 \sim c_0 \begin{pmatrix} z^{-\beta g} \\ 0 \end{pmatrix}, \quad \widetilde{\Xi}_1 \sim c_1 \begin{pmatrix} 0 \\ z^{\beta g} \end{pmatrix}, \quad \text{as } |z| \to 0. \tag{3.3.4}$$

To find the small-$|z|$ solutions of the original linear system (3.2.4), we recall that solutions to the original linear system can be recovered from gauge transformed solutions by applying $U^{-1} = e^{\beta\phi H/2}$ to the gauge transformed solutions $\Psi = U^{-1}\widetilde{\Psi}$.

In the small-$|z|$ limit,

$$U^{-1} = e^{\beta(g \log z\bar{z})H/2} = \begin{pmatrix} (z\bar{z})^{\beta g/2} & 0 \\ 0 & (z\bar{z})^{-\beta g/2} \end{pmatrix}, \tag{3.3.5}$$

where we have used the definition of the matrix exponential (3.2.11). The small-$|z|$

solutions of the original linear system are then

$$\Xi_0 = U^{-1}\widetilde{\Xi}_0 \sim c_0 \begin{pmatrix} e^{-i\beta g\varphi} \\ 0 \end{pmatrix}, \quad \text{as } |z| \to 0, \tag{3.3.6}$$

$$\Xi_1 = U^{-1}\widetilde{\Xi}_1 \sim c_1 \begin{pmatrix} 0 \\ e^{i\beta g\varphi} \end{pmatrix}, \quad \text{as } |z| \to 0, \tag{3.3.7}$$

where we use polar coordinates $z = |z|e^{i\varphi}$. We are free to choose the arbitrary constants c_0 and c_1; we choose $c_0 = e^{-\theta\beta g}$ and $c_1 = e^{\theta\beta g}$. This has the effect of ensuring the solutions Ξ_i are invariant under Symanzik rotation (3.2.15). The small-$|z|$ solutions to the linear system $(\partial_z + A)\Psi = 0$ are then given by

$$\Xi_0 \sim \begin{pmatrix} e^{-(\theta+i\varphi)\beta g} \\ 0 \end{pmatrix}, \quad \Xi_1 \sim \begin{pmatrix} 0 \\ e^{(\theta+i\varphi)\beta g} \end{pmatrix}, \quad \text{as } |z| \to 0. \tag{3.3.8}$$

The solutions Ξ_0, Ξ_1 form a basis of the solution space of the Lax pair (3.2.4)-(3.2.5) in the neighbourhood of $|z| = 0$. In this way, any solution Ψ can be expressed as a linear combination of these two solutions. The same solutions would have been found if we began with the conjugate linear problem $(\partial_{\bar{z}} + \bar{A})\widetilde{\Psi} = 0$, applied the gauge transformation (3.2.10) with $U = e^{\beta\phi H/2}$ to remove the exponential terms, analysed the small-$|z|$ asymptotics, and then reverted the gauge transformation in that limit. The gauge transformation was merely an aid to our calculations.

3.3.2 Large-$|z|$ asymptotics of the linear systems

To consider the large-$|z|$ behaviour of the linear system (3.2.17), we substitute the large-$|z|$ behaviour of ϕ defined in (3.2.19), with the result

$$\begin{pmatrix} \partial_z + \frac{M}{2z} & me^{\theta} \\ me^{\theta}p(z) & \partial_z - \frac{M}{2z} \end{pmatrix} \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} = 0. \tag{3.3.9}$$

We consider this system of equations in the large-$|z|$ limit. The $O(1/z)$ terms become irrelevant in this limit, and the resulting system can be collected into a single equation for $\widetilde{\psi}_1$

$$-\partial_z^2\widetilde{\psi}_1 + m^2e^{2\theta}p(z)\widetilde{\psi}_1 = 0. \tag{3.3.10}$$

Solving this equation for $\widetilde{\psi}_1$ in the large-$|z|$ limit allows us to compute $\widetilde{\psi}_2$ and hence a solution to the linear system in the large-$|z|$ limit. To do this, we apply the WKB approximation [13] to (3.3.10), with the result

$$\widetilde{\psi}_1 \sim b_+p(z)^{-1/4}\exp\left(me^{\theta}\int^z \sqrt{p(t)}\,\mathrm{d}t\right) + b_-p(z)^{-1/4}\exp\left(-me^{\theta}\int^z \sqrt{p(t)}\,\mathrm{d}t\right), \tag{3.3.11}$$

in the large-$|z|$ limit, and here $b_\pm$ are arbitrary functions of $\bar{z}$. We require our large-$|z|$ solution $\widetilde{\Psi}$ of the linear system (3.2.17) to have the most rapid decay on the positive real axis of all the possible solutions (we call this the *subdominant* solution of the linear system). To achieve this, we set $b_+ = 0$ in (3.3.11), and we find

$$\widetilde{\psi}_1 \sim b_-z^{-M/2}\exp\left(-me^{\theta}\frac{z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty, \tag{3.3.12}$$

where, as we are working in the large-$|z|$ limit, we approximate $p(z) \sim z^{2M}$, which allows us to perform the integral over $\sqrt{p(t)}$ in (3.3.11).

The expression (3.3.12), along with the linear system (3.2.17) in the large-$|z|$ limit, defines a large-$|z|$ solution for the gauge-transformed linear system $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$,

$$\widetilde{\Psi} \sim b_- \begin{pmatrix} z^{-M/2} \\ z^{M/2} \end{pmatrix} \exp\left(-me^{\theta}\frac{z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty, \tag{3.3.13}$$

which is then mapped into a large-$|z|$ solution of the original linear system (3.2.4) by applying the inverse $U^{-1} = e^{\beta\phi H/2}$ of the gauge transformation matrix U

$$\begin{aligned} \Psi = U^{-1}\widetilde{\Psi} &\sim \begin{pmatrix} (z\bar{z})^{M/4} & 0 \\ 0 & (z\bar{z})^{-M/4} \end{pmatrix} \widetilde{\Psi}, & (3.3.14) \\ &\sim b_- \begin{pmatrix} (z\bar{z})^{M/4} & 0 \\ 0 & (z\bar{z})^{-M/4} \end{pmatrix} \begin{pmatrix} z^{-M/2} \\ z^{M/2} \end{pmatrix} \exp\left(-me^{\theta}\frac{z^{M+1}}{M+1}\right), & (3.3.15) \\ &\sim b_- \begin{pmatrix} (z/\bar{z})^{-M/4} \\ (z/\bar{z})^{M/4} \end{pmatrix} \exp\left(-me^{\theta}\frac{z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty. & (3.3.16) \end{aligned}$$

The constant b_- is chosen by recalling that Ψ must also satisfy the conjugate linear problem (3.2.5) in the large-$|z|$ limit. Repeating the above large-$|z|$ analysis on the conjugate linear problem we arrive at a similar expression for Ψ

$$\Psi \sim \bar{b}_- \begin{pmatrix} (z/\bar{z})^{-M/4} \\ (z/\bar{z})^{M/4} \end{pmatrix} \exp\left(-me^{-\theta}\frac{\bar{z}^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty, \tag{3.3.17}$$

where $\bar{b}_-$ is an arbitrary function of z. The two expressions for Ψ are reconciled by choosing b_-, $\bar{b}_-$ to be

$$b_- = \exp\left(-me^{\theta}\frac{\bar{z}^{M+1}}{M+1}\right), \quad \bar{b}_- = \exp\left(-me^{-\theta}\frac{z^{M+1}}{M+1}\right), \tag{3.3.18}$$

so that in polar coordinates $z = |z|e^{i\varphi}$, $\bar{z} = |z|e^{-i\varphi}$, the large-$|z|$ solution Ψ is written

$$\Psi \sim \begin{pmatrix} e^{-iM\varphi/2} \\ e^{iM\varphi/2} \end{pmatrix} \exp\left(-\frac{2m|z|^{M+1}}{M+1}\cosh(\theta + i(M+1)\varphi)\right), \text{ as } |z| \to \infty, \tag{3.3.19}$$

which matches the large-$|z|$ solution for the A_1 linear system in [2].

3.3.3 Taking the conformal limit

We have calculated small-$|z|$ and large-$|z|$ asymptotics for the solution Ψ of the Lax pair (3.2.4)-(3.2.5). What remains unclear, however, is the connection between these systems of differential equations and the eigenvalue problem (1.1.5) that was discussed in section 1.1.2. In this section we explain this connection, and thus define the massive analogues of the spectral determinants $D_\pm(E, l)$ we discussed previously.

We begin with the gauge-transformed linear system $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$,

$$\begin{pmatrix} \partial_z + \beta\partial_z\phi & me^{\theta} \\ me^{\theta}p(z) & \partial_z - \beta\partial_z\phi \end{pmatrix} \begin{pmatrix} \widetilde{\psi}_1 \\ \widetilde{\psi}_2 \end{pmatrix} = 0, \tag{3.3.20}$$

and rewrite it as a single equation in $\widetilde{\psi}_1$

$$(\partial_z - \beta\partial_z\phi)(\partial_z + \beta\partial_z\phi)\widetilde{\psi}_1 - m^2e^{2\theta}p(z)\widetilde{\psi}_1 = 0. \tag{3.3.21}$$

We then send $\bar{z} \to 0$, (treating z and $\bar{z}$ as independent complex coordinates) which allows us to replace ϕ with its small-$|z|$ asymptotics (3.2.20),

$$\left(\partial_z - \frac{\beta g}{z}\right)\left(\partial_z + \frac{\beta g}{z}\right)\widetilde{\psi}_1 - m^2 e^{2\theta}(z^{2M} - s^{2M})\widetilde{\psi}_1 = 0. \tag{3.3.22}$$

We then take the conformal limit $z \to 0$, $\theta \to +\infty$, with

$$x = z(me^{\theta})^{\frac{1}{M+1}}, \quad E = s^{2M}(me^{\theta})^{\frac{2M}{M+1}} \tag{3.3.23}$$

held finite. With $\beta g = l$, the differential equation becomes

$$-\partial_x^2\widetilde{\psi}_1 + \left(x^{2M} + \frac{l(l+1)}{x^2}\right)\widetilde{\psi}_1 = E\widetilde{\psi}_1, \tag{3.3.24}$$

which is exactly the same differential equation as (1.1.5). We recall that the subdominant large-$|x|$ solution $y(x, E, l)$ was written as a linear combination of the two small-$|x|$ solutions $\psi^{\pm}(x, E, l)$

$$y(x, E, l) = \frac{D_-(E, l)}{2l+1}\psi^-(x, E, l) - \frac{D_+(E, l)}{2l+1}\psi^+(x, E, l) \tag{3.3.25}$$

The functions y and $\psi^{\pm}$ are simply the conformal limit counterparts to the first components of the solutions $\widetilde{\Psi}$, $\widetilde{\Xi}_0$ and $\widetilde{\Xi}_1$ of the gauge-transformed linear system (3.2.17). We then define the massive analogues of the spectral determinants $D_{\pm}(E, l)$ as functions $Q_0(\theta, g)$ and $Q_1(\theta, g)$

$$\widetilde{\Psi} = Q_0(\theta, g)\widetilde{\Xi}_0 + Q_1(\theta, g)\widetilde{\Xi}_1. \tag{3.3.26}$$

The choice of gauge does not affect this definition of the Q-functions. Multiplying both sides of (3.3.26) by $U^{-1} = e^{\beta\phi H/2}$ we see that

$$\Psi = Q_0(\theta, g)\Xi_0 + Q_1(\theta, g)\Xi_1. \tag{3.3.27}$$

The Q-functions are written in terms of the solutions Ψ, Ξ_0 and Ξ_1 by taking particular determinants of (3.3.27)

$$Q_0(\theta, g) = \det(\Psi, \Xi_1), \quad Q_1(\theta, g) = \det(\Xi_0, \Psi), \tag{3.3.28}$$

where we have used $\det(\Xi_0, \Xi_1) = 1$, derived from the asymptotics of Ξ_i in the small-$|z|$ limit. The process of writing Q-functions in terms of determinants in the massive case is equivalent to the process of taking Wronskians in the massless case to define the spectral determinants $D_{\pm}(E, l)$ (1.1.12). To see this, we write the general solution of the linear system (3.3.20) in terms of a solution $\widetilde{\psi}_1$ of (3.3.21):

$$\widetilde{\Psi}_1 = \begin{pmatrix} \widetilde{\psi}_1 \\ -(me^{\theta})^{-1}(\partial_z + \beta\partial_z\phi)\widetilde{\psi}_1 \end{pmatrix} \tag{3.3.29}$$

We then take the determinant of two such solutions $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$

$$\det\left(\widetilde{\Psi}_1, \widetilde{\Psi}_2\right) = -(me^{\theta})^{-1} \begin{vmatrix} \widetilde{\psi}_1 & \widetilde{\psi}_2 \\ (\partial_z + \beta\partial_z\phi)\widetilde{\psi}_1 & (\partial_z + \beta\partial_z\phi)\widetilde{\psi}_2 \end{vmatrix} \tag{3.3.30}$$

$$= -(me^{\theta})^{-1} W[\widetilde{\psi}_1, \widetilde{\psi}_2], \tag{3.3.31}$$

which demonstrates the equivalence, up to rescaling, of taking determinants in the massive case and taking Wronskians in the massless case to define the relevant spectral determinants.

From (3.3.28) and the relationship with the massless spectral determinants (1.1.12) it is clear that the Q-functions are indeed spectral determinants of the linear systems (3.2.4)-(3.2.5); points θ_k where $Q_i(\theta_k, g) = 0$ are precisely the points where Ψ and Ξ_i coincide and become the same solution up to normalisation. We therefore consider the linear systems (3.2.4)-(3.2.5) as eigenvalue problems with boundary conditions given by the asymptotic solutions Ψ, Ξ_i. The properties of these Q-functions are our main concern for the rest of this chapter.

3.4 Q-functions

We now demonstrate some useful properties of the Q-functions. We will see that the Q-functions satisfy a quasiperiodicity property and a particular functional relation known as the quantum Wronskian. We also give an expression for the asymptotics for $Q_0(\theta, g)$ in the limits $\operatorname{Re}\theta \to \pm\infty$, following [21, 45].

In the calculations that follow it will often be convenient to omit the g-dependence of the Q-functions, with $Q_i(\theta, g) = Q_i(\theta)$.

3.4.1 Quasiperiodicity

The Q-functions satisfy the following quasiperiodicity properties

$$Q_0\left(\theta + \frac{i\pi(M+1)}{M}\right) = e^{-i\pi\gamma} Q_0\left(\theta\right), \tag{3.4.1}$$

$$Q_1\left(\theta + \frac{i\pi(M+1)}{M}\right) = e^{i\pi\gamma} Q_1\left(\theta\right), \tag{3.4.2}$$

where $\gamma = -(\beta g + 1/2)$. To prove this, we define the matrix

$$S = e^{i\pi H/2} = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}, \tag{3.4.3}$$

and firstly prove the following identities

$$S\Xi_0\left(\varphi+\frac{\pi}{M}\middle|\,\theta-\frac{i\pi}{M}-i\pi\right)=e^{-i\pi\gamma}\,\Xi_0\left(\varphi|\,\theta\right), \tag{3.4.4}$$

$$S\Xi_1\left(\varphi+\frac{\pi}{M}\middle|\,\theta-\frac{i\pi}{M}-i\pi\right)=e^{i\pi\gamma}\,\Xi_1\left(\varphi|\,\theta\right), \tag{3.4.5}$$

$$S\Psi\left(\varphi+\frac{\pi}{M}\middle|\,\theta-\frac{i\pi}{M}-i\pi\right)=\Psi\left(\varphi|\,\theta\right). \tag{3.4.6}$$

Proof of (3.4.4)**-**(3.4.5)

Using the small-$|z|$ asymptotics (3.3.8) and the definition of S,

$$S\Xi_0\left(\varphi+\frac{\pi}{M}\middle|\,\theta-\frac{i\pi}{M}-i\pi\right)\sim\begin{pmatrix}e^{i\pi/2}&0\\0&e^{-i\pi/2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e^{-(\theta+i\varphi-i\pi)\beta g}, \tag{3.4.7}$$

$$\sim e^{-i\pi\gamma}\,\Xi_0\left(\varphi|\,\theta\right),\quad \text{as } |z|\to 0. \tag{3.4.8}$$

This identity holds away from $z=0$ as S is a constant matrix, unaffected by the limit. (3.4.5) follows similarly.

Proof of (3.4.6)

We evaluate the left-hand side of (3.4.6) in the large-$|z|$ limit

$$S\Psi\left(\varphi+\frac{\pi}{M}\middle|\,\theta-\frac{i\pi}{M}-i\pi\right) \tag{3.4.9}$$

$$\sim\begin{pmatrix}e^{i\pi/2}&0\\0&e^{-i\pi/2}\end{pmatrix}\begin{pmatrix}e^{-i\pi/2}e^{-iM\varphi/2}\\e^{i\pi/2}e^{iM\varphi/2}\end{pmatrix}\exp\left(-\frac{2m|z|^{M+1}}{M+1}\cosh(\theta+i(M+1)\varphi)\right)$$

$$\sim\Psi\left(\varphi|\,\theta\right),\quad \text{as } |z|\to\infty. \tag{3.4.10}$$

Similarly to the small-$|z|$ identities, this asymptotics matching is enough for the identity (3.4.6) to hold everywhere, as S is a constant matrix.

Proof of the quasiperiodicity properties

We now use the identities (3.4.4)-(3.4.6) to demonstrate the quasiperiodicity properties of Q_0 and Q_1. We begin with the determinant definition of Q_0,

$$Q_0(\theta) = \det\left(\Psi\left(\varphi\,|\theta\right), \Xi_1\left(\varphi\,|\theta\right)\right), \tag{3.4.11}$$

and invoke the identities (3.4.5) and (3.4.6),

$$Q_0(\theta) = \det\left(S\Psi\left(\varphi + \frac{\pi}{M}\middle|\, \theta - \frac{i\pi}{M} - i\pi\right), e^{-i\pi\gamma} S\Xi_1\left(\varphi + \frac{\pi}{M}\middle|\, \theta - \frac{i\pi}{M} - i\pi\right)\right) \tag{3.4.12}$$

We extract S from the determinant in (3.4.12) by exploiting the linear algebra identity

$$\det(Sv_1, \dots, Sv_n) = \det S \det(v_1, \dots, v_n), \tag{3.4.13}$$

which is satisfied for any matrix S and collection of vectors $\{v_1, \dots, v_n\} \subset \mathbb{R}^n$. Using $\det S = 1$, equation (3.4.12) then reduces to

$$Q_0(\theta) = e^{-i\pi\gamma} \det\left(\Psi\left(\varphi + \frac{\pi}{M}\middle|\, \theta - \frac{i\pi}{M} - i\pi\right), \Xi_1\left(\varphi + \frac{\pi}{M}\middle|\, \theta - \frac{i\pi}{M} - i\pi\right)\right), \tag{3.4.14}$$

$$= e^{-i\pi\gamma} Q_0\left(\theta - \frac{i\pi}{M} - i\pi\right). \tag{3.4.15}$$

where we have used the independence of the Q-functions from φ, which follows from their definition (3.3.27), which must hold at any values of $z, \bar{z}$. Shifting $\theta \to \theta + \frac{i\pi(M+1)}{M}$, we find the quasiperiodicity property (3.4.1). The analogous identity (3.4.2) for Q_1 follows similarly.

3.4.2 Asymptotics of $Q_0(\theta)$ as $\mathrm{Re}\,\theta \to \pm\infty$

For calculations later in this chapter, it will be useful to have available an asymptotic expression for $Q_0(\theta)$ in the limits $\mathrm{Re}\,\theta \to \pm\infty$, similar to the asymptotics given in equation (3.12) of [45]. We begin by considering the asymptotics of $Q_0(\theta)$ in the limit $\mathrm{Re}\,\theta \to +\infty$, and recall the definition of $Q_0(\theta)$ as the determinant

$$Q_0(\theta) = \det\Big(\widetilde{\Psi}, \widetilde{\Xi}_1\Big), \tag{3.4.16}$$

with $\widetilde{\Psi}$ being the subdominant solution of the gauge transformed linear system $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ in the large-$|z|$ limit, and $\widetilde{\Xi}_1$ is one of the solutions of that linear system in the small-$|z|$ limit. To find the large-θ asymptotics of $Q_0(\theta)$, we consider the general solution of $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ in the large-θ limit. We recast $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ into an equation for the top component $\widetilde{\psi}_1$ of $\widetilde{\Psi}$

$$(\partial_z - \beta\partial_z\phi)(\partial_z + \beta\partial_z\phi)\widetilde{\psi}_1 - m^2 e^{2\theta} p(z)\widetilde{\psi}_1 = 0. \tag{3.4.17}$$

We use the WKB approximation [13] to consider this equation in the $\theta \to +\infty$ limit; the general solution in that limit is

$$\widetilde{\psi}_1 \sim b_+ p(z)^{-1/4} \exp\left(m e^{\theta} \int^z \sqrt{p(t)}\,\mathrm{d}t\right) + b_- p(z)^{-1/4} \exp\left(-m e^{\theta} \int^z \sqrt{p(t)}\,\mathrm{d}t\right), \tag{3.4.18}$$

which induces the vector solution for the linear system $\widetilde{\Psi}$

$$\begin{aligned} \widetilde{\Psi} \sim b_- &\begin{pmatrix} p(z)^{-1/4} \\ p(z)^{1/4} \end{pmatrix} \exp\left(-m e^{\theta} \int^z \sqrt{p(t)}\,\mathrm{d}t\right) \\ + b_+ &\begin{pmatrix} p(z)^{-1/4} \\ -p(z)^{1/4} \end{pmatrix} \exp\left(m e^{\theta} \int^z \sqrt{p(t)}\,\mathrm{d}t\right), \text{ as } \mathrm{Re}\,\theta \to +\infty. \end{aligned} \tag{3.4.19}$$

We require this solution to be compatible with the required subdominant behaviour for $\widetilde{\Psi}$ in the large-$|z|$ limit

$$\widetilde{\Psi} \sim \begin{pmatrix} z^{-M/2} \\ z^{M/2} \end{pmatrix} \exp\left(-\frac{me^{\theta} z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty, \tag{3.4.20}$$

and so set $b_+ = 0$ and redefine b_- so that $\widetilde{\Psi}$ becomes

$$\widetilde{\Psi} \sim b_- \begin{pmatrix} p(z)^{-1/4} \\ p(z)^{1/4} \end{pmatrix} \exp\left(me^{\theta} \int_z^{\infty} \left\{(t^{2M} - s^{2M})^{1/2} - t^M\right\} \mathrm{d}t - \frac{me^{\theta} z^{M+1}}{M+1}\right), \quad \text{as } \mathrm{Re}\,\theta \to +\infty. \tag{3.4.21}$$

In the small-$|z|$ limit, $\widetilde{\Psi}$ must be a linear combination of $\widetilde{\Xi}_0$ and $\widetilde{\Xi}_1$, as these solutions span the solution space in that limit. Furthermore, from the asymptotics of $\widetilde{\Xi}_0$ and $\widetilde{\Xi}_1$ in the small-$|z|$ limit (3.3.4) imply that the constants c_0, c_1 in the redefinition

$$b_+ \begin{pmatrix} p(z)^{-1/4} \\ p(z)^{1/4} \end{pmatrix} = c_0\, \widetilde{\Xi}_0 + c_1\, \widetilde{\Xi}_1, \tag{3.4.22}$$

are independent of θ. Therefore, in the small-$|z|$ limit,

$$\widetilde{\Psi} \sim \exp\left(me^{\theta} \int_0^{\infty} \left\{(t^{2M} - s^{2M})^{1/2} - t^M\right\} \mathrm{d}t\right) (c_0\, \widetilde{\Xi}_0 + c_1\, \widetilde{\Xi}_1), \quad \text{as } \mathrm{Re}\,\theta \to \infty, \tag{3.4.23}$$

which combined with the determinant definition of $Q_0(\theta)$, gives an asymptotic expression for $Q_0(\theta)$ in the limit $\mathrm{Re}\,\theta \to +\infty$,

$$Q_0(\theta) \sim c_0 \exp\left(me^{\theta} \int_0^{\infty} \left\{(t^{2M} - s^{2M})^{1/2} - t^M\right\} \mathrm{d}t\right). \tag{3.4.24}$$

It remains to evaluate the integral in (3.4.24); firstly reparameterise $s = \tilde{s}(-1)^{\frac{M+1}{2M}}$, so that $s^{2M} = -\tilde{s}^{2M}$, and then change variables $t = \tilde{s}u$. The integral becomes

$$\int_0^\infty \left\{(t^{2M} - s^{2M})^{1/2} - t^M\right\} \mathrm{d}t = (-1)^{\frac{M+1}{2M}} s^{M+1} \int_0^\infty \left\{(u^{2M} + 1)^{1/2} - u^M\right\} \mathrm{d}u. \tag{3.4.25}$$

A more general form of the integral in (3.4.25) was given in [21, 23]:

$$\tau(h, M) = \int_0^\infty \left\{(u^{hM} + 1)^{1/h} - u^M\right\} \mathrm{d}u = \frac{\Gamma(1 + \frac{1}{hM})\Gamma(-\frac{1}{h} - \frac{1}{hM})}{\Gamma(\frac{1}{h})}. \tag{3.4.26}$$

$Q_0(\theta)$ then has the asymptotic expression

$$Q_0(\theta) \sim c_0 \exp\left(s^{M+1} m e^\theta (-1)^{\frac{M+1}{2M}} \tau(2, M)\right), \quad \text{as } \operatorname{Re}\theta \to +\infty. \tag{3.4.27}$$

We must ensure this asymptotic expression is compatible with the quasiperiodicity relation (3.4.1) satisfied by $Q_0(\theta)$. Following [45], define $H_\pm$ to be the strips in the complex θ-plane satisfying

$$H_+ : 0 < \operatorname{Im}\theta < \frac{\pi(M+1)}{M}, \quad H_- : -\frac{\pi(M+1)}{M} < \operatorname{Im}\theta < 0. \tag{3.4.28}$$

Then, rescaling the constant c_0 appropriately,

$$Q_0(\theta) \sim c_0 e^{\mp i\pi\gamma/2} \exp\left(s^{M+1} m e^{\theta \mp \frac{i\pi(M+1)}{2M}} \tau(2, M)\right), \quad \theta \in H_\pm. \tag{3.4.29}$$

An exactly analogous argument to the above leads to the $Q_1(\theta)$ asymptotics in the same limit

$$Q_1(\theta) \sim c_1 e^{\pm i\pi\gamma/2} \exp\left(s^{M+1} m e^{\theta \mp \frac{i\pi(M+1)}{2M}} \tau(2, M)\right), \quad \theta \in H_\pm. \tag{3.4.30}$$

We will also require the asymptotics for $Q_0(\theta)$ in the limit $\operatorname{Re}\theta \to -\infty$. To recover these, we begin with the gauge transformed conjugate linear system $(\partial_{\bar{z}} + \widetilde{\bar{A}})\widetilde{\Psi} = 0$ and recast this system of equations as a single equation in the second component $\widetilde{\psi}_2$ of $\widetilde{\Psi}$

$$(\partial_{\bar{z}} + \beta\partial_{\bar{z}}\phi)(\partial_{\bar{z}} - \beta\partial_{\bar{z}}\phi)\widetilde{\psi}_2 - m^2 e^{-2\theta} p(\bar{z})\widetilde{\psi}_2 = 0. \tag{3.4.31}$$

The same procedure as for the $\operatorname{Re}\theta \to +\infty$ limit is then followed; equation (3.4.31) is considered in the limit $\operatorname{Re}\theta \to -\infty$, and its solutions induce a particular solution of $(\partial_{\bar{z}} + \widetilde{\bar{A}})\widetilde{\Psi} = 0$ in that limit. The limit $\bar{z} \to 0$ is then taken, and the determinant definition of $Q_0(\theta)$ (3.4.16) is used to determine the asymptotics of $Q(\theta)$ as $\operatorname{Re}\theta \to -\infty$:

$$Q_0(\theta) \sim c_0 e^{\mp i\pi\gamma/2} \exp\left(s^{M+1} m e^{-\theta \pm \frac{i\pi(M+1)}{2M}} \tau(2, M)\right), \quad \theta \in H_{\pm}. \tag{3.4.32}$$

3.4.3 The quantum Wronskian

The Q-functions $Q_0(\theta)$ and $Q_1(\theta)$ also satisfy a particular functional relation, known as a quantum Wronskian [7, 45]. This relation follows naturally from the definition of the Q-functions (3.3.26) and the Symanzik rotation Ω_k, given by (3.2.15). Under a Symanzik rotation Ω_k, (3.3.27) becomes

$$\Omega_k[\widetilde{\Psi}] = Q_0\left(\theta - \frac{i\pi k}{M}\right)\widetilde{\Xi}_0 + Q_1\left(\theta - \frac{i\pi k}{M}\right)\widetilde{\Xi}_1, \tag{3.4.33}$$

where we have used the invariance of the small-$|z|$ solutions $\widetilde{\Xi}_i$ under Symanzik rotation. We then take the determinant of $\widetilde{\Psi}$ with $\Omega_1[\widetilde{\Psi}]$, and use $\det\left(\widetilde{\Xi}_0, \widetilde{\Xi}_1\right) = 1$

and the antisymmetry of determinants to find

$$\det\left(\widetilde{\Psi}, \Omega_1[\widetilde{\Psi}]\right) = Q_0\,(\theta)\,Q_1\left(\theta - \frac{i\pi}{M}\right) - Q_0\left(\theta - \frac{i\pi}{M}\right) Q_1\,(\theta)\,. \tag{3.4.34}$$

As the Q-functions were defined as $z, \bar{z}$-independent coefficients, we may take the large-$|z|$ limit of (3.4.34) without affecting the Q-functions. In that limit, the rotated solutions $\Omega_k[\widetilde{\Psi}]$ are given by

$$\Omega_k[\widetilde{\Psi}] \sim e^{-i\pi k/2} \begin{pmatrix} z^{-M/2} \\ e^{i\pi k} z^{M/2} \end{pmatrix} \exp\left(-e^{i\pi k} \frac{m e^{\theta} z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty. \tag{3.4.35}$$

Using standard properties of determinants, the large-$|z|$ limit of the determinant $\det\left(\widetilde{\Psi}, \Omega_1[\widetilde{\Psi}]\right)$ is then given by

$$\det\left(\widetilde{\Psi}, \Omega_1[\widetilde{\Psi}]\right) \sim e^{-i\pi/2} \begin{vmatrix} 1 & 1 \\ 1 & e^{i\pi} \end{vmatrix} = 2i. \tag{3.4.36}$$

Taking the large-$|z|$ limit of (3.4.34) and substituting (3.4.36), we find the quantum Wronskian

$$Q_0\,(\theta)\,Q_1\left(\theta - \frac{i\pi}{M}\right) - Q_0\left(\theta - \frac{i\pi}{M}\right) Q_1\,(\theta) = 2i. \tag{3.4.37}$$

This quantum Wronskian almost exactly matches the quantum Wronskian found in [45]; the absence of the factor $-\cos\pi l$ is due to Lukyanov and Zamolodchikov's different choice for the normalisation and ordering of their small-$|z|$ solutions $\Psi_\pm$. Our choice of normalisation will generalise more readily to the cases of the massive ODE/IM correspondence for more elaborate Lie algebras.

3.4.4 Bethe ansatz equations

A useful algebraic relation satisfied by $Q_0(\theta)$ follows immediately from the quantum Wronskian (3.4.37). Define the zeroes $\theta_j,\ j \in \mathbb{Z}$ of $Q_0(\theta)$ satisfying

$$Q_0(\theta_j) = 0. \tag{3.4.38}$$

We then substitute $\theta = \theta_j \pm \frac{i\pi}{M}$ into the quantum Wronskian (3.4.37), giving two relations

$$Q_0\left(\theta_j + \frac{i\pi}{M}\right) Q_1\left(\theta_j\right) = 2i, \tag{3.4.39}$$

$$Q_0\left(\theta_j - \frac{i\pi}{M}\right) Q_1\left(\theta_j\right) = -2i. \tag{3.4.40}$$

We divide (3.4.39) by (3.4.40) and find a set of *Bethe ansatz equations* (BAEs)

$$\frac{Q_0\left(\theta_j + \frac{i\pi}{M}\right)}{Q_0\left(\theta_j - \frac{i\pi}{M}\right)} = -1, \tag{3.4.41}$$

which match the Bethe ansatz equations found in [45]. (In [45] the authors combine Q_0 and Q_1 into a single Q-function and derive identical BAEs for that new function. It is currently unclear how to generalise their method for the cases of more elaborate Lie algebras; we will only require BAEs for the first Q-function, Q_0.) For later calculations it is useful to 'twist' these Bethe ansatz equations using the quasiperiodicity relation (3.4.1):

$$e^{-2i\pi\gamma}\frac{Q\left(\theta_j - i\pi\right)}{Q\left(\theta_j + i\pi\right)} = -1. \tag{3.4.42}$$

where we set $Q_0(\theta) = Q(\theta)$ as we will mainly be concerned with this single Q-function for the remainder of the chapter. These new twisted BAEs more closely resemble the BAEs found in [21], for the case of the Lie algebra $A_1 = \mathfrak{su}(2)$. The resemblance is not exact, as the twists in the BAEs in that paper are powers of $\omega = e^{i\pi/(M+1)}$, rather than $e^{-i\pi}$. To pass between the two sets of BAEs, set

$$Q(\theta) = A^{(1)}(s^{2M} e^{\frac{2M\theta}{M+1}}) e^{-\frac{\gamma M\theta}{M+1}}, \quad E = s^{2M} e^{\frac{2M\theta}{M+1}}, \tag{3.4.43}$$

with the BAEs (3.4.42) becoming

$$\omega^{2\gamma} \frac{A^{(1)}(\omega^{2M} E_j)}{A^{(1)}(\omega^{-2M} E_j)} = -1, \quad \omega = e^{\frac{2\pi i}{2M+2}}, \quad E_j = s^{2M} e^{\frac{2M}{M+1}\theta_j}, \tag{3.4.44}$$

matching the BAEs in [21], with $n = 2$, $\gamma = \beta_1$ and $C_{11} = 2$.

The BAEs in the original twisted form (3.4.42) will be the most useful out of all these various forms of BAEs in the calculation of an equivalent non-linear integral equation (NLIE). We will then derive an expression for $\log Q$, and the asymptotic expansion of that expression will contain the ground-state eigenvalues of the integrals of motion of the quantum sine-Gordon integrable field theory.

3.5 The non-linear integral equation and integrals of motion

3.5.1 The non-linear integral equation

We begin our construction of the non-linear integral equation for the massive integrable field theory related to $\mathfrak{su}(2)$, following the construction given in [21] for the case of the massless ODE/IM correspondence for the Lie algebras $\mathfrak{su}(n)$. This construction is the first step in deriving expressions for the integrals of motion of the integrable field theory. We begin by defining the function

$$a(\theta) = e^{-2i\pi\gamma} \frac{Q(\theta - i\pi)}{Q(\theta + i\pi)}, \tag{3.5.1}$$

which due to the BAEs (3.4.42), satisfy $a(\theta_j) = -1$. We then expand $Q(\theta)$-functions in (3.5.1) using an infinite product expansion. A standard result for defining an infinite product expansion for an entire function is the Hadamard factorisation theorem [15]. We recall the order of an entire function $f(z)$ is defined to be the infimum of the set of numbers a such that $|f(z)| < \exp(|z|^a)$ for $|z|$ sufficiently large. The Hadamard factorisation theorem then states that if f has finite order a, then $f(z)$ can be written in the form

$$f(z) = z^m e^{g(z)} \prod_j \left(1 - \frac{z}{z_j}\right), \tag{3.5.2}$$

where $g(z)$ is a polynomial. From the $\mathrm{Re}\,\theta \to \pm\infty$ asymptotics (3.4.29)-(3.4.32) of $Q(\theta)$ and the Hadamard factorisation theorem, an infinite product expansion of $Q(\theta)$ over its zeroes θ_j exists for $M > 1$, given by

$$Q(\theta) = Q(0) e^{-\frac{\gamma M \theta}{M+1}} \prod_{j=0}^{\infty} \left(1 - e^{\frac{2M}{M+1}(\theta - \theta_j)}\right) \left(1 - e^{-\frac{2M}{M+1}(\theta - \theta_{-j-1})}\right), \tag{3.5.3}$$

where the prefactor $e^{-\frac{\gamma M \theta}{M+1}}$ is inserted to ensure the infinite product expansion of $Q(\theta) = Q_0(\theta)$ satisfies the quasiperiodicity relation (3.4.1). We then substitute (3.5.3) into the definition of $a(\theta)$ (3.5.1)

$$a(\theta) = e^{-\frac{2i\pi\gamma}{M+1}} \prod_{j=0}^{\infty} \frac{\left(1 - e^{\frac{2M}{M+1}(\theta-\theta_j)} e^{-\frac{2i\pi M}{M+1}}\right)\left(1 - e^{-\frac{2M}{M+1}(\theta-\theta_{-j-1})} e^{\frac{2i\pi M}{M+1}}\right)}{\left(1 - e^{\frac{2M}{M+1}(\theta-\theta_j)} e^{\frac{2i\pi M}{M+1}}\right)\left(1 - e^{-\frac{2M}{M+1}(\theta-\theta_{-j-1})} e^{-\frac{2i\pi M}{M+1}}\right)}. \tag{3.5.4}$$

$$= e^{-\frac{2i\pi\gamma}{M+1}} \prod_{j=-\infty}^{\infty} \frac{1 - e^{\frac{2M}{M+1}(\theta-\theta_j - i\pi)}}{1 - e^{\frac{2M}{M+1}(\theta-\theta_j + i\pi)}}. \tag{3.5.5}$$

We then take the logarithm of (3.5.5),

$$\log a(\theta) = \frac{-2i\pi\gamma}{M+1} + \sum_{j=-\infty}^{\infty} F(\theta - \theta_j), \tag{3.5.6}$$

where

$$F(\theta) = \log\left(\frac{1 - e^{\frac{2M}{M+1}(\theta - i\pi)}}{1 - e^{\frac{2M}{M+1}(\theta + i\pi)}}\right). \tag{3.5.7}$$

To progress, we now must consider the locations of the zeroes θ_j. As in [21], where the BAEs related to the ground state of a massless integrable field theory related to the Lie algebra $\mathfrak{su}(n)$ were considered, we work with the assumption that all the zeroes are along the real axis of the complex θ-plane. We then use Cauchy's integral theorem to rewrite the infinite sum over the zeroes θ_j as a contour integral

$$\sum_{j=-\infty}^{\infty} F(\theta - \theta_j) = \int_{\xi} \frac{\mathrm{d}\theta'}{2i\pi} F(\theta - \theta')\partial_{\theta'} \log(1 + a(\theta')), \tag{3.5.8}$$

where ξ is a contour in the θ'-plane consisting of two parallel lines enclosing the real axis, with the direction of integration along ξ chosen such that the real axis remains on the left of the contour. The logarithm of $a(\theta)$ is then given by

$$\log a(\theta) = \frac{-2i\pi\gamma}{M+1} + \int_{\xi} \frac{\mathrm{d}\theta'}{2i\pi} F(\theta - \theta')\partial_{\theta'} \log(1 + a(\theta')). \tag{3.5.9}$$

We then integrate (3.5.9) by parts and consider the two contributions of the contour γ from above and below the real axis separately. We then rewrite $\log a(\theta)$

$$\begin{aligned} \log a(\theta) &= \frac{-2i\pi\gamma}{M+1} \\ &+ \int_{-\infty}^{\infty} R(\theta - \theta' + i0)\left\{\log\left(1 + a(\theta' + i0)\right) - \log\left(1 + a(\theta' - i0)\right)\right\} \mathrm{d}\theta', \end{aligned} \tag{3.5.10}$$

where $R(\theta) = (i/2\pi)\partial_\theta F(\theta)$. We next use the identity $a(\theta)^* = a(\theta^*)^{-1}$, which follows from the product expansion of $a(\theta)$ (3.5.5) and the reality of the zeroes θ_j, to rewrite (3.5.10) as

$$\log a(\theta) = \frac{-2i\pi\gamma}{M+1} + \int_{-\infty}^{\infty} R(\theta - \theta' + i0) \left\{\log\left(a(\theta' - i0)\right) - 2i \operatorname{Im} \log\left(1 + a(\theta' - i0)\right)\right\} \mathrm{d}\theta'. \tag{3.5.11}$$

The next step is to take a Fourier transform to both sides of (3.5.11). In this thesis, the Fourier transform is defined as

$$\mathcal{F}[f](k) = \tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ik\theta} f(\theta)\, \mathrm{d}\theta, \tag{3.5.12}$$

and its inverse is given by

$$\mathcal{F}^{-1}[\tilde{f}](\theta) = f(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\theta} \tilde{f}(k)\, \mathrm{d}k. \tag{3.5.13}$$

Applying a Fourier transform to both sides of (3.5.11), we find

$$\mathcal{F}[\log a] = \frac{-4i\pi^2\gamma}{M+1}\delta(k) + \tilde{R}(k)\{\mathcal{F}[\log a] - 2i\mathcal{F}[\operatorname{Im}\log(1+a)]\}, \tag{3.5.14}$$

where we have used the definition of the Dirac delta function $\delta(k)$ in integral form

$$\int_{-\infty}^{\infty} e^{-ik\theta}\, \mathrm{d}\theta = 2\pi\delta(k). \tag{3.5.15}$$

We collect the $\mathcal{F}[\log a]$ terms in (3.5.14)

$$(1 - \tilde{R}(k))\mathcal{F}[\log a] = \frac{-4i\pi^2\gamma}{M+1}\delta(k) - 2i\tilde{R}(k)\mathcal{F}[\operatorname{Im}\log(1+a)], \tag{3.5.16}$$

and then divide both sides of (3.5.16) by $(1-\tilde{R}(k))$

$$\mathcal{F}[\log a] = \frac{-4i\pi^2\delta(k)}{(M+1)(1-\tilde{R}(k))} + \sum_{p=-\infty}^{\infty} b^{(2p+1)}\mathcal{F}[e^{(2p+1)\theta}] \tag{3.5.17}$$

$$- \frac{2i\tilde{R}(k)}{1-\tilde{R}(k)}\mathcal{F}[\operatorname{Im}\log(1+a)], \tag{3.5.18}$$

where the terms proportional to the arbitrary constants $b^{(2p+1)}$ arise from the points $k=\pm(2p+1)i$ (where $p\in\mathbb{Z}$) where the inverse of $(1-\tilde{R}(k))$ is not well defined. The constants $b^{(2p+1)}$ will be chosen to match the asymptotics of $\log Q(\theta)$ (3.4.29)-(3.4.32) in the limits $\operatorname{Re}\theta \to \pm\infty$; this choice is best made when an expression for $\log Q(\theta)$ is found. For now, we apply the inverse Fourier transform (3.5.13) to (3.5.18)

$$\log a(\theta) = \lim_{k\to 0} \frac{-2i\pi\gamma}{(M+1)(1-\tilde{R}(k))} + \sum_{p=-\infty}^{\infty} b^{(2p+1)}e^{(2p+1)\theta} - 2i\int_{-\infty}^{\infty} \varphi(\theta-\theta'+i0)\operatorname{Im}\log(1+a(\theta'-i0))\,\mathrm{d}\theta', \tag{3.5.19}$$

where $\varphi(\theta) = \mathcal{F}^{-1}[(1-\tilde{R}(k))^{-1}\tilde{R}(k)]$. To simplify (3.5.19), we must calculate $\tilde{R}(k) = (i/2\pi)\mathcal{F}[\partial_\theta F(\theta)]$ explicitly. We rewrite $F(\theta)$ as given by (3.5.7) using the identity

$$\log\left(\frac{1-e^{X-Y}}{1-e^{X+Y}}\right) = -Y + \log\left(\frac{\sinh\frac{X-Y}{2}}{\sinh\frac{X+Y}{2}}\right), \tag{3.5.20}$$

which implies

$$F(\theta) = -\frac{2i\pi M}{M+1} + \log\left(\frac{\sinh\left(\frac{M\theta}{M+1}-\frac{i\pi M}{M+1}\right)}{\sinh\left(\frac{M\theta}{M+1}+\frac{i\pi M}{M+1}\right)}\right). \tag{3.5.21}$$

The Fourier transform of $R(\theta) = (i/2\pi)\partial_\theta F(\theta)$ is then calculated using equations (D.53) and (D.54) from [23]

$$i\partial_\theta \log \frac{\sinh(\sigma\theta + i\pi\tau)}{\sinh(\sigma\theta - i\pi\tau)} = \frac{2\sigma \sin 2\pi\tau}{\cosh 2\sigma\theta - \cos 2\pi\tau}, \tag{3.5.22}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2\pi} e^{-ik\theta} \frac{2\sigma \sin 2\pi\tau}{\cosh 2\sigma\theta - \cos 2\pi\tau} = \frac{\sinh(1-2\tau)\frac{\pi k}{2\sigma}}{\sinh \frac{\pi k}{2\sigma}}. \tag{3.5.23}$$

We then find

$$\tilde{R}(k) = \frac{\sinh \frac{(M-1)\pi k}{2M}}{\sinh \frac{(M+1)\pi k}{2M}}. \tag{3.5.24}$$

We now use (3.5.24) to simplify the integral equation (3.5.19), evaluating the limit term

$$\lim_{k\to 0} \frac{-2i\pi\gamma}{(M+1)(1-\tilde{R}(k))} = -i\pi\gamma, \tag{3.5.25}$$

so that the non-linear integral equation becomes

$$\begin{aligned} \log a(\theta) = -i\pi\gamma + \sum_{p=-\infty}^{\infty} b^{(2p+1)} e^{(2p+1)\theta} \\ - 2i \int_{-\infty}^{\infty} \varphi(\theta - \theta' + i0) \operatorname{Im} \log(1 + a(\theta' - i0))\, \mathrm{d}\theta', \end{aligned} \tag{3.5.26}$$

where

$$\varphi(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\theta} \frac{\sinh \frac{(M-1)\pi k}{2M}}{2\cosh \frac{\pi k}{2} \sinh \frac{\pi k}{2M}}\, \mathrm{d}k. \tag{3.5.27}$$

In the next subsection, we will use the non-linear integral equation (3.5.26) and combine it with the definition of $a(\theta)$ (3.5.1) to produce an integral expression for the logarithm of the Q-function. This expression will then be expanded in the limits $\operatorname{Re}\theta \to \pm\infty$, with the coefficients of the resulting power series containing the integrals of motion of the $A_1^{(1)}$ massive integrable field theory.

3.5.2 Integral form of $\log Q\left(\theta + \frac{i\pi(M+1)}{2M}\right)$

We begin by taking logarithms of (3.5.1)

$$\log a(\theta) = -2i\pi\gamma + \log Q\,(\theta - i\pi) - \log Q\,(\theta + i\pi)\,, \tag{3.5.28}$$

and invoke the logarithm of the quasiperiodicity relation (3.4.1)

$$\log Q\left(\theta + \frac{i\pi n(M+1)}{M}\right) = -i\pi n\gamma + \log Q(\theta), \quad n \in \mathbb{Z}, \tag{3.5.29}$$

to rewrite (3.5.28) as

$$\log a(\theta) = -i\pi\gamma + \log Q\left(\theta + \frac{i\pi}{M}\right) - \log Q\,(\theta + i\pi)\,. \tag{3.5.30}$$

We then set the NLIE (3.5.26) and (3.5.30) equal to one another

$$\begin{aligned} &\log Q\left(\theta + \frac{i\pi}{M}\right) - \log Q\,(\theta + i\pi) \\ &= \sum_{p=-\infty}^{\infty} b^{(2p+1)} e^{(2p+1)\theta} - 2i\int_{-\infty}^{\infty} \varphi(\theta - \theta' + i0)\, \mathrm{Im}\log(1 + a(\theta' - i0))\, \mathrm{d}\theta'. \end{aligned} \tag{3.5.31}$$

The next step is to take Fourier transforms of (3.5.31), simplifying that expression using the Fourier transform identity

$$\mathcal{F}[f(\theta + \alpha)] = e^{ik\alpha}\mathcal{F}[f(\theta)]. \tag{3.5.32}$$

The result is

$$\begin{aligned} &2\sinh\frac{\pi k(M-1)}{2M}\mathcal{F}\left[\log Q\left(\theta + \frac{i\pi(M+1)}{2M}\right)\right] \\ &= \sum_{p=-\infty}^{\infty} b^{(2p+1)}\mathcal{F}[e^{(2p+1)\theta}] - 2i\mathcal{F}[\varphi](k)\mathcal{F}[\mathrm{Im}\log(1+a)](k). \end{aligned} \tag{3.5.33}$$

We then isolate the $\log Q$ term, with an extra constant term $b^{(0)}$ appearing due to the pole of $\left(2\sinh\frac{\pi k(M-1)}{2M}\right)^{-1}$ at $k=0$. We then take the inverse Fourier transform, with the result

$$\log Q\left(\theta+\frac{i\pi(M+1)}{2M}\right)=b^{(0)}+\sum_{p=-\infty}^{\infty}\frac{b^{(2p+1)}e^{(2p+1)\theta}}{2i\sin\frac{(2p+1)(M-1)\pi}{2M}} \tag{3.5.34}$$

$$-2i\int_{-\infty}^{\infty}H(\theta-\theta'+i0)\,\mathrm{Im}\log(1+a(\theta'-i0))\,\mathrm{d}\theta'. \tag{3.5.35}$$

where

$$H(\theta)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik\theta}}{4\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2M}}\,\mathrm{d}k. \tag{3.5.36}$$

The final integral expression for $\log Q\left(\theta+\frac{i\pi(M+1)}{2M}\right)$ is then found by choosing the constants $b^{(p)}$ to match the earlier derived asymptotics (3.4.29)-(3.4.32) for $Q(\theta)$ in the limits $\mathrm{Re}\,\theta\to\pm\infty$. We set

$$\frac{b^{(1)}}{2i\sin\frac{(M-1)\pi}{2M}}=ms^{M+1}\tau(2,M),\qquad\frac{-b^{(-1)}}{2i\sin\frac{(M-1)\pi}{2M}}=ms^{M+1}\tau(2,M),\qquad b^{(0)}=-\frac{i\pi\gamma}{2}, \tag{3.5.37}$$

with all other constants $b^{(p)}$ set equal to zero. $\log Q\left(\theta+\frac{i\pi(M+1)}{2M}\right)$ is then given by

$$\log Q\left(\theta+\frac{i\pi(M+1)}{2M}\right)=-\frac{i\pi\gamma}{2}+2m\tau(2,M)s^{M+1}\cosh\theta \tag{3.5.38}$$
$$-2i\int_{-\infty}^{\infty}H(\theta-\theta'+i0)\,\mathrm{Im}\log(1+a(\theta'-i0))\,\mathrm{d}\theta'.$$

3.5.3 Integrals of motion

Recasting the Bethe ansatz equations as a non-linear integral equation, we rewrote the logarithm of the Q-function in the integral form (3.5.38). The coefficients of the expansion of this integral form in the limits $\mathrm{Re}\,\theta \to \pm\infty$ are the integrals of motion for the massive $A_1^{(1)}$ integrable field theory. The $H(\theta - \theta' + i0)$ term in (3.5.38) is itself an integral as defined by (3.5.36). We evaluate this integral using Cauchy's residue theorem, resulting in an infinite series. Closing the integration contour in the upper or lower half k-plane leads to two different expansions for $H(\theta)$. Closing in the upper half plane,

$$H(\theta) = i\sum_{p=1}^{\infty} \mathrm{Res}\left[\frac{e^{ik\theta}}{4\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2M}}, k = (2p-1)i\right] \tag{3.5.39}$$

$$+ i\sum_{q=0}^{\infty} \mathrm{Res}\left[\frac{e^{ik\theta}}{4\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2M}}, k = 2qMi\right],$$

$$= i\sum_{p=1}^{\infty} \frac{(-1)^p e^{-(2p-1)\theta}}{2\pi\sin\frac{(2p-1)\pi}{2M}} + i\sum_{q=0}^{\infty}\frac{(-1)^q M e^{-2qM\theta}}{2\pi\cos qM\pi}, \tag{3.5.40}$$

and in the lower half plane,

$$H(\theta) = -i\sum_{p=1}^{\infty} \mathrm{Res}\left[\frac{e^{ik\theta}}{4\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2M}}, k = -(2p-1)i\right] \tag{3.5.41}$$

$$- i\sum_{q=0}^{\infty} \mathrm{Res}\left[\frac{e^{ik\theta}}{4\cosh\frac{\pi k}{2}\sinh\frac{\pi k}{2M}}, k = -2qMi\right],$$

$$= i\sum_{p=1}^{\infty} \frac{(-1)^p e^{(2p-1)\theta}}{2\pi\sin\frac{(2p-1)\pi}{2M}} + i\sum_{q=0}^{\infty}\frac{(-1)^q M e^{2qM\theta}}{2\pi\cos qM\pi}, \tag{3.5.42}$$

We then write $\log Q\left(\theta + \frac{i\pi(M+1)}{2M}\right)$ as a pair of asymptotic series for $M > 1$ and $|\,\mathrm{Im}\,\theta| < \frac{i\pi(1+M)}{2M}$

$$\log Q\left(\theta + \frac{i\pi(M+1)}{2M}\right) = \frac{-i\pi\gamma}{2} + 2m\tau(2,M)s^{M+1}e^{\theta}$$

$$+ \sum_{p=1}^{\infty} \mathfrak{I}_{2p-1} e^{-(2p-1)\theta} + \sum_{q=0}^{\infty} \mathfrak{S}_q\, e^{-2qM\theta}, \quad \text{as } \mathrm{Re}\,\theta \to +\infty, \tag{3.5.43}$$

$$
\log Q\left(\theta + \frac{i\pi(M+1)}{2M}\right) = \frac{-i\pi\gamma}{2} + 2m\tau(2,M)s^{M+1}e^{-\theta}
+ \sum_{p=1}^{\infty} \bar{\mathfrak{I}}_{2p-1} e^{(2p-1)\theta} + \sum_{q=0}^{\infty} \bar{\mathfrak{S}}_q \, e^{2qM\theta}, \quad \text{as } \operatorname{Re}\theta \to -\infty, \tag{3.5.44}
$$

where

$$
\mathfrak{I}_{2p-1} = 2m\tau(2,M)s^{M+1}\delta_{p,1} + \int_{-\infty}^{\infty} \frac{(-1)^p e^{(2p-1)(\theta'-i0)}}{\pi \sin\frac{(2p-1)\pi}{2M}} \operatorname{Im}\log\left(1 + a(\theta' - i0)\right) \, \mathrm{d}\theta', \tag{3.5.45}
$$

$$
\mathfrak{S}_q = \int_{-\infty}^{\infty} \frac{(-1)^q M e^{2qM(\theta'-i0)}}{\pi \cos qM\pi} \operatorname{Im}\log\left(1 + a(\theta' - i0)\right) \, \mathrm{d}\theta', \tag{3.5.46}
$$

and

$$
\bar{\mathfrak{I}}_{2p-1} = 2m\tau(2,M)s^{M+1}\delta_{p,1} - \int_{-\infty}^{\infty} \frac{(-1)^p e^{-(2p-1)(\theta'-i0)}}{\pi \sin\frac{(2p-1)\pi}{2M}} \operatorname{Im}\log\left(1 + a(\theta' - i0)\right) \, \mathrm{d}\theta', \tag{3.5.47}
$$

$$
\bar{\mathfrak{S}}_q = -\int_{-\infty}^{\infty} \frac{(-1)^q M e^{-2qM(\theta'-i0)}}{\pi \cos qM\pi} \operatorname{Im}\log\left(1 + a(\theta' - i0)\right) \, \mathrm{d}\theta', \tag{3.5.48}
$$

are the ground state eigenvalues of the local and non-local integrals of motion for the massive integrable field theory, matching the integrals of motion found in [45] (with the terms proportional to $\log \mathfrak{S}$ in [45] being absorbed into $\mathfrak{S}_0$ and $\bar{\mathfrak{S}}_0$ here).

3.6 Conclusions

In this chapter, we reproduced the results of [45], beginning with the classical modified sinh-Gordon equation and studying the asymptotic solutions of the associated Lax pair to recover information about the related massive integrable field theory. In the remainder of the thesis, we will apply the procedure we have detailed here to the affine Toda field theory equations of motion; these are partial differential equations associated with various Lie algebras that generalise the modified sinh-Gordon equation we have considered in this chapter. With slight alterations (the derivation of the Bethe ansatz equations requires additional information about the representation theory of the relevant Lie algebra encoded in a Ψ-system) this procedure, outlined in Figure 1, will be followed for all other cases.

Chapter 4

Lie algebras and systems of differential equations

4.1 Introduction

In order to generalise the analysis of the ODE/IM correspondence for the Lie algebra $A_1^{(1)}$ in chapter 3, we first require a brief overview of the basic concepts in the theory of Lie algebras, and in doing so set up notation that will be used throughout the remainder of the thesis. Many texts exist to provide a far more thorough introduction to Lie algebras; the brief notes here are based largely on [17, 30, 35].

The systems of differential equations we will consider for the massive ODE/IM correspondence for simply-laced Lie algebras are constructed from representations of the Lie algebras we will consider in section 4.2. In section 4.3, we consider general properties of such systems of differential equations, giving a general method of rewriting them as pseudo-differential equations, and introduce a general method for the study of their asymptotics.

4.2 Notes on Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field F endowed with a Lie bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following:

$$\forall\, x, y \in \mathfrak{g},\ [x, y] = -[y, x], \quad \text{(antisymmetry)} \tag{4.2.1}$$

$$\forall\, x, y, z \in \mathfrak{g}, \lambda, \mu \in F, [x, \lambda y + \mu z] = \lambda[x, y] + \mu[x, z], \quad \text{(linearity)} \tag{4.2.2}$$

$$\forall\, x, y, z \in \mathfrak{g}, [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \quad \text{(Jacobi identity)} \tag{4.2.3}$$

For all Lie algebras we shall consider, the field $F = \mathbb{C}$. The *dimension of the algebra* $\mathfrak{g}$ is the dimension of $\mathfrak{g}$ when considered as a vector space. A subspace $\mathfrak{h} \leq \mathfrak{g}$ is said to be a *subalgebra* of $\mathfrak{g}$ if $\forall\, x, y \in \mathfrak{h}, [x, y] \in \mathfrak{h}$. An *ideal* is a subalgebra satisfying the stronger property $[x, y] \in \mathfrak{h}\ \forall\, x \in \mathfrak{h}, y \in \mathfrak{g}$. The Lie algebras we will be chiefly interested in have no non-trivial ideals; such algebras are *simple* Lie algebras. A direct sum of simple Lie algebras is a *semisimple* Lie algebra. For the remainder of this section $\mathfrak{g}$ will be a simple Lie algebra of finite type.

A particularly important subalgebra of $\mathfrak{g}$ is the *Cartan subalgebra* $\mathfrak{h}$, which is the subspace of largest possible dimension spanned by generators H_i $(i = 1, \ldots, r)$ that satisfy

$$[H_i, H_j] = 0. \tag{4.2.4}$$

The number of generators r (the dimension of the subspace $\mathfrak{h}$) is the *rank* of the Lie algebra.

A *matrix representation* of a Lie algebra is a homomorphism $\pi : \mathfrak{g} \to End(V)$ (where $End(V)$ is the endomorphism algebra defined on a vector space V) such that $\forall\, g_1, g_2 \in \mathfrak{g}$,

$$\pi([g_1, g_2]) = [\pi(g_1), \pi(g_2)], \tag{4.2.5}$$

where $[\pi(g_1), \pi(g_2)] = \pi(g_1)\pi(g_2) - \pi(g_2)\pi(g_1)$ is the standard matrix commutator. The *dimension of the representation* is the dimension of the vector space V. A representation is *irreducible* if there are no non-trivial proper subspaces $W \leq V$ such that $\pi(x)W \subseteq W$. In this thesis, the differential equations with which we will be concerned are related to irreducible matrix representations of certain Lie algebras.

The adjoint representation

An important representation for the study of the structure of Lie algebras is the *adjoint representation*; this is the representation where we choose the vector space $V = \mathfrak{g}$. The adjoint map $\mathrm{ad} : \mathfrak{g} \to End(\mathfrak{g})$ takes an element $x \in \mathfrak{g}$ and maps it to a linear map on $\mathfrak{g}$ with the action

$$\mathrm{ad}(x)(y) = [x, y] \quad \forall\, y \in \mathfrak{g}. \tag{4.2.6}$$

Using the Jacobi identity, it can be seen that this map is indeed a homomorphism from $\mathfrak{g}$ to $End(\mathfrak{g})$, and so it is a representation. The adjoint representation provides information about the nature of the remaining generators of $\mathfrak{g}$. Consider $H_i, H_j \in \mathfrak{h}$. From the definition of the adjoint map and its nature as a homomorphism,

$$[\mathrm{ad}(H_i), \mathrm{ad}(H_j)] = \mathrm{ad}([H_i, H_j]) = \mathrm{ad}(0) = 0. \tag{4.2.7}$$

As all of the matrices $\mathrm{ad}(H_i)$ are commuting, and are diagonalisable [30], a standard result in linear algebra (see Lemma 16.7 in [30]) implies that the matrices $\mathrm{ad}(H_i)$ may be simultaneously diagonalised. Hence we construct an eigenbasis of the matrices $\mathrm{ad}(H_i)$ composed of elements $H_j, E_\alpha \in \mathfrak{g}$, such that

$$\mathrm{ad}(H_i)(H_j) = [H_i, H_j] = 0, \tag{4.2.8}$$

$$\mathrm{ad}(H_i)(E_\alpha) = [H_i, E_\alpha] = \alpha^i E_\alpha. \tag{4.2.9}$$

The eigenbasis $\{H_j, E_\alpha\}$ is the *Cartan-Weyl basis*. The eigenvalues α^i are components of r-dimensional vectors α which are the *roots* of the Lie algebra. The set of roots is denoted by Δ, and the classification of these sets of roots is equivalent to the classification of simple Lie algebras. For now, it remains only to discuss the remaining commutators in $\mathfrak{g}$, which are of the form $[E_\alpha, E_\beta]$ where $\alpha, \beta \in \Delta$. Consider the commutator

$$[H_i, [E_\alpha, E_\beta]] = [E_\alpha, [H_i, E_\beta]] + [[H_i, E_\alpha], E_\beta] = (\alpha^i + \beta^i)[E_\alpha, E_\beta], \tag{4.2.10}$$

where the first equality is due to the Jacobi identity (4.2.3). It is clear that $[E_\alpha, E_\beta] = N(\alpha, \beta)E_{\alpha+\beta}$ for some *structure constant* $N(\alpha, \beta)$, where $\alpha + \beta \in \Delta$. If $\beta = -\alpha$, then (4.2.10) implies that $[E_\alpha, E_{-\alpha}]$ must commute with all the commuting generators H_i. As the H_i span the subspace of commuting elements of $\mathfrak{g}$, $[E_\alpha, E_{-\alpha}]$ must then be a linear combination of H_i, up to a free choice of normalisation of E_α. We will follow the convention in [17] to set $[E_\alpha, E_{-\alpha}] = 2\alpha \cdot H/|\alpha|^2$, where

$$\alpha \cdot H = \sum_{i=1}^{r} \alpha^i H_i, \quad |\alpha|^2 = \sum_{i=1}^{r} \alpha^i \alpha^i. \tag{4.2.11}$$

Lastly, by (4.2.10), if $\alpha+\beta \notin \Delta$, then $[E_\alpha, E_\beta] = 0$. We now summarise the above commutation relations:

$$[H_i, H_j] = 0, \tag{4.2.12}$$

$$[H_i, E_\alpha] = \alpha^i E_\alpha, \tag{4.2.13}$$

$$[E_\alpha, E_\beta] = \begin{cases} N(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta \\ \dfrac{2\alpha \cdot H}{|\alpha|^2} & \text{if } \beta = -\alpha \\ 0 & \text{if } \alpha+\beta \notin \Delta. \end{cases} \tag{4.2.14}$$

Roots

We have described the structure of simple Lie algebras in terms of roots α in a set of roots Δ. To discover more about the structure of simple Lie algebras, we will need to learn more about the structure of these roots. To begin, we derive a useful result that severely constrains the inner products of roots. We consider a subalgebra of $\mathfrak{g}$ generated by $\{E_\alpha, E_{-\alpha},\ \alpha \cdot H/|\alpha|^2\}$. This subalgebra is isomorphic to the Lie algebra $\mathfrak{su}(2)$:

$$E_\alpha = J_+, \tag{4.2.15}$$

$$E_{-\alpha} = J_-, \tag{4.2.16}$$

$$\alpha \cdot H/|\alpha|^2 = J_3, \tag{4.2.17}$$

where

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3. \tag{4.2.18}$$

We then invoke a pair of results from the representation theory of $\mathfrak{su}(2)$:

- If a representation of $\mathfrak{su}(2)$ is finite-dimensional, then the eigenvalues of J_3 are integers or half-integers.

- Let the dimension of an irreducible representation of $\mathfrak{su}(2)$ be given by $2j+1$, where j is an integer or a half-integer. Then the highest and lowest eigenvalues of J_3 are j and $-j$.

Any finite-dimensional representation of $\mathfrak{g}$ must induce a finite-dimensional representation of the $\mathfrak{su}(2)$ subalgebra $\{E_\alpha, E_{-\alpha},\ \alpha\cdot H/|\alpha|^2\}$. Consider repeatedly acting with the operator $\mathrm{ad}(E_\alpha)$ on E_β. Then there exists a maximal integer $p \geq 0$ such that $\mathrm{ad}(E_\alpha)^p E_\beta$ is an eigenvector of J_3, with eigenvalue j, and similarly, there exists an maximal integer $q \geq 0$ such that $\mathrm{ad}(E_{-\alpha})^q E_\beta$ is an eigenvector of J_3 with eigenvalue $-j$. From (4.2.10) and the definition of the adjoint map, $\beta + p\alpha$ and $\beta - q\alpha$ are roots which satisfy

$$\frac{\alpha\cdot(\beta+p\alpha)}{|\alpha|^2} = j, \quad \frac{\alpha\cdot(\beta-q\alpha)}{|\alpha|^2} = -j, \tag{4.2.19}$$

$$\implies \frac{2\alpha\cdot\beta}{|\alpha|^2} = -(p-q) \in \mathbb{Z}. \tag{4.2.20}$$

The result (4.2.20) is one of the defining properties of a *root system* (defined fully in Definition 11.1 of [30]), and it can be shown that there is a one-to-one correspondence between root systems and semisimple Lie algebras. We only require a notion of positivity of roots; a set of positive roots Δ_+ is a subset of roots Δ such that exactly one of $\pm\alpha \in \Delta_+$, and for $\alpha, \beta \in \Delta_+, \alpha+\beta \in \Delta_+$. Given a set of positive roots, there exists a unique set of simple roots $\{\alpha_i\}_{i=1}^r$ with the following properties:

- $\alpha_i \in \Delta_+$.

- α_i form a basis for the vector space containing the roots.

- α_i cannot be written as a sum of two positive roots.

- Any positive root can be written as a sum of simple roots with non-negative integer coefficients.

The simple roots and their inner products define the elements of the *Cartan matrix*

$$C_{ij} = \frac{2\alpha_i \cdot \alpha_j}{|\alpha_j|^2}. \tag{4.2.21}$$

It is clear from (4.2.20) that the elements of the Cartan matrix are integers. As $\alpha_i - \alpha_j \notin \Delta$ (else α_i could be written as a sum of two positive roots), we set $q = 0$ in the result (4.2.20) to see $\alpha_i \cdot \alpha_j \leq 0$ for $i \neq j$. This means that the off-diagonal elements of C are non-positive integers. The diagonal elements are equal to 2 from the definition of C. Lastly, we gain more information about the possible Cartan matrices using the Cauchy-Schwarz inequality:

$$(\alpha_i \cdot \alpha_j)^2 < |\alpha_i|^2 |\alpha_j|^2, \quad (i \neq j). \tag{4.2.22}$$

From (4.2.22) we see that $C_{ij}C_{ji} < 4$ for $i \neq j$. As the elements C_{ij} are non-positive, this implies $C_{ij} = 0, -1, -2$ or -3. This strongly constrains the possible angles between simple roots. Let θ_{ij} be the angle between α_i and α_j, so that $\alpha_i \cdot \alpha_j = |\alpha_i||\alpha_j| \cos\theta_{ij}$. Substituting this into the Cartan matrix definition (4.2.21) and applying $\alpha_i \cdot \alpha_j \leq 0$ for $i \neq j$ we find

$$\cos\theta_{ij} = -\frac{1}{2}\sqrt{C_{ij}C_{ji}}. \tag{4.2.23}$$

Along with the constraint that C_{ij} must be integers, θ_{ij} may only take a handful of possible values: $\theta_{ij} = \pi/2, 2\pi/3, 3\pi/4$ or $5\pi/6$. Dividing entries of the Cartan matrix also leads to a constraint on the ratios of the magnitudes of the roots. If $\theta_{ij} = \pi/2$, the lengths are unrestricted; if $\theta_{ij} = 2\pi/3$, the lengths are the same; if $\theta_{ij} = 3\pi/4$, the ratio of the lengths is $\sqrt{2}$, and if $\theta_{ij} = 5\pi/6$, the ratio of the lengths is $\sqrt{3}$. We are free to choose an overall normalisation for the roots, and we will choose the longest root to have length $\sqrt{2}$. If all the roots are the same length, the Lie algebra is said to be *simply-laced*.

The above constraints on the simple roots allows a complete classification of the allowed Cartan matrices, and hence a classification of the simple Lie algebras. Another way of representing the data encoded in the Cartan matrices is through *Dynkin diagrams*. A Dynkin diagram is a graph with a vertex associated to each simple root α_i. The vertices corresponding to α_i and α_j are connected by $C_{ij}C_{ji}$ lines. Shorter roots (in the case of non simply-laced Lie algebras) are represented by filled-in vertices.

Figure 2 shows the Dynkin diagrams for the simple Lie algebras. There are four infinite families A_r, B_r, C_r, D_r and five exceptional Lie algebras E_6, E_7, E_8, F_4 and G_2. The A, D and E algebras, having diagrams with no filled-in vertices, are simply-laced and these will be our main algebras of interest.

Other useful definitions

The Cartan matrix (or equivalently, the associated Dynkin diagram) encodes the inner products of the simple roots α_i. The remaining roots of the Lie algebra $\mathfrak{g}$ are found in terms of the simple roots by the action of the *Weyl group* of the algebra. The Weyl group has r generators s_i, which act as reflections in the space of roots:

$$s_i(\beta) = \beta - \frac{2\alpha_i \cdot \beta}{\alpha_i \cdot \alpha_i}\alpha_i \quad \forall\, \beta \in \Delta. \tag{4.2.24}$$

The generators s_i are constrained by the relations

$$(s_i s_j)^{m_{ij}} = 1, \quad \text{where } m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \dfrac{\pi}{\pi - \theta_{ij}} & \text{otherwise.} \end{cases} \tag{4.2.25}$$

The roots Δ of the Lie algebra $\mathfrak{g}$ are then the orbit of a simple root, say α_1, under the Weyl group. From the result (4.2.20), all elements of Δ are integer sums of simple roots. A particularly important root is the *highest root* $-\alpha_0 = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r$, where the coefficients n_i are maximised. The coefficients n_i of the highest root are the *Kac labels*. Defining $n_0 = 1$, we have $\sum_{i=0}^{r} n_i\alpha_i = 0$.

There is a notion of duality for roots. We define the *co-root* $\alpha^\vee$ of a root α:

$$\alpha^\vee = \frac{2\alpha}{|\alpha|^2}. \tag{4.2.26}$$

The Cartan matrix elements can then be written $C_{ij} = \alpha_i \cdot \alpha_j^\vee$. The *dual Kac labels* $n_i^\vee$ satisfy $\sum_{i=0}^{r} n_i^\vee \alpha_i^\vee = 0$. The *Coxeter number* h and its dual $h^\vee$ are given by

$$h = \sum_{i=0}^{r} n_i, \quad h^\vee = \sum_{i=0}^{r} n_i^\vee. \tag{4.2.27}$$

As we have chosen a normalisation for the simple roots such that the longest root satisfies $|\alpha_i|^2 = 2$, for simply-laced Lie algebras, co-roots are the same as roots. To be consistent with the notation in the literature [37, 38] we will retain the distinction between roots and co-roots, although for the simply-laced ADE Lie algebras we will be primarily concerned with, this will be redundant.

The Chevalley generators

It is occasionally useful to perform a relabelling of the generators H_i, replacing them with a different set of generators $\tilde{H}_i$ in the Cartan subalgebra $\mathfrak{h}$:

$$H_i \to \tilde{H}_i = \frac{2\alpha_i \cdot H}{|\alpha_i|^2}, \tag{4.2.28}$$

with $\alpha_i \cdot H$ defined as in (4.2.11):

$$\alpha_i \cdot H = \sum_{j=1}^{r} \alpha_i^j H_j, \tag{4.2.29}$$

where α_i^j is the j^{th} component of the simple root α_i.

With this redefinition of the generators of the Cartan subalgebra the commutation relations of $\mathfrak{g}$ related to the simple roots α_i take the elegant form

$$[H_i, H_j] = 0, \tag{4.2.30}$$

$$[H_i, E_{\alpha_j}] = C_{ji} E_{\alpha_j}, \tag{4.2.31}$$

$$[H_i, E_{-\alpha_j}] = -C_{ji} E_{-\alpha_j}, \tag{4.2.32}$$

$$[E_{\alpha_i}, E_{-\alpha_j}] = \delta_{ij} H_j, \tag{4.2.33}$$

(where δ_{ij} is the Kronecker delta) which along with the *Serre relations*

$$\mathrm{ad}(E_{\alpha_i})^{(1-C_{ji})}(E_{\alpha_j}) = 0, \tag{4.2.34}$$

$$\mathrm{ad}(E_{-\alpha_i})^{(1-C_{ji})}(E_{-\alpha_j}) = 0, \quad i \neq j, \tag{4.2.35}$$

generate the entire Lie algebra $\mathfrak{g}$. The use of Chevalley generators also ensures the structure constants are all integers, which simplifies analysis of the representations of $\mathfrak{g}$.

4.2.1 Representation theory of simple Lie algebras

We recall the definition of a matrix representation of a Lie algebra: a homomorphism $\pi : \mathfrak{g} \to End(V)$ that preserves the Lie algebra structure as defined in (4.2.5). The adjoint representation (4.2.6) is an example of a matrix representation of a simple Lie algebra $\mathfrak{g}$. This representation has the distinguishing feature of its associated vector space V being the Lie algebra $\mathfrak{g}$ itself. We now discuss general finite-dimensional representations of $\mathfrak{g}$.

As the generators H_i are commuting and are diagonalisable, a basis for V $\{|\lambda\rangle\}$ exists that simultaneously diagonalises the H_i [30]:

$$H_i \, |\lambda\rangle = \lambda^i \, |\lambda\rangle \tag{4.2.36}$$

where the vector λ is a *weight*. We define the *fundamental weights* ω_i and the *co-fundamental weights* $\omega_i^\vee$ to satisfy

$$\omega_i \cdot \alpha_j^\vee = \frac{2\omega_i \cdot \alpha_j}{|\alpha_j|^2} = \delta_{ij}, \quad \omega_i^\vee \cdot \alpha_j = \delta_{ij}. \tag{4.2.37}$$

Using this definition, we find a useful identity for the simple root α_i in terms of the fundamental weights ω_j. We write

$$\alpha_i = \sum_{s=1}^{r} k_s \omega_s, \tag{4.2.38}$$

and dot with $\alpha_j^\vee$, simplifying using the definition of the fundamental weights ω_s:

$$\alpha_i \cdot \alpha_j^\vee = \sum_{s=1}^{r} k_s \omega_s \cdot \alpha_j^\vee = k_j. \tag{4.2.39}$$

By definition of the Cartan matrix, $k_j = C_{ij}$, and so the Cartan matrix can be thought of as a change of basis from the weight basis to the root basis:

$$\alpha_i = \sum_{j=1}^{r} C_{ij}\omega_j. \tag{4.2.40}$$

It can be shown that any weight λ in a finite-dimensional representation of $\mathfrak{g}$ can be written in the form $\lambda = \lambda^1\omega_1 + \lambda^2\omega_2 + \cdots + \lambda^r\omega_r$ and that the coefficients λ^i are all integers. It is also straightforward to show from the commutation relations (4.2.30)-(4.2.33) that the E_{α_i} and $E_{-\alpha_i}$ act as raising and lowering operators respectively:

$$H_i E_{\alpha_j} |\lambda\rangle = (\lambda_i + C_{ji}) E_{\alpha_j} |\lambda\rangle , \tag{4.2.41}$$

$$H_i E_{-\alpha_j} |\lambda\rangle = (\lambda_i - C_{ji}) E_{-\alpha_j} |\lambda\rangle . \tag{4.2.42}$$

In order to construct a finite-dimensional representation it is necessary for there to exist a *highest weight* vector $|\lambda\rangle$, such that $E_{\alpha_i} |\lambda\rangle = 0$ for $i = 1, \ldots, r$. The integer coefficients λ^i of the weight λ of this vector are freely chosen, and they serve to label the representation. Other weight vectors in the representation (themselves eigenvectors of the H_i) are generated by acting on $|\lambda\rangle$ with lowering operators $E_{-\alpha_j}$ using the following algorithm:

- Start with the highest weight eigenvector $|\lambda\rangle$. If the j^{th} component λ_j is positive, add the states $E_{-\alpha_j} |\lambda\rangle , E^2_{-\alpha_j} |\lambda\rangle , \ldots , E^{\lambda_j}_{-\alpha_j} |\lambda\rangle$ to the representation space. Do this for each j.
- Continue with the same procedure for each of the newly generated states.
- Repeat this process until all the newly generated states have negative (or zero) weight components.

This algorithm produces a spanning set for the vector space V, however, it is not a basis as many redundant vectors are generated. We may cull the redundant vectors using a Gram-Schmidt procedure with respect to an inner product τ on the states

$$\tau\left(E_{-\alpha_{a_1}}E_{-\alpha_{a_2}}\dots E_{-\alpha_{a_k}}\left|\lambda\right\rangle, E_{-\alpha_{b_1}}E_{-\alpha_{b_2}}\dots E_{-\alpha_{b_l}}\left|\lambda\right\rangle\right) \tag{4.2.43}$$

$$= \left\langle\lambda\right| E_{\alpha_{a_k}}\dots E_{\alpha_{a_1}}E_{-\alpha_{b_1}}\dots E_{-\alpha_{b_l}}\left|\lambda\right\rangle, \tag{4.2.44}$$

(where $1 \leq a_p, b_q \leq r$) which is evaluated using the commutation relations, the action of H_i on the highest weight vector $|\lambda\rangle$, and the normalisation $\langle\lambda|\lambda\rangle = 1$. We denote the representation with highest weight λ by $L(\lambda)$.

The representations $L(\omega_i)$ are *fundamental representations* of $\mathfrak{g}$, denoted by $V^{(i)}$ to match the convention in [38]. The sum of the (co-)fundamental weights ρ ($\rho^\vee$) is the *(co-)Weyl vector*. The Weyl vector ρ and the co-Weyl vector $\rho^\vee$ can also be defined by

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha = \sum_{i=1}^{r}\omega_i, \quad \rho^\vee = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha^\vee = \sum_{i=1}^{r}\omega_i^\vee. \tag{4.2.45}$$

All the Lie algebras we will be concerned with are simply-laced, ensuring roots and co-roots are the same, and $\rho^\vee = \rho$. Using the Weyl vector, it is also possible to determine the dimension of the representation generated by the highest weight λ without computing the representation explicitly. To do this, we invoke the *Weyl dimension formula*:

$$\dim(L(\lambda)) = \prod_{\alpha\in\Delta_+}\frac{(\lambda+\rho)\cdot\alpha}{\rho\cdot\alpha}. \tag{4.2.46}$$

We will use this formula to compute the dimensions of the fundamental representations of the Lie algebras with which we will be concerned. All of the relevant data for the Lie algebras will be given at the end of this chapter.

Summing the weights of a Lie algebra representation

We will now justify a result which will be very useful in the following chapters. Consider a semisimple Lie algebra $\mathfrak{g}$ and its fundamental representations $V^{(a)}$. Let the weights of the representation be given by $\lambda_i^{(a)}, i = 1, \ldots, \dim V^{(a)}$, such that

$$H_j^{(a)} \, |\lambda_i^{(a)}\rangle = (\lambda_i^{(a)})_j \, |\lambda_i^{(a)}\rangle \tag{4.2.47}$$

where the vectors $\{|\lambda_i^{(a)}\rangle\}$ satisfy $\langle\lambda_i^{(a)}|\lambda_i^{(a)}\rangle = 1$. We will often require that the sum of the weight vectors $\lambda_i^{(a)}$ is zero:

$$\sum_{i=1}^{\dim V^{(a)}} \lambda_i^{(a)} = 0. \tag{4.2.48}$$

This result is a corollary of a more general result: the generators of a finite-dimensional representation of a semisimple Lie algebra are traceless. We demonstrate this using the Cartan-Weyl basis $\{H_i, E_\alpha\}$. Recall that H_i and E_α satisfy the commutator

$$[H_i, E_\alpha] = \alpha^i E_\alpha. \tag{4.2.49}$$

Take the trace of both sides of (4.2.49). By the cyclic property of traces the left hand side of (4.2.49) is zero, which immediately implies $\mathrm{tr}(E_\alpha) = 0$. It remains to show that the traces of the generators H_i of the Cartan subalgebra are zero. Recall for a general root $\alpha \in \Delta$ the commutator of E_α and $E_{-\alpha}$ is given by

$$[E_\alpha, E_{-\alpha}] = \frac{2\alpha \cdot H}{|\alpha|^2}, \tag{4.2.50}$$

and we once again take the trace of this commutator. As the trace is a linear operator on matrices we find the following result which must hold for all roots

$\alpha \in \Delta$:

$$\sum_{j=1}^{r} \alpha^j \, \mathrm{tr}(H_j) = 0. \tag{4.2.51}$$

We now employ the basis of simple roots for our Lie algebra $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$. Substituting each of these into (4.2.51) yields a system of linear equations

$$\sum_{j=1}^{r} (\alpha_i)^j \, \mathrm{tr}(H_j) = 0. \tag{4.2.52}$$

As our simple roots form a basis, the matrix with elements $(\alpha_i)^j$ has a kernel containing only the zero vector. We then immediately find $\mathrm{tr}(H_i) = 0$. As a linear combination of traceless matrices is itself traceless, we have that any element of a finite-dimensional matrix representation of a semisimple Lie algebra $\mathfrak{g}$ is traceless.

We now apply this to find our desired result (4.2.48). We write the trace of a generator $H_j^{(a)}$ (where $j = 1, \ldots, r$) as a sum over the eigenbasis $\{|\lambda_i^{(a)}\rangle\}$ and our result immediately follows:

$$\mathrm{tr}\left(H_j^{(a)}\right) = \sum_{i=1}^{\dim V^{(a)}} \langle\lambda_i^{(a)}| \, H_j^{(a)} \, |\lambda_i^{(a)}\rangle = \sum_{i=1}^{\dim V^{(a)}} (\lambda_i^{(a)})_j \, \langle\lambda_i^{(a)}|\lambda_i^{(a)}\rangle = \sum_{i=1}^{\dim V^{(a)}} (\lambda_i^{(a)})_j = 0. \tag{4.2.53}$$

Products of representations

Many of the representations of $\mathfrak{g}$ we will be concerned with are constructed from particular products of smaller representations. The first such product of is a *tensor product* of representations. Let V and W be representations of a simple Lie algebra $\mathfrak{g}$. Then the tensor product of V and W is denoted by $V \otimes W$ and is the vector space generated by the elements $v \otimes w$, for $v \in V$ and $w \in W$, where $\otimes$ is a bilinear operation. We then establish a new representation of $\mathfrak{g}$ on the vector space $V \otimes W$ by defining the action of $A \in \mathfrak{g}$ on elements $v \otimes w \in V \otimes W$ in the following way:

$$A(v \otimes w) = A(v) \otimes w + v \otimes A(w) \quad \forall\, v \in V, w \in W. \tag{4.2.54}$$

Given a vector space V (over a field $F = \mathbb{C}$) and the tensor product operator $\otimes$, we construct the tensor algebra $T(V)$ as a direct sum of tensor products of V:

$$T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots \tag{4.2.55}$$

We then construct the exterior algebra $\Lambda(V)$ of a vector space V as the quotient of the tensor algebra $T(V)$ by the ideal $I = \{v \otimes v | v \in V\}$. The *wedge product* $\wedge$ (or *exterior product*) is then the product on elements of $\Lambda(V)$ induced by the tensor product $\otimes$ on $T(V)$. The wedge product on V satisfies $v \wedge v = 0$ for all $v \in V$, and inherits bilinearity from the tensor product $\otimes$. By setting $v = x + y$ for $x, y \in V$ and using bilinearity to expand $(x + y) \wedge (x + y)$, we find the wedge product is antisymmetric: $x \wedge y = -y \wedge x$ for all $x, y \in V$.

Using the wedge product, we then construct new vector spaces $\bigwedge^a V$

$$\bigwedge^a V = \{v_1 \wedge v_2 \wedge \dots \wedge v_a |\, v_1, \dots, v_a \in V\}, \tag{4.2.56}$$

and then establish new representations of $\mathfrak{g}$ on $\bigwedge^a V$ by defining the action of $A \in \mathfrak{g}$ on a-vectors $v_1 \wedge v_2 \wedge \dots \wedge v_a \in \bigwedge^a V$:

$$\begin{aligned} &A(v_1 \wedge v_2 \wedge \dots \wedge v_a) \\ &= A(v_1) \wedge v_2 \wedge \dots \wedge v_a + v_1 \wedge A(v_2) \wedge \dots \wedge v_a + \dots + v_1 \wedge v_2 \wedge \dots \wedge A(v_a). \end{aligned} \tag{4.2.57}$$

Many of the representations of interest in this thesis will be products of evaluation representations V_k, which will be defined below. The evaluation representations are defined over the same vector space V and so the product representations constructed in this section remain well-defined.

4.2.2 Affine Lie algebras and Lie algebra data

The differential equations and integrable models we will consider are related to *affine Lie algebras*. We now briefly detail their construction from a given simple Lie algebra, and note the particular representations of affine Lie algebras that we will find useful in the following chapters. More complete details of the construction of affine Lie algebras are found in chapter 14 of [17].

We let $\mathbb{C}[t, t^{-1}]$ be the space of Laurent polynomials in the variable t. We then define an affine Lie algebra $\hat{\mathfrak{g}}$ to be the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ with the Lie bracket

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + \frac{m}{h^\vee}\kappa(a, b)\delta_{m+n,0}\, c, \tag{4.2.58}$$

$$[c, \hat{\mathfrak{g}}] = 0, \quad (a, b \in \mathfrak{g},\ m, n \in \mathbb{Z}) \tag{4.2.59}$$

where c is the central element of the affine Lie algebra and $\kappa(a, b)$ is the Killing form of the Lie algebra $\mathfrak{g}$. These affine Lie algebras retain the simple root structure of the simple Lie algebras, with an additional simple root α_0 equal to the lowest root of $\mathfrak{g}$ plus an imaginary root δ [17]. Its corresponding Chevalley generator is then given by $E_{\alpha_0} \otimes t$, which by an abuse of notation we shall denote E_{α_0}.

Certain finite-dimensional representations of $\hat{\mathfrak{g}}$ will be of interest to us. We follow [47] and define the *evaluation representation* of $\hat{\mathfrak{g}}$. Let V be some finite-dimensional representation of the simple Lie algebra $\mathfrak{g}$, and let $\zeta \in \mathbb{C} \setminus \{0\}$. Then the evaluation representation $V(\zeta)$ of $\hat{\mathfrak{g}}$ has the following action on $v \in V$:

$$(a \otimes t^n)v = \zeta^n av, \quad (a \in \mathfrak{g}). \tag{4.2.60}$$

As in [47] we denote $V_k = V(e^{2\pi ik})$ to be the evaluation representation of $\hat{\mathfrak{g}}$ corresponding to the representation V of $\mathfrak{g}$ and $\zeta = e^{2\pi ik}$. For our purposes, k will be an integer or a half-integer. If k is an integer, these evaluation representations are exactly equivalent to the original representation V. If k is a half-integer, the representation of the Chevalley generator $E_{\alpha_0} \otimes t \in \hat{\mathfrak{g}}$ becomes $E_{\alpha_0} \otimes (-1) = -E_{\alpha_0}$.

In this thesis, the affine Lie algebras we will consider are denoted by $A_r^{(1)} = \hat{A}_r$, $D_r^{(1)} = \hat{D}_r$, $E_6^{(1)} = \hat{E}_6$, $E_7^{(1)} = \hat{E}_7$ and $E_8^{(1)} = \hat{E}_8$.

Lie algebra data

As stated at the beginning of this chapter, the above sketch of the theory of Lie algebras is by no means comprehensive. Further details and proofs may be found in the texts [30], [35] and Chapter 13 of [17]. We conclude this chapter with a collection of relevant data for the simply-laced Lie algebras A_r, D_r, E_6, E_7 and E_8. The Weyl vectors and dimensions of fundamental representations for E_6, E_7 and E_8 were calculated using the *LieART* package for Mathematica.

A_r

- Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

- Coxeter number $h = r + 1$
- Highest root $-\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_{r-1} + \alpha_r$
- Weyl vector $\rho = \rho^\vee = \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{i=1}^r i(r - i + 1)\alpha_i$
- Kac labels: $n_0 = n_1 = \cdots = n_r = 1$
- Dimensions of fundamental representations $V^{(a)} = L(\omega_a)$: $\binom{r+1}{a}$ for $1 \le a \le r - 2$, 2^{r-1} for $a = r - 1, r$.

D_r

- Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 & 2 \end{pmatrix}$$

- Coxeter number $h = 2r - 2$

- Highest root $-\alpha_0 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$

- Weyl vector $\rho = \rho^\vee = \sum_{i=1}^r \omega_i = \frac{1}{2}\sum_{i=1}^r i(2r - i - 1)\alpha_i$

- Kac labels: $n_0 = n_1 = 1$, $n_2 = \cdots = n_{r-2} = 2$, $n_{r-1} = n_r = 1$

- Dimensions of fundamental representations $V^{(a)}$: $\binom{2r}{a}$ for $1 \leq a \leq r-2$, 2^{r-1} for $a = r-1, r$.

E_6

- Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

- Coxeter number $h = 12$

- Highest root $-\alpha_0 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$

- Weyl vector $\rho = \rho^\vee = \sum_{i=1}^{6} \omega_i = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 15\alpha_4 + 8\alpha_5 + 11\alpha_6$

- Kac labels: $n_0 = n_1 = 1$, $n_2 = 2$, $n_3 = 3$, $n_4 = 2$, $n_5 = 1$, $n_6 = 2$

- Dimensions of fundamental representations $V^{(a)}$:

  (27, 351, 2925, 351, 27, 78).

E_7

- Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

- Coxeter number $h = 18$

- Highest root $-\alpha_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7$.

- Weyl vector $\rho = \rho^\vee = \sum_{i=1}^{7} \omega_i = 17\alpha_1 + 33\alpha_2 + 48\alpha_3 + \frac{75}{2}\alpha_4 + 52\alpha_5 + \frac{27}{2}\alpha_6 + \frac{49}{2}\alpha_7$

- Kac labels: $n_0 = 1$, $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, $n_4 = 3$, $n_5 = 2$, $n_6 = 1$, $n_7 = 2$

- Dimensions of fundamental representations $V^{(a)}$:
  (133, 8645, 365750, 27664, 1539, 56, 912).

E_8

- Cartan matrix

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

- Coxeter number $h = 30$

- Highest root $-\alpha_0 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + 3\alpha_8$

- Weyl vector $\rho = \rho^\vee = \sum_{i=1}^{8} \omega_i = 46\alpha_1 + 91\alpha_2 + 135\alpha_3 + 110\alpha_4 + 84\alpha_5 + 57\alpha_6 + 29\alpha_7 + 68\alpha_8$

- Kac labels: $n_0 = 1$, $n_1 = 2$ $n_2 = 4$, $n_3 = 6$, $n_4 = 5$, $n_5 = 4$, $n_6 = 3$, $n_7 = 2$, $n_8 = 3$

- Dimensions of fundamental representations $V^{(a)}$:
(3875, 6696000, 6899079264, 146325270, 2450240, 30380, 248, 147250).

4.3 Systems of differential equations

The differential equations that form one side of the ODE/IM correspondence are often most elegantly written in the form of a system of coupled differential equations. In this section, we will study these systems of differential equations. In subsection 4.3.1 we will describe two methods of converting such systems into pseudo-differential equations, involving the inverse differential operator ∂_z^{-1}. We apply both methods to a relevant example, demonstrating that they are in agreement. Lastly, in subsection 4.3.2 we consider the asymptotics of systems of differential equations directly, using a generalisation of the WKB approximation.

A similar procedure is performed in [28] to construct analogues of the Korteweg-de Vries (KdV) equation from a pair of matrices known as a Lax pair, constructed from a representation of a simple Lie algebra. In this section, although we construct similar pseudo-differential operators, we do not consider them in the context of Lax pairs and integrable equations. We will consider them merely as systems of differential equations, manipulating them directly to find pseudo-differential operators acting on a single function $\psi(x)$.

4.3.1 From systems of differential equations to pseudo-differential equations

For the remainder of the chapter, we will consider systems of differential equations of the form

$$(\partial_z + A(z))\Psi(z) = 0, \tag{4.3.1}$$

where $A(z)$ is an n-by-n matrix and $\Psi(z) = (\psi_1(z), \ldots, \psi_n(z))^T$ is a column vector. In component form this system of equations is given by

$$\partial_z\psi_i + \sum_{j=1}^{n} A(z)_{ij}\psi_j = 0. \tag{4.3.2}$$

For a general matrix $A(z)$, the system of equations (4.3.1) does not have a closed-form solution. In the special case $[A(z_1), A(z_2)] = 0$ for $z_1 \neq z_2$, the solution can be written explicitly

$$\Psi(z) = \exp\left(-\int_0^z A(t)\,\mathrm{d}t\right)\Psi(0). \tag{4.3.3}$$

However, if $[A(z_1), A(z_2)] \neq 0$, the simple solution (4.3.3) no longer holds; the solution may then only be written in terms of a Magnus series [46]:

$$\Psi(z) = \exp\left(-\sum_{k=1}^{\infty}\mathcal{A}_k(z)\right)\Psi(0), \tag{4.3.4}$$

where the first few terms in (4.3.4) are given by

$$\mathcal{A}_1(z) = \int_0^z A(\tau_1)\,\mathrm{d}\tau_1, \tag{4.3.5}$$

$$\mathcal{A}_2(z) = \frac{1}{2}\int_0^z\int_0^{\tau_1}[A(\tau_1), A(\tau_2)]\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1, \tag{4.3.6}$$

$$\mathcal{A}_3(z) = \frac{1}{6}\int_0^z\int_0^{\tau_1}\int_0^{\tau_2}\left([A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]]\right)\,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1. \tag{4.3.7}$$

The Magnus series is effectively a re-ordering of the terms in the more standard path-ordered exponential, defined as

$$\begin{aligned}&\mathcal{T}\left\{\exp\left(-\int_0^z A(s)\,\mathrm{d}s\right)\right\} \\ &= \left(I - \int_0^z A(s_1)\,\mathrm{d}s_1 + \int_0^z\int_0^{s_1} A(s_1)A(s_2)\,\mathrm{d}s_2\,\mathrm{d}s_1 + \dots\right).\end{aligned} \tag{4.3.8}$$

Evaluated as full series, the Magnus series (4.3.4) and the expansion of the path-ordered exponential (4.3.8) are equivalent. The form of the Magnus series, evaluated up to its first term, is more useful for our purposes than the first-order expansion of the path-ordered exponential.

When we consider systems of equations of the form (4.3.1) in the small-$|z|$ limit, only the first term in this expansion will be relevant. Our matrices $A(z)$

will also be diagonal in the small-$|z|$ limit, simplifying the form of the asymptotics considerably.

In the context of the ODE/IM correspondence, the systems of differential equations (4.3.1) are more commonly presented in the form of pseudo-differential equations [20] by combining all the component equations of (4.3.1) into one equation in ψ_1. To discuss pseudo-differential equations and related pseudo-differential operators, we briefly define the principal characteristic of such operators: the presence of an inverse differential operator ∂_z^{-1}, with the properties defined in section 3 of [20]:

$$\partial_z^{-1}(z^n) = \frac{z^{n+1}}{n+1}, \tag{4.3.9}$$

$$\partial_z^{-1}\partial_z(z^n) = \partial_z\partial_z^{-1}(z^n) = z^n, \tag{4.3.10}$$

$$\partial_z^{-1}\left(f(z)\partial_z(g(z))\right) = f(z)g(z) - \partial_z^{-1}\left(\partial_z(f(z))g(z)\right). \tag{4.3.11}$$

A pseudo-differential operator can then be considered to be an element of an algebra generated by analytic functions $f(z)$ and differential operators ∂_z^a for $a \in \mathbb{Z}$. We will now demonstrate two different methods of converting systems of differential equations into pseudo-differential equations, thus exhibiting an example of a pseudo-differential operator.

Method of repeated differentiation

We will consider a system of ten coupled differential equations, related to the second fundamental representation (the representation with highest weight ω_2) of the Lie algebra $A_4^{(1)}$:

$$
\begin{aligned}
\partial\psi_1 + \psi_2 &= 0, \\
\partial\psi_2 + \psi_3 + \psi_4 &= 0, \\
\partial\psi_3 + \psi_5 &= 0, \\
\partial\psi_4 + \psi_5 + \psi_6 &= 0, \\
\partial\psi_5 + \psi_7 + \psi_8 &= 0, \\
\partial\psi_6 + \psi_8 &= 0, \\
\partial\psi_7 + \psi_9 &= 0, \\
\partial\psi_8 + \psi_9 + p(z)\psi_1 &= 0, \\
\partial\psi_9 + \psi_{10} + p(z)\psi_2 &= 0, \\
\partial\psi_{10} + p(z)\psi_4 &= 0,
\end{aligned}
\tag{4.3.12}
$$

where we have used the truncated notation $\partial = \partial_z$, and introduced the z-dependent function $p(z)$. The exact form of $p(z)$ does not concern us in this section. (The systems of differential equations we will consider in later chapters will contain constants of the form me^{θ}, but these constants do not affect the calculation of pseudo-differential equations.)

To construct a single equation in ψ_1, we repeatedly differentiate the first equation in (4.3.12), and find

$$
\partial^7\psi_1 = -5p\psi_4 + 5\partial(p\psi_2) - 3\partial^2(p\psi_1). \tag{4.3.13}
$$

To produce a pseudo-differential equation, we must rewrite ψ_2 and ψ_4 in terms of ψ_1. The ψ_2 term is dealt with straightforwardly using the first equation in

(4.3.12):

$$\psi_2 = -\partial\psi_1. \tag{4.3.14}$$

We rewrite the ψ_4 term in (4.3.13) by using the system of equations (4.3.12) to derive the following useful identities

$$\partial^3\psi_4 = -3\psi_9 - 2p\psi_1, \tag{4.3.15}$$

$$\partial^5\psi_1 = -5\psi_9 - 3p\psi_1. \tag{4.3.16}$$

We then combine (4.3.15) and (4.3.16), removing the ψ_9 terms, to find an equation for ψ_4 in terms of ψ_1:

$$5\psi_4 = 3\partial^2\psi_1 - \partial^{-3}(p\psi_1). \tag{4.3.17}$$

The pseudo-differential equation that is equivalent to the system of equations (4.3.12) is then found by substituting (4.3.14) and (4.3.17) into (4.3.13):

$$\partial^7\psi_1 + 3\,\partial^2(p\psi_1) + 5\partial(p\,\partial\psi_1) + 3p\,\partial^2\psi_1 - p\,\partial^{-3}(p\psi_1) = 0. \tag{4.3.18}$$

This pseudo-differential equation matches the tenth-order ODE studied in section 3.4 of [1], after removing the inverse differential term by dividing (4.3.18) through by p and differentiating a further three times.

Method of loop-counting

We also present a diagrammatic method, given in section 2.3 of [50], of computing the equivalent pseudo-differential equation from a system of differential equations. For a given system of n equations $(\partial + A)\Psi = 0$ (again setting $\partial = \partial_z$), we construct a directed graph with n vertices, connecting two vertices i and j with an arrow from i to j if $A_{ij} \neq 0$. For the system of equations (4.3.12), the directed graph constructed in this way is shown in Figure 3. We then construct a pseudo-differential equation corresponding to the system of differential equations (4.3.12) using its directed graph and the following procedure:

- Find all distinct loops (closed paths of the form $i \to \cdots \to i$) in the directed graph.
- Each loop contributes a term in the pseudo-differential equation. For each loop, start from the lowest numbered node in the loop, and for each arrow $i \to j$, write down $(-\partial^{-1} A_{ij})$.
- Compute the distance of the lowest numbered node from node 1. Let this distance be a positive integer d. Multiply the product of $(-\partial^{-1} A_{ij})$ terms by ∂^{-d} on the left and by $\partial^{d} \psi_1$.
- Sum all such expressions from each of the loops. Set this sum equal to ψ_1. Simplify as necessary.

We apply this procedure to (4.3.12) by firstly counting all the distinct loops in Figure 3. All the distinct loops have lowest numbered nodes 1, 2 or 4, and we categorise them in that way. The loops with lowest node 1 are:

$$\begin{aligned}
&1 \to 2 \to 3 \to 5 \to 8 \to 1, \\
&1 \to 2 \to 4 \to 5 \to 8 \to 1, \\
&1 \to 2 \to 4 \to 6 \to 8 \to 1, \\
&1 \to 2 \to 3 \to 5 \to 7 \to 9 \to 10 \to 4 \to 6 \to 8 \to 1.
\end{aligned} \tag{4.3.19}$$

The loops with lowest node 2 are:

$$\begin{aligned}
&2 \to 3 \to 5 \to 7 \to 9 \to 2, \\
&2 \to 3 \to 5 \to 8 \to 9 \to 2, \\
&2 \to 4 \to 5 \to 7 \to 9 \to 2, \\
&2 \to 4 \to 5 \to 8 \to 9 \to 2, \\
&2 \to 4 \to 6 \to 8 \to 9 \to 2.
\end{aligned} \tag{4.3.20}$$

and the loops with lowest node 4 are

$$\begin{aligned}
&4 \to 5 \to 7 \to 9 \to 10 \to 4, \\
&4 \to 5 \to 8 \to 9 \to 10 \to 4, \\
&4 \to 6 \to 8 \to 9 \to 10 \to 4.
\end{aligned} \tag{4.3.21}$$

We next convert these loops into terms in the pseudo-differential equation. As an example, the loop $4 \to 5 \to 7 \to 9 \to 10 \to 4$ corresponds to the term

$$\begin{aligned}
&(\partial^{-2})(-\partial^{-1})(-\partial^{-1})(-\partial^{-1})(-\partial^{-1})(-\partial^{-1}p)\partial^2\psi_1 \\
&= -\partial^{-7}(p\partial^2\psi_1).
\end{aligned} \tag{4.3.22}$$

Performing the same conversion to each of the loops, and then adding the results and setting them equal to ψ_1, we find

$$\psi_1 = -3\partial^{-5}(p\psi_1) - 5\partial^{-6}(p\partial\psi_1) - 3\partial^{-7}(p\partial^2\psi_1) + \partial^{-7}(p\partial^{-3}(p\psi_1)). \tag{4.3.23}$$

We differentiate this expression seven times and rearrange terms to recover the pseudo-differential equation (4.3.18)

$$\partial^7\psi_1 + 3\,\partial^2(p\psi_1) + 5\partial(p\,\partial\psi_1) + 3p\,\partial^2\psi_1 - p\,\partial^{-3}(p\psi_1) = 0. \tag{4.3.24}$$

The system (4.3.12) is a special case of the system of differential equations related to the second fundamental representation of $A_4^{(1)}$. In general, the derivative operators ∂_z are replaced by more general differential operators of the form $D(\lambda)$, where λ is an r-component weight vector associated with the particular representation of a simple Lie algebra. The system (4.3.12) becomes

$$\begin{aligned} D(\lambda_1^{(2)})\psi_1 + \psi_2 &= 0, \\ D(\lambda_2^{(2)})\psi_2 + \psi_3 + \psi_4 &= 0, \\ D(\lambda_3^{(2)})\psi_3 + \psi_5 &= 0, \\ D(\lambda_4^{(2)})\psi_4 + \psi_5 + \psi_6 &= 0, \\ D(\lambda_5^{(2)})\psi_5 + \psi_7 + \psi_8 &= 0, \\ D(\lambda_6^{(2)})\psi_6 + \psi_8 &= 0, \\ D(\lambda_7^{(2)})\psi_7 + \psi_9 &= 0, \\ D(\lambda_8^{(2)})\psi_8 + \psi_9 + p(z)\psi_1 &= 0, \\ D(\lambda_9^{(2)})\psi_9 + \psi_{10} + p(z)\psi_2 &= 0, \\ D(\lambda_{10}^{(2)})\psi_{10} + p(z)\psi_4 &= 0. \end{aligned} \tag{4.3.25}$$

where $\lambda_i^{(2)}$ are the weight vectors associated with second fundamental representation of $A_4^{(1)}$. The presence of the more general operators $D(\lambda)$ makes this system of equations much more difficult to simplify into a single pseudo-differential equation. The equation will contain terms of the form $D(\lambda_3^{(2)})^{-1} + D(\lambda_4^{(2)})^{-1}$, which cannot be simplified except in special cases of the weight vectors. In chapter 6 we will see the symmetry of the weight vectors $\lambda_i^{(1)}$ in the first fundamental representation of $D_r^{(1)}$ allow for the construction of a compact pseudo-differential equation in that case.

4.3.2 Asymptotics of systems of differential equations

In the following chapters, we will often be concerned with the small-$|z|$ and large-$|z|$ behaviour of systems of differential equations of the form $(\partial_z + A(z))\Psi = 0$. Particular solutions of these systems in the small-$|z|$ and large-$|z|$ regimes will define Q-functions which encode information about the related integrable quantum field theory. In this section, we will briefly discuss methods of analysing the asymptotics of systems of differential equation in these regimes, setting up these methods for use in later chapters.

The small-$|z|$ asymptotics of $(\partial_z + A)\Psi = 0$ are easily studied using the first term of the Magnus series (4.3.4). For small-$|z|$, the solution is given by

$$\Psi(z) = \exp\left(-\int_0^z A(\tau_1)\,\mathrm{d}\tau_1\right), \tag{4.3.26}$$

which is further simplified by taking the $|z| \to 0$ limit. For the systems of differential equations we are concerned with, the matrices $A(z)$ will be diagonal in this limit. The matrix exponential in (4.3.26) is then a diagonal matrix, and the small-$|z|$ asymptotics are easily extracted.

The large-$|z|$ asymptotics are more involved; to study these, we will employ a slight generalisation of the WKB approximation (discussed in [13]) given in [49].

The WKB approximation for systems of differential equations

We perform a slight change of notation, working with the equation

$$(\varepsilon\partial_z + A(z))\Psi = 0, \tag{4.3.27}$$

and considering the $\varepsilon \to 0$ limit. The following 'abelianisation' procedure, similar to that found in section 3.8 of [3], will yield an asymptotic expansion for $\Psi(z)$, which we will see through an example matches the large-$|z|$ asymptotic expansion of solutions of $(\partial_z + A)\Psi = 0$ when $\varepsilon = 1$. We also must assume that the matrix $A(z)$ is a diagonalisable matrix in the large-$|z|$ limit, so that a basis of eigenvectors exists in the neighbourhood of $z = \infty$. All of the systems of differential equations that we will consider satisfy this property.

We begin by performing a change of variables from Ψ to $\hat{\Psi}$

$$\Psi = (P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)\hat{\Psi}, \tag{4.3.28}$$

(where the P_i are z-dependent matrices to be determined) so that the linear system (4.3.27) becomes

$$\begin{aligned}
&\varepsilon\partial_z((P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)\hat{\Psi}) + A(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)\hat{\Psi} \\
&= \varepsilon(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)\partial_z\hat{\Psi} + \varepsilon(\partial_z P_0 + \varepsilon\partial_z P_1 + \dots)\hat{\Psi} \\
&+ A(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)\hat{\Psi} = 0.
\end{aligned} \tag{4.3.29}$$

We then multiply this expression on the left with the matrix $(P_0 + \varepsilon P_1 + \dots)^{-1}$:

$$\begin{aligned}
\varepsilon\partial_z\hat{\Psi} = &-\varepsilon(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)^{-1}(\partial_z P_0 + \varepsilon\partial_z P_1 + \dots)\hat{\Psi} \\
&- (P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)^{-1}A(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)\hat{\Psi}.
\end{aligned} \tag{4.3.30}$$

(P_0 will be a matrix of eigenvectors for the matrix A. P_0 is then invertible and for sufficiently small $\varepsilon > 0$, $(P_0+\varepsilon P_1+\dots)^{-1}$ exists.) We then expand the right-hand side of equation (4.3.30) in powers of ε. We first note

$$\begin{aligned}(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \dots)^{-1} &= (I + \varepsilon P_0^{-1} P_1 + \varepsilon^2 P_0^{-1} P_2 + \dots)^{-1} P_0^{-1}, \\ &= (I + \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots)^{-1} P_0^{-1}, \\ &\sim (I - \varepsilon Q_1 + \varepsilon^2 (Q_1^2 - Q_2) + \dots) P_0^{-1} \text{ as } \varepsilon \to 0, \end{aligned} \tag{4.3.31}$$

where for convenience we have set $Q_i = P_0^{-1} P_i$. We have also used the identity

$$(I + X)^{-1} = \sum_{j=0}^{\infty} (-1)^j X^j, \tag{4.3.32}$$

for $X = \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots$. This identity holds when $\|X\| < 1$, which is true for $X = \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots$ for sufficiently small $\varepsilon > 0$. We substitute (4.3.31) into (4.3.30):

$$\begin{aligned}\varepsilon \partial_z \hat{\Psi} = &-\varepsilon (I - \varepsilon Q_1 + \varepsilon^2 (Q_1^2 - Q_2) + \dots) P_0^{-1} (\partial_z P_0 + \varepsilon \partial_z P_1 + \dots) \hat{\Psi} \\ &- (I - \varepsilon Q_1 + \varepsilon^2 (Q_1^2 - Q_2) + \dots) P_0^{-1} A P_0 (I + \varepsilon Q_1 + \varepsilon^2 Q_2 + \dots) \hat{\Psi}, \end{aligned} \tag{4.3.33}$$

and consider terms proportional to powers of ε on the right-hand side of equation (4.3.33).

The only $O(\varepsilon^0)$ term on the right-hand side of (4.3.33) is

$$\Lambda_0 = -P_0^{-1} A P_0, \tag{4.3.34}$$

and we choose P_0 such that Λ_0 is diagonal. We can always do this given the starting assumption that A was a diagonalisable matrix.

The $O(\varepsilon)$ term on the right-hand side of (4.3.33) is given by

$$\Lambda_1 = -[\Lambda_0, Q_1] - P_0^{-1}\partial_z P_0. \tag{4.3.35}$$

We choose Q_1 so that Λ_1 is a diagonal matrix. As Λ_0 is diagonal, $[\Lambda_0, Q_1]$ has zeroes along its diagonal. Choosing the elements of Q_1 to cancel the off-diagonal elements, we have

$$(\Lambda_1)_{ii} = -(P_0^{-1}\partial_z P_0)_{ii}. \tag{4.3.36}$$

The linear system then becomes

$$\varepsilon\partial_z\hat{\Psi} = (\Lambda_0 + \varepsilon\Lambda_1 + \dots)\hat{\Psi}, \tag{4.3.37}$$

which decouples into n separate first-order ordinary differential equations

$$\begin{aligned}
&\varepsilon\partial_z\hat{\psi}_i = (\Lambda_0 + \varepsilon\Lambda_1 + \dots)_{ii}\hat{\psi}_i \\
\implies &\hat{\psi}_i = A_i \exp\left(\frac{1}{\varepsilon}\int^z (\Lambda_0(s) + \varepsilon\Lambda_1(s) + \dots)_{ii}\,\mathrm{d}s\right).
\end{aligned} \tag{4.3.38}$$

To recover the asymptotic expansion to the original problem (in Ψ) we act on $\hat{\Psi}$ with the matrix $(P_0 + \varepsilon P_1 + \dots)$:

$$\begin{aligned}
\Psi &= (P_0 + \varepsilon P_1 + \dots)\hat{\Psi} \qquad (4.3.39)\\
&\sim (P_0 + \varepsilon P_1 + \dots)\exp\left(\frac{1}{\varepsilon}\int^z \Lambda_0(s) + \varepsilon\Lambda_1(s) + \dots\ \mathrm{d}s\right)\hat{\Psi}_0 \text{ as } \varepsilon \to 0,
\end{aligned}$$

where $\hat{\Psi}_0$ is a constant vector.

Example: the Airy equation

We illustrate the above method on a well-known system. The Airy equation

$$\varepsilon^2 \partial_z^2 \psi = z\psi \tag{4.3.40}$$

can be written as a system of equations

$$\varepsilon \partial_z \Psi = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \Psi, \tag{4.3.41}$$

where $\Psi = (\psi_1, \psi_2)^T$. Let

$$A = \begin{pmatrix} 0 & -1 \\ -z & 0 \end{pmatrix}, \tag{4.3.42}$$

and choose the matrix P_0 so that $P_0^{-1} A P_0$ is diagonal:

$$P_0 = \begin{pmatrix} -z^{-1/2} & z^{-1/2} \\ 1 & 1 \end{pmatrix} \implies \Lambda_0 = \begin{pmatrix} -z^{1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix}. \tag{4.3.43}$$

Using (4.3.36), we find

$$\Lambda_1 = \frac{1}{4z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.3.44}$$

Substitute P_0, Λ_0 and Λ_1 into equation (4.3.39) to find asymptotics for Ψ:

$$\Psi \sim P_0 \exp\left(\frac{1}{\varepsilon} \int^z \Lambda_0(s) + \varepsilon \Lambda_1(s) \, \mathrm{d}s \right) \hat{\Psi}_0, \text{ as } \varepsilon \to 0, \tag{4.3.45}$$

implies

$$\Psi \sim \begin{pmatrix} -z^{-1/2} & z^{-1/2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -z^{1/4}e^{-\frac{2}{3\varepsilon}z^{3/2}} & 0 \\ 0 & z^{1/4}e^{\frac{2}{3\varepsilon}z^{3/2}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.3.46}$$

Multiplying out these matrices,

$$\Psi \sim \begin{pmatrix} -az^{-1/4}e^{-\frac{2}{3\varepsilon}z^{3/2}} + bz^{-1/4}e^{\frac{2}{3\varepsilon}z^{3/2}} \\ az^{1/4}e^{-\frac{2}{3\varepsilon}z^{3/2}} + bz^{1/4}e^{\frac{2}{3\varepsilon}z^{3/2}} \end{pmatrix} \text{ as } \varepsilon \to 0, \tag{4.3.47}$$

where we set the constant vector $\hat{\Psi}_0 = (a, b)^T$. These asymptotics match, up to a rescaling, the first-order asymptotic expansions for two solutions to the Airy equation $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ in the large-$|z|$ limit:

$$\begin{aligned} \mathrm{Ai}(z) &\sim z^{-1/4}e^{-\frac{2}{3}z^{3/2}}, \\ \mathrm{Bi}(z) &\sim z^{-1/4}e^{\frac{2}{3}z^{3/2}}, \\ \mathrm{Ai}'(z) &\sim z^{1/4}e^{-\frac{2}{3}z^{3/2}}, \\ \mathrm{Bi}'(z) &\sim z^{1/4}e^{\frac{2}{3}z^{3/2}}, \text{ as } |z| \to \infty. \end{aligned} \tag{4.3.48}$$

4.4 Conclusions

In this chapter, we have provided a brief introduction to Lie algebra theory, constructing the representations of the Lie algebras that we will work with for the remainder of the thesis. We have also introduced some techniques for studying systems of differential equations. We discussed two methods of converting systems of differential equations into single pseudo-differential equations, which is useful for the discussion of eigenvalue problems. We have also introduced a generalisation of the WKB approximation to certain well-behaved systems of differential equations, which will be invaluable for the study of the asymptotic solutions of the differential equations that constitute one side of the ODE/IM correspondence.

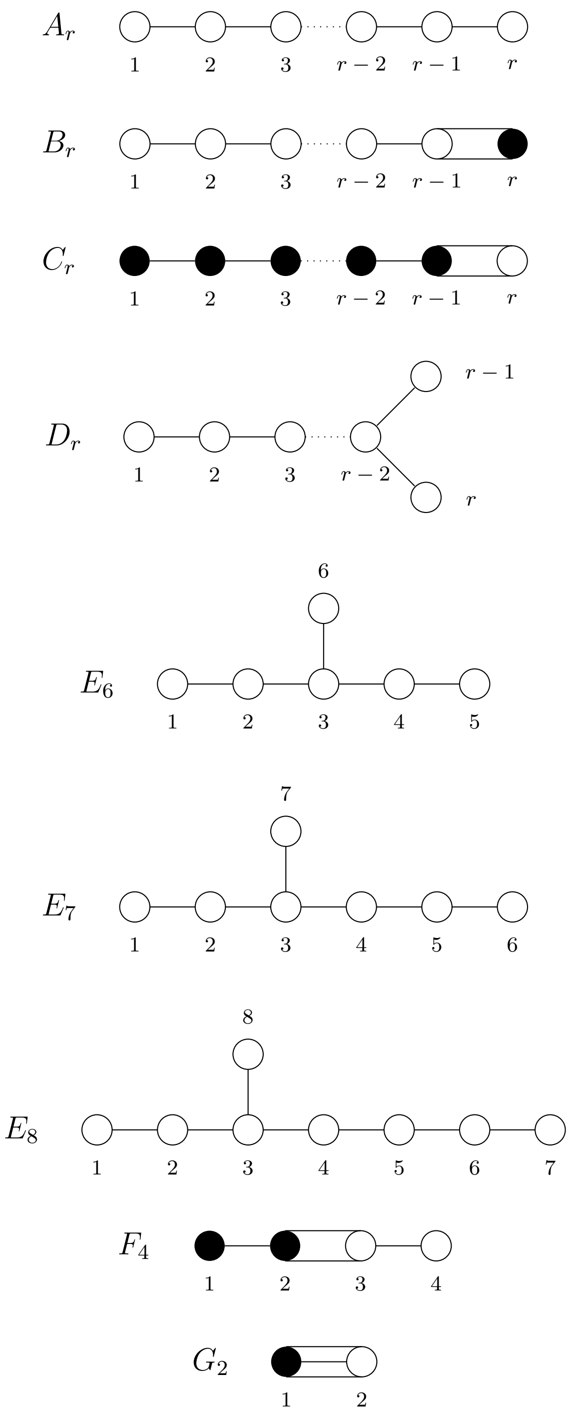


Figure 2: Dynkin diagrams of the simple Lie algebras. The labels on the vertices correspond to the fundamental roots and weights related to that vertex.

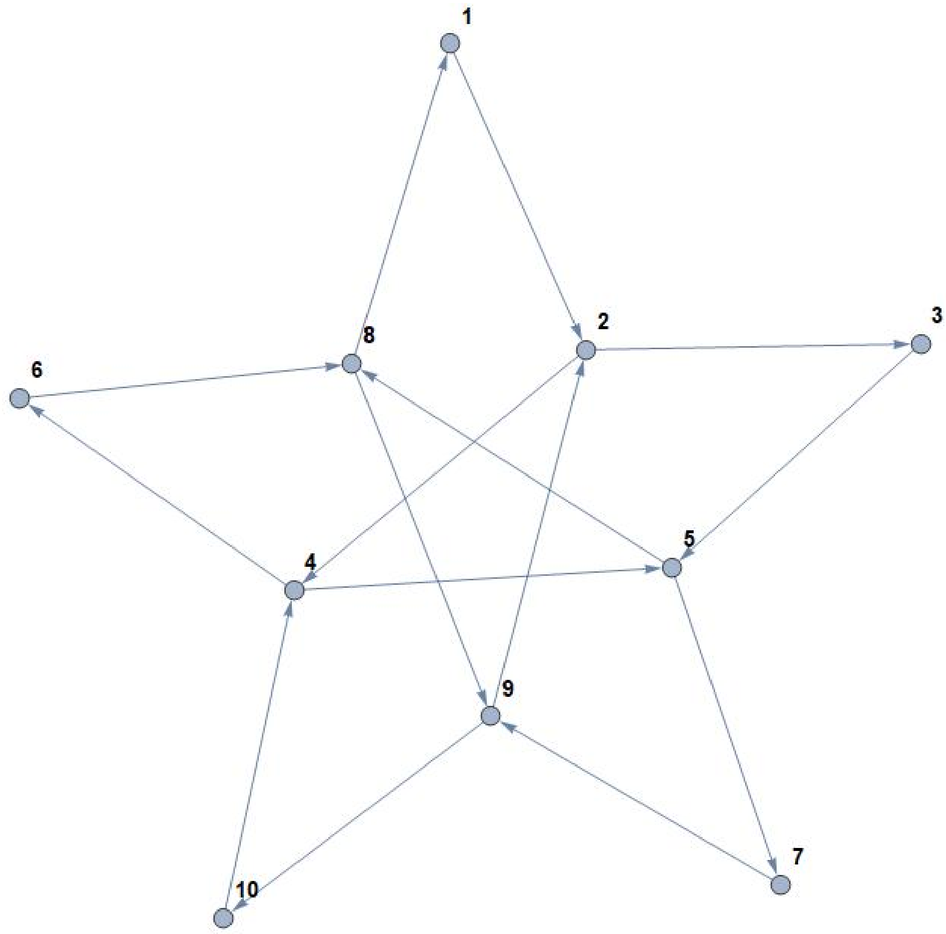


Figure 3: Directed graph associated with the A-matrix of the second fundamental representation of $A_4^{(1)}$.

Chapter 5

On the $A_r^{(1)}$ case of the massive ODE/IM correspondence

5.1 Introduction

In this chapter, we will find features of an integrable quantum field theory associated with the Lie algebra $A_r^{(1)}$ arising from the differential equations associated with a classical $A_r^{(1)}$ Toda field theory. This work will be a direct generalisation of the $A_1^{(1)}$ case of the massive ODE/IM correspondence studied in chapter 3. In section 5.2 we follow the notation given in [37, 38] and define the differential equations of interest in terms of the generators of the Lie algebra $A_r^{(1)}$. In section 5.3 we work in particular representations of $A_r^{(1)}$ and define the differential equations of interest. We then use these differential equations to define Q-functions, and derive certain useful properties of these equations in 5.4. The solutions of the differential equations and the representations of $A_r^{(1)}$ are then as in [38, 47] used to define Ψ-systems, which imply functional relations on the Q-functions known as Bethe ansatz equations. In section 5.6 these are then used to derive integrals of motion associated with a massive integrable theory with $A_r^{(1)}$ symmetry. In section 5.7 the integrals of motion are used to exhibit the spectral equivalence between Lastly, in section 5.8 the Q-functions are used to derive other functional relations which arise in the study of integrable models.

5.2 Affine Toda field theory

5.2.1 Definitions

We begin with the affine Toda field theory Lagrangian associated with the affine Lie algebra $\hat{\mathfrak{g}}$ [37]

$$\mathcal{L} = \partial_w \phi \cdot \partial_{\bar{w}} \phi + \frac{m^2}{\beta^2} \sum_{i=0}^{r} n_i \exp(\beta \alpha_i \cdot \phi), \tag{5.2.1}$$

where we work in light-cone coordinates w, $\bar{w}$. ϕ is an r-component vector field and m and β are constants. We also note the choice of signature to match [38, 45]. The associated equations of motion are

$$\partial_w \partial_{\bar{w}} \phi - \frac{m^2}{\beta} \sum_{i=0}^{r} n_i \alpha_i \exp(\beta \alpha_i \cdot \phi) = 0. \tag{5.2.2}$$

The differential equations that we are interested in are the modified affine Toda field equations, related to the equations of motion (5.2.2) by a change of variables $w \to z$ and a shift in the field ϕ:

$$w = \int^z p(\tilde{z})^{1/h} \, \mathrm{d}\tilde{z}, \quad \phi \to \phi - \frac{\rho^\vee}{\beta h} \log(p(z) p(\bar{z})). \tag{5.2.3}$$

Applying this transformation to (5.2.2), using properties of the co-Weyl vector $\rho^\vee$:

$$\rho^\vee \cdot \alpha_i = 1, \quad \rho^\vee \cdot \alpha_0 = 1 - h, \tag{5.2.4}$$

we find the modified affine Toda field equations

$$\partial_z \partial_{\bar{z}} \phi - \frac{m^2}{\beta} \left[\sum_{i=1}^{r} n_i \alpha_i \exp(\beta \alpha_i \cdot \phi) + p(z) p(\bar{z}) n_0 \alpha_0 \exp(\beta \alpha_0 \cdot \phi) \right] = 0. \tag{5.2.5}$$

We set $p(z) = z^{hM} - s^{hM}$, where M is real and positive and s is complex. We recover the modified sinh-Gordon equation (3.2.1) by setting $r = 1$ and substituting the roots of the Lie algebra $A_1^{(1)}$ in (5.2.5). The modified affine Toda field equations (5.2.5) can be written as the compatibility condition

$$\partial_{\bar{z}} A - \partial_z \bar{A} + [A, \bar{A}] = 0, \tag{5.2.6}$$

where A and $\bar{A}$ are elements of a representation of the Lie algebra $\mathfrak{g}$, given by [38]:

$$A = \frac{\beta}{2} \partial_z \phi \cdot H + m e^{\theta} \left[\sum_{i=1}^{r} \sqrt{n_i^\vee} e^{\beta \alpha_i \cdot \phi / 2} E_{\alpha_i} + p(z) \sqrt{n_0^\vee} e^{\beta \alpha_0 \cdot \phi / 2} E_{\alpha_0} \right], \tag{5.2.7}$$

$$\bar{A} = -\frac{\beta}{2} \partial_{\bar{z}} \phi \cdot H + m e^{-\theta} \left[\sum_{i=1}^{r} \sqrt{n_i^\vee} e^{\beta \alpha_i \cdot \phi / 2} E_{-\alpha_i} + p(\bar{z}) \sqrt{n_0^\vee} e^{\beta \alpha_0 \cdot \phi / 2} E_{-\alpha_0} \right], \tag{5.2.8}$$

where we have introduced the spectral parameter θ. Note the dual Kac labels $n_i^\vee = \frac{n_i |\alpha_i|^2}{2}$. The modified affine Toda field equations then have an associated

Lax pair representation

$$(\partial_z + A)\Psi = 0, \tag{5.2.9}$$

$$(\partial_{\bar{z}} + \bar{A})\Psi = 0. \tag{5.2.10}$$

We will mainly consider only the first of these equations; henceforth we call this *the linear problem*. We will be concerned with the asymptotics of the various solutions of this set of differential equations and the relationships between them.

It will be useful to define polar co-ordinates $z = |z|e^{i\varphi}$. The linear problem (5.2.9) is then invariant under a Symanzik rotation

$$\Omega_k : \varphi \to \varphi + \frac{2\pi k}{hM}, \quad \theta \to \theta - \frac{2\pi i k}{hM}, \quad s \to s e^{\frac{2\pi i k}{hM}}, \tag{5.2.11}$$

for any integer k. The linear problem (5.2.9) is also invariant under a gauge transformation. Set

$$\tilde{A} = UAU^{-1} + U\partial_z U^{-1}, \quad \tilde{\Psi} = U\Psi, \tag{5.2.12}$$

for an arbitrary matrix U. Then, using $\partial_z(UU^{-1}) = U\partial_z U^{-1} + \partial_z U U^{-1} = 0$, we find

$$(\partial_z + \tilde{A})\tilde{\Psi} = U(\partial_z + A)\Psi = 0, \tag{5.2.13}$$

hence the linear problem is unchanged. It is useful to set $U = e^{-\beta H\cdot\phi/2}$, where the exponential of an operator X was defined in (3.2.11). Gauge transforming A using this matrix U has the effect of removing the inconvenient exponential terms from A:

$$A \to \tilde{A} = \beta \partial_z \phi \cdot H + me^{\theta} \left[\sum_{i=1}^{r} \sqrt{n_i^{\vee}} E_{\alpha_i} + p(z) \sqrt{n_0^{\vee}} E_{\alpha_0} \right] \tag{5.2.14}$$

where the derivation of (5.2.14) uses the identity (3.2.13). The conjugate linear problem (5.2.10) is itself invariant under a similar gauge transformation, with $U = e^{\beta H \cdot \phi/2}$.

5.2.2 Asymptotics of ϕ

To define the solutions to the linear problem (5.2.9) uniquely, we will need to specify constraints on the solution $\phi(|z|, \varphi)$ of the modified affine Toda field equations (5.2.5). Our required solution for the modified affine Toda field equations exists only on the subspace of $(z, \bar{z})$ where $z = \bar{z}$. Following [37] we impose the following:

1. $\phi(|z|, \varphi)$ should be real and finite everywhere except at $|z| = 0$.
2. Periodicity:
$$\phi\left(|z|, \varphi + \frac{2\pi}{hM}\right) = \phi(|z|, \varphi). \tag{5.2.15}$$
3. Large-$|z|$ asymptotics:
$$\phi(|z|, \varphi) = \frac{2M\rho^{\vee}}{\beta} \log|z| + o(1) \quad \text{as } |z| \to \infty. \tag{5.2.16}$$
4. Small-$|z|$ asymptotics:
$$\phi(|z|, \varphi) = 2g \log|z| + O(1) \quad \text{as } |z| \to 0. \tag{5.2.17}$$

We find the asymptotics in the limit $|z| \to 0$ of the solution to (5.2.5) satisfying the above constraints. We substitute the ansatz

$$\phi(z,\bar{z}) = g\log(z\bar{z}) + f(z,\bar{z}) \tag{5.2.18}$$

into the modified affine Toda field equations (5.2.5). The result is an equation for f:

$$\begin{aligned} z\bar{z}\partial_z\partial_{\bar{z}} f = & \frac{m^2}{\beta}\sum_{i=1}^{r} n_i\alpha_i (z\bar{z})^{\beta\alpha_i\cdot g+1} e^{\beta\alpha_i\cdot f} \\ & + \frac{m^2}{\beta} n_0\alpha_0 (z^{hM} - s^{hM})(\bar{z}^{hM} - s^{hM})(z\bar{z})^{\beta\alpha_0\cdot g+1} e^{\beta\alpha_0\cdot f}. \end{aligned} \tag{5.2.19}$$

To ensure that the leading order asymptotics (5.2.18) are preserved in the limit $|z| \to 0$, we require a constraint on g:

$$\beta\alpha_\mu \cdot g + 1 > 0, \quad \mu = 0, \ldots, r. \tag{5.2.20}$$

To aid further calculations, we define some more concise notation. Let

$$\mathcal{D} = z\bar{z}\partial_z\partial_{\bar{z}}, \tag{5.2.21}$$

$$u_\mu = (z\bar{z})^{\beta\alpha_\mu\cdot g+1}, \tag{5.2.22}$$

$$B_\mu = \frac{m^2}{\beta} n_\mu \alpha_\mu, \quad (\mu = 0, 1, \ldots, r) \tag{5.2.23}$$

$$v = z^{hM}, \tag{5.2.24}$$

$$\bar{v} = \bar{z}^{hM}, \tag{5.2.25}$$

$$S = s^{hM}, \tag{5.2.26}$$

so (5.2.19) becomes

$$\mathcal{D}f = \sum_{i=1}^{r} B_i u_i e^{\beta\alpha_i \cdot f} + B_0 u_0 (v - S)(\bar{v} - S) e^{\beta\alpha_0 \cdot f}. \tag{5.2.27}$$

We then expand f as a power series in the variables $u_0, u_1, \ldots, u_r, v, \bar{v}$:

$$f(u_\mu, v, \bar{v}) = \sum_{\substack{\vec{a} \in \mathbb{N}_0^{r+1} \\ b=0,\, c=0}}^{\infty} F(\vec{a}; b, c) \vec{u}^{\vec{a}} v^b \bar{v}^c \tag{5.2.28}$$

where we have defined the multi-index $\vec{a} = (a_0, a_1, \ldots, a_r)$ where $a_i = 0, 1, \ldots$ and the compact notation $\vec{u}^{\vec{a}} = u_0^{a_0} u_1^{a_1} \ldots u_r^{a_r}$. Noting that $\mathcal{D}$ acts as a dilation operator on powers of z and $\bar{z}$:

$$\mathcal{D}(z^p \bar{z}^q) = pq z^p \bar{z}^q, \tag{5.2.29}$$

it is therefore straightforward to substitute (5.2.28) into (5.2.27):

$$\begin{aligned}
&\sum_{\vec{a},b,c}^{\infty} (bhM + \sum_{\mu=0}^{r} (\beta\alpha_\mu \cdot g + 1) a_\mu)(chM + \sum_{\mu=0}^{r} (\beta\alpha_\mu \cdot g + 1) a_\mu) F(\vec{a}; b, c) \vec{u}^{\vec{a}} v^b \bar{v}^c \\
&= \sum_{i=1}^{r} B_i u_i \exp\left(\sum_{\vec{a},b,c} \beta\alpha_i \cdot F(\vec{a}; b, c) \vec{u}^{\vec{a}} v^b \bar{v}^c \right) \\
&+ B_0 u_0 (v - S)(\bar{v} - S) \exp\left(\sum_{\vec{a},b,c} \beta\alpha_0 \cdot F(\vec{a}; b, c) \vec{u}^{\vec{a}} v^b \bar{v}^c \right)
\end{aligned} \tag{5.2.30}$$

Expanding this expression term by term in the powers $\vec{u}^{\vec{a}} v^b \bar{v}^c$ yields a set of recurrence relations for the constants $F(\vec{a}, b, c)$. In this way, the asymptotics of ϕ can in principle be calculated to arbitrarily high orders. We will check that the first-order terms match those in [37].

We notice that $\mathcal{D}(v^k) = \mathcal{D}(\bar{v}^k) = 0$ and there are no $O(v^k)$ or $O(\bar{v}^k)$ terms on the right-hand side of (5.2.30). This means that the values $F(\vec{0}; k, 0)$ and

$F(\vec{0};0,k)$ for $k \geq 0$ are not defined by the recurrence relations for F. They arise from the field redefinition

$$\phi \to \phi - \frac{\rho^\vee}{\beta h} \log(p(z)p(\bar{z})), \tag{5.2.31}$$

and we expand this as a function of v and $\bar{v}$ to find the constants $F(\vec{0};k,0)$ and $F(\vec{0};0,k)$:

$$F(\vec{0};k,0) = F(\vec{0};0,k) = \frac{\rho^\vee}{\beta h k s^{hM}}, \quad k \in \mathbb{Z}_{\geq 0}. \tag{5.2.32}$$

This power series expansion of (5.2.31) leaves behind stray constants which are absorbed into $F(\vec{0};0,0) = \phi^{(0)}$.

All the other constants $F(\vec{a};b,c)$ are defined by recurrence relations found by expanding (5.2.30) term-by-term. We denote $\vec{a} = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \cdots + a_r\mathbf{e}_r$ where $\{\mathbf{e}_j\}_{j=0}^r$ is the standard orthonormal basis. By considering the u_0 term in (5.2.30) we find

$$F(\mathbf{e}_0;0,0) = \frac{B_0 S^2 e^{\beta\alpha_0 \cdot F(\vec{0};0,0)}}{(\beta\alpha_0 \cdot g + 1)^2} = \frac{m^2 s^{2hM} n_0 \alpha_0 e^{\beta\alpha_0\cdot\phi^{(0)}}}{\beta(\beta\alpha_0 \cdot g + 1)^2}. \tag{5.2.33}$$

Similarly, considering the u_i term for $i = 1, \ldots, r$ we find

$$F(\mathbf{e}_i;0,0) = \frac{B_i e^{\beta\alpha_i \cdot F(\vec{0};0,0)}}{(\beta\alpha_i \cdot g + 1)^2} = \frac{m^2 n_i \alpha_i e^{\beta\alpha_i\cdot\phi^{(0)}}}{\beta(\beta\alpha_i \cdot g + 1)^2}. \tag{5.2.34}$$

Putting it all together, the first terms of the small-$|z|$ expansion of our chosen solution to (5.2.5) is

$$\begin{aligned}\phi \sim\, & g \log z\bar{z} + \phi^{(0)} + \frac{\rho^\vee}{\beta h} \sum_{k=1}^{\infty} \frac{z^{hkM} + \bar{z}^{hkM}}{k s^{hkM}} \\ & + \frac{m^2 s^{2hM} n_0 \alpha_0 e^{\beta\alpha_0\cdot\phi^{(0)}}}{\beta(\beta\alpha_0 \cdot g + 1)^2} (z\bar{z})^{\beta\alpha_0\cdot g+1} + \sum_{i=1}^{r} \frac{m^2 n_i \alpha_i e^{\beta\alpha_i\cdot\phi^{(0)}}}{\beta(\beta\alpha_i \cdot g + 1)^2} (z\bar{z})^{\beta\alpha_i\cdot g+1} + \ldots\end{aligned} \tag{5.2.35}$$

This asymptotic expansion matches those found in [37, 45, 2]. With the formalism above, we can compute arbitrarily high orders of the asymptotic expansion by solving the recurrence relations given by (5.2.30). This rapidly becomes difficult, as the exponentials of multinomial series in (5.2.30) produce a plethora of terms. To second-order, there are ten types of terms to calculate in the power series (5.2.28); those corresponding to the powers $u_0^2, u_0u_i, u_0v, u_0\bar{v}, u_iu_j, u_iv, u_i\bar{v}, v^2, v\bar{v}, \bar{v}^2$. To demonstrate the calculation, we calculate the 'cross term' $F(\mathbf{e}_0 + \mathbf{e}_i; 0, 0)$.

The $O(u_0u_i)$ term of the left-hand side of (5.2.30) is

$$(\beta\alpha_0 \cdot g + 1)(\beta\alpha_i \cdot g + 1)F(\mathbf{e}_0 + \mathbf{e}_i; 0, 0), \tag{5.2.36}$$

and the $O(u_0u_i)$ term of the right-hand side is

$$B_i e^{\beta\alpha_i\cdot\phi^{(0)}}\beta\alpha_i \cdot F(\mathbf{e}_0; 0, 0) + B_0 S^2 e^{\beta\alpha_i\cdot\phi^{(0)}}\beta\alpha_0 \cdot F(\mathbf{e}_i; 0, 0). \tag{5.2.37}$$

We then set these expressions equal to one another, substitute in our expressions for $F(\mathbf{e}_0; 0, 0)$ and $F(\mathbf{e}_i; 0, 0)$ (5.2.33) and (5.2.34), and then solve for $F(\mathbf{e}_0 + \mathbf{e}_i; 0, 0)$:

$$\begin{aligned} F(\mathbf{e}_0 + \mathbf{e}_i; 0, 0) &= \frac{m^4}{\beta} s^{2hM} e^{\beta(\alpha_0+\alpha_i)\cdot\phi^{(0)}} n_0 n_1 \\ &\cdot \left(\frac{\alpha_i(\alpha_i \cdot \alpha_0)}{(\beta\alpha_0 \cdot g + 1)^3(\beta\alpha_i \cdot g + 1)} + \frac{\alpha_0(\alpha_0 \cdot \alpha_i)}{(\beta\alpha_0 \cdot g + 1)(\beta\alpha_i \cdot g + 1)^3} \right). \end{aligned} \tag{5.2.38}$$

A full asymptotic expansion of ϕ must contain many such cross terms.

5.3 $A_r^{(1)}$ linear problems

5.3.1 Representations of $A_r^{(1)}$

The results of the previous section are valid for the affine Toda field theory associated with any affine Lie algebra $\hat{\mathfrak{g}}$. To make further progress, we consider a particular case of the modified affine Toda field equations, corresponding to a choice of Lie algebra and a representation of that algebra. For the rest of this section, we will be concerned with certain representations of the affine Lie algebra $A_r^{(1)}$. To construct these representations, we begin by explicitly constructing the smallest non-trivial representation of A_r, $L(\omega_1)$. This is the first of the fundamental representations of A_r, with highest weight ω_1. We then define the associated evaluation representations of $A_r^{(1)}$, following section 4.2.2. The representations of interest will then be wedge products of particular evaluation representations, as defined in section 4.2.1.

The representation $L(\omega_1)$ is $(r+1)$-dimensional, and with the conventions in [37], the weights $\lambda_i^{(1)}$ satisfy

$$\lambda_1^{(1)} = \omega_1, \tag{5.3.1}$$

$$\lambda_{i+1}^{(1)} = \lambda_i^{(1)} - \alpha_i, \quad (i = 1, \ldots, r). \tag{5.3.2}$$

We choose an orthonormal basis $\{\mathbf{e}_j^{(1)}\}_{j=0}^r$ for the vector space associated with $L(\omega_1)$. The generators of the Lie algebra act on this vector space in the following way:

$$E_{\alpha_i}\mathbf{e}_j^{(1)} = \delta_{i,j}\mathbf{e}_{j-1}^{(1)}, \tag{5.3.3}$$

$$E_{-\alpha_i}\mathbf{e}_j^{(1)} = \delta_{i-1,j}\mathbf{e}_{j+1}^{(1)}, \tag{5.3.4}$$

$$H_i\mathbf{e}_j^{(1)} = [E_{\alpha_i}, E_{-\alpha_i}]\mathbf{e}_j^{(1)} = \delta_{i,j+1}\mathbf{e}_j^{(1)} - \delta_{i,j}\mathbf{e}_j^{(1)}, \tag{5.3.5}$$

$$E_{\alpha_0}\mathbf{e}_j^{(1)} = \delta_{0,j}\mathbf{e}_r^{(1)}, \tag{5.3.6}$$

where $i = 1, \ldots, r$. The weight spaces of $L(\omega_1)$ are one-dimensional, and for each weight $\lambda_{j+1}^{(1)}$ (where $j = 0, \ldots, r$) the weight space is spanned by the vector $\mathbf{e}_j^{(1)}$, with the normalisations of these vectors fixed by the commutation relations (5.3.3)-(5.3.6).

The representation $L(\omega_1)$ naturally extends to an evaluation representation of $A_r^{(1)}$, as detailed in section 4.2.2. As in that section, denote $L(\omega_1)_k$ to be the evaluation representation with $\zeta = e^{2\pi ik}$. We will only be concerned with the cases where k is an integer or a half-integer. In the integer case, the evaluation representation is equivalent to the original representation of A_r; in the half-integer case, the evaluation representation has the effect of changing the sign of the generator $E_{\alpha_0} \to -E_{\alpha_0}$.

As described in 4.2.1, we can construct larger representations of $A_r^{(1)}$ by taking the wedge product of several copies of $L(\omega_1)_k$ (for various values of k.) The r representations with which we shall be concerned with are then given by

$$V^{(a)} = \bigwedge_{i=1}^{a} L(\omega_1)_{i-\frac{a+1}{2}}. \tag{5.3.7}$$

For each of these representations we may write the gauge-transformed linear problem

$$(\partial_z + \tilde{A})\tilde{\Psi} = 0, \tag{5.3.8}$$

in terms of the matrices H_i, E_α. We recall $\widetilde{A}$ is given by

$$\widetilde{A} = \beta\partial_z\phi \cdot H + me^\theta \left[\sum_{i=1}^{r} E_{\alpha_i} + p(z)E_{\alpha_0}\right]. \tag{5.3.9}$$

(We note for $A_r^{(1)}$, $n_i^\vee = n_i = 1$.) The linear problem (5.3.8) may then be written as a system of differential equations. To demonstrate this, we choose the representation $V^{(1)} = L(\omega_1)$, and then set the vector $\tilde{\Psi} = \tilde{\psi}_1\mathbf{e}_0^{(1)} + \tilde{\psi}_2\mathbf{e}_1^{(1)} + \cdots + \tilde{\psi}_{r+1}\mathbf{e}_r^{(1)}$. We then write the equation (5.3.8) in its components:

$$D(\lambda_1^{(1)})\tilde{\psi}_1 + me^\theta\tilde{\psi}_2 = 0, \tag{5.3.10}$$

$$D(\lambda_2^{(1)})\tilde{\psi}_2 + me^\theta\tilde{\psi}_3 = 0, \tag{5.3.11}$$

$$\vdots \tag{5.3.12}$$

$$D(\lambda_r^{(1)})\tilde{\psi}_r + me^\theta\tilde{\psi}_{r+1} = 0, \tag{5.3.13}$$

$$D(\lambda_{r+1}^{(1)})\tilde{\psi}_{r+1} + me^\theta p(z)\tilde{\psi}_1 = 0, \tag{5.3.14}$$

where the differential operator D is defined as

$$D(\lambda) = \partial_z + \beta\lambda \cdot \partial_z\phi. \tag{5.3.15}$$

Following [21, 37], we combine the equations (5.3.10)-(5.3.14) into a single differential equation for ψ_1. We apply the operators $D(\lambda_2^{(1)})$, $D(\lambda_3^{(1)})$, $\ldots$, $D(\lambda_{r+1}^{(1)})$ (in that order) to both sides of (5.3.10), and use the other equations (5.3.11)-(5.3.14) to simplify the resulting expression. The result is

$$D(\lambda_{r+1}^{(1)})D(\lambda_r^{(1)})\dots D(\lambda_1^{(1)})\tilde{\psi}_1 + (-1)^r(me^\theta)^{r+1}p(z)\widetilde{\psi}_1 = 0. \tag{5.3.16}$$

This differential equation is the generalisation of (3.3.21) to the affine Lie algebra $A_r^{(1)}$. As in section 3.3.3, we recover the massless analogue of (5.3.16) by taking the conformal limit, making the change of variables

$$x = (me^\theta)^{\frac{1}{M+1}}z, \quad E = s^{hM}(me^\theta)^{\frac{hM}{M+1}}, \tag{5.3.17}$$

$$\bar{x} = (me^{-\theta})^{\frac{1}{M+1}}\bar{z}, \quad \bar{E} = s^{hM}(me^{-\theta})^{\frac{hM}{M+1}} \tag{5.3.18}$$

(where $h = r+1$) and send $z, \bar{z} \to 0$, $\theta \to \infty$ so that x and E remain finite. In this limit, the operator $D(\lambda)$ becomes

$$D(\lambda) \to D_x(\lambda) = \left(\frac{E}{s}\right)^{\frac{1}{hM}}\left(\partial_x + \frac{\beta\lambda \cdot g}{x}\right), \tag{5.3.19}$$

and (5.3.16) becomes

$$((-1)^r D(\mathbf{g}) + p(x,E))\,\widetilde{\psi}_1 = 0, \tag{5.3.20}$$

where

$$D_r(\mathbf{g}) = D_x(\lambda_{r+1}^{(1)}) \cdot D_x(\lambda_1^{(1)}), \quad p(x,E) = x^{hM} - E. \tag{5.3.21}$$

The equation (5.3.20) and the properties of its solutions are the main focus of [21]. We could proceed with the analysis of the massive analogue (5.3.16) of this equation, but it is easier to generalise the procedure to more general representations $V^{(a)}$ by considering the associated linear systems (5.2.9)-(5.2.10) directly, using the techniques we developed in chapter 4.3.

5.3.2 $V^{(1)}$ linear problem asymptotics

We will be interested in solutions of the original linear problem (5.2.9) with particular small-$|z|$ and large-$|z|$ asymptotics. To this end, we will now consider the more straightforward gauge-transformed system of equations (5.3.8) in the representation $V^{(1)}$. We briefly classify the solutions of this system of equations in the small-$|z|$ and large-$|z|$ limits, and then undo the gauge transform (5.2.12) by applying the matrix U^{-1} to $\tilde{\Psi}$, where $U = e^{-\beta\phi\cdot H/2}$.

Small-$|z|$ asymptotics

In the representation $V^{(1)}$ the matrix $\tilde{A}$ is given by

$$\tilde{A} = \begin{pmatrix} \beta\lambda_1^{(1)}\cdot\partial_z\phi & me^{\theta} & 0 & \cdots & 0 & 0 \\ 0 & \beta\lambda_2^{(1)}\cdot\partial_z\phi & me^{\theta} & \cdots & 0 & 0 \\ 0 & 0 & \beta\lambda_3^{(1)}\cdot\partial_z\phi & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta\lambda_r^{(1)}\cdot\partial_z\phi & me^{\theta} \\ me^{\theta}p(z) & 0 & 0 & \cdots & 0 & \beta\lambda_{r+1}^{(1)}\cdot\partial_z\phi \end{pmatrix}. \tag{5.3.22}$$

In the limit $|z| \to 0$, we recall from (5.2.35) the leading order behaviour of ϕ in the small-$|z|$ limit $\phi \sim g\log z\bar{z}$. We consider $\tilde{A}$ in the small-$|z|$ limit, substituting the small-$|z|$ behaviour of ϕ. The terms proportional to me^{θ} become irrelevant in this limit and so the matrix $\tilde{A}$ becomes diagonal:

$$\tilde{A} \sim \begin{pmatrix} \frac{\beta\lambda_1^{(1)}\cdot g}{z} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\beta\lambda_2^{(1)}\cdot g}{z} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\beta\lambda_r^{(1)}\cdot g}{z} & 0 \\ 0 & 0 & \cdots & 0 & \frac{\beta\lambda_{r+1}^{(1)}\cdot g}{z} \end{pmatrix} \tag{5.3.23}$$

The system of equations $(\partial_z + \tilde{A})\tilde{\Psi} = 0$ then decouples in the small-$|z|$ limit and is easily solved component-by-component:

$$\partial_z \tilde{\psi}_i + \frac{\beta\lambda_i^{(1)} \cdot g}{z}\tilde{\psi}_i = 0 \tag{5.3.24}$$

$$\implies \tilde{\psi}_i = c_i z^{-\beta\lambda_i^{(1)}\cdot g}, \tag{5.3.25}$$

where c_{i+1} are arbitrary constants. Choosing a standard orthonomal basis $\{\mathbf{e}_i^{(1)}\}_{i=0}^r$ for the space $V^{(1)}$ we then have a basis of solutions $\{\tilde{\Xi}_i\}_{i=0}^r$ to the system of equations $(\partial_z + \tilde{A})\tilde{\Psi} = 0$:

$$\tilde{\Xi}_i \sim \mathbf{e}_i^{(1)} c_{i+1} z^{-\beta\lambda_{i+1}^{(1)}\cdot g} \quad \text{as } |z| \to 0. \tag{5.3.26}$$

We now apply the matrix $U^{-1} = e^{\beta\phi\cdot H/2}$ to these solutions in the small-z limit.

$$\begin{aligned}
\Xi_i &= U^{-1}\widetilde{\Xi}_i && (5.3.27)\\
&= e^{\beta\phi\cdot H/2}\mathbf{e}_i^{(1)}c_{i+1}z^{-\beta\lambda_{i+1}^{(1)}\cdot g} && (5.3.28)\\
&= e^{\beta\phi\cdot\lambda_{i+1}^{(1)}/2}\mathbf{e}_i^{(1)}c_{i+1}z^{-\beta\lambda_{i+1}^{(1)}\cdot g} && (5.3.29)\\
&\sim (z\bar{z})^{\beta\lambda_{i+1}^{(1)}\cdot g/2}\mathbf{e}_i^{(1)}c_{i+1}z^{-\beta\lambda_{i+1}^{(1)}\cdot g} && (5.3.30)\\
&\sim \left(\frac{\bar{z}}{z}\right)^{\beta\lambda_{i+1}^{(1)}\cdot g/2}\mathbf{e}_i^{(1)}c_{i+1} && (5.3.31)\\
&\sim e^{-i\beta\lambda_{i+1}^{(1)}\cdot g\varphi}\mathbf{e}_i^{(1)}c_{i+1} && (5.3.32)\\
&\sim e^{-(\theta+i\varphi)\beta\lambda_{i+1}^{(1)}\cdot g}\mathbf{e}_i^{(1)}, \quad \text{as } |z|\to 0. && (5.3.33)
\end{aligned}$$

where we use polar co-ordinates $z = |z|e^{i\varphi}$ and choose the constants $c_{i+1} = e^{-\theta\beta\lambda_{i+1}^{(1)}\cdot g}$. This is to ensure the solutions Ξ_i are invariant under Symanzik rotation (5.2.11).

Large-$|z|$ asymptotics

We now consider the linear problem (5.3.8) in the limit $|z|\to\infty$. We recall the large-$|z|$ behaviour of ϕ, which we imposed in equation (5.2.16):

$$\phi \sim \frac{M\rho^\vee}{\beta}\log z\bar{z} + o(1) \quad \text{as } |z|\to\infty. \qquad (5.3.34)$$

From this, it is easy to see $\partial_z \phi \sim O(1/z)$ and is therefore dominated in the large-$|z|$ limit by the me^θ and $me^\theta p(z)$. The matrix $\tilde{A}$ then becomes

$$\tilde{A} \sim \begin{pmatrix} 0 & me^\theta & 0 & \cdots & 0 & 0 \\ 0 & 0 & me^\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & me^\theta \\ me^\theta p(z) & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{5.3.35}$$

We then apply a generalisation of the WKB approximation [13] to systems of differential equations, adapted from [49]. Full details are given in section 4.3.2; here we will state the results of applying the generalised WKB approximation to the system $(\partial_z + \tilde{A})\tilde{\Psi} = 0$.

In the large-$|z|$ limit, there are $r + 1 = h$ linearly independent solutions of the system $(\partial_z + \tilde{A})\tilde{\Psi} = 0$, with asymptotic behaviour

$$\left(\sum_{j=0}^{r} \omega^j z^{M(j-r/2)} \mathbf{e}_j^{(1)} \right) \exp\left(-\omega \frac{me^\theta z^{M+1}}{M+1} \right), \tag{5.3.36}$$

where $\omega^{r+1} = 1$. A particularly important member of this set of solutions is the solution that decays to zero most rapidly on the positive real axis: this is the solution (5.3.36) with $\omega = 1$. Denote this subdominant solution $\tilde{\Psi}^{(1)}$. We then undo the effect of the gauge transformation (5.2.12) by multiplying $\tilde{\Psi}^{(1)}$ by $U^{-1} = e^{\beta\phi\cdot H/2}$. Using the large-$|z|$ behaviour of ϕ (5.2.16) the matrix U^{-1} in the large-$|z|$ limit is given by

$$U^{-1} \sim (z\bar{z})^{M\rho^\vee \cdot H/2}. \tag{5.3.37}$$

We work in an orthonormal basis $\{\mathbf{e}_i^{(1)}\}_{i=0}^r$ of $V^{(1)}$ with $H_{i+1}\mathbf{e}_i^{(1)} = (\lambda_{i+1}^{(1)})_i \, \mathbf{e}^{(1)}$.

The effect of U^{-1} on this basis is then

$$U^{-1}\mathbf{e}_i^{(1)} = (z\bar{z})^{M\rho^\vee\cdot\lambda_{i+1}^{(1)}/2}\mathbf{e}_i^{(1)}. \tag{5.3.38}$$

To progress, we must compute the dot products of the co-Weyl vector $\rho^\vee$ with the weights $\lambda_{i+1}^{(1)}$. From the definition of the co-Weyl vector for A_r given in section 4.2.2 and the definitions of the weights $\lambda_{i+1}^{(1)}$ of the representation $V^{(1)}$ (5.3.1)-(5.3.2) it is straightforward to compute

$$\rho^\vee \cdot \lambda_{i+1}^{(1)} = \frac{r}{2} - i. \tag{5.3.39}$$

The elements of the matrix U^{-1} in the basis $\{\mathbf{e}_i^{(1)}\}_{i=0}^r$ are then given by

$$U_{ij}^{-1} \sim \left((z\bar{z})^{M\rho^\vee\cdot H/2}\right)_{ij} = (z\bar{z})^{\frac{M}{2}(\frac{r}{2}-j)}\delta_{ij}, \quad i,j = 0,\ldots,r. \tag{5.3.40}$$

We now apply this matrix to the subdominant solution of the gauge-transformed linear problem $\tilde{\Psi}^{(1)}$:

$$U^{-1}\tilde{\Psi}^{(1)} \sim \left(\sum_{j=0}^{r}(z/\bar{z})^{(2j-r)M/4}\mathbf{e}_j^{(1)}\right)\exp\left(-\frac{me^\theta z^{M+1}}{M+1}\right). \tag{5.3.41}$$

The large-$|z|$ asymptotics of the conjugate linear problem (5.2.10) can also be analysed in a similar manner to the above. To ensure compatibility with this calculation (5.2.10) we premultiply $U^{-1}\tilde{\Psi}^{(1)}$ by a suitable $\bar{z}$-dependent function (at this stage treating z and $\bar{z}$ as independent complex co-ordinates as discussed at the beginning of section 3.2):

$$
\begin{aligned}
\Psi^{(1)} &= \exp\left(-\frac{me^{\theta}}{M+1}\bar{z}^{M+1}\right) U^{-1}\tilde{\Psi}^{(1)}, \\
\implies \Psi^{(1)} &\sim \left(\sum_{j=0}^{r} e^{iM\varphi(j-r/2)}\mathbf{e}_j^{(1)}\right) \exp\left(-\frac{2|z|^{M+1}}{M+1} m\cosh(\theta + i\varphi(M+1))\right),
\end{aligned}
\tag{5.3.42}
$$

where we have subsequently restricted $\Psi^{(1)}$ to the subset of $\mathbb{C}^2$ where $\bar{z} = z^*$ by choosing polar co-ordinates $z = |z|e^{i\varphi}$, $\bar{z} = |z|e^{-i\varphi}$.

5.3.3 $V^{(a)}$ linear problem asymptotics

We now construct the asymptotic solutions to the $V^{(a)}$ linear problem from the asymptotic solutions to the $V^{(1)}$ linear problem. We recall that the representation $V^{(a)}$ of $A_r^{(1)}$ is constructed from wedge products of copies of the representation $V^{(1)}$:

$$
V^{(a)} = \bigwedge_{i=1}^{a} V^{(1)}_{i-\frac{a+1}{2}}, \quad a = 1, \ldots, r. \tag{5.3.43}
$$

The representations $V_k^{(1)}$ correspond to different evaluation representations of $A_r^{(1)}$. These are related to $V_0^{(1)}$ by applying a Symanzik rotation (5.2.11) Ω_k to the relevant linear problem. The solutions of this new linear problem are then Symanzik rotations of the original solutions. Once these are computed, finding the asymptotics of the solutions of the $V^{(a)}$ is then reduced to taking the wedge product of the correct Symanzik rotated $V^{(1)}$ solutions.

Small-$|z|$ asymptotics

We recall the small-$|z|$ solutions for the $V^{(1)}$ linear problem:

$$\Xi_i^{(1)} \sim e^{-(\theta+i\varphi)\beta\lambda_{i+1}^{(1)}\cdot g}\mathbf{e}_i^{(1)}, \quad (i = 0, \ldots, r). \tag{5.3.44}$$

These solutions were normalised to be invariant under any Symanzik rotation. The calculation of the small-$|z|$ asymptotics of the $V^{(a)}$ linear problem is then straightforward; we simply take the wedge product of a copies of (5.3.44):

$$\begin{aligned} \Xi_{i_1 i_2 \ldots i_a}^{(a)} &= \Xi_{i_1}^{(1)} \wedge \Xi_{i_2}^{(1)} \wedge \cdots \wedge \Xi_{i_a}^{(1)} \qquad (5.3.45)\\ &\sim \exp\left(-(\theta+i\varphi)\beta(\lambda_{i_1+1}^{(1)} + \lambda_{i_2+1}^{(1)} + \cdots + \lambda_{i_a+1}^{(1)})\cdot g\right)\mathbf{e}_{i_1}^{(1)} \wedge \mathbf{e}_{i_2}^{(1)} \wedge \cdots \wedge \mathbf{e}_{i_a}^{(1)}, \\ &= \exp\left(-(\theta+i\varphi)\beta\lambda_{I+1}^{(a)}\cdot g\right)\mathbf{e}_I^{(a)}, \qquad (5.3.46) \end{aligned}$$

where in the last line we represent the ordered subset $\{0 \leq i_1 < i_2 < \cdots < i_a \leq r\}$ of $\{0, 1, \ldots, r\}$ by the integer I, where $I = 0, 1, \ldots, \binom{r+1}{a} - 1$. This integer is chosen using the standard lexicographic ordering of subsets:

$$\begin{aligned} \{0, 1, \ldots, a-2, a-1\} &\to I = 0, \\ \{0, 1, \ldots, a-2, a\} &\to I = 1, \\ &\vdots \\ \{r-a+1, \ldots, r\} &\to I = \binom{r+1}{a} - 1. \end{aligned} \tag{5.3.47}$$

We also have chosen a basis for $V^{(a)}$, denoted with the same ordering:

$$\mathbf{e}_{i_1}^{(1)} \wedge \mathbf{e}_{i_2}^{(1)} \wedge \cdots \wedge \mathbf{e}_{i_a}^{(1)} = \mathbf{e}_I^{(a)}. \tag{5.3.48}$$

The weights of the representation $V^{(a)}$ are also labelled the same way:

$$\lambda_{I+1}^{(a)} = \lambda_{i_1+1}^{(1)} + \lambda_{i_2+1}^{(1)} + \cdots + \lambda_{i_a+1}^{(1)}. \tag{5.3.49}$$

As with the small-$|z|$ solutions to the $V^{(1)}$ linear problem, the small-$|z|$ solutions (5.3.45) to the $V^{(a)}$ linear problem form a basis of the vector space $V^{(a)}$.

Large-$|z|$ asymptotics

As with the small-$|z|$ solutions, we begin by recalling the large-$|z|$ solutions for the $V^{(1)}$ linear problem:

$$\Psi^{(1)} \sim \left(\sum_{j=0}^{r} e^{iM\varphi(j-r/2)}\mathbf{e}_j^{(1)}\right) \exp\left(-\frac{2|z|^{M+1}}{M+1} m \cosh(\theta + i\varphi(M+1))\right) \tag{5.3.50}$$

It is easier, however, to work with the gauge-transformed solution

$$\tilde{\Psi}^{(1)} = U\Psi \sim \left(\sum_{j=0}^{r} z^{M(j-r/2)}\mathbf{e}_j^{(1)}\right) \exp\left(-\frac{me^{\theta}z^{M+1}}{M+1}\right). \tag{5.3.51}$$

Working with this gauge-transformed solution does not affect any of the forthcoming calculations as the matrix U in the large-$|z|$ limit is invariant under Symanzik rotation (5.2.11):

$$U \sim (z\bar{z})^{M\rho^\vee \cdot H/2} \to (ze^{\frac{2\pi ik}{hM}}\bar{z}e^{-\frac{2\pi ik}{hM}})^{M\rho^\vee \cdot H/2} = (z\bar{z})^{M\rho^\vee \cdot H/2}. \tag{5.3.52}$$

As U is diagonal in the representation $V^{(1)}$, U^{-1} remains diagonal in the new representations $V^{(a)}$. At the end of our calculation, just as with the earlier defined $V^{(1)}$ case, we then multiply our result by U^{-1} to find the subdominant asymptotic solution $\Psi^{(a)}$ to the $V^{(a)}$ linear problem in the large-$|z|$ limit.

The wedge product definition of the representations $V^{(a)}$ implies the subdominant solution $\tilde{\Psi}^{(a)}$ is the wedge product of certain Symanzik rotated solutions of the $V^{(1)}$ linear problem:

$$\tilde{\Psi}^{(a)} = \bigwedge_{i=1}^{a} \tilde{\Psi}_{i-\frac{1+a}{2}}. \tag{5.3.53}$$

The asymptotics of $\tilde{\Psi}^{(1)}$ after a Symanzik rotation are given by

$$\tilde{\Psi}^{(1)}_k = \Omega_k \tilde{\Psi}^{(1)} \sim \left(\sum_{j=0}^{r} e^{\frac{2\pi i j k}{h}} z^{M(j-r/2)} \mathbf{e}^{(1)}_j \right) \exp\left(-\frac{m e^{\frac{2\pi i k}{h}} e^{\theta} z^{M+1}}{M+1} \right). \tag{5.3.54}$$

We substitute this expression into (5.3.53):

$$\begin{aligned}
\tilde{\Psi}^{(a)} &= \tilde{\Psi}_{\frac{1-a}{2}} \wedge \tilde{\Psi}_{\frac{3-a}{2}} \wedge \cdots \wedge \tilde{\Psi}_{\frac{a-1}{2}} \\
&\sim \exp\left(-(\omega^{\frac{1-a}{2}} + \cdots + \omega^{\frac{a-1}{2}}) \frac{m e^{\theta} z^{M+1}}{M+1} \right) \cdot \left(\sum_{j_1=0}^{r} \omega^{\left(\frac{1-a}{2}\right) j_1} z^{M(j_1 - r/2)} \mathbf{e}^{(1)}_{j_1} \right) \\
&\wedge \left(\sum_{j_2=0}^{r} \omega^{\left(\frac{3-a}{2}\right) j_2} z^{M(j_2 - r/2)} \mathbf{e}^{(1)}_{j_2} \right) \wedge \cdots \wedge \left(\sum_{j_a=0}^{r} \omega^{\left(\frac{a-1}{2}\right) j_a} z^{M(j_a - r/2)} \mathbf{e}^{(1)}_{j_a} \right),
\end{aligned} \tag{5.3.55}$$

where $\omega = e^{2\pi i/h}$. The prefactor contains a geometric series which we simplify

$$\begin{aligned}
\omega^{\frac{1-a}{2}} + \cdots + \omega^{\frac{a-1}{2}} &= \omega^{\frac{1-a}{2}} \left(1 + \omega + \omega^2 + \cdots + \omega^{a-1} \right), \\
&= \frac{\omega^{\frac{1-a}{2}} (1 - \omega^a)}{1 - \omega}, \\
&= \frac{\sin \frac{\pi a}{h}}{\sin \frac{\pi}{h}}.
\end{aligned} \tag{5.3.56}$$

The large-$|z|$ asymptotics of the gauge-transformed solution $\tilde{\Psi}^{(a)}$ are then given by

$$\tilde{\Psi}^{(a)} \sim \exp\left(-\frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}}\frac{me^{\theta}z^{M+1}}{M+1}\right)$$
$$\cdot\left(\sum_{j_1,j_2,\dots,j_a=0}^{r}\omega^{\frac{1-a}{2}j_1+\dots+\frac{a-1}{2}j_a}z^{M(j_1+\dots+j_a-ar/2)}\mathbf{e}_{j_1}^{(1)}\wedge\dots\wedge\mathbf{e}_{j_a}^{(1)}\right). \tag{5.3.57}$$

As discussed previously, we finally undo the gauge transformation by acting with U^{-1} on this solution. As with the $V^{(1)}$ case we premultiply by a $\bar{z}$-dependent factor to ensure compatibility with the conjugate linear problem (5.2.10):

$$\Psi^{(a)} \sim \exp\left(-\frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}}\cdot\frac{2|z|^{M+1}}{M+1}m\cosh(\theta+i\varphi(M+1))\right)$$
$$\cdot\left(\sum_{j_1,j_2,\dots,j_a=0}^{r}\omega^{\frac{1-a}{2}j_1+\dots+\frac{a-1}{2}j_a}e^{i\varphi M(j_1+\dots+j_a-ar/2)}\mathbf{e}_{j_1}^{(1)}\wedge\dots\wedge\mathbf{e}_{j_a}^{(1)}\right). \tag{5.3.58}$$

As we have seen, the small-$|z|$ solutions for $V^{(a)}$, $\Xi_J^{(a)}$ (where we relabel the solutions with a new lexicographical index J which runs from 0 to $\dim V^{(a)}-1$) form a basis of the solution space in the same way as for $V^{(1)}$. The suitably defined subdominant large-$|z|$ solution $\Psi^{(a)}$ may then be written in this basis:

$$\Psi^{(a)} = \sum_{J=0}^{\dim V^{(a)}-1} Q_J^{(a)}(\theta)\Xi_J^{(a)}. \tag{5.3.59}$$

These Q-functions will be the main objects of study for the rest of this chapter. Information about the quantum integrable associated with the affine Toda field theory associated with the Lie algebra $A_r^{(1)}$ symmetry is encoded in the various properties of these Q-functions.

5.4 Q-functions

Just as for the $A_1^{(1)}$ case in section 3.4, we now demonstrate some useful properties of the $A_r^{(1)}$ Q-functions. The Q-functions satisfy a quasiperiodicity property and particular determinants of Q-functions are constant (the quantum Wronskian). We also give an expression for the asymptotics for particular Q-functions, following [21, 45].

5.4.1 Quasiperiodicity

For a given representation $V^{(a)}$, we define the operator $S = e^{\frac{2\pi i}{h}\rho^\vee \cdot H}$, which is a diagonal operator with respect to a basis of weight vectors

$$S = e^{\frac{2\pi i}{h}\rho^\vee \cdot H} = \mathrm{diag}\left(e^{\frac{2\pi i}{h}\rho^\vee \cdot \lambda_1^{(a)}}, e^{\frac{2\pi i}{h}\rho^\vee \cdot \lambda_2^{(a)}}, \ldots, e^{\frac{2\pi i}{h}\rho^\vee \cdot \lambda^{(a)}_{\dim V^{(a)}}} \right). \tag{5.4.1}$$

We will prove the following identities:

$$S\Xi_J^{(a)}\left(\varphi + \frac{2\pi}{hM}\middle|\, \theta - \frac{2\pi i}{hM} - \frac{2\pi i}{h}\right) = \exp\left(\frac{2\pi i}{h}(\rho^\vee + \beta g)\cdot \lambda_{J+1}^{(a)}\right)\Xi_J^{(a)}\left(\varphi |\, \theta\right), \tag{5.4.2}$$

$$S\Psi^{(a)}\left(\varphi + \frac{2\pi}{hM}\middle|\, \theta - \frac{2\pi i}{hM} - \frac{2\pi i}{h}\right) = \Psi^{(a)}\left(\varphi |\, \theta\right). \tag{5.4.3}$$

Proof of (5.4.2)

We begin by recalling the invariance of $\Xi^{(a)}$ under a Symanzik rotation:

$$\Xi_J^{(a)}\left(\varphi + \frac{2\pi}{hM}\middle|\, \theta - \frac{2\pi i}{hM} - \frac{2\pi i}{h}\right) = \Xi_J^{(a)}\left(\varphi |\, \theta - \frac{2\pi i}{h}\right). \tag{5.4.4}$$

Applying the matrix S to the right hand side of this expression and using the asymptotic expansion of $\Xi_J^{(a)}$ (5.3.45):

$$S\Xi_J^{(a)}(\varphi|\theta-\frac{2\pi i}{h}) = e^{\frac{2\pi i}{h}\rho^\vee\cdot H}\Xi_J^{(a)}(\varphi|\theta-\frac{2\pi i}{h}), \tag{5.4.5}$$

$$\sim e^{\frac{2\pi i}{h}\rho^\vee\cdot\lambda_{J+1}^{(a)}}e^{-(\theta-\frac{2\pi i}{h}+i\varphi)\beta\lambda_{J+1}^{(a)}\cdot g}\mathbf{e}_J^{(a)}, \tag{5.4.6}$$

$$\sim e^{\frac{2\pi i}{h}(\rho^\vee+\beta g)\cdot\lambda_{J+1}^{(a)}}e^{-(\theta+i\varphi)\beta\lambda_{J+1}^{(a)}\cdot g}\mathbf{e}_J^{(a)}, \tag{5.4.7}$$

$$\sim e^{\frac{2\pi i}{h}(\rho^\vee+\beta g)\cdot\lambda_{J+1}^{(a)}}\Xi_J^{(a)}(\varphi|\theta), \tag{5.4.8}$$

as required.

Proof of (5.4.3)

We apply S to the vector part of the twisted large-$|z|$ solution $\Psi^{(a)}$ (5.3.58), noting that the exponential prefactor is invariant under S:

$$e^{\frac{2\pi i}{h}\rho^\vee\cdot H}\Psi^{(a)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right) \tag{5.4.9}$$

$$= \sum_{j_1,j_2,\dots,j_a=0}^{r}\Big(\omega^{\frac{1-a}{2}j_1+\dots+\frac{a-1}{2}j_a}e^{i\varphi M(j_1+\dots+j_a-ar/2)}$$

$$\cdot e^{\frac{2\pi i}{h}(j_1+\dots+j_a-ar/2)+\rho^\vee\cdot\lambda_{j_1j_2\cdots j_a}^{(a)}}\Big)\,\mathbf{e}_J^{(a)}$$

To proceed, we must evaluate expressions of the form $\rho^\vee\cdot\lambda_{j_1\cdots j_a}^{(a)}$; the dot products of the co-Weyl vector $\rho^\vee$ with the weights of the representation $V^{(a)}$. We recall the weights of the representation $V^{(a)}$ are given by sums of a distinct weights of $V^{(1)}$:

$$\lambda_{J+1}^{(a)} = \lambda_{j_1+1}^{(1)}+\lambda_{j_2+1}^{(1)}+\dots+\lambda_{j_a+1}^{(1)}, \quad (0\le j_k\le r,\ J=0,\dots,\dim V^{(a)}-1). \tag{5.4.10}$$

We also recall the result of taking the dot product of $\rho^\vee$ with the weights of $V^{(1)}$:

$$\rho^\vee \cdot \lambda^{(1)}_{i+1} = \frac{r}{2} - i, \quad (0 \leq i \leq r). \tag{5.4.11}$$

The dot product $\rho^\vee \cdot \lambda^{(a)}_{J+1}$ is then

$$\rho^\vee \cdot \lambda^{(a)}_{J+1} = \frac{ar}{2} - j_1 - \ldots - j_a, \tag{5.4.12}$$

which we substitute into (5.4.9) to find the required result:

$$\begin{aligned}
&S\Psi^{(a)}\left(\varphi + \frac{2\pi}{hM}\middle|\, \theta - \frac{2\pi i}{hM} - \frac{2\pi i}{h}\right) \\
&= \sum_{j_1, j_2, \ldots, j_a = 0}^{r} \left(\omega^{\frac{1-a}{2}j_1 + \cdots + \frac{a-1}{2}j_a} e^{i\varphi M(j_1 + \ldots + j_a - ar/2)}\right) \mathbf{e}^{(1)}_{j_1} \wedge \cdots \wedge \mathbf{e}^{(1)}_{j_a}, \\
&= \Psi^{(a)}(\varphi|\theta).
\end{aligned} \tag{5.4.13}$$

We will also require the main result of section 4.2.1, namely that the sum of the weights $\lambda^{(a)}_J$ of any finite-dimensional representation of a Lie algebra is zero:

$$\sum_{J=1}^{\dim V^{(a)}} \lambda^{(a)}_J = 0. \tag{5.4.14}$$

From the form of S this immediately implies $\det S = 1$. We also invoke a determinant identity for any set of n-dimensional vectors $\{\mathbf{v}_i\}_{i=1}^n$:

$$\det\left(S\mathbf{v}_1, \ldots, S\mathbf{v}_n\right) = \det S \det\left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \tag{5.4.15}$$

where the notation $\det(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ represents the determinant of the matrix with columns $\mathbf{v}_i$.

We now apply all these various results to a proof of a quasiperiodicity property for the functions $Q_j^{(a)}(\theta)$. Recall the definitions of the Q-functions:

$$\Psi^{(a)} = \sum_{K=0}^{\dim V^{(a)}-1} Q_K^{(a)}(\theta)\Xi_K^{(a)}. \tag{5.4.16}$$

We isolate a Q-function $Q_J^{(a)}(\theta)$ by taking determinants of (5.4.16):

$$\begin{aligned}&\det\left(\Xi_0^{(a)},\ldots,\Xi_{J-1}^{(a)},\Psi^{(a)},\Xi_{J+1}^{(a)},\ldots,\Xi_{\dim V^{(a)}-1}^{(a)}\right)\\ &= Q_J^{(a)}(\theta)\det\left(\Xi_0^{(a)},\ldots,\Xi_{\dim V^{(a)}-1}^{(a)}\right) = Q_J^{(a)}(\theta),\end{aligned} \tag{5.4.17}$$

where we use the definition of the small-$|z|$ asymptotics $\Xi^{(a)}$ and (5.4.14) to find $\det\left(\Xi_0^{(a)},\ldots,\Xi_{\dim V^{(a)}-1}^{(a)}\right) = 1$.

We now use this determinant form of $Q_J^{(a)}(\theta)$, the determinant identity (5.4.15) and the identities (5.4.2)-(5.4.3) to derive our desired quasiperiodicity result:

$$\begin{aligned}Q_J^{(a)}(\theta) &= \det\left(\Xi_0^{(a)}(\varphi|\theta),\ldots,\Xi_{J-1}^{(a)}(\varphi|\theta),\Psi^{(a)}(\varphi|\theta),\Xi_{J+1}^{(a)}(\varphi|\theta),\ldots,\Xi_{\dim V^{(a)}-1}^{(a)}(\varphi|\theta)\right)\\
&= \det\left(e^{-\frac{2\pi i}{h}(\beta g+\rho^\vee)\cdot\lambda_1^{(a)}} S\Xi_0^{(a)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right),\ldots,\right.\\
&e^{-\frac{2\pi i}{h}(\beta g+\rho^\vee)\cdot\lambda_J^{(a)}} S\Xi_{J-1}^{(a)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right),\\
&S\Psi^{(a)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right),\\
&e^{-\frac{2\pi i}{h}(\beta g+\rho^\vee)\cdot\lambda_{J+2}^{(a)}} S\Xi_{J+1}^{(a)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right),\ldots,\\
&\left.e^{-\frac{2\pi i}{h}(\beta g+\rho^\vee)\cdot\lambda_{\dim V^{(a)}}^{(a)}} S\Xi_{\dim V^{(a)}-1}^{(a)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right)\right)\\
&= \exp\left(\frac{2\pi i}{h}(\beta g+\rho^\vee)\cdot\lambda_{J+1}^{(a)}\right) Q_J^{(a)}\left(\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right),\end{aligned}$$

where in the last line we have used (5.4.14). A shift in θ gives the desired quasiperiodicity result for the Q-functions

$$Q_J^{(a)}\left(\theta + \frac{2\pi i}{hM} + \frac{2\pi i}{h}\right) = \exp\left(\frac{2\pi i}{h}(\beta g + \rho^\vee)\cdot\lambda_{J+1}^{(a)}\right) Q_J^{(a)}(\theta), \tag{5.4.18}$$

which matches the quasiperiodicity relation in [38]. The quasiperiodicity result for $Q_0^{(a)}(\theta)$ will be particularly important for the remainder of the chapter

$$Q_0^{(a)}\left(\theta + \frac{2\pi i}{hM} + \frac{2\pi i}{h}\right) = \exp\left(-\frac{2\pi i}{h}\gamma_a\right) Q_0^{(a)}(\theta), \tag{5.4.19}$$

where

$$\gamma_a = -(\rho^\vee + \beta g)\cdot\lambda_1^{(a)}. \tag{5.4.20}$$

5.4.2 Asymptotics of $Q_0^{(a)}(\theta)$ as $\mathrm{Re}\,\theta \to \pm\infty$

We next require an asymptotic expansion for $Q_0^{(a)}(\theta)$ in the limits $\mathrm{Re}\,\theta \to \pm\infty$. We begin with the determinant definition of $Q_0^{(a)}(\theta)$ which follows from (5.4.16) and (5.4.15)

$$Q_0^{(a)}(\theta) = \det\left(\widetilde{\Psi}^{(a)}, \widetilde{\Xi}_1^{(a)}, \ldots, \widetilde{\Xi}_{\dim V^{(a)}-1}^{(a)}\right). \tag{5.4.21}$$

We consider the solution $\widetilde{\Psi}^{(a)}$ to the gauge-transformed linear problem (5.3.8) in the limit $\mathrm{Re}\,\theta \to \infty$. Using the WKB approximation discussed in chapter 4.3, we choose a solution of the gauge-transformed linear problem in the $\mathrm{Re}\,\theta \to +\infty$

limit

$$\Psi^{(a)} \sim \mathbf{v}_0^{(a)}(p(z)) \tag{5.4.22}$$
$$\cdot \exp\left(-\frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}} me^\theta \int_\infty^z \left\{(t^{hM} - s^{hM})^{1/h} - t^M\right\} \mathrm{d}t - \frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}} \frac{me^\theta z^{M+1}}{M+1}\right),$$

where $\mathbf{v}_0^{(a)}(p(z))$ is the eigenvector of $\widetilde{A}$ with eigenvalue $me^\theta p(z)^{1/h}$, with the expansion in the basis $\{\mathbf{e}_{j_1}^{(1)} \wedge \cdots \wedge \mathbf{e}_{j_a}^{(1)}\}$ of $V^{(a)}$

$$\mathbf{v}_0^{(a)}(p(z)) = \sum_{j_1, j_2, \ldots, j_a = 0}^{r} \omega^{\frac{1-a}{2} j_1 + \cdots + \frac{a-1}{2} j_a} p(z)^{(j_1 + \cdots + j_a - ar/2)/h} \mathbf{e}_{j_1}^{(1)} \wedge \ldots \mathbf{e}_{j_a}^{(1)}. \tag{5.4.23}$$

The large-θ solution is chosen to match the required subdominant behaviour for $\Psi^{(a)}$ in the large-$|z|$ limit

$$\Psi^{(a)} \sim \mathbf{v}_0^{(a)}(z^{hM}) \exp\left(-\frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}} \frac{me^\theta z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty. \tag{5.4.24}$$

In the small-$|z|$ limit, $\Psi^{(a)}$ must be a linear combination of the small-$|z|$ solutions $\widetilde{\Xi}_J^{(a)}$ of (5.3.8). The solutions $\widetilde{\Xi}_J^{(a)}$ have the asymptotic behaviour

$$\widetilde{\Xi}_J^{(a)} \sim z^{-\beta \lambda_J^{(a)} \cdot g} \mathbf{e}_J^{(a)} \tag{5.4.25}$$

which is independent of θ. This implies that the coefficients c_J of $\mathbf{v}_0^{(a)}(p(z))$ in the small-$|z|$ basis of solutions $\widetilde{\Xi}_J^{(a)}$

$$\mathbf{v}_0^{(a)}(p(z)) \sim \sum_{J=0}^{\dim V^{(a)} - 1} c_J \widetilde{\Xi}_J^{(a)}, \tag{5.4.26}$$

are independent of θ. Combining (5.4.26) with (5.4.21) gives an asymptotic expression for $Q_0^{(a)}(\theta)$ in the limit $\operatorname{Re}\theta \to +\infty$

$$Q_0^{(a)}(\theta) \sim c_0^{(a)} \exp\left(me^\theta \int_0^\infty \left\{(t^{hM} - s^{hM})^{1/h} - t^M\right\} \mathrm{d}t\right). \tag{5.4.27}$$

The integral in (5.4.27) is evaluated using (3.4.26),

$$\int_0^\infty \left\{(t^{hM} - s^{hM})^{1/h} - t^M\right\} \, \mathrm{d}t = (-1)^{\frac{M+1}{hM}} s^{M+1} \tau(h, M). \tag{5.4.28}$$

$Q_0^{(a)}(\theta)$ then has the asymptotic expression

$$Q_0^{(a)}(\theta) \sim c_0^{(a)} \exp\left(s^{M+1} m e^{\theta} (-1)^{\frac{M+1}{hM}} \tau(h, M)\right), \quad \text{as } \operatorname{Re}\theta \to +\infty. \tag{5.4.29}$$

In order for this expression to be compatible with the quasiperiodicity relation (5.4.19), we first define the strips $H_\pm$ in the complex plane

$$H_+ : 0 < \operatorname{Im}\theta < \frac{2\pi(M+1)}{hM}, \quad H_- : -\frac{2\pi(M+1)}{hM} < \operatorname{Im}\theta < 0. \tag{5.4.30}$$

and rescale the constant c_0 appropriately

$$Q_0^{(a)}(\theta) \sim c_0^{(a)} e^{\mp i\pi\gamma_a/h} \exp\left(s^{M+1} m e^{\theta \mp \frac{i\pi(M+1)}{hM}} \tau(h, M)\right), \quad \theta \in H_\pm. \tag{5.4.31}$$

The $\operatorname{Re}\theta \to -\infty$ limit of $Q_0^{(a)}(\theta)$ is recovered by considering the gauge-transformed conjugate linear problem $(\partial_{\bar{z}} + \widetilde{\bar{A}})\Psi = 0$ in the $\operatorname{Re}\theta \to -\infty$ limit. A solution in that limit is constructed to match the required large-$|z|$ behaviour

$$\begin{aligned}\Psi^{(a)} \sim{}& \mathbf{v}_0^{(a)}(p(\bar{z})) \\ &\cdot \exp\left(-\frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}} m e^{-\theta} \int_\infty^{\bar{z}} \left\{(t^{hM} - s^{hM})^{1/h} - t^M\right\} \, \mathrm{d}t - \frac{\sin\frac{\pi a}{h}}{\sin\frac{\pi}{h}} \frac{m e^{-\theta} \bar{z}^{M+1}}{M+1}\right),\end{aligned} \tag{5.4.32}$$

where $\mathbf{v}_0^{(a)}(p(\bar{z}))$ is the eigenvector of $\widetilde{\bar{A}}$ with eigenvalue $m e^{-\theta} p(\bar{z})^{1/h}$. $\mathbf{v}_0^{(a)}(p(\bar{z}))$ is then expanded as a sum of small-$|z|$ solutions, as in (5.4.26), and then the definition of $Q_0^{(a)}(\theta)$ (5.4.21) is used to determine the asymptotics of $Q_0^{(a)}(\theta)$ in the limit $\mathrm{Re}\,\theta \to -\infty$:

$$Q_0^{(a)}(\theta) \sim c_0^{(a)} e^{\mp i\pi\gamma_a/h} \exp\left(s^{M+1} m e^{-\theta \pm \frac{i\pi(M+1)}{hM}} \tau(h,M)\right), \quad \theta \in H_{\pm}. \tag{5.4.33}$$

5.4.3 The quantum Wronskian

The Q-functions associated with the linear problem on the representation $V^{(1)}$ satisfy a particular determinant relation, known as a quantum Wronskian. The Q-functions for this linear problem are defined by the expansion of the large-$|z|$ solution $\tilde{\Psi}^{(1)}$ in the basis of small-$|z|$ solutions $\tilde{\Xi}_j$:

$$\tilde{\Psi}^{(1)} = \sum_{j=0}^{r} Q_j^{(1)}(\theta)\tilde{\Xi}_j^{(1)}. \tag{5.4.34}$$

We apply a Symanzik rotation Ω_k to both sides of this expression, recalling that the solutions $\tilde{\Xi}_j^{(1)}$ are invariant under Ω_k:

$$\Omega_k \tilde{\Psi}^{(1)} = \sum_{j=0}^{r} Q_j^{(1)}(\theta - \frac{2\pi i k}{hM})\tilde{\Xi}_j^{(1)} \tag{5.4.35}$$

We then take the determinant of $r+1=h$ copies of the expression (5.4.35):

$$\det\left(\tilde{\Psi}^{(1)}, \Omega_1\tilde{\Psi}^{(1)}, \ldots, \Omega_r\tilde{\Psi}^{(1)}\right) \tag{5.4.36}$$

$$= \sum_{j_0,j_1,\ldots,j_r=0}^{r} \det\left(\tilde{\Xi}_{j_0}, \tilde{\Xi}_{j_1}, \ldots, \tilde{\Xi}_{j_r}\right) \prod_{k=0}^{r} Q_{j_k}^{(1)}(\theta - \frac{2\pi ik}{hM}) \tag{5.4.37}$$

$$= \sum_{j_0,j_1,\ldots,j_r=0}^{r} \varepsilon_{j_0 j_1 \ldots j_r} \prod_{k=0}^{r} Q_{j_k}^{(1)}(\theta - \frac{2\pi ik}{hM}) \tag{5.4.38}$$

$$= \begin{vmatrix} Q_0^{(1)}(\theta) & Q_0^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_0^{(1)}(\theta - \frac{2\pi ir}{hM}) \\ Q_1^{(1)}(\theta) & Q_1^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_1^{(1)}(\theta - \frac{2\pi ir}{hM}) \\ \vdots & \vdots & \ddots & \vdots \\ Q_r^{(1)}(\theta) & Q_r^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_r^{(1)}(\theta - \frac{2\pi ir}{hM}) \end{vmatrix}. \tag{5.4.39}$$

As (5.4.39) does not depend on z, we may use the large-$|z|$ asymptotics of the solutions $\Omega_k\tilde{\Psi}^{(1)}$ to compute (5.4.36). From the large-$|z|$ asymptotics (5.3.36) and the definition of the Symanzik rotation (5.2.11) it is straightforward to show

$$\Omega_k\tilde{\Psi}^{(1)} \sim \omega^{-kr/2} \begin{pmatrix} z^{-rM/2} \\ \omega^k z^{M(1-r/2)} \\ \vdots \\ \omega^{k(r-1)} z^{M(r/2-1)} \\ \omega^{kr} z^{rM/2} \end{pmatrix} \exp\left(-\omega^k \frac{me^\theta z^{M+1}}{M+1}\right) \tag{5.4.40}$$

where $\omega = e^{\frac{2\pi i}{h}} = e^{\frac{2\pi i}{r+1}}$. We then substitute (5.4.40) into (5.4.36):

$$\det\left(\tilde{\Psi}^{(1)}, \Omega_1\tilde{\Psi}^{(1)}, \ldots, \Omega_r\tilde{\Psi}^{(1)}\right) \tag{5.4.41}$$
$$= \omega^{-r^2(r+1)/4} \begin{vmatrix} z^{-rM/2} & z^{-rM/2} & \ldots & z^{-rM/2} & z^{-rM/2} \\ z^{M(1-r/2)} & \omega z^{M(1-r/2)} & \ldots & \omega^{r-1} z^{M(1-r/2)} & \omega^r z^{M(1-r/2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z^{M(r/2-1)} & \omega^{r-1} z^{M(r/2-1)} & \ldots & \omega^{(r-1)^2} z^{M(r/2-1)} & \omega^{(r-1)r} z^{M(r/2-1)} \\ z^{rM/2} & \omega^r z^{rM/2} & \ldots & \omega^{r(r-1)} z^{rM/2} & \omega^{r^2} z^{rM/2} \end{vmatrix},$$

where we have used the property of roots of unity

$$\sum_{j=0}^{r} \omega^j = 0, \tag{5.4.42}$$

to remove the exponential factors. Using row operations to remove powers of z we find this determinant is proportional to a Vandermonde matrix:

$$\det\left(\tilde{\Psi}^{(1)}, \Omega_1\tilde{\Psi}^{(1)}, \ldots, \Omega_r\tilde{\Psi}^{(1)}\right) = \omega^{-r^2(r+1)/4} \begin{vmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & \omega & \ldots & \omega^{r-1} & \omega^r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{r-1} & \ldots & \omega^{(r-1)^2} & \omega^{(r-1)r} \\ 1 & \omega^r & \ldots & \omega^{r(r-1)} & \omega^{r^2} \end{vmatrix}$$
$$= \omega^{-r^2(r+1)/4} \prod_{0\leq j<k\leq r} (\omega^k - \omega^j), \tag{5.4.43}$$

where in (5.4.43) we have used the standard expression for the determinant of a Vandermonde matrix. We now evaluate this product. Firstly, we expand:

$$\omega^{-r^2(r+1)/4} \prod_{0\leq j<k\leq r} (\omega^k - \omega^j) = (-1)^{r(r+1)/2} \omega^{-\frac{1}{12}r(r+1)(r+2)} \prod_{s=1}^{r} (1-\omega^s)^{r+1-s}. \tag{5.4.44}$$

Substitute the identity

$$1-\omega^s = -2i\omega^{s/2}\sin\frac{\pi s}{h}, \tag{5.4.45}$$

into (5.4.44) and collect powers of ω and -1:

$$\det\left(\tilde{\Psi}^{(1)}, \Omega_1\tilde{\Psi}^{(1)}, \dots, \Omega_r\tilde{\Psi}^{(1)}\right) = (2i)^{h(h-1)/2}\prod_{s=1}^{h-1}\left(\sin\frac{\pi s}{h}\right)^{h-s} \tag{5.4.46}$$

$$= (2i)^{h(h-1)/2}\left(\prod_{s=1}^{h-1}\sin\frac{\pi s}{h}\right)^{h/2} \tag{5.4.47}$$

$$= i^{h(h-1)/2}h^{h/2}, \tag{5.4.48}$$

where in the last equality we have used the product identity [36]

$$\prod_{s=1}^{h-1}\sin\frac{\pi s}{h} = \frac{h}{2^{h-1}}. \tag{5.4.49}$$

We have therefore found the quantum Wronskian:

$$\begin{vmatrix} Q_0^{(1)}(\theta) & Q_0^{(1)}(\theta-\frac{2\pi i}{hM}) & \cdots & Q_0^{(1)}(\theta-\frac{2\pi i r}{hM}) \\ Q_1^{(1)}(\theta) & Q_1^{(1)}(\theta-\frac{2\pi i}{hM}) & \cdots & Q_1^{(1)}(\theta-\frac{2\pi i r}{hM}) \\ \vdots & \vdots & \ddots & \vdots \\ Q_r^{(1)}(\theta) & Q_r^{(1)}(\theta-\frac{2\pi i}{hM}) & \cdots & Q_r^{(1)}(\theta-\frac{2\pi i r}{hM}) \end{vmatrix} = i^{h(h-1)/2}h^{h/2}. \tag{5.4.50}$$

This is a generalisation of the quantum Wronskian given in [45] for $r=1$ ($h=2$).

5.5 Ψ-system and the Bethe ansatz equations

The Q-functions discussed in the previous section satisfy certain useful algebraic relations known as *Bethe ansatz equations*. We will derive these equations from relations between large-$|z|$ solutions of linear problems for the representations

$V^{(a)}$, known as the Ψ-system. These, in turn, arise from an embedding of a representation into another representation.

5.5.1 Embedding of representations

We begin by defining the two main representations of interest. We recall the definition of the representation $V^{(a)}$ of $A_r^{(1)}$ as a wedge product of a copies of $V^{(1)}$:

$$V^{(a)} = \bigwedge_{i=1}^{a} V^{(1)}_{\frac{1-a}{2}+i} \tag{5.5.1}$$

We now define a generalisation of these representations: related evaluation representations of $A_r^{(1)}$ defined by a parameter k:

$$V_k^{(a)} = \bigwedge_{i=1}^{a} V^{(1)}_{\frac{1-a}{2}+i+k} \tag{5.5.2}$$

We are then concerned with the wedge product of two of these representations: the representation $V^{(a)}_{-1/2} \wedge V^{(a)}_{1/2}$. For a more explicit construction, we let $\{\mathbf{e}_I^{(a)}\}_{I=0}^{\dim V^{(a)}-1}$ be a basis for the vector space $V^{(a)}_{-1/2}$ and $V^{(a)}_{1/2}$. The vector space $V^{(a)}_{-1/2} \wedge V^{(a)}_{1/2}$ is then spanned by the set of bivectors

$$\{\mathbf{e}_I^{(a)} \wedge \mathbf{e}_J^{(a)} | 0 \leq I < J \leq \dim V^{(a)} - 1\}. \tag{5.5.3}$$

We note that the vectors $\mathbf{e}_I^{(a)}$ are, in this context, not themselves wedge products. We treat the spaces $V_k^{(a)}$ as vector spaces in their own right and define a wedge product on copies of that space, rather than $V^{(1)}$.

We order the basis $\{\mathbf{e}_I^{(a)}\}_{I=0}^{\dim V^{(a)}-1}$ of $V_k^{(a)}$ such that $\mathbf{e}_0^{(a)}$ is the highest weight state, with

$$H_i \mathbf{e}_0^{(a)} = (\lambda_1^{(a)})^i \mathbf{e}_0^{(a)} = (\omega_a)^i \mathbf{e}_0^{(a)}. \tag{5.5.4}$$

Beginning with the highest weight ω_a, we apply the algorithm in section 4.2.1 to construct the remaining weights. We find that $\mathbf{e}_1^{(a)}$ is associated with the weight $\omega_a - \alpha_a$. The highest weight state of $V_{-1/2}^{(a)} \wedge V_{1/2}^{(a)}$ is then given by $\mathbf{e}_0^{(a)} \wedge \mathbf{e}_1^{(a)}$, associated with the weight $2\omega_a - \alpha_a$. We rewrite this weight in the weight basis using the identity

$$\alpha_i = \sum_{i=1}^{r} C_{ij}\omega_j \tag{5.5.5}$$

$$\implies 2\omega_a - \alpha_a = \sum_{b=1}^{r} (2\delta_{ab} - C_{ab})\omega_b \tag{5.5.6}$$

$$= \sum_{b=1}^{r} B_{ab}\omega_b \tag{5.5.7}$$

$$= \omega_{a-1} + \omega_{a+1}, \tag{5.5.8}$$

where C is the A_r Cartan matrix and $B = 2I - C$ is the *incidence matrix*. The representation $V_{-1/2}^{(a)} \wedge V_{1/2}^{(a)}$ therefore has highest weight $\omega_{a-1} + \omega_{a+1}$.

We recall the definition of a *tensor product* of representations, given in section 4.2.1. Let $\{\mathbf{e}_I^{(a-1)}\}_{I=0}^{\dim V^{(a-1)}-1}$ be a basis for $V^{(a-1)}$, with $\mathbf{e}_0^{(a-1)}$ as the highest weight state with weight ω_{a-1}. Similarly, let $\{\mathbf{e}_J^{(a+1)}\}_{J=0}^{\dim V^{(a+1)}-1}$ be a basis for $V^{(a+1)}$, with $\mathbf{e}_0^{(a+1)}$ as its highest weight state with weight ω_{a+1}. We then construct the tensor product representation $V^{(a-1)} \otimes V^{(a+1)}$ with basis

$$\left\{ \mathbf{e}_I^{(a-1)} \otimes \mathbf{e}_J^{(a+1)} \,\middle|\, 0 \le I \le \dim V^{(a-1)}, 0 \le J \le \dim V^{(a+1)} \right\}. \tag{5.5.9}$$

The highest weight of this representation is $\omega_{a-1} + \omega_{a+1}$, which is the same as the representation $V_{-1/2}^{(a)} \wedge V_{1/2}^{(a)}$. There exists an embedding ι between these two representations [38, 47]

$$\iota : V^{(a)}_{-1/2} \wedge V^{(a)}_{1/2} \to V^{(a-1)} \otimes V^{(a+1)}, \tag{5.5.10}$$

which maps the highest weight in $V^{(a)}_{-1/2} \wedge V^{(a)}_{1/2}$ to the highest weight in $V^{(a-1)} \otimes V^{(a+1)}$:

$$\iota\left(\mathbf{e}^{(a)}_0 \wedge \mathbf{e}^{(a)}_1\right) = \mathbf{e}^{(a-1)}_0 \otimes \mathbf{e}^{(a+1)}_0. \tag{5.5.11}$$

We will now use the embedding ι to connect the solutions of the linear problem $(\partial_z + A)\Psi = 0$ in different representations $V^{(a)}_k$.

5.5.2 From the Ψ-system to the Bethe ansatz equations

The solutions $\Psi^{(a)}_k$ to the linear problem associated with $V^{(a)}_k$ are equivalent to Symanzik rotated solutions $\Omega_k \Psi^{(a)}$ of the linear problem associated with $V^{(a)}$. The embedding (5.5.10) then defines the Ψ-system

$$\iota\left(\Psi^{(a)}_{-1/2} \wedge \Psi^{(a)}_{1/2}\right) = \Psi^{(a-1)} \otimes \Psi^{(a+1)}. \tag{5.5.12}$$

The small-$|z|$ expansion of the solutions $\Psi^{(a)}_k$ is given by

$$\Psi^{(a)}_k(\varphi|\theta) = Q^{(a)}_0(\theta_{-k})\Xi^{(a)}_0(\varphi_k|\theta_{-k}) + Q^{(a)}_1(\theta_{-k})\Xi^{(a)}_1(\varphi_k|\theta_{-k}) + \dots \tag{5.5.13}$$

where we have set

$$\theta_p = \theta + \frac{2i\pi p}{hM}, \quad \varphi_p = \varphi + \frac{2\pi p}{hM}, \tag{5.5.14}$$

for brevity. We now substitute (5.5.13) into the Ψ-system (5.5.12):

$$\begin{aligned} &\iota\left(\Psi^{(a)}_{-1/2}\wedge\Psi^{(a)}_{1/2}\right)=\Psi^{(a-1)}\otimes\Psi^{(a+1)}\\ \Longrightarrow\ &\iota\left(\left(Q^{(a)}_0(\theta_{1/2})\Xi^{(a)}_0+Q^{(a)}_1(\theta_{1/2})\Xi^{(a)}_1\right)\wedge\left(Q^{(a)}_0(\theta_{-1/2})\Xi^{(a)}_0+Q^{(a)}_1(\theta_{-1/2})\Xi^{(a)}_1\right)\right)\\ &=Q^{(a-1)}_0(\theta)Q^{(a+1)}_0(\theta)\Xi^{(a-1)}_0\otimes\Xi^{(a+1)}_0. \end{aligned} \tag{5.5.15}$$

where we have used the Symanzik rotation invariance of the small-$|z|$ solutions $\Xi^{(a)}_j$. We expand (5.5.15)

$$\begin{aligned} &\left(Q^{(a)}_0(\theta_{1/2})Q^{(a)}_1(\theta_{-1/2})-Q^{(a)}_0(\theta_{-1/2})Q^{(a)}_1(\theta_{1/2})\right)\iota\left(\Xi^{(a)}_0\wedge\Xi^{(a)}_1\right)\\ &=Q^{(a-1)}_0(\theta)Q^{(a+1)}_0(\theta)\ \Xi^{(a-1)}_0\otimes\Xi^{(a+1)}_0, \end{aligned} \tag{5.5.16}$$

and then take the $|z|\to 0$ limit, applying the asymptotic expansion

$$\Xi^{(a)}_J\sim e^{-\beta(\theta+i\varphi)\lambda^{(a)}_{J+1}\cdot g}\mathbf{e}_J\quad (J=0,\ldots,\dim V^{(a)}-1). \tag{5.5.17}$$

The result is

$$\begin{aligned} &\left(Q^{(a)}_0(\theta_{1/2})Q^{(a)}_1(\theta_{-1/2})-Q^{(a)}_0(\theta_{-1/2})Q^{(a)}_1(\theta_{1/2})\right)e^{-\beta(\theta+i\varphi)(\omega_{a-1}+\omega_{a+1})\cdot g}\iota\left(\mathbf{e}^{(a)}_0\wedge\mathbf{e}^{(a)}_1\right)\\ &=Q^{(a-1)}_0(\theta)Q^{(a+1)}_0(\theta)e^{-\beta(\theta+i\varphi)(\omega_{a-1}+\omega_{a+1})\cdot g}\mathbf{e}^{(a-1)}_0\otimes\mathbf{e}^{(a+1)}_0 \end{aligned} \tag{5.5.18}$$

Substituting (5.5.11) and simplifying leads to a relation between Q-functions

$$\begin{aligned} &Q^{(a)}_0\left(\theta+\frac{i\pi}{hM}\right)Q^{(a)}_1\left(\theta-\frac{i\pi}{hM}\right)-Q^{(a)}_0\left(\theta-\frac{i\pi}{hM}\right)Q^{(a)}_1\left(\theta+\frac{i\pi}{hM}\right)\\ &=Q^{(a-1)}_0(\theta)Q^{(a+1)}_0(\theta). \end{aligned} \tag{5.5.19}$$

We denote the zeroes of the Q_0-functions by $\theta_j^{(a)}$, so that $Q_0^{(a)}(\theta_j^{(a)}) = 0$. Substituting $\theta = \theta_j^{(a)} \pm i\pi/hM$ into (5.5.19) yields two equations:

$$\begin{aligned}
Q_0^{(a)}\left(\theta_j^{(a)} + \frac{2i\pi}{hM}\right) Q_1^{(a)}\left(\theta_j^{(a)}\right) &= Q_0^{(a-1)}\left(\theta_j^{(a)} + \frac{i\pi}{hM}\right) Q_0^{(a+1)}\left(\theta_j^{(a)} + \frac{i\pi}{hM}\right), \\
Q_0^{(a)}\left(\theta_j^{(a)} - \frac{2i\pi}{hM}\right) Q_1^{(a)}\left(\theta_j^{(a)}\right) &= -Q_0^{(a-1)}\left(\theta_j^{(a)} - \frac{i\pi}{hM}\right) Q_0^{(a+1)}\left(\theta_j^{(a)} - \frac{i\pi}{hM}\right).
\end{aligned} \tag{5.5.20}$$

Eliminating $Q_1^{(a)}$ from these equations yields the *untwisted Bethe ansatz equations*:

$$\prod_{b=1}^{r} \frac{Q^{(b)}\left(\theta_j^{(a)} + \frac{i\pi}{hM} C_{ab}\right)}{Q^{(b)}\left(\theta_j^{(a)} - \frac{i\pi}{hM} C_{ab}\right)} = -1, \quad (a = 1, \ldots, r), \tag{5.5.21}$$

where we now abbreviate the Q-functions $Q_0^{(a)}(\theta) = Q^{(a)}(\theta)$, as these 'leading order' Q-functions will be our main concern for this section and the next.

5.5.3 Twisting the Bethe ansatz equations

The Bethe ansatz equations (BAEs) (5.5.21) we have derived are untwisted; BAEs in the literature for the $A_r^{(1)}$ case [21, 37, 38, 45] contain twists in the form of extra constant prefactors. To extract information about the integrals of motion of the associated massive Toda field theory for $A_r^{(1)}$, we must twist the BAEs.

We begin with the newly-derived BAEs (5.5.21) and then shift the Q-functions and their zeroes

$$Q^{(b)}(\theta) = \hat{Q}^{(b)}(\theta + \nu_b), \quad \theta_j^{(b)} = \hat{\theta}_j^{(b)} - \nu_b, \quad \hat{Q}^{(b)}(\hat{\theta}_j^{(b)}) = 0. \tag{5.5.22}$$

The shifts ν_b are defined (up to an overall constant) by the antisymmetric matrix

Λ

$$\nu_a - \nu_b = \frac{i\pi(M+1)}{hM}\Lambda_{ab} \tag{5.5.23}$$

where Λ is chosen to ensure $C-\Lambda$ is upper triangular and $C+\Lambda$ is lower triangular. $C-\Lambda$ and $C+\Lambda$ are then matrices with even entries. The BAEs become

$$\prod_{b=1}^{r} \frac{\hat{Q}^{(b)}\left(\hat{\theta}_j^{(a)} + \frac{i\pi}{hM}(C-\Lambda)_{ab} - \frac{i\pi}{h}\Lambda_{ab}\right)}{\hat{Q}^{(b)}\left(\hat{\theta}_j^{(a)} + \frac{i\pi}{hM}(C+\Lambda)_{ab} - \frac{i\pi}{h}\Lambda_{ab}\right)} = -1. \tag{5.5.24}$$

We then apply the quasiperiodicity relation (5.4.19) to (5.5.24). The BAEs (5.5.24) become

$$\prod_{b=1}^{r} \frac{e^{-i\pi(C-\Lambda)_{ab}\gamma_b/h}\,\hat{Q}^{(b)}\left(\hat{\theta}_j^{(a)} - \frac{i\pi}{h}(C-\Lambda)_{ab} - \frac{i\pi}{h}\Lambda_{ab}\right)}{e^{i\pi(C-\Lambda)_{ab}\gamma_b/h}\,\hat{Q}^{(b)}\left(\hat{\theta}_j^{(a)} + \frac{i\pi}{h}(C+\Lambda)_{ab} - \frac{i\pi}{h}\Lambda_{ab}\right)} = -1, \tag{5.5.25}$$

which reduces to our final set of BAEs which are consistent with [45]

$$\prod_{b=1}^{r} e^{-\frac{2i\pi}{h}C_{ab}\gamma_b} \frac{\hat{Q}^{(b)}(\hat{\theta}_j^{(a)} - \frac{i\pi}{h}C_{ab})}{\hat{Q}^{(b)}(\hat{\theta}_j^{(a)} + \frac{i\pi}{h}C_{ab})} = -1. \tag{5.5.26}$$

We will be concerned with the shifted $\hat{Q}^{(a)}$-functions for the remainder of the chapter. The original $Q^{(a)}$-functions are recovered by undoing the shift (5.5.22), where for consistency with (5.5.23) and the matrix Λ we set

$$\nu_a = -(a-1)\frac{i\pi(M+1)}{hM}. \tag{5.5.27}$$

5.6 Non-linear integral equations and integrals of motion

5.6.1 The non-linear integral equation

The integrals of motion for the $A_r^{(1)}$ massive Toda theory are the coefficients of the higher-order terms of the asymptotic expansion of the logarithm of the Q-functions [45, 5]. The first step in deriving this asymptotic expansion is to construct a useful non-linear integral equation (NLIE). The construction here will follow that given in [21] for the massless A_r case, and is a direct generalisation of the derivation given in section 3.5. We begin by defining the function

$$a^{(m)}(\theta) = \prod_{t=1}^{r} e^{-\frac{2i\pi}{h}C_{mt}\gamma_t} \frac{\hat{Q}^{(t)}(\theta - \frac{i\pi}{h}C_{mt})}{\hat{Q}^{(t)}(\theta + \frac{i\pi}{h}C_{mt})}. \tag{5.6.1}$$

As a consequence of the BAEs (5.5.26), $a^{(m)}(\hat{\theta}_j^{(m)}) = -1$. We expand the $\hat{Q}^{(t)}$ functions in this expression using an infinite product expansion:

$$\hat{Q}^{(t)}(\theta) = \hat{Q}^{(t)}(0)e^{-\frac{\gamma_t M\theta}{M+1}} \prod_{j=0}^{\infty} \left(1 - e^{\frac{hM}{M+1}(\theta - \hat{\theta}_j^{(t)})}\right)\left(1 - e^{-\frac{hM}{M+1}(\theta - \hat{\theta}_{-j-1}^{(t)})}\right). \tag{5.6.2}$$

We then substitute this infinite product expansion into $a^{(m)}(\theta)$:

$$\begin{aligned}
&a^{(m)}(\theta) \\
&= \prod_{t=1}^{r} e^{-\frac{2i\pi}{h(M+1)}C_{mt}\gamma_t} \prod_{j=0}^{\infty} \frac{\left(1 - e^{\frac{hM}{M+1}(\theta - \hat{\theta}_j^{(t)})} e^{-\frac{i\pi M C_{mt}}{M+1}}\right)\left(1 - e^{-\frac{hM}{M+1}(\theta - \hat{\theta}_{-j-1}^{(t)})} e^{\frac{i\pi M C_{mt}}{M+1}}\right)}{\left(1 - e^{\frac{hM}{M+1}(\theta - \hat{\theta}_j^{(t)})} e^{\frac{i\pi M C_{mt}}{M+1}}\right)\left(1 - e^{-\frac{hM}{M+1}(\theta - \hat{\theta}_{-j-1}^{(t)})} e^{-\frac{i\pi M C_{mt}}{M+1}}\right)}, \\
&= \prod_{t=1}^{r} e^{-\frac{2i\pi}{h(M+1)}C_{mt}\gamma_t} \prod_{j=-\infty}^{\infty} \frac{1 - e^{\frac{hM}{M+1}(\theta - \hat{\theta}_j^{(t)} - \frac{i\pi C_{mt}}{h})}}{1 - e^{\frac{hM}{M+1}(\theta - \hat{\theta}_j^{(t)} + \frac{i\pi C_{mt}}{h})}}.
\end{aligned} \tag{5.6.3}$$

We then take the logarithm of (5.6.3):

$$\log a^{(m)}(\theta) = \frac{-2i\pi}{h(M+1)} \sum_{t=1}^{r} C_{mt}\gamma_t + \sum_{t=1}^{r} \sum_{j=-\infty}^{\infty} F_{mt}(\theta - \hat{\theta}_j^{(t)}), \tag{5.6.4}$$

where

$$F_{mt}(\theta) = \log \left(\frac{1 - e^{\frac{hM}{M+1}(\theta - \frac{i\pi}{h} C_{mt})}}{1 - e^{\frac{hM}{M+1}(\theta + \frac{i\pi}{h} C_{mt})}} \right). \tag{5.6.5}$$

As in [21], we use Cauchy's integral theorem to rewrite the infinite sum over the zeroes $\theta_j^{(t)}$ as a contour integral:

$$\sum_{t=1}^{r} \sum_{j=0}^{\infty} F_{mt}(\theta - \theta_j^{(t)}) = \sum_{t=1}^{r} \int_{\xi} \frac{\mathrm{d}\theta'}{2i\pi} F_{mt}(\theta - \theta') \partial_{\theta'} \log\left(1 + a^{(t)}(\theta')\right), \tag{5.6.6}$$

where ξ is a contour enclosing the zeroes anticlockwise. As in [21], we assume all the zeroes are along the real axis. The contour ξ is then chosen to be two parallel lines enclosing the real axis, with the direction of integration along ξ chosen such that the real axis remains on the left of the contour. For brevity of notation, we now omit the components of the matrices C and F and the vectors $a(\theta), \gamma$ from this calculation. The logarithm of $a(\theta)$ is then given by

$$\log a(\theta) = \frac{-2i\pi}{h(M+1)} C\gamma + \int_{\xi} \frac{\mathrm{d}\theta'}{2i\pi} F(\theta - \theta') \partial_{\theta'} \log\left(1 + a(\theta)\right), \tag{5.6.7}$$

where $\log(1 + a(\theta))$ represents the column vector

$$\left(\log\left(1 + a^{(1)}(\theta)\right), \log\left(1 + a^{(2)}(\theta)\right), \ldots, \log\left(1 + a^{(r)}(\theta)\right)\right)^T. \tag{5.6.8}$$

Integrating (5.6.7) by parts and considering the two contributions from above and below the real axis from γ separately, we rewrite $\log a(\theta)$:

$$\begin{aligned}
&\log a(\theta) = \frac{-2i\pi}{h(M+1)} C\gamma \\
&+ \int_{-\infty}^{\infty} R(\theta - \theta') \left\{ \log\left(1 + a(\theta' + i0)\right) - \log\left(1 + a(\theta' - i0)\right) \right\} \mathrm{d}\theta' \\
&= \frac{-2i\pi}{h(M+1)} C\gamma \\
&+ \int_{-\infty}^{\infty} R(\theta - \theta' + i0) \left\{ \log a(\theta' - i0) - 2i \operatorname{Im} \log\left(1 + a(\theta' - i0)\right) \right\} \mathrm{d}\theta', \qquad (5.6.9)
\end{aligned}$$

where $R(\theta) = (i/2\pi)\partial_\theta F(\theta)$. We now apply a Fourier transform $\mathcal{F}$ to both sides of (5.6.9), where the Fourier transform is defined as

$$\mathcal{F}[f](k) = \tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ik\theta} f(\theta) \, \mathrm{d}\theta \qquad (5.6.10)$$

and its inverse is given by

$$\mathcal{F}^{-1}[\tilde{f}](\theta) = f(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik\theta} \tilde{f}(k) \, \mathrm{d}k. \qquad (5.6.11)$$

Applying the Fourier transform (5.6.10) to both sides of (5.6.9), we find

$$\mathcal{F}[\log a] = \frac{-2i\pi}{h(M+1)} C\gamma \cdot 2\pi\delta(k) + \tilde{R}(k) \left\{ \mathcal{F}[\log a] - 2i\mathcal{F}[\operatorname{Im}\log(1+a)] \right\}, \qquad (5.6.12)$$

where we have used the definition of the Dirac delta function $\delta(k)$ in the form of an integral

$$\int_{-\infty}^{\infty} e^{-ik\theta} \, \mathrm{d}\theta = 2\pi\delta(k). \qquad (5.6.13)$$

We collect the $\mathcal{F}[\log a]$ terms in (5.6.12)

$$(I-\tilde{R}(k))\mathcal{F}[\log a] = \frac{-2i\pi}{h(M+1)} C\gamma\cdot 2\pi\delta(k) - 2i\tilde{R}(k)\mathcal{F}[\operatorname{Im}\log(1+a)], \quad (5.6.14)$$

and then pre-multiply both sides of (5.6.14) by the operator $(1-\tilde{R}(k))^{-1}$

$$\begin{aligned}\mathcal{F}[\log a] = &\frac{-2i\pi}{h(M+1)}(I-\tilde{R}(k))^{-1}C\gamma\cdot 2\pi\delta(k)\\ &+ b^{(1)}\mathcal{F}[e^{\theta}] + b^{(2)}\mathcal{F}[e^{-\theta}] - 2i(I-\tilde{R}(k))^{-1}\tilde{R}(k)\mathcal{F}[\operatorname{Im}\log(1+a)]. \quad (5.6.15)\end{aligned}$$

where the terms proportional to the arbitrary constants $b^{(1)}$ and $b^{(2)}$ arise from the points $k=\pm i$ where the inverse of the operator $(I-\tilde{R}(k))$ is not well-defined. $\mathcal{F}^{-1}(I-\tilde{R}(k))$ can be thought of as a differential operator in θ, with e^{θ} and $e^{-\theta}$ in the kernel of this operator.

We then take the inverse Fourier transform (5.6.11) of (5.6.15):

$$\begin{aligned}\log a(\theta) = &\frac{-2i\pi}{h(M+1)}(I-\tilde{R}(0))^{-1}C\gamma\\ &+ b^{(1)}e^{\theta} + b^{(2)}e^{-\theta} - 2i\int_{-\infty}^{\infty}\varphi(\theta-\theta'+i0)\operatorname{Im}\log(1+a(\theta-i0))\mathrm{d}\theta',\end{aligned} \quad (5.6.16)$$

where $\varphi(\theta) = \mathcal{F}^{-1}[(I-\tilde{R}(k))^{-1}\tilde{R}(k)]$. The constant vectors b_1 and b_2 will be chosen to match the Q-asymptotics (5.4.31)-(5.4.33) for $\operatorname{Re}\theta\to\pm\infty$. We now simplify (5.6.16) by computing $R(k)$ explicitly. This is done using the following relation

$$\log\left(\frac{1-e^{X-Y}}{1-e^{X+Y}}\right) = -Y + \log\left(\frac{\sinh\frac{X-Y}{2}}{\sinh\frac{X+Y}{2}}\right), \quad (5.6.17)$$

and equations (D.53) and (D.54) from [23]:

$$i\partial_\theta \log \frac{\sinh \sigma\theta + i\pi\tau}{\sinh \sigma\theta - i\pi\tau} = \frac{2\sigma \sin 2\pi\tau}{\cosh 2\sigma\theta - \cos 2\pi\tau}, \tag{5.6.18}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2\pi} e^{-ik\theta} \frac{2\sigma \sin 2\pi\tau}{\cosh 2\sigma\theta - \cos 2\pi\tau} = \frac{\sinh(1-2\tau)\frac{\pi k}{2\sigma}}{\sinh \frac{\pi k}{2\sigma}}. \tag{5.6.19}$$

We then see the elements of $R(k)$ are given by

$$\tilde{R}_{<mt>}(k) = \frac{\sinh \frac{\pi k}{hM}}{\sinh \frac{\pi(M+1)k}{hM}}, \quad \tilde{R}_{mm}(k) = \frac{\sinh \frac{\pi(M-1)k}{hM}}{\sinh \frac{\pi(M+1)k}{hM}}, \tag{5.6.20}$$

where $< mt >$ indicates the nodes m and t on the Dynkin diagram of A_r are connected, and all other elements of R are zero. Taking the limit of these elements as $k \to 0$ it can be shown that

$$(I - \tilde{R}(0))^{-1} C = (M+1) I, \tag{5.6.21}$$

so that the integral equation (5.6.16) becomes

$$\begin{aligned} \log a^{(m)}(\theta) = &\frac{-2i\pi}{h}\gamma_m + b_m^{(1)} e^{\theta} + b_m^{(2)} e^{-\theta} \\ &- 2i \sum_{t=1}^{r} \int_{-\infty}^{\infty} \varphi_{mt}(\theta - \theta' + i0) \, \mathrm{Im} \log \left(1 + a^{(t)}(\theta' - i0)\right) \mathrm{d}\theta', \end{aligned} \tag{5.6.22}$$

where we have restored component notation.

5.6.2 Integral form of $\log \hat{Q}^{(m)}\left(\theta + \frac{i\pi(M+1)}{hM}\right)$

As stated previously, the integrals of motion are coefficients in an asymptotic expansion of the logarithm of the $\hat{Q}^{(m)}$-functions. The next step in deriving this expansion is to find an expression for $\log \hat{Q}^{(m)}$, using the definition of the a-function (5.6.1) and the newly derived non-linear integral equation (5.6.22).

We begin by taking logarithms of (5.6.1):

$$\log a^{(m)}(\theta) = \sum_{t=1}^{r} \left[\frac{-2i\pi}{h} C_{mt}\gamma_t + \log \hat{Q}^{(t)} \left(\theta - \frac{i\pi}{h} C_{mt} \right) - \log \hat{Q}^{(t)} \left(\theta + \frac{i\pi}{h} C_{mt} \right) \right], \tag{5.6.23}$$

We then invoke the logarithm of the quasiperiodicity relation (5.4.19) applied to $\hat{Q}_0^{(t)}(\theta)$:

$$\log \hat{Q}^{(t)} \left(\theta + \frac{2i\pi(M+1)}{hM} n \right) = -\frac{2i\pi}{h} n\gamma_t + \log \hat{Q}^{(t)}(\theta), \tag{5.6.24}$$

where $n \in \mathbb{Z}$. Using quasiperiodicity to rewrite $\log a^{(m)}$:

$$\begin{aligned} \log a^{(m)}(\theta) = \sum_{t=1}^{r} \Bigg[& \frac{2i\pi}{h} (p_{mt} - q_{mt} - C_{mt})\gamma_t \\ & + \log \hat{Q}^{(t)} \left(\theta - \frac{i\pi}{h} C_{mt} + \frac{2i\pi(M+1)}{hM} p_{mt} \right) \\ & - \log \hat{Q}^{(t)} \left(\theta + \frac{i\pi}{h} C_{mt} + \frac{2i\pi(M+1)}{hM} q_{mt} \right) \Bigg], \end{aligned} \tag{5.6.25}$$

we choose the matrices p and q with integer entries to satisfy

$$p_{mt} - q_{mt} - C_{mt} = -\delta_{mt}. \tag{5.6.26}$$

We then set the NLIE (5.6.22) and (5.6.25) equal to one another. The choice of p and q ensures that the constant terms proportional to γ_t are eliminated:

$$\begin{aligned} & \sum_{t=1}^{r} \Bigg[\log \hat{Q}^{(t)} \left(\theta - \frac{i\pi}{h} C_{mt} + \frac{2i\pi(M+1)}{hM} p_{mt} \right) \\ & - \log \hat{Q}^{(t)} \left(\theta + \frac{i\pi}{h} C_{mt} + \frac{2i\pi(M+1)}{hM} q_{mt} \right) \Bigg] \\ & = b_m^{(1)} e^{\theta} + b_m^{(2)} e^{-\theta} - 2i \sum_{t=1}^{r} \int_{-\infty}^{\infty} \varphi_{mt}(\theta - \theta' + i0) \, \mathrm{Im} \log \left(1 + a^{(t)}(\theta' - i0) \right) \mathrm{d}\theta'. \end{aligned} \tag{5.6.27}$$

The next step is to take Fourier transforms of (5.6.27), simplifying that expression using the Fourier transform identity

$$\mathcal{F}[f(\theta+\alpha)] = e^{ik\alpha}\mathcal{F}[f(\theta)]. \tag{5.6.28}$$

The result is

$$\begin{aligned}&\sum_{t=1}^{r}\tilde{G}_{mt}(k)\mathcal{F}\left[\log\hat{Q}^{(t)}\left(\theta+\frac{i\pi(M+1)}{hM}\right)\right]\\ &= b_m^{(1)}\cdot 2\pi\delta(k+i) + b_m^{(2)}\cdot 2\pi\delta(k-i) - 2i\sum_{t=1}^{r}\mathcal{F}[\varphi_{mt}](k)\mathcal{F}[\operatorname{Im}\log\bigl(1+a^{(t)}\bigr)](k),\end{aligned} \tag{5.6.29}$$

where $\tilde{G}(k)$ is the matrix with elements

$$\tilde{G}_{<mt>}(k) = 2\sinh\frac{\pi k}{hM}, \quad \tilde{G}_{mm}(k) = 2\sinh\frac{\pi k(M-1)}{hM}. \tag{5.6.30}$$

As with $\tilde{R}$, $< mt >$ indicates the nodes m and t are connected on the Dynkin diagram of A_r. We rewrite (5.6.29) in component-free notation:

$$\begin{aligned}&\tilde{G}(k)\mathcal{F}\left[\log\hat{Q}\left(\theta+\frac{i\pi(M+1)}{hM}\right)\right]\\ &= 2\pi b^{(1)}\delta(k+i) + 2\pi b^{(2)}\delta(k-i) - 2i\mathcal{F}[\varphi]\mathcal{F}[\operatorname{Im}\log(1+a)],\end{aligned} \tag{5.6.31}$$

and then premultiply both sides of (5.6.31) by $G(k)^{-1}$, adding an extra term on the right-hand side of this expression due to the pole of G^{-1} at $k=0$:

$$\begin{aligned}\mathcal{F}\left[\log\hat{Q}\left(\theta+\frac{i\pi(M+1)}{hM}\right)\right] &= 2\pi\tilde{G}(k)^{-1}b^{(1)}\delta(k+i)\\ &+ 2\pi\tilde{G}(k)^{-1}b^{(2)}\delta(k-i) - 2i\tilde{H}(k)\mathcal{F}[\operatorname{Im}\log(1+a)] + 2\pi b^{(3)}\delta(k),\end{aligned} \tag{5.6.32}$$

where $b^{(3)}$ is an arbitrary constant vector and

$$\tilde{H}(k) = \tilde{G}(k)^{-1}\mathcal{F}[\varphi] = \tilde{G}(k)^{-1}(I - \tilde{R}(k))^{-1}\tilde{R}(k). \tag{5.6.33}$$

We now take inverse Fourier transforms of (5.6.32):

$$\begin{aligned}\log \hat{Q}\left(\theta + \frac{i\pi(M+1)}{hM}\right) &= \tilde{G}(-i)^{-1}b^{(1)}e^{\theta} + \tilde{G}(i)^{-1}b^{(2)}e^{-\theta} + b^{(3)} \\ &\quad - 2i\int_{-\infty}^{\infty} H(\theta - \theta' + i0)\,\mathrm{Im}\log\left(1 + a(\theta' - i0)\right)\mathrm{d}\theta'.\end{aligned} \tag{5.6.34}$$

Finally, we undo the shift in the Q-functions (5.5.22) and choose $b^{(1)}$, $b^{(2)}$ and $b^{(3)}$ to match the earlier derived Q-asymptotics (5.4.31)-(5.4.33)

$$\tilde{G}(-i)^{-1}b^{(1)} = s^{1+M}m\tau(h,M)w, \quad \tilde{G}(i)^{-1}b^{(2)} = s^{1+M}m\tau(h,M)w, \quad b^{(3)} = -\frac{i\pi}{h}\gamma, \tag{5.6.35}$$

where w is a vector with components

$$w_a = \frac{\sin\frac{a\pi}{h}}{\sin\frac{\pi}{h}}. \tag{5.6.36}$$

The $\log Q^{(m)}$ functions are then given by

$$\begin{aligned}\log Q^{(m)}\left(\theta + \frac{i\pi(M+1)}{hM}\right) &= 2m\tau(h,M)w_m\cosh\theta - \frac{i\pi}{h}\gamma_m \\ &\quad - 2i\sum_{t=1}^{r}\int_{-\infty}^{\infty} H_{mt}(\theta - \theta' + i0)\,\mathrm{Im}\log\bigl(1 + a^{(t)}(\theta' - i0)\bigr)\mathrm{d}\theta'.\end{aligned} \tag{5.6.37}$$

5.6.3 Integrals of motion

The final step in the calculation of the integrals of motion of the $A_r^{(1)}$ Toda theory is the computation of the asymptotic expansion of the matrix $H(\theta)$. H is given in terms of the inverse Fourier transform of $\tilde{H}(k)$:

$$H(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{H}(k) e^{ik\theta} \mathrm{d}k. \tag{5.6.38}$$

We then need to calculate $\tilde{H}(k)$. We recall

$$\tilde{H}(k) = \tilde{G}^{-1}(I - \tilde{R})^{-1}\tilde{R}, \tag{5.6.39}$$

and from the definitions of $\tilde{R}$ (5.6.20) and $\tilde{G}$ (5.6.30),

$$\tilde{R} = \frac{1}{2\sinh\frac{\pi(M+1)k}{hM}} \tilde{G}. \tag{5.6.40}$$

We use this relation between $\tilde{R}$ and $\tilde{G}$ to compute $\tilde{H}^{-1}$:

$$\begin{aligned} \tilde{H}^{-1} &= \tilde{R}^{-1}(I - \tilde{R})\tilde{G} \\ &= \left(2\sinh\frac{\pi(M+1)k}{hM} I - \tilde{G} \right), \end{aligned} \tag{5.6.41}$$

which has components

$$\tilde{H}^{-1}_{<mt>} = -2\sinh\frac{\pi k}{hM}, \quad \tilde{H}^{-1}_{mm} = 4\sinh\frac{\pi k}{hM}\cosh\frac{\pi k}{h}. \tag{5.6.42}$$

Following [21], define the deformed Cartan matrix $\tilde{C}$:

$$\tilde{C}_{<mt>}(k) = -\frac{1}{\cosh\frac{\pi k}{h}}, \quad \tilde{C}_{mm}(k) = 2. \tag{5.6.43}$$

We then write $\tilde{H}^{-1}$ and hence $\tilde{H}$ in terms of $\tilde{C}$:

$$\begin{aligned} \tilde{H}^{-1} &= 2\sinh\frac{\pi k}{hM}\cosh\frac{\pi k}{h}\tilde{C} \\ \implies \tilde{H} &= \frac{1}{2\sinh\frac{\pi k}{hM}\cosh\frac{\pi k}{h}} \tilde{C}^{-1}. \end{aligned} \tag{5.6.44}$$

The inverse of the deformed Cartan matrix is given in [21]:

$$\tilde{C}^{-1}_{tm}(k) = \tilde{C}^{-1}_{mt}(k) = \frac{\coth\frac{\pi k}{h}\sinh\frac{\pi(h-m)k}{h}\sinh\frac{\pi tk}{h}}{\sinh\pi k}, \quad m \geq t. \tag{5.6.45}$$

$$\implies \tilde{H}_{tm}(k) = \tilde{H}_{mt}(k) = \frac{\sinh\frac{\pi(h-m)k}{h}\sinh\frac{\pi tk}{h}}{2\sinh\frac{\pi k}{h}\sinh\frac{\pi k}{hM}\sinh\pi k}. \quad m \geq t. \tag{5.6.46}$$

With $\tilde{H}$ calculated, we now substitute $\tilde{H}$ into the integral term in (5.6.37):

$$\left(-2i\int_{-\infty}^{\infty} H(\theta-\theta'+i0)\,\mathrm{Im}\log\left(1+a(\theta'-i0)\right)\mathrm{d}\theta'\right)_m \tag{5.6.47}$$

$$= -2i\sum_{t=1}^{r}\int_{-\infty}^{\infty} H_{mt}(\theta-\theta'+i0)\,\mathrm{Im}\log\left(1+a^{(t)}(\theta')\right)\mathrm{d}\theta' \tag{5.6.48}$$

$$= -2i\sum_{t=1}^{r}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}\mathrm{d}k\right]\mathrm{Im}\log\left(1+a^{(t)}(\theta')\right)\mathrm{d}\theta'. \tag{5.6.49}$$

To proceed, we evaluate the integral over k using Cauchy's residue theorem. The poles of $\tilde{H}(k)$ are at $k = pi$ and $k = hqMi$, with $p, q \in \mathbb{Z}$. Closing the integration contour in the upper half plane, the integral over k is given by a sum of residues:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}\mathrm{d}k \tag{5.6.50}$$

$$= i\sum_{p=1}^{\infty}\mathrm{Res}\Big[\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}, k=pi\Big] + i\sum_{q=1}^{\infty}\mathrm{Res}\Big[\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}, k=hqMi\Big].$$

The residues of $\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}$ are given by

$$\mathrm{Res}\Big[\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}, k=pi\Big] = \frac{(-1)^p\sin\left(\frac{p\pi m}{h}\right)\sin\left(\frac{p\pi t}{h}\right)}{2\pi\sin\left(\frac{p\pi}{h}\right)\sin\frac{p\pi}{hM}}e^{-p(\theta-\theta'+i0)}, \tag{5.6.51}$$

$$\mathrm{Res}\Big[\tilde{H}_{mt}(k)e^{ik(\theta-\theta'+i0)}, k=hqMi\Big] = \frac{(-1)^q hM}{2\pi\sin(qM\pi)\sin(hqM\pi)}g_{mt}e^{-hqM(\theta-\theta'+i0)}, \tag{5.6.52}$$

where g_{mt} is a symmetric function of m and t:

$$g_{mt} = \begin{cases} \sin(qtM\pi)\sin(q(h-m)M\pi) & m \geq t \\ \sin(qmM\pi)\sin(q(h-t)M\pi) & m < t. \end{cases} \tag{5.6.53}$$

Using these residues, the integral (5.6.48) is then expressed as an infinite series in powers of e^{θ}:

$$-2i\sum_{t=1}^{r}\int_{-\infty}^{\infty} H_{mt}(\theta-\theta'+i0)\,\mathrm{Im}\log\left(1+a^{(t)}(\theta'-i0)\right)\mathrm{d}\theta' \tag{5.6.54}$$

$$= \sum_{p=1}^{\infty}\mathfrak{I}_p^{(m)}e^{-p\theta} + \sum_{q=0}^{\infty}\mathfrak{S}_q^{(m)}e^{-hqM\theta}. \tag{5.6.55}$$

The asymptotic expansion of $\log Q^{(m)}(\theta+\frac{i\pi(M+1)}{hM})$ is then given by

$$\log Q^{(m)}\left(\theta+\frac{i\pi(M+1)}{hM}\right)$$
$$= 2m\tau(h,M)w_m\cosh\theta - \frac{i\pi}{h}\gamma_m + \sum_{p=1}^{\infty}\mathfrak{I}_p^{(m)}e^{-p\theta} + \sum_{q=0}^{\infty}\mathfrak{S}_q^{(m)}e^{-hqM\theta}, \tag{5.6.56}$$

where

$$\mathfrak{I}_p^{(m)} = \sum_{t=1}^{r}\int_{-\infty}^{\infty}\frac{(-1)^p\sin\left(\frac{p\pi m}{h}\right)\sin\left(\frac{p\pi t}{h}\right)}{\pi\sin\left(\frac{p\pi}{h}\right)\sin\left(\frac{p\pi}{hM}\right)}e^{p(\theta'-i0)}\,\mathrm{Im}\log\left(1+a^{(t)}(\theta'-i0)\right)\mathrm{d}\theta', \tag{5.6.57}$$

$$\mathfrak{S}_q^{(m)} = \sum_{t=1}^{r}\int_{-\infty}^{\infty}\frac{(-1)^q hMg_{mt}}{\pi\sin(qM\pi)\sin(hqM\pi)}e^{hqM(\theta'-i0)}\,\mathrm{Im}\log\left(1+a^{(t)}(\theta'-i0)\right)\mathrm{d}\theta', \tag{5.6.58}$$

are the integrals of motion for the massive $A_r^{(1)}$ Toda field theory. The calculation of the conjugate integrals of motion $\bar{\mathfrak{I}}_p^{(a)}$, $\bar{\mathfrak{S}}_q^{(a)}$ is exactly analogous, and is done by closing the contour integral (5.6.38) in the lower-half complex k-plane and evaluating the coefficients of the resulting expansion in powers of e^{θ} and $e^{hM\theta}$.

5.7 Spectral equivalence

In [22], the authors considered a pair of differential equations

$$\left(-\frac{d^2}{dx^2} + x^6 + \frac{l(l+1)}{x^2}\right)\psi(x) = E\psi(x), \tag{5.7.1}$$

and

$$\left(\frac{d}{dx} - \frac{g_2 - 2}{x}\right)\left(\frac{d}{dx} - \frac{g_1 - 1}{x}\right)\left(\frac{d}{dx} - \frac{g_0}{x}\right)\phi(x) + x^3\phi(x) = E\phi(x), \tag{5.7.2}$$

where

$$g_0 = (1 - 6l)/4, \quad g_1 = 1, \quad g_2 = (7 + 6l)/4. \tag{5.7.3}$$

These differential equations are the conformal limits of the equations involved with the $A_1^{(1)}$ and $A_2^{(1)}$ cases of the massive ODE/IM correspondence, with certain choices of the parameters M, g. In [22] it was shown that eigenvalue problems associated with these equations have spectral determinants that satisfy the same set of Bethe ansatz equations. This implies that their spectra should be equivalent, up to a rescaling of eigenvalues. In this section, we consider the massive analogoues of the equations (5.7.1)-(5.7.2), demonstrating that the suitably defined spectral determinants satisfy the same set of BAEs, and that the integrals of motion for these two cases coincide.

5.7.1 The equivalence of $A_1^{(1)}$ and $A_1^{(2)}$ Bethe ansatz equations

The equation (5.7.1) is the conformal limit of (5.3.16), with $r = 1$ ($h = r+1 = 2$) and $M = 3$. We also set

$$\beta\lambda_1^{(1)} \cdot g = l, \quad \beta\lambda_2^{(1)} \cdot g = -l, \quad \gamma_1 = -(\rho^\vee + \beta g) \cdot \lambda_1^{(1)} = -l - \frac{1}{2}. \tag{5.7.4}$$

The corresponding $A_1^{(1)}$ BAEs (5.5.26) are then

$$e^{i\pi(2l+1)} \frac{Q^{(1)}\left(\theta_j^{(1)} - i\pi\right)}{Q^{(1)}\left(\theta_j^{(1)} + i\pi\right)} = -1. \tag{5.7.5}$$

$Q^{(1)}(\theta)$ also satisfies a quasiperiodicity relation (5.4.19)

$$Q^{(1)}\left(\theta + \frac{4i\pi}{3}\right) = e^{i\pi(l+1/2)} Q^{(1)}(\theta). \tag{5.7.6}$$

We now consider equation (5.7.2), which is the conformal limit of (5.3.16), with $r = 2$, $h = 3$ and $M = 1$. The constants g_0, g_1 and g_2 are related to the weights $\lambda_i^{(1)}$, in the following way [37]:

$$g_0 = -\beta\lambda_1^{(1)} \cdot g, \quad g_1 = 1 - \beta\lambda_2^{(1)} \cdot g, \quad g_2 = 2 - \beta\lambda_3^{(1)} \cdot g. \tag{5.7.7}$$

We also define the related constants $\hat{\gamma}_a = -(\rho^\vee + \beta g) \cdot \lambda_1^{(a)}$ in terms of the constants g_i

$$\hat{\gamma}_1 = g_0 - 1, \quad \hat{\gamma}_2 = g_0 + g_1 - 2. \tag{5.7.8}$$

Defining $R^{(1)}(\theta)$ and $R^{(2)}(\theta)$ as the spectral determinants associated with this equation, the BAEs are

$$e^{-\frac{2i\pi}{3}(2\hat{\gamma}_1-\hat{\gamma}_2)}\frac{R^{(1)}\left(\theta_j^{(1)}-\frac{2i\pi}{3}\right)R^{(2)}\left(\theta_j^{(1)}+\frac{i\pi}{3}\right)}{R^{(1)}\left(\theta_j^{(1)}+\frac{2i\pi}{3}\right)R^{(2)}\left(\theta_j^{(1)}-\frac{i\pi}{3}\right)}=-1, \tag{5.7.9}$$

$$e^{-\frac{2i\pi}{3}(-\hat{\gamma}_1+2\hat{\gamma}_2)}\frac{R^{(1)}\left(\theta_j^{(2)}+\frac{i\pi}{3}\right)R^{(2)}\left(\theta_j^{(2)}-\frac{2i\pi}{3}\right)}{R^{(1)}\left(\theta_j^{(2)}-\frac{i\pi}{3}\right)R^{(2)}\left(\theta_j^{(2)}+\frac{2i\pi}{3}\right)}=-1, \tag{5.7.10}$$

where $R^{(1)}(\theta)$ and $R^{(2)}(\theta)$ satisfy the quasiperiodicity relations

$$R^{(a)}\left(\theta+\frac{4i\pi}{3}\right)=e^{-\frac{2i\pi}{3}\hat{\gamma}_a}R^{(a)}(\theta), \quad a=1,2. \tag{5.7.11}$$

We apply (5.7.11) to the $A_2^{(1)}$ BAEs (5.7.9)-(5.7.10). The BAEs then become

$$e^{i\pi(2l+1)}\frac{R^{(2)}\left(\theta_j^{(1)}-i\pi\right)}{R^{(2)}\left(\theta_j^{(1)}+i\pi\right)}=-1, \tag{5.7.12}$$

$$e^{i\pi(2l+1)}\frac{R^{(1)}\left(\theta_j^{(2)}-i\pi\right)}{R^{(1)}\left(\theta_j^{(2)}+i\pi\right)}=-1. \tag{5.7.13}$$

where we have applied the choice of constants (5.7.3) to match the $A_2^{(1)}$ BAEs (5.7.12)-(5.7.13) with the $A_1^{(1)}$ BAE (5.7.5). The BAEs (5.7.12)-(5.7.13) become two copies of the $A_1^{(1)}$ BAE (5.7.5) under the identification

$$R^{(2)}(\theta)=i\sqrt{3}R^{(1)}(\theta), \tag{5.7.14}$$

which follows from the definitions of $R^{(1)}(\theta)$ and $R^{(2)}(\theta)$

$$R^{(1)}(\theta)=\det\left(\Psi^{(1)},\Xi_1^{(1)},\Xi_2^{(1)}\right), \quad R^{(2)}(\theta)=\det\left(\Psi^{(2)},\Xi_1^{(2)},\Xi_2^{(2)}\right), \tag{5.7.15}$$

where the vectors $\Xi_i^{(a)}$ and $\Psi^{(a)}$ are the small-$|z|$ and large-$|z|$ solutions of the linear system (5.2.9) in the representation $V^{(a)}$, as defined in section 5.3. With the choice of constants (5.7.3), these asymptotic solutions have the behaviour

$$\Xi_1^{(1)} \sim \mathbf{e}_1^{(1)}, \quad \Xi_2^{(1)} \sim z^{\frac{-1+6l}{4}} \mathbf{e}_2^{(1)}, \quad \text{as } |z| \to 0, \tag{5.7.16}$$

$$\Xi_1^{(2)} \sim \mathbf{e}_1^{(2)}, \quad \Xi_2^{(2)} \sim z^{\frac{-1+6l}{4}} \mathbf{e}_2^{(2)}, \quad \text{as } |z| \to 0, \tag{5.7.17}$$

$$\Psi^{(1)} \sim \exp\left(-\frac{me^{\theta}z^2}{2}\right)\left(z^{-1}\mathbf{e}_0^{(1)} + \mathbf{e}_1^{(1)} + z\mathbf{e}_2^{(1)}\right), \quad \text{as } |z| \to \infty, \tag{5.7.18}$$

$$\Psi^{(2)} \sim i\sqrt{3}\exp\left(-\frac{me^{\theta}z^2}{2}\right)\left(z^{-1}\mathbf{e}_0^{(2)} + \mathbf{e}_1^{(2)} + z\mathbf{e}_2^{(2)}\right), \quad \text{as } |z| \to \infty, \tag{5.7.19}$$

where $\{\mathbf{e}_J^{(a)}\}_{J=0}^{\dim V^{(a)}-1}$ is an orthonormal basis for the vector space $V^{(a)}$. The definitions of $R^{(1)}(\theta)$ and $R^{(2)}(\theta)$ then imply the identification (5.7.14). The $A_2^{(1)}$ BAEs then reduce to two copies of the $A_1^{(1)}$ BAE (5.7.5).

5.7.2 Equivalence of the integrals of motion

We define the a-function (5.6.1) from the $A_1^{(1)}$ BAEs (5.7.5)

$$a(\theta) = e^{i\pi(2l+1)} \frac{Q^{(1)}(\theta - i\pi)}{Q^{(1)}(\theta + i\pi)}. \tag{5.7.20}$$

The R-functions, as they satisfy the same BAEs, also have the same associated a-function. The procedure we followed in section 5.6 is now applied to derive the $A_1^{(1)}$ and $A_2^{(1)}$ integrals of motion for the respective values of M we have considered in this section.

The $A_1^{(1)}$ local integrals of motion $\mathfrak{I}_p$ for $M = 3$ are

$$\mathfrak{I}_p = \int_{-\infty}^{\infty} \frac{(-1)^p \sin\frac{p\pi}{2}}{\pi \sin\frac{p\pi}{6}} e^{p(\theta'-i0)} \operatorname{Im}\log\left(1 + a(\theta' - i0)\right) \mathrm{d}\theta', \quad p \in \mathbb{N}, \tag{5.7.21}$$

and the $A_1^{(1)}$ non-local integrals of motion $\mathfrak{S}_q$ for $M = 1$ are

$$\mathfrak{S}_q = \int_{-\infty}^{\infty} \frac{6(-1)^q \sin 3q\pi}{\pi \sin 3q\pi \sin 6q\pi} e^{6q(\theta'-i0)} \operatorname{Im} \log\left(1 + a(\theta' - i0)\right) \mathrm{d}\theta', \quad q \in \mathbb{N}. \quad (5.7.22)$$

The $A_2^{(1)}$ integrals of motion $\widehat{\mathfrak{I}}_p^{(m)}$, $\widehat{\mathfrak{S}}_q^{(m)}$ for $M = 1$ are found by setting $a^{(1)}(\theta) = a^{(2)}(\theta)$ and $M = 1$ in the expressions (5.6.57) and (5.6.58). We find

$$\widehat{\mathfrak{I}}_p^{(m)} = \int_{-\infty}^{\infty} \frac{(-1)^p \sin \frac{pm\pi}{3} \left(\sin \frac{p\pi}{3} + \sin \frac{2p\pi}{3}\right)}{\pi \sin^2 \frac{p\pi}{3}} e^{p(\theta'-i0)} \operatorname{Im} \log\left(1 + a(\theta' - i0)\right) \mathrm{d}\theta', \quad (5.7.23)$$

$$\widehat{\mathfrak{S}}_q^{(m)} = \int_{-\infty}^{\infty} \frac{3(-1)^q \left(\sin q\pi + \sin 2q\pi\right)}{\pi \sin 3q\pi} e^{3q(\theta'-i0)} \operatorname{Im} \log\left(1 + a(\theta' - i0)\right) \mathrm{d}\theta', \quad (5.7.24)$$

where $3 \nmid p$, and $q \in \mathbb{N}$. Comparing the $A_1^{(1)}$ and $A_2^{(1)}$ integrals of motion, we find

$$\mathfrak{I}_{3p+1} = \widehat{\mathfrak{I}}_{3p+1}^{(m)}, \quad \mathfrak{I}_{3p+2} = \widehat{\mathfrak{I}}_{3p+2}^{(m)}, \quad \mathfrak{S}_q = \widehat{\mathfrak{S}}_{2q}^{(m)}, \quad (5.7.25)$$

the $A_1^{(1)}$ $M = 3$ integrals of motion are completely contained in the set of $A_2^{(1)}$ $M = 1$ integrals of motion.

5.8 T-functions: fusion relations and TQ-relations

The Q-functions discussed in the previous sections define T-functions, which satisfy certain functional identities. In this section, we shall define these T-functions and demonstrate that they satisfy two such identities- the fusion relations and the TQ-relations.

5.8.1 Definitions

The main building block of the functions defined in this section will be the Q-functions for the linear problem in the first fundamental representation, $Q_j^{(1)}(\theta)$. It will be convenient to rescale the spectral parameter θ in the arguments of these Q-functions. The effect on the Q-functions is given by:

$$Q_j^{(1)}\left(\theta + \frac{i\pi\alpha}{hM}\right) \to Q_j^{(1)}\left(u+\alpha\right). \qquad (5.8.1)$$

It will also be useful to have a notation for a column vector of Q-functions:

$$\vec{Q}(u) = \left(Q_0^{(1)}(u), Q_1^{(1)}(u), \dots, Q_r^{(1)}(u)\right)^T. \qquad (5.8.2)$$

We define a particular determinant constructed from these vectors of Q-functions using notation given in [5]:

$$T_{(\mu_0,\mu_1,\dots,\mu_r)}(u) = \frac{1}{z_0}\sum_{i_0,\dots,i_r=0}^{r} \varepsilon_{i_0 i_1 \dots i_r}\prod_{k=0}^{r} Q_{i_k}^{(1)}(u+2(\mu_k+r/2-k))$$
$$= \frac{1}{z_0}\det\left(\vec{Q}(u+2\mu_0+r), \vec{Q}(u+2\mu_1+r-2), \dots, \vec{Q}(u+2\mu_r-r)\right), \qquad (5.8.3)$$

where

$$z_0 = \det\left(\vec{Q}(u), \vec{Q}(u-2), \dots, \vec{Q}(u-2r)\right) = i^{r(r+1)/2}(r+1)^{\frac{r+1}{2}}, \qquad (5.8.4)$$

which is the quantum Wronskian as discussed in section 5.4.3. We note that the quantum Wronskian (5.8.4) holds for all values of u.

The T-functions are then defined as certain values of the determinant (5.8.3):

$$T_m^{(a)}(u) = T_{(m,m,\dots,m,0,\dots,0)}(u-m-h/2+a), \qquad (5.8.5)$$

where $0 \le a \le h$, $h = r+1$ is the Coxeter number of A_r, and there are a components containing m in the vector m in the vector $(m, m, \dots, m, 0, \dots, 0)$. We also note

$$\forall\, m, u,\ T_m^{(0)}(u) = T_m^{(r+1)}(u) = 1, \tag{5.8.6}$$

as a consequence of the quantum Wronskian (5.8.4).

The T-functions (5.8.5) may also be written in terms of the $Q^{(a)}$ functions associated with the other fundamental representations with highest weights ω_a. We recall the definition of the $Q^{(a)}$ functions

$$\Psi^{(a)} = \sum_{I=0}^{\binom{r+1}{a}-1} Q_I^{(a)}(u)\Xi_I^{(a)}, \tag{5.8.7}$$

and recall the wedge product construction of the solutions $\Psi^{(a)}$ and $\Xi^{(a)}$:

$$\Psi^{(a)} = \Psi^{(1)}_{\frac{1-a}{2}} \wedge \dots \wedge \Psi^{(1)}_{\frac{a-1}{2}}, \tag{5.8.8}$$

$$\Xi_I^{(a)} = \Xi_{i_1}^{(1)} \wedge \dots \Xi_{i_a}^{(1)}, \tag{5.8.9}$$

where $\{i_1, \dots, i_a\}$ is a subset of $\{0, 1, \dots, r\}$, and the subsets are associated with an integer I by the standard lexicographical ordering (5.3.47). We combine (5.8.7) and the wedge product constructions of $\Psi^{(a)}$ and $\Xi^{(a)}$ with the original definition of the $Q^{(1)}$ functions

$$\Psi^{(1)} = \sum_{j=0}^{r} Q_j^{(a)}(u)\Xi_j^{(a)}, \tag{5.8.10}$$

to find

$$Q_I^{(a)}(u) = \begin{vmatrix} Q_{i_1}^{(1)}(u+a-1) & Q_{i_1}^{(1)}(u+a-3) & \cdots & Q_{i_1}^{(1)}(u-a+1) \\ Q_{i_2}^{(1)}(u+a-1) & Q_{i_2}^{(1)}(u+a-3) & \cdots & Q_{i_2}^{(1)}(u-a+1) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{i_a}^{(1)}(u+a-1) & Q_{i_a}^{(1)}(u+a-3) & \cdots & Q_{i_a}^{(1)}(u-a+1) \end{vmatrix}. \quad (5.8.11)$$

Applying a generalised Laplace expansion to the determinants $T_m^{(a)}(u)$ by expanding over a-by-a minors, the T-functions $T_m^{(a)}(u)$ may be written in the convenient form

$$T_m^{(a)}(u) = \sum_{I,J=0}^{\binom{r+1}{a}-1} \epsilon_{IJ} Q_I^{(a)}(u+m+h/2) Q_J^{(h-a)}(u-m-h/2) \quad (5.8.12)$$

where ϵ_{IJ} is a truncated notation for the Levi-Civita symbol $\epsilon_{i_1 i_2 \dots i_a j_1 \dots j_{h-a}}$. This form of the T-function will be particularly useful in the study of TQ-relations.

5.8.2 Fusion relations

The T-functions satisfy the fusion relations

$$T_m^{(a)}(u+1) T_m^{(a)}(u-1) = T_{m+1}^{(a)}(u) T_{m-1}^{(a)}(u) + T_m^{(a+1)}(u) T_m^{(a-1)}(u). \quad (5.8.13)$$

To verify this, we write the T-functions explicitly in determinant form:

$$\begin{aligned} &z_0 T_m^{(a)}(u) \\ &= \det\Bigg(\underbrace{\vec{Q}(u+h/2+m+a-1), \dots, \vec{Q}(u+h/2+m-a+1),}_{a \text{ terms}} \\ &\qquad \underbrace{\vec{Q}(u+h/2-m-a-1), \dots, \vec{Q}(u-3h/2-m+a+1)}_{h-a \text{ terms}}\Bigg). \quad (5.8.14) \end{aligned}$$

Let

$$A_k = \vec{Q}(u + h/2 + m + a - 2k), \tag{5.8.15}$$

$$B_k = \vec{Q}(u + h/2 - m - a - 2k). \tag{5.8.16}$$

The fusion relations (5.8.13) are then equivalent to the determinant identity

$$\begin{aligned}
&\det(A_0, A_1, \ldots, A_{a-1}, B_0, B_1, \ldots, B_{h-a-1}) \det(A_1, \ldots, A_a, B_1, \ldots, B_{h-a}) \\
&- \det(A_0, A_1, \ldots, A_{a-1}, B_1, \ldots, B_{h-a}) \det(A_1, \ldots, A_a, B_0, B_1, \ldots, B_{h-a-1}) \\
&- \det(A_0, A_1, \ldots, A_a, B_1, \ldots, B_{h-a-1}) \det(A_1, \ldots, A_{a-1}, B_0, B_1, \ldots, B_{h-a}) = 0.
\end{aligned} \tag{5.8.17}$$

To prove the fusion relations (5.8.13), it remains only to demonstrate the identity (5.8.17) for all sets of vectors $\{A_0, \ldots, A_a, B_0, \ldots, B_{h-a}\} \subset \mathbb{R}^h$. To do this, we consider the vector space $\mathbb{R}^{h+2}$ with basis $\{e_i\}_{i=0}^{h+1}$. Using the wedge product defined on this vector space, we construct the h-vector

$$x = \sum_{0 \le i < j \le h+1} \hat{C}_{ij} e_0 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_{h+1}, \tag{5.8.18}$$

where the hats on the vectors e_i, e_j indicate those vectors are to be omitted from the product. We also choose the coefficients $\hat{C}_{ij}$ to be determinants of vectors C_i in $\mathbb{R}^h$:

$$\hat{C}_{ij} = \det\left(C_0, \ldots, \hat{C}_i, \ldots, \hat{C}_j, \ldots, C_{h+1}\right), \tag{5.8.19}$$

where the hats on the vectors C_i, C_j indicate these vectors are omitted from the determinant.

Given a vector space V of dimension n and its associated spaces of k-vectors $\bigwedge^k V$, there exists an isomorphism between the space of k-vectors known as the

Hodge star operator, defined as

$$\star : \bigwedge^{k} V \to \bigwedge^{h-k} V,$$
$$\star \left(e_{i_1} \wedge \cdots \wedge e_{i_k}\right) = \epsilon_{i_1 i_2 \dots i_a j_1 \dots j_{h-k}} e_{j_1} \wedge \dots e_{j_{h-k}}, \tag{5.8.20}$$

where $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_{h-k}$. We apply this isomorphism to the h-vector x defined in (5.8.18), producing a 2-vector:

$$\star x = \sum_{0 \leq i < j \leq h+1} \hat{C}_{ij} (-1)^{i+j-1} e_i \wedge e_j. \tag{5.8.21}$$

As the wedge product is an antisymmetric operation, we immediately find the 4-vector $\star x \wedge \star x = 0$:

$$\star x \wedge \star x = \sum_{\substack{0 \leq i < j \leq h+1, \\ 0 \leq k < l \leq h+1}} \hat{C}_{ij} \hat{C}_{kl} (-1)^{i+j+k+l} e_i \wedge e_j \wedge e_k \wedge e_l = 0. \tag{5.8.22}$$

Each component of this 4-vector is then equal to zero; we then consider the coefficient of the $e_0 \wedge e_a \wedge e_{a+1} \wedge e_{h+1}$ term in this 4-vector to find:

$$\hat{C}_{0a} \hat{C}_{(a+1)(h+1)} - \hat{C}_{0(a+1)} \hat{C}_{a(h+1)} + \hat{C}_{0(h+1)} \hat{C}_{a(a+1)} = 0. \tag{5.8.23}$$

Now choose the arbitrary vectors C_k to be

$$C_k = \begin{cases} A_k & \text{if } 0 \leq k \leq a \\ B_k & \text{if } a+1 \leq k \leq h+1. \end{cases} \tag{5.8.24}$$

Rewrite the $\hat{C}_{ij}$ terms in their full form to recover the determinant identity (5.8.17), and then apply the substitutions (5.8.15) and (5.8.16) to recover the fusion relations

$$T_m^{(a)}(u+1)T_m^{(a)}(u-1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a+1)}(u)T_m^{(a-1)}(u). \tag{5.8.25}$$

5.8.3 TQ-relations

Another set of functional relations satisfied by the T-functions and the Q-functions are the TQ-relations. TQ-relations arose originally in Baxter's solution of the six-vertex ice-type integrable model [4] and have since become a staple of integrable model theory, as the TQ-relations contain information on the allowed energy levels of the system in question. We will further demonstrate the ODE/IM correspondence for the $A_r^{(1)}$ Toda theory by constructing the TQ-relations associated with this model.

The TQ-relations were constructed from the relevant differential equation for the $A_1^{(1)}$ case in [45], and similar relations were found for the $A_2^{(1)}$ case in [5]. In this section, we will follow the analysis in [42] to exhibit analogous TQ-relations for the remaining algebras $A_r^{(1)}$ where $r > 2$. We will also check that the TQ-relations we derive here match those found in [45, 5].

We recall the Q-functions for the linear problem in the representation $V^{(a)}$ are given by $Q_I^{(a)}(u)$, where $I = 0, 1, \ldots, \binom{h}{a} - 1$. With these, we define a column vector of values of the function $Q_J^{(a)}$:

$$\vec{X}^{(a)}(J) = \left(Q_J^{(a)}(u), Q_J^{(a)}(u+2), \ldots, Q_J^{(a)}\left(u + 2\binom{h}{a}\right)\right)^T, \tag{5.8.26}$$

and then, following [42], we define a function of indices J_k, K as a $\binom{h}{a} + 1 \times \binom{h}{a} + 1$

determinant of certain vectors of the form (5.8.26):

$$\mathcal{Q}\left(I_1,\ldots I_{\binom{h}{a}},K\right)=\det\left(\vec{X}^{(a)}(I_1),\vec{X}^{(a)}(I_2),\ldots,\vec{X}^{(a)}(I_{\binom{h}{a}}),\vec{X}^{(a)}(K)\right). \quad (5.8.27)$$

For any choice of the indices I_k, K, $\mathcal{Q}=0$. We exploit this by contracting $\mathcal{Q}$ with specially chosen $Q^{(h-a)}$-functions:

$$\sum_{I,J}\prod_{k=1}^{\binom{h}{a}}\frac{1}{z_0}\left(\epsilon_{I_kJ_k}Q^{(h-a)}(u+h-2k)\right)\mathcal{Q}(I_1,\ldots,I_{\binom{h}{a}},K)=0, \qquad (5.8.28)$$

where the sum is over all indices $\{I_1,\ldots,I_{\binom{h}{a}},J_1,\ldots,J_{\binom{h}{a}}\}$. By expanding the determinant $\mathcal{Q}$ and using the definition of $T_m^{(a)}(u)$ (5.8.12), we find the TQ-relations in determinant form:

$$\begin{vmatrix} T^{(a)}_{1-h}(u+\frac{h}{2}-1) & \ldots & T^{(a)}_{\binom{h}{a}-h}(u+\frac{h}{2}-\binom{h}{a}) & Q^{(a)}_K(u) \\ T^{(a)}_{2-h}(u+\frac{h}{2}) & \ldots & T^{(a)}_{\binom{h}{a}-h+1}(u+\frac{h}{2}-\binom{h}{a}+1) & Q^{(a)}_K(u+2) \\ \vdots & \ddots & \vdots & \vdots \\ T^{(a)}_{\binom{h}{a}-h+1}(u+\frac{h}{2}+\binom{h}{a}-1) & \ldots & T^{(a)}_{2\binom{h}{a}-h}(u+\frac{h}{2}) & Q^{(a)}_K(u+2\binom{h}{a}) \end{vmatrix}=0. \qquad (5.8.29)$$

We check these TQ-relations agree with the $h=2$ case in [45] and the $h=3$ case in [5].

$h=2$

In the case $h=2$, (5.8.29) becomes

$$\begin{vmatrix} T^{(1)}_{-1}(u) & T^{(1)}_{0}(u-1) & Q^{(1)}_k(u) \\ T^{(1)}_{0}(u+1) & T^{(1)}_{1}(u) & Q^{(1)}_k(u+2) \\ T^{(1)}_{1}(u+2) & T^{(1)}_{2}(u+1) & Q^{(1)}_k(u+4) \end{vmatrix}=0. \qquad (5.8.30)$$

We note from the definition of $T_m^{(a)}(u)$ (5.8.12),

$$T_{-1}^{(1)}(u) = 0, \quad T_0^{(1)}(u-1) = T_0^{(1)}(u+1) = 1. \tag{5.8.31}$$

Expanding (5.8.30) along the rightmost column, we find

$$Q_k^{(1)}(u) \begin{vmatrix} 1 & T_1^{(1)}(u) \\ T_1^{(1)}(u+2) & T_2^{(1)}(u+1) \end{vmatrix} - Q_k^{(1)}(u+2) \begin{vmatrix} 0 & 1 \\ T_1^{(1)}(u+2) & T_2^{(1)}(u+1) \end{vmatrix}$$
$$+ Q_k^{(1)}(u+4) \begin{vmatrix} 0 & 1 \\ 1 & T_1^{(1)}(u) \end{vmatrix} = 0. \tag{5.8.32}$$

The fusion relations (5.8.13) for the case $h = 2, a = 1$ are

$$T_m^{(1)}(u+1)T_m^{(1)}(u-1) = 1 + T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u). \tag{5.8.33}$$

Substituting these into (5.8.32) and shifting $u \to u-2$, we find the TQ-relation in [45]:

$$T_1^{(1)}(u)Q_k^{(1)}(u) = Q_k^{(1)}(u+2) + Q_k^{(1)}(u-2). \tag{5.8.34}$$

$h = 3$**,** $a = 1$

Equation (5.8.29) in the case $h = 3$, $a = 1$ is given by:

$$\begin{vmatrix} T_{-2}^{(1)}(u+1/2) & T_{-1}^{(1)}(u-1/2) & T_0^{(1)}(u-3/2) & Q_k^{(1)}(u) \\ T_{-1}^{(1)}(u+3/2) & T_0^{(1)}(u+1/2) & T_1^{(1)}(u-1/2) & Q_k^{(1)}(u+2) \\ T_0^{(1)}(u+5/2) & T_1^{(1)}(u+3/2) & T_2^{(1)}(u+1/2) & Q_k^{(1)}(u+4) \\ T_1^{(1)}(u+7/2) & T_2^{(1)}(u+5/2) & T_3^{(1)}(u+3/2) & Q_k^{(1)}(u+6) \end{vmatrix} = 0. \tag{5.8.35}$$

Shifting $u \to u-3$ to match [5] and expanding the determinant (5.8.35) along the rightmost columns, we recover the following relation after a rearranging of Q-functions:

$$\begin{aligned}
&Q_k^{(1)}(u+3)T_1^{(0)}(u+3/2)Q^{(3)}(u-3)Q^{(3)}(u-5)\\
&-Q_k^{(1)}(u+1)T_1^{(1)}(u+1/2)Q^{(3)}(u-3)Q^{(3)}(u-5)\\
&+Q_k^{(1)}(u-1)T_1^{(2)}(u-1/2)Q^{(3)}(u-3)Q^{(3)}(u-5)\\
&-Q_k^{(1)}(u-3)T_1^{(3)}(u-3/2)Q^{(3)}(u-3)Q^{(3)}(u-5)=0.
\end{aligned} \tag{5.8.36}$$

Noting that the $Q^{(3)}$, $T^{(0)}$ and $T^{(3)}$ functions are all quantum Wronskians, they can be removed from this expression. We then find

$$\begin{aligned}
&Q_k^{(1)}(u+3)-Q_k^{(1)}(u+1)T_1^{(1)}(u+1/2)\\
&+Q_k^{(1)}(u-1)T_1^{(2)}(u-1/2)-Q_k^{(1)}(u-3)=0,
\end{aligned} \tag{5.8.37}$$

which is exactly equivalent to equation (5.7) in [5].

5.9 Conclusions

In this chapter, we began with a differential equation related to the $A_r^{(1)}$ affine Toda field theory. This differential equation had an associated linear problem, which we used to construct Q-functions. These Q-functions contained information on the associated quantum integrable model; a set of Bethe ansatz equations, the integrals of motion associated with this model, fusion relations between T-functions (related to the associated transfer matrices of this model) and a set of TQ-relations. We have therefore demonstrated an example of the ODE/IM correspondence between differential equations related to $A_r^{(1)}$ affine Toda field theories and massive quantum integrable models with $A_r^{(1)}$ symmetry.

Chapter 6

On the massive ODE/IM correspondence for the simply-laced Lie algebras

6.1 Introduction

In this chapter, we will generalise the analysis of the massive ODE/IM correspondence in Chapter 5 to the remaining simply-laced Lie algebras; the classical family $D_r^{(1)}$ for $r \geq 3$, and the exponential Lie algebras $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$.

The Lie algebra notation [37, 38] we have adopted now begins to pay dividends, as many of the calculations we will need to perform in our analysis of the $D_r^{(1)}$ case will follow immediately from the relevant calculations in the $A_r^{(1)}$ case.

6.2 The $D_r^{(1)}$ massive ODE/IM correspondence

6.2.1 The $D_r^{(1)}$ linear problem

Representations of $D_r^{(1)}$

We begin with the gauge-transformed linear problem $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$, with $\widetilde{A}$ defined as in (5.2.14)

$$\widetilde{A} = \beta\partial\phi \cdot H + me^{\theta}\left[\sum_{i=1}^{r} \sqrt{n_i^\vee} E_{\alpha_i} + p(z)\sqrt{n_0^\vee} E_{\alpha_0}\right]. \tag{6.2.1}$$

Since we are now concerned with the affine Lie algebras $D_r^{(1)}$, we substitute the relevant dual Kac labels $n_0^\vee, n_i^\vee$ as given in section 4.2.2, giving the $D_r^{(1)}$ linear problem

$$\left(\partial + \beta\partial\phi \cdot H + me^{\theta}\left(p(z)E_{\alpha_0} + E_{\alpha_1} + \sqrt{2}\sum_{i=2}^{r-2} E_{\alpha_i} + E_{\alpha_{r-1}} + E_{\alpha_r}\right)\right)\widetilde{\Psi} = 0. \tag{6.2.2}$$

To make further progress, we must choose a particular representation of the Lie algebra $D_r^{(1)}$, making the linear problem (6.2.2) into a system of differential equations. We will be concerned with representations of $D_r^{(1)}$ constructed from fundamental representations of the simple Lie algebra D_r; these are the representations with highest weight ω_i $(i = 1, \ldots, r)$, where ω_i are the fundamental weights of the simple Lie algebra D_r. As in chapter 4, we denote these representations by $L(\omega_i)$.

As described in [47] and section 4.2.2, we construct evaluation representations $L(\omega_i)_k$ of the affine Lie algebra $D_r^{(1)}$ by mapping the generator $E_{\alpha_0} \to e^{2\pi ik}E_{\alpha_0}$, where $k \in \mathbb{Z}/2$. We then define the representations $V^{(a)}$ of the affine Lie algebra

$D_r^{(1)}$ by the following:

$$V^{(a)} = L(\omega_a)_{\frac{a-1}{2}}, \quad (a = 1, \ldots, r-2), \quad V^{(r-1)} = L(\omega_{r-1})_{r/2}, \quad V^{(r)} = L(\omega_r)_{r/2}. \tag{6.2.3}$$

The representations $L(\omega_1), \ldots, L(\omega_{r-2})$ can also be generated by wedge products of elements of the vector space $L(\omega_1)$, as for the fundamental representations of A_r. The provenance of the remaining representations $L(\omega_{r-1})$ and $L(\omega_r)$ is slightly more complicated; they are known as the *half spin representations* of D_r. The elements of these representations correspond to generators of the action of the rotation group on spinors, which arise naturally in the discussion of fermions in quantum field theory. For more details on their explicit construction in terms of Pauli matrices see Appendix B of [57].

In this chapter, we will mostly be concerned with the smallest non-trivial representation $V^{(1)} = L(\omega_1)_0 = L(\omega_1)$. By the Weyl dimension formula (4.2.46), $L(\omega_1)$ is $2r$-dimensional, and its weights $\lambda_i^{(1)}$ are found using the algorithm described in section 4.2.1. They satisfy

$$\lambda_1^{(1)} = \omega_1, \tag{6.2.4}$$

$$\lambda_{i+1}^{(1)} = \lambda_i^{(1)} - \alpha_i, \quad (i = 1, \ldots, r-1), \tag{6.2.5}$$

$$\lambda_{2r+1-i}^{(1)} = -\lambda_i^{(1)}, \quad (i = 1, \ldots, r). \tag{6.2.6}$$

We choose a basis $\{\mathbf{e}_i^{(1)}\}_{i=1}^{2r}$ of the vector space $V^{(1)}$ such that the generators of the Cartan subalgebra are diagonalised

$$\begin{aligned} H_i \mathbf{e}_j^{(1)} &= (\lambda_j^{(1)})^i \mathbf{e}_i^{(1)}, \\ &= (\delta_{i,j} + \delta_{2r-i,j} - \delta_{i+1,j} - \delta_{2r+1-i,j}) \mathbf{e}_j^{(1)}, \end{aligned} \tag{6.2.7}$$

and the generators E_{α_i}, $(i = 1, \ldots, r)$, E_{α_0} of the Lie algebra $D_r^{(1)}$ are represented by the matrices with elements

$$E_{\alpha_i}\mathbf{e}_j^{(1)} = \delta_{i+1,j}\mathbf{e}_i^{(1)} + \delta_{2r+1-i,j}\mathbf{e}_{2r-i}^{(1)}, \tag{6.2.8}$$

$$E_{-\alpha_i}\mathbf{e}_j^{(1)} = \delta_{i,j}\mathbf{e}_{i+1}^{(1)} + \delta_{2r-i,j}\mathbf{e}_{2r+1-i}^{(1)}, \tag{6.2.9}$$

$$E_{\alpha_r}\mathbf{e}_j^{(1)} = \delta_{r+1,j}\mathbf{e}_{r-1}^{(1)} + \delta_{r+2,j}\mathbf{e}_r^{(1)}, \tag{6.2.10}$$

$$E_{-\alpha_r}\mathbf{e}_j^{(1)} = \delta_{r-1,j}\mathbf{e}_{r+1}^{(1)} + \delta_{r,j}\mathbf{e}_{r+2}^{(1)}, \tag{6.2.11}$$

$$E_{\alpha_0}\mathbf{e}_j^{(1)} = \delta_{1,j}\mathbf{e}_{2r-1}^{(1)} + \delta_{2,j}\mathbf{e}_{2r}^{(1)}. \tag{6.2.12}$$

We note that the above definitions (6.2.8)-(6.2.12) hold only for $r \geq 3$, as D_2 is no longer a semi-simple Lie algebra (specifically, $D_2 = A_1 \oplus A_1$.)

Using the algorithm in section 4.2.1, a similar procedure may be followed to generate explicit matrices for the representations $V^{(a)}$. We shall not demonstrate this here, as by the Weyl dimension formula (4.2.46), the dimensions of the 'higher' representations rapidly become large. The representation $V^{(1)}$ is sufficient to demonstrate all the major features of the $D_r^{(1)}$ case of the massive ODE/IM correspondence.

The linear problem in the representation $V^{(1)}$ and the pseudo-differential equation

Now that we have constructed an explicit representation of $D_r^{(1)}$ in the form of the matrices (6.2.7)-(6.2.12), we may write the gauge-transformed linear problem

$(\partial + \widetilde{A})\widetilde{\Psi} = 0$ as an explicit system of $2r$ coupled differential equations:

$$D(\lambda_1^{(1)})\widetilde{\psi}_1 + me^{\theta}\ \widetilde{\psi}_2 = 0, \tag{6.2.13}$$

$$D(\lambda_2^{(1)})\widetilde{\psi}_2 + me^{\theta}\sqrt{2}\ \widetilde{\psi}_3 = 0, \tag{6.2.14}$$

$$\vdots$$

$$D(\lambda_{r-2}^{(1)})\widetilde{\psi}_{r-2} + me^{\theta}\sqrt{2}\ \widetilde{\psi}_{r-1} = 0, \tag{6.2.15}$$

$$D(\lambda_{r-1}^{(1)})\widetilde{\psi}_{r-1} + me^{\theta}\ \widetilde{\psi}_r + me^{\theta}\ \widetilde{\psi}_{r+1} = 0, \tag{6.2.16}$$

$$D(\lambda_r^{(1)})\widetilde{\psi}_r + me^{\theta}\ \widetilde{\psi}_{r+2} = 0, \tag{6.2.17}$$

$$D(-\lambda_r^{(1)})\widetilde{\psi}_{r+1} + me^{\theta}\ \widetilde{\psi}_{r+2} = 0, \tag{6.2.18}$$

$$\vdots$$

$$D(-\lambda_3^{(1)})\widetilde{\psi}_{2r-2} + me^{\theta}\sqrt{2}\ \widetilde{\psi}_{2r-1} = 0, \tag{6.2.19}$$

$$D(-\lambda_2^{(1)})\widetilde{\psi}_{2r-1} + me^{\theta}\ \widetilde{\psi}_{2r} + me^{\theta}\ p(z)\widetilde{\psi}_1 = 0, \tag{6.2.20}$$

$$D(-\lambda_1^{(1)})\widetilde{\psi}_{2r} + me^{\theta}\ p(z)\widetilde{\psi}_2 = 0, \tag{6.2.21}$$

where the differential operator D is defined as

$$D(\lambda) = \partial + \beta\lambda \cdot \partial\phi, \tag{6.2.22}$$

and we have used the symmetry property of the weights of $V^{(1)}$ (6.2.6) to write $D(\lambda_{r+k}^{(1)})$ as $D(-\lambda_k^{(1)})$ for $k = 1, \ldots, r$.

Just as with the representation $V^{(1)}$ of $A_r^{(1)}$, the equations (6.2.13)-(6.2.21) can be combined into a single pseudo-differential equation involving $\widetilde{\psi}_1$, following the method in [37]. Let

$$D(\boldsymbol{\lambda}) = D(\lambda_r^{(1)}) \ldots D(\lambda_1^{(1)}), \tag{6.2.23}$$

and apply this differential operator to $\widetilde{\psi}_1$. Using (6.2.13)-(6.2.17)

$$\begin{aligned} D(\boldsymbol{\lambda})\widetilde{\psi}_1 &= D(\lambda_r^{(1)})\dots D(\lambda_1^{(1)})\widetilde{\psi}_1 && (6.2.24)\\ &= -me^{\theta} D(\lambda_r^{(1)})\dots D(\lambda_2^{(1)})\widetilde{\psi}_2 && (6.2.25)\\ &\vdots \\ &= (-me^{\theta})^{r-1} 2^{\frac{r-3}{2}} \left((-me^{\theta}\widetilde{\psi}_{r+2}) + D(\lambda_r^{(1)})\widetilde{\psi}_{r+1} \right). && (6.2.26) \end{aligned}$$

Applying (6.2.18) to (6.2.26),

$$D(\boldsymbol{\lambda})\widetilde{\psi}_1 = (-me^{\theta})^{r-1} 2^{\frac{r-3}{2}} \left(D(-\lambda_r^{(1)})\widetilde{\psi}_{r+1} + D(\lambda_r^{(1)})\widetilde{\psi}_{r+1} \right). \tag{6.2.27}$$

Using the definition of the differential operator D (6.2.22), we find

$$D(\boldsymbol{\lambda})\widetilde{\psi}_1 = (-me^{\theta})^{r-1} 2^{\frac{r-1}{2}} \partial \widetilde{\psi}_{r+1}. \tag{6.2.28}$$

Similarly, we combine the equations (6.2.18)-(6.2.21) to form another equation in terms of $\widetilde{\psi}_1$ and $\widetilde{\psi}_{r+1}$. Following the notation in [37], let

$$D(\boldsymbol{\lambda}^{\dagger}) = D(-\lambda_1^{(1)})\dots D(-\lambda_r^{(1)}), \tag{6.2.29}$$

and apply this operator to $\widetilde{\psi}_{r+1}$. Using (6.2.18)-(6.2.20) we find:

$$\begin{aligned} D(\boldsymbol{\lambda}^{\dagger})\widetilde{\psi}_{r+1} &= D(-\lambda_1^{(1)})\dots D(-\lambda_r^{(1)})\widetilde{\psi}_{r+1} && (6.2.30)\\ &= (-me^{\theta})^{r-1} 2^{\frac{r-3}{2}} D(-\lambda_1^{(1)})(\widetilde{\psi}_{2r} + p(z)\widetilde{\psi}_1). && (6.2.31) \end{aligned}$$

To simplify the right hand side of (6.2.31), we use equations (6.2.13) and (6.2.21) and the definition of the differential operator $D(-\lambda_1^{(1)})$ (6.2.22):

$$D(-\lambda_1^{(1)})(\widetilde{\psi}_{2r} + p(z)\widetilde{\psi}_1) = 2p(z)\,\partial\widetilde{\psi}_1 + p'(z)\widetilde{\psi}_1 = 2\sqrt{p(z)}\,\partial\Big(\sqrt{p(z)}\widetilde{\psi}_1\Big). \tag{6.2.32}$$

We substitute (6.2.32) into (6.2.31) to find

$$D(\boldsymbol{\lambda}^{\dagger})\widetilde{\psi}_{r+1} = (-me^{\theta})^{r-1}2^{\frac{r-1}{2}}\sqrt{p(z)}\;\partial\Big(\sqrt{p(z)}\widetilde{\psi}_{1}\Big)\,. \tag{6.2.33}$$

The final step is apply the inverse differential operator ∂^{-1} to both sides of (6.2.28) and then apply the differential operator $D(\boldsymbol{\lambda}^{\dagger})$:

$$D(\boldsymbol{\lambda}^{\dagger})\partial^{-1}D(\boldsymbol{\lambda})\widetilde{\psi}_{1} = (-me^{\theta})^{r-1}2^{\frac{r-1}{2}}D(\boldsymbol{\lambda}^{\dagger})\widetilde{\psi}_{r+1}, \tag{6.2.34}$$

and then finally apply (6.2.33) to derive a pseudo-differential equation for the top component $\widetilde{\psi}_1$:

$$D(\boldsymbol{\lambda}^{\dagger})\partial^{-1}D(\boldsymbol{\lambda})\widetilde{\psi}_{1} = (-me^{\theta})^{2r-2}2^{r-1}\sqrt{p(z)}\;\partial\Big(\sqrt{p(z)}\widetilde{\psi}_{1}\Big)\,, \tag{6.2.35}$$

which is the equation (4.7) found in [37].

The asymptotic behaviour of the pseudo-differential equation (6.2.35) in the small-$|z|$ and large-$|z|$ limits could now be studied to define the $D_r^{(1)}$ Q-functions, which as for the $A_r^{(1)}$ case will be our main objects of study. The inverse differential operator ∂_z^{-1} obscures the analysis of the asymptotics of the solutions of this equation, and for this reason, we will consider the equivalent gauge-transformed linear problem $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$, using the WKB method for systems of linear equations given in section 4.3.2.

The massless limit

We now show equation (6.2.35) matches the the $D_r^{(1)}$ equation found in [19] for the massless limit of the $D_r^{(1)}$ ODE/IM correspondence. We change variables in the equation (6.2.35) and take the massless limit. Following [37], we make the change of variables

$$x = (me^{\theta})^{\frac{1}{M+1}} z, \quad E = s^{hM} (me^{\theta})^{\frac{hM}{M+1}}, \tag{6.2.36}$$

$$\bar{x} = (me^{-\theta})^{\frac{1}{M+1}} \bar{z}, \quad \bar{E} = s^{hM} (me^{-\theta})^{\frac{hM}{M+1}}. \tag{6.2.37}$$

and the change of notation

$$D_x(\lambda) = (me^{\theta})^{\frac{1}{M+1}} (\partial_x + \beta\lambda \cdot \partial_x \phi). \tag{6.2.38}$$

We rewrite (6.2.35) in these new variables, taking the massless limit. We first set $\bar{z} \to 0$ and $\theta \to \infty$ and let $z \to 0$ so that x and E remain finite. Using the small-$|z|$ limit of ϕ (5.2.17), $D_x(\lambda)$ becomes

$$D_x(\lambda) = \left(\frac{E}{s}\right)^{\frac{1}{hM}} \left(\partial_x + \frac{\beta\lambda \cdot g}{x}\right). \tag{6.2.39}$$

Setting

$$D_r(\mathbf{g}) := D_x(\lambda_r^{(1)}) \cdot D_x(\lambda_1^{(1)}), \quad D_r(\mathbf{g}^{\dagger}) := D_x(-\lambda_1^{(1)}) \cdot D_x(-\lambda_r^{(1)}), \tag{6.2.40}$$

to match the notation in [18, 19], the massless pseudo-differential equation is then given by

$$D_r(\mathbf{g}^{\dagger}) \partial_x^{-1} D_r(\mathbf{g}) \widetilde{\psi}_1 = 2^{r-1} \sqrt{p(x,E)}\, \partial_x \left(\sqrt{p(x,E)}\, \widetilde{\psi}_1\right) \tag{6.2.41}$$

where $p(x,E) = x^{hM} - E$. This equation corresponds to the $D_r^{(1)}$ equation found in [19] for the massless limit of the $D_r^{(1)}$ ODE/IM correspondence, up to factors of powers of 2 that can be absorbed into $p(x,E)$.

6.2.2 Asymptotics of the $V^{(1)}$ linear problem

We will need to consider the small-$|z|$ and large-$|z|$ solutions of the gauge-transformed linear problem (6.2.2). After undoing the gauge transform, the solutions $\{\Xi_i^{(1)}\}_{i=1}^{2r}$ with particular small-$|z|$ behaviour form a basis of the space of solutions of the linear problem. We write the large-$|z|$ solution Ψ with the most rapid decay to zero on the positive real axis as $|z| \to \infty$ as a sum of small-$|z|$ solutions. The Q-functions are defined as the coefficients of this expansion:

$$\Psi^{(1)}(\varphi|\theta) = \sum_{i=1}^{2r} Q_i^{(1)}(\theta)\, \Xi_i^{(1)}(\varphi|\theta). \tag{6.2.42}$$

We could also consider the linear problems in the larger representations $V^{(a)}$ of $D_r^{(1)}$. The small-$|z|$ and large-$|z|$ solutions of these linear problems then define the $Q^{(a)}$ functions as coefficients of a similar expansion to (6.2.42), as we saw in the $A_r^{(1)}$ case. We will define these asymptotics in terms of constants w_a which will satisfy a certain linear system of equations derived from the Ψ-system, found in equation (3.8) of [38].

$V^{(1)}$ small-$|z|$ asymptotics

In the small-$|z|$ limit, the gauge-transformed linear problem (6.2.2) becomes

$$\left(\partial_z + \frac{\beta g \cdot H}{z}\right) \widetilde{\Psi} = 0, \tag{6.2.43}$$

where we have used the small-$|z|$ behaviour of the solution to the modified affine Toda field equation ϕ given in (5.2.17). Recalling the action of the Cartan subalgebra generators H_i on the basis vectors of $V^{(1)}$ $\{\mathbf{e}_j^{(1)}\}_{j=1}^{2r}$, $H_i \mathbf{e}_j^{(1)} = (\lambda_j^{(1)})^i \mathbf{e}_i^{(1)}$, the solutions of the small-$|z|$ linear problem (6.2.43) are given by

$$\widetilde{\Psi} = z^{-\beta g\cdot\lambda_j^{(1)}} c_j \mathbf{e}_j^{(1)}, \tag{6.2.44}$$

where c_j are arbitrary constants. We therefore choose a basis of solutions to the original gauge-transformed linear problem (6.2.2) to be the set $\{\widetilde{\Xi}_j\}_{j=1}^{2r}$, with $\widetilde{\Xi}_j$ defined to have the asymptotic behaviour

$$\widetilde{\Xi}_j^{(1)} \sim z^{-\beta g\cdot\lambda_j^{(1)}} e^{-\beta\theta g\cdot\lambda_j^{(1)}} \mathbf{e}_j^{(1)} \quad \text{as } |z| \to 0. \tag{6.2.45}$$

We have chosen $c_j = e^{-\beta\theta g\cdot\lambda_j^{(1)}}$ to ensure invariance of these solutions under Symanzik rotation (5.2.11). It will also be useful to find the small-$|z|$ solutions to the original linear problem $(\partial_z + A)\Psi = 0$ given in (5.2.9) (setting all constants in (5.2.9) to their relevant $D_r^{(1)}$ values), which are accessed by applying the matrix $U^{-1} = e^{\beta\phi\cdot H/2}$ to the gauge-transformed solutions $\widetilde{\Xi}_j^{(1)}$. As we are working in the small-$|z|$ limit, we simplify U^{-1} by applying the small-$|z|$ behaviour of ϕ (5.2.17):

$$U^{-1} \sim (z\bar{z})^{\beta g\cdot H/2} \quad \text{as } |z| \to 0. \tag{6.2.46}$$

Applying this matrix to (6.2.45), we find the small-$|z|$ solutions to the original $D_r^{(1)}$ linear problem (5.2.9) behave as

$$\Xi_j^{(1)} \sim e^{-(\theta+i\varphi)\beta g\cdot\lambda_j^{(1)}} \mathbf{e}_j^{(1)}, \quad \text{as } |z| \to 0, \tag{6.2.47}$$

where we work in polar co-ordinates $z = |z|e^{i\varphi}$.

$V^{(1)}$ large-$|z|$ asymptotics

The term proportional to H in the gauge-transformed linear problem (6.2.2) is dropped in the large-$|z|$ limit. (6.2.2) then becomes

$$\left(\partial_z + me^{\theta}\left(p(z)E_{\alpha_0} + E_{\alpha_1} + \sqrt{2}\sum_{i=2}^{r-2} E_{\alpha_i} + E_{\alpha_{r-1}} + E_{\alpha_r}\right)\right)\widetilde{\Psi} = 0, \qquad (6.2.48)$$

due to the large-$|z|$ behaviour of ϕ (5.2.16):

$$\beta\partial_z\phi \cdot H \sim \frac{M\rho^{\vee}\cdot H}{z}, \quad \text{as } |z| \to \infty. \qquad (6.2.49)$$

We are interested in the particular (subdominant) solution of (6.2.48) that decays most rapidly to zero on the positive real axis. In section 4.3.2 solutions of linear systems of the form $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ have asymptotic solutions in the limit $|z| \to \infty$ of the form

$$\widetilde{\Psi} \sim \mathbf{v}(z)\exp\left(-\int^{z}\sigma(u)\mathrm{d}u\right), \quad \text{as } |z| \to \infty, \qquad (6.2.50)$$

where $\mathbf{v}(z)$ is a particular normalisation of an eigenvector of $\widetilde{A}$ and $\sigma(z)$ is its associated eigenvalue. The subdominant solution corresponds to the eigenvalue with largest real part of the sum of the matrices in (6.2.48). Following the method in section 4.3.2, the subdominant solution is given by

$$\widetilde{\Psi}^{(1)} \sim f(\bar{z})\exp\left(-m\sqrt{2}e^{\theta}\frac{z^{M+1}}{M+1}\right)\left(z^{-(r-1)M}\mathbf{e}_1^{(1)} + \sqrt{2}\sum_{j=2}^{r-1} z^{-(r-j)M}\mathbf{e}_j^{(1)}\right.$$
$$\left. + \mathbf{e}_r^{(1)} + \mathbf{e}_{r+1}^{(1)} + \sqrt{2}\sum_{j=2}^{r-1} z^{(j-1)M}\mathbf{e}_{r+j}^{(1)} + z^{(r-1)M}\mathbf{e}_{2r}^{(1)}\right), \quad \text{as } |z| \to \infty,$$
$$(6.2.51)$$

where $f(\bar{z})$ is an arbitrary function in the conjugate variable $\bar{z}$, which is fixed by considering the analogous subdominant solution to the conjugate linear problem $(\partial_{\bar{z}} + \bar{A})\Psi = 0$. Firstly, as for the $A_r^{(1)}$ case we undo the gauge transform by applying the matrix $U^{-1} = e^{\beta\phi\cdot H/2}$ to the subdominant solution (6.2.51). Recalling the large-$|z|$ behaviour of ϕ,

$$U^{-1}\mathbf{e}_j^{(1)} \sim (z\bar{z})^{M\rho^\vee\cdot H/2}\mathbf{e}_j^{(1)}, \quad \text{as } |z| \to \infty, \tag{6.2.52}$$

$$\sim (z\bar{z})^{M\rho^\vee\cdot\lambda_j^{(1)}/2}\mathbf{e}_j^{(1)}. \tag{6.2.53}$$

To calculate the subdominant solution to the original linear problem $\Psi = U^{-1}\widetilde{\Psi}$ it remains to compute the values $\rho^\vee\cdot\lambda_j^{(1)}$, where $\rho^\vee$ is the co-Weyl vector (equal to the Weyl vector as D_r is a simply-laced Lie algebra), defined in equation (4.2.45) as the sum of fundamental weights or as half the sum of the positive roots of D_r. To compute $\rho^\vee \cdot \lambda_j^{(1)}$ we use the definition of the Weyl vector (4.2.45), the definitions of the weights of the representation $V^{(1)}$ (6.2.4)-(6.2.6) and the orthogonality of weights and roots $\omega_i\cdot\alpha_j = \delta_{ij}$ (again this holds as D_r is a simply-laced Lie algebra.)

From the D_r Weyl vector sum found in section 4.2.2, we have

$$\rho^\vee = \omega_1 + \cdots + \omega_r = \frac{1}{2}\sum_{i=1}^{r} i(2r-i-1)\alpha_i, \tag{6.2.54}$$

therefore

$$\rho^\vee \cdot \lambda_1^{(1)} = \rho^\vee \cdot \omega_1 = \frac{1}{2}\sum_{i=1}^{r} i(2r-i-1)\alpha_i \cdot \omega_1 = r-1. \tag{6.2.55}$$

Using the definitions of the weights (6.2.4)-(6.2.6) the dot products of the Weyl vector with the remaining weights are easily generated:

$$\rho^\vee \cdot \lambda_{i+1}^{(1)} = \rho^\vee \cdot (\lambda_i^{(1)} - \alpha_i), \quad (i = 1, \ldots, r-1), \tag{6.2.56}$$

$$\rho^\vee \cdot \lambda_{2r+1-j}^{(1)} = -\rho^\vee \cdot \lambda_j^{(1)}, \quad (j = 1, \ldots, r). \tag{6.2.57}$$

From the definition of the Weyl vector as a sum of fundamental weights and the orthogonality of weights and roots,

$$\rho^\vee \cdot \alpha_i = 1, \tag{6.2.58}$$

which allows us to compute all the dot products (6.2.56)-(6.2.57):

$$\rho^\vee \cdot \lambda_i^{(1)} = r - i, \quad (i = 1, \ldots, r), \tag{6.2.59}$$

$$\rho^\vee \cdot \lambda_{r+j}^{(1)} = -(j-1), \quad (j = 1, \ldots, r). \tag{6.2.60}$$

We now apply the matrix U^{-1} in the large-$|z|$ limit to the large-$|z|$ subdominant solution (6.2.51):

$$\begin{aligned} \Psi^{(1)} = U^{-1}\widetilde{\Psi}^{(1)} &\sim f(\bar{z}) \exp\left(-m\sqrt{2}e^{\theta}\frac{z^{M+1}}{M+1}\right) \cdot \Big(e^{-i\varphi(r-1)M}\mathbf{e}_1^{(1)} \\ &+\sqrt{2}\sum_{j=2}^{r-1} e^{-i\varphi(r-j)M}\mathbf{e}_j^{(1)} + \mathbf{e}_r^{(1)} + \mathbf{e}_{r+1}^{(1)} + \sqrt{2}\sum_{j=2}^{r-1} e^{i\varphi(j-1)M}\mathbf{e}_{r+j}^{(1)} \\ &+e^{i\varphi(r-1)M}\mathbf{e}_{2r}^{(1)}\Big), \quad \text{as } |z| \to \infty. \end{aligned} \tag{6.2.61}$$

Similarly to chapters 3 and 5, we require $\Psi^{(1)}$ to be the large-$|z|$ asymptotic solution for both the linear problem (5.2.9) and the conjugate linear problem (5.2.10). To ensure this, we set

$$f(\bar{z}) = \exp\left(-me^{-\theta}\sqrt{2}\frac{\bar{z}^{M+1}}{M+1}\right), \tag{6.2.62}$$

so that the large-$|z|$ asymptotics to the $D_r^{(1)}$ linear problem are given by

$$\begin{aligned}\Psi^{(1)} &\sim \exp\left(-\frac{2\sqrt{2}|z|^{M+1}}{M+1} m\cosh(\theta + i\varphi(M+1))\right)\\ &\left(e^{-i\varphi(r-1)M}\mathbf{e}_1^{(1)} + \sqrt{2}\sum_{j=2}^{r-1} e^{-i\varphi(r-j)M}\mathbf{e}_j^{(1)} + \mathbf{e}_r^{(1)} + \mathbf{e}_{r+1}^{(1)}\right.\\ &\left.+\sqrt{2}\sum_{j=2}^{r-1} e^{i\varphi(j-1)M}\mathbf{e}_{r+j}^{(1)} + e^{i\varphi(r-1)M}\mathbf{e}_{2r}^{(1)}\right), \quad \text{as } |z| \to \infty.\end{aligned} \tag{6.2.63}$$

The asymptotics of the larger linear problems in the representations $V^{(a)}$ can be determined via an explicit construction of the representations $V^{(a)}$. Here we only sketch the $V^{(a)}$ asymptotics; their general structure is all that we require.

The small-$|z|$ asymptotic solutions $\Xi_J^{(a)}$ for the $V^{(a)}$ linear problem are given by

$$\Xi_J^{(a)} \sim e^{-\beta(\theta+i\varphi)\lambda_J^{(a)}\cdot g}\mathbf{e}_J^{(a)} \quad \text{as } |z| \to 0, \tag{6.2.64}$$

where $\{\mathbf{e}_j^{(a)}\}_{J=1}^{\dim V^{(a)}}$ is a basis for the representation $V^{(a)}$ with $H_i\mathbf{e}_J^{(a)} = (\lambda_J^{(a)})^i\mathbf{e}_J^{(a)}$, and $J = 1, \ldots, \dim V^{(a)}$ is a standard index with no lexicographic interpretation as in section 5.3.3.

The large-$|z|$ asymptotics of the subdominant solution of the linear problems (5.2.9)-(5.2.10) are given (for general Lie algebras) in equation (2.23) of [38] in the general form

$$\Psi^{(a)} \sim \exp\left(-2w_a\frac{|z|^{M+1}}{M+1} m\cosh(\theta + i\varphi(M+1))\right)(z\bar{z})^{M\rho^\vee\cdot H/2}\mathbf{v}^{(a)}(z) \tag{6.2.65}$$

where $\mathbf{v}^{(a)}(z)$ is the eigenvector corresponding to the eigenvalue $me^\theta z^M w_a$ of A in (5.2.9) with largest positive real part, and the w_a are constants determined from a set of linear equations which arise from the associated Ψ-system, which we will discuss in section 6.2.4. From our discussion of the large-$|z|$ asymptotics of the $V^{(1)}$ linear problem we see that $w_1 = \sqrt{2}$. The expressions (6.2.64) and (6.2.65) will be useful in the discussion of the properties of the $Q^{(a)}$ functions.

6.2.3 Q-functions

Having derived the asymptotics of solutions to the $D_r^{(1)}$ linear problem in the representation $V^{(1)}$, we now derive some properties of the Q-functions defined in equation (6.2.42). The arguments establishing these properties are almost identical to those in section 5.4, only differing in the details of the asymptotic solutions we have derived for the $D_r^{(1)}$ case. For brevity we will only consider the cases where the details are non-trivially different from the $A_r^{(1)}$ case.

Notes on quasiperiodicity and the asymptotics of the Q-functions

Recall from section 5.4.1 the definition of the matrix $S = e^{\frac{2\pi i}{h}\rho^\vee \cdot H}$ (5.4.1) which is independent of the choice of Lie algebra. Quasiperiodicity of the $Q^{(1)}$ functions follows from the following identities

$$S\Xi_j^{(1)}\left(\varphi + \frac{2\pi}{hM}\middle|\, \theta - \frac{2\pi i}{hM} - \frac{2\pi i}{h}\right) = \exp\left(\frac{2\pi i}{h}(\rho^\vee + \beta g)\cdot\lambda_j^{(1)}\right)\Xi_j^{(1)}\left(\varphi|\, \theta\right), \tag{6.2.66}$$

$$S\Psi^{(1)}\left(\varphi + \frac{2\pi}{hM}\middle|\, \theta - \frac{2\pi i}{hM} - \frac{2\pi i}{h}\right) = \Psi^{(1)}\left(\varphi|\, \theta\right), \tag{6.2.67}$$

just as for the $A_r^{(1)}$ case. The proof of (6.2.66) is identical to the proof of (5.4.2), due to the identical form of the small-$|z|$ asymptotics. We briefly demonstrate the proof of the second identity (6.2.67). This is done by computing the left-hand side of (6.2.67) directly:

$$
\begin{aligned}
&S\Psi^{(1)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right)\\
&\sim \exp\left(-\frac{2\sqrt{2}|z|^{M+1}}{M+1}m\cosh(\theta+i\varphi(M+1))\right)e^{\frac{2\pi i}{h}\rho^{\vee}\cdot H} \qquad (6.2.68)\\
&\left(e^{-i\varphi(r-1)M}e^{-\frac{2\pi i}{h}(r-1)}\mathbf{e}_1^{(1)}++\sqrt{2}\sum_{j=2}^{r-1}e^{-i\varphi(r-j)M}e^{-\frac{2\pi i}{h}(r-j)}\mathbf{e}_j^{(1)}+\ \mathbf{e}_r^{(1)}+\mathbf{e}_{r+1}^{(1)}\right.\\
&\left.+\sqrt{2}\sum_{j=2}^{r-1}e^{i\varphi(j-1)M}e^{\frac{2\pi i}{h}(j-1)}\mathbf{e}_{r+j}^{(1)}+e^{i\varphi(r-1)M}e^{\frac{2\pi i}{h}(r-1)}\mathbf{e}_{2r}^{(1)}\right),\quad \text{as } |z|\to\infty.
\end{aligned}
$$

Applying the operator $e^{\frac{2\pi i}{h}\rho^{\vee}\cdot H}$ we find

$$
\begin{aligned}
&S\Psi^{(1)}\left(\varphi+\frac{2\pi}{hM}\middle|\,\theta-\frac{2\pi i}{hM}-\frac{2\pi i}{h}\right)\\
&\sim\left(e^{-i\varphi(r-1)M}\mathbf{e}_1^{(1)}+\sqrt{2}\sum_{j=2}^{r-1}e^{-i\varphi(r-j)M}\mathbf{e}_j^{(1)}+\ \mathbf{e}_r^{(1)}+\mathbf{e}_{r+1}^{(1)}\right. \qquad (6.2.69)\\
&\left.+\sqrt{2}\sum_{j=2}^{r-1}e^{i\varphi(j-1)M}\mathbf{e}_{r+j}^{(1)}+e^{i\varphi(r-1)M}\mathbf{e}_{2r}^{(1)}\right),\quad \text{as } |z|\to\infty.
\end{aligned}
$$

These asymptotics are exactly the same as the large-$|z|$ asymptotics of $\Psi^{(1)}(\varphi|\theta)$. Since the subdominant solution on the positive real axis for a given linear problem is unique relation (6.2.67) holds.

The identities (6.2.66) and (6.2.67), together with the definition of the $Q^{(1)}$ functions (6.2.42) imply the quasiperiodicity of the $Q^{(1)}$ functions:

$$
Q_j^{(1)}\left(\theta+\frac{2\pi i}{hM}+\frac{2\pi i}{h}\right)=\exp\left(-\frac{2\pi i}{h}(\beta g+\rho^{\vee})\cdot\lambda_j^{(1)}\right)Q_j^{(1)}(\theta). \qquad (6.2.70)
$$

This quasiperiodicity relation extends to all the $Q^{(a)}$-functions associated with the linear problems $(\partial_z+A)\Psi=0$ in the representations $V^{(a)}$ of $D_r^{(1)}$. This could in principle be demonstrated using the asymptotics of the subdominant solution in the large-$|z|$ limit of these larger linear problems, but we do not have an explicit expression valid for any r for the eigenvector $\mathbf{v}(z)$ in (6.2.65). This eigenvector can be computed in all cases once the matrices in the representation are known, but a general expression for it is complicated by the presence of the half-spin representations in the $D_r^{(1)}$ case.

The argument leading to the asymptotics of the first of the $Q^{(a)}$-functions in the $\operatorname{Re}\theta \to \pm\infty$ limit is identical to that found in section 5.4.2. The forms of the $D_r^{(1)}$ small-$|z|$ and large-$|z|$ asymptotic solutions of (5.2.9)-(5.2.10) are similar to the $A_r^{(1)}$ asymptotic solutions, and the resulting calculation is unchanged. In the $\operatorname{Re}\theta \to \infty$ limit the asymptotics of $Q_1^{(a)}(\theta)$ are

$$Q_1^{(a)}(\theta) \sim c_1^{(a)} e^{\mp i\pi\gamma_a/h} \exp\left(s^{M+1} m w_a e^{\theta \mp \frac{i\pi(M+1)}{hM}} \tau(h,M)\right), \quad \theta \in H_\pm, \qquad (6.2.71)$$

and similarly the $\operatorname{Re}\theta \to -\infty$ asymptotics are

$$Q_1^{(a)}(\theta) \sim c_1^{(a)} e^{\mp i\pi\gamma_a/h} \exp\left(s^{M+1} m w_a e^{-\theta \pm \frac{i\pi(M+1)}{hM}} \tau(h,M)\right), \quad \theta \in H_\pm. \quad (6.2.72)$$

where $\gamma_a = -(\beta g + \rho^\vee) \cdot \lambda_1^{(a)}$, and $H_\pm$ are defined as strips in the complex θ-plane as in (5.4.30).

The quantum Wronskian

An analogue to the quantum Wronskian identity discussed in section 5.4.3 also holds for the $D_r^{(1)}$ $Q^{(1)}$ functions. As we did there, we consider the determinant of subdominant solutions of the gauge-transformed linear problem $\widetilde{\Psi}^{(1)}$ twisted by Symanzik rotations (5.2.11). Using the definition of the $Q^{(1)}$ functions (6.2.42) and the argument followed for the analogous $A_r^{(1)}$ case (5.4.36)-(5.4.39), we find

$$
\begin{aligned}
&\det\left(\widetilde{\Psi}^{(1)}(\varphi|\theta), \Omega_1\widetilde{\Psi}^{(1)}(\varphi|\theta), \ldots, \Omega_{2r-1}\widetilde{\Psi}^{(1)}(\varphi|\theta)\right) \\
&= \begin{vmatrix}
Q_1^{(1)}(\theta) & Q_1^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_1^{(1)}(\theta - \frac{2\pi i(2r-1)}{hM}) \\
Q_2^{(1)}(\theta) & Q_2^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_2^{(1)}(\theta - \frac{2\pi i(2r-1)}{hM}) \\
\vdots & \vdots & \ddots & \vdots \\
Q_{2r}^{(1)}(\theta) & Q_{2r}^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_{2r}^{(1)}(\theta - \frac{2\pi i(2r-1)}{hM})
\end{vmatrix}.
\end{aligned}
\tag{6.2.73}
$$

In precisely the same manner as in section 5.4.3, the left-hand side of (6.2.73) can be considered in the large-$|z|$ limit, with the result being the numerical value of the determinant of the $Q^{(1)}$-functions in (6.2.73). Applying a Symanzik rotation (5.2.11) to the large-$|z|$ asymptotics (6.2.51) we find

$$
\begin{aligned}
\Omega_k\widetilde{\Psi}^{(1)} \sim \exp\left(-m\omega^k\sqrt{2}e^\theta\frac{z^{M+1}}{M+1}\right)&\Big(\omega^{-k(r-1)}z^{-(r-1)M}\mathbf{e}_1^{(1)} \\
&+\sqrt{2}\sum_{j=2}^{r-1}\omega^{-k(r-j)}z^{-(r-j)M}\mathbf{e}_j^{(1)} + \mathbf{e}_r^{(1)} + \mathbf{e}_{r+1}^{(1)} \\
&+\sqrt{2}\sum_{j=2}^{r-1}\omega^{k(j-1)}z^{(j-1)M}\mathbf{e}_{r+j}^{(1)} + \omega^{k(r-1)}z^{(r-1)M}\mathbf{e}_{2r}^{(1)}\Big), \quad \text{as } |z| \to \infty,
\end{aligned}
\tag{6.2.74}
$$

where $\omega = e^{\frac{2\pi i}{h}}$, recalling $h = 2r - 2$ for the Lie algebra $D_r^{(1)}$. The determinant on the left-hand side of (6.2.73) then becomes

$$\det\left(\widetilde{\Psi}^{(1)}(\varphi|\theta), \Omega_1\widetilde{\Psi}^{(1)}(\varphi|\theta), \ldots, \Omega_{2r-1}\widetilde{\Psi}^{(1)}(\varphi|\theta)\right) \tag{6.2.75}$$
$$= \begin{vmatrix} z^{-(r-1)M} & \omega^{-(r-1)}z^{-(r-1)M} & \cdots & 1 & 1 & \cdots & \omega^{-(2r-1)(r-1)}z^{-(r-1)M} \\ \sqrt{2}z^{-(r-2)M} & \omega^{-(r-2)}z^{-(r-2)M} & \cdots & 1 & 1 & \cdots & \omega^{-(2r-1)(r-2)}z^{-(r-2)M} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \sqrt{2}z^{-M} & \sqrt{2}\omega^{-1}z^{-M} & \cdots & 1 & 1 & \cdots & \omega^{-(2r-1)}z^{-M} \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \sqrt{2}z^{M} & \sqrt{2}\omega z^{M} & \cdots & 1 & 1 & \cdots & \omega^{(2r-1)}z^{M} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \sqrt{2}z^{(r-2)M} & \omega^{(r-2)}z^{(r-2)M} & \cdots & 1 & 1 & \cdots & \omega^{(2r-1)(r-2)}z^{(r-2)M} \\ z^{(r-1)M} & \omega^{(r-1)}z^{(r-1)M} & \cdots & 1 & 1 & \cdots & \omega^{(2r-1)(r-1)}z^{(r-1)M} \end{vmatrix} = 0,$$

hence we have found the $D_r^{(1)}$ quantum Wronskian

$$\begin{vmatrix} Q_1^{(1)}(\theta) & Q_1^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_1^{(1)}(\theta - \frac{2\pi i(2r-1)}{hM}) \\ Q_2^{(1)}(\theta) & Q_2^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_2^{(1)}(\theta - \frac{2\pi i(2r-1)}{hM}) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{2r}^{(1)}(\theta) & Q_{2r}^{(1)}(\theta - \frac{2\pi i}{hM}) & \cdots & Q_{2r}^{(1)}(\theta - \frac{2\pi i(2r-1)}{hM}) \end{vmatrix} = 0. \tag{6.2.76}$$

This result points to a certain redundancy in the matrix description of the linear system. The rank of the matrix in (6.2.76) is $2r - 2$, meaning that our $2r$ vector solutions overcount the number of truly independent solutions of the linear system by 2. The $2r$-dimensional $D(\omega_1)$ representation of D_r was irreducible by construction, however, so it is unclear how to cull two of the dimensions to make the solutions linearly independent, if this is indeed possible or desirable.

This zero causes another problem in the construction of suitable T-functions for the $D_r^{(1)}$ case of the ODE/IM correspondence. For the definition of the T-functions in the $A_r^{(1)}$ case (5.8.3)-(5.8.5), the constant z_0, defined using the $A_r^{(1)}$ quantum Wronskian, was used to normalise these T-functions. As for $D_r^{(1)}$ $z_0 = 0$, this determinant definition for T-functions will need to be reconsidered.

6.2.4 The Ψ-system, the Bethe ansatz equations and integrals of motion

We now continue to follow the path to deriving the integrals of motion for the massive integrable field theory associated with the $D_r^{(1)}$ Toda field equations. As we did for the $A_r^{(1)}$ case, we define an embedding of particular representations, which leads to the D_r Ψ-system. (See, for example, [47].) This Ψ-system, just as in section 5.5.2, leads to associated Bethe ansatz equations. These will be of the same structure as the Bethe ansatz equations defined in section 5.5.2. With these BAEs and the Q-asymptotics (6.2.71)-(6.2.72) being the same structurally as for the $A_r^{(1)}$ case, the entire integrals of motion calculation in 5.6 carries over with only a different Cartan matrix. We may therefore write down the analogous expression for $\log Q^{(a)}(\theta)$ (5.6.37), expanding it in the $\mathrm{Re}\,\theta \to \infty$ limit to recover the integrals of motion.

The Ψ-system and the Bethe ansatz equations

We begin by defining the representations that will be related via an embedding ι. We recall the definitions (6.2.3) of the representations $V^{(a)}$. We define the rotated representations $V_k^{(a)}$ with a parameter $k \in \mathbb{R}$:

$$V_k^{(a)} = L(\omega_a)_{\frac{a-1}{2}+k}, \quad (a = 1, \dots, r-2),$$
$$V_k^{(r-1)} = L(\omega_{r-1})_{r/2+k}, \quad V_k^{(r)} = L(\omega_r)_{r/2+k}. \qquad (6.2.77)$$

As for the $A_r^{(1)}$ case, the representation $V_{-1/2}^{(a)} \wedge V_{1/2}^{(a)}$ has highest weight $2\omega_a - \alpha_a$. We also consider the representation

$$\bigotimes_{b=1}^{r} (V^{(b)})^{B_{ab}}, \tag{6.2.78}$$

where $B = 2I - C$ is the D_r incidence matrix and C is the D_r Cartan matrix. The representation (6.2.78) has highest weight

$$\sum_{b=1}^{r} B_{ab}\omega_b = \sum_{b=1}^{r} (2I - C)_{ab}\omega_b = 2\omega_a - \alpha_a, \tag{6.2.79}$$

and therefore there exists an embedding ι given by [51]:

$$\iota : V_{-1/2}^{(a)} \wedge V_{1/2}^{(a)} \to \bigotimes_{b=1}^{r} (V^{(b)})^{B_{ab}}, \quad (a = 1, \ldots, r-2), \tag{6.2.80}$$

$$\iota : V^{(r-2)} \to V_{-1/2}^{(r-1)} \wedge V_{1/2}^{(r-1)} \simeq V_{-1/2}^{(r)} \wedge V_{1/2}^{(r)}, \tag{6.2.81}$$

The two representations $V_{-1/2}^{(r-1)} \wedge V_{1/2}^{(r-1)}$ and $V_{-1/2}^{(r)} \wedge V_{1/2}^{(r)}$ have the same highest weight $2\omega_{r-1} - \alpha_{r-1} = 2\omega_r - \alpha_r = \omega_{r-2}$ and have the same dimension $\binom{2^{r-1}}{2}$. They are therefore isomorphic. We also note the order of the representations in the embedding ι is swapped for $a = r-1, r$ as $\dim\left(V^{(r-2)}\right) \leq \dim\left(V_{-1/2}^{(r-1)} \wedge V_{1/2}^{(r-1)}\right)$, or equivalently

$$\binom{2r}{r-2} \leq \binom{2^{r-1}}{2}, \quad \text{for } r \geq 3. \tag{6.2.82}$$

For each representation $V_k^{(a)}$ define $\Psi_k^{(a)}$ to be the subdominant solution on the positive real axis of the linear problem associated with $V_k^{(a)}$. The embedding ι

defines the Ψ-system associated with the $D_r^{(1)}$ case:

$$\begin{aligned}
\iota\left(\Psi^{(a)}_{-1/2}\wedge\Psi^{(a)}_{1/2}\right) &= \Psi^{(a-1)}\otimes\Psi^{(a+1)}, \quad (a=1,\ldots,r-3),\\
\iota\left(\Psi^{(r-2)}_{-1/2}\wedge\Psi^{(r-2)}_{1/2}\right) &= \Psi^{(r-3)}\otimes\Psi^{(r-1)}\otimes\Psi^{(r)},\\
\iota\left(\Psi^{(r-2)}\right) &= \Psi^{(r-1)}_{-1/2}\wedge\Psi^{(r-1)}_{1/2} = \Psi^{(r)}_{-1/2}\wedge\Psi^{(r)}_{1/2}.
\end{aligned} \tag{6.2.83}$$

With this Ψ-system, we can derive a linear system for the constants w_a in the asymptotics (6.2.65) of the large-$|z|$ solutions $\Psi^{(a)}$. We substitute the large-$|z|$ asymptotics into the Ψ-system (6.2.83). We have

$$\begin{aligned}
&\Psi^{(a)}_{-1/2}\wedge\Psi^{(a)}_{1/2} \\
&\sim \exp\left\{-\frac{2mw_a|z|^{M+1}}{M+1}\cdot\right.\\
&\qquad\left.\left(\cosh\left(\theta+i\varphi(M+1)-\frac{i\pi}{h}\right)+\cosh\left(\theta+i\varphi(M+1)+\frac{i\pi}{h}\right)\right)\right\}\mathbf{v}_1,
\end{aligned} \tag{6.2.84}$$

and

$$\begin{aligned}
&\bigotimes_{b=1}^{r}(\Psi^{(b)})^{B_{ab}} \\
&\sim \exp\left\{-\frac{2m|z|^{M+1}}{M+1}\cdot\left(\sum_{b=1}^{r}B_{ab}w_b\right)\cdot\cosh\left(\theta+i\varphi(M+1)\right)\right\}\mathbf{v}_2,
\end{aligned} \tag{6.2.85}$$

where $\mathbf{v}_1\in V^{(a)}_{-1/2}\wedge V^{(a)}_{1/2}$, and $\mathbf{v}_2\in\bigotimes_{b=1}^{r}(V^{(b)})^{B_{ab}}$. The large-$|z|$ asymptotics of both sides of all the components of the Ψ-system (6.2.83) must be the same, implying the equality of all exponential prefactors of (6.2.84) and (6.2.85). Simplifying the sum of hyperbolic cosines in (6.2.84), we have

$$2\cos\frac{\pi}{h}w_a = \sum_{b=1}^{r}B_{ab}w_b, \tag{6.2.86}$$

as in equation (3.8) of [38]. The vector $w = (w_1, \dots, w_r)$ lies in the kernel of the matrix $2\cos\frac{\pi}{h}I - B$, with normalisation of w_1 determined by the eigenvalue of the matrix A in the representation $V^{(1)}$ as determined in section 6.2.1. For the $D_r^{(1)}$ case, $w_1 = \sqrt{2}$, as seen in equation (6.2.65). This normalisation and the system of equations (6.2.86) is sufficient to uniquely define the constants $w_2, \dots, w_r$.

For each representation $V_k^{(a)}$, there exists a basis of solutions $\{\Xi_J^{(a)}\}_{J=1}^{\dim V^{(a)}}$ with small-$|z|$ asymptotics given by (6.2.64). These solutions are defined to be invariant under Symanzik rotation Ω_k, defined in (5.2.11). Each subdominant solution $\Psi_k^{(a)} = \Omega_k[\Psi^{(a)}]$ can then be written in terms of this basis of solutions:

$$\Psi_k^{(a)}(\varphi|\theta) = \sum_{J=1}^{\dim V^{(a)}} Q_J^{(a)}\left(\theta - \frac{2\pi i k}{hM}\right)\Xi_J^{(a)}(\varphi|\theta). \tag{6.2.87}$$

We substitute the expansions (6.2.87) into the Ψ-system (6.2.83) following the same procedure as in section 5.5.2 to find the untwisted D_r Bethe ansatz equations:

$$\prod_{b=1}^{r} \frac{Q^{(b)}\left(\theta_j^{(a)} + \frac{i\pi}{hM}C_{ab}\right)}{Q^{(b)}\left(\theta_j^{(a)} - \frac{i\pi}{hM}C_{ab}\right)} = -1, \quad (a = 1, \dots, r), \tag{6.2.88}$$

where C is the D_r Cartan matrix, the leading order Q-functions $Q_1^{(a)}(\theta)$ have been abbreivated by $Q_1^{(a)}(\theta) = Q^{(a)}(\theta)$ and $Q^{(a)}(\theta_j^{(a)}) = 0$.

Calculation of the integrals of motion

The Bethe ansatz equations (6.2.88) have been written in Lie algebra notation and take on the same form as the $A_r^{(1)}$ BAEs (5.5.21). The $\operatorname{Re}\theta \to \infty$ asymptotics of the $Q^{(a)}$-functions (6.2.65) also have the same structure as the analogous asymptotics of the $A_r^{(1)}$ $Q^{(a)}$-functions. The calculation of the non-linear integral equation and the integrals of motion detailed in section 5.6 then carries over completely to the $D_r^{(1)}$ case.

As in section 5.6, we follow [21] and define the deformed Cartan matrix $\widetilde{C}(k)$:

$$\widetilde{C}_{<mt>}(k) = -\frac{1}{\cosh\frac{\pi k}{h}}, \quad \widetilde{C}_{mm}(k) = 2, \tag{6.2.89}$$

where $<mt>$ indicates the nodes corresponding to m and t are connected on the Dynkin diagram of D_r given in section 4.2. We then define the matrix $\widetilde{H}(k)$ by

$$\widetilde{H}(k) = \frac{1}{2\sinh\frac{\pi k}{hM}\cosh\frac{\pi k}{h}}\widetilde{C}^{-1} \tag{6.2.90}$$

The expression for $\log Q^{(a)}$ was derived in section 5.6 and is given by

$$\begin{aligned}\log Q^{(a)}\left(\theta + \frac{i\pi(M+1)}{hM}\right) &= 2m\tau(h,M)w_a\cosh\theta - \frac{i\pi}{h}\gamma_a \\ &- 2i\sum_{b=1}^{r}\int_{-\infty}^{\infty} H_{ab}(\theta-\theta'+i0)\,\mathrm{Im}\log\bigl(1+a^{(b)}(\theta'-i0)\bigr)\mathrm{d}\theta',\end{aligned} \tag{6.2.91}$$

where $H_{ab}(\theta)$ is the inverse Fourier transform of $\widetilde{H}_{ab}(k)$, $\tau(h,M)$ is the integral given by equation (3.4.26), and γ_a is given by equation (5.4.20). As for the integrals of motion calculation for the $A_r^{(1)}$ case given in section 5.6, we expand (6.2.91) in the limit $\mathrm{Re}\,\theta \to +\infty$. This is done by considering the integral form of $H(\theta)$ in terms of $\widetilde{H}(k)$

$$H(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{H}(k)e^{ik\theta}\,\mathrm{d}k, \tag{6.2.92}$$

and closing this integral in the upper half of the k-plane. Summing over the residues produces a power series in $e^{-\theta}$, which forms the asymptotic expansion of $\log Q^{(a)}$ in the $\mathrm{Re}\,\theta \to +\infty$ limit. The coefficients of this expansion are the integrals of motion, which further subdivide into local and non-local integrals of motion.

To perform the integral (6.2.92), we require an expression for $\widetilde{H}(k)$ (6.2.90). The inverse of the deformed Cartan matrix $\widetilde{C}$ for D_r is given in [58]:

$$\widetilde{C}^{-1}_{ab}(k) = \begin{cases} \frac{\coth(\pi k/h)\cosh((r-1-a)\pi k/h)\sinh(\pi bk/h)}{\cosh(\pi k/2)} & a \leq r-2 \\ \frac{\coth(\pi k/h)\sinh(b\pi k/h)}{2\cosh(\pi k/2)} & a \geq r-1, b \leq r-2 \\ \frac{\sinh(r\pi k/h)}{4\cosh(\pi k/2)\sinh(\pi k/h)} & a = b \geq r-1 \\ \frac{\sinh((r-2)\pi k/h)}{4\cosh(\pi k/2)\sinh(\pi k/h)} & a = r, b = r-1 \end{cases} \qquad (a \geq b), \tag{6.2.93}$$

with $\widetilde{C}^{-1}_{ba}(k) = \widetilde{C}^{-1}_{ab}(k)$. From (6.2.93) and the definition of $\widetilde{H}(k)$ (6.2.90) we see that the poles of $\widetilde{H}(k)$ are at $k = (2p_1 - 1)i$ and $k = qhMi$, where $p_1, q \in \mathbb{N}$. If r is odd there exists an additional family of poles at $k = (2p_2 - 1)hi/2$, where $p_2 \in \mathbb{N}$. We therefore expand the integral (6.2.92) as a sum of the residues at these poles:

$$\begin{aligned} H(\theta) = i \Bigg(& \sum_{p_1=1}^{\infty} e^{-(2p_1-1)\theta} \operatorname{Res}\Big[\widetilde{H}(k), k = (2p_1 - 1)i\Big] \\ & + \sum_{q=1}^{\infty} e^{-qhM\theta} \operatorname{Res}\Big[\widetilde{H}(k), k = qhMi\Big] \Bigg) \quad (r \text{ is even}), \end{aligned}$$

$$\begin{aligned} H(\theta) = i \Bigg(& \sum_{p_1=1}^{\infty} e^{-(2p_1-1)\theta} \operatorname{Res}\Big[\widetilde{H}(k), k = (2p_1 - 1)i\Big] \\ & + \sum_{p_2=1}^{\infty} e^{-(2p_2-1)h\theta/2} \operatorname{Res}\Big[\widetilde{H}(k), k = (2p_2 - 1)hi/2\Big] \\ & + \sum_{q=1}^{\infty} e^{-qhM\theta} \operatorname{Res}\Big[\widetilde{H}(k), k = qhMi\Big] \Bigg) \quad (r \text{ is odd}), \end{aligned} \tag{6.2.94}$$

Using (6.2.93) we calculate the residues of $\widetilde{H}(k)$:

$$\operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = (2p_1 - 1)i\Big] \tag{6.2.95}$$
$$= \begin{cases} \frac{(-1)^{p_1}\cos((2p_1-1)\pi(r-a-1)/h)\sin((2p_1-1)\pi b/h)}{\pi\sin((2p_1-1)\pi/h)\sin((2p_1-1)\pi/hM)} & a \leq r-2 \\ \\ \frac{(-1)^{p_1}\sin((2p_1-1)\pi b/h)}{2\pi\sin((2p_1-1)\pi/h)\sin((2p_1-1)\pi/hM)} & a \geq r-1,\ b \leq r-2 \\ \\ \frac{(-1)^{p_1}\sin((2p_1-1)\pi r/h)}{2\pi\sin(2(2p_1-1)\pi/h)\sin((2p_1-1)\pi/hM)} & a = b \geq r-1 \\ \\ \frac{(-1)^{p_1}\sin((2p_1-1)(r-2)\pi/h)}{2\pi\sin(2(2p_1-1)\pi/h)\sin((2p_1-1)\pi/hM)} & a = r,\ b = r-1 \end{cases} \qquad (a \geq b).$$

$$\text{If } r \text{ is odd, } \operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = (2p_2 - 1)hi/2\Big] \tag{6.2.96}$$
$$= \begin{cases} 0 & a \leq r-2 \\ \\ 0 & a \geq r-1,\ b \leq r-2 \\ \\ \frac{-h\sin((2p_2-1)\pi r/2)}{8\pi\sin((2p_2-1)\pi/(2M))\cos((2p_2-1)h\pi/4)} & a = b \geq r-1 \\ \\ \frac{-h\sin((2p_2-1)\pi(r-2)/2)}{8\pi\sin((2p_2-1)\pi/(2M))\cos((2p_2-1)h\pi/4)} & a = r,\ b = r-1 \end{cases} \qquad (a \geq b),$$

$$
\begin{aligned}
&\operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = qhMi\Big] && (6.2.97)\\
&= \begin{cases}
\frac{(-1)^q Mh\cos(qM\pi(r-a-1))\sin(qM\pi b)}{2\pi\sin(qM\pi)\cos(qhM\pi/2)} & a \leq r-2 \\
\frac{(-1)^q Mh\sin(qM\pi b)}{4\pi\sin(qM\pi)\cos(qhM\pi/2)} & a \geq r-1,\ b \leq r-2 \\
\frac{(-1)^q Mh\sin(qM\pi r)}{4\pi\sin(2qM\pi)\cos(qhM\pi/2)} & a = b \geq r-1 \\
\frac{(-1)^q Mh\sin(qM\pi(r-2))}{4\pi\sin(2qM\pi)\cos(qhM\pi/2)} & a = r,\ b = r-1
\end{cases} \qquad (a \geq b),
\end{aligned}
$$

where if $a \leq b$, we permute a and b. With these residues and the integral expression for $H(\theta)$, we take the $\operatorname{Re}\theta \to \infty$ limit of the $\log Q$ expression (6.2.91) to find the integrals of motion:

$$
\begin{aligned}
&\log Q^{(a)}\left(\theta + \frac{i\pi(M+1)}{hM}\right) = 2m\tau(h,M)w_a\cosh\theta - \frac{i\pi}{h}\gamma_a && (6.2.98)\\
&+ \sum_{p_1=1}^{\infty} \mathfrak{I}^{(a)}_{2p_1-1}\ e^{-(2p_1-1)\theta} + (r \bmod 2)\cdot \sum_{p_2=1}^{\infty} \mathfrak{I}^{(a)}_{(2p_2-1)h/2} e^{-(2p_2-1)h\theta/2} + \sum_{q=1}^{\infty} \mathfrak{S}^{(a)}_q\ e^{-qhM\theta},
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathfrak{I}^{(a)}_{2p_1-1} && (6.2.99)\\
&= 2\sum_{b=1}^{r}\int_{-\infty}^{\infty} e^{(2p_1-1)(\theta'-i0)}\operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = (2p_1-1)i\Big]\operatorname{Im}\log\big(1 + a^{(b)}(\theta'-i0)\big)\ \mathrm{d}\theta', \\
&\mathfrak{I}^{(a)}_{(2p_2-1)h/2} = 2\sum_{b=1}^{r}\int_{-\infty}^{\infty}\Big\{e^{(2p_2-1)h(\theta'-i0)/2}\ \cdot && (6.2.100)\\
&\operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = (2p_2-1)hi/2\Big]\operatorname{Im}\log\big(1 + a^{(b)}(\theta'-i0)\big)\ \Big\}\,\mathrm{d}\theta',
\end{aligned}
$$

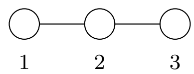
are the local integrals of motion for the $D_r^{(1)}$ massive integrable field theory and

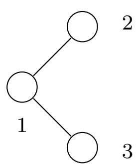
$$\mathfrak{S}_q^{(a)} = 2\sum_{b=1}^{r}\int_{-\infty}^{\infty} e^{qhM(\theta'-i0)}\,\mathrm{Res}\Big[\widetilde{H}_{ab}(k), k = qhMi\Big]\,\mathrm{Im}\log\big(1 + a^{(b)}(\theta' - i0)\big)\;\mathrm{d}\theta'. \tag{6.2.101}$$

are the non-local integrals of motion. The conjugate integrals of motion $\bar{\mathfrak{I}}_p^{(a)}$, $\bar{\mathfrak{S}}_q^{(a)}$ are found as in chapter 3 by closing the contour integral (6.2.92) in the lower-half complex k-plane and evaluating the coefficients of the resulting expansion in powers of e^{θ} and $e^{hM\theta}$.

Integrals of motion for $D_3^{(1)} \simeq A_3^{(1)}$

As a check on the form of the $D_r^{(1)}$ integrals of motion (6.2.99)-(6.2.101), we note that the Lie algebras A_3 and D_3 have the same Dynkin diagram and are therefore isomorphic Lie algebras. The only difference between those two algebras in our notation is the labelling of the nodes on their Dynkin diagrams:

A_3 

D_3 

In principle, the $D_3^{(1)}$ integrals of motion corresponding to the first node $\mathfrak{I}_p^{(1)}, \mathfrak{S}_q^{(1)}$ should match those the $A_3^{(1)}$ integrals of motion corresponding to the second node $\mathfrak{I}_p^{(2)}$, $\mathfrak{S}_q^{(2)}$, up to a relabelling of p and q. Similar equivalences should exist for the other two nodes as well. We demonstrate these equivalences by writing the $A_3^{(1)}$ integrals of motion in a vector form and finding the $D_3^{(1)}$ integrals of motion from these using a change of basis matrix.

Define the $\widetilde{H}$-matrix in the case $A_3^{(1)}$

$$\widetilde{H}_A(k) = \frac{1}{2\sinh\frac{\pi k}{4M}\cosh\frac{\pi k}{4}} \begin{pmatrix} 2 & \frac{-1}{\cosh\frac{\pi k}{4}} & 0 \\ \frac{-1}{\cosh\frac{\pi k}{4}} & 2 & \frac{-1}{\cosh\frac{\pi k}{4}} \\ 0 & \frac{-1}{\cosh\frac{\pi k}{4}} & 2 \end{pmatrix}^{-1}, \tag{6.2.102}$$

and the $\widetilde{H}$-matrix in the case $D_3^{(1)}$

$$\widetilde{H}_D(k) = \frac{1}{2\sinh\frac{\pi k}{4M}\cosh\frac{\pi k}{4}} \begin{pmatrix} 2 & \frac{-1}{\cosh\frac{\pi k}{4}} & \frac{-1}{\cosh\frac{\pi k}{4}} \\ \frac{-1}{\cosh\frac{\pi k}{4}} & 2 & 0 \\ \frac{-1}{\cosh\frac{\pi k}{4}} & 0 & 2 \end{pmatrix}^{-1}. \tag{6.2.103}$$

These matrices are related by the change of basis matrix P:

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P\widetilde{H}_A(k)P^{-1} = \widetilde{H}_D(k). \tag{6.2.104}$$

The matrix P corresponds to the relabelling of the nodes of the Dynkin diagrams for A_3 and D_3, specifically swapping the labels 1 and 2. As P is a constant matrix in k, we have at any pole in the complex k-plane

$$P\,\mathrm{Res}\Big[\widetilde{H}_A(k)\Big]P^{-1} = \mathrm{Res}\Big[P\widetilde{H}_A(k)P^{-1}\Big] = \mathrm{Res}\Big[\widetilde{H}_D(k)\Big]. \tag{6.2.105}$$

This notation will be useful in writing the integrals of motion succinctly. We also recall the vector notation for the $A_3^{(1)}$ a-functions defined in equation (5.6.8):

$$\begin{aligned} &\mathrm{Im}\log\left(1 + a(\theta' - i0)\right) \qquad (6.2.106)\\ &= \Big(\mathrm{Im}\log\big(1 + a^{(1)}(\theta' - i0)\big), \mathrm{Im}\log\big(1 + a^{(2)}(\theta' - i0)\big), \mathrm{Im}\log\big(1 + a^{(3)}(\theta' - i0)\big)\Big)^T. \end{aligned}$$

The $D_3^{(1)}$ a-functions (defined similarly to the a-functions (5.6.1) in the $A_r^{(1)}$ case) are then recovered by relabelling, represented by pre-multiplying the vector (6.2.106) by the matrix P. Finally, we set

$$\mathfrak{I}_p(A) = (\mathfrak{I}_p^{(1)}, \mathfrak{I}_p^{(2)}, \mathfrak{I}_p^{(3)})^T, \tag{6.2.107}$$

$$\mathfrak{S}_q(A) = (\mathfrak{S}_q^{(1)}, \mathfrak{S}_q^{(2)}, \mathfrak{S}_q^{(3)})^T \tag{6.2.108}$$

to be vectors constructed from the $A_3^{(1)}$ local and non-local integrals of motion. With all this notation, we can write the $A_3^{(1)}$ integrals of motion in the compact form

$$\mathfrak{I}_p(A) = 2\int_{-\infty}^{\infty} \mathrm{Res}\Big[\widetilde{H}_A(k), k = ip\Big]\,\mathrm{Im}\{\log(1 + a(\theta' - i0))\}e^{p(\theta' - i0)}\mathrm{d}\theta'. \tag{6.2.109}$$

Now we consider the $D_3^{(1)}$ integrals of motion

$$\begin{aligned}
\mathfrak{I}_p(D) &= 2\int_{-\infty}^{\infty} \mathrm{Res}\Big[\widetilde{H}_D(k), k = ip\Big]\,P\,\mathrm{Im}\{\log(1 + a(\theta' - i0))\}e^{p(\theta' - i0)}\mathrm{d}\theta', && (6.2.110)\\
&= 2\int_{-\infty}^{\infty} P\,\mathrm{Res}\Big[\widetilde{H}_A(k), k = ip\Big]\,P^{-1}P\,\mathrm{Im}\{\log(1 + a(\theta' - i0))\}e^{p(\theta' - i0)}\mathrm{d}\theta' && (6.2.111)\\
&= 2P\int_{-\infty}^{\infty} \mathrm{Res}\Big[\widetilde{H}_A(k), k = ip\Big]\,\mathrm{Im}\{\log(1 + a(\theta' - i0))\}e^{p(\theta' - i0)}\mathrm{d}\theta' && (6.2.112)\\
&= P\mathfrak{I}_p(A). && (6.2.113)
\end{aligned}$$

This calculation demonstrates the equivalence of the $A_3^{(1)}$ and $D_3^{(1)}$ local integrals of motion, up to a relabelling of the nodes of the Dynkin diagram, here facilitated by the change of basis matrix P. The non-local integrals of motion for $A_3^{(1)}$ and

$D_3^{(1)}$ are related in the same way:

$$\mathfrak{S}_q(D) = P\mathfrak{S}_q(A). \tag{6.2.114}$$

The next step in the analysis of the massive ODE/IM correspondence in the $D_r^{(1)}$ case would be to define the T-functions $T_m^{(a)}(u)$ and construct a set of fusion relations [43]:

$$T_m^{(a)}(u+1)T_m^{(a)}(u-1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + \prod_{b=1}^{r}(T_m^{(b)}(u))^{B_{ab}}. \tag{6.2.115}$$

However, the determinant definition of $T_m^{(a)}(u)$ (5.8.3)-(5.8.5) seen in the analysis of the $A_r^{(1)}$ case does not immediately generalise to the $D_r^{(1)}$ case. The right-hand side of the quantum Wronskian (6.2.76) arises in the normalisation constant z_0 in (5.8.3). For the $D_r^{(1)}$ case, the quantum Wronskian (6.2.76) is zero, which causes the normalisation of (5.8.3) to be undefined. The definition of $T_m^{(a)}(u)$ as seen in the $A_r^{(1)}$ case does not then immediately generalise to the $D_r^{(1)}$ case. This is an interesting open problem.

6.3 The $E_6^{(1)}$ massive ODE/IM correspondence

We continue our study of the massive ODE/IM correspondence for the simply-laced Lie algebras by considering the case of the exceptional Lie algebra $E_6^{(1)}$.

6.3.1 The linear problem in the representation $V^{(1)}$ and its massless limit

We begin our discussion of the $E_6^{(1)}$ massive ODE/IM correspondence by defining the relevant representations. We follow the definition in [47], defining $V^{(a)}$ as the evaluation representations

$$V^{(1)} = L(\omega_1)_0, \quad V^{(2)} = L(\omega_2)_{1/2}, \quad V^{(3)} = L(\omega_3)_1, \tag{6.3.1}$$

$$V^{(4)} = L(\omega_4)_{1/2}, \quad V^{(5)} = L(\omega_5)_0, \quad V^{(6)} = L(\omega_6)_{1/2}, \tag{6.3.2}$$

where the notation $L(\omega_k)$ was defined in section 4.2.2. As for the $D_r^{(1)}$ case, we will mostly be concerned with the representation $V^{(1)}$. The $E_6^{(1)}$ gauge-transformed linear problem is in the form $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$, where $\widetilde{A}$ is the matrix given by

$$\widetilde{A} = \beta \partial_z \phi \cdot H + m e^{\theta} \left[p(z) \sqrt{n_0^{\vee}} E_{\alpha_0} + \sum_{i=1}^{6} \sqrt{n_i^{\vee}} E_{\alpha_i} \right] \tag{6.3.3}$$

$$= \beta \partial_z \phi \cdot H + m e^{\theta} \left[E_{\alpha_1} + \sqrt{2} E_{\alpha_2} + \sqrt{3} E_{\alpha_3} + \sqrt{2} E_{\alpha_4} + E_{\alpha_5} + \sqrt{2} E_{\alpha_6} \right] + m e^{\theta} p(z) E_{\alpha_0}, \tag{6.3.4}$$

where to pass from (6.3.3) to (6.3.4) we have substituted the dual Kac labels for E_6 found in section 4.2.2.

Let $e_{a,b}$ be the matrix with elements $(e_{a,b})_{ij} = \delta_{ia}\delta_{jb}$. Using the basis for the vector space $V^{(1)}$ implicit in [39], the generators E_{α_i} in the representation $V^{(1)}$ are given by

$$E_{\alpha_1} = e_{1,2} + e_{12,15} + e_{14,17} + e_{16,19} + e_{18,21} + e_{20,22}, \tag{6.3.5}$$

$$E_{\alpha_2} = e_{2,3} + e_{10,12} + e_{11,14} + e_{13,16} + e_{21,23} + e_{22,24}, \tag{6.3.6}$$

$$E_{\alpha_3} = e_{3,4} + e_{8,10} + e_{9,11} + e_{16,18} + e_{19,21} + e_{24,25}, \tag{6.3.7}$$

$$E_{\alpha_4} = e_{4,5} + e_{6,8} + e_{11,13} + e_{14,16} + e_{17,19} + e_{25,26}, \tag{6.3.8}$$

$$E_{\alpha_5} = e_{5,7} + e_{8,9} + e_{10,11} + e_{12,14} + e_{15,17} + e_{26,27}, \tag{6.3.9}$$

$$E_{\alpha_6} = e_{4,6} + e_{5,8} + e_{7,9} + e_{18,20} + e_{21,22} + e_{23,24}, \tag{6.3.10}$$

$$E_{\alpha_0} = e_{20,1} + e_{22,2} + e_{24,3} + e_{25,4} + e_{26,5} + e_{27,7}, \tag{6.3.11}$$

where we have corrected the expression for E_{α_0} in [39]. The gauge-transformed linear problem $(\partial_z + \widetilde{A})\Psi = 0$ can then be written in the representation $V^{(1)}$ as a set of 27 coupled differential equations of the form

$$D(\lambda_j^{(1)})\ \widetilde{\psi}_j + me^{\theta} \sum_{k=1}^{27} \left[p(z)\sqrt{n_0^{\vee}} E_{\alpha_0} + \sum_{i=1}^{6} \sqrt{n_i^{\vee}} E_{\alpha_i} \right]_{jk} \widetilde{\psi}_k = 0, \quad j = 1, \ldots, 27, \tag{6.3.12}$$

where the differential operator $D(\lambda_j^{(1)})$ was defined in equation (5.3.15). The linear problem (6.3.12) has asymptotic solutions that can be used to define Q-functions, and hence Bethe ansatz equations and integrals of motion. However, we will first consider the pseudo-differential equation formulation of the linear problem (6.3.12) in the massless limit, for a particular choice of the vector parameter g.

6.3.2 The $V^{(1)}$ $E_6^{(1)}$ pseudo-differential equation in the massless limit with $g = 0$

We take the massless limit of the linear problem (6.3.12) by making the now familiar change of variables

$$x = (me^{\theta})^{\frac{1}{M+1}} z, \quad E = s^{12M} (me^{\theta})^{\frac{12M}{M+1}}, \tag{6.3.13}$$

and sending $\theta \to \infty$ and $z, s \to 0$ so that x and E remain finite. In this limit, ϕ is represented by its small-$|z|$ behaviour (5.2.17), so that the differential operators $D(\lambda_j^{(1)})$ are replaced by the operators $D_x(\lambda_j^{(1)})$, given by

$$D_x(\lambda_j^{(1)}) = \left(\partial_x + \frac{\beta g \cdot \lambda_j^{(1)}}{x} \right). \tag{6.3.14}$$

The authors of [39] consider the linear problem (6.3.12) in the massless limit in the special case $g = 0$, so that

$$D_x(\lambda_j^{(1)}) = \partial_x. \tag{6.3.15}$$

This simplifies the linear problem significantly, allowing the derivation of a pseudo-differential equation. Explicitly, the linear problem in the massless limit with $g = 0$ is given by the 27 differential equations

$$\begin{aligned}
\partial_x\widetilde{\psi}_1 + \widetilde{\psi}_2 &= 0, & \partial_x\widetilde{\psi}_2 + \sqrt{2}\widetilde{\psi}_3 &= 0,\\
\partial_x\widetilde{\psi}_3 + \sqrt{3}\widetilde{\psi}_4 &= 0, & \partial_x\widetilde{\psi}_4 + \sqrt{2}\widetilde{\psi}_5 + \sqrt{2}\widetilde{\psi}_6 &= 0,\\
\partial_x\widetilde{\psi}_5 + \widetilde{\psi}_7 + \sqrt{2}\widetilde{\psi}_8 &= 0, & \partial_x\widetilde{\psi}_6 + \sqrt{2}\widetilde{\psi}_8 &= 0,\\
\partial_x\widetilde{\psi}_7 + \sqrt{2}\widetilde{\psi}_9 &= 0, & \partial_x\widetilde{\psi}_8 + \widetilde{\psi}_9 + \sqrt{3}\widetilde{\psi}_{10} &= 0,\\
\partial_x\widetilde{\psi}_9 + \sqrt{3}\widetilde{\psi}_{11} &= 0, & \partial_x\widetilde{\psi}_{10} + \widetilde{\psi}_{11} + \sqrt{2}\widetilde{\psi}_{12} &= 0,\\
\partial_x\widetilde{\psi}_{11} + \sqrt{2}\widetilde{\psi}_{13} + \sqrt{2}\widetilde{\psi}_{14} &= 0, & \partial_x\widetilde{\psi}_{12} + \widetilde{\psi}_{14} + \widetilde{\psi}_{15} &= 0,\\
\partial_x\widetilde{\psi}_{13} + \sqrt{2}\widetilde{\psi}_{16} &= 0, & \partial_x\widetilde{\psi}_{14} + \sqrt{2}\widetilde{\psi}_{16} + \widetilde{\psi}_{17} &= 0,\\
\partial_x\widetilde{\psi}_{15} + \widetilde{\psi}_{17} &= 0, & \partial_x\widetilde{\psi}_{16} + \sqrt{3}\widetilde{\psi}_{18} + \widetilde{\psi}_{19} &= 0,\\
\partial_x\widetilde{\psi}_{17} + \sqrt{2}\widetilde{\psi}_{19} &= 0, & \partial_x\widetilde{\psi}_{18} + \sqrt{2}\widetilde{\psi}_{20} + \widetilde{\psi}_{21} &= 0,\\
\partial_x\widetilde{\psi}_{19} + \sqrt{3}\widetilde{\psi}_{21} &= 0, & \partial_x\widetilde{\psi}_{20} + p(x)\widetilde{\psi}_1 + \widetilde{\psi}_{22} &= 0,\\
\partial_x\widetilde{\psi}_{21} + \sqrt{2}\widetilde{\psi}_{22} + \sqrt{2}\widetilde{\psi}_{23} &= 0, & \partial_x\widetilde{\psi}_{22} + p(x)\widetilde{\psi}_2 + \sqrt{2}\widetilde{\psi}_{24} &= 0,\\
\partial_x\widetilde{\psi}_{23} + \sqrt{2}\widetilde{\psi}_{24} &= 0, & \partial_x\widetilde{\psi}_{24} + p(x)\widetilde{\psi}_3 + \sqrt{3}\widetilde{\psi}_{25} &= 0,\\
\partial_x\widetilde{\psi}_{25} + p(x)\widetilde{\psi}_4 + \sqrt{2}\widetilde{\psi}_{26} &= 0, & \partial_x\widetilde{\psi}_{26} + p(x)\widetilde{\psi}_5 + \widetilde{\psi}_{27} &= 0,\\
\partial_x\widetilde{\psi}_{27} + p(x)\widetilde{\psi}_7 &= 0.
\end{aligned} \tag{6.3.16}$$

We combine these into a single equation in $\widetilde{\psi}_1$ by repeated differentiation and substitution. Repeatedly differentiating the first equation in (6.3.16), simplifying derivatives of $\widetilde{\psi}_i$ in favour of derivatives of $\widetilde{\psi}_1$, we find

$$\begin{aligned}\partial^{17}\widetilde{\psi}_1 = 288\sqrt{3}\,\partial^5(p\widetilde{\psi}_1) - 1080\sqrt{3}\,\partial^4(p\widetilde{\psi}_2) + 1872\sqrt{6}\partial^3(p\widetilde{\psi}_3)\\ - 5616\sqrt{2}\partial^2(p\widetilde{\psi}_4) + 11232\partial(p\widetilde{\psi}_5) - 11232p\widetilde{\psi}_7,\end{aligned} \tag{6.3.17}$$

where for brevity of notation we have set $\partial_x = \partial$ and $p(x) = p$. To derive a pseudo-differential equation in terms of $\widetilde{\psi}_1$, we need to write $\widetilde{\psi}_2, \ldots, \widetilde{\psi}_7$ in terms of $\widetilde{\psi}_1$. $\widetilde{\psi}_2, \widetilde{\psi}_3$ and $\widetilde{\psi}_4$ are easily dealt with, as they are straightforward derivatives of $\widetilde{\psi}_1$:

$$\widetilde{\psi}_2 = -\partial\widetilde{\psi}_1, \quad \widetilde{\psi}_3 = \frac{1}{\sqrt{2}}\partial^2\widetilde{\psi}_1, \quad \widetilde{\psi}_4 = -\frac{1}{\sqrt{6}}\partial^3\widetilde{\psi}_1, \tag{6.3.18}$$

so that (6.3.17) becomes

$$\begin{aligned}\partial^{17}\widetilde{\psi}_1 = 288\sqrt{3}\,\partial^5(p\widetilde{\psi}_1) + 1080\sqrt{3}\,\partial^4(p\,\partial\widetilde{\psi}_1) + 1872\sqrt{6}\,\partial^3(p\,\partial^2\widetilde{\psi}_1)\\ + 1872\sqrt{3}\,\partial^2(p\,\partial^3\widetilde{\psi}_1) + 11232\,\partial(p\widetilde{\psi}_5) - 11232p\widetilde{\psi}_7.\end{aligned} \tag{6.3.19}$$

To deal with the $\widetilde{\psi}_5$ and $\widetilde{\psi}_7$ terms, we consider certain derivatives of $\widetilde{\psi}_1$, $\widetilde{\psi}_5$ and $\widetilde{\psi}_7$:

$$\partial^{13}\widetilde{\psi}_1 = 288\sqrt{3}\,\partial(p\widetilde{\psi}_1) - 1080\sqrt{3}p\,\partial\widetilde{\psi}_1 - 1872\sqrt{6}\widetilde{\psi}_{24}, \tag{6.3.20}$$

$$\partial^9\widetilde{\psi}_5 = 84\,\partial(p\widetilde{\psi}_1) + 312p\,\partial\widetilde{\psi}_1 - 540\sqrt{2}\widetilde{\psi}_{24}, \tag{6.3.21}$$

$$\partial^8\widetilde{\psi}_7 = -24\,\partial(p\widetilde{\psi}_1) - 84p\,\partial\widetilde{\psi}_1 + 144\sqrt{2}\widetilde{\psi}_{24}. \tag{6.3.22}$$

Next, we remove the $\widetilde{\psi}_{24}$ terms in (6.3.21) and (6.3.22) using (6.3.20). Integrating and simplifying, we find:

$$52\sqrt{3}\,\widetilde{\psi}_5 = 15\,\partial^4\widetilde{\psi}_1 + 24\sqrt{3}\,\partial^{-9}(p\partial\widetilde{\psi}_1) + 48\sqrt{3}\,\partial^{-8}(p\widetilde{\psi}_1), \tag{6.3.23}$$

$$13\sqrt{3}\,\widetilde{\psi}_7 = -\partial^5\widetilde{\psi}_1 - 12\sqrt{3}\,\partial^{-8}(p\partial\widetilde{\psi}_1) - 24\sqrt{3}\,\partial^{-7}(p\widetilde{\psi}_1). \tag{6.3.24}$$

We substitute the expressions (6.3.23) and (6.3.24) into (6.3.19), which gives us a pseudo-differential equation in $\widetilde{\psi}_1$:

$$\frac{1}{864}\partial^{17}\widetilde{\psi}_1 = \frac{1}{\sqrt{3}}\partial^5 p\,\widetilde{\psi}_1 + \frac{35\sqrt{3}}{12}\,\partial^4 p\,\partial\widetilde{\psi}_1 + \frac{21\sqrt{3}}{2}\partial^3 p\,\partial^2\widetilde{\psi}_1 + \frac{39\sqrt{3}}{2}\partial^2 p\,\partial^3\widetilde{\psi}_1$$
$$+ \frac{75\sqrt{3}}{4}\partial p\,\partial^4\widetilde{\psi}_1 + \frac{15\sqrt{3}}{2}p\partial^5\widetilde{\psi}_1 + 18p\partial^{-8}(p\partial\widetilde{\psi}_1) + 36p\partial^{-7}(p\widetilde{\psi}_1)$$
$$+ 6\partial p\,\partial^{-9}(p\partial\widetilde{\psi}_1) + 12\partial p\,\partial^{-8}(p\widetilde{\psi}_1). \qquad (6.3.25)$$

Using integration by parts, the integral terms can be collected together:

$$18p\,\partial^{-8}(p\partial\widetilde{\psi}_1) + 36p\,\partial^{-7}(p\widetilde{\psi}_1) + 6\partial p\,\partial^{-9}(p\partial\widetilde{\psi}_1) + 12\partial p\,\partial^{-8}(p\widetilde{\psi}_1)$$
$$= 6(\partial p + 3p\partial)\partial^{-9}(2\partial p + 3p\partial)\widetilde{\psi}_1 \qquad (6.3.26)$$

The pseudo-differential equation for the $V^{(1)}$ $E_6^{(1)}$ linear problem in the massless limit and with $g = 0$ is then given by

$$\frac{1}{864}\partial^{17}\widetilde{\psi}_1 = \frac{1}{\sqrt{3}}\partial^5 p\,\widetilde{\psi}_1 + \frac{35\sqrt{3}}{12}\,\partial^4 p\,\partial\widetilde{\psi}_1 + \frac{21\sqrt{3}}{2}\partial^3 p\,\partial^2\widetilde{\psi}_1 + \frac{39\sqrt{3}}{2}\partial^2 p\,\partial^3\widetilde{\psi}_1$$
$$+ \frac{75\sqrt{3}}{4}\partial p\,\partial^4\widetilde{\psi}_1 + \frac{15\sqrt{3}}{2}p\,\partial^5\widetilde{\psi}_1 + 6(\partial p + 3p\partial)\partial^{-9}(2\partial p + 3p\partial)\widetilde{\psi}_1. \qquad (6.3.27)$$

As a check on the calculation of (6.3.27), we employ the loop counting method found in [50] which was discussed in section 4.3.1. To employ this method, we first construct the weight diagram, shown in Figure 4, for the representation $V^{(1)}$ from the matrix $\widetilde{A}$. We then proceed by counting all the loops in this diagram. There are 297 distinct loops that contribute to the calculation of the $E_6^{(1)}$ $V^{(1)}$ pseudo-differential equation. We will not write a list of all the loops in the diagram above; we will merely list the contributions resulting from the loops through particular

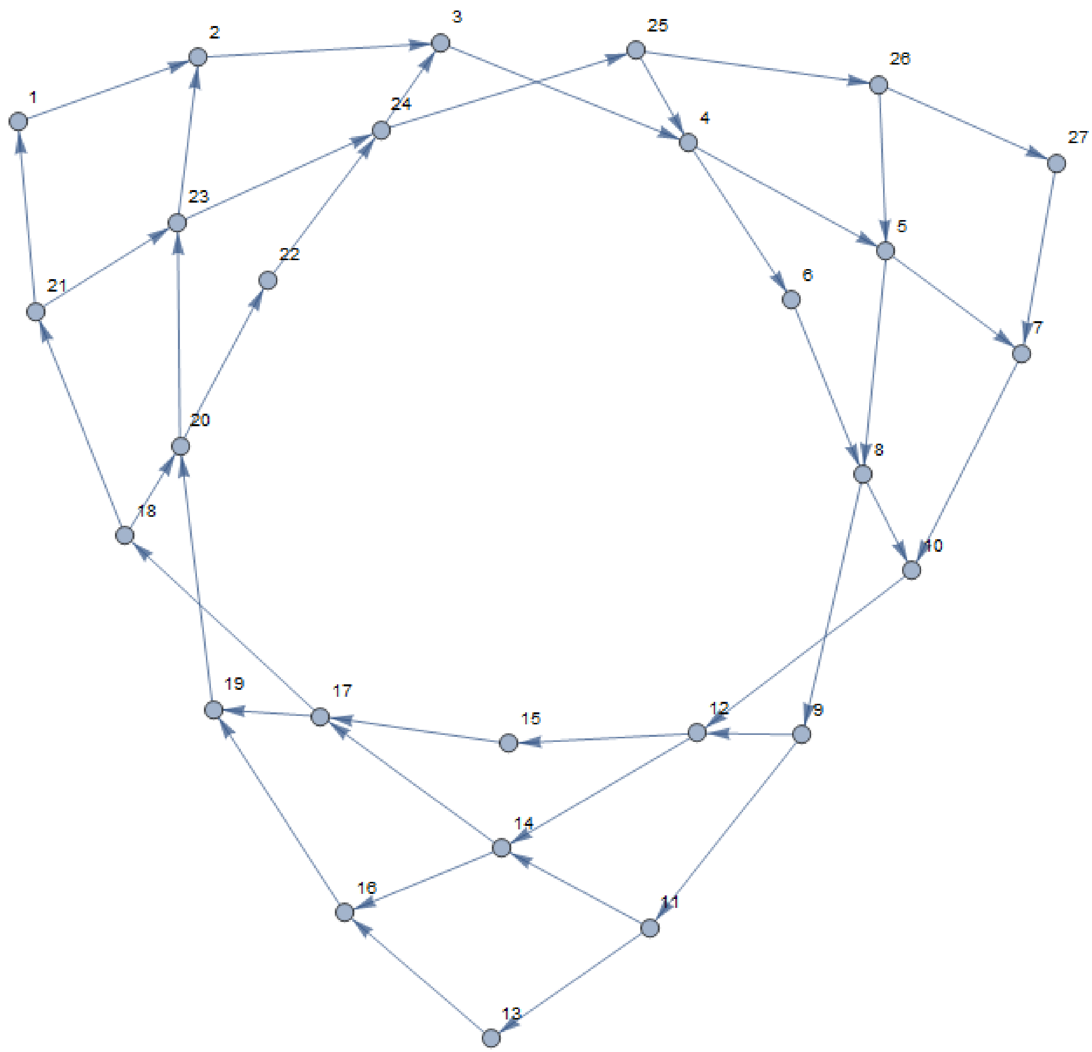


Figure 4: Weight diagram of the first fundamental representation of $E_6^{(1)}$.

nodes. The loops through node 1 contribute the terms

$$288\sqrt{3}\partial^{-12}(p\widetilde{\psi}_1) + 10368\partial^{-16}(p\partial^{-8}(p\widetilde{\psi}_1)) + 20736\partial^{-17}(p\partial^{-7}(p\widetilde{\psi}_1)), \qquad (6.3.28)$$

the loops through node 2 and not through node 1 contribute the terms

$$1080\sqrt{3}\partial^{-13}(p\partial\widetilde{\psi}_1) + 5184\partial^{-16}(p\partial^{-9}(p\partial\widetilde{\psi}_1)) + 10368\partial^{-17}(p\partial^{-8}(p\partial\widetilde{\psi}_1)), \quad (6.3.29)$$

the loops through node 3 and not through 1 and 2 contribute the term

$$1872\sqrt{3}\partial^{-14}(p\partial^2\widetilde{\psi}_1), \tag{6.3.30}$$

the loops through node 4, not through 1, 2 and 3 contribute the term

$$1872\sqrt{3}\partial^{-15}(p\partial^3\widetilde{\psi}_1), \tag{6.3.31}$$

the loops through node 5, not through 1, 2, 3 and 4 contribute the term

$$1080\sqrt{3}\partial^{-16}(p\partial^4\widetilde{\psi}_1), \tag{6.3.32}$$

and finally, the remaining loops, which all pass through node 7, contribute the term

$$288\sqrt{3}\partial^{-17}(p\partial^5\widetilde{\psi}_1). \tag{6.3.33}$$

Using (6.3.28)-(6.3.33), the $E_6^{(1)}\ V^{(1)}$ pseudo-differential equation becomes

$$\begin{aligned}\widetilde{\psi}_1 &= 288\sqrt{3}\,\partial^{-12}(p\widetilde{\psi}_1) + 10368\,\partial^{-16}(p\,\partial^{-8}(p\widetilde{\psi}_1)) + 20736\,\partial^{-17}(p\,\partial^{-7}(p\widetilde{\psi}_1)) \\ &\quad + 1080\sqrt{3}\,\partial^{-13}(p\,\partial\widetilde{\psi}_1) + 5184\,\partial^{-16}(p\,\partial^{-9}(p\partial\widetilde{\psi}_1)) + 10368\,\partial^{-17}(p\,\partial^{-8}(p\partial\widetilde{\psi}_1)) \\ &\quad + 1872\sqrt{3}\,\partial^{-14}(p\,\partial^2\widetilde{\psi}_1) + 1080\sqrt{3}\,\partial^{-16}(p\,\partial^4\widetilde{\psi}_1) + 288\sqrt{3}\,\partial^{-17}(p\,\partial^5\widetilde{\psi}_1).\end{aligned} \tag{6.3.34}$$

We act on both sides of this expression with the operator $\frac{1}{864}\partial^{17}$. We collect integral terms using integration by parts and expand any derivatives to find the pseudo-differential equation

$$\frac{1}{864}\partial^{17}\widetilde{\psi}_1 = \frac{1}{\sqrt{3}}\partial^5 p\,\widetilde{\psi}_1 + \frac{35\sqrt{3}}{12}\,\partial^4 p\,\partial\widetilde{\psi}_1 + \frac{21\sqrt{3}}{2}\partial^3 p\,\partial^2\widetilde{\psi}_1 + \frac{39\sqrt{3}}{2}\partial^2 p\,\partial^3\widetilde{\psi}_1 + \frac{75\sqrt{3}}{4}\partial p\,\partial^4\widetilde{\psi}_1 + \frac{15\sqrt{3}}{2}p\,\partial^5\widetilde{\psi}_1 + 6(\partial p + 3p\partial)\partial^{-9}(2\partial p + 3p\partial)\widetilde{\psi}_1, \tag{6.3.35}$$

which matches the pseudo-differential equation (6.3.27). This almost matches the pseudo-differential equation for $E_6^{(1)}$ given in [39], save for the coefficient of the $\partial^4 p\,\partial\widetilde{\psi}_1$ term, which in [39] is given by $\frac{367\sqrt{3}}{24}$, rather than $\frac{35\sqrt{3}}{12}$.

6.3.3 Asymptotics of the $V^{(1)}$ linear problem and Q-functions

Just as for the previously considered $A_r^{(1)}$ and $D_r^{(1)}$ cases, we define Q-functions related to the $E_6^{(1)}$ case of the massive ODE/IM correspondence by considering small and large-$|z|$ asymptotic solutions of the linear problem $(\partial_z + A)\Psi = 0$. Solutions of this linear problem can be found from the simpler gauge-transformed linear problem $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ (with $\widetilde{A}$ defined in (6.3.3)) by applying the matrix $U^{-1} = e^{\beta\phi\cdot H/2}$ to $\widetilde{\Psi}$. In this section, we will state the small and large-$|z|$ asymptotic solutions to both the original and gauge-transformed linear problems.

Small-$|z|$ asymptotics

Setting $\{\mathbf{e}_j^{(1)}\}_{j=1}^{27}$ to be a basis of $V^{(1)}$, with $H_i\mathbf{e}_j^{(1)} = (\lambda_j^{(1)})^i\mathbf{e}_j^{(1)}$, the basis of solutions $\{\widetilde{\Xi}_j^{(1)}\}_{j=1}^{27}$ of the gauge-transformed linear problem $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ have the small-$|z|$ behaviour

$$\widetilde{\Xi}_j^{(1)} \sim z^{-\beta g\cdot\lambda_j^{(1)}} e^{-\beta\theta g\cdot\lambda_j^{(1)}}\mathbf{e}_j^{(1)}, \quad \text{as } |z| \to 0. \tag{6.3.36}$$

In the small-$|z|$ limit, the matrix U^{-1} takes the form

$$U^{-1} = e^{\beta\phi\cdot H/2} \sim (z\bar{z})^{g\cdot H/2}, \quad \text{as } |z| \to 0, \tag{6.3.37}$$

therefore the small-$|z|$ solutions to the original linear problem $(\partial_z + A)\Psi = 0$ are

$$U^{-1}\widetilde{\Xi}_j^{(1)} = \Xi_j^{(1)} \sim e^{-\beta(\theta+i\varphi)g\cdot\lambda_j^{(1)}}\mathbf{e}_j^{(1)}, \quad \text{as } |z| \to 0. \tag{6.3.38}$$

Large-$|z|$ asymptotics

In the large-$|z|$ limit, the matrix $\widetilde{A}$ is given by

$$\begin{aligned}\widetilde{A} \sim me^{\theta}\Big[z^{12M}E_{\alpha_0} + E_{\alpha_1} + \sqrt{2}\,E_{\alpha_2} + \sqrt{3}\,E_{\alpha_3} \\ + \sqrt{2}\,E_{\alpha_4} + E_{\alpha_5} + \sqrt{2}\,E_{\alpha_6}\Big], \quad \text{as } |z| \to \infty.\end{aligned} \tag{6.3.39}$$

The large-$|z|$ solutions of the linear problem $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ then take the form

$$\mathbf{v}^{(1)}(z)\exp\left(-\int^z \sigma(u)\mathrm{d}u\right), \tag{6.3.40}$$

where, following section 4.3.2, $\mathbf{v}(z)$ is a particular eigenvector of $\widetilde{A}$ in the large-$|z|$ limit given by (6.3.39), and $\sigma(z)$ is its associated eigenvalue. The subdominant solution (the solution with the fastest decay to zero as $|z| \to \infty$ on the positive real axis) is associated with the eigenvalue of (6.3.39) with largest positive real part. The eigenvalues of (6.3.39) are

$$0,\, 0,\, 0,\, me^{\theta}z^M\sqrt{3+\sqrt{3}}\,e^{p\pi i/6}, \quad me^{\theta}z^M\sqrt{3-\sqrt{3}}\,e^{(2p+1)\pi i/12}, \quad p = 0, 1, \ldots, 11. \tag{6.3.41}$$

The eigenvalue with largest real part in (6.3.41) is $me^{\theta}z^M\sqrt{3+\sqrt{3}}$. Computing its associated eigenvector we find the subdominant solution to the linear problem

$(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ is given by

$$\widetilde{\Psi}^{(1)} \sim \mathbf{v}^{(1)}(z) \exp\left(-me^{\theta}\sqrt{3+\sqrt{3}}\frac{z^{M+1}}{M+1}\right), \quad \text{as } |z| \to \infty, \tag{6.3.42}$$

where

$$\begin{aligned}
\mathbf{v}^{(1)}(z) = {} & z^{-8M}\mathbf{e}_1^{(1)} + \sqrt{3+\sqrt{3}}\, z^{-7M}\mathbf{e}_2^{(1)} + \sqrt{3(2+\sqrt{3})}\, z^{-6M}\mathbf{e}_3^{(1)} \\
& + \sqrt{9+5\sqrt{3}}\, z^{-5M}\mathbf{e}_4^{(1)} + (2+\sqrt{3})\, z^{-4M}\mathbf{e}_5^{(1)} + (1+\sqrt{3})\, z^{-4M}\mathbf{e}_6^{(1)} \\
& + \sqrt{3+\sqrt{3}}\, z^{-3M}\mathbf{e}_7^{(1)} + \sqrt{9+5\sqrt{3}}\, z^{-3M}\mathbf{e}_8^{(1)} + \sqrt{3(2+\sqrt{3})}\, z^{-2M}\mathbf{e}_9^{(1)} \\
& + \sqrt{3(2+\sqrt{3})}\, z^{-2M}\mathbf{e}_{10}^{(1)} + \sqrt{3+\sqrt{3}}\, z^{-M}\mathbf{e}_{11}^{(1)} + \sqrt{9+5\sqrt{3}}\, z^{-M}\mathbf{e}_{12}^{(1)} \\
& + \mathbf{e}_{13}^{(1)} + (2+\sqrt{3})\mathbf{e}_{14}^{(1)} + (1+\sqrt{3})\mathbf{e}_{15}^{(1)} \qquad (6.3.43) \\
& + \sqrt{3+\sqrt{3}}\, z^{M}\mathbf{e}_{16}^{(1)} + \sqrt{9+5\sqrt{3}}\, z^{M}\mathbf{e}_{17}^{(1)} + \sqrt{3(2+\sqrt{3})}\, z^{2M}\mathbf{e}_{18}^{(1)} \\
& + \sqrt{3(2+\sqrt{3})}\, z^{2M}\mathbf{e}_{19}^{(1)} + \sqrt{9+5\sqrt{3}}\, z^{3M}\mathbf{e}_{20}^{(1)} + \sqrt{3+\sqrt{3}}\, z^{3M}\mathbf{e}_{21}^{(1)} \\
& + (1+\sqrt{3})\, z^{4M}\mathbf{e}_{22}^{(1)} + (2+\sqrt{3})\, z^{4M}\mathbf{e}_{23}^{(1)} + \sqrt{9+5\sqrt{3}}\, z^{5M}\mathbf{e}_{24}^{(1)} \\
& + \sqrt{3(2+\sqrt{3})}\, z^{6M}\mathbf{e}_{25}^{(1)} + \sqrt{3+\sqrt{3}}\, z^{7M}\mathbf{e}_{26}^{(1)} + z^{8M}\mathbf{e}_{27}^{(1)}.
\end{aligned}$$

We now apply the large-$|z|$ limit of the matrix $U^{-1} = e^{\beta\phi\cdot H/2}$ to $\widetilde{\Psi}^{(1)}$ to find the large-$|z|$ solution of the original linear problem $(\partial_z + A)\Psi = 0$. In the large-$|z|$ limit, $U^{-1} \sim (z\bar{z})^{M\rho^\vee\cdot H/2}$, with

$$U^{-1}\mathbf{e}_j^{(1)} = (z\bar{z})^{M\rho^\vee\cdot\lambda_j^{(1)}/2}\mathbf{e}_j^{(1)} \text{ as } |z| \to \infty. \tag{6.3.44}$$

To explicitly define this operator we require the dot products of the Weyl vector $\rho^\vee$ with the weights $\lambda_i^{(1)}$. The Weyl vector $\rho^\vee$ is given by

$$\rho^\vee = \sum_{i=1}^{6} \omega_i = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 15\alpha_4 + 8\alpha_5 + 11\alpha_6, \tag{6.3.45}$$

and the weight vectors $\lambda_i^{(1)}$ are calculated using the algorithm given in section 4.2.1, starting with $\lambda_1^{(1)} = \omega_1$. Using $\omega_i \cdot \alpha_j = \delta_{ij}$ for simply-laced Lie algebras, we compute $\rho^\vee \cdot \lambda_i^{(1)}$, written here in a list from $i = 1$ to $i = 27$:

$$\begin{aligned} \rho^\vee \cdot \lambda_i^{(1)} = &(8, 7, 6, 5, 4, 4, 3, 3, 2, 2, 1, 1, 0, 0, \\ &\qquad 0, -1, -1, -2, -2, -3, -3, -4, -4, -5, -6, -7, -8)\,. \end{aligned} \tag{6.3.46}$$

Applying U^{-1} in the large-$|z|$ limit to (6.3.42) and pre-multiplying by a factor depending on $\bar{z}$ to match the asymptotics of the conjugate linear problem $(\partial_{\bar{z}} + \bar{A})\Psi = 0$, we find the subdominant large-$|z|$ solution

$$\Psi^{(1)} \sim \mathbf{v}^{(1)}(e^{-i\varphi}) \exp\left(-\frac{2\sqrt{3+\sqrt{3}}\,|z|^{M+1}}{M+1} m \cosh(\theta + i\varphi(M+1)) \right), \text{ as } |z| \to \infty, \tag{6.3.47}$$

with $\mathbf{v}^{(1)}(z)$ defined as in equation (6.3.43). The large-$|z|$ solution Ψ and the basis of small-$|z|$ solutions $\Xi_j^{(1)}$ define $Q^{(1)}$-functions through the expansion

$$\Psi^{(1)}(\theta|\varphi) = \sum_{j=1}^{27} Q_j^{(1)}(\theta)\Xi_j^{(1)}(\theta|\varphi). \tag{6.3.48}$$

The linear problems for the other representations $V^{(a)}$ can be analysed in a similar way, allowing the definition of analogous $Q^{(a)}$-functions from the subdominant solution $\Psi^{(a)}$ and a basis of small-$|z|$ solutions $\{\Xi_J^{(a)}\}_{J=1}^{\dim V^{(a)}}$:

$$\Psi^{(a)}(\theta|\varphi) = \sum_{J=1}^{\dim V^{(a)}} Q_J^{(a)}(\theta)\Xi_J^{(a)}(\theta|\varphi). \tag{6.3.49}$$

The $V^{(a)}$ large-$|z|$ subdominant solutions $\Psi^{(a)}$ have the general asymptotic structure

$$\Psi^{(a)} \sim \mathbf{v}^{(a)}(e^{-i\varphi}) \exp\left(-\frac{2|z|^{M+1}}{M+1} m w_a \cosh(\theta + i\varphi(M+1))\right), \text{ as } |z| \to \infty, \tag{6.3.50}$$

where the constants w_a satisfy the constraints

$$2\cos\frac{\pi}{12} w_a = \sum_{b=1}^{6} B_{ab} w_b, \quad w_1 = \sqrt{3+\sqrt{3}}, \tag{6.3.51}$$

where $B = 2I - C$ is the E_6 incidence matrix. These constraints arise from an identical argument to that found in section 6.2.4, i.e. by consideration of the asymptotics of both sides of the E_6 Ψ-system [47]:

$$\begin{aligned}
\iota\left(\Psi^{(1)}_{-1/2} \wedge \Psi^{(1)}_{1/2}\right) &= \Psi^{(2)}, \\
\iota\left(\Psi^{(2)}_{-1/2} \wedge \Psi^{(2)}_{1/2}\right) &= \Psi^{(1)} \otimes \Psi^{(3)}, \\
\iota\left(\Psi^{(3)}_{-1/2} \wedge \Psi^{(3)}_{1/2}\right) &= \Psi^{(2)} \otimes \Psi^{(4)} \otimes \Psi^{(6)}, \\
\iota\left(\Psi^{(4)}_{-1/2} \wedge \Psi^{(4)}_{1/2}\right) &= \Psi^{(3)} \otimes \Psi^{(5)}, \\
\iota\left(\Psi^{(5)}_{-1/2} \wedge \Psi^{(5)}_{1/2}\right) &= \Psi^{(4)}, \\
\iota\left(\Psi^{(3)}\right) &= \Psi^{(6)}_{-1/2} \wedge \Psi^{(6)}_{1/2}.
\end{aligned} \tag{6.3.52}$$

6.3.4 Bethe ansatz equations and the integrals of motion

Using the same method as in section 5.5.2, the E_6 Ψ-system (6.3.52) and the definition of the $Q^{(a)}$-functions (6.3.49) imply the E_6 Bethe ansatz equations

$$\prod_{b=1}^{6} \frac{Q^{(b)}\left(\theta_j^{(a)} + \frac{i\pi}{hM} C_{ab}\right)}{Q^{(b)}\left(\theta_j^{(a)} - \frac{i\pi}{hM} C_{ab}\right)} = -1, \quad (a = 1, \ldots, 6), \tag{6.3.53}$$

where once again we use the truncated notation

$$Q^{(a)}(\theta) = Q_1^{(a)}(\theta). \tag{6.3.54}$$

The identical form of the Bethe ansatz equations and the leading order asymptotics of the $Q^{(a)}$-functions ensures the integrals of motion calculation given in section 5.6 holds in the case of $E_6^{(1)}$. As we did for $A_r^{(1)}$ and $D_r^{(1)}$ we define the deformed Cartan matrix $\widetilde{C}(k)$ with the elements

$$\widetilde{C}_{<ab>}(k) = \frac{-1}{\cosh \frac{\pi k}{12}}, \quad \widetilde{C}_{aa}(k) = 2, \tag{6.3.55}$$

where, as in the $A_r^{(1)}$ and $D_r^{(1)}$ cases, here $< ab >$ implies the nodes a and b are connected on the E_6 Dynkin diagram. We then define the H-matrix via its Fourier transform $\widetilde{H}(k)$:

$$\widetilde{H}(k) = \frac{1}{2\sinh \frac{\pi k}{12M} \cosh \frac{\pi k}{12}} \widetilde{C}(k)^{-1}. \tag{6.3.56}$$

The H-matrix occurs in the expression for $\log Q^{(a)}$ derived in section 5.6:

$$\begin{aligned}\log Q^{(a)}\left(\theta + \frac{i\pi(M+1)}{12M}\right) &= 2m\tau(12,M) w_a \cosh\theta - \frac{i\pi}{12}\gamma_a \\ &- 2i \sum_{b=1}^{6} \int_{-\infty}^{\infty} H_{ab}(\theta - \theta' + i0) \operatorname{Im} \log\bigl(1 + a^{(b)}(\theta' - i0)\bigr) \mathrm{d}\theta',\end{aligned} \tag{6.3.57}$$

with $\tau(12, M)$ and γ_a defined in section 5.6. The integrals of motion arise from a series expansion of (6.3.57) in $e^{-\theta}$. To calculate this, we apply Cauchy's theorem to rewrite the integral

$$H(\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{H}(k) e^{ik\theta} \,\mathrm{d}k \tag{6.3.58}$$

as a sum of residues:

$$H(\theta) = i\sum_{p=1}^{\infty} \mathrm{Res}\Big[\widetilde{H}(k), k = pi\Big] e^{-p\theta} + i\sum_{q=1}^{\infty} \mathrm{Res}\Big[\widetilde{H}(k), k = 12qMi\Big] e^{-12qM\theta}. \tag{6.3.59}$$

Substituting this expansion into (6.3.57) we find

$$\begin{aligned}\log Q^{(a)}\left(\theta + \frac{i\pi(M+1)}{12M}\right) &= 2m\tau(12, M)w_a \cosh\theta - \frac{i\pi}{12}\gamma_a \\ &+ \sum_{p=1}^{\infty} \mathfrak{I}_p^{(a)} e^{-p\theta} + \sum_{q=1}^{\infty} \mathfrak{S}_q^{(a)} e^{-12qM\theta},\end{aligned} \tag{6.3.60}$$

where the local integrals of motion $\mathfrak{I}_p^{(a)}$ are given by

$$\mathfrak{I}_p^{(a)} = 2\sum_{b=1}^{6} \int_{-\infty}^{\infty} e^{p(\theta'-i0)} \,\mathrm{Res}\Big[\widetilde{H}_{ab}(k), k = ip\Big] \,\mathrm{Im}\Big\{\log\big(1 + a^{(b)}(\theta' - i0)\big)\Big\}\, \mathrm{d}\theta', \tag{6.3.61}$$

and the non-local integrals of motion $\mathfrak{S}_q^{(a)}$ are given by

$$\begin{aligned}\mathfrak{S}_q^{(a)} = 2\sum_{b=1}^{6} \int_{-\infty}^{\infty} e^{12qM(\theta'-i0)} \,\mathrm{Res}\Big[\widetilde{H}_{ab}(k), k = 12qMi\Big]\cdot \\ \mathrm{Im}\Big\{\log\big(1 + a^{(b)}(\theta' - i0)\big)\Big\}\, \mathrm{d}\theta'.\end{aligned} \tag{6.3.62}$$

All that remains is to calculate the residues of the matrix $\widetilde{H}(k)$. Its poles are located at $k = (12n \pm 1)i, (12n \pm 4)i, (12n \pm 5)i, 12qMni$, for $n \in \mathbb{Z}$. As we have closed the integration contour in the upper-half k-plane, we only consider of the residues of the poles with positive imaginary part.

As a demonstration, we will now calculate the first local integral of motion

$\mathfrak{I}_1^{(1)}$ and the first non-local integral of motion $\mathfrak{S}_1^{(1)}$. $\mathfrak{I}_1^{(1)}$ is given by

$$\mathfrak{I}_1^{(1)} = -\frac{3\csc\left(\frac{\pi}{12M}\right)}{(3+\sqrt{3})\pi}\sum_{k=1}^{6}\int_{-\infty}^{\infty} \mathrm{d}\theta'\, e^{(\theta'-i0)} R_k \,\mathrm{Im}\big\{\log\big(1+a^{(k)}(\theta'-i0)\big)\big\}, \quad (6.3.63)$$

where

$$\begin{aligned} R_1 &= \sqrt{2}(1+\sqrt{3}), \\ R_2 &= 2(2+\sqrt{3}), \\ R_3 &= 2\sqrt{2}(2+\sqrt{3}), \\ R_4 &= 2(2+\sqrt{3}), \\ R_5 &= \sqrt{2}(1+\sqrt{3}), \\ R_6 &= 2(1+\sqrt{3}). \end{aligned} \quad (6.3.64)$$

The non-local integral of motion $\mathfrak{S}_1^{(1)}$ is given by

$$\mathfrak{S}_1^{(1)} = -\frac{24M}{\pi}\sum_{k=1}^{6}\int_{-\infty}^{\infty} \mathrm{d}\theta'\, e^{12M(\theta'-i0)} S_k \,\mathrm{Im}\big\{\log\big(1+a^{(k)}(\theta'-i0)\big)\big\}, \quad (6.3.65)$$

where

$$\begin{aligned} S_1 &= \frac{4\cos(M\pi)\cos(4M\pi)}{2\cos(6M\pi)+2\cos(4M\pi)-1}, \\ S_2 &= \frac{1+2\cos(2M\pi)+2\cos(4M\pi)}{2\cos(6M\pi)+2\cos(4M\pi)-1}, \\ S_3 &= \frac{2\cos(M\pi)}{2\cos(4M\pi)-1}, \\ S_4 &= \frac{2(1+\cos(2M\pi))}{2\cos(6M\pi)+2\cos(4M\pi)-1}, \\ S_5 &= \frac{2\cos(M\pi)}{2\cos(6M\pi)+2\cos(4M\pi)-1}, \\ S_6 &= \frac{1}{2\cos(4M\pi)-1}. \end{aligned} \quad (6.3.66)$$

The conjugate integrals of motion $\bar{\mathfrak{I}}_p^{(a)}$, $\bar{\mathfrak{S}}_q^{(a)}$ are calculated in a similar way by closing the contour of integration in (6.3.58) in the lower-half k-plane.

6.4 The massive ODE/IM correspondence for $E_7^{(1)}$ and $E_8^{(1)}$

We now conclude our study of the massive ODE/IM correspondence for the simply-laced Lie algebras by considering the correspondence for the exceptional Lie algebras $E_7^{(1)}$ and $E_8^{(1)}$. The smallest representations of these exceptional Lie algebras are quite large, making explicit calculations rather unwieldy. The representation $V^{(1)}$ has dimension 56 and 248 in $E_7^{(1)}$ and $E_8^{(1)}$ respectively. We exhibit the large-$|z|$ eigenvalues of the $E_7^{(1)}$ and $E_8^{(1)}$ A-matrices in the representation $V^{(1)}$, Bethe ansatz equations for suitably defined Q-functions, and integrals of motion.

6.4.1 $E_7^{(1)}$

The 56-dimensional representation $V^{(6)}$ was given in the Appendix of [39]:

$$E_{\alpha_1} = e_{7,8} + e_{9,10} + e_{11,12} + e_{13,15} + e_{16,18} + e_{19,22} \tag{6.4.1}$$
$$+ e_{35,38} + e_{39,41} + e_{42,44} + e_{45,46} + e_{47,48} + e_{49,50},$$
$$E_{\alpha_2} = e_{5,6} + e_{7,9} + e_{8,10} + e_{20,23} + e_{24,26} + e_{27,29} \tag{6.4.2}$$
$$+ e_{28,30} + e_{31,33} + e_{34,37} + e_{47,49} + e_{48,50} + e_{51,52},$$
$$E_{\alpha_3} = e_{5,7} + e_{6,9} + e_{12,14} + e_{15,17} + e_{18,21} + e_{22,25} \tag{6.4.3}$$
$$+ e_{32,35} + e_{36,39} + e_{40,42} + e_{43,45} + e_{48,51} + e_{50,52},$$
$$E_{\alpha_4} = e_{4,5} + e_{9,11} + e_{10,12} + e_{17,20} + e_{21,24} + e_{25,28} \tag{6.4.4}$$
$$+ e_{29,32} + e_{33,36} + e_{37,40} + e_{45,47} + e_{46,48} + e_{52,53},$$
$$E_{\alpha_5} = e_{3,4} + e_{11,13} + e_{12,15} + e_{14,17} + e_{24,27} + e_{26,29} \tag{6.4.5}$$
$$+ e_{28,31} + e_{30,33} + e_{40,43} + e_{42,45} + e_{44,46} + e_{53,54},$$
$$E_{\alpha_6} = e_{2,3} + e_{13,16} + e_{15,18} + e_{17,21} + e_{20,24} + e_{23,26} \tag{6.4.6}$$
$$+ e_{31,34} + e_{33,37} + e_{36,40} + e_{39,42} + e_{41,44} + e_{54,55},$$
$$E_{\alpha_7} = e_{1,2} + e_{16,19} + e_{18,22} + e_{21,25} + e_{24,28} + e_{26,30} \tag{6.4.7}$$
$$+ e_{27,31} + e_{29,33} + e_{32,36} + e_{35,39} + e_{38,41} + e_{55,56},$$
$$E_{\alpha_0} = e_{38,1} + e_{41,2} + e_{44,3} + e_{46,4} + e_{48,5} + e_{50,6} \tag{6.4.8}$$
$$+ e_{51,7} + e_{52,9} + e_{53,11} + e_{54,13} + e_{55,16} + e_{56,19},$$

where $e_{a,b}$ is the matrix with elements $(e_{a,b})_{ij} = \delta_{ia}\delta_{jb}$. $V^{(6)}$ is considered as it is the smallest non-trivial representation of $E_7^{(1)}$; $V^{(1)}$, the adjoint representation, is 133-dimensional. The gauge-transformed linear problem is given by $(\partial_z + \widetilde{A})\Psi = 0$, with

$$\widetilde{A} = \beta \partial_z \phi \cdot H + m e^{\theta} \left[p(z) \sqrt{n_0^{\vee}} E_{\alpha_0} + \sum_{i=1}^{7} \sqrt{n_i^{\vee}} E_{\alpha_i} \right] \tag{6.4.9}$$

where the dual Kac labels $n_0^{\vee}$, $n_i^{\vee}$ were given in section 4.2.2. The small-$|z|$ asymptotic solutions $\{\widetilde{\Xi}_J^{(6)}\}_{J=1}^{56}$ to this linear problem are given by

$$\widetilde{\Xi}_J^{(6)} \sim z^{-\beta g \cdot H} \mathbf{e}_J^{(6)}, \quad \text{as } |z| \to 0, \tag{6.4.10}$$

and the large-$|z|$ subdominant solution $\widetilde{\Psi}^{(6)}$ is given by

$$\widetilde{\Psi}^{(6)} \sim \mathbf{v}^{(6)}(z) \exp\left(- \int^{z} \sigma(u) \, \mathrm{d}u \right), \tag{6.4.11}$$

where $\sigma(u)$ is the eigenvalue of the matrix (6.4.9) in the large-$|z|$ limit with largest positive real part, and $\mathbf{v}^{(6)}(z)$ is the corresponding eigenvector. To find $\sigma(z)$, we consider the characteristic polynomial of the $\widetilde{A}$ matrix in the large-$|z|$ limit:

$$\begin{aligned} \det\left(\widetilde{A} - \sigma I \right) \sim \sigma^{56} &- 50,969,088 z^{18M} m e^{\theta} \sigma^{38} \\ &+ 1,199,792,259,072 \, z^{36M} m^2 e^{2\theta} \sigma^{20} \\ &- 330,225,942,528 \, z^{54M} m^3 e^{3\theta} \sigma^2 \quad \text{as } |z| \to \infty. \end{aligned} \tag{6.4.12}$$

The roots of this polynomial are

$$\sigma = 0, 0, m e^{\theta} z^M \xi_1^{1/18} e^{n\pi i/9}, \tag{6.4.13}$$

$$m e^{\theta} z^M \xi_2^{1/18} e^{(2n+1)\pi i/18}, \tag{6.4.14}$$

$$m e^{\theta} z^M \xi_3^{1/18} e^{n\pi i/9}, \tag{6.4.15}$$

where $n = 0, 1, \ldots, 17$ and $\xi_1 < \xi_2 < \xi_3$ are the three real solutions of the polynomial equation

$$\xi^3 - 50,969,088\xi^2 + 1,199,792,259,072\xi - 330,225,942,528 = 0. \qquad (6.4.16)$$

The eigenvalue with largest positive real part is then $me^{\theta} z^M \xi_3^{1/18}$, where $\xi_3^{1/18} = 2.68023308478642....$

The definition of the asymptotic solutions $\widetilde{\Xi}_J^{(a)}$ and $\widetilde{\Psi}^{(a)}$ in the various fundamental representations $V^{(a)}$ of $E_7^{(1)}$ leads to the definition of $Q^{(a)}$-functions

$$\Psi^{(a)}(\theta|\varphi) = \sum_{J=1}^{\dim V^{(a)}} Q_J^{(a)}(\theta)\Xi_J^{(a)}(\theta|\varphi), \qquad (6.4.17)$$

which then satisfy Bethe ansatz equations

$$\prod_{b=1}^{7} \frac{Q^{(b)}\left(\theta_j^{(a)} + \frac{i\pi}{hM} C_{ab}\right)}{Q^{(b)}\left(\theta_j^{(a)} - \frac{i\pi}{hM} C_{ab}\right)} = -1, \quad (a = 1, \ldots, 7), \qquad (6.4.18)$$

where C is the E_7 Cartan matrix given in section 4.2.2 and $Q^{(a)}(\theta) = Q_1^{(a)}(\theta)$. We once again consider integrals of motion by computing the residues of the matrix $\widetilde{H}(k)$

$$\widetilde{H}(k) = \frac{1}{2\sinh\frac{\pi k}{18M}\cosh\frac{\pi k}{18}} \widetilde{C}(k)^{-1}. \qquad (6.4.19)$$

where $\widetilde{C}(k)$ is the E_7 deformed Cartan matrix. The poles of $\widetilde{H}(k)$ are at $k = (18n \pm 1)i, (18n \pm 5)i, (18n \pm 7)i, (18n \pm 9)i$, and $k = 18nMi$, where $n \in \mathbb{Z}$. The local integrals of motion are then given by

$$\mathfrak{I}_p^{(a)} = 2\sum_{b=1}^{7}\int_{-\infty}^{\infty} e^{p(\theta'-i0)}\,\mathrm{Res}\Big[\widetilde{H}_{ab}(k), k = pi\Big]\,\mathrm{Im}\Big\{\log\big(1 + a^{(b)}(\theta' - i0)\big)\Big\}\,\mathrm{d}\theta', \tag{6.4.20}$$

and the non-local integrals of motion are given by

$$\mathfrak{S}_q^{(a)} = 2\sum_{b=1}^{7}\int_{-\infty}^{\infty} e^{12qM(\theta'-i0)}\,\mathrm{Res}\Big[\widetilde{H}_{ab}(k), k = 18qMi\Big]\cdot$$
$$\mathrm{Im}\Big\{\log\big(1 + a^{(b)}(\theta' - i0)\big)\Big\}\,\mathrm{d}\theta'. \tag{6.4.21}$$

6.4.2 $E_8^{(1)}$

The 248-dimensional representation of $E_8^{(1)}$, $V^{(1)}$, is given in the Appendix of [39]. From the matrices defined there and the dual Kac labels in section 4.2.2, we define the matrix $\widetilde{A}$:

$$\widetilde{A} = \beta\partial_z\phi\cdot H + me^{\theta}\left[p(z)\sqrt{n_0^\vee}E_{\alpha_0} + \sum_{i=1}^{8}\sqrt{n_i^\vee}E_{\alpha_i}\right] \tag{6.4.22}$$

The large-$|z|$ solutions to the linear problem $(\partial_z + \widetilde{A})\widetilde{\Psi} = 0$ are described using the eigenvalues and eigenvectors of (6.4.22) in the large-$|z|$ limit. The characteristic polynomial in this limit is

$$
\begin{aligned}
&\sigma^{248} + 9,845,667,127,296,000,000\sqrt{5}(me^{\theta}z^{M})^{30}\sigma^{218} \\
&-997,429,956,592,632,574,022,516,736,000,000,000(me^{\theta}z^{M})^{60}\sigma^{188} \\
&+ 62,106,889,173,106,930,566,304,048, \\
&\quad 704,022,118,400,000,000,000,000\sqrt{5}(me^{\theta}z^{M})^{90}\sigma^{158} \\
&+ 1,783,320,600,763,454,867,346,450,898,335,637, \\
&\quad 305,559,860,903,936,000,000,000,000,000(me^{\theta}z^{M})^{120}\sigma^{128} \\
&- 19,422,643,883,849,883,504,740,159,769,364,196, \\
&\quad 361,424,820,433,321,984,000,000,000,000,000,000,000\sqrt{5}(me^{\theta}z^{M})^{150}\sigma^{98} \\
&+ 3,518,345,698,492,137,878,835,967,728,970, \\
&\quad 575,127,409,632,826,324,441,628,672,000,000, \\
&\quad 000,000,000,000,000,000(me^{\theta}z^{M})^{180}\sigma^{68} \qquad (6.4.23) \\
&+ 262,380,855,963,325,641,992,292,565,498,191, \\
&\quad 833,239,248,829,760,256,186,777,600,000,000, \\
&\quad 000,000,000,000,000,000,000\sqrt{5}(me^{\theta}z^{M})^{210}\sigma^{38} \\
&+ 52,477,712,140,573,920,113,791,072,551,142,890,519,592, \\
&\quad 132,233,368,961,024,000,000,000,000,000,000,000,000,000,000(me^{\theta}z^{M})^{240}\sigma^{8}
\end{aligned}
$$

The roots of (6.4.23) form eight rings of shifted 30^{th} roots of unity. From the form of (6.4.23) the root at $\sigma = 0$ has multiplicity eight. Figure 5 is a plot of the roots of (6.4.23) with $me^{\theta}z^{M} = 1$ on the σ-plane. From this plot it is apparent there is no unique root with maximally positive real part. It is therefore an interesting open problem to obtain the $Q^{(a)}$-functions. Nevertheless, we can assume the Bethe

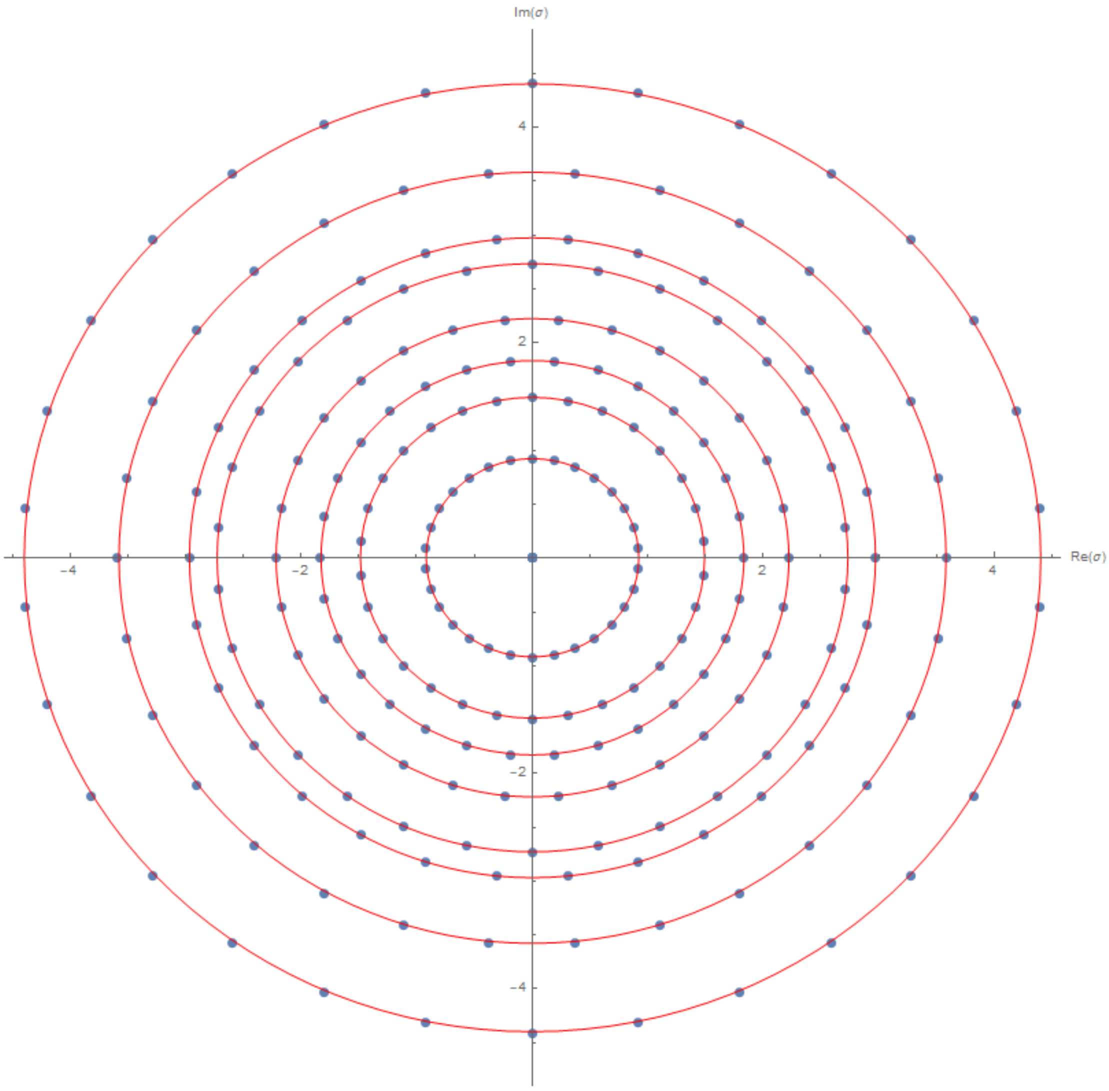


Figure 5: Eigenvalues of the A-matrix associated with the first fundamental representation of $E_8^{(1)}$ in the large-$|z|$ limit.

ansatz equations take the same form

$$\prod_{b=1}^{8} \frac{Q^{(b)}\left(\theta_j^{(a)} + \frac{i\pi}{hM} C_{ab}\right)}{Q^{(b)}\left(\theta_j^{(a)} - \frac{i\pi}{hM} C_{ab}\right)} = -1, \quad (a = 1, \ldots, 8), \tag{6.4.24}$$

where C is the E_8 Cartan matrix given in section 4.2.2. Integrals of motion are then calculated from the residues of $\widetilde{H}(k)$:

$$\widetilde{H}(k) = \frac{1}{2 \sinh \frac{\pi k}{30M} \cosh \frac{\pi k}{30}} \widetilde{C}(k)^{-1}, \tag{6.4.25}$$

where $\widetilde{C}(k)$ is the E_8 deformed Cartan matrix. The poles of $\widetilde{H}(k)$ are at $k = (30n \pm 1)i, (30n \pm 7)i, (30n \pm 11)i, (30n \pm 13)i$ and $k = 30nMi$, where $n \in \mathbf{Z}$. Integrals of motion are then defined identically to earlier cases, with the local integrals of motion

$$\mathfrak{I}_p^{(a)} = 2 \sum_{b=1}^{8} \int_{-\infty}^{\infty} e^{p(\theta' - i0)} \operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = ip\Big] \operatorname{Im}\big\{\log\big(1 + a^{(b)}(\theta' - i0)\big)\big\} \, \mathrm{d}\theta', \tag{6.4.26}$$

and the non-local integrals of motion are given by

$$\mathfrak{S}_q^{(a)} = 2 \sum_{b=1}^{8} \int_{-\infty}^{\infty} e^{30qM(\theta' - i0)} \operatorname{Res}\Big[\widetilde{H}_{ab}(k), k = 30qMi\Big] \cdot \\ \operatorname{Im}\big\{\log\big(1 + a^{(b)}(\theta' - i0)\big)\big\} \, \mathrm{d}\theta'. \tag{6.4.27}$$

6.5 Conclusions

In this chapter we have extended the massive ODE/IM correspondence to systems of differential equations associated with the affine Toda field theories for the Lie algebras $D_r^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$. We applied the same procedure used to calculate the integrals of motion for the $A_r^{(1)}$ case to do the same for the remaining simply-laced Lie algebras. For the $E_6^{(1)}$ case, we also constructed the associated $g = 0$ pseudo-differential equation in the massless limit. In [39], this was done for $E_7^{(1)}$ as well, converting the linear problem into three coupled pseudo-differential equations.

Some open questions regarding the treatment of the exceptional Lie algebras remain. We have not discussed the larger representations $V^{(a)}$ with $a > 1$ for the exceptional Lie algebras and their corresponding linear problems. It would be interesting to compute the asymptotics of these linear problems and define $Q^{(a)}$-functions exactly. The quantum Wronskians for the Lie algebras we considered in this chapter are all zero, leading to difficulty defining T-functions using determinants as we saw in section 5.8 for the $A_r^{(1)}$ case. The linear problem associated with the representation $V^{(1)}$ of $E_8^{(1)}$ also seems to have no subdominant solution; this causes the definition of the $Q^{(a)}$-functions for $E_8^{(1)}$ to become an interesting open problem.

Chapter 7

Conclusions and outlook

This thesis was focused on two major generalisations of the ODE/IM correspondence as presented in [26, 9]: the connection between excited states of a conformal field theory (CFT) and suitable second-order ODEs, and the extension of the correspondence to certain massive integrable models with more general Lie algebra symmetries. We began in the introduction by introducing the massless $A_1^{(1)}$ ODE/IM correspondence, matching spectral determinants constructed from eigenvalue problems related to second-order ODEs, and the ground-state eigenvalues of **Q**-operators defined on certain CFTs.

In chapter 2, we studied an extension of the massless $A_1^{(1)}$ ODE/IM correspondence to the excited states of the CFTs. The related ODEs were defined by a set of parameters $\{z_i\}_{i=1}^{L}$, $z_i \neq z_j$ which were constrained by a set of algebraic locus equations. Each solution of these locus equations is conjectured to correspond to an excited state of the CFT. The appearance of singular vectors in the CFT is telegraphed by the disappearance of one or more of the solutions of the locus equations. While investigating the solutions of the locus equations, we noticed that for particular values of the ODE parameters l and M, one of the solutions of the locus equations disappeared as three of the parameters z_i converged on the same point, but there were no singular vectors at these values of l and M in the corresponding CFT. This discrepancy was resolved by constructing a more general set of locus equations that were valid at these triple points and solving these for the location of said triple points.

It would be interesting to extend the work in chapter 2 on the $A_1^{(1)}$ excited states to more general Lie algebras. The derivation of the algebraic locus equations in section 2.3 hinged on a set of conditions due to Duistermaat [29] that ensured single-valuedness of the solutions of a particular second-order ODE. To extend the ODE/IM correspondence to the excited states of an integrable model with, for example, $A_r^{(1)}$ symmetry, a similar set of single-valuedness condition for certain r^{th}-order ODEs will need to be found. These conditions would then induce a new set of locus equations which would define a new family of ODEs corresponding to the excited states of this integrable model.

After considering the ODE/IM correspondence for the excited states of a massless integrable field theory (described by a conformal field theory) we then introduced in chapter 3 the ODE/IM correspondence as applied to a massive integrable field theory with $A_1^{(1)}$ symmetry. This case was considered in [45], and we introduced it as an indicative example of the general procedure we followed for the other simply-laced Lie algebras. Beginning with the affine Toda field equations, we defined an equivalent pair of systems of differential equations. The asymptotic solutions of these systems in the small-$|z|$ and large-$|z|$ limits were used to define Q-functions which are the massive analogue to the spectral determinants encountered in the massless case. These Q-functions satisfy certain functional relations called quantum Wronskians, and a set of Bethe ansatz equations. These Bethe ansatz equations, along with the asymptotic behaviour of the Q-functions in the limits $\operatorname{Re}\theta \to \pm\infty$, are used to construct a set of non-linear integral equations, which themselves facilitated an expression for the logarithm of the Q-functions.

The coefficients of this expression expanded in the $\mathrm{Re}\,\theta \to \pm\infty$ limits were the ground-state eigenvalues of the integrals of motion associated with the massive integrable field theory.

After codifying Lie algebra notations and outlining some general methods for analysing systems of differential equations in chapter 4, the massive ODE/IM correspondence was defined for the simply-laced Lie algebras in chapters 5 and 6. In the $A_r^{(1)}$ case we additionally constructed T-functions that were found to satisfy fusion relations and TQ-relations. In this way, features of the integrable models were found to have analogues in the regime of classical partial differential equations.

An immediate avenue for further research would be to extend the massive ODE/IM correspondence to the affine Toda field theories described by the non-simply-laced Lie algebras. This line of inquiry has begun to be explored, with some results in [18, 38, 48]. Many of the techniques we have applied in this thesis have depended on the Langlands self-duality of the simply-laced Lie algebras; the algebras we have seen are invariant under sending roots to co-roots and vice versa. Without this self-duality, the treatment of the non-simply-laced cases Lie algebras is rather more subtle. It would be interesting to generalise the methods for finding functional relations given in [40] to different non-simply-laced Lie algebras.

Bibliography

[1] P. Adamopoulou, *Differential Equations and Quantum Integrable Systems*, PhD thesis, University of Kent, October 2013.

[2] P. Adamopoulou and C. Dunning, *Bethe ansatz equations for the classical $A_n^{(1)}$ affine Toda field theories*, Journal of Physics A Mathematical General **47** (2014), no. 20, 205205.

[3] O. Babelon, D. Bernard, and M. Talon, *Introduction to classical integrable systems*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2003.

[4] R.J Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, 1982.

[5] V. V. Bazhanov, A. N. Hibberd, and S. M. Khoroshkin, *Integrable structure of W_3 Conformal Field Theory, Quantum Boussinesq Theory and Boundary Affine Toda Theory*, Nuclear Physics B **622** (2002), 475–547.

[6] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, *Integrable structure of conformal field theory, quantum KdV theory and Thermodynamic Bethe Ansatz*, Communications in Mathematical Physics **177** (1996), 381–398.

[7] ———, *Integrable Structure of Conformal Field Theory II. Q-operator and*

DDV equation, Communications in Mathematical Physics **190** (1997), 247–278.

[8] ______, *Integrable Structure of Conformal Field Theory III. The Yang-Baxter Relation*, Communications in Mathematical Physics **200** (1999), 297–324.

[9] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, *Spectral Determinants for Schrödinger Equation and Q-Operators of Conformal Field Theory*, Journal of Statistical Physics **102** (2001), 567–576.

[10] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, *Higher-level eigenvalues of Q-operators and Schrödinger equation*, Advances in Theoretical and Mathematical Physics **7** (2004), 711–725.

[11] V. V. Bazhanov and V. V. Mangazeev, *Eight-vertex model and non-stationary Lamé equation*, Journal of Physics A Mathematical General **38** (2005), L145–L153.

[12] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Physics B **241** (1984), 333–380.

[13] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I*, Springer, 1999, chapter on WKB Theory.

[14] J.-S. Caux and J. Mossel, *Remarks on the notion of quantum integrability*, Journal of Statistical Mechanics: Theory and Experiment **2** (2011), 02023.

[15] J. B. Conway, *Functions of one complex variable*, 2 ed., Academic Press, 1978.

[16] C. Destri and H. J. de Vega, *New thermodynamic Bethe ansatz equations without strings*, Physical Review Letters **69** (1992), 2313–2317.

[17] P. Di Francesco, P. Mathieu, and S. A. Sénéchal, *Conformal Field Theory*, Springer, 1997.

[18] P. Dorey, C. Dunning, D. Masoero, J. Suzuki, and R. Tateo, *ABCD and ODEs*, ICMP, Rio de Janeiro, Brazil, 2007.

[19] ______, *Pseudo-differential equations, and the Bethe ansatz for the classical Lie algebras*, Nuclear Physics B **772** (2007), 249–289.

[20] ______, *Pseudo-differential equations, and the Bethe ansatz for the classical Lie algebras*, Nuclear Physics B **772** (2007), 249–289.

[21] P. Dorey, C. Dunning, and R. Tateo, *Differential equations for general SU(n) Bethe ansatz systems*, Journal of Physics A Mathematical General **33** (2000), 8427–8441.

[22] ______, *Spectral equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics*, Journal of Physics A Mathematical General **34** (2001), 5679–5704.

[23] ______, *The ODE/IM correspondence*, Journal of Physics A Mathematical General **40** (2007), 205.

[24] P. Dorey, A. Millican-Slater, and R. Tateo, *Beyond the WKB approximation in {\cal PT} -symmetric quantum mechanics*, Journal of Physics A Mathematical General **38** (2005), 1305–1331.

[25] P. Dorey and R. Tateo, *Anharmonic oscillators, the thermodynamic Bethe ansatz and nonlinear integral equations*, Journal of Physics A Mathematical General **32** (1999), L419–L425.

[26] ______, *On the relation between Stokes multipliers and the T-Q systems of conformal field theory*, Nucl. Phys. **B563** (1999), 573–602.

[27] ______, *Differential equations and integrable models: the SU(3) case*, Nuclear Physics B **571** (2000), 583–606.

[28] V. G. Drinfel'd and V. V. Sokolov, *Lie Algebras and Equations of KORTEWEG-DE Vries Type*, W-Symmetry. Series: Advanced Series in Mathematical Physics, ISBN: ¡ISBN¿978-981-02-1762-4¡/ISBN¿. WORLD SCIENTIFIC, Edited by P Bouwknegt and K Schoutens, vol. 22, pp. 25-86 **22** (1995), 25–86.

[29] J. J. Duistermaat and F. A. Grünbaum, *Differential equations in the spectral parameter*, Communications in Mathematical Physics **103** (1986), 177–240.

[30] K. Erdmann and M. J. Wildon, *Introduction to Lie Algebras*, Springer, 2006.

[31] B. Feigin and E. Frenkel, *Quantization of soliton systems and Langlands duality*, ArXiv e-prints (2007).

[32] G. Felder, A. D. Hemery, and A. P. Veselov, *Zeros of Wronskians of Hermite polynomials and Young diagrams*, Physica D Nonlinear Phenomena **241** (2012), 2131–2137.

[33] D. Fioravanti, *Geometrical loci and CFTs via the Virasoro symmetry of the mKdV-SG hierarchy: an excursus*, Physics Letters B **609** (2005), 173–179.

[34] E. Frenkel and D. Hernandez, *Spectra of Quantum KdV Hamiltonians, Langlands Duality, and Affine Opers*, Communications in Mathematical Physics (2018).

[35] H. Georgi, *Lie Algebras in Particle Physics: From Isospin to Unified Theories*, Westview Press, 1999.

[36] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, 5 ed., Academic Press, 1994.

[37] K. Ito and C. Locke, *ODE/IM correspondence and modified affine Toda field equations*, Nuclear Physics B **885** (2014), 600–619.

[38] ———, *ODE/IM correspondence and Bethe ansatz for affine Toda field equations*, Nuclear Physics B **896** (2015), 763–778.

[39] K. Ito and H. Shu, *ODE/IM correspondence and the Argyres-Douglas theory*, Journal of High Energy Physics **8** (2017), 71.

[40] ———, *ODE/IM correspondence for modified $B_2^{(1)}$ affine Toda field equation*, Nuclear Physics B **916** (2017), 414–429.

[41] ———, *Massive ODE/IM correspondence and nonlinear integral equations for {A_r^{(1)}} -type modified affine Toda field equations*, Journal of Physics A Mathematical General **51** (2018), 385401.

[42] I. Krichever, O. Lipan, P. Wiegmann, and A. Zabrodin, *Quantum Integrable Models and Discrete Classical Hirota Equations*, Communications in Mathematical Physics **188** (1997), 267–304.

[43] A. Kuniba, T. Nakanishi, and J. Suzuki, *T-systems and Y-systems in integrable systems*, Journal of Physics A Mathematical General **44** (2011), no. 10, 103001.

[44] E. H. Lieb, *Exact Solution of the Problem of the Entropy of Two-Dimensional Ice*, Physical Review Letters **18** (1967), 692–694.

[45] S. L. Lukyanov and A. B. Zamolodchikov, *Quantum sine(h)-Gordon model and classical integrable equations*, Journal of High Energy Physics **07** (2010), 008.

[46] W. Magnus, *On the exponential solution of differential equations for a linear operator*, Commun. Pure Appl. Math. **7** (1954), 649–673.

[47] D. Masoero, A. Raimondo, and D. Valeri, *Bethe Ansatz and the Spectral Theory of Affine Lie Algebra-Valued Connections I. The simply-laced Case*, Communications in Mathematical Physics **344** (2016), 719–750.

[48] ———, *Bethe Ansatz and the Spectral Theory of Affine Lie algebra-Valued Connections II: The Non Simply-Laced Case*, Communications in Mathematical Physics **349** (2017), 1063–1105.

[49] M. Nakano, M. Namiki, and T. Nishimoto, *On the WKB method for certain third order ordinary differential equations*, Kodai Mathematical Journal **14** (1991), no. 3, 432462.

[50] S. Negro, *ODE/IM Correspondence in Toda Field Theories and Fermionic Basis in sin(h)-Gordon Model*, PhD thesis, University of Turin, June 2014.

[51] J. Sun, *Polynomial Relations for q-Characters via the ODE/IM Correspondence*, SIGMA **8** (2012), 028.

[52] B. Sutherland, *Exact Solution of a Two-Dimensional Model for Hydrogen-Bonded Crystals*, Physical Review Letters **19** (1967), 103–104.

[53] J. Suzuki, *Functional relations in Stokes multipliers and solvable models related to $U_q(A^{(1)}{}_n)$*, Journal of Physics A Mathematical General **33** (2000), 3507–3521.

[54] ———, *Functional Relations in Stokes Multipliers - Fun with $x^6 + \alpha x^2$ Potential*, Journal of Statistical Physics **102** (2001), 1029–1047.

[55] A. Voros, *Spectral zeta functions*, Zeta Functions in Geometry (Tokyo, Japan), Mathematical Society of Japan, 1992, pp. 327–358.

[56] A. Voros, *Airy function - exact WKB results for potentials of odd degree*, Journal of Physics A Mathematical General **32** (1999), 1301–1311.

[57] A. Zee, *Quantum Field Theory in a Nutshell*, 2 ed., Princeton University Press, 2010.

[58] P. Zinn-Justin, *Nonlinear integral equations for complex affine Toda models associated with simply laced Lie algebras*, Journal of Physics A Mathematical General **31** (1998), 6747–6770.