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Robust Observers for Nonlinear Large Scale Interconnected Systems

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By

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ABSTRACT

Some control design schemes need system states to develop controllers. In real applications, it might not be possible to measure all system states due to the cost of sensors or any technical issues. Therefore, Observers are needed to estimate unavailable states and then use the estimated states in control design schemes.

This thesis focuses on the development of robust observers for nonlinear large-scale interconnected systems. System structures of nonlinear interconnected systems have been considered in the design to reduce conservatism and different techniques have been utilised in observer design to enhance the robustness. The main developments in this thesis include:

- A robust observer is designed for nonlinear interconnected systems with uncertainties where both the nominal isolated subsystems and interconnections are nonlinear. A novel variable structure dynamic system is designed to estimate the state variables of the interconnected systems asymptotically.

- Sliding mode techniques have been utilised to design observers for a class of interconnected systems with both structured and unstructured uncertainties. For structured uncertainties, a robust sliding mode observer is designed using the distribution matrix of the uncertainties to guarantee that the error dynamics are asymptotically stable. For unstructured uncertainties, an ultimately bounded approximate observer is proposed to estimate the system states.

- In the case that the interconnected systems have unknown time varying parameters, adaptive observers are developed to estimate both the system states and the unknown time varying parameters simultaneously. A variable structure control technique is utilized to guarantee that the corresponding error dynamical system is convergent asymptotically while the parameter estimation error is uniformly ultimately bounded. Moreover, adaptive sliding mode observer is proposed to estimate the inaccessible states in presence of unknown time varying parameters.

This thesis also provides case studies on applications of the proposed methods. A case study of a coupled inverted pendulum system shows the practicality of the variable

structure observer. The designed sliding mode observer is applied to a multi-machine power system, and the simulation is provided to demonstrate the validity and effectiveness of the designed observer. A coupled inverted pendulum system and a quarter-car suspension system are employed to explain the designed observation scheme and to demonstrate that the designed adaptive observers can be applied to practical systems effectively.

NOTATION AND SYMBLOS

$\equiv$	Identically equal
$<$ $(>)$	Less (greater) than
$\leq$ $(\geq)$	Less (greater) than or equal to
$\forall$	For all
$\in$	Belongs to
$\rightarrow$	Tends to
$\Rightarrow$	Implies
$\Leftrightarrow$	Equivalent to, if and only if
$\emptyset$	The empty set
R	The set of real numbers
R^+	The set of nonnegative real numbers
R^n	The n dimensional Euclidean space
$R^{n\times m}$	The set of $n \times m$ matrices with elements in R
a	The absolute value of a scaler a
$\\|\cdot\\|$	The Euclidean norm or its induced norm
I_n	The unit matrix with dimension n
$Im(A)$	The range space of matrix A
B_r	The ball $\{x \mid \\|x\\| < r\}$ with radius r where $r \in (0, +\infty)$
A^T	The transpose of matrix A
$A > 0$	A is a symmetric positive definite matrix
$A < 0$	A is a symmetric negative definite matrix
λ_{min}	The minimum eigenvalue of the squar matrix A
λ_{max}	The maximum eigenvalue of the squar matrix A
L_f	The Lipschitz constant of the function $f(\cdot)$
L_fh	Derivative of the mapping $h(x) : R^n \mapsto R^p$, along the vector field $f(x,u) : R^n \times R^m \mapsto R^n$ defined by $L_{f(x,u)}h(x) := \frac{\partial h}{\partial x} f(x,u)$

$J_f(x)$ *or* $\frac{\partial f(x)}{\partial x}$	The Jacobian matrix of the function $f(x)$
$[f, g]$	Lie bracket (product) of the vector fields $f(x)$ and $g(x)$, defined by $[f, g](x) = J_g(x)f(x) - J_f(x)g(x)$
$L_f^r h$	The rth order Lie derivative of the mapping $h(x) : R^n \mapsto R^p$, along vector field $f(x, u) : R^n \times R^m \mapsto R^n$
$f(x, u)$	$f(x_1, x_2, \cdots, x_{n_1}, y_1, y_2, \cdots, y_{n_2})$ where $x = [x_1, x_2, \cdots, x_{n_1}]^T \in R^{n_1}$ and $y = [y_1, y_2, \cdots, y_{n_2}]^T \in R^{n_2}$
$\dot{y}$	The first derivative of y with respect to time
$\frac{\partial f(x,y)}{\partial x)}$	Jacobian matrix of function $f(x, y)$ relating the variable x where $x \in R^{n_1}$ and $y =\in R^{n_2}$
col$(x_1, x_2, \cdots, x_n)$	The coordinates $[x_1, x_2, \cdots, x_n]^T$ where $x_i \in R$ for $i = 1, 2, \cdots, n$
col(x_1, x_2)	The coordinates $[x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{2n}]^T$ where $x_1 = [x_{11}, x_{12}, \cdots, x_{1n}]^T \in R^{n_1}$ and $x_2 = [x_{21}, x_{22}, \cdots, x_{2n}]^T \in R^{n_2}$
$A := B$	A is defined by B
$A \Rightarrow B$	A implies B
$A \Leftrightarrow B$	A is equivalent to B
$sgn(\cdot)$	The signum function

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CHAPTER. 1

INTRODUCTION

Control theory which mainly focuses on control systems, has become an important topic in a modern technical society since the beginning of the twentieth century. Control system is a system that could improve a behaviour of a particular process to achieve the desired performance. An automatic control system is a system which is made to be self-regulating. World War II was the main motivation behind the significant developments in the automatic control field [2]. Then, the needs of modern life, such as power systems, water treatment, industry applications and space technologies etc, required accuracy and reliability of automatic control systems. Since then, control techniques have been significantly developed to tackle control and estimation problems. In some designed robust feedback control strategies, the information of all system states is required for design. However, in reality, only a subset of the system state variables are available for design. This provides a requirement to estimate all the system states to form a feedback control system if possible. In this thesis, studies on designing observers/estimators for nonlinear interconnected systems have been carried out with detailed analysis and background description.

1.1. Background and Motivation

Large-scale interconnected systems widely exist in the real world, for example, power networks, ecological systems, transportation networks, biological systems and information technology networks [3, 4]. A large-scale system is composed of several subsystems with interconnections, whereby the dynamics interact with each other [5]. The application of centralised control [6] to prescribe stability of an interconnected system may require additional costs for implementation and careful consideration of the required information sharing between subsystems particularly when the system is spread over a wide geographical area. This motivates the design of decentralized control strategies whereby each subsystem has a local controller which is based only upon local available information.

Increasing requirements for system performance have resulted in increasing complexity within system modelling and it becomes of interest to consider nonlinear large-scale interconnected systems. Such models are then used for controller design to sort out the practical problems. Early work focused on linear systems [7, 8]. However, due to the uncertainties and disturbances present in large-scale interconnected systems, study of such systems is a very challenging task [9]. Series of results have been developed using decentralised control framework for nonlinear large-scale interconnected systems. The study of such decentralised controllers has stimulated a wide literature (e.g. [10, 11]) and recently [12, 13].

In much of this work, however, it is assumed that all the system state variables are available and the system parameters are known for control design. [14, 3, 15, 16]. However, this assumption can limit practical application as usually only a subset of state variables is available/measurable [17]. It becomes of interest to establish observers to estimate the system states and then use the estimated states to replace the true system states in order to implement state feedback decentralised controllers. Moreover, in many physical systems, unforeseen changes in the plant may occur such as changing stator and rotor resistance in an electric drive/motor due to the temperature rising or load changes. In order to avoid poor control performance and even instability caused by parametric uncertainties, an adaptive observer is mostly needed to estimate the unknown constant/time varying parameters and the system states simultaneously [2]. The two kinds of observers: Robust Observers and Adaptive observers will be developed in this thesis.

1.1.1. OBSERVER

The concept of an observer was first introduced by Luenberger (1964, 1966) where the difference between the output measurements from the actual plant and the output measurements of a corresponding dynamical model was used to develop an injection signal to force the resulting output error to be zero [18, 19]. Since then, observer theory attracts a huge amount of attention to extend Luenberger observer to be applicable to a wide range of different classes of systems. For nonlinear systems, many approaches have been developed to design observers. Early results have been presented in [20, 21]. In these studies, the techniques that used are based on Lyapunov theory, and the nonlinearities are required to satisfy the well known Lipschitz condition. In [22, 23], the transformation of the nonlinear system to observable canonical form is needed, and it is assumed that the information of the nonlinearities exists which is not applicable in many cases. The technique of extended linearization has been used in [24] to design an observer such that the family of linearized closed-loop systems have locally constant eigenvalues. The extended Luenberger observer has been given in [25] using linearisation technique. Although the algorithm proposed in [25] is easy to implement, the convergence of the observer is not guaranteed. Sufficient Lyapunov-like conditions are presented in [26] for the existence of a nonlinear observer which guarantees the convergence of the error system to zero. An exponential observer for a nonlinear system has been presented in [27] where an appropriate Lyapunov-like equation is needed, and the existence of global Lipschitz change of coordinates is required.

Variable structure control and sliding mode techniques have been used widely in literature to design observers for linear and nonlinear systems due to its robustness against uncertainties and disturbances in input channels and the finite-time convergence [17, 28]. The main difference between sliding mode observers and linear Luenberger observers is that the sliding mode observer includes a nonlinear discontinuous term. The discontinuous term is designed to drive the trajectory of the estimation error towards a surface and remains on it thereafter. Most of the researchers design the sliding surface as the difference between the output of the observer and the output of the system and then force the error system to zero. The earliest contributions in this area are presented in [29, 30, 31, 32]. The authors in [29, 30] designed an observer which has the output error being fed back in both a linear and a discontinuous manner for nonlinear systems. A variable structure observer is developed in [32] and an asymptotic state observer for uncertain systems satisfying the matching conditions was developed. Utkin designed an observer, with the discontinuous term being fed back through an appropriate gain [33]. A sliding mode observer based on a canonical form presented in [34] where specific conditions are imposed on the invariant zeros of the system. The authors in [35] designed sliding mode observers for a class of uncertain systems using linear matrix inequalities to exploit the degrees of freedom available in the design to overcome the limitation in [34]. It is shown in [36] that "The work in [32] was instrumental in defining the structural conditions for existence of sliding mode observers for linear systems and laid important foundations for subsequent contributions which formulated constructive design methodologies [28, 34]".

Sliding mode techniques have been used to design observers for nonlinear interconnected power systems in [89]. In [90] state estimation and sliding mode control for a special class of stochastic dynamic systems which is semi Markovian jump systems is presented. The authors designed a state observer to generate the estimate of unmeasured state components, and then synthesize a sliding mode control law based on the state estimates. The authors in [91] discussed the position regulation problem of permanent magnet synchronous motor (PMSM) servo system based on adaptive fuzzy sliding mode control (AFSMC) method. They used adaptive method to estimate the upper bound of the approximation error between the equivalent control law and the fuzzy controller are utilized in the paper.

Observer-based controller design has been studied extensively for power systems. An observer-based controller proposed in [112] by combining a variable structure control with a reduced-order observer, which is then applied to a power system stabilizer. However, the observer-based controller is designed for a linear system and the system considered incorporates one power system. In addition, there are no unstructured uncertainties considered in the system. The research in [113] considers controller design for nonlinear systems and a nonlinear observer is used to estimate the unmeasurable states. This requires that the system can be represented in a Hamiltonian and triangular form. The authors in [114] designed a decentralized controller, i.e., for each subsystem a local controller is designed, using sliding mode techniques. This work does not involve observer design. The authors in [115] develop a functional observer approach for load frequency control of highly interconnected power networks. A quasi-decentralized functional observer is used to generate the control signal rather than estimate all the system states without considering any uncertainties. A load frequency control strategy based on sliding mode techniques and a disturbance observer is proposed in [116]. Although the authors consider uncertainties in the structure of the power system model, the observer designed is just for a power system instead of a multimachine power system. In [117] a controller which uses a nonlinear observer is developed for multimachine power systems to improve the transient stability. However, the authors did not consider the impact of disturbances. In [118] an unknown-input observer is deployed which can estimate the system states as well as perform fault detection and isolation. This is applied to a three-bus power system with one generator and two loads. It should be pointed out that the power system model considered in [118] is a differential algebraic model which is called a singular system (see [119]). Moreover, from the point of view of observation, observer based controller design imposes strong requirements on the considered system as the designed observer is for a specific task. In addition, observer design in the presence of unknown signals is challenging. An extended Kalman filter is used in [120, 121] to enhance frequency estimation of distorted power signals. However, in real time, it is difficult to implement this Kalman filter due to the poor flexibility in dealing with higher order systems. In addition, sliding mode techniques have advantages over Kalman filter approaches for electric power systems. One of these advantages is that robustness of the sliding-mode observer to parameter uncertainties and external noise can be guaranteed [122, 123]. State estimation and sliding mode control for a special class of stochastic dynamic systems, semi Markovian jump systems, is presented in [90]. The authors designed a state observer to estimate unmeasured state components, and then synthesize a sliding mode control law based on the state estimates. The exact feedback linearization technique is used to design a nonlinear observer in [124] when the power system can be fully linearized. A sliding mode observer is presented in [125] to develop a robust observer-based nonlinear controller. This is used to construct the state variables of the system and estimate the perturbation. A sliding mode observer is designed in [86] where the observation error dynamics are ultimately stable instead of asymptotically stable as the structure of the uncertainties is not available. Moreover, results relating to sliding mode observer design for multimachine power systems are limited. It should be noted that when the structure of the interconnections is known and when the interconnections have certain properties, it is possible to design an asymptotic observer to obtain estimates with high accuracy. However, when the structure of the interconnections is not available, to design an asymptotic observer is challenging. In this case an approximate observer for large scale interconnected systems may satisfy the practical requirements.

In the 1970s, the problem of designing observers for estimating system states for large-scale interconnected systems was addressed in [7]. Subsequently, many methods have been developed to design observers for large-scale interconnected linear systems [37, 38, 39, 40]. Observer schemes for interconnected systems are proposed in [92, 84, 93, 85] where the obtained results are unavoidably conservative as it is required that the designed observer can be used for certain fault detection and isolation problems. For example, it is required that the uncertainty can be decoupled with faults in [85] and the considered system is not interconnected systems. Robust observer design is considered in [94] for a class of linear large scale dynamical systems where it is required that the interconnections satisfy quadratic constraints. In [95] a new decentralized control scheme which uses estimated states from a decentralised observer within a feedback controller is proposed. This uses a design framework based on linear matrix inequalities and is thus applicable for linear systems. A robust observer for nonlinear interconnected systems based on a constrained Lyapunov equation has been developed in [96].

A Proportional Integral observer is utilized for nonlinear interconnected systems for disturbance attenuation in [97] and interconnected nonlinear dynamical systems are considered in [98] where the authors combine the advantages of input-to-state dynamical stability and use reduced order observers to obtain quantitative information about the state estimation error. This work does not, however, consider uncertainties. It should be noted that in all the existing work relating to observer design for large scale interconnected systems, it is required that either the isolated subsystems are linear or the interconnections are linear. Moreover, most of the designed observers are used for special purposes such as fault detection or stabilization and thus they impose specific requirements on the class of interconnected systems considered.

MOTIVATION 1

It should be noted that in all of the existing work relating to observer design for large scale interconnected systems, it is required that either the isolated subsystems are linear or the interconnections are linear. Moreover, most of the designed observers are used for special purposes such as fault detection or stabilization and thus some specific limitations are unavoidably required on the interconnected systems as a result of the problems considered. It should be noted that results concerning observer design for nonlinear interconnected systems are very few when compared with the results available on observer design for systems without interconnections. This is also the case for controller design. The observer design for nonlinear interconnected systems with uncertainties is an interesting but challenging topic for researchers.

1.1.2. ADAPTIVE OBSERVER

In the real world, many practical control systems involve unknown parameters due to the mechanical wearing and modeling errors. Therefore the adaptive observer is developed to estimate the unavailable states and the unknown parameters simultaneously. It is well known that adaptive technique is powerful for parameter estimation. Over the last few decades, much literature has been devoted to the design of adaptive observers for linear and nonlinear systems. The early results are mainly for linear systems [41, 42, 43, 44, 45] and later some results are presented in [46]. An adaptive observer is designed for a class of interconnected systems in [87] in which it is required that the isolated nominal subsystems are linear. Many results have been developed for nonlinear systems since the late of 1980s [47, 48, 49, 50, 51, 52, 53, 54]. The authors in [47] designed an adaptive observer for a single input single output nonlinear system based on a transformation of the system into a certain canonical form.

The work in [48] is a general class of nonlinear systems presented in [49] in which the nonlinearities are assumed to be Lipschitz. In the presence of bounded disturbances, robust adaptive observers are developed in [52]. The results for linear systems presented in [44] are extended to the case of nonlinear systems in [54] and arbitrarily fast exponential convergence of both parameter and state estimates to their actual values respectively is guaranteed. The parameter convergence is guaranteed under certain conditions of persistent excitation in most of the previously mentioned works. Some results on design of adaptive observers deal with nonlinear parameterization, see for example [55, 56, 57, 58]. Recently, much literature has devoted to design adaptive observers for nonlinear systems and many different methods have been developed in order to obtain high estimation performance in the presence of parametric uncertainty and/or unstructural uncertainty. Boizot *et al* in [129] developed an adaptive observer by using extended Kalman filter to reduce the effect of perturbations. However, in terms of the parameter estimation for nonlinear systems, it is usually very difficult to analysis the stability of the extended Kalman filter.

Adaptive observers using different techniques have been proposed in (see e.g. [146, 147, 148]) where the unknown parameters are limited to be constant. Compared with much existing work in adaptive observer design with unknown constant parameters, the corresponding observation results for unknown time varying parameters (TVPs) are very limited. The approach for nonlinear time varying systems proposed in [47] is based on the fact that the nonlinear systems can be transformed to a particular observable canonical form, and the unknown parameters are bounded. The authors in [58] proposed a sampled output high gain observer for a class of uniformly observable nonlinear systems where the unknown parameters are bounded. An adaptive estimator is proposed in [149] to estimate TVPs for nonlinear systems. However, all the system states are assumed available. The H_-/H_∞ fault detection observer in the finite frequency domain has been designed in [150] for a class of linear parameter varying descriptor systems. In [151] an adaptive observer for a class of nonlinear interconnected systems with uncertain TVPs has been developed. It is required to solve the well known constrained Lyapunov problem (CLP) (see e.g. [100, 99]). The authors in [152] used an adaptive unscented Kalman filter approach to estimate the time varying parameters and system states of a class of nonlinear high-speed objects. This technique requires the assumption that the additive noise vectors are Gaussian uncorrelated white noises. If the assumption is not satisfied, the estimation process accuracy will be significantly affected [153, 154].

At the beginning of this century, sliding mode techniques have been successfully combined with adaptive observer design. In the early work in [59], an adaptive sliding mode observer was designed to reconstruct the state from the output measurements and to estimate the uncertain part of dynamics by using the nonlinear adaptive gain. The authors in [60] combined sliding mode techniques and adaptive techniques to identify the unknown parameter and to estimate the system state variables. Then the designed observer was used in a fault estimation scheme. Sliding mode techniques have been applied in [130] to enhance the performance of the adaptive observer proposed by [60]. It should be noted that unknown parameters considered in these papers are constant. An observer for linear time-varying systems with known time-varying matrices affected by unknown input is designed in [131] to estimate the systems states using high order sliding mode techniques. Adaptive observer has been considered in [132] to estimate just the synthetical perturbation with unknown bounds in order to achieve a fast and accurate reusable launch vehicle attitude tracking with chattering attenuation in presence of knowing the system states and its parameters. The authors in [133] designed a state observer and an adaptive disturbance observer to estimate the system state and the disturbance, simultaneously. However, the unknown parameters and interconnected systems are not considered. An adaptive redesign of reduced order nonlinear observers is presented in [134] where the solution of a partial differential equation is required, which may not be possible in most of cases. In order to improve the quality of the current drawn from the utility grid, an adaptive nonlinear observer is designed in [135] to estimate the inductor current which is required in the closed-loop control system of power factor correction as an essential part of AC/DC converters. Adaptive sliding-mode observer-based approaches have been widely applied to many real systems (see e.g. [61, 62, 63, 64, 65]).

An adaptive observer is designed for a class of MIMO uniformly observable nonlinear systems with linear and nonlinear parametrizations in [55] and the exponential convergence of the error dynamics for both types of parametrization is guaranteed under the persistent excitation condition. Tyukin *et al* in [57] considered the problem of asymptotic reconstruction of the state and parameter. However, in both [55] and [57], it is required that the unknown parameters are constant. The literature in [136] proposed an adaptive state estimator for a class of multi-input and multi-output non-linear systems with uncertainties in the state and the output equations, in which the systems considered are not interconnected systems. The work in [137] proposed an adaptive observer which expands the extended state observer to nonlinear disturbed systems. However, the adaptive extended state observer is linear and requires that the error dynamics can be transformed into a canonical form.

Observer design for interconnected systems has been widely studied. Observers have been proposed in [138] for linear large scale systems, where the unknown parameters are not considered. Sliding mode observers have been presented for interconnected systems in [139] where a few coordinates are required to obtain the regular form, and the parameter uncertainty is not considered. Adaptive sliding mode observer based fault reconstruction for nonlinear systems with parameters uncertainties is proposed in [140]. However, the unknown parameters vector considered is constant. An adaptive interconnected observer is proposed for sensorless control of a synchronous motor in [141] where the system considered includes only two subsystems. In addition, the observer designed is mainly used to implement a special control task. Therefore, strong limitation is unavoidably imposed on the considered interconnected systems. Moreover, in most of the existing work, it is required that either the unknown parameters are constant (see e.g [130, 52]) or the nominal values of the unknown parameters are known [142]. The corresponding observation results for large scale nonlinear interconnected systems are very limited, particularly when uncertain time varying parameters are involved. Many adaptive observers have been developed using sliding mode techniques for particular applications and for particular purposes (see e.g. [157, 158, 159]) and thus corresponding specific conditions need to be imposed on the systems considered. Sliding mode techniques with super twisting algorithm are used in [160] to design adaptive observers for nonlinear systems where the unknown parameter vector is assumed to be constant. Sliding mode synchronization method is combined with adaptive techniques in [161] to estimate the unknown parameters for multiple chaotic systems where the system states are assumed to be known and the unknown parameters are constant.

MOTIVATION 2

It should be noted that the vast majority of existing work related to the adaptive observer design scheme assumed that the unknown parameters are constant. The problem of designing an adaptive observer for nonlinear interconnected systems with time varying parameters is full of challenge. To the best of my knowledge, this thesis provides the first contribution where sliding mode techniques and adaptive technique are combined together to design an adaptive sliding mode observer for nonlinear interconnected systems with unknown time varying parameters.

Motivated by the existing work mentioned above, this thesis studies the issue of observer design for nonlinear interconnected systems with uncertainties and adaptive observer design for nonlinear interconnected systems with unknown time varying parameters. Variable structure control and/or sliding mode techniques will be used to form these observer design schemes. Novel approaches developed in this work have considered designing observers to be applicable to a wide range of real interconnected systems. Moreover, the adaptive observer designed could estimate the unknown time varying parameters and the unavailable states of the system simultaneously to certain accuracy.

1.2. CONTRIBUTIONS AND THESIS ORGANISATION

This thesis contributes to the knowledge and research of observer design for nonlinear interconnected systems. The approaches established in this thesis are applied to several practical systems through case studies which include inverted pendulum systems, multi-machine power systems, and quarter-car suspension systems. The contribution of this thesis can be summarised as follows:

- A variable structure observer for nonlinear interconnected systems is proposed. Using the structure property of the interconnected systems, novel variable structure dynamics are designed to estimate the state variables of the interconnected systems asymptotically with low conservatism.
- A robust sliding mode observer has been developed to estimate states of a nonlinear interconnected systems in the presence of both structured and unstructured uncertainties. For structured uncertainties, a robust sliding mode observer is designed using the distribution matrix of the uncertainties and a set of sufficient conditions is developed to guarantee that the error dynamics are asymptotically stable. For unstructured uncertainties, an ultimately bounded approximate observer is proposed to estimate the system states.
- Unknown constant/time varying parameters exist in many practical systems. Therefore, adaptive observers have been developed to estimate the unknown time varying parameters and system states simultaneously. In addition, adaptive sliding mode observer for nonlinear interconnected systems has been proposed for the first time to estimate the unmeasurable states in the presence of the unknown time varying parameters.

The rest of this thesis is structured as follows:

Chapter 2 gives some mathematical preliminaries required for the subsequent analysis and design. Particularly, the necessary definitions and fundamental results of the Lyapunov stability theory and elementary theory of nonlinear feedback for single input single output systems required for the following chapters are provided in this chapter.

Chapter 3 states basic concepts and fundamental knowledge to help readers to understand this thesis. Basic knowledge of the observability concept and the Luneburger observer are presented. Definition and structure of interconnected systems are discussed in details and supported by three paratactical examples. Then, theory of variable structure control including sliding mode control methodology is introduced in details. After that, the concept of sliding mode observer and its history are explained clearly. Some adaptive theories and techniques used in the observer design schemes are also provided in this chapter.

Chapter 4 represents a novel approach to design an observer for nonlinear interconnected systems with disturbances using a variable structure control technique. A simulation example and a case study are presented to demonstrated the effectiveness and the feasibility of the developed results.

Chapter 5 is focused on observer design for nonlinear interconnected systems with structured and unstructured uncertainties using sliding mode techniques. The states of the system converges to the states of the designed observer asymptotically in the case of structured disturbances. An ultimately bounded converge is guaranteed for interconnected systems with unstructured uncertainties. The results obtained are applied to a multimachine power system, and simulation results are presented to demonstrate the feasibility and effectiveness of the developed methods.

Chapter 6 interests in designing adaptive observers for nonlinear interconnected systems that have unknown time varying parameters. An adaptive variable structure observer has been developed to estimate unknown time varying parameters and unmeasurable states simultaneously. Two practical systems are employed for case studies to show the effectiveness and the practicability of the obtained results.

Chapter 7 is focused on a class of systems that has unknown time varying parameters and inaccessible states. An adaptive sliding mode observer is developed to estimate the unmeasurable states in the presence of the unknown time varying parameters. The simulation examples show that the developed results are effective.

Chapter 8 presents a summary of the main conclusions in this thesis. Discussion for the potential future work is also provided in this chapter.

Finally, **Appendix** presents an explanation example of the Simulink software used in this thesis to get simulation results for all practical examples as well as numerical examples.

CHAPTER. 2

FUNDAMENTAL KNOWLEDGE AND BASIC CONCEPTS

Chapter 1 has briefly introduced the background and motivation of this research, which includes the challenges of designing observers for nonlinear interconnected systems. In order to establish a basic understanding of this work, a necessary background knowledge and basic concepts related to this thesis are to be discussed in this chapter.

To be specific, a basic knowledge of Observability and Luenburger Observer is given in Section 3.1. The definitions and the structure of interconnected systems with three practical examples are introduced in Section 3.2. Variable structure control and sliding mode technique will be explained in Section 3.3. The literature in the area of sliding mode observer is then reviewed in Section 3.4. In Section 3.5. some relevant adaptive tools will be presented.

2.1. OBSERVABILITY AND LUENBURGER OBSERVER

2.1.1. OBSERVABILITY

A system is said to be completely observable if every initial state $x(t_0)$ can be exactly determined from the measurements of the output $y(t)$ over the finite interval of time $t_0 \leq t \leq t_f$. This implies that every state of $x(t)$ affects the output $y(t)$ [67].

Consider a linear system

$$\dot{x} = Ax \tag{2.1}$$

$$y = Cx \tag{2.2}$$

where $x \in R^n$ and $y \in R^p$ are system states and output respectively. $A \in R^{n\times n}$ and $C \in R^{p\times n}$ are constant matrices.

The system (2.1) is observable if and only if the observability matrix O has full rank. i.e. rank $O = n$.

$$\text{Rank } O = \text{Rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{2.3}$$

2.1.2. LUENBURGER OBSERVER

The concept of observer was first introduced by Luenberger (1964) where the difference between the output measurements from the actual plant and the output measurements of a corresponding dynamical model were used to develop an injection signal to force the resulting output error to zero [18].

Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.4}$$

where $x \in R^n$, $y \in R^p$ and $u \in R^m$ are system states, output and input respectively. $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{p\times n}$ are constant matrices. The flow chart for the system (2.4) is shown at the upper of Figure 2.1.

The observer that has been proposed by Luenberger is

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \dot{\hat{x}} &= (A - LC)\hat{x} + \boldsymbol{Bu} + \boldsymbol{Ly} \\ \hat{y} &= C\hat{x} \end{aligned} \tag{2.5}$$

Subtracting equation (2.5) from equation (2.4) yield the observer error state equation

$$\dot{e} = (A - LC)e$$

By an appropriate choice of the observer gain L, all of the eigenvalues of $A - LC$ can be assigned to the left side plane, so that the steady state value of $e(t)$ for any initial condition is zero.

$$\lim_{t\to\infty} e(t) = 0 \tag{2.6}$$

2.2. INTERCONNECTED SYSTEMS

The advancement of modern technologies has produced many complex systems. An important class of such systems, which is frequently called a system of systems or large-scale system, can be expressed by sets of lower-order ordinary differential equations which are linked through interconnections. Such models are typically called large-scale interconnected systems [14, 3, 68, 69]. Large-scale interconnected systems have been studied since the 1960s (see [70] and references therein) due to their close relevance with a number of practical application areas and the availability of pertinent theoretical results.

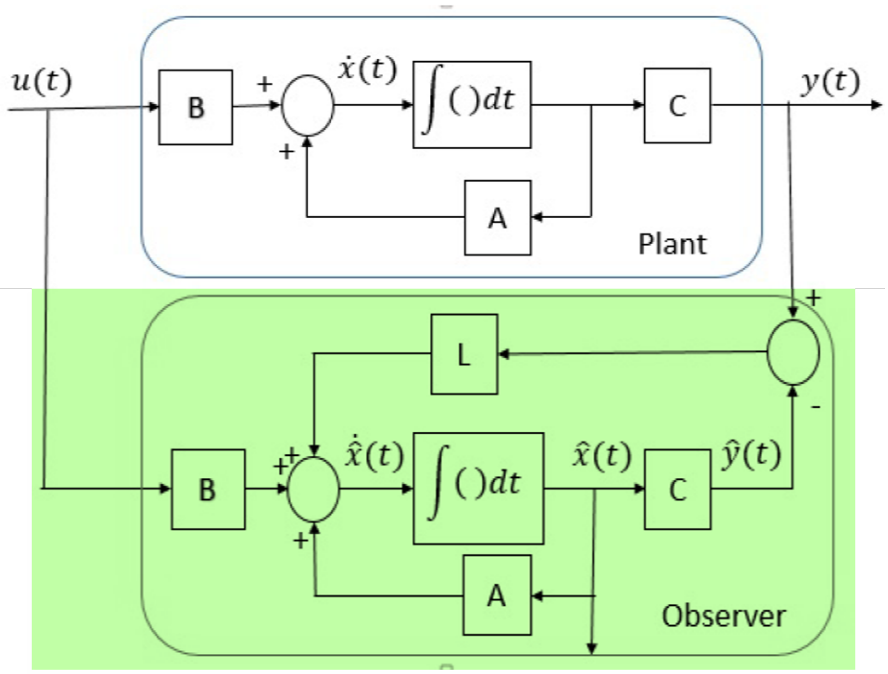

Figure 2.1: Plant with inaccessible states and an observer that estimates the unmeasurable states.

2.2.1. Structure of Interconnected System

A large-scale interconnected system usually can be modelled by the following ordinary differential equations

$$\dot{x}_i = f_i(t, x_i, u_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}(x_j) \tag{2.7}$$

where $x_i \in R^{n_i}$ and $u_i \in R^{m_i}$ denote the state and input signal of ith subsystems. The whole system state $x :=$ col$(x_1, x_2, \cdots, x_N)$ denote the states of the whole system. $H_{ij}(\cdot)$ represent the interconnection linked with the j-th subsystems for $i = 1, \cdots, N$ and $i \neq j$. The term $\sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}(x_j)$ is called the interconnection term. System

$$\dot{x}_i = f_i(t, x_i, u_i) \tag{2.8}$$

is called the ith isolated subsystem of system (2.7). In such systems, although each isolated subsystem may exhibit desired performance, each interaction dynamic due to the interconnections may significantly affect on the performance of the whole system plant.

Consider the form of system (2.7) in

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x_i)(u_i + \phi_i(t, x_i)) + \sum_{\substack{j=1 \\ j \neq i}}^{N} H_{ij}(x_j) \tag{2.9}$$

where $\phi_i(\cdot)$ denote the matched uncertainties (uncertainties in input channels) for $i = 1, 2, \cdots, N$. Then system

$$\dot{x}_i = f_i(t, x_i) + g_i(t, x_i)(u_i + \phi_i(, x_i)) \tag{2.10}$$

is called the ith nominal isolated subsystems of system (2.9) for $i = 1, 2, \cdots, N$.

There are three main architectures, i.e. centralised structure, distributed structure and decentralised structure which will be briefly introduced.

Centralised Structure

For a relative small-scale system, a system structure shown in Figure 2.2, in which the control utilises all the state information to stabilise the overall system, is the so-called centralised structure. The local controller with centralised structure is often in the form

$$u_i = u_i(t, x_1, x_2, \cdots, x_N) \tag{2.11}$$

For small scale interconnected systems, a centralised strategy might provide better control performance than a complete decentralised control strategy since the central coordinator has much more information than a local controller to deal with the interactions. However, as the system nowadays has become increasingly complicated, the system inevitably requires computing units with much larger memory and much faster computation capability. The extensive amount of information fed into the coordinator and the massive computing tasks may make the centralised scheme difficult or even impossible to be implemented [14].

Distributed Structure

In distributed control structure see Figure 2.3, it is assumed that some information can be transmitted among those local regulators through a network so that regulators in each subsystem can have some knowledge on the behaviour of the others [71]. This system structure is widely used in power grid regulation [72], intelligent transportation

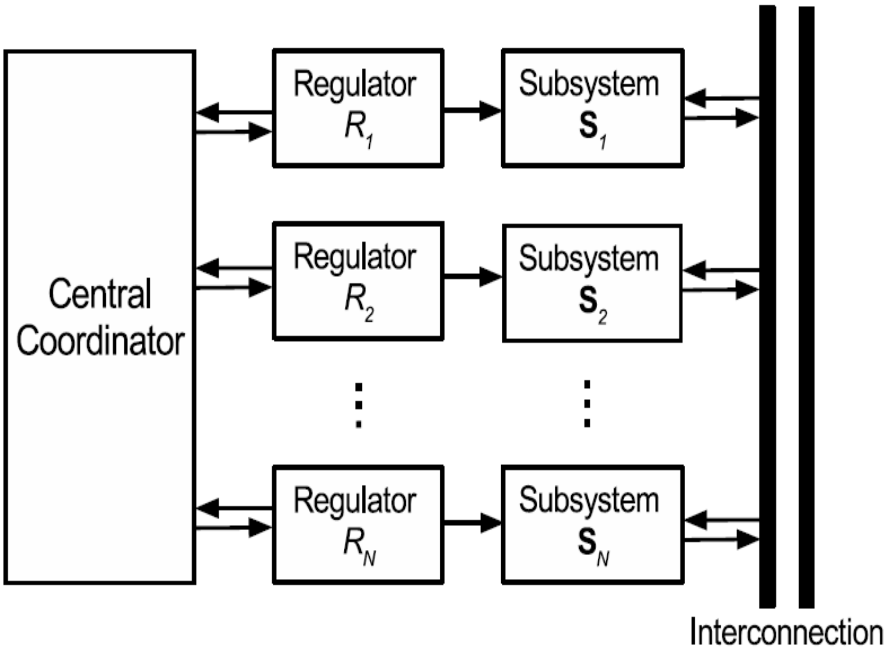

Figure 2.2: Centralised system structure

system [73]. Since a subsystem may not need all the information of the system plant, therefore, the information that is transmitted from the local regulators is usually fed in a given subset of the others, which is also known as *partially connected* [74]. However, the delay of the network is still an essential problem for the distributed control system. Furthermore, the network failure may greatly degrade the control performance of a local regulator or even devastate the stability of the subsystem.

Decentralised Structure In a decentralised control system shown in Figure 2.4, the local controllers of each subsystem for system (2.7) are usually in the form

$$u_i = u_i(t, x_i) \tag{2.12}$$

in which only local state information x_i is used for local controller in the ith subsystem. Since the controller is based only on local information, the reliability of system performance only depends on the control performance of the local controllers. This system structure is also economical since the network for information transmitting is not required and thus the cost of the implementation and complexity of the overall system is reduced. However, due to the lack of information from the other subsystems, to maintain desired control performance in the presence of unknown interconnection usually is full of chal-

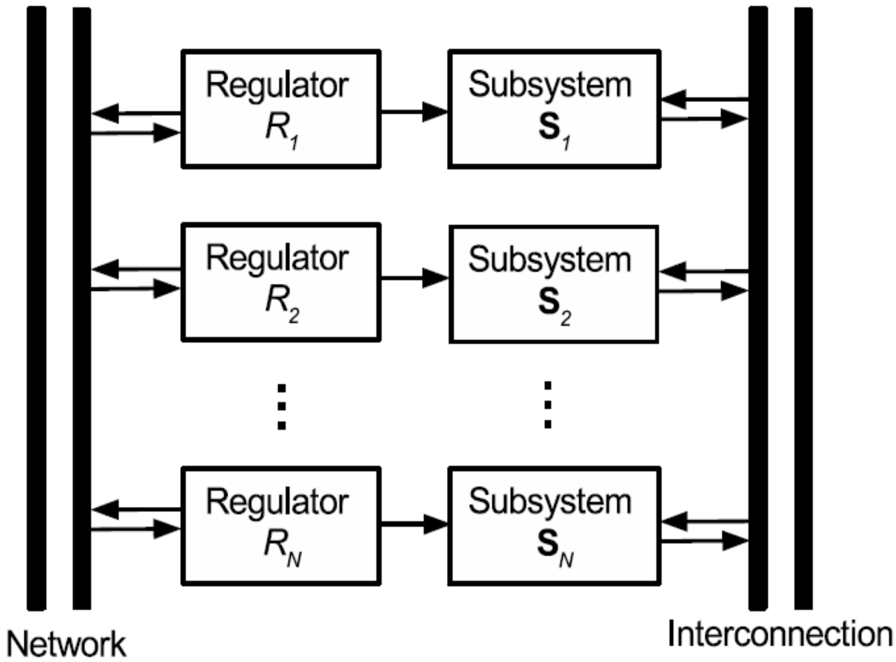

Figure 2.3: Distributed system structure

lenges. Furthermore, disturbances, modelling errors and parameter variations also widely exist in most practical systems. Specifically, uncertainties experienced by one subsystem not only affect its performance but affect the other subsystems' performance as well due to the interactions among subsystems [17]. In the real world, it is not safe to assume that all state variables are available due to the cost of sensors or technical difficulties. Moreover, many practical systems have unknown constants/time varying parameters affecting system performance. To over come such a problem, observers and adaptive observers have been developed since 1960s. Therefore, observers/adaptive observers for nonlinear interconnected systems will be considered in this thesis.

2.2.2. Practical Examples of Interconnected Systems

Large-scale interconnected systems widely exist in the real world. In this section, the models of several practical interconnected systems will be presented to show that the system (2.7) can be well used to express practical interconnected systems.

Power Systems

Nowadays, electricity plays a crucial role in our daily lives. Apparently, with the

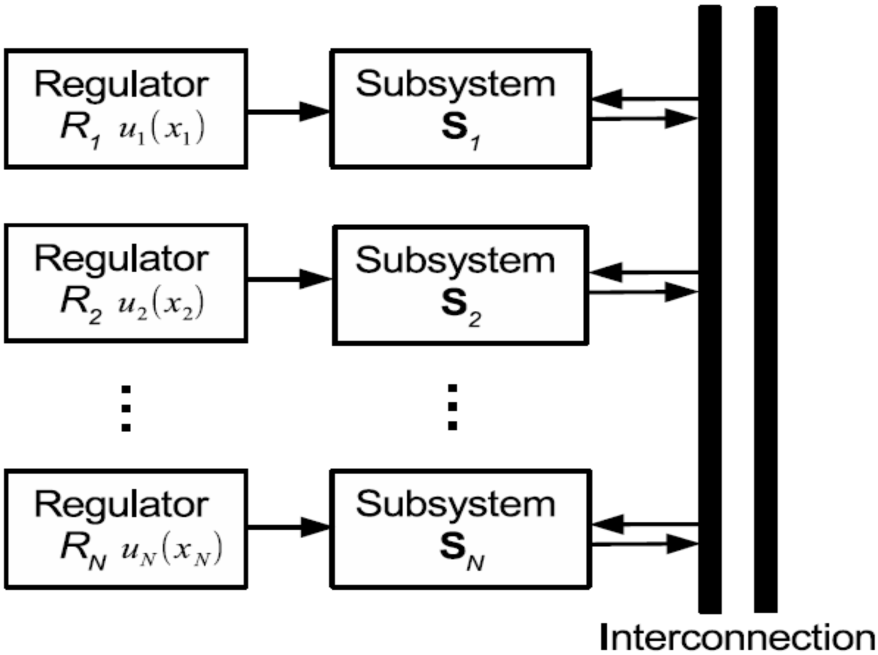

Figure 2.4: Decentralised system structure

increasing complexity of power distribution systems and demands of users, it has become ever more important to develop the automation of some tasks, such as generation, transmission, and distribution etc. Among these tasks, a multi-machine power system with N synchronous generators as shown in Figure 2.5 is a typical large-scale interconnected system [17]. The interconnection between each local generator through a transmission network greatly increases the system complexity

- The mechanical equation of the generator is described by

$$\dot{\delta}_i = \omega_i \tag{2.13}$$

$$\dot{\omega}_i = -\frac{D_i}{2H_i}\omega_i + \frac{\omega_0}{2H_i}(P_{mi0} - P_{ei}) \tag{2.14}$$

- The electrical dynamics of the generator are described by

$$\dot{E}'_{qi} = \frac{1}{T'_{doi}}(E_{fi} - E_{qi}) \tag{2.15}$$

- The electrical equations

$$
\begin{aligned}
E_{qi} &= E'_{qi} - (x_{di} - x'_{di})I_{di}, && (2.16)\\
E_{fi} &= K_{ci}u_{fi} && (2.17)\\
P_{ei} &= \sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\sin(\delta_i - \delta_j) && (2.18)\\
Q_{ei} &= -\sum_{j=1}^{N} E'_{qi}E'_{qj}B_{ij}\cos(\delta_i - \delta_j) && (2.19)\\
I_{qi} &= \sum_{j=1}^{N} E'_{qj}B_{ij}\sin(\delta_i - \delta_j) && (2.20)\\
I_{di} &= \sum_{j=1}^{N} E'_{qj}B_{ij}\cos(\delta_i - \delta_j) && (2.21)\\
E_{qi} &= x_{adi}I_{fi} && (2.22)\\
V_{ti} &= \sqrt{(E'_{qi} - x'_{di}I_{di})^2 + (x'_{di}I_{di})^2} && (2.23)
\end{aligned}
$$

where δ_i is the ith generator power angle [rad], and ω_i s the relative speed [rad/s], E'_{qi} represents the transient EMF in the quadrature axis [p.u.], Q_{ei} and T'_{doi} denote the reactive power and the direct axis trasient short circuit time constant respectively. x_{di} and x'_{di} represent the reactance and the transient reactance in direct axis respectively, I_{di} and I_{qi} represent current in direct axis and quadrature axis respectively. K_{ci} and B_{ij} denote the gain of the excitation amplifier and the $i-$th row and $j-$th column element of nodal susceptance matrix at internal nodes respectively. u_{fi} is the input of the amplifier of ith generator for $i = 1, 2, \cdots, N$. This model has been widely used for the research of multi-machine power system (see e.g. [75, 76, 77]).

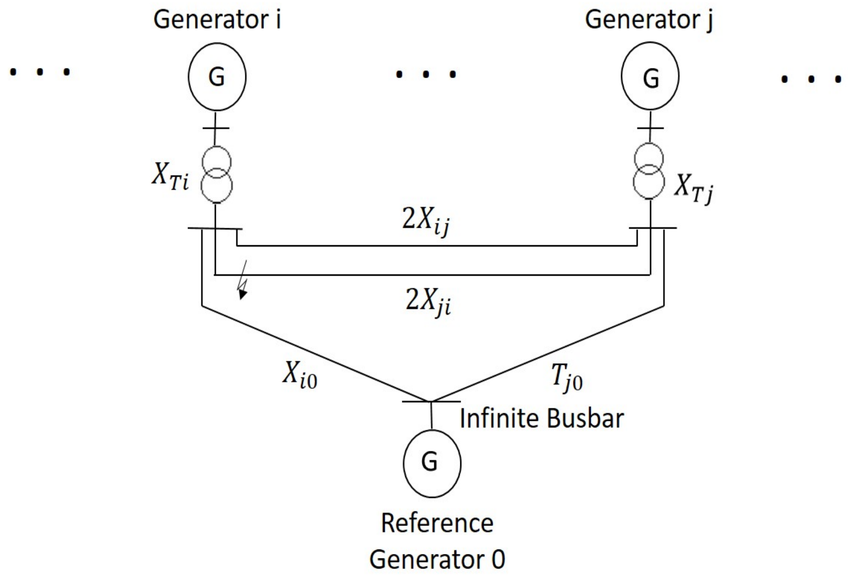

Figure 2.5: Over view of multi-machine power systems with an infinite busbar

δ_i	generator power angle [rad]
P_{ei}	electrical power [p.u.]
w_i	relative speed [rad/s]
w_0	synchronous machine speed [rad/s]
D_i	per unit damping constant
H_i	inertia constant [s]
E'_{qi}	transient EMF in the quadrature axis [p.u.]
T'_{doi}	direct axis transient short circuit time constant [s]
x_{di}	direct axis reactance [p.u.]
x'_{di}	direct axis transient reactance [p.u.]
B_{ij}	i-th row and j-th column element of nodal susceptance matrix at internal nodes after eliminating all physical buses[p.u.]
I_{qi}	quadrature axis current [p.u.]
K_{ci}	gain of the excitation amplifier [p.u.]
u_{fi}	input of the SCR amplifier [p.u.]
Q_{ei}	reactive power [p.u.]
I_{di}	direct axis current [p.u.]
P_{mi0}	mechanical input power [p.u.]

Coupled Inverted Pendula on Carts [17]

Consider a coupled inverted pendulum connected by a moving spring mounted on two carts as shown in Figure 2.6. It is assumed that the pivot position of the moving spring is a function of time which can change along the length l of pendula. The input to each pendulum is the torque u_i applied at the pivot which is produced by the external forces F_1 and F_2 applied to the carts.

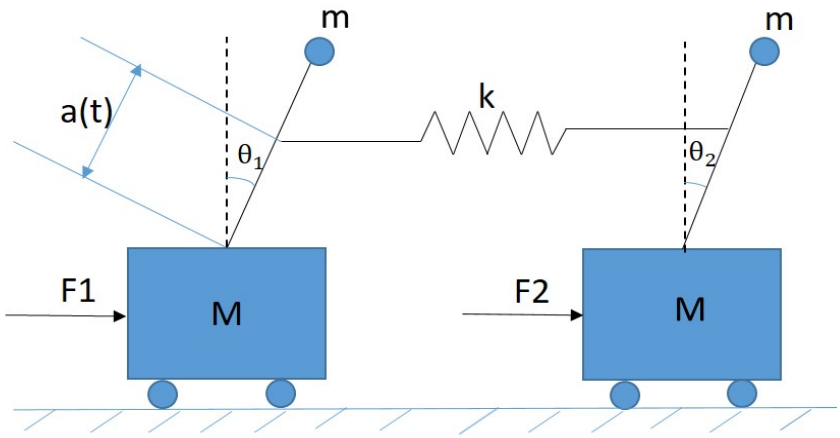

Figure 2.6: Two coupled inverted pendula on carts

Let

$$x_1 = \text{col}(\theta_1, \dot{\theta}_1)^T \quad \text{and} \quad x_2 = \text{col}(\theta_2, \dot{\theta}_2)^T \tag{2.24}$$

Then the dynamical model for the coupled inverted pendulum systems is described by (see [78]):

$$\begin{aligned}
\dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{cl} - \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{cml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 \\ \frac{m}{M}(\sin\theta_1)\dot{\theta}_1^2 + \frac{ka(t)(a(t)-cl)}{cml^2}(s_1 - s_2) \end{bmatrix} \qquad (2.25) \\
\dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{cl} - \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{cml^2} \end{bmatrix} u_2 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)(a(t)-cl)}{cml^2} & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 \\ \frac{m}{M}(\sin\theta_2)\dot{\theta}_2^2 + \frac{ka(t)(a(t)-cl)}{cml^2}(s_2 - s_1) \end{bmatrix} \qquad (2.26)
\end{aligned}$$

where s_1 and s_2 are positions of the two carts,

$$c = \frac{M}{(M+m)} \tag{2.27}$$

and k and g are the spring and gravity constants respectively.

The Model of the 2 DOF Robot Arm [1]

Consider the two degree of freedom robot arm moving in the vertical plane, see the block scheme in Figure 2.7

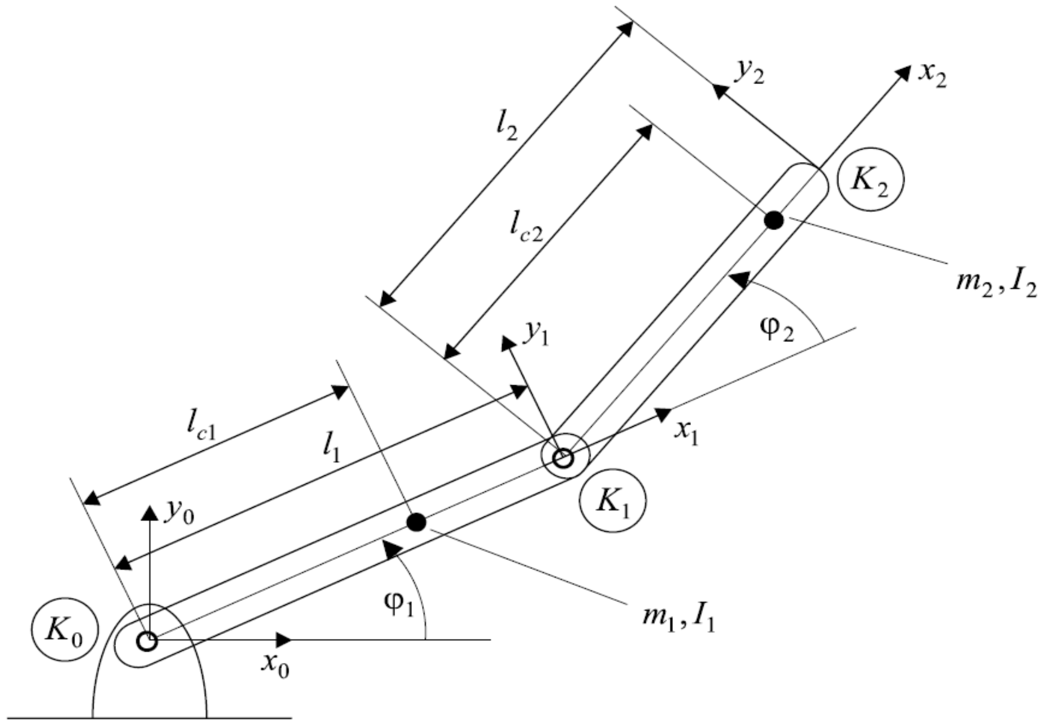

Figure 2.7: The model of a 2 DOF robot moving in vertical plane [1]

Let the generalized coordinates are

$$q = (q_1, q_2)^T := (\varphi_1, \varphi_2)^T \tag{2.28}$$

and the generalized forces are the driving torques

$$\tau = (\tau_1, \tau_2)^T \tag{2.29}$$

The position vectors pointing from the origin of the fixed frame (coordinate system) K_0

into the origins of the moving frames K_1 and K_2 are respectively

$$\begin{aligned} p_{01} &= (l_1C_1, l_1S_1, 0)^T, \\ p_{02} &= (l_1C_1 + l_2C_{12}, l_1S_1 + l_2S_{12}, 0)^T \end{aligned} \tag{2.30}$$

where the notations $C_1 := \cos(q_1), \cdots, S_{12} = \sin(q_1 + q_2)$ are used. The velocities in the basis of fixed frame K_0 can be obtained by formal differentiation of the position vectors:

$$\begin{aligned} Ov_1 &= \begin{bmatrix} -l_1S1 \\ l_1C_1 \\ 0 \end{bmatrix} \dot{q}_1 \\ Ov_2 &= \begin{bmatrix} -l_1S_1 - l_2S_{12} & -l_2S_{12} \\ l_1C_1 + l_2C_{12} & l_2C_{12} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \end{aligned} \tag{2.31}$$

The inertia matrix is constant in the frame of the moving body but is nonconstant in the fixed frame. Hence we transform the velocities into the basis of the moving frame:

$$\begin{aligned} v_1 &= v_1^1 \begin{bmatrix} C1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_1^0 = \begin{bmatrix} c_0 \\ l_1 \\ 0 \end{bmatrix} \dot{q}_1 \\ v_2 &= v_2^2 \begin{bmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} v_2^0 = \begin{bmatrix} l_1S_2 & 0 \\ l_1C_2 + l_2 & l_2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \end{aligned} \tag{2.32}$$

If each moving frame is parallelly transferred to the origin of the centre of mass, then the velocity v_c and the inertia matrix belonging to the centre of mass will have to be used in the motion equation. In the block scheme, I_1 and I_2 are already the right lower elements of the inertia matrices I_{c1} and I_{c2}, respectively. The velocities v_{c1} and v_{c2} can be immediately written down using simple geometrical considerations:

$$
\begin{aligned}
v_{c_1} &= \begin{bmatrix} 0 \\ l_{c_1} \\ 0 \end{bmatrix} \dot{q}_1 \\
v_{c_2} &= \begin{bmatrix} l_1 S_2 & 0 \\ l_1 C_2 + l_{c_2} & l_{c_2} \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
\end{aligned}
\tag{2.33}
$$

From the geometrical image, the angular velocities are simply given by

$$
\begin{aligned}
\omega_1 &= (0, 0, \dot{q}_1)^T, \\
\omega_2 &= (0, 0, \dot{q}_1 + \dot{q}_2)^T
\end{aligned}
$$

which can also be written in a matrix-vector form:

$$
\begin{aligned}
v_{c_1} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{q}_1 \\
\omega_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}
\end{aligned}
\tag{2.34}
$$

Hence, the kinetic and potential energy of the robot arm are respectively described by,

$$
\begin{aligned}
K &= \frac{1}{2}(v_{c_1}, v_{c_1})m_1 + \frac{1}{2}(I_1\omega_1, \omega_1) \\
&\quad \frac{1}{2}(v_{c_2}, v_{c_2})m_2 + \frac{1}{2}(I_2\omega_2, \omega_2) \\
P &= m_1 g l_{c_1} S_1 + m_2 g (l_1 S_1 + l_{c_2} S_{12})
\end{aligned}
\tag{2.35}
$$

After simple algebraic manipulations, the kinetic energy can be formulated in the quadratic form as

$$
K = \frac{1}{2}(H(q)\dot{q}, \dot{q}) = \frac{1}{2}\sum_j \sum_k D_{jk}\dot{q}_j\dot{q}_k, \tag{2.36}
$$

where

$$
\begin{aligned}
H(q) &= \begin{bmatrix} D_{11}(q) & D_{12}(q) \\ D_{12}(q) & D_{22}(q) \end{bmatrix} \\
D_{11} &= m_1 l_{c_1}^2 + m_2(l_1^2 + l_{c_2}^2 + 2l_1 l_{c_2} C_2) + I_1 + I_2, \\
D_{12} &= m_2(l_1 C_2 + l_{c_2}) l_{c_2} + I_2, \\
D_{22} &= m_2 l_{c_2}^2 + I_2 \qquad (2.37)
\end{aligned}
$$

The dynamic model of the robot arm can be found using the Lagrange equations. Performing the differentiations and introducing the notations

$$
\begin{aligned}
D_{112}(q) &= -m_2 l_1 l_{c_2} S_2 \\
D_1(q) &= \frac{\partial P}{\partial q_1} = m_1 g l_{c_1} C_1 + m_2 g(l_1 C_1 + l_{c_2} C_{12}) \\
D_1(q) &= \frac{\partial P}{\partial q_2} = m_2 g l_{c_2} C_{12} \qquad (2.38)
\end{aligned}
$$

The following dynamic model of the two degree of freedom robot arm is obtained:

$$
\begin{aligned}
D_{11}(q)\ddot{q}_1 + D_{12}\ddot{q}_2 + D_{112}(q)(2\dot{q}_1\dot{q}_2 + \dot{q}_2^2) + D_1(q) &= \tau_1 \\
D_{12}(q)\ddot{q}_1 + D_{22}\ddot{q}_2 - D_{112}(q)\dot{q}_1^2 + D_2(q) &= \tau_2 \qquad (2.39)
\end{aligned}
$$

where $x := (q_1, q_2, \dot{q}_1, \dot{q}_2)^T$ is system states and $u = (\tau_1, \tau_2)$ is the control signal (the input of the model).

2.3. THEORY OF VARIABLE STRUCTURE CONTROL SYSTEMS (VSCS)

Variable structure control systems (VSCS) evolved from the pioneering work in Russia of Emel'yanov and Barbashin in the early 1960s. The concepts of VSCS have appeared in the mid of 1970s when a book by Itkis [79] and a survey paper by Utkin [80] were published in English. VSC techniques have subsequently been extensively studied, and widely applied to theoretical research and practical linear/nonlinear systems due to their high robustness. Specifically, as one special case of variable structure controllers, sliding mode controllers are completely robust to matched uncertainties (uncertainties acting in the input channels). For the purpose of illustration, consider a dynamical control system expressed by the following differential equation

$$\dot{x} = f(t, x, u) \tag{2.40}$$

where $x \in R^n$ denotes the system state, $u \in R^q$ represents the system input/control and $t \in R^+$ is time.

2.3.1. Variable Structure Control

Consider the control system (2.40) in the domain $D \subset R^n$ [17]. A corresponding *variable structure control* can be expressed as

$$u = \begin{cases} u_1(t,x), & (t,x) \in R^+ \times D \\ u_2(t,x), & (t,x) \in R^+ \times D \\ \quad\vdots & \quad\vdots \\ u_q(t,x), & (t,x) \in R^+ \times D \end{cases} \tag{2.41}$$

where the functions $u_i(t,x)$ are continuous for $i = 1, 2, \cdots, q$. The structures of the functions $u_i(t,x)$ and $u_j(t,x)$ are different for $i \neq j$, $j = 1, 2, \cdots q$ $(q \geq 2)$. When the variable structure control in (2.41) is applied to the system (2.40), the corresponding closed-loop system becomes a *variable structure system*. Literally speaking, variable structure control is a control whose structure is changed or keeps changing in order to obtain and maintain the desired system performance during the control process.

For example, in real control design, when the response error/accuracy $e(t)$ is over the threshold, a proportional control is used to increase the response speed;

2.3.2. Sliding Mode Control Methodology

A typical Variable Structure Control which leads to a sliding motion, has underpinned the development of a systematic research methodology, which is the well-known sliding mode control [17]. Sliding mode control changes the system dynamics by employing a discontinuous control signal. This approach has been well developed and extensively used in theoretical research and practical design. It has been successfully employed to solve various control problems in combination with other control approaches.

The sliding mode control method consists of two steps:

- The design of sliding surface such that the system considered possesses the desired performance when it is restricted to the surface;
- The design of a variable structure control which drives the system trajectory to the sliding surface in finite time and maintain a sliding motion on it thereafter.

In view of these two steps, the system motion can be separated into two phases: the *reaching phase* and the *sliding phase*. The former refers to the motion when the system trajectory moves towards the sliding surface and the latter concerns the motion when the system trajectory moves on the sliding surface.

3.3.2.1 Sliding Phase

Consider system (2.40). In order to design a proper switching/sliding function

$$s = s(x) \tag{2.42}$$

such that the resulting sliding motion has the desired performance, one way is to find the dynamical equations governing the sliding motion, which is called sliding mode. Then the sliding surface based on the characteristics of the sliding mode dynamics or sliding motion can be . It is assumed that the sliding motion exists. The following two approaches are usually employed to find the sliding mode dynamics and in this way the stability of the sliding motion is transformed to the problem of ensuring stability of an unforced system.

- **Equivalent control**: When the considered system (2.40) is limited to and moving on the sliding surface

  $$s(x) = 0, \quad \text{and} \quad \dot{s}(x) = 0. \tag{2.43}$$

  The time derivative of $s(x)$ along the system (2.40) is given by

  $$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = \frac{\partial s}{\partial x} f(t, x, u) \tag{2.44}$$

In the sliding motion,

$$\frac{\partial s}{\partial x} f(t, x, u) = 0 \tag{2.45}$$

Suppose there is a solution for u to the Eq. (2.45) denoted by

$$u_{eq} = u_{eq}(t, x) \tag{2.46}$$

which is the so-called *equivalent control* . Then, the sliding mode dynamics governing the sliding motion may be obtained by

$$\begin{cases} \dot{x} &= f(t, x, u_{eq}(t, x)) \\ s(x) &= 0 \end{cases}$$

Now, assume that system (2.40) is in the following affine nonlinear form,

$$\dot{x} = F(t, x) + G(t, x)u \tag{2.47}$$

Then, for the sliding surface $s(x) = 0$, it follows from $\dot{x} = 0$ that the corresponding equivalent control is given by

$$u_{eq} = -(s(x)G(x, t))^{-1} s(x)F(t, x) \tag{2.48}$$

where $s(x)$ should be chosen such that $s(x)G(t, x)$ is nonsingular for all x in the considered domain and $t \in R^+$. Substitute u_{eq} in (2.48) into the system (2.40), it follows that the corresponding sliding motion can be described by

$$\begin{cases} \dot{x} &= F(t, x) - G(t, x)(s(x)G(x, t))^{-1} s(x)F(t, x) \\ s(x) &= 0 \end{cases}$$

- **Regular form**: Another approach to find the sliding mode dynamics relating to the sliding function $s = s(x)$ for the system (2.40) is to employ the well-known regular form. Suppose that there exists a coordinate transformation $z = T(x)$ such that in the new coordinate system z, the sliding surface $s(x) = 0$ can be described in the form

$$z_2 = \sigma(z_1) \tag{2.49}$$

where $z_1 \in R^{n-m}$, $z_2 \in R^m$, $z := col(z_1, z_2)$ and system (2.40) can be described by

$$\dot{z}_1 = F_1(t, z_1, z_2) \tag{2.50}$$

$$\dot{z}_2 = F_2(t, z_1, z_2, u) \tag{2.51}$$

where $u \in R^m$ is the control. The Jacobian $\frac{\partial F_2(t,z_1,z_2,u)}{\partial u}$ is assumed to be non-singular in the considered domain. Note the system (2.50) is independent of the control signal and the dimension of z_2 is the same as the dimension of the control u. System (2.50)-(2.51) is the so-called *regular form*. Based on the regular form in (2.50)-(2.51), it is clear to see that the corresponding sliding mode dynamics of system (2.40) is described by

$$\dot{z}_1 = F_1(t, z_1, \sigma(z_1)) \tag{2.52}$$

which is a reduced-order system when compared with system (2.40).

Note, if system (2.40) is in the form as given in (2.47), then the corresponding regular form can be described by

$$\dot{z}_1 = F_1(t, z_1, z_2) \tag{2.53}$$

$$\dot{z}_2 = F_2(t, z_1, z_2) + G_2(t, z_1, z_2)u \tag{2.54}$$

where the functions $F_1(\cdot)$, $F_2(\cdot)$ and $G_2(\cdot)$ are dependent on the coordinate transformation $z = T(x)$ and the functions $F(\cdot)$ and $G(\cdot)$ respectively.

3.3.2.2 Reaching Phase In order to guarantee that the system trajectory can be driven to the sliding surface $s(x) = 0$ in finite time and a sliding motion can be maintained on it thereafter, a proper discontinuous control

$$u = u(t, x) \tag{2.55}$$

needs to be designed such that the following condition is satisfied

$$s^T(x)\dot{s}(x) \leq -\eta \|s(x)\| \tag{2.56}$$

for some constant $\eta > 0$ which is called *reachability constant*. The inequality (2.56) is the so-called *reachability condition* which guarantees that sliding motion occurs in finite time.

From eq. (2.40), it follows that

$$\dot{s} = \frac{\partial s}{\partial x}\dot{x} = \frac{\partial s}{\partial x} f(t, x, u) \tag{2.57}$$

Therefore, from (2.56) and (2.57)

$$s^T(x)\frac{\partial s}{\partial x} f(t, x, u) \leq -\eta \|s(x)\| \tag{2.58}$$

which explicitly contains the variable u. The sliding mode controller guaranteeing reachability can usually be synthesised from (2.58).

The following condition

$$s^T(x)\dot{s}(x) < 0 \tag{2.59}$$

is called a reachability condition but it cannot guarantee that a sliding motion takes place in finite time and thus a sliding motion may not occur if (2.59) is satisfied.

It should be emphasised that, when the designed sliding/switching function is time varying, for example,

$$s = s(t, x) \tag{2.60}$$

it is straightforward to see that the condition (2.58) used to synthesis the sliding mode control law should be updated to

$$s^T(t, x)\left(\frac{\partial s}{\partial t} + \frac{\partial s}{\partial x} f(t, x, u)\right) \leq -\eta \|s(t, x)\| \tag{2.61}$$

In this case, the study becomes more complex.

2.4. Sliding Mode Observer

Many of the theoretical developments in the area of sliding mode control systems assume that the system state vector is available for use in the control design. This assumption is not valid for real systems in many cases. Therefore, study of the observer design to estimate system states becomes very important.

Consider initially the linear system described by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.62}$$

$$y(t) = Cx(t) \tag{2.63}$$

where $x \in R^n$, $y \in R^p$ and $u \in R^m$ are system states, output and input respectively. $A \in R^{n\times n}, B \in R^{n\times m}$ and $C \in R^{p\times n}$ are constant matrices and $p \geq m$. Assume that the matrices B and C are full rank and the pair (A, C) is observable.

2.4.1. AN UTKIN OBSERVER

As the outputs are to be considered, it is logical to change of coordinates so that the outputs appear as components of the states [28]. One possibility is to consider the transformation $x \mapsto T_c x$ where

$$T_c = \begin{bmatrix} N_c^T \\ C \end{bmatrix} \tag{2.64}$$

where the columns of $N_c \in R^{n\times(n-p)}$ span the null space of C. This transformation is nonsingular, and with respect to this new coordinate system, the new output distribution matrix is

$$C_i = \begin{bmatrix} 0 & I_{p_i} \end{bmatrix} \tag{2.65}$$

Split the matrices A and B in a compatible way as

$$T_c A T_c^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{and} \quad T_c B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{2.66}$$

Then, the nominal system can be written as

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}y(t) + B_1u(t) \tag{2.67}$$

$$\dot{y}(t) = A_{21}x_1(t) + A_{22}y(t) + B_2u(t) \tag{2.68}$$

where

$$T_c x = \begin{bmatrix} x_1 \\ y \end{bmatrix}$$

and $x_1 \in R^{n-p}$ and $y \in R^p$. The observer proposed by Utkin (1981) has the form

$$\dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + A_{12}\hat{y}(t) + B_1u(t) + Lv \quad (2.69)$$

$$\dot{\hat{y}}(t) = A_{21}\hat{x}_1(t) + A_{22}\hat{y}(t) + B_2u(t) - v \quad (2.70)$$

where $(\hat{x}_1, \hat{y})$ represent the estimation of (x_1, y), $L \in R^{(n-p)\times p}$ is a constant feedback gain matrix and the discontinuous vector v is a discontinuous term defined component-wise by

$$v_i = Msgn(\hat{y}_i - y_i), \quad i = 1, 2, \cdots, p \quad (2.71)$$

where $M \in R^+$. If the errors between the estimates and the true states are written as $e_1 = \hat{x}_1 - x_1$ and $e_y = \hat{y} - y$ then from equations (2.67) to (2.70) the following error system is obtained

$$\dot{e}_1(t) = A_{11}e_1(t) + A_{12}e_y(t) + Lv \quad (2.72)$$

$$\dot{e}_y(t) = A_{21}e_1(t) + A_{22}e_y(t) - v \quad (2.73)$$

Since the pair (A, C) is observable, the pair (A_{11}, A_{21}) is observable (see Proposition 3.3 in [28]). As a consequence, L can be chosen to make the spectrum of $A_{11} + LA_{21}$ has a negative eigenvalues. Define a further change of coordinates, dependent on L, by

$$\tilde{T} = \begin{bmatrix} I_{n-p} & L \\ 0 & I_p \end{bmatrix} \quad (2.74)$$

and let $\tilde{e}_1 = e_1 + Le_y$. The error system with respect to the new coordinates can be written as

$$\dot{\tilde{e}}_1(t) = \tilde{A}_{11}\tilde{e}_1(t) + \tilde{A}_{12}e_y(t) \quad (2.75)$$

$$\dot{e}_y(t) = \tilde{A}_{21}\tilde{e}_1(t) + \tilde{A}_{22}e_y(t) - v \quad (2.76)$$

where $\tilde{A}_{11} = A_{11} + LA_{21}$, $\tilde{A}_{12} = A_{12} + LA_{22} - \tilde{A}_{11}L$ and $\tilde{A}_{22} = A_{22} - A_{21}L$. It follows from (2.76) that in the domain

$$\Omega = \{(e_1, e_y) : \|A_{21}e_1\| + \frac{1}{2}\lambda_{max}(\tilde{A}_{22} + \tilde{A}_{22}^T)\|e_y\| < M - \eta\} \quad (2.77)$$

where $\eta < M$ is some small positive scalar, the reachability condition

$$e_y^T\dot{e}_y < -\eta\|e_y\| \quad (2.78)$$

is satisfied. Consequently, an ideal sliding motion will take place on the surface

$$S_o = \{(e_1, e_y) : e_y = 0\} \tag{2.79}$$

It follows that after some finite time t_s, for all subseqent time $e_y = 0$ and $\dot{e} = 0$. Equation (2.75) then reduced to

$$\dot{\tilde{e}}_1(t) \;=\; \tilde{A}_{11}\tilde{e}_1(t) \tag{2.80}$$

which by choice of L, represents a stable system and $\dot{\tilde{e}}_1(t) \to 0$ and consequently $\hat{x} \to x_1$ as $t \to \infty$. Equation (2.80) represents the reduced order sliding mode error dynamics, and thus the corresponding sliding motion is stable.

2.4.2. A Discontinuous Observer for A System with Disturbances

Consider the dynamical system [28]

$$\dot{x}(t) \;=\; Ax(t) + Bu(t) + D\xi(t, x, u) \tag{2.81}$$

$$y(t) \;=\; Cx(t) \tag{2.82}$$

where $x \in R^n$, $y \in R^p$ and $u \in R^m$ are system states, output and input respectively. $A \in R^{n\times n}, B \in R^{n\times m}, C \in R^{P\times n}$ and $D \in R^{n\times q}$ where $p \geq q$. Assume that the matrices B, C and D are full rank and the function $\xi : R^+ \times R^n \times R^m \to R^q$ is unknown but bounded so that

$$\|\xi(t, x, u)\| \leq r_1\|u\| + \alpha(t, y) \tag{2.83}$$

where r_1 is a known scalar and $\alpha : R^+ \times R^p \to R^+$ is a known function.

- **A Canonical Form for Observer Design**

  Suppose that there exists a linear change of coordinates T_o so that the system (2.81)-(2.82) can be written as

$$\dot{x}_1(t) \;=\; A_{11}x_1(t) + A_{12}y(t) + B_1u(t) \tag{2.84}$$

$$\dot{y}(t) \;=\; A_{21}x_1(t) + A_{22}y(t) + B_2u(t) + D\xi \tag{2.85}$$

where $x_1 \in R^{n-p}, y \in R^p$ and the matrix A_{11} has stable eigenvalues. Consider an observer of the form

$$\dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + A_{12}\hat{y}(t) + B_1u(t) - A_{12}e_y(t) \quad (2.86)$$

$$\dot{\hat{y}}(t) = A_{21}\hat{x}_1(t) + A_{22}\hat{y}(t) + B_2u(t) - (A_{22} - A_{22}^s)e_y + v \quad (2.87)$$

where A_{22}^s is a stable design matrix and $e_y = \hat{y} - y$. Let $P_2 \in R^{p\times p}$ be symmetric positive definite Lyapunov matrix for A_{22}^s then the discontinuous vector v is defined by

$$v = \begin{cases} -\rho(t,y,u)\|D_2\|\frac{P_2e_y}{\|P_2e_y\|} & \text{if } e_y \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.88)$$

where the scalar function $\rho : R^+ \times R^p \times R^m \to R^+$ satisfies

$$\rho(t,y,u) \geq r_1\|u\| + \alpha(t,y) + \gamma_o \quad (2.89)$$

and γ_o is a positive scalar. If the state estimation error $e_1 = \hat{x}_1 - x_1$, then it is straightforward to show

$$\dot{e}_1(t) = A_{11}e_1(t) \quad (2.90)$$

$$\dot{e}_y(t) = A_{21}e_1(t) + A_{22}^s e_y(t) + v - D_2\xi \quad (2.91)$$

The lower block triangular structure has been shown to occur quite naturally as a result of the state space representation chosen and the output error feedback gains employed.

Proposition 3.4.2.1 *There exists a family of symmetric positive definite matrices P_2 such that the uncertain dynamical error system (2.90)-(2.91) is quadratically stable.*

Proof see [28].

2.5. ADAPTIVE CONTROL

The term *adaptive system* was first introduced in control theory by Drenick and Shahbender in 1957 [2, 81]. It is used to refer to control systems that monitor their own performance and adjust their parameters in the direction of improving that performance. The authors in [82] introduced the *Adaptive Control* term as a set of techniques which provide a systematic approach for automatic adjustment of controllers in real time, in order to achieve, or to maintain, a desired level of control system performance when the parameters of the plant dynamic model are unknown and/or change over time.

2.5.1. TOOLS FOR ADAPTIVE CONTROL

In this section, some tools for adaptive control are introduced which are used or mentioned in this thesis, including the Barbalat's Lemma, LaSalle-Yoshizawa Theorem, persistent excitation criteria, and a parameter convergence lemma [83].

Lemma 3.5.1 (Barbalat's Lemma) *Let $\alpha : [t_0, \infty) \mapsto R$ be continuously differentiable scalar function. If $\alpha(t)$ has a finite limit as $t \to \infty$, and $\dot{\alpha}(t)$ is uniformly continuous over $[t_0, \infty)$, then*

$$\lim_{t\to\infty} \dot{\alpha}(t) = 0 \tag{2.92}$$

As an application of the Barbalat's Lemma, the LaSalle-Yoshizawa Theorem has been obtained

Theorem 3.5.1 (LaSalle-Yoshizawa Theorem). Consider

$$\dot{x} = f(x, d(t)) \tag{2.93}$$

where $d : [t_0, \infty) \mapsto D \subset R$ with D a non-empty set represents external unpredictable disturbance and/or internal parameter variation, and $f(x, d(t))$ is locally Lipschitz in x uniformly in t. If there exists a continuously differentiable function $V(x,t) : R^n \times [t_0, \infty) \mapsto R^+$ such that

$$W_1(x) \leq V(x,t) \leq W_2(x) \tag{2.94}$$

$$\dot{V}(x,t) \leq -\alpha(x) \leq 0 \quad \forall\, x \in R^n, \forall t \geq t_0 \tag{2.95}$$

where $W_1(x)$ and $W_2(x)$ are continuous positive definite and radially unbounded functions and $\alpha(x)$ is a continuous positive semi definite function, then the state is bounded and satisfies

$$\lim_{t\to\infty} \alpha(x(t)) = 0 \tag{2.96}$$

Moreover, if $\alpha(x)$ is positive definite, then the equilibrium $x = 0$ is uniformly globally asymptotically stable.

Remark 3.5.1. By stating that $f(x, d(t))$ is locally Lipschitz in x uniformly in t, it means that, for any $x^\star \in R^n$,

$$\|f(x, d(t)) - f(y, d(t))\| \le L\|x - y\| \tag{2.97}$$

is satisfied for all $x, y \in \{x \in R^n \mid \|x - x^\star\| \le r\}$ for some $r > 0$ and for all $t \ge 0$. The Lipschitz constant L depends on $x^\star$, but is independent of t. If $f(x, d(t))$ is locally Lipschitz in x, and D is a compact set, then clearly, $f(x, d(t))$ is locally Lipschitz in x uniformly in t.

Remark 3.5.2. Theorem 3.5.1 holds with $W_1(x)$ and $W_2(x)$ replaced by two class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\|x\|)$ and $\bar{\alpha}(\|x\|)$, respectively.

The following material is concerned with the so-called persistent exciting property of a signal, which is widely used in the parameters convergence analysis in adaptive control, and it has been mentioned for a few times in this thesis.

Definition 3.5.1 A bounded piecewise continuous function $f : [0, \infty) \mapsto D \subset R^n$ is said to be persistent exciting (PE) if there exist positive constants ϵ, t_0 and T_0 such that, for any unit row vector c of dimension n

$$\frac{1}{T_0}\int_t^{t+T_0} |cf(s)|ds \ge \epsilon, \forall t \ge t_0. \tag{2.98}$$

Lemma 3.5.2 *A bounded piecewise continuous function $f : [0, \infty) \mapsto D \subset R^n$ is PE if and only if there exist positive constants ϵ, t_0 and T_0 such that*

$$\frac{1}{T_0}\int_t^{t+T_0} f(s)f^T(s)ds \ge \epsilon^2 I, \qquad \forall t \ge t_0. \tag{2.99}$$

Proof: By the Jensen's inequality, i.e.,

$$(b - a)\int_a^b [g(s)]^2 ds \ge \left(\int_a^b g(s)ds\right)^2, \tag{2.100}$$

for any integrable real valued function g, we have

$$c\left[\frac{1}{T_0}\int_t^{t+T_0} f(s)f^T(s)ds\right]c^T = \frac{1}{T_0}\int_t^{t+T_0}[cf(s)]^2ds \geq \left(\frac{1}{T_0}\int_t^{t+T_0}|cf(s)|ds\right)^2 \quad (2.101)$$

for any unit vector c of dimension n. As f is PE, one has (2.98), and hence

$$c\left[\frac{1}{T_0}\int_t^{t+T_0} f(s)f^T(s)ds\right]c^T \geq \epsilon^2 = c(\epsilon^2 I)c^T \quad (2.102)$$

which implies (2.99).

"If": From (2.99). one has

$$\frac{1}{T_0}\int_t^{t+T_0} cf(s)f^T(s)c^T ds \geq \epsilon^2 \quad (2.103)$$

or

$$\int_t^{t+T_0}[cf(s)]^2 ds \geq T_0\epsilon^2 \quad (2.104)$$

for any unit row vector c of dimension n. Since the function f is bounded, so is $cf(s)$, i.e.,

$$|cf(s)| \leq R, \qquad \forall s \geq 0 \quad (2.105)$$

for a constant R. Let $R_1 = \frac{\epsilon}{\sqrt{2}}, S_1 = \{s \mid |cf(s)|\} \geq R_1, t \leq s \leq T + t_0$ and $\{S_2 = \{s \mid |cf(s)|\} \geq R_1, t \leq s \leq T + t_0\}$. Then

$$S_1 \cup S_2 = [t, t+T_0], \;\; S_1 \cap S_2 = \varnothing \quad (2.106)$$

Moreover, since $cf(s)$ is bounded and piecewise continuous in $[0,\infty)$, both S_1 and S_2 are Lebesgue measurable. Denote the length of a Lebesgue measurable set $S \subset [t, t+T_0]$ by $|S|$. Then $0 \leq |S_i| \leq T_0, i = 1, 2$, and $|S_1 \cup S_2| = T_0$. Furthermore,

$$\begin{aligned} T_0\epsilon^2 &\leq \int_t^{t+T_0}[cf(s)]^2ds = \int_{S_1}[cf(s)]^2ds + \int_{S_2}[cf(s)]^2ds \\ &\leq \int_{S_1}[cf(s)]^2ds + R_1^2|S_2|. \end{aligned} \quad (2.107)$$

The above inequality implies

$$|S_1|R^2 \geq \int_{S_1}[cf(s)]^2ds \geq T_0\epsilon^2 - R_1^2|S_2| = (T_0 - |S_2|/2)\epsilon^2 \geq T_0\epsilon^2/2 > 0 \quad (2.108)$$

and

$$|S_1| \geq T_0\epsilon^2/(2R^2) > 0 \tag{2.109}$$

Then, we have

$$\frac{1}{T_0}\int_t^{t+T_0} |cf(s)|ds \geq \frac{1}{T_0}\int_{S_1} |cf(s)|ds \geq \frac{|S_1|R_1}{T_0} \geq \frac{\epsilon^3}{2\sqrt{2}R^2} \tag{2.110}$$

which is (2.98) with ϵ replaced by another constant $\frac{\epsilon^3}{2\sqrt{2}R^2}$. From the definition, f is PE. The proof is thus completed. $\triangle$

2.6. SUMMARY

In this chapter, the basic knowledge of the observer concept has been provided. The fundamental ideas of designing observers have been discussed. The structure of the interconnected systems has been studies with support of a few practical systems. The basic ideas of the techniques that used in this thesis which are Variable Structure Control and Sliding Mode Control have been given in details in this chapter. Then, the literature and the concept of the sliding mode observer has been provided. After that, adaptive tools used or mentioned in this thesis have been presented in this chapter. As a result, Sliding Mode Control and Variable Structure Control techniques are considered as the main methodologies in designing observer schemes for nonlinear interconnected systems in the following chapters.

CHAPTER. 3

VARIABLE STRUCTURE OBSERVERS FOR NONLINEAR INTERCONNECTED SYSTEMS

In this chapter, a class of nonlinear interconnected systems with disturbances is considered where both the nominal isolated subsystems and interconnections are nonlinear. It is not required that either the nominal isolated subsystems or the interconnections are linearisable. A robust variable structure observer is established based on a simplified system structure by using a Lyapunov analysis methodology. The structure of the internal dynamics, the structure of uncertainties and the bounds on uncertainties are fully used in the observer design to reduce conservatism. These bounds are allowed to have a general nonlinear form. The observer states converge to the system states asymptotically. A numerical simulation example is given to demonstrate the application of the proposed approach. A case study of a coupled inverted pendulum system shows the practicality of the designed observer.

3.1. PRELIMINARIES

Consider the single-input-single-output nonlinear system

$$\dot{x}(t) = f(x) + g(x)u \tag{3.1}$$

$$y(t) = h(x) \tag{3.2}$$

where $x \in \Omega \subset R^n$ (Ω is a neighborhood of the origin), $y \in R$ and $u \in U \subset R$ (U is an admissible control set) are the state, output and input respectively, $f(x)$, $g(x) \in R^n$ are smooth vector fields defined in the domain Ω, and $h(x) \in R^m$ is a smooth vector in the domain Ω.

First, some key elements of the geometric approach in [66] are recalled. These will be used in the later analysis.

Definition 4.1 ([66]) System $(3.1) - (3.2)$ is said to have uniform relative degree r in the domain Ω if for any $x \in \Omega$,

(i) $L_g L_f^k h(x) = 0, \quad \text{for} \quad k = 1, 2, \cdots, r-2$

(ii) $L_g L_f^{r-1} h(x) \neq 0$

Now consider system $(3.1) - (3.2)$. It is assumed that system $(3.1) - (3.2)$ has uniform relative degree r in the domain Ω. Construct a mapping $\phi : x \to z$ as follows:

$$\phi(\cdot) : \begin{cases} z_1 = h(x) \\ z_2 = L_f h(x) \\ \vdots \\ z_r = L_f^{r-1} h(x) \\ z_{r+1} = \phi_{r+1} \\ \vdots \\ z_n = \phi_n(x) \end{cases} \tag{3.3}$$

where $\phi(\cdot) = \text{col}(\phi_1(x), \phi_2(x), \cdots, \phi_n(x))$, $\phi_1(x) = h(x)$, $\phi_2(x) = L_f h(x), \cdots,$ $\phi_r(x) = L_f^{r-1} h(x)$ and the functions $\phi_{r+1}(x), \cdots, \phi_n(x)$ need to be selected such that

$$L_g \phi_i(x) = 0, \qquad i = r+1, r+2, \cdots, n$$

and the Jacobian matrix

$$J_\phi := \frac{\partial \phi(x)}{\partial x}$$

is nonsingular in the domain Ω. Then the mapping $\phi : x \to z$ forms a diffeomorphism in the domain Ω (see Proposition 4.1.3 in [66]). For the sake of simplicity, let

$$\begin{aligned} \zeta &= \begin{bmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_r \end{bmatrix}^T := \begin{bmatrix} z_1 & z_2 & \cdots & z_r \end{bmatrix}^T \\ \eta &= \begin{bmatrix} \zeta_{r+1} & \zeta_{r+2} & \cdots & \zeta_n \end{bmatrix}^T := \begin{bmatrix} z_{r+1} & z_{r+2} & \cdots & z_n \end{bmatrix}^T \end{aligned}$$

Then, from [66], it follows that in the new coordinates z, system $(3.1) - (3.2)$ can be described by

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \zeta_3 \\ &\vdots \\ \dot{\zeta}_{r-1} &= \zeta_r \\ \dot{\zeta}_r &= a(\zeta, \eta) + b(\zeta, \eta)u \\ \dot{\eta} &= q(\zeta, \eta) \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} a(\zeta, \eta) &= L_f^r h(\phi^{-1}(\zeta, \eta)) \\ b(\zeta, \eta) &= L_g L_f^{r-1} h(\phi^{-1}(\zeta, \eta)) \end{aligned}$$

and

$$q(\zeta, \eta) = \begin{bmatrix} q_{r+1}(\zeta, \eta) \\ q_{r+2}(\zeta, \eta) \\ \vdots \\ q_n(\zeta, \eta) \end{bmatrix} = \begin{bmatrix} L_f \phi_{r+1}(\phi^{-1}(\zeta, \eta)) \\ L_f \phi_{r+2}(\phi^{-1}(\zeta, \eta)) \\ \vdots \\ L_f \phi_n(\phi^{-1}(\zeta, \eta)) \end{bmatrix}$$

It should be noted that the coordinate transformation (3.3) will be available if $\phi_i(x)$ are available for $i = r + 1, \cdots, n$, and in this case, the system (3.4) can be obtained directly.

3.2. PROBLEM FORMULATION

Consider the set of nonlinear interconnected systems

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i + \Delta f_i(x_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} D_{ij}(x_j) \tag{3.5}$$

$$y_i(t) = h_i(x_i), \quad i = 1, 2, \cdots, N \tag{3.6}$$

where $x_i \in \Omega_i \subset R^{n_i}$ (Ω_i is a neighborhood of the origin), $y_i \in R$ and $u_i \in U_i \subset R$ (U_i is an admissible control set) are the state, output and input of the i-th subsystem respectively, $f_i(x_i) \in R^{n_i}$ and $g_i(x_i) \in R^{n_i}$ are smooth vector fields defined in the domain Ω_i, and $h_i(x_i) \in R^{m_i}$ are smooth in the domain Ω_i for $i = 1, 2, \cdots, N$. The term $\Delta f_i(x_i)$ includes all the uncertainties experienced by the i-th subsystem. The term $\sum_{\substack{j=1 \\ j\neq i}}^{N} D_{ij}(x_j)$ is the nonlinear interconnection of the i-th subsystem.

Definition 4.2 The systems

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i + \Delta f_i(x_i) \tag{3.7}$$

$$y_i(t) = h_i(x_i), \qquad i = 1, 2, \cdots, N \tag{3.8}$$

are called the isolated subsystems of the systems (3.5)-(3.6), and the systems

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i \tag{3.9}$$

$$y_i(t) = h_i(x_i), \qquad i = 1, 2, \cdots, N \tag{3.10}$$

are called the nominal isolated subsystems of the systems (3.5)-(3.6).

In this chapter, under the assumption that the isolated subsystems (3.9)-(3.10) have uniform relative degree r_i in the considered domain Ω_i, the interconnected systems (3.5)-(3.6) are to be analysed. The objective is to explore the system structure based on a geometric transformation to design a robust asymptotic observer for the interconnected system (3.5)-(3.6).

It should be noted that the following results can be extended to the case where the isolated subsystems are multi-input and multi-output using the corresponding framework to Section 3.1 for the multi-input and multi-output case provided in [66].

3.3. System Analysis and Assumptions

In this section, some assumptions are imposed on the system (3.5)–(3.6) to facilitate the observer design.

Assumption 4.1. The nominal isolated subsystem (3.9)-(3.10) has uniform relative degree r_i in the domain $x_i \in \Omega_i$ for $i = 1, 2, \cdots, N$.

The practical meaning of Assumption 4.1 is that the nominal isolated subsystem (3.9)-(3.10) with well defined relative degree r_i are exactly those for which input/output feedback linearization is possible in the domain $x_i \in \Omega_i$ for $i = 1, 2, \cdots, N$. The basic approach of input-output linearization is simply to differentiate the output function y_i repeatedly until the input u_i appears, and then design u_i to cancel the nonlinearity. In which case no relative degree can be defined at any point in the domain $x_i \in \Omega_i$ for $i = 1, 2, \cdots, N$, then the output of the system is not affected by the input. i.e. that $y(t)$ is a function depending only on the initial state and not on the input.

Under Assumption 4.1, it follows from Section 3.1 that there exists a coordinate transformation

$$T_i : x_i \rightarrow \text{col}(\zeta_i, \eta_i) \tag{3.11}$$

where

$$\zeta_i = \begin{bmatrix} \zeta_{i1} \\ \zeta_{i2} \\ \vdots \\ \zeta_{ir_i} \end{bmatrix} = \begin{bmatrix} h_i(x_i) \\ L_f h_i(x_i) \\ \vdots \\ L_f^{r_i-1} h_i(x_i) \end{bmatrix} \in R^{r_i} \tag{3.12}$$

and $\eta_i \in R^{n_i - r_i}$ is defined by

$$\eta_i = \begin{bmatrix} \eta_{i1} \\ \eta_{i2} \\ \vdots \\ \eta_{n_i - r_i} \end{bmatrix} = \begin{bmatrix} \phi_{i(r_i+1)}(x_i) \\ \phi_{i(r_i+2)}(x_i) \\ \vdots \\ \phi_{in_i}(x_i) \end{bmatrix} \in R^{n_i - r_i} \tag{3.13}$$

for $i = 1, 2, \cdots, N$. The functions $\phi_{i(r_i+1)}(x_i)$, $\phi_{i(r_i+2)}(x_i), \cdots, \phi_{in_i}(x_i)$ can be obtained by solving the following partial differential equations:

$$L_{g_i}\phi_i(x_i) = 0, \quad x_i \in \Omega_i, \quad i = 1, 2, \cdots, N. \tag{3.14}$$

From Section 3.1, it follows that in the new coordinate system (ζ_i, η_i), the nominal isolated subsystem (3.9)-(3.10) can be described by

$$\dot{\zeta}_i = A_i\zeta_i + \beta_i(\zeta_i, \eta_i, u_i) \tag{3.15}$$

$$\dot{\eta}_i = q_i(\zeta_i, \eta_i) \tag{3.16}$$

$$y_i = C_i\zeta_i \tag{3.17}$$

where

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in R^{r_i \times r_i}, \qquad C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in R^{1 \times r_i} \tag{3.18}$$

$$\beta_i(\zeta_i, \eta_i, u_i) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ L_{f_i}^{r_i} h_i(T_i^{-1}(\zeta_i, \eta_i)) + L_{g_i} L_{f_i}^{r_i - 1} h_i(T_i^{-1}(\zeta_i, \eta_i)) u_i \end{bmatrix} \tag{3.19}$$

It is clear to see that the pair (A_i, C_i) is observable. Thus, there exists a matrix L_i such that $A_i - L_iC_i$ is Hurwitz stable. This implies that, for any positive definite matrix $Q_i \in R^{r_i \times r_i}$, the Lyapunov equation

$$(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i) = -Q_i \tag{3.20}$$

has a unique positive-definite solution $P_i \in R^{r_i \times r_i}$ for $i = 1, 2, \cdots, N$.

Assumption 4.2. The uncertainty $\Delta f_i(x_i)$ in (3.5) satisfies

$$\frac{\partial T_i}{\partial x_i} \Delta f_i(x_i) = \begin{bmatrix} E_i \Delta \Psi_i(x_i) \\ 0 \end{bmatrix} \tag{3.21}$$

where $T_i(\cdot)$ is defined in (3.11), $E_i \in R^{r_i \times r_i}$ is a constant matrix satisfying

$$E_i^T P_i = H_i C_i \tag{3.22}$$

for some matrix H_i, with P_i satisfying (3.20), and $\|\Delta\Psi_i(x_i)\| \leq \kappa_i(x_i)$, where $\kappa_i(x_i)$ is continuous and Lipschitz about x_i in the domain Ω_i for $i = 1, 2, \cdots, N$.

Remark 4.1. Solving the Lyapunov equation (3.20) in the presence of the constraint (3.22) constitutes the well known constrained Lyapunov problem [99]. Although there is no general solution available for this problem, associated discussion and an algorithm can be found in [100].

Remark 4.2. Assumption 4.2 is a limitation on the uncertainty $\Delta f_i(x_i)$, which is used to guarantee the existence of asymptotic observers. Denote the nonlinear uncertain term $\Delta\Psi_i(x_i)$ in (3.21) in the new coordinate frame (ζ_i, η_i) by $\Delta\Phi_i(\zeta_i, \eta_i)$ i.e.

$$\Delta\Phi_i(\zeta_i, \eta_i) = [\Delta\Psi_i(x_i)]_{x_i = T_i^{-1}(\zeta_i, \eta_i)} \tag{3.23}$$

From Assumption 2, there exists a function $\rho_i(\zeta_i, \eta_i)$ such that

$$\|\Delta\Phi_i(\zeta_i, \eta_i)\| \leq \rho_i(\zeta_i, \eta_i) \tag{3.24}$$

and $\rho_i(\zeta_i, \eta_i)$ satisfies the Lipschitz condition in $T_i(\Omega_i)$. Thus for any (ζ_i, η_i) and $(\hat{\zeta}_i, \hat{\eta}_i) \in T_i(\Omega_i)$,

$$\|\rho_i(\zeta_i, \eta_i) - \rho_i(\hat{\zeta}_i, \hat{\eta}_i)\| \leq l_i^a \|\zeta_i - \hat{\zeta}_i\| + l_i^b \|\eta_i - \hat{\eta}_i\| \tag{3.25}$$

where both l_i^a and l_i^b are nonnegative constants.

Consider the interconnections $D_{ij}(x_j)$ in system (3.5). Partition $\frac{\partial T_i}{\partial x_i} D_{ij}(x_j)$ as follows

$$\frac{\partial T_i}{\partial x_i} D_{ij}(x_j)\big|_{x_j = T_j^{-1}(\zeta_j, \eta_j)} = \begin{bmatrix} \Gamma_{ij}^a(\zeta_j, \eta_j) \\ \Gamma_{ij}^b(\zeta_j, \eta_j) \end{bmatrix} \tag{3.26}$$

where $\Gamma_{ij}^a(\zeta_j, \eta_j) \in R^{r_i}$, $\Gamma_{ij}^b(\zeta_j, \eta_j) \in R^{n_i - r_i}$ for $i = 1, 2, \cdots, N$ and $i \neq j$.

Assumption 4.3. The nonlinear terms $\Gamma_{ij}^a(\zeta_j, \eta_j) \in R^{r_i}$ and $\Gamma_{ij}^b(\zeta_j, \eta_j) \in R^{n_i - r_i}$ in (3.26) satisfy the Lipschitz condition in $T_i(\Omega_i)$.

Assumption 3 implies that there exist nonnegative constants α_{ij}^{a}, α_{ij}^{b}, μ_{ij}^{a} and μ_{ij}^{b} such that

$$\|\Gamma_{ij}^{a}(\zeta_i,\eta_i)-\Gamma_{ij}^{a}(\hat{\zeta}_i,\hat{\eta}_i)\| \leq \alpha_{ij}^{a}\|\zeta_j-\hat{\zeta}_j\|+\alpha_{ij}^{b}\parallel \eta_j-\hat{\eta}_j\| \tag{3.27}$$

$$\|\Gamma_{ij}^{b}(\zeta_i,\eta_i)-\Gamma_{ij}^{b}(\hat{\zeta}_i,\hat{\eta}_i)\| \leq \mu_{ij}^{a}\|\zeta_j-\hat{\zeta}_j\|+\mu_{ij}^{b}\parallel \eta_j-\hat{\eta}_j\| \tag{3.28}$$

for $i=1,2,\cdots,N$ and $i\neq j$.

Remark 4.3. It should be noted that Assumption 4.2 and 4.3 have limitations to the coordinate transformation T_i in (3.11). However, in local case, the corresponding limitation is trivial and the Lipschitz conditions in Assumption 4.2 and 4.3 can be satisfied if all the associated functions are smooth.

From $(3.15)-(3.17)$ and the analysis above, it follows that under Assumption 2, in the new coordinate system (ζ_i,η_i), the system (3.5)-(3.6) can be described by

$$\dot{\zeta}_i = A_i\zeta_i+\beta_i(\zeta_i,\eta_i,u_i)+E_i\Delta\Psi_i(\zeta_i,\eta_i)+\sum_{\substack{j=1\\ j\neq i}}^{N}\Gamma_{ij}^{a}(\zeta_j,\eta_j) \tag{3.29}$$

$$\dot{\eta}_i = q_i(\zeta_i,\eta_i)+\sum_{\substack{j=1\\ j\neq i}}^{N}\Gamma_{ij}^{b}(\zeta_j,\eta_j) \tag{3.30}$$

$$y_i = C_i\zeta_i \tag{3.31}$$

where A_i and C_i are given in (3.18), $\beta_i(\cdot)$ is defined in (3.19) and $\Gamma_{ij}^{a}(\cdot)$ and $\Gamma_{ij}^{b}(\cdot)$ are defined in (3.26).

Remark 4.4. Since $\beta_i(\cdot)$ is continuous in the domain $T_i(\Omega_i)$, it is straightforward to see that there exists a subset in the domain $T_i(\Omega_i)$ such that the function $\beta_i(\cdot)$ is Lipschitz in the subset

$$\parallel \beta_i(\zeta_i,\eta_i,u_i)-\beta_i(\hat{\zeta}_i,\hat{\eta}_i,u_i)\parallel\leq v_i^{a}(u_i)\parallel \zeta_i-\hat{\zeta}_i\parallel +v_i^{b}(u_i)\parallel \eta_i-\hat{\eta}_i\parallel \tag{3.32}$$

where $v_i^{a}(u_i)$ and $v_i^{b}(u_i)$ are known nonnegative functions of u_i for $i=1,2,\cdots,N$.

Assumption 4.4. The function $q_i(\zeta_i,\eta_i)$ in equation (3.30) has the following decomposition

$$q_i(\zeta_i,\eta_i)=M_i\eta_i+\theta_i(\zeta_i,\eta_i) \tag{3.33}$$

where $M_i\in R^{(n_i-r_i)\times(n_i-r_i)}$ is a Hurwitz matrix and $\theta_i(\zeta_i,\eta_i)$ are Lipschitz in the domain $T_i(\Omega_i)$.

Under Assumption 4, there exist nonnegative constants τ_i^a and τ_i^b such that.

$$\| \theta_i(\zeta_i, \eta_i) - \theta_i(\hat{\zeta}_i, \hat{\eta}_i) \| \leq \tau_i^a \| \zeta_i - \hat{\zeta}_i \| + \tau_i^b \| \eta_i - \hat{\eta}_i \| \tag{3.34}$$

for $i = 1, 2, \cdots, N$.. Further, from the fact that M_i is Hurwitz stable for $\Lambda_i > 0$, the following Lyapunov equation has a unique solution $\Pi_i > 0$

$$M_i^T \Pi_i + \Pi_i M_i = -\Lambda_i, \qquad i = 1, 2, \cdots, N. \tag{3.35}$$

3.4. NONLINEAR OBSERVER SYNTHESIS

In this section, an observer is designed for the transformed systems $(3.29) - (3.31)$ which provides asymptotic estimation of the states of the interconnected systems $(3.29) - (3.31)$. For the system $(3.29) - (3.31)$, construct dynamical systems

$$\begin{aligned} \dot{\hat{\zeta}}_i &= A_i\hat{\zeta}_i + L_i(y_i - C_i\hat{\zeta}_i) + \beta_i(\hat{\zeta}_i, \hat{\eta}_i, u_i) + K_i(y, \hat{\zeta}_i, \hat{\eta}_i) \\ &\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Gamma_{ij}^a(\hat{\zeta}_j, \hat{\eta}_j) \end{aligned} \tag{3.36}$$

$$\dot{\hat{\eta}}_i = M_i\hat{\eta}_i + \theta_i(\hat{\zeta}_i, \hat{\eta}_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} \Gamma_{ij}^b(\hat{\zeta}_j, \hat{\eta}_j) \tag{3.37}$$

where the term $K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i)$ is defined by

$$K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i) = \begin{cases} \frac{P_i^{-1}C_i^T(y_i - C_i\hat{\zeta}_i)}{\|y_i - C_i\hat{\zeta}_i\|} \| H_i \| \rho_i(\hat{\zeta}_i, \hat{\eta}_i), & y_i - C_i\hat{\zeta}_i \neq 0 \\ 0, & y_i - C_i\hat{\zeta}_i = 0 \end{cases} \tag{3.38}$$

where P_i and H_i satisfy (3.20) and (3.22) respectively.

The following results are ready to be presented.

Theorem 4.1. Suppose Assumptions $1-4$ hold. Then, the dynamical system (3.36)–(3.37) is a robust asymptotic observer of system (3.29)-(3.31), if the function matrix $W^T(\cdot) + W(\cdot)$ is positive definite in the domain $T(\Omega) \times U := T(\Omega_1) \times U_1 \times T(\Omega_2) \times U_2 \times \cdots \times T(\Omega_N) \times U_N$, where the matrix $W(\cdot) = [w_{ij}(\cdot)]_{2N\times 2N}$, and its entries $w_{ij}(\cdot)$ are defined by

$$w_{ij} = \begin{cases} \lambda_{\min}(Q_i) - 2\lambda_{\max}(P_i)v_i^a(\cdot) - 2l_i^a\|C_i\|\|H_i\|, & i = j,\ 1 \le i \le N \\ -2\lambda_{\max}(P_i)\alpha_{ij}^a, & i \ne j,\ 1 \le i \le N, 1 \le j \le N \\ \lambda_{\min}(\Lambda_{i-N}) - 2\lambda_{\max}(\Pi_{i-N})\tau_{i-N}^b, & i = j,\ N+1 \le i \le 2N \\ -2\lambda_{\max}(\Pi_{(i-N)})\mu_{(i-N)(j-N)}^b, & \\ \qquad i \ne j,\ N+1 \le i \le 2N, N+1 \le j \le 2N & \\ -2[\lambda_{\max}(P_i)v_i^b(\cdot) + l_i^b\|C_i\|\|H_i\| + \lambda_{\max}(\Pi_i)\tau_i^a], & \\ \qquad j - i = N,\ 1 \le i \le N, N+1 \le j \le 2N & \\ -2\lambda_{\max}(P_i)\alpha_{i(j-N)}^b, & j - i \ne N, \quad 1 \le i \le N, N+1 \le j \le 2N \\ 0, & i - j = N, \quad N+1 \le i \le 2N, 1 \le j \le N \\ -2\lambda_{\max}(\Pi_{i-N})\mu_{(i-N)j}^a, & i - j \ne N, \quad N+1 \le i \le 2N, 1 \le j \le N \end{cases}$$

Proof. Let $e_{\zeta_i} = \zeta_i - \hat{\zeta}_i$ and $e_{\eta_i} = \eta_i - \hat{\eta}_i$ for $i = 1, 2, \cdots, N$. Compare systems $(3.29) - (3.30)$ and $(3.36) - (3.37)$. It follows that the error dynamical systems are described by

$$\begin{aligned} \dot{e}_{\zeta_i} &= (A_i - L_iC_i)e_{\zeta_i} + \beta_i(\zeta_i, \eta_i, u_i) - \beta_i(\hat{\zeta}_i, \hat{\eta}_i, u_i) + E_i\Delta\Psi_i(\zeta_i, \eta_i) \\ &\quad -K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i) + \sum_{\substack{j=1 \\ j\ne i}}^{N} \Gamma_{ij}^a(\zeta_j, \eta_j) - \sum_{\substack{j=1 \\ j\ne i}}^{N} \Gamma_{ij}^a(\hat{\zeta}_j, \hat{\eta}_j) \end{aligned} \tag{3.39}$$

$$\begin{aligned} \dot{e}_{\eta_i} &= M_ie_{\eta_i} + \theta_i(\zeta_i, \eta_i) - \theta_i(\hat{\zeta}_i, \hat{\eta}_i) + \sum_{\substack{j=1 \\ j\ne i}}^{N} \Gamma_{ij}^b(\zeta_j, \eta_j) \\ &\quad - \sum_{\substack{j=1 \\ j\ne i}}^{N} \Gamma_{ij}^b(\hat{\zeta}_j, \hat{\eta}_j) \end{aligned} \tag{3.40}$$

Now, for the system (3.39) and (3.40) consider the following candidate Lyapunov function

$$V = \sum_{i=1}^{N} e_{\zeta_i}^T P_i e_{\zeta_i} + \sum_{i=1}^{N} e_{\eta_i}^T \Pi_i e_{\eta_i} \tag{3.41}$$

Then, the time derivative of the candidate Lyapunov function can be described by

$$\dot{V} = \sum_{i=1}^{N} [(\dot{e}_{\zeta_i}^T P_i e_{\zeta_i} + e_{\zeta_i}^T P_i \dot{e_{\zeta_i}}) + (\dot{e}_{\eta_i}^T \Pi_i e_{\eta_i} + e_{\eta_i}^T \Pi_i \dot{e}_{\eta_i})] \tag{3.42}$$

Substituting both $\dot{e}_{\zeta_i}$ in (3.39) and $\dot{e}_{\eta_i}$ in (3.40) into equation (3.42), it follows by direct computation that the time derivative of the function V in (3.41) can be described by

$$\begin{aligned}
\dot{V} \;=\; & \sum_{i=1}^{N} \Big\{ e_{\zeta_i}^T[(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i)]e_{\zeta_i} + 2e_{\zeta_i}^T P_i[\beta_i(\zeta_i, \eta_i, u_i) \\
& -\beta_i(\hat{\zeta}_i, \hat{\eta}_i, u_i)] + 2[e_{\zeta_i}^T P_i E_i \Delta\Psi_i(\zeta_i, \eta_i) - e_{\zeta_i}^T P_i K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i)] + 2e_{\zeta_i}^T P_i \\
& \times \sum_{\substack{j=1 \\ j\neq i}}^{N} [\Gamma_{ij}^a(\zeta_j, \eta_j) - \Gamma_{ij}^a(\hat{\zeta}_j, \hat{\eta}_j)] + e_{\eta_i}^T(M_i^T \Pi_i + \Pi_i M_i)e_{\eta_i} + 2e_{\eta_i}^T \Pi_i \\
& \times[\theta_i(\zeta_i, \eta_i) - \theta_i(\hat{\zeta}_i, \hat{\eta}_i)] + 2e_{\eta_i}^T \Pi_i \sum_{\substack{j=1 \\ j\neq i}}^{N} [\Gamma_{ij}^b(\zeta_j, \eta_j) - \Gamma_{ij}^b(\hat{\zeta}_j, \hat{\eta}_j)] \Big\} \qquad (3.43)
\end{aligned}$$

From (3.22), (3.24), (3.25) and (3.38), it follows that:

(i) If $y_i - C_i\hat{\zeta}_i = 0$, then from (3.22) and $e_{\zeta_i}^T C_i^T = (y_i - C_i\hat{\zeta}_i)^T$

$$\begin{aligned}
e_{\zeta_i}^T P_i E_i \Delta\Phi_i(\zeta_i, \eta_i) \;-\; & e_{\zeta_i}^T P_i K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i) = e_{\zeta_i}^T C_i^T H_i^T \Delta\Phi_i(\zeta_i, \eta_i) \qquad (3.44) \\
& = (H_i(y_i - C_i\hat{\zeta}_i))^T \Delta\Phi_i(\zeta_i, \eta_i) = 0 \qquad (3.45)
\end{aligned}$$

(ii) If $y_i - C_i\hat{\zeta}_i \neq 0$, then from (3.22), (3.24), (3.25) and (3.38)

$$\begin{aligned}
& e_{\zeta_i}^T P_i E_i \Delta\Phi_i(\zeta_i, \eta_i) - e_{\zeta_i}^T P_i K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i) \\
= \;& e_{\zeta_i}^T C_i^T H_i^T \Delta\Phi_i(\zeta_i, \eta_i) - e_{\zeta_i}^T P_i \frac{P_i^{-1} C_i^T (y_i - C_i\hat{\zeta}_i)}{\|y_i - C_i\hat{\zeta}_i\|} \parallel H_i \parallel \rho_i(\hat{\zeta}_i, \hat{\eta}_i) \\
= \;& (C_i e_{\zeta_i})^T H_i^T \Delta\Phi_i(\zeta_i, \eta_i) - \frac{e_{\zeta_i}^T C_i^T C_i e_{\zeta_i}}{\|C_i e_{\zeta_i}\|} \|H_i\| \rho_i(\hat{\zeta}_i, \hat{\eta}_i) \\
\leq \;& \|C_i e_{\zeta_i}\| \|H_i\| \big\{ \rho_i(\zeta_i, \eta_i) - \rho_i(\hat{\zeta}_i, \hat{\eta}_i) \big\} \\
\leq \;& \|C_i e_{\zeta_i}\| \|H_i\| \big\{ l_i^a \|\zeta_i - \hat{\zeta}_i\| + l_i^b \|\eta_i - \hat{\eta}_i\| \big\}
\end{aligned}$$

Then, from (i) and (ii) above, it follows that

$$\begin{aligned}
e_{\zeta_i}^T P_i E_i \Delta\Phi_i(\zeta_i, \eta_i) \;-\; & e_{\zeta_i}^T P_i K_i(y_i, \hat{\zeta}_i, \hat{\eta}_i) \\
\leq \;& \|C_i e_{\zeta_i}\| \|H_i\| \big(l_i^a \|e_{\zeta_i}\| + l_i^b \|e_{\eta_i}\| \big) \qquad (3.46)
\end{aligned}$$

Substituting (3.27), (3.28), (3.32), (3.34), and (3.46) into (3.43) yields

$$
\begin{aligned}
\dot{V} \;\leq\;& \sum_{i=1}^{N}\Big\{ -e_{\zeta_i}^T Q_i e_{\zeta_i} + 2\|e_{\zeta_i}\|\|P_i\|\big[v_i^a\|e_{\zeta_i}\| + v_i^b\|e_{\eta_i}\|\big] + 2\|e_{\zeta_i}\|\|C_i\|\|H_i\| \\
& \times\big[l_i^a\|e_{\zeta_i}\| + l_i^b\|e_{\eta_i}\|\big] + 2\|e_{\zeta_i}\|\|P_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}\big[\alpha_{ij}^a\|e_{\zeta_j}\| + \alpha_{ij}^b\|e_{\eta_j}\|\big] - e_{\eta_i}^T\Lambda_i e_{\eta_i} \\
+\;& 2e_{\eta_i}^T\|\Pi_i\|\big[\tau_i^a\|e_{\zeta_i}\| + \tau_i^b\|e_{\eta_i}\|\big] + 2e_{\eta_i}^T\|\Pi_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}\big[\mu_{ij}^a\|e_{\zeta_j}\| + \mu_{ij}^b\|e_{\eta_j}\|\big]\Big\} \\
\leq\;& \sum_{i=1}^{N}\Big\{ -e_{\zeta_i}^T Q_i e_{\zeta_i} + 2v_i^a\|e_{\zeta_i}\|^2\|P_i\| + 2v_i^b\|e_{\zeta_i}\|\|e_{\eta_i}\|\|P_i\| + 2l_i^a\|e_{\zeta_i}\|^2\|C_i\| \\
& \times\|H_i\| + 2l_i^b\|e_{\zeta_i}\|\|e_{\eta_i}\|C_i\|\|H_i\| + \sum_{\substack{j=1\\ j\neq i}}^{N}\big[2\alpha_{ij}^a\|e_{\zeta_i}\|\|e_{\zeta_j}\|\|P_i\| + 2\alpha_{ij}^b\|e_{\zeta_i}\| \\
& \times\|e_{\eta_j}\|\|P_i\|\big] - e_{\eta_i}^T\Lambda_i e_{\eta_i} + 2\tau_i^a\|\Pi_i\|\|e_{\zeta_i}\|\|e_{\eta_i}\| + 2\tau_i^b\|\Pi_i\|\|e_{\eta_i}\|^2 \\
& +\sum_{\substack{j=1\\ j\neq i}}^{N}\big[2\mu_{ij}^a\|\Pi_i\|\|e_{\zeta_j}\|\|e_{\eta_i}\| + 2\mu_{ij}^b\|\Pi_i\|\|e_{\eta_i}\|\|e_{\eta_j}\|\big]\Big\} \\
\leq\;& \sum_{i=1}^{N}\Big\{ -\big[\lambda_{\min}(Q_i) - 2\lambda_{\max}(P_i)v_i^a - 2l_i^a\|C_i\|\|H_i\|\big]\|e_{\zeta_i}\|^2 \\
& +\big[\sum_{\substack{j=1\\ j\neq i}}^{N}2\lambda_{\max}(P_i)\alpha_{ij}^a\big]\|e_{\zeta_i}\|\|e_{\zeta_j}\| + \big[2\lambda_{\max}(P_i)v_i^b + 2l_i^b\|C_i\|\|H_i\| \\
& +2\lambda_{\max}(\Pi_i)\tau_i^a\big]\|e_{\zeta_i}\|\|e_{\eta_i}\| + 2\sum_{\substack{j=1\\ j\neq i}}^{N}\lambda_{\max}(P_i)\alpha_{ij}^b\|e_{\zeta_i}\|\|e_{\eta_j}\| \\
& +\sum_{\substack{j=1\\ j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^a\|e_{\zeta_j}\|\|e_{\eta_i}\| - \big[\lambda_{\min}(\Lambda_i) - 2\lambda_{\max}(\Pi_i)\tau_i^b\big]\|e_{\eta_i}\|^2 \\
& +\big[\sum_{\substack{j=1\\ j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^b\big]\|e_{\eta_i}\|\|e_{\eta_j}\|\Big\}
\end{aligned}
$$

$$
\begin{aligned}
\le\ & -\sum_{i=1}^{N}\Big\{\big[\lambda_{\min}(Q_i)-2\lambda_{\max}(P_i)v_i^a-2l_i^a\|C_i\|\|H_i\|\big]\|e_{\zeta_i}\|^2 \\
& -\big[\sum_{\substack{j=1\\ j\neq i}}^{N}2\lambda_{\max}(P_i)\alpha_{ij}^a\big]\|e_{\zeta_i}\|\|e_{\zeta_j}\|-\big[2\lambda_{\max}(P_i)v_i^b+2l_i^b\|C_i\|\|H_i\| \\
& +2\lambda_{\max}(\Pi_i)\tau_i^a\big]\|e_{\zeta_i}\|\|e_{\eta_i}\big]\|-2\sum_{\substack{j=1\\ j\neq i}}^{N}\lambda_{\max}(P_i)\alpha_{ij}^b\|e_{\zeta_i}\|\|e_{\eta_j}\|\big] \\
& -\sum_{\substack{j=1\\ j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^a\|e_{\zeta_j}\|\|e_{\eta_i}\|+\big[\lambda_{\min}(\Lambda_i)-2\lambda_{\max}(\Pi_i)\tau_i^b\big]\|e_{\eta_i}\|^2 \\
& -\big[\sum_{\substack{j=1\\ j\neq i}}^{N}2\lambda_{\max}(\Pi_i)\mu_{ij}^b\big]\|e_{\eta_i}\|\|e_{\eta_j}\|\Big\}
\end{aligned}
$$

Then, from the definition of the matrix $W(\cdot)$ and the inequality above, it follows that

$$\dot V \le -\frac{1}{2}X^T[W^T(\cdot)+W(\cdot)]X$$

where $X=[\|e_{\zeta_1}\|,\|e_{\zeta_2}\|,\cdots,\|e_{\zeta_N}\|,\|e_{\eta_1}\|,\|e_{\eta_2}\|,\cdots,\|e_{\eta_N}\|]^T$. Since $W^T(\cdot)+W(\cdot)$ is positive definite in the domain $T(\Omega)\times U$, it is clear that $\dot V|_{(3.36)-(3.37)}$ is negative definite. Therefore, the error system $(3.39)-(3.40)$ is asymptotically stable, that is,

$$\lim_{t\to\infty}\|\zeta_i(t)-\hat\zeta_i(t)\|=0 \qquad \text{and} \qquad \lim_{t\to\infty}\|\eta_i(t)-\hat\eta_i(t)\|=0 \tag{3.47}$$

Hence, the conclusion follows. △

Remark 4.5. It is clear that the structure of system (3.36)-(3.37) is variable due to the term in (3.38). Theorem 4.1 shows that system (3.36)-(3.37) is an asymptotic observer of the interconnected system (3.29)-(3.31). Therefore, system (3.36)-(3.37) is called a variable structure observer throughout this chapter.

Now, consider the interconnected system $(3.5)-(3.6)$. Assume that $\frac{\partial T_i^{-1}(\zeta_i,\eta_i)}{\partial(\zeta_i,\eta_i)}$ is bounded in $T_i(\Omega_i)$ for $i=1,2,\cdots,N$. There exists a positive constant γ_i such that

$$\left\|\frac{\partial T_i^{-1}(\zeta_i,\eta_i)}{\partial(\zeta_i,\eta_i)}\right\|\le\gamma_i,\qquad (\zeta_i,\eta_i)\in T_i(\Omega_i),\ i=1,2,\cdots,N$$

Define $\hat x_i=T_i^{-1}(\hat\zeta_i,\hat\eta_i),\ i=1,2,\cdots,N$. Then,

$$\|x_i-\hat x_i\|=\|T_i^{-1}(\zeta_i,\eta_i)-T_i^{-1}(\hat\zeta_i,\hat\eta_i)\|\le\gamma_i(\|\zeta_i-\hat\zeta_i\|+\|\eta_i-\hat\eta_i\|) \tag{3.48}$$

From (3.47) and (3.48), it follows that

$$\lim_{t\to\infty} \|x_i(t) - \hat{x}_i(t)\| = 0$$

This implies that $\hat{x}_i$ is an asymptotic estimate of x_i for $i = 1, 2, \cdots, N$. Therefore,

$$\hat{x}_i = T_i^{-1}(\hat{\zeta}_i, \hat{\eta}_i)$$

provides an asymptotic estimate of the states x_i of the system $(3.5) - (3.6)$, where $\hat{\zeta}_i$ and $\hat{\eta}_i$ are given by (3.36)-(3.37) for $i = 1, 2, \ldots, N$.

Remark 4.6. From the analysis above, it is clear to see that, in this chapter, it is not required that either the nominal isolated subsystems or the interconnections are linearisable. The uncertainties are bounded by nonlinear functions and are fully used in the observer design in order to reject the effects of the uncertainties, and thus robustness is enhanced. Although the designed observer is a local asymptotic observer, the developed results can be extended to the global case if the associated conditions hold globally.

3.5. Simulation Examples

3.5.1. A Numerical Example

Consider the nonlinear interconnected systems:

$$\begin{aligned}
\dot{x}_1 &= \underbrace{\begin{bmatrix} x_{12} \\ -0.1\sin x_{12} \\ -3x_{11}^2 - 3.25x_{13} - 2x_{12} \end{bmatrix}}_{f_1(x_1)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{g_1(x_1)} u_1 + \underbrace{\begin{bmatrix} \Delta\sigma_1 \\ 0.5\Delta\sigma_1 \\ -2\Delta\sigma_1 \end{bmatrix}}_{\Delta f_1(x_1)} \\
&\quad + \underbrace{\begin{bmatrix} 0.2(x_{21}^2 + x_{22}) \\ 0 \\ 0.1\sin x_{21} \end{bmatrix}}_{D_{12}(x_2)} \qquad (3.49) \\
y_1 &= \underbrace{x_{11}}_{h_1(x_1)} \qquad (3.50)
\end{aligned}$$

$$
\begin{aligned}
\dot{x}_2 &= \underbrace{\begin{bmatrix} -x_{21} \\ -x_{21}^2 - 3x_{22} + \cos(x_{21}^2 + x_{22}) - 1 \\ -2x_{23} + 0.2x_{21}^2 \end{bmatrix}}_{f_2(x_2)} + \underbrace{\begin{bmatrix} 1 \\ -2x_{21} \\ 0 \end{bmatrix}}_{g_2(x_2)} u_2 + \underbrace{\begin{bmatrix} -\Delta\sigma_2 \\ 2x_{21}\Delta\sigma_2 \\ 0 \end{bmatrix}}_{\Delta f_2(x_2)} \\
&\quad + \underbrace{\begin{bmatrix} 0 \\ 0.1\sin(x_{13} + 2x_{11}) \\ 0 \end{bmatrix}}_{D_{21}(x_1)} \qquad (3.51) \\
y_2 &= \underbrace{x_{21}}_{h_2(x_2)} \qquad (3.52)
\end{aligned}
$$

where $x_1 = \text{col}(x_{11}, x_{12}, x_{13})$ and $x_2 = \text{col}(x_{21}, x_{22}, x_{23})$, $h_1(x_1)$ and $h_2(x_2)$, and $u_1(t)$ and $u_2(t)$ are the system state, output and input respectively, $D_{12}(\cdot)$ and $D_{21}(\cdot)$ are interconnected terms and $\Delta f_1(x_1)$ and $\Delta f_2(x_2)$ are the uncertainties experienced by the system which satisfy

$$
\begin{aligned}
||\Delta f_1(x_1)|| &= 0.1|x_{13} + 2x_{11}|\sin^2 t \qquad (3.53) \\
||\Delta f_2(x_2)|| &= 0.1x_{21}^2|\cos t| \qquad (3.54)
\end{aligned}
$$

The domain considered is

$$
\begin{aligned}
\Omega &= \Big\{(x_{11},\ x_{12},\ x_{13},\ x_{21},\ x_{22},\ x_{23}),\ \Big|\ |x_{11}| < 3, \\
&\quad |x_{21}| \leq 1.3,\ x_{12},\ x_{13},\ x_{22},\ x_{23} \in R\Big\} \qquad (3.55)
\end{aligned}
$$

By direct computation, it follows that the first subsystem has uniform relative degree 2, and the second subsystem has uniform relative degree 1. The corresponding transformations are obtained as follows:

$$
T_1 : \begin{cases} \zeta_{11} = x_{11} \\ \zeta_{12} = x_{12} \\ \eta_1 = x_{13} + 2x_{11} \end{cases}, \qquad T_2 : \begin{cases} \zeta_2 = x_{21} \\ \eta_{21} = x_{21}^2 + x_{22} \\ \eta_{22} = x_{23} \end{cases}
$$

In the new coordinates, the system $(3.49) - (3.52)$ can be described by:

$$\begin{aligned}
\dot{\zeta}_1 &= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_1} \begin{bmatrix} \zeta_{11} \\ \zeta_{12} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -0.1\sin\zeta_{11} + u_1 \end{bmatrix}}_{\beta_1} + \underbrace{\begin{bmatrix} \Delta\sigma_1(\zeta_1,\eta_1) \\ 0.5\Delta\sigma_1(\zeta_1,\eta_1) \end{bmatrix}}_{E_1\Delta\Psi(\zeta_1,\eta_1)} \\
&\quad + \underbrace{\begin{bmatrix} 0.2\eta_{21} \\ 0 \end{bmatrix}}_{\Gamma_{12}^a} & (3.56) \\
\dot{\eta}_1 &= \underbrace{-3.25\eta_1 + 0.25\zeta_{11}^2}_{q_1(\zeta_1,\eta_1)} + \underbrace{0.4\eta_{21} + 0.1\sin\zeta_2}_{\Gamma_{12}^b} & (3.57) \\
y_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_{11} \\ \zeta_{12} \end{bmatrix} & (3.58) \\
\dot{\zeta}_2 &= \underbrace{-}_{A_2}\zeta_2 + \underbrace{u_2}_{\beta_2} - \underbrace{\Delta\sigma_2(\zeta_2,\eta_2)}_{E_2\Delta\Psi(\zeta_2,\eta_2)} & (3.59) \\
\dot{\eta}_2 &= \underbrace{\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \eta_{21} \\ \eta_{22} \end{bmatrix} + \begin{bmatrix} \cos\eta_{21} - 1 \\ 0.2\zeta_2^2 \end{bmatrix}}_{q_2(\zeta_2,\eta_2)} + \underbrace{\begin{bmatrix} 0.1\sin\eta_1 \\ 0 \end{bmatrix}}_{\Gamma_{21}^b} & (3.60) \\
y_2 &= \zeta_2 & (3.61)
\end{aligned}$$

where $\zeta_1 = (\zeta_{11},\ \zeta_{12})^T,\ \eta_1 \in R,\ \zeta_2 \in R,$ and $\eta_2 = (\eta_{21}, \eta_{22})^T$.

From (3.53) and (3.54)

$$\begin{aligned}
||\Delta\Psi_1(\zeta_1,\eta_1)|| &\le ||\Delta\sigma_1(\zeta_1,\eta_1)|| \le \underbrace{0.1|\eta_1|^2\cos t}_{\rho_1(\cdot)} \\
||\Delta\Psi_2(\zeta_2,\eta_2)|| &\le ||\Delta\sigma_2(\zeta_2,\eta_2)|| \le \underbrace{0.1\zeta_2|\sin t|^2}_{\rho_2(\cdot)}
\end{aligned}$$

Then, for the first subsystem, choose $L_1 = \begin{bmatrix} 3 & 2 \end{bmatrix}^T$ and $Q = I$. It follows that the Lyapunov equation (3.20) has a unique solution:

$$P_1 = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

and the solution to equation (3.22) is $H_1 = 0.25$. As $M_1 = -3.25$, let $\Lambda_1 = 3.25$. Thus the solution of equation (3.35) is $\Pi_1 = 0.5$. Now, for the second subsystem, choose

$L_2 = 0$ and $Q_2 = 2$. It follows that the Lyapunov equation (3.20) has a unique solution $P_2 = 1$ and the solution to equation(3.22) is $H_2 = -1$. As

$$M_2 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \text{let} \qquad \Lambda_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\Pi_2 = \begin{bmatrix} 0.1667 & 0 \\ 0 & 0.25 \end{bmatrix}$$

By direct computation, it follows that the matrix $W^T + W$ is positive definite in the domain Ω defined in (3.55). Thus, all the conditions of Theorem 4.1 are satisfied. This implies that the dynamical system

$$\dot{\hat{\zeta}}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\zeta}_{11} \\ \hat{\zeta}_{12} \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} (y_1 - C_1 \hat{\zeta}_1) + \begin{bmatrix} 0 \\ u_1 \end{bmatrix} + K_1(\cdot) + \begin{bmatrix} 0.2\hat{\eta}_{21} \\ 0 \end{bmatrix} \tag{3.62}$$

$$\dot{\hat{\eta}}_1 = -3.25\hat{\eta}_1 + 0.25\hat{\zeta}_{11}^2 + 0.4\hat{\eta}_{21} + 0.1 \sin \hat{\zeta}_2 \tag{3.63}$$

$$\dot{\hat{\zeta}}_2 = -\hat{\zeta}_2 + u_2 + K_2(\cdot) \tag{3.64}$$

$$\dot{\hat{\eta}}_2 = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \hat{\eta}_{21} \\ \hat{\eta}_{22} \end{bmatrix} + \begin{bmatrix} \cos \hat{\eta}_{21} - 1 \\ 0.2\hat{\zeta}_2^2 \end{bmatrix} + \begin{bmatrix} 0.1 \sin \hat{\eta}_1 \\ 0 \end{bmatrix} \tag{3.65}$$

is a robust observer of the system $(3.56)-(3.61)$ where $\hat{\zeta}_1 = \text{col}(\hat{\zeta}_{11}, \hat{\zeta}_{12})$, $\hat{\eta}_2 = \text{col}(\hat{\eta}_{21}, \hat{\eta}_{22})$, and $K_1(\cdot)$ and $K_2(\cdot)$ defined in (3.38) are as follows

$$K_1(y_1, \hat{\zeta}_1, \hat{\eta}_1) = \begin{cases} \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix} \frac{(\zeta_{11} - \hat{\zeta}_{11})}{\|\zeta_{11} - \hat{\zeta}_{11}\|} |\eta_1| \sin^2 t), & \zeta_{11} - \hat{\zeta}_{11} \neq 0 \\ 0, & \zeta_{11} - \hat{\zeta}_{11} = 0 \end{cases}$$

$$K_2(y_2, \hat{\zeta}_2, \hat{\eta}_2) = \begin{cases} 0.1 \frac{(\zeta_2 - \hat{\zeta}_2)}{\|\zeta_2 - \hat{\zeta}_2\|} \zeta_2^2 |\cos t|, & \zeta_2 - \hat{\zeta}_2 \neq 0 \\ 0, & \zeta_2 - \hat{\zeta}_2 = 0 \end{cases}$$

Therefore,

$$\begin{aligned} \hat{x}_{11} &= \hat{\zeta}_{11} \\ \hat{x}_{12} &= \hat{\zeta}_{12} \\ \hat{x}_{13} &= \hat{\eta}_1 - 2\hat{\zeta}_{11} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \hat{x}_{21} &= \hat{\zeta}_2 \\ \hat{x}_{22} &= \hat{\eta}_{21} - \hat{\zeta}_2^2 \\ \hat{x}_{23} &= \hat{\eta}_{22} \end{aligned}$$

with $\hat{\zeta}_1 = \text{col}(\hat{\zeta}_{11},\ \hat{\zeta}_{12}),\ \hat{\eta}_1,\ \hat{\zeta}_2$ and $\hat{\eta}_2 = \text{col}(\hat{\eta}_{21}, \hat{\eta}_{22})$ given by system $(3.62) - (3.65)$, provide an asymptotic estimate for x_1 and x_2 of system (3.49)–(3.52).

For simulation purposes, the controllers are chosen as:

$$u_1 = -\zeta_{11} - 2\zeta_{12} \quad \text{and} \quad u_2 = -\zeta_{21} \tag{3.66}$$

The simulation results in Figures 3.1 and 3.2 show that the designed observer estimates the states of the interconnected system in $(3.49) - (3.52)$, $x_1 = \text{col}(x_{11}, x_{12}, x_{13})$ and $x_2 = \text{col}\ (x_{21},\ x_{22}, x_{23})$ respectively, very well. Figure 3.3 shows the error between the actual states and the estimated states in both subsystems.

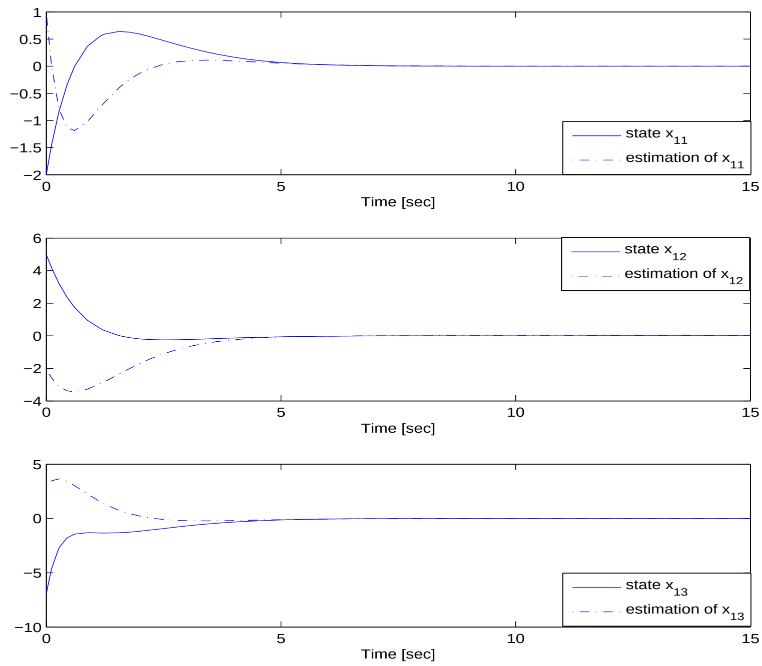

Figure 3.1: The time responses of the states of the first subsystem, $x_1 = \text{col}\ (x_{11}, x_{12}, x_{13})$, and their estimates $\hat{x}_1 = \text{col}\ (\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{13})$

3.5.2. Case Study: Observer Design for Coupled Inverted Pendula

Consider the system given in Figure 3.4 formed by two identical inverted pendula which are connected by a spring and subject to distinct inputs u_1 and u_2 (see, e.g [101]).

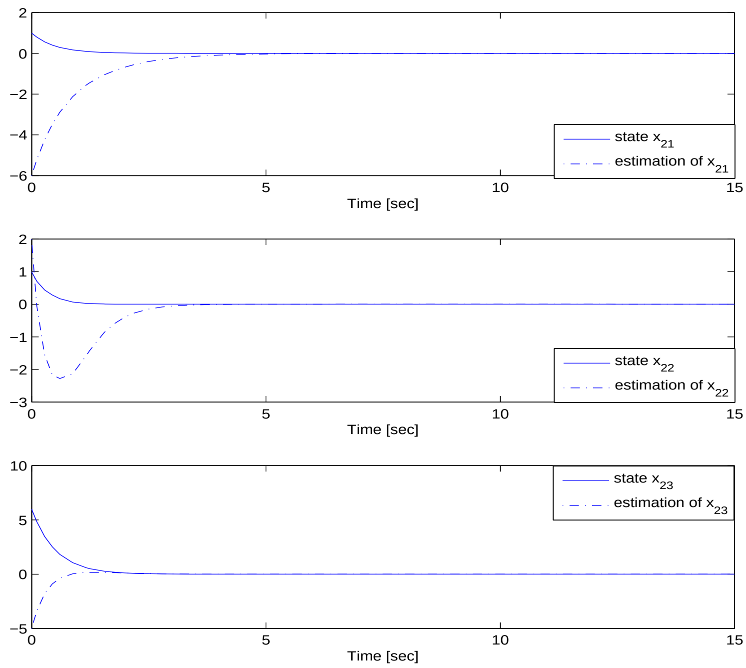

Figure 3.2: The time responses of the states of the second subsystem, x_2 = col (x_{21}, x_{22}, x_{23}), and their estimates $\hat{x}_2$ = col $(\hat{x}_{21}, \hat{x}_{22}, \hat{x}_{23})$

A salient feature of the system is that the point of attachment of the spring can change along the full length (l) of the pendulum. The input to each pendulum is the torque u_i applied at the pivot point. The pay-loads are assumed to be both known and equal to m. Let x_i = col(x_{i1}, x_{i2}) = col$(\theta_i, \theta_i - \omega_i)$ for $i = 1, 2$ where $\omega_i := \dot{\theta}_i$ is the corresponding angle velocity. From [101], the dynamic equations of the system can be described as

$$\begin{aligned}\dot{x}_1 &= \underbrace{\begin{bmatrix} 1 & -1 \\ 1 - \frac{g}{l} & -1 \end{bmatrix}}_{A_1} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{-1}{ml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ \frac{ka^2}{ml^2} x_{11} \end{bmatrix}}_{\beta_1} + \underbrace{\begin{bmatrix} 0 \\ 1.8033\ \Delta\sigma_1(\zeta_1, \eta_1) \end{bmatrix}}_{E_1 \Delta\Psi(x_1)} \\ &\quad + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{-ka^2}{ml^2} & 0 \end{bmatrix}}_{\Gamma_{12}^a} x_2 \end{aligned} \tag{3.67}$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \tag{3.68}$$

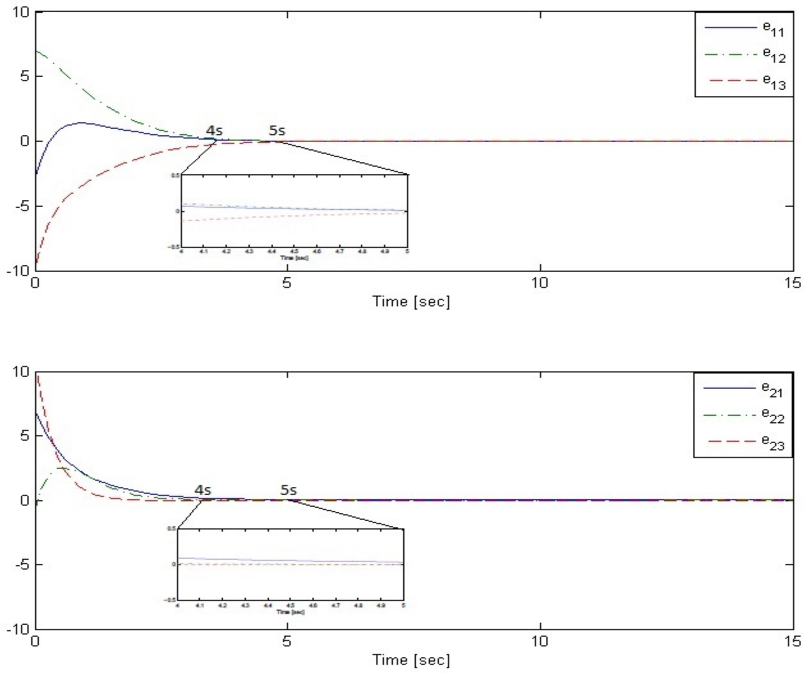

Figure 3.3: The time response of the estimates errors

$$\dot{x}_2 = \underbrace{\begin{bmatrix} 1 & -1 \\ 1-\frac{g}{l} & -1 \end{bmatrix}}_{A_2} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{-1}{ml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ -\frac{ka^2}{ml^2} x_{21} \end{bmatrix}}_{\beta_2} + \underbrace{\begin{bmatrix} 0 \\ 1.8033\ \Delta\sigma_2(\zeta_1, \eta_1) \end{bmatrix}}_{E_2 \Delta\Psi(x_1)} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{ka^2}{ml^2} & 0 \end{bmatrix}}_{\Gamma_{21}^a} x_1 \quad (3.69)$$

$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \quad (3.70)$$

where k and g are the spring and gravity constants, and a is a parameter explained in Figure 3.4. It is assumed the only measurable state is

$$y_i = \begin{bmatrix} 1 & 0 \end{bmatrix} x_i, \qquad i = 1, 2 \quad (3.71)$$

and the parameters are chosen as

$$\frac{g}{l} = 9.8, \qquad \frac{1}{ml^2} = 0.5, \qquad \frac{ka^2}{ml^2} = 2.268 \quad (3.72)$$

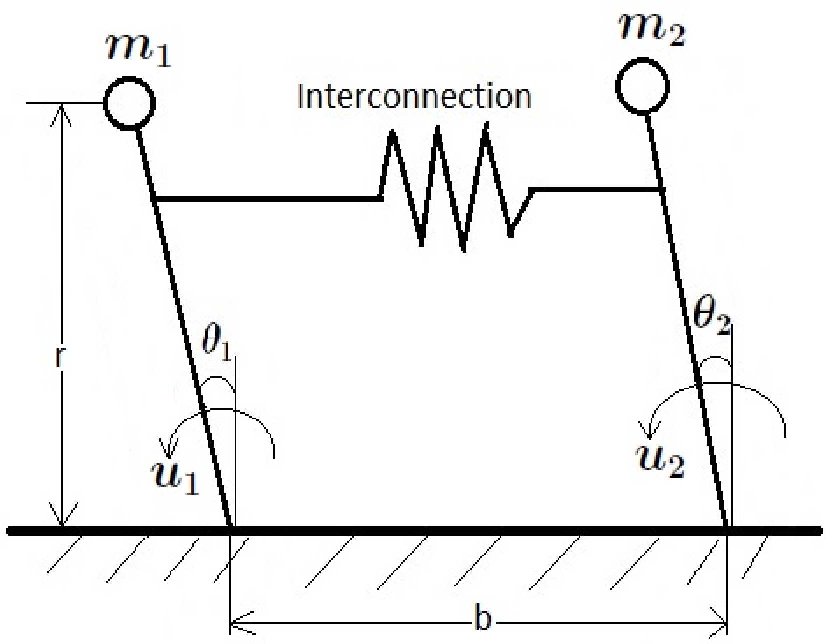

Figure 3.4: Coupled inverted pendula

$$\begin{aligned}
\|\Delta\Psi_1(x_1)\| &\leq ||\Delta\sigma_1(x_1)|| \leq \underbrace{0.5\ \mathrm{cos(t)}}_{\rho_1(\cdot)} \\
\|\Delta\Psi_2(x_2)\| &\leq ||\Delta\sigma_2(x_2)|| \leq \underbrace{0.1\ \mathrm{sin(t)}}_{\rho_2(\cdot)}
\end{aligned}$$

Then, for both subsystems, choose $L_i = [11 \quad -8.8]^T$ and $Q_i = I$ where $i = 1, 2$. It follows that the Lyapunov equation (3.20) has unique solutions:

$$P_i = \begin{bmatrix} 0.0545 & -0.0455 \\ -0.0455 & 0.5 \end{bmatrix}, \qquad i = 1, 2 \tag{3.73}$$

and the solutions to equation (3.22) are $H_i = [0 \ \ 1]^T$ for $i = 1, 2$.

By direct computation, it follows that the matrix $W^T + W$ is symmetric positive definite. Thus, all the conditions of Theorem 4.1 are satisfied. This implies that the dynamical system

$$\begin{aligned}
\dot{\hat{x}}_1 &= \begin{bmatrix} 1 & -1 \\ -8.81 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} + \begin{bmatrix} 11 \\ -8.8 \end{bmatrix} (y_1 - C_1\hat{x}_1) + \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} u_1 \\
&+ \begin{bmatrix} 0 \\ 2.268\hat{x}_{11} \end{bmatrix} + K_1(\cdot) + \begin{bmatrix} 0 & 0 \\ -2.268 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix}
\end{aligned} \tag{3.74}$$

$$\dot{\hat{x}}_2 = \begin{bmatrix} 1 & -1 \\ -8.81 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix} + \begin{bmatrix} 11 \\ -8.8 \end{bmatrix} (y_2 - C_2\hat{x}_2) + \begin{bmatrix} 0 \\ -0.5 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ -2.268\hat{x}_{21} \end{bmatrix} + K_2(\cdot) + \begin{bmatrix} 0 & 0 \\ 2.268 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} \tag{3.75}$$

is a robust observer of the system $(3.67) - (3.70)$ where $\hat{x}_1 = \text{col}(\hat{x}_{11}, \hat{x}_{12})$, $\hat{x}_2 = \text{col}(\hat{x}_{21}, \hat{x}_{22})$, and the $K_1(\cdot)$ and $K_2(\cdot)$ defined in (3.38) are

$$K_1(y_1, \hat{x}_1) = \begin{cases} \begin{bmatrix} 9.918 \\ 0.9016 \end{bmatrix} \frac{(x_{11}-\hat{x}_{11})}{\|x_{11}-\hat{x}_{11}\|} \cos t), & x_{11} - \hat{x}_{11} \neq 0 \\ 0, & x_{11} - \hat{x}_{11} = 0 \end{cases}$$

$$K_2(y_2, \hat{x}_2) = \begin{cases} \begin{bmatrix} 1.9836 \\ 0.1803 \end{bmatrix} \frac{(x_{21}-\hat{x}_{21})}{\|x_{21}-\hat{x}_{21}\|} \sin t), & x_{21} - \hat{x}_{21} \neq 0 \\ 0, & x_{21} - \hat{x}_{21} = 0 \end{cases}$$

For simulation purposes, the controllers are chosen as:

$$u_i = -61.6x_{i1} + 22x_{i2}, \qquad i = 1, 2 \tag{3.76}$$

The simulation results in Figures 3.5 and 3.6 show that the states $\hat{x}_1$ and $\hat{x}_2$ of the designed dynamics approximate the states of the interconnected system (3.67)–(3.70), $x_1 = \text{col}\ (x_{11}, x_{12})$ and $x_2 = \text{col}\ (x_{21},\ x_{22})$ respectively, very well.

3.6. CONCLUSIONS

In this chapter, observer design for a class of nonlinear large scale interconnected systems with uniform relative degree has been considered. An asymptotic observer has been developed for nonlinear interconnected systems with uncertainties using the Lyapunov approach together with a geometric transformation. It is not required that either the isolated nominal subsystems or the interconnections are linearisable. Robustness to uncertainties is enhanced by using the system structure and the structure of the uncertainties within the design framework. The developed results are applicable to a wide class of nonlinear interconnected systems.

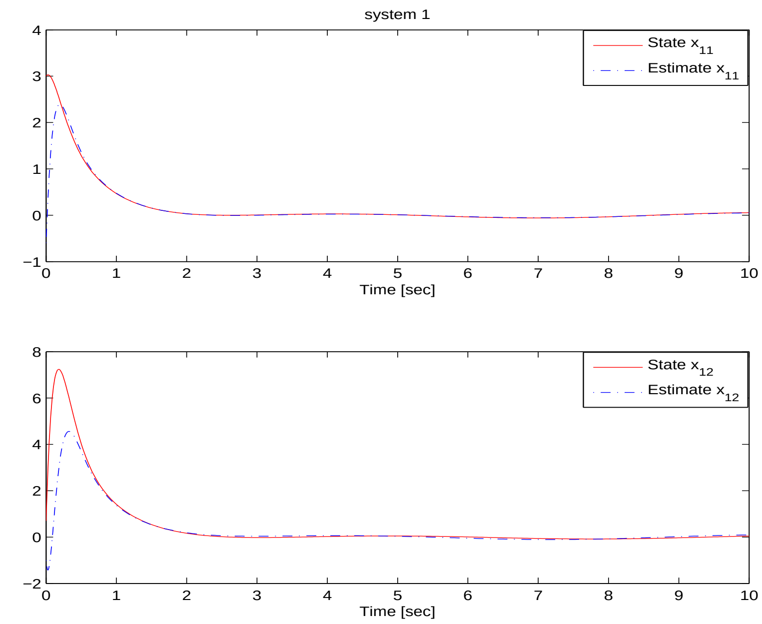

Figure 3.5: The time responses of the states of the first subsystem, $x_1 = \text{col}\,(x_{11}, x_{12})$, and their estimates $\hat{x}_1 = \text{col}\,(\hat{x}_{11}, \hat{x}_{12})$

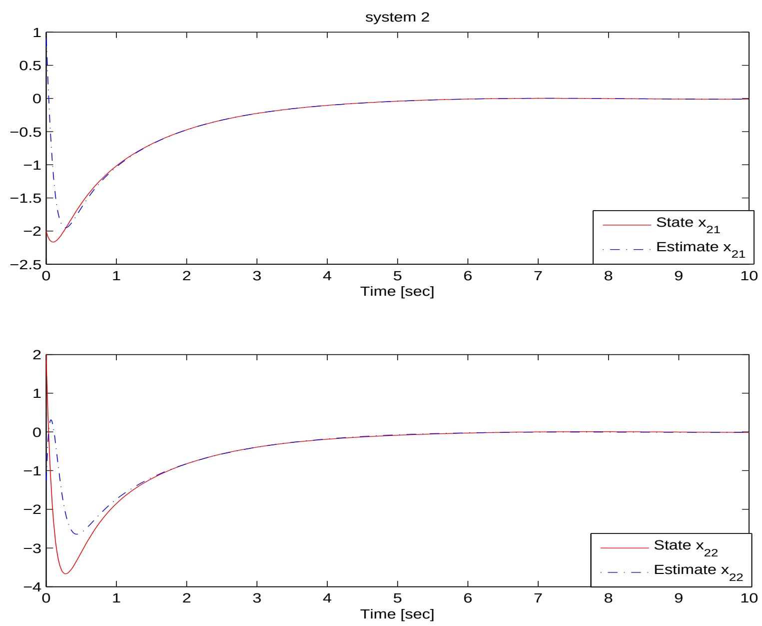

Figure 3.6: The time responses of the states of the second subsystem, $x_2 = \text{col}\,(x_{21}, x_{22})$, and their estimates $\hat{x}_2 = \text{col}\,(\hat{x}_{21}, \hat{x}_{22})$

CHAPTER. 4

ROBUST SLIDING MODE OBSERVERS FOR LARGE SCALE SYSTEMS WITH APPLICATION TO A MULTIMACHINE POWER SYSTEM

In this chapter, a class of interconnected systems with structured and unstructured uncertainties is considered where the known interconnections and uncertain interconnections are nonlinear. The bounds on the uncertainties are employed in the observer design to enhance the robustness when the structure of the uncertainties is available for design. Under the condition that the structure distribution matrices of the uncertainties are known, a robust sliding mode observer is designed and a set of sufficient conditions is developed to guarantee that the error dynamics are asymptotically stable. In the case that the structure of uncertainties is unknown, an ultimately bounded approximate observer is developed to estimate the system states using sliding mode techniques. The results obtained are applied to a multimachine power system, and simulation for a two machine power system is presented to demonstrate the feasibility and effectiveness of the developed methods.

4.1. SYSTEM DESCRIPTION AND PRELIMINARIES

Consider a nonlinear interconnected system composed of N subsystems as follows

$$\dot{x}_i = A_i x_i + B_i u_i + \Delta\phi_i(x_i, u_i) + M_i(x) + \Delta M_i(x) \tag{4.1}$$

$$y_i = C_i x_i \tag{4.2}$$

where $x_i \in R^{n_i}$, $u_i \in U \in R^{m_i}$ (U is the admissible control set) and $y_i \in R^{p_i}$ with $m_i \leq p_i \leq n_i$ are the state variables, inputs and outputs of the i-th subsystem respectively. The matrix triples (A_i, B_i, C_i) are constant with appropriate dimensions and C_i are full row rank. The terms $\Delta\phi_i(x_i, u_i)$ and $\Delta M_i(x)$ are the uncertainties in the i-th isolated subsystems and interconnections respectively. The terms $M_i(x)$ are the known interconnections for $i = 1, \cdots, N$.

The aim is to design a robust sliding mode observer for the above system to estimate inaccessible states in the presence of both structured and unstructured uncertainties.

Assumption 5.1. The uncertainties $\Delta\phi_i(x_i, u_i)$ and $\Delta M_i(x)$ have the decomposition

$$\Delta\phi_i(x_i, u_i) = H_i^a \Delta\xi_i(x_i, u_i), \qquad \Delta M_i(x) = H_i^b \Delta E_i(x) \tag{4.3}$$

where $H_i^a \in R^{n_i \times k_i}$ and $H_i^b \in R^{n_i \times r_i}$ are the distribution matrices of the uncertainties, and

$$\|\Delta\xi_i(x_i, u_i)\| \leq \rho_i(x_i, u_i) \qquad \text{and} \qquad \|\Delta E_i(x)\| \leq \sigma_i(x) \tag{4.4}$$

where $\rho_i(x_i, u_i)$ is known and Lipshitz about x_i uniformly for $u_i \in U$, and $\sigma_i(x)$ is known and Lipshitz about x.

Since the C_i are full row rank, there exist nonsingular matrices T_{c_i} such that

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i1} & \bar{A}_{i2} \\ \bar{A}_{i3} & \bar{A}_{i4} \end{bmatrix} := T_{c_i} A_i T_{c_i}^{-1}, \tag{4.5}$$

$$\bar{B}_i = \begin{bmatrix} \bar{B}_{i1} \\ \bar{B}_{i2} \end{bmatrix} := T_{c_i} B_i, \qquad \bar{C}_i = \begin{bmatrix} 0 & I_{p_i} \end{bmatrix} := C_i T_{c_i}^{-1} \tag{4.6}$$

where $\bar{A}_{i1} \in R^{(n_i-p_i)\times(n_i-p_i)}$, $\bar{B}_{i1} \in R^{(n_i-p_i)\times m_i}$ and $\bar{B}_{i2} \in R^{p_i\times m_i}$ for $i = 1, \cdots, N$.

Then in the new coordinates

$$\bar{x}_i = T_{c_i} x_i \tag{4.7}$$

system (4.1)-(4.2) can be rewritten as

$$\dot{\bar{x}}_{i1} = \bar{A}_{i1}\bar{x}_{i1} + \bar{A}_{i2}\bar{x}_{i2} + \bar{B}_{i1}u_i + \bar{H}_{i1}^a \Delta\bar{\phi}_i(\bar{x}_i, u_i) + \bar{M}_{i1}(\bar{x}) + \bar{H}_{i1}^b \Delta\bar{M}_i(\bar{x}) \tag{4.8}$$

$$\dot{\bar{x}}_{i2} = \bar{A}_{i3}\bar{x}_{i1} + \bar{A}_{i4}\bar{x}_{i2} + \bar{B}_{i2}u_i + \bar{H}_{i2}^a \Delta\bar{\phi}_i(\bar{x}_i, u_i) + \bar{M}_{i2}(\bar{x}) + \bar{H}_{i2}^b \Delta\bar{M}_i(\bar{x}) \tag{4.9}$$

$$y_i = \bar{x}_{i2} \tag{4.10}$$

where $\bar{x} = \text{col}(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$, $\bar{x}_i = \text{col}(\bar{x}_{i1}, \bar{x}_{i2})$, $\bar{x}_{i1} \in R^{n_i-p_i}$, $\bar{x}_{i2} \in R^{p_i}$, $\bar{A}_{ij}$ and $\bar{B}_{il}$ are defined in (4.5)-(4.6) for $j = 1, 2, 3, 4$, $l = 1, 2$, $i = 1, 2, \cdots, N$, and

$$\begin{bmatrix} \bar{H}_{i1}^a \\ \bar{H}_{i2}^a \end{bmatrix} := T_{c_i} H_i^a, \qquad \begin{bmatrix} \bar{H}_{i1}^b \\ \bar{H}_{i2}^b \end{bmatrix} := T_{c_i} H_i^b \tag{4.11}$$

$$\begin{bmatrix} \bar{M}_{i1}(x) \\ \bar{M}_{i2}(x) \end{bmatrix} := T_{c_i} M_i(x) \tag{4.12}$$

$$\Delta\bar{\phi}_i(\bar{x}_i, u_i) = \Delta\xi_i(T_{c_i}^{-1}\bar{x}_i, u_i) \tag{4.13}$$

$$\Delta\bar{M}_i(\bar{x}) = \Delta E_i(T_{c_i}^{-1}\bar{x}) \tag{4.14}$$

where $\bar{H}_{i1}^a \in R^{(n_i-p_i)\times k_i}$, $\bar{H}_{i1}^b \in R^{(n_i-p_i)\times r_i}$, and $\bar{M}_{i1}(\cdot) \in R^{(n_i-p_i)}$ for $i = 1, 2, \cdots, N$.

Assumption 5.2. The matrix pair $(\bar{A}_i, \bar{C}_i)$ in (4.5)-(4.6) is observable for $i = 1, 2, \cdots, N$.

Under Assumption 5.2, there exists a matrix L_i such that $\bar{A}_i - L_i\bar{C}_i$ is stable, and thus for any $Q_i > 0$ the Lyapunov equation

$$(\bar{A}_i - L_i\bar{C}_i)^T P_i + P_i(\bar{A}_i - L_i\bar{C}_i) = -Q_i \tag{4.15}$$

has an unique solution $P_i > 0$ for $i = 1, 2, \cdots, N$.

Assumption 5.3. There exist matrices $F_i^a \in R^{k_i \times p_i}$ and $F_i^b \in R^{r_i \times p_i}$ such that the solution P_i to the Lyapunov equation (4.15) satisfies the constraint

$$\bar{H}_i^{aT} P_i = F_i^a \bar{C}_i \tag{4.16}$$

$$\bar{H}_i^{bT} P_i = F_i^b \bar{C}_i \tag{4.17}$$

For further analysis, introduce partitions of P_i and Q_i which are conformable with the decomposition in (4.8)-(4.10) as follows

$$P_i = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i2}^T & P_{i3} \end{bmatrix}, \qquad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} \tag{4.18}$$

where $P_{i1} \in R^{(n_i - p_i) \times (n_i - p_i)}$ and $Q_{i1} \in R^{(n_i - p_i) \times (n_i - p_i)}$. Then, from $P_i > 0$ and $Q_i > 0$, it follows that $P_{i1} > 0$, $P_{i3} > 0$, $Q_{i1} > 0$ and $Q_{i3} > 0$.

The following results are required for further analysis.

Lemma 5.1. If P_i and Q_i have the partition in (4.18), then under Assumption 5.3, the following results hold

(i). $P_{i1}^{-1} P_{i2} \bar{H}_{i2}^a + \bar{H}_{i1}^a = 0$ if (4.16) is satisfied.

(ii). $P_{i1}^{-1} P_{i2} \bar{H}_{i2}^b + \bar{H}_{i1}^b = 0$ if (4.17) is satisfied.

(iii). The matrix $A_{i1} + P_{i1}^{-1} P_{i2} A_{i3}$ is Hurwitz stable if the Lyapunov equation (4.15) is satisfied.

Proof. See Lemma 2.1 in [126].

4.2. SLIDING MODE OBSERVER DESIGN

Consider the system in (4.8)-(4.10). Introduce a linear coordinate transformation

$$z_i = \underbrace{\begin{bmatrix} I_{n_i - p_i} & P_{i1}^{-1} P_{i2} \\ 0 & I_{p_i} \end{bmatrix}}_{T_i} \bar{x}_i \tag{4.19}$$

In the new coordinate system z_i, system (4.8)-(4.10) has the following form

$$\begin{aligned}
\dot{z}_{i1} &= (\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})z_{i1} + (\bar{A}_{i2} - \bar{A}_{i1}P_{i1}^{-1}P_{i2} + P_{i1}^{-1}P_{i2}(\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2}))z_{i2} \\
&\quad + \bar{B}_{i1}u_i + P_{i1}^{-1}P_{i2}\bar{B}_{i2}u_i + \bar{M}_{i1}(T^{-1}z) + P_{i1}^{-1}P_{i2}\bar{M}_{i2}(T^{-1}z) \qquad (4.20) \\
\dot{z}_{i2} &= \bar{A}_{i3}z_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2})z_{i2} + \bar{B}_{i2}u_i + \bar{H}_{i2}^{a}\Delta\bar{\phi}_i(T_i^{-1}z_i, u_i) \\
&\quad + \bar{M}_{i2}(T^{-1}z) + \bar{H}_{i2}^{b}\Delta\bar{M}_i(T^{-1}z) \qquad (4.21) \\
y_i &= z_{i2} \qquad (4.22)
\end{aligned}$$

where $z_i = \text{col}(z_{i1}, z_{i2})$ with $z_{i1} \in R^{n_i - p_i}$. From Assumption 5.1, (4.13) and (4.14)

$$\begin{aligned}
\|\Delta\bar{\phi}_i(T_i^{-1}z_i, u_i)\| &\leq \rho_i((T_iT_{c_i})^{-1}z_i, u_i) := \bar{\rho}_i(z_i, u_i) \qquad (4.23) \\
\|\Delta\bar{M}_i(T^{-1}z)\| &\leq \sigma_i((TT_c)^{-1}z) := \bar{\sigma}_i(z) \qquad (4.24)
\end{aligned}$$

and $\bar{\rho}_i(z_i, u_i), \bar{\sigma}_i(z)$ satisfy the Lipschitz condition

$$\begin{aligned}
\|\bar{\rho}_i(z_i, u_i) - \bar{\rho}_i(\hat{z}_i, u_i)\| &\leq \ell_{\bar{\rho}_i}\|z_i - \hat{z}_i\| \qquad (4.25) \\
\|\bar{\sigma}_i(z) - \bar{\sigma}_i(\hat{z})\| &\leq \ell_{\bar{\sigma}_i}\|z - \hat{z}\| \qquad (4.26)
\end{aligned}$$

Here $\ell_{\bar{\rho}_i}$ may be a function of u_i.

For system (4.20)-(4.22), consider a dynamical system

$$\begin{aligned}
\dot{\hat{z}}_{i1} &= (\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})\hat{z}_{i1} + (\bar{A}_{i2} - \bar{A}_{i1}P_{i1}^{-1}P_{i2} + P_{i1}^{-1}P_{i2}(\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2}))y_i \\
&\quad + \bar{B}_{i1}u_i + P_{i1}^{-1}P_{i2}\bar{B}_{i2}u_i + \bar{M}_{i1}(T^{-1}\hat{z}) + P_{i1}^{-1}P_{i2}\bar{M}_{i2}(T^{-1}\hat{z}) \qquad (4.27) \\
\dot{\hat{z}}_{i2} &= \bar{A}_{i3}\hat{z}_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2})\hat{z}_{i2} + \bar{B}_{i2}u_i + \bar{M}_{i2}(T^{-1}\hat{z}) + d_i(\cdot) \qquad (4.28) \\
\hat{y}_i &= \hat{z}_{i2} \qquad (4.29)
\end{aligned}$$

where $\hat{z} = \text{col}(\hat{z}_1, y)$, and the injection term $d_i(\cdot)$ is defined by

$$\begin{aligned}
d_i(\cdot) &= (\|\bar{H}_{i2}^{a}\|\bar{\rho}_i(\hat{z}_i, u_i) + \|\bar{H}_{i2}^{b}\|\bar{\sigma}_i(\hat{z}) + \|\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2}\|\|y_i - \hat{y}_i\| \\
&\quad + k_i)\text{sgn}(y_i - \hat{y}_i) \qquad (4.30)
\end{aligned}$$

where $\bar{\rho}_i(\hat{z}_i, u_i) = \bar{\rho}_i(\hat{z}_{i1}, y_i, u_i)$ and $\bar{\sigma}_i(\hat{z}) = \bar{\sigma}_i(\hat{z}_{11}, y_1, \hat{z}_{21}, y_2, \cdots, \hat{z}_{N1}, y_N)$.

Let $e_{i1} = z_{i1} - \hat{z}_{i1}$ and $e_{y_i} = y_i - \hat{y}_i$. Then from (4.20)-(4.22) and (4.27)-(4.29), the error dynamics are described by

$$\begin{aligned}\dot{e}_{i1} =& (\bar{A}_{i1} + P_{i1}^{-1} P_{i2} \bar{A}_{i3}) e_{i1} + [\bar{M}_{i1}(T^{-1}z) - \bar{M}_{i1}(T^{-1}\hat{z})] \\ &+ P_{i1}^{-1} P_{i2} [\bar{M}_{i2}(T^{-1}z) - \bar{M}_{i2}(T^{-1}\hat{z})] \qquad (4.31)\\ \dot{e}_{y_i} =& \bar{A}_{i3} e_{i1} + (\bar{A}_{i4} - \bar{A}_{i3} P_{i1}^{-1} P_{i2}) e_{y_i} + [\bar{M}_{i2}(T^{-1}z) - \bar{M}_{i2}(T^{-1}\hat{z})] \\ &+ \bar{H}_{i2}^{a} \Delta\bar{\phi}_i(T_i^{-1} z_i, u_i) + \bar{H}_{i2}^{b} \Delta\bar{M}_i(T^{-1}z) - d_i(\cdot) \qquad (4.32)\end{aligned}$$

where $d_i(\cdot)$ is given in (4.30) for $i = 1, 2, \cdots, N$.

From the structure of the transformation matrix T_i in (4.19) and the fact that $\hat{z}_i = \text{col}(\hat{z}_{i1}, y_i)$, it follows that

$$\|T_i^{-1} z_i - T_i^{-1} \hat{z}_i\| = \|T_i^{-1}(z_i - \hat{z}_i)\| = \left\| T_i^{-1} \begin{bmatrix} e_{i1} \\ 0 \end{bmatrix} \right\| = \|e_{i1}\|$$

From the analysis above, it is straightforward to see

$$\|T^{-1}z - T^{-1}\hat{z}\| = \|e_1\| \qquad (4.33)$$

where

$$e_1 := \text{col}(e_{11}, e_{21}, \cdots, e_{N1}) \qquad (4.34)$$

Therefore,

$$\|\bar{M}_{i1}(T^{-1}z) - \bar{M}_{i1}(T^{-1}\hat{z})\| \leq \ell_{\bar{M}_{i1}} \|e_1\| \qquad (4.35)$$

$$\|\bar{M}_{i2}(T^{-1}z) - M_{i2}(T^{-1}\hat{z})\| \leq \ell_{\bar{M}_{i2}} \|e_1\| \qquad (4.36)$$

The following conclusion is ready to be presented:

Theorem 5.1. Under Assumptions $5.1 - 5.3$, the error system (4.31) is asymptotically stable if the matrix $W^T + W$ is positive definite, where the matrix $W = [w_{ij}]_{N\times N}$, and its entries w_{ij} are defined by

$$w_{ij} = \begin{cases} \lambda_{min}(Q_{i1}) - 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}}\big], & i = j \\ -2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}}\big], & i \neq j \end{cases} \qquad (4.37)$$

where P_{i1}, P_{i2} and Q_{i1} are given in (4.18).

Proof. For system (4.31), consider a Lyapunov function candidate

$$V = \sum_{i=1}^{N} e_{i1}^T P_{i1} e_{i1}$$

Then, the time derivative of V along the trajectories of system (4.31) is given by

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N} \Big\{ e_{i1}^T [P_{i1}(\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})^T + (\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})P_{i1}]e_{i1} \\
&\quad + 2\|P_{i1}\|\|e_{i1}\| \big\{ \big[\ell_{\bar{M}_{i1}} + \|P_{i1}^{-1}P_{i2}\|\ell_{\bar{M}_{i2}} \big] \|e_1\| \big\} \Big\} \\
&\le \sum_{i=1}^{N} \Big\{ -e_{i1}^T Q_{i1} e_{i1} + 2\|e_{i1}\| \big\{ \big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}} \big] \|e_1\| \big\} \Big\}
\end{aligned} \tag{4.38}$$

From the definition of e_1 in (4.34)

$$\|e_1\| \le \sum_{j=1}^{N} \|e_{j1}\| = \|e_{i1}\| + \sum_{\substack{j=1 \\ j\neq i}}^{N} \|e_{j1}\| \tag{4.39}$$

Then, from (4.38) and (4.39)

$$\begin{aligned}
\dot{V} &\le \sum_{i=1}^{N} \Big\{ -e_{i1}^T Q_{i1} e_{i1} + 2\|e_{i1}\| \big\{ \big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}} \big] \big[\|e_{i1}\| + \sum_{\substack{j=1 \\ j\neq i}}^{N} \|e_{j1}\| \big] \big\} \Big\} \\
&\le \sum_{i=1}^{N} \Big\{ -e_{i1}^T Q_{i1} e_{i1} + 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{M_{i2}} \big] \|e_{i1}\|^2 \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}} \big] \|e_{i1}\|\|e_{j1}\| \Big\} \\
&\le -\sum_{i=1}^{N} \Big\{ \big\{ \lambda_{min}(Q_{i1}) - 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}} \big] \big\} \|e_{i1}\|^2 \\
&\quad - \sum_{\substack{j=1 \\ j\neq i}}^{N} 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}} \big] \|e_{i1}\|\|e_{j1}\| \Big\}
\end{aligned} \tag{4.40}$$

Then, from the definition of the matrix W in (4.37) and the inequality above, it follows that

$$\dot{V} \le -\frac{1}{2} X^T [W^T + W] X$$

where $X = [\|e_{11}\|, \|e_{21}\|, \cdots, \|e_{N1}\|]^T$. Hence, the conclusion follows from $W^T + W > 0$. $\triangle$

Remark 5.1. From the error dynamics (4.31)-(4.32), it is clear to see that the e_{i1} dynamics interact with the dynamics e_{y_i} through the interconnection terms $\bar{M}_{i1}(\cdot)$ and $\bar{M}_{i2}(\cdot)$. From the inequalities (4.35) and (4.36), it follows that the interconnections on the right-hand side of equation (4.31) are bounded by functions of e_1 only. The proof of Theorem 5.1 further shows that the stability of the error dynamics (4.31) are actually independent of e_{y_i}. This fact will be used to show the stability of the sliding motion later.

Remark 5.2. From the stability of Theorem 1, it follows that e_1 is bounded and thus there exists a constant $\beta > 0$ such that

$$\|e_1\| \leq \beta, \tag{4.41}$$

where β can be estimated using the approach given in [126].

For system (4.31)-(4.32), consider a sliding surface

$$S = \{(e_{11}, e_{y_1}, e_{21}, e_{y_2}, \cdots, e_{N1}, e_{y_N})\big| e_{y_1} = 0, e_{y_2} = 0, \cdots, e_{y_N} = 0\} \tag{4.42}$$

From the structure of the error dynamical system (4.31)-(4.32), it follows that the sliding mode of the error system (4.31)-(4.32) with respect to the sliding surface (4.42) is the system (4.31) when limited to the sliding surface (4.42). From Remark 5.1 and Theorem 5.1, the sliding mode associated with the sliding surface S given in (4.42) is asymptotically stable if the the conditions of Theorem 5.1 hold. All that remains is to determine the gains k_i in (4.30) such that the system (4.31)-(4.32) can be driven to the sliding surface S in finite time and a sliding motion maintained thereafter.

Theorem 5.2. Under Assumptions 5.1-5.3, system (4.31)-(4.32) is driven to the sliding surface (4.42) in finite time and remains on it if

$$k_i \geq (\|\bar{A}_{i3}\| + \ell_{\bar{M}_{i2}} + \|\bar{H}^a_{i2}\|\ell_{\bar{\rho}} + \|\bar{H}^b_{i2}\|\ell_{\bar{\sigma}})\beta + \eta \tag{4.43}$$

where β is determined by (4.41) and η is a positive constant.

Proof. From (4.32)

$$\begin{aligned}\sum_{i=1}^{N} e^T_{y_i}\dot{e}_{y_i} &= \sum_{i=1}^{N} e^T_{y_i}\Big\{\bar{A}_{i3}e_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}P^{-1}_{i1}P_{i2})e_{y_i} + [\bar{M}_{i2} - \hat{\bar{M}}_{i2}] \\ &\quad +\bar{H}^a_{i2}\Delta\bar{\phi}_i(T^{-1}_i z_i, u_i) + \bar{H}^b_{i2}\Delta\bar{M}_i(T^{-1}z) - d_i(\cdot)\Big\}\end{aligned}$$

$$\begin{aligned} \leq \;& \sum_{i=1}^{N}\Big\{\|\bar{A}_{i3}\|\|e_{i1}\|\|e_{y_i}\| + \ell_{\bar{M}_{i2}}\|e_{y_i}\|\|e_1\| + \|\bar{H}^a_{i2}\|\bar{\rho}_i(z_i,u_i)\|e_{y_i}\| \\ & +\|\bar{H}^b_{i2}\|\bar{\sigma}_i(z)\|e_{y_i}\| + \|(\bar{A}_{i4}-\bar{A}_{i3}P_{i1}^{-1}P_{i2})\|\|e_{y_i}\|^2 - \|e_{y_i}\|\big\{\|\bar{H}^a_{i2}\|\bar{\rho}_i(\hat{z}_{i1},y_i,u_i) \\ & +\|\bar{H}^b_{i2}\|\bar{\sigma}_i(\hat{z}) + \|\bar{A}_{i4}-\bar{A}_{i3}P_{i1}^{-1}P_{i2}\|\|e_{y_i}\| + k_i)\text{sgn}(e_{y_i})\big\}\Big\} \end{aligned} \tag{4.44}$$

From (4.41), $\|e_{i1}\| \leq \beta$. Applying (4.41) to (4.44), it follows that

$$\sum_{i=1}^{N} e_{y_i}^T\dot{e}_{y_i} \;\leq\; \sum_{i=1}^{N}\Big\{\big\{(\|\bar{A}_{i3}\| + \ell_{\bar{M}_{i2}} + \|\bar{H}^a_{i2}\|\ell_{\bar{\rho}_i} + \|\bar{H}^b_{i2}\|\ell_{\bar{\sigma}_i})\beta - k_i\big\}\|e_{y_i}\|\Big\} \tag{4.45}$$

Applying (4.43) to (4.45)

$$\sum_{i=1}^{N} e_{y_i}^T\dot{e}_{y_i} \leq -\eta\sum_{i=1}^{N}\|e_{y_i}\| \tag{4.46}$$

which implies that

$$e_y^T\dot{e}_y \leq -\eta\|e_y\|$$

where $e_y = \text{col}(e_{y_1}, e_{y_2}, \cdots, e_{y_N})$ and the inequality $\|e_y\| \leq \sum_{i=1}^{N}\|e_{y_i}\|$ is applied to obtain the inequality above. This shows that the reachability condition is satisfied. Hence the conclusion follows. $\triangle$

The study above shows that (4.27)-(4.29) is an asymptotic observer of the system (4.20)-(4.22).

If the structure of the uncertainties $\Delta\phi_i(x_i,u_i)$ and $\Delta M_i(x)$ in the system (4.1)-(4.2) are unknown, which implies that Assumption 5.1 does not hold, then an asymptotic observer usually is not available. In this case, an ultimately bounded observer will be designed. The following Assumption is required.

Assumption 5.4. The uncertainties $\Delta\phi_i(x_i,u_i)$ and $\Delta M_i(x)$ in system (4.1)-(4.2) satisfy

$$\|\Delta\phi_i(x_i,u_i)\| \;\leq\; \varepsilon_i \tag{4.47}$$

$$\|\Delta\bar{M}_i(x)\| \;\leq\; \Upsilon_i \tag{4.48}$$

where ε_i and Υ_i are positive constants.

In this case, in the new coordinate z the system (4.1)-(4.2) is described by

$$\begin{aligned}\dot{z}_{i1} =& (\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})z_{i1} + (\bar{A}_{i2} - \bar{A}_{i1}P_{i1}^{-1}P_{i2} + P_{i1}^{-1}P_{i2}(\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2}))z_{i2} \\ &+\bar{B}_{i1}u_i + P_{i1}^{-1}P_{i2} \times \bar{B}_{i2}u_i + \bar{M}_{i1}(T^{-1}z) + P_{i1}^{-1}P_{i2}\bar{M}_{i2}(T^{-1}z) \\ &+\Delta\tilde{\phi}_{i1}(T_i^{-1}z_i, u_i) + \Delta\tilde{M}_{i1}(T^{-1}z) \qquad (4.49)\end{aligned}$$

$$\begin{aligned}\dot{z}_{i2} =& \bar{A}_{i3}z_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2})z_{i2} + \bar{B}_{i2}u_i + \bar{M}_{i2}(T^{-1}z) \\ &+\Delta\tilde{\phi}_{i2}(T_i^{-1}z_i, u_i) + \Delta\tilde{M}_{i2}(T^{-1}z) \qquad (4.50)\end{aligned}$$

$$y_i = z_{i2} \qquad (4.51)$$

where

$$\begin{bmatrix} \Delta\tilde{\phi}_{i1}(T_i^{-1}z_i, u_i) \\ \Delta\tilde{\phi}_{i2}(T_i^{-1}z_i, u_i) \end{bmatrix} = T_i^{-1} \begin{bmatrix} \Delta\phi_{i1}(T_i^{-1}z_i, u_i) \\ \Delta\phi_{i2}(T_i^{-1}z_i, u_i) \end{bmatrix} \qquad (4.52)$$

$$\begin{bmatrix} \Delta\tilde{M}_{i1}(T^{-1}z) \\ \Delta\tilde{M}_{i2}(T^{-1}z) \end{bmatrix} = T_i^{-1} \begin{bmatrix} \Delta\bar{M}_{i1}(T^{-1}z) \\ \Delta\bar{M}_{i2}(T^{-1}z) \end{bmatrix} \qquad (4.53)$$

and $z_i = \text{col}(z_{i1}, z_{i2})$ with $z_{i1} \in R^{n_i - p_i}$. From (4.47)-(4.48), there are constants $\varepsilon_i^a,\ \varepsilon_i^b,\ \Upsilon_i^a$ and Υ_i^b such that

$$\|\Delta\tilde{\phi}_{i1}(T_i^{-1}z_i, u_i)\| \leq \varepsilon_i^a \qquad (4.54)$$

$$\|\Delta\tilde{\phi}_{i2}(T_i^{-1}z_i, u_i)\| \leq \varepsilon_i^b \qquad (4.55)$$

$$\|\Delta\tilde{M}_{i1}(T^{-1}z)\| \leq \Upsilon_i^a \qquad (4.56)$$

$$\|\Delta\tilde{M}_{i2}(T^{-1}z)\| \leq \Upsilon_i^b \qquad (4.57)$$

Now consider the dynamical systems

$$\begin{aligned}\dot{\hat{z}}_{i1} =& (\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})\hat{z}_{i1} + (\bar{A}_{i2} - \bar{A}_{i1}P_{i1}^{-1}P_{i2} + P_{i1}^{-1}P_{i2}(\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2}))y_i \\ &+\bar{B}_{i1}u_i + P_{i1}^{-1}P_{i2}\bar{B}_{i2}u_i + \bar{M}_{i1}(T^{-1}\hat{z}) + P_{i1}^{-1}P_{i2}\bar{M}_{i2}(T^{-1}\hat{z}) \qquad (4.58)\end{aligned}$$

$$\dot{\hat{z}}_{i2} = \bar{A}_{i3}\hat{z}_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2})\hat{z}_{i2} + \bar{B}_{i2}u_i + \bar{M}_{i2}(T^{-1}\hat{z}) + d_i(\cdot) \qquad (4.59)$$

$$\hat{y}_i = \hat{z}_{i2} \qquad (4.60)$$

where $\hat{z} = \text{col}(\hat{z}_1, y)$. The injection term $d_i(\cdot)$ is defined by

$$\begin{aligned}d_i(\cdot) =& (\|\Delta\tilde{\phi}_{i2}(T_i^{-1}\hat{z}_i, u_i)\| + \|\Delta\tilde{M}_{i2}(T^{-1}\hat{z})\| \\ &+\|\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2}\|\|y_i - \hat{y}_i\| + k_i)\text{sgn}(y_i - \hat{y}_i) \qquad (4.61)\end{aligned}$$

Let $e_{i1} = z_{i1} - \hat{z}_{i1}$ and $e_{y_i} = y_i - \hat{y}_i$. Then from (4.49)-(4.51) and (4.58)-(4.60), the error dynamical equation is described by

$$\begin{aligned}\dot{e}_{i1} = &(\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3})e_{i1} + [\bar{M}_{i1}(T^{-1}z) - \bar{M}_{i1}(T^{-1}\hat{z})] + P_{i1}^{-1}P_{i2}[\bar{M}_{i2}(T^{-1}z) \\ &-\bar{M}_{i2}(T^{-1}\hat{z})] + \Delta\tilde{\phi}_{i1}(T_i^{-1}z_i, u_i) + \Delta\tilde{M}_{i1}(T^{-1}z) \qquad (4.62)\\ \dot{e}_{y_i} = &\bar{A}_{i3}e_1 + (\bar{A}_{i4} - \bar{A}_{i3}P_{i1}^{-1}P_{i2})e_{y_i} + [\bar{M}_{i2}(T^{-1}z) - \bar{M}_{i2}(T^{-1}\hat{z})] \\ &+\Delta\tilde{\phi}_{i2}(T_i^{-1}z_i, u_i) + \Delta\tilde{M}_{i2}(T^{-1}z) - d_i(\cdot) \qquad (4.63)\end{aligned}$$

The following result is ready to be presented:

Theorem 5.3. Under Assumptions 5.2 and 5.4, the system (4.62) is ultimately bounded stable if the function matrix $W^T + W$ is positive definite, where the matrix $W = [w_{ij}]_{N\times N}$, and its entries w_{ij} are defined by

$$w_{ij} = \begin{cases} \lambda_{min}(Q_{i1}) - 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}}\big], & i = j \\ -2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}}\big], & i \neq j \end{cases} \qquad (4.64)$$

where P_{i1}, P_{i2} and Q_{i1} are from (4.18).

Proof. Consider a Lyapunov function candidate for the system (4.62)

$$V = \sum_{i=1}^{N} e_{i1}^T P_{i1} e_{i1}$$

where P_{i1} is defined in (4.18).

Following a similar proof as in Theorem 1, it is obtained

$$\begin{aligned}\dot{V} \le &-\sum_{i=1}^{N}\Big\{\big\{\lambda_{min}(Q_{i1}) - 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}}\big]\big\}\|e_{i1}\| \\ &-\sum_{\substack{j=1\\ j\neq i}}^{N} 2\big[\|P_{i1}\|\ell_{\bar{M}_{i1}} + \|P_{i2}\|\ell_{\bar{M}_{i2}}\big]\|e_{j1}\|\Big\}\|e_{i1}\| + 2\sum_{i=1}^{N}\|P_{i1}\|\big[\varepsilon_i^a + \Upsilon_i^a\big]\|e_{i1}\| \qquad (4.65)\end{aligned}$$

Then, from the definition of the matrix W in Theorem 5.2 and the inequality above, it follows that

$$\begin{aligned}\dot{V} &\le -\frac{1}{2}X^T[W^T + W]X + \mu X \\ &= -(\frac{1}{2}\lambda_{min}(W^T + W)\|X\| - \mu)\|X\| \qquad (4.66)\end{aligned}$$

where $\mu = 2\sqrt{\sum_{i=1}^{N}(\|P_{i1}\|\left[\varepsilon_i^a + \Upsilon_i^a\right])^2}$ and $X = [\|e_{11}\|, \|e_{21}\|, \cdots, \|e_{N1}\|]^T$. It is clear to see that $\dot{V}$ is negative definite if $\mu < \frac{1}{2}\lambda_{min}(W^T + W)$. Therefore system (4.62) is ultimately bounded. Hence the result follows. $\triangle$

For system (4.62)-(4.63), consider the same sliding surface S given in (4.42). It is straightforward to see that Theorem 5.3 implies that the sliding mode of the system (4.62)-(4.63) associated with the sliding surface S given in (4.42) is ultimately bounded.

The objective now is to determine the gains k_i in (4.61) such that the system can be driven to the sliding surface S in (4.42) in finite time and a sliding motion maintained thereafter.

Theorem 5.4. Under Assumptions 5.2 and 5.4, the system (4.62)-(4.63) is driven to the sliding surface (4.42) in finite time and remains on it if

$$k_i \geq (\|\bar{A}_{i3}\| + \ell_{\bar{M}_{i2}} + \ell_{\Delta\tilde{\phi}_{i2}} + \ell_{\Delta\tilde{M}_{i2}})\beta + \eta \tag{4.67}$$

where β is determined by (4.41) and η is a positive constant.

Proof. The proof of Theorem 5.4 can be obtained directly by following the proof of Theorem 5.2. It is omitted here.

Remark 5.3. The results above show that the sliding mode observers of the interconnected system (4.1)-(4.2) in z coordinates are given by (4.27)-(4.29) or (4.58)-(4.60). Let

$$\hat{x}_i = (T_i T_{c_i})^{-1}\hat{z}_i, \qquad i = 1, 2, \ldots, N \tag{4.68}$$

where T_{c_i} and T_i are given in (4.7) and (4.19) respectively and $\hat{z}_i$ are given in (4.27)-(4.29) or (4.58)-(4.60) for $i = 1, 2, \ldots, N$. Therefore the varibles $\hat{x}_i$ given in (4.68) are the estimate of the states x_i of the interconnected system (4.1)-(4.2) for $i = 1, 2, \ldots, N$.

4.3. Case Study: Multimachine Power System

In this section, a case study on a multimachine power system is developed. In this case, the state variable of each machine is given by $x_i = [x_{i1}\ x_{i2}\ x_{i3}] = [\delta_i - \delta_{i0}\ \ \omega_i\ \ \Delta P_{ei}]$

with $\Delta P_{ei} \equiv: P_{ei} - P_{mi0}$ where δ_i is the generator power angle [rad], P_{ei} is electrical power [p.u.], and ω_i is relative speed [rad/s] for $i = 1, 2, \cdots, N$. It is assumed that $P_{mi} = P_{mi0} =$ constant. All the symbols and terms are the same as in [127]. Then by using direct feedback linearsation compensation for the power system as in [77], the multimachine power system can be described by the system (4.1)-(4.2) with

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{D_i}{2H_i} & -\frac{\omega_0}{2H_i} \\ 0 & 0 & -\frac{1}{T'_{doi}} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ T'_{doi} \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.69}$$

From the matrix C_i, it is clear to see that the measured states are the generator power angle δ_i [rad] and the electrical power P_{ei} [p.u.]. The objective is to mainly estimate the relative speed ω_i [rad/s] for $i = 1, 2, \cdots, N$.

The known and uncertain interconnections are given by

$$\bar{M}_i(x) = 0, \qquad \Delta\bar{M}_i(x) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{H_i^b} \Phi_i(x)$$

where

$$|\Phi_i(x)| \le \sum_{j=1}^{N} (\gamma_{ij}^I |\sin\delta_j| + \gamma_{ij}^{II} |\omega_j|)$$

with the constants γ_{ij}^I and γ_{ij}^{II} defined by

$$\begin{aligned} \gamma_{ij}^I &= \frac{4}{|T'_{doj}|_{min}} |P_{ei}|_{max} \\ \gamma_{ij}^{II} &= |Q_{ei}|_{max} \end{aligned}$$

for $i = 1, 2, \cdots, N$, and

$$\|\Delta\bar{M}_i(x)\| = |\Phi_i(x)| \le \sum_{j=1}^{N} (\gamma_{ij}^I |\sin x_{j1}| + \gamma_{ij}^{II} |x_{j2}|) \tag{4.70}$$

The input control variables are

$$v_{fi} = I_{qi} K_{ci} u_{fi} - (x_{di} - x'_{di}) I_{qi} I_{di} - P_{mi0} - T'_{doi} Q_{ei} \omega_i$$

Choose

$$T_{c_i} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for} \quad i =, 1, 2, \cdots, N.$$

Following the transformation $\bar{x}_i = T_{c_i} x_i$, the system matrices become

$$\bar{A}_i = T_c A_i T_c^{-1} = \begin{bmatrix} -\frac{D_i}{2H_i} & 0 & -\frac{\omega_0}{2H_i} \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{T'_{doi}} \end{bmatrix} \tag{4.71}$$

$$\bar{B}_i = T_c B_i = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{T'_{doi}} \end{bmatrix}, \qquad \bar{C}_i = \begin{bmatrix} 0 & I_{p_i} \end{bmatrix} \tag{4.72}$$

and

$$\bar{M}_i(\bar{x}) = 0, \ \Delta\bar{M}_i(\bar{x}) = T_c \Delta\bar{M}_i(x) = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{H_i^b} \Phi_i(x) \tag{4.73}$$

Comparing $(4.5) - (4.6)$, it follows that

$$\bar{A}_{i1} = -\frac{D_i}{2H_i}, \qquad \bar{A}_{i2} = \begin{bmatrix} 0 & -\frac{\omega_0}{2H_i} \end{bmatrix}, \qquad \bar{A}_{i3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \bar{A}_{i4} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{T'_{doi}} \end{bmatrix}$$

$$\bar{B}_{i1} = 0, \qquad \bar{B}_{i2} = \begin{bmatrix} 0 \\ \frac{1}{T'_{doi}} \end{bmatrix}, \qquad \Delta\bar{M}_{i1} = 0, \qquad \Delta\bar{M}_{i2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Phi_i(x)$$

For simulation purposes, consider a two machine power system where all the parameters are chosen as in [127]. In order to illustrate the obtained results, the following uncertainties are added to the isolated subsystems

$$\Delta\phi_1(x_1,u_1) \;=\; \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}}_{H_1^a} \underbrace{x_{11}\sin u_1}_{\Delta\xi_1(x_1,u_1)} \tag{4.74}$$

$$\Delta\phi_2(x_2,u_2) \;=\; \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix}}_{H_2^a} \underbrace{\sin^2(x_{21}+x_{23})}_{\Delta\xi_2(x_2,u_2)} \tag{4.75}$$

It is straightforward to see

$$|\Delta\xi_1(x_1,u_1)| \;\leq\; |x_{11}|\,|\sin u_1| := \rho_1(x_1,u_1) \tag{4.76}$$

$$|\Delta\xi_2(x_2,u_2)| \;\leq\; |\sin^2(x_{21}+x_{23})| := \rho_2(x_2,u_2) \tag{4.77}$$

Then, let $Q_1 = Q_2 = I_3$. The solutions of Lyapunov equation (4.15) are given by

$$P_1 \;=\; \begin{bmatrix} 0.5841 & -0.135 & 0 \\ -0.135 & 0.2304 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.6799 & -0.3 & 0 \\ -0.3 & 0.3485 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

The transformation matrix T_i in the equation $z_i = T_i\bar{x}_i$ is given by

$$T_1 = \begin{bmatrix} 1 & -0.2311 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ T_2 = \begin{bmatrix} 1 & -0.4412 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.78}$$

Therefore, under the transformation $x_i = (T_iT_{c_i})^{-1}z_i$ with T_{c_i} and T_i defined in (4.71) and (4.78), the two machine power system can be described in z coordinates as follows

$$\dot{z}_{11} \;= -0.704z_{11} + \begin{bmatrix} -0.0555 & -39.27 \end{bmatrix} z_{12} \tag{4.79}$$

$$\dot{z}_{12} \;=\; \begin{bmatrix} 1 \\ 0 \end{bmatrix} z_{11} + \begin{bmatrix} 0.0788535 & 0 \\ 0 & -0.1449 \end{bmatrix} z_{12} + \begin{bmatrix} 0 \\ 0.1449 \end{bmatrix} u_1$$

$$+\begin{bmatrix}0\\0.5\end{bmatrix}\Delta\bar{\phi}_1(T^{-1}z_1,u_1)+\bar{M}_{12}(T^{-1}z)+\begin{bmatrix}0\\1\end{bmatrix}\Delta\bar{M}_{12}(T^{-1}z) \quad (4.80)$$

$$y_1 = z_{12} \quad (4.81)$$

$$\dot{z}_{21} = -0.4941z_{21}+\begin{bmatrix}-0.1 & -30.8\end{bmatrix}z_{22} \quad (4.82)$$

$$\begin{aligned}\dot{z}_{22} &= \begin{bmatrix}1\\0\end{bmatrix}z_{21}+\begin{bmatrix}0.2 & 0\\0 & -0.1256\end{bmatrix}z_{22}+\begin{bmatrix}0\\0.1256\end{bmatrix}u_2\\ &+\begin{bmatrix}0\\0.2\end{bmatrix}\Delta\bar{\phi}_2(T^{-1}z_2,u_2)+\bar{M}_{22}(T^{-1}z)+\begin{bmatrix}0\\1\end{bmatrix}\Delta\bar{M}_{22}(T^{-1}z) \quad (4.83)\end{aligned}$$

$$y_2 = z_{22} \quad (4.84)$$

where $z_{i1}\in R,\ \ z_{i2}:=\text{col}(z_{i21},z_{i22})\in R^2$. From $(4.74)-(4.75)$.

$$\|\bar{\rho}_1(z_1,u_1)\| \leq |z_{121}||\sin u_1| \quad (4.85)$$

$$\|\bar{\rho}_2(z_2,u_2)\| \leq |\sin^2(z_{221}+z_{222})| \quad (4.86)$$

By direct calculation $\ell_{\bar{\rho}_1}=1,\ \ \ell_{\bar{\rho}_2}=2$. From (4.24) and (4.70).

$$\begin{aligned}|\bar{\sigma}_1(z)| &\leq(\gamma_{11}^{I}|\sin z_{121}|+\gamma_{11}^{II}|(z_{11}+0.2311z_{121}|))\\ &\quad+(\gamma_{12}^{I}|\sin z_{221}|+\gamma_{12}^{II}|(z_{21}+0.4412z_{221}|)) \quad (4.87)\\ |\bar{\sigma}_2(z)| &\leq(\gamma_{21}^{I}|\sin z_{121}|+\gamma_{21}^{II}|(z_{11}+0.2311z_{121}|))\\ &\quad+(\gamma_{22}^{I}|\sin z_{221}|+\gamma_{22}^{II}|(z_{21}+0.4412z_{221}|)) \quad (4.88)\end{aligned}$$

Therefore,

$$|\bar{\sigma}_1(z)-\bar{\sigma}_1(\hat{z})| = \begin{bmatrix}\gamma_{11}^{II} & \gamma_{11}^{I}+0.2311\gamma_{11}^{II} & 0 & \gamma_{12}^{II} & \gamma_{12}^{I}+0.4412\gamma_{12}^{II} & 0\end{bmatrix}\begin{bmatrix}|z_{11}-\hat{z}_{11}|\\ \|z_{12}-\hat{z}_{12}\|\\ |z_{21}-\hat{z}_{21}|\\ \|z_{22}-\hat{z}_{22}\|\end{bmatrix}$$

$$|\bar{\sigma}_2(z) - \bar{\sigma}_2(\hat{z})| =$$

$$\begin{bmatrix} \gamma_{21}^{II} & \gamma_{21}^{I} + 0.2311\gamma_{21}^{II} & 0 & \gamma_{22}^{II} & \gamma_{22}^{I} + 0.4412\gamma_{22}^{II} & 0 \end{bmatrix} \begin{bmatrix} |z_{11} - \hat{z}_{11}| \\ \|z_{12} - \hat{z}_{12}\| \\ |z_{21} - \hat{z}_{21}| \\ \|z_{22} - \hat{z}_{22}\| \end{bmatrix}$$

where $\gamma_{11}^I = 0.9,\ \gamma_{12}^I = 0.7355,\ \gamma_{11}^{II} = \gamma_{12}^{II} = 1.4$ and $\gamma_{21}^I = 0.966,\ \gamma_{22}^I = 0.788,\ \gamma_{21}^{II} = \gamma_{22}^{II} = 1.5$. Thus $\ell_{\bar{\sigma}_1} = 2.69224$ and $\ell_{\bar{\sigma}_2} = 2.88532$.

By direct computation, it follows that the matrix $W^T + W$ is positive definite. Thus, all the conditions of Theorem 5.1 are satisfied. Therefore the following dynamical system is an asymptotic observer of the system (4.79)-(4.84)

$$\dot{\hat{z}}_{11} = -0.704\hat{z}_{11} + \begin{bmatrix} -0.0555 & -39.27 \end{bmatrix} \hat{z}_{12} \tag{4.89}$$

$$\dot{\hat{z}}_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{z}_{11} + \begin{bmatrix} 0.0788535 & 0 \\ 0 & -0.1449 \end{bmatrix} \hat{z}_{12} + \begin{bmatrix} 0 \\ 0.1449 \end{bmatrix} u_1 + d_1(\cdot) \tag{4.90}$$

$$\hat{y}_1 = \hat{z}_{12} \tag{4.91}$$

$$\dot{\hat{z}}_{21} = -0.4941\hat{z}_{21} + \begin{bmatrix} -0.1 & -30.8 \end{bmatrix} \hat{z}_{22} \tag{4.92}$$

$$\dot{\hat{z}}_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \hat{z}_{21} + \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1256 \end{bmatrix} \hat{z}_{22} + \begin{bmatrix} 0 \\ 0.1256 \end{bmatrix} u_2 + d_2(\cdot) \tag{4.93}$$

$$\hat{y}_2 = \hat{z}_{22} \tag{4.94}$$

where the terms $d_1(\cdot)$ and $d_1(\cdot)$ are defined by

$$\begin{aligned} d_1(\cdot) = {} & \Big(\left\| \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right\| \bar{\rho}_1(T^{-1}\hat{z}_1, u_1) + \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \bar{\sigma}_1(T^{-1}\hat{z}) \\ & + \left\| \begin{bmatrix} 0.0788535 & 0 \\ 0 & -0.1449 \end{bmatrix} \right\| \|y_1 - \hat{y}_1\| + k_1\Big)\mathrm{sgn}(y_1 - \hat{y}_1) \end{aligned} \tag{4.95}$$

$$\begin{aligned} d_2(\cdot) = {} & \Big(\left\| \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \right\| \bar{\rho}_2(T^{-1}\hat{z}_2, u_2) + \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \bar{\sigma}_2(T^{-1}\hat{z}) \\ & + \left\| \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1256 \end{bmatrix} \right\| \|y_2 - \hat{y}_2\| + k_2\Big)\mathrm{sgn}(y_2 - \hat{y}_2) \end{aligned} \tag{4.96}$$

$$k_1 \geq \left(\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| + \ell_{\bar{M}_{12}} + \left\| \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \right\| \ell_{\bar{\rho}_1} + \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \ell_{\bar{\sigma}_1} \right) \beta + \eta \qquad (4.97)$$

$$k_2 \geq \left(\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| + \ell_{\bar{M}_{22}} + \left\| \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \right\| \ell_{\bar{\rho}_2} + \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \ell_{\bar{\sigma}_2} \right) \beta + \eta \qquad (4.98)$$

Therefore, $\hat{x}_i = (T_i T_{c_i})^{-1} \hat{z}_i$ is an estimate of $x_i = [x_{i1} \;\; x_{i2} \;\; x_{i3}] = [\delta_i - \delta_{i0} \;\; \omega_i \;\; \Delta P_{ei}]$ where T_{c_i} and T_i are defined in (4.71) and (4.78) respectively. The simulation results presented in Figures 4.1 and 4.2 show the effectiveness of the designed observer. It should be noted that the estimation process is implemented on-line.

Remark 5.4. As in existing work in [127, 110, 111, 117], both the multimachine power system considered and the interconnections are nonlinear. However, most work focuses on control design or observer-based control design. In this work, the observer can be applied to the multimachine power system as shown in the example. Specifically the interconnections are nonlinear and all the uncertainties are bounded by nonlinear functions which encompasses a large class of disturbances.

4.4. CONCLUSION

In this chapter, robust sliding mode observers have been designed for a class of nonlinear interconnected systems with uncertainties. The known nonlinear interconnections and uncertain nonlinear interconnections have been dealt with separately to reduce the effects of the interconnections without introducing unnecessary conservatism. A set of sufficient conditions has been provided such that the error dynamics are asymptotically stable if the structure of the uncertainties is known. All the bounds on the uncertainties involved are nonlinear and are employed in the observer design to reject/reduce the effect of uncertainties. An ultimately bounded sliding mode observer is proposed to estimate the states of the interconnected system if the structure of the uncertainties is not available. A case study relating to a multimachine power system has been used to demonstrate the proposed approach.

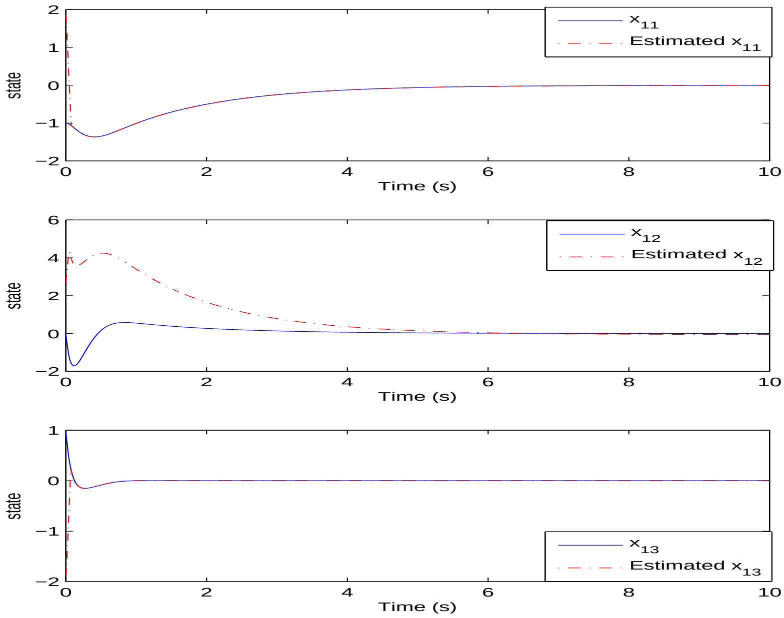

Figure 4.1: The time responses of the 1st subsystem states $x_1 = \text{col}\,(x_{11}, x_{12}, x_{13})$ and their estimation $\hat{x}_1 = \text{col}\,(\hat{x}_{11}, \hat{x}_{12}, \hat{x}_{13})$

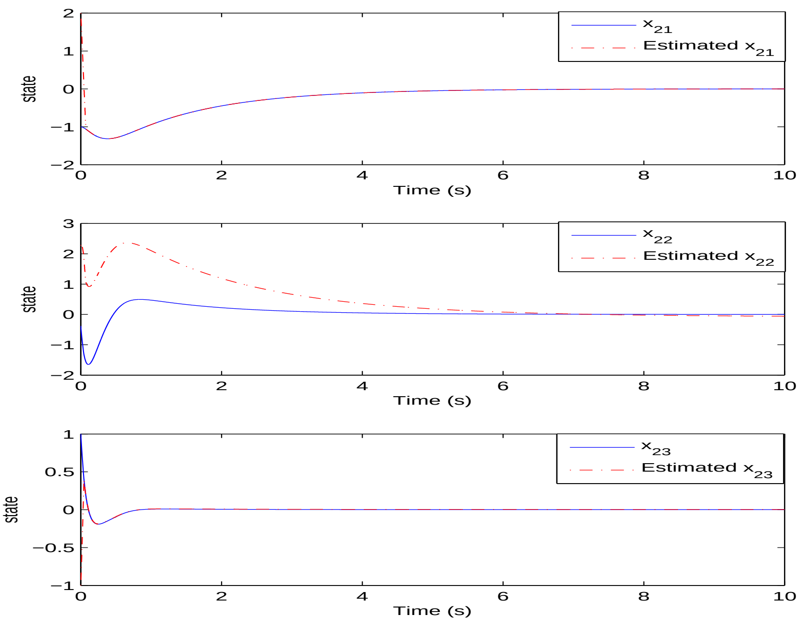

Figure 4.2: The time responses of the 2nd subsystem states $x_2 = \text{col}\,(x_{21}, x_{22}, x_{23})$ and their estimation $\hat{x}_2 = \text{col}\,(\hat{x}_{21}, \hat{x}_{22}, \hat{x}_{23})$

CHAPTER. 5

ADAPTIVE OBSERVER DESIGN FOR NONLINEAR INTERCONNECTED SYSTEMS

In this chapter, a class of nonlinear interconnected systems with uncertain time varying parameters (TVPs) is considered. Both the interconnections and the isolated subsystems are nonlinear. The differences between the unknown TVPs and their corresponding nominal values are assumed to be bounded where the nominal value is not required to be known. A dynamical system is proposed and then, the error systems between the original interconnected system and the designed dynamical system are analysed. A set of conditions is developed such that the augmented systems formed by the error dynamical systems and the designed adaptive laws, are uniformly ultimately bounded. The state observation errors are asymptotically convergent to zero based on the LaSalle's Theorem while the parameter estimation errors are uniformly ultimately bounded. Case study on a coupled inverted pendulum system is presented to demonstrate the developed methodology, and simulation shows that the proposed approach is effective and practicable.

5.1. System Description and Problem Formulation

Consider a nonlinear interconnected system composed of N subsystems described as follows

$$\dot{x}_i = A_i x_i + f_i(x_i, u_i) + B_i \theta_i(t) \xi_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} H_{ij}(x_j) \tag{5.1}$$

$$y_i = C_i x_i \tag{5.2}$$

where $x_i \in R^{n_i}$, $u_i \in U_i \in R^{m_i}$ (U_i are the admissible control set) and $y_i \in R$ are the state variables, inputs and outputs of the i-th subsystem respectively. The functions $f_i(\cdot)$ are known continuous, the scalars $\theta_i(t) \in R$ are unknown time varying parameters and $\xi_i(t) \in R$ are known regressor signals. The matrices $A_i \in R^{n_i \times n_i}, B_i \in R^{n_i \times m_i}$ and $C_i \in R^{1 \times n_i}$ are constants, and C_i are of full rank. The terms

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} H_{ij}(x_j)$$

are the known interconnections of the i-th subsystems for $i = 1, \cdots, N$.

Assumption 6.1. The matrix pairs (A_i, C_i) are observable for $i = 1, \cdots, N$.

From Assumption 1, there exist matrices L_i such that $A_i - L_i C_i$ are Hurwitz stable. This implies that, for any positive-definite matrices $Q_i \in R^{n_i \times n_i}$, the Lyapunov equations

$$(A_i - L_i C_i)^T P_i + P_i (A_i - L_i C_i) = -Q_i \tag{5.3}$$

have unique positive-definite solutions $P_i \in R^{n_i \times n_i}$.

Assumption 6.2. There exist matrices $F_i \in R^{m_i \times 1}$ such that solutions P_i to the Lyapunov equations (5.3) satisfy the constraints

$$B_i^T P_i = F_i C_i \tag{5.4}$$

where B_i and C_i are given in system (5.1)-(5.2) for $i = 1, \cdots, N$.

Remark 6.1. To solve the Lyapunov equations (A.11) in the presence of the constraints (A.14) is the well known constrained Lyapunov problem (CLP) [99]. Although there is no general solution available for this problem, associated discussion and an algorithm can be found in [100] which may help to solve the CLP for a specific system.

Assumption 6.3. The uncertain time varying parameters $\theta_i(t)$ satisfy

$$|\theta_i(t) - \theta_{0_i}| \leq \epsilon_{0_i} \tag{5.5}$$

where θ_{0_i} are unknown constant, and ϵ_{0_i} are known constant for $i = 1, \cdots, N$.

Remark 6.2. Assumption 6.3 is to specify a class of uncertainties tolerated in the observer design. The unknown constants θ_{0_i} given in (5.5) are called the nominal value of the uncertain TVPs $\theta_i(t)$ throughout this work. Different from the existing work (see e.g.[60, 142]), the unknown parameters $\theta_i(t)$ are time varying and the nominal values θ_{0_i} are not required to be known.

For further analysis, the terms $B_i\theta_i(t)\xi_i(t)$ in system (5.1) are rewritten as

$$B_i\theta_i(t)\xi_i(t) = B_i[\theta_{0_i} + \epsilon_i(t)]\xi_i(t) \tag{5.6}$$

where the scalers $\epsilon_i(t) = \theta_i(t) - \theta_{0_i}$.

Assumption 6.4. The nonlinear terms $f_i(x_i, u_i)$ satisfy the Lipschitz condition with respect to $x_i \in R^{n_i}$, and uniformly for $u_i \in U_i \in R^{m_i}$, and $H_{ij}(x_j)$ satisfy the Lipschitz condition in $x_j \in \Omega_j$ for $i = 1, 2, \cdots, N$ and $i \neq j$.

Assumption 6.4 implies that there exist nonnegative function ℓ_{f_i} and constant $\ell_{H_{ij}}$ such that

$$\|f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)\| \leq \ell_{f_i}(u_i)\|\hat{x}_i - x_i\| \tag{5.7}$$

$$\|H_{ij}(\hat{x}_j) - H_{ij}(x_j)\| \leq \ell_{H_{ij}}\|\hat{x}_j - x_j\| \tag{5.8}$$

for $i = 1, 2, \cdots, N$ and $i \neq j$.

Remark 6.3. The Assumption 6.4 is the limitation to the nonlinear terms and the interconnections which is necessary to achieve the asymptotic stability of the observation error dynamics. It should be noted that in the Assumption 6.4, it is required that $f_i(x_i, u_i)$ satisfy the Lipschitz condition with respect to the variable x_i only.

For nonlinear interconnected system (5.1)–(5.2) satisfying Assumptions 6.1-6.4, the objective of this work is to design an observer with appropriate adaptive laws such that the states of the system (5.1)–(5.2) can be estimated asymptotically, and the estimation errors of the unknown parameters $\theta_i(t)$ in (5.1) are uniformly ultimately bounded.

5.2. Adaptive Observer Design with Parameters Estimation

In this section, an asymptotic observer is to be designed and the proposed adaptive laws are to be presented.

From equation (5.6), system (5.1) can be rewritten as

$$\begin{aligned} \dot{x}_i &= A_i x_i + f_i(x_i, u_i) + B_i[\theta_{0_i} + \epsilon_i(t)]\xi_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} H_{ij}(x_j) && (5.9) \\ y_i &= C_i x_i && (5.10) \end{aligned}$$

For system (A.30)-(A.31), construct dynamical systems

$$\begin{aligned} \dot{\hat{x}}_i &= A_i\hat{x}_i + f_i(\hat{x}_i, u_i) + L_i(y_i - \hat{y}_i) + B_i\hat{\theta}_i(t)\xi_i(t) - 2P_i^{-1}(F_iC_i)^T|\xi_i(t)|\ \epsilon_{0_i}\psi_i(\hat{y}_i, y_i) \\ &\quad -B_i\hat{\epsilon}_i(t)\xi_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^{N} H_{ij}(\hat{x}_j) && (5.11) \\ \hat{y}_i &= C_i\hat{x}_i && (5.12) \end{aligned}$$

where P_i and C_i satisfy equations (5.3) and (5.4) and the known constant ϵ_{0_i} satisfies the inequality in Assumption 6.3.

$$\psi_i(\hat{y}_i, y_i) = \begin{cases} \frac{F_i(\hat{y}_i - y_i)}{\|F_i(\hat{y}_i - y_i)\|}, & F_i(\hat{y}_i - y_i) \neq 0 \\ 0, & F_i(\hat{y}_i - y_i) = 0 \end{cases} \qquad (5.13)$$

for $i = 1, 2, \cdots, N$, and $\hat{\theta}_i(t)$ is given by the adaptive law as follows

$$\dot{\hat{\theta}}_i(t) = -2\delta_i(F_i(\hat{y}_i - y_i))^T\xi_i(t) \qquad (5.14)$$

where δ_i is a positive constant which is design parameter and $\hat{\epsilon}_i(t)$ is defined by

$$\hat{\epsilon}_i(t) = -\frac{1}{\delta_i}\hat{\theta}_i(t) \qquad (5.15)$$

for $i = 1, 2, \cdots, N$.

Let $e_{x_i} = \hat{x}_i - x_i$. Then, from systems (5.9)-(5.10) and (5.11)-(5.12), the error dynamical systems can be described by

$$\begin{aligned} \dot{e}_{x_i} &= (A_i - L_iC_i)e_{x_i} + [f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)] + \sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}(\hat{x}_j) - H_{ij}(x_j)] + B_i\tilde{\theta}_i(t)\xi_i(t) \\ &\quad -B_i\hat{\epsilon}_i(t)\xi_i(t) - B_i\epsilon_i(t)\xi_i(t) - 2P_i^{-1}(F_iC_i)^T|\xi_i(t)|\epsilon_{0_i}\psi_i(\hat{y}_i, y_i) \end{aligned} \tag{5.16}$$

where $\tilde{\theta}_i(t)$ is defined by

$$\tilde{\theta}_i(t) = \hat{\theta}_i(t) - \theta_{0_i} \tag{5.17}$$

for $i = 1, 2, \cdots, N$.

For the convenience of further analysis , let

$$\tilde{\epsilon}_i(t) = \hat{\epsilon}_i(t) - \epsilon_{0_i} \tag{5.18}$$

where the known constant ϵ_{0_i} satisfies the inequality (5.5) in Assumption 6.3 and $\hat{\epsilon}_i(t)$ is defined in (5.15), for $i = 1, 2, \cdots, N$.

The following result is ready to be presented:

Theorem 6.1. Under Assumptions $6.1-6.4$, the error dynamical systems (5.16) with adaptive law (5.14) are uniformly ultimately bounded if the matrix $W^T + W$ is positive definite, where the matrix $W = [w_{ij}]_{N\times N}$ and its entries w_{ij} are defined by

$$w_{ij} = \begin{cases} \lambda_{\min}(Q_i) - 2\ell_{f_i}\|P_i\|, & i = j \\ \\ -2\|P_i\|\ell_{H_{ij}}, & i \neq j \end{cases} \tag{5.19}$$

where P_i and Q_i satisfy Lyapunov equation in (5.3) and $\lambda_{\min}(Q_i)$ represents the minimum eigenvalue of the matrix Q_i for $i = 1, 2, \ldots, N$. Further, the error e_{x_i} given in (5.16) satisfies

$$\lim_{t\to\infty}\|e_{x_i}(t)\| = 0, \qquad i = 1, 2, \ldots, N \tag{5.20}$$

Proof. For system (5.14) and (5.16), consider the candidate Lyapunov function

$$V = \sum_{i=1}^{N} e_{x_i}^T P_i e_{x_i} + \frac{1}{2}\sum_{i=1}^{N}(\frac{1}{\delta_i}\tilde{\theta}_i^2(t) + \tilde{\epsilon}_i^2(t)) \tag{5.21}$$

where $\delta_i > 0$ are design parameters given in (5.14) for $i = 1, 2, \cdots, N$. Note that, in (5.21) $\tilde{\epsilon}_i(t)$ is dependent on $\tilde{\theta}_i(t)$. From (5.15), (5.17) and (5.18) it can be seen that the relationship between $\tilde{\epsilon}_i(t)$ and $\tilde{\theta}_i(t)$ is given by

$$\begin{aligned}
\tilde{\epsilon}_i(t) &= \hat{\epsilon}_i(t) - \epsilon_{0_i} \\
&= -\frac{1}{\delta_i}\hat{\theta}_i(t) - \epsilon_{0_i} \\
&= -\frac{1}{\delta_i}(\tilde{\theta}_i(t) + \theta_{0_i}) - \epsilon_{0_i}
\end{aligned}$$

Then, from (5.16)

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N}(\dot{e}_{x_i}^T P_i e_{x_i} + e_{x_i}^T P_i \dot{e}_{x_i}) + \sum_{i=1}^{N}(\frac{1}{\delta_i}\tilde{\theta}_i(t)\dot{\tilde{\theta}}_i(t) + \tilde{\epsilon}_i(t)\dot{\tilde{\epsilon}}_i(t)) \\
&= \sum_{i=1}^{N}\Big\{ e_{x_i}^T[(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i)]e_{x_i} + 2e_{x_i}^T P_i[f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)] \\
&\quad +2e_{x_i}^T P_i \sum_{\substack{j=1 \\ j\neq i}}^{N}[H_{ij}(\hat{x}_j) - H_{ij}(x_j)] + 2e_{x_i}^T P_i B_i \tilde{\theta}_i(t)\xi_i(t) - 2e_{x_i}^T P_i B_i \epsilon_i(t)\xi_i(t) \\
&\quad -2e_{x_i}^T P_i B_i \hat{\epsilon}_i(t)\xi_i(t) + \frac{1}{\delta_i}\tilde{\theta}_i(t)\dot{\tilde{\theta}}_i(t) + \tilde{\epsilon}_i(t)\dot{\tilde{\epsilon}}_i(t) \\
&\quad -4e_{x_i}^T P_i P_i^{-1}(F_iC_i)^T|\xi_i(t)|\epsilon_{0_i}\psi_i(\hat{y}_i, y_i)\Big\}
\end{aligned} \tag{5.22}$$

By using condition (A.14) and $C_i e_{x_i} = \hat{y}_i - y_i$,

$$\begin{aligned}
e_{x_i}^T P_i B_i &= ((P_iB_i)^T e_{x_i})^T = (B_i^T P_i e_{x_i})^T \\
&= (F_iC_ie_{x_i})^T = (F_i(\hat{y}_i - y_i))^T
\end{aligned} \tag{5.23}$$

Substituting (5.23) into (5.22), it follows that

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{N}\Big\{ e_{x_i}^T[(A_i - L_iC_i)^T P_i + P_i(A_i - L_iC_i)]e_{x_i} + 2e_{x_i}^T P_i[f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)] \\
&\quad +2e_{x_i}^T P_i \sum_{\substack{j=1 \\ j\neq i}}^{N}[H_{ij}(\hat{x}_j) - H_{ij}(x_j)] + [2(F_i(\hat{y}_i - y_i))^T\xi_i(t) + \frac{1}{\delta_i}\dot{\tilde{\theta}}_i(t)]\tilde{\theta}_i(t) \\
&\quad -2(F_i(\hat{y}_i - y_i))^T\epsilon_i(t)\xi_i(t) - 2(F_i(\hat{y}_i - y_i))^T\hat{\epsilon}_i(t)\xi_i(t) \\
&\quad +\tilde{\epsilon}_i(t)\dot{\tilde{\epsilon}}_i(t) - 4(F_i(\hat{y}_i - y_i))^T|\xi_i(t)|\epsilon_{0_i}\psi_i(\hat{y}_i, y_i)\Big\}
\end{aligned} \tag{5.24}$$

From (5.17), it can be seen that $\dot{\tilde{\theta}}_i(t) = \dot{\hat{\theta}}_i(t)$ because θ_{0_i} is constant. Substituting (5.13) and (5.14) into (2.47) gives

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \Big\{ e_{x_i}^T [(A_i - L_i C_i)^T P_i + P_i (A_i - L_i C_i)] e_{x_i} + 2 e_{x_i}^T P_i [f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)] \\ &\quad + 2 e_{x_i}^T P_i \sum_{\substack{j=1 \\ j \neq i}}^{N} [H_{ij}(\hat{x}_j) - H_{ij}(x_j)] - 2(F_i(\hat{y}_i - y_i))^T \epsilon_i(t) \xi_i(t) \\ &\quad - 2(F_i(\hat{y}_i - y_i))^T \hat{\epsilon}_i(t) \xi_i(t) + \tilde{\epsilon}_i(t)) \dot{\tilde{\epsilon}}_i(t) - 4 \|F_i(\hat{y}_i - y_i)\| \, |\xi_i(t)| \epsilon_{0_i} \Big\} \end{aligned}$$

From (5.18), it can be seen that $\dot{\tilde{\epsilon}}_i(t) = \dot{\hat{\epsilon}}_i(t)$.

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \Big\{ e_{x_i}^T [(A_i - L_i C_i)^T P_i + P_i (A_i - L_i C_i)] e_{x_i} + 2 e_{x_i}^T P_i [f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)] \\ &\quad + 2 e_{x_i}^T P_i \sum_{\substack{j=1 \\ j \neq i}}^{N} [H_{ij}(\hat{x}_j) - H_{ij}(x_j)] - 2(F_i(\hat{y}_i - y_i))^T \epsilon_i(t) \xi_i(t) \\ &\quad - [2(F_i(\hat{y}_i - y_i))^T \xi_i(t) - \dot{\hat{\epsilon}}_i(t)] \hat{\epsilon}_i(t) - \epsilon_{0_i} \dot{\hat{\epsilon}}_i(t) - 4 \|F_i(\hat{y}_i - y_i)\| \, |\xi_i(t)| \epsilon_{0_i} \Big\} \end{aligned} \tag{5.25}$$

Substituting (5.15) into (5.25) yields

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{N} \Big\{ e_{x_i}^T [(A_i - L_i C_i)^T P_i + P_i (A_i - L_i C_i)] e_{x_i} + 2 e_{x_i}^T P_i [f_i(\hat{x}_i, u_i) - f_i(x_i, u_i)] \\ &\quad + 2 e_{x_i}^T P_i \sum_{\substack{j=1 \\ j \neq i}}^{N} [H_{ij}(\hat{x}_j) - H_{ij}(x_j)] - 2(F_i(\hat{y}_i - y_i))^T \epsilon_i(t) \xi_i(t) \\ &\quad - 2\epsilon_{0_i} (F_i(\hat{y}_i - y_i))^T \xi_i(t) - 4 \|F_i(\hat{y}_i - y_i)\| \|\xi_i(t)\| \epsilon_{0_i} \Big\} \end{aligned}$$

It is clear from (5.3) that

$$\begin{aligned}
\dot{V} &\leq \sum_{i=1}^{N}\Big\{-e_{x_i}^T Q_i e_{x_i} + 2\|e_{x_i}\|\|P_i\|[f_i(\hat{x}_i,u_i) - f_i(x_i,u_i)] + 2\|e_{x_i}\|\|P_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}(\hat{x}_j) - H_{ij}(x_j)] \\
&\quad -2(F_i(\hat{y}_i - y_i))^T\xi_i(t)[\epsilon_i(t) + \epsilon_{0_i}] - 4\|F_i(\hat{y}_i - y_i)\|\ \|\xi_i(t)\|\epsilon_{0_i}\Big\} \\
&\leq \sum_{i=1}^{N}\Big\{-e_{x_i}^T Q_i e_{x_i} + 2\|e_{x_i}\|\|P_i\|[f_i(\hat{x}_i,u_i) - f_i(x_i,u_i)] + 2\|e_{x_i}\|\|P_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}(\hat{x}_j) \\
&\quad -H_{ij}(x_j)] + 4\|F_i(\hat{y}_i - y_i)\|\|\xi_i(t)\|\epsilon_{0_i} - 4\|F_i(\hat{y}_i - y_i)\|\|\xi_i(t)\|\epsilon_{0_i}\} \\
&\leq \sum_{i=1}^{N}\Big\{-e_{x_i}^T Q_i e_{x_i} + 2\|e_{x_i}\|\|P_i\|[\ell_{f_i}\|\hat{x}_i - x_i\|] + 2\|e_{x_i}\|\|P_i\|\sum_{\substack{j=1\\ j\neq i}}^{N}[\ell_{H_{ij}}\|\hat{x}_j - x_j\|]\Big\} \\
&\leq -\sum_{i=1}^{N}\Big\{(\lambda_{\min}(Q_i) - 2\|P_i\|\ell_{f_i})\|e_{x_i}\|^2 - \sum_{\substack{j=1\\ j\neq i}}^{N}(2\|P_i\|\ell_{H_{ij}}\|e_{x_i}\|\|e_{x_j}\|)\Big\}
\end{aligned} \tag{5.26}$$

Then, from the definition of the matrix W in (5.19) and the inequality above, it follows that

$$\dot{V} \leq -\frac{1}{2}X^T[W^T + W]X \tag{5.27}$$

where $X = [\|e_{x_1}\|, \|e_{x_2}\|, \cdots, \|e_{x_N}\|]^T$. From the LaSalle's Theorem (see. e.g. [5]), all the solutions of (5.16) are uniformly ultimately bounded and satisfy

$$\lim_{t\to\infty} X^T[W^T + W]X = 0 \tag{5.28}$$

Further, from the facts

$$\lambda_{\min}(W^T + W)\|X\|^2 \leq X^T(W^T + W)X$$

and

$$\|X\|^2 = \|e_{x_1}\|^2 + \|e_{x_2}\|^2 + \cdots + \|e_{x_N}\|^2$$

it is straightforward to see from (5.28) and the condition $W^T + W > 0$ that

$$\lim_{t\to\infty}\|e_{x_i}(t)\| = 0, \qquad i = 1, 2, \ldots, N$$

Hence the conclusion follows. $\triangle$

Remark 6.4. It should be noted that the constructed Lyapunov function (5.21) is a function of variables e_{x_i}, $\tilde{\theta}_i$ and $\tilde{\epsilon}_i$ while the right hand side of inequality (5.27) is a function of variables e_{x_i} only. Therefore, the condition $W^T + W$ is positive definite in Theorem 1 implies that $\dot{V}$ is semi-positive definite instead of positive definite.

Remark 6.5. Theorem 6.1 shows that the augmented systems formed by (5.16) and the adaptive law (5.14) are uniformly ultimately bounded. It should be noted that the estimated states $\hat{x}_i$ given by the observer (5.11) converge to the system states x_i in (5.1) asymptotically although the estimate error for the parameters may not be asymptotically convergent. As the uncertain parameters θ_i in system (5.1) are time-varying, the approaches developed in [60, 140] cannot be applied to the systems considered in this work.

Remark 6.6. The designed observer is a variable structure interconnected system but it may not produce a sliding motion, which is different from the work in [60]. In addition, the unknown parameters are considered as constants in [60] while in this work they are TVPs.

5.3. CASE STUDY 1: A COUPLED INVERTED PENDULUM

In order to illustrate the method developed in this work, case study on a coupled pendulum system is carried out in this section.

Consider a system formed by two inverted pendulums connected by a spring as given in Figure 5.1. There are two balls are attached at the end of the two rigid rods respectively. The symbol u_1 and u_2 denote external torques imposed on the two pendulums respectively which are the control inputs. The distance b between the two pendulum hinges are assumed to be changeable with respect to time t.

Let $\varphi_1 = x_{11}, \varphi_2 = x_{21}, \dot{\varphi}_1 = x_{12}$, and $\dot{\varphi}_2 = x_{22}$. The coupled inverted pendulums can be modelled as (see e.g. [143])

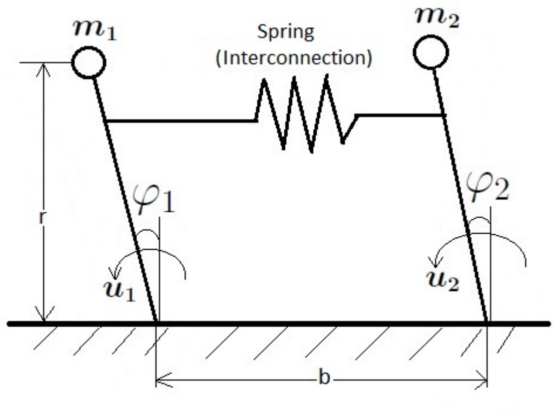

Figure 5.1: Coupled inverted pendulums

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ (\frac{m_1 g r}{J_1} - \frac{kr^2}{4J_1}) \sin(x_{11}) + \frac{1}{J_1} u_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{kr}{2J_1} \end{bmatrix} (l - b) \\ &\quad + \begin{bmatrix} 0 \\ \frac{kr^2}{4J_1} \sin(x_{21}) \end{bmatrix} \end{aligned} \tag{5.29}$$

$$y_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \tag{5.30}$$

$$\begin{aligned} \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ (\frac{m_2 g r}{J_2} - \frac{kr^2}{4J_2}) \sin(x_{21}) + \frac{1}{J_2} u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{kr}{2J_2} \end{bmatrix} (l - b) \\ &\quad + \begin{bmatrix} 0 \\ \frac{kr^2}{4J_2} \sin(x_{11}) \end{bmatrix} \end{aligned} \tag{5.31}$$

$$y_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \tag{5.32}$$

The end masses of pendulums are $m_1 = 0.7$ kg and $m_2 = 0.6$ kg, the moments of inertia are $J_1 = 5$ kg and $J_2 = 4$ kg, the constant of connecting spring is $k = 90$ N/m, the pendulum height is $r = 0.25$ m, and the gravitational acceleration is $g = 9.81$ m/s^2. In order to illustrate the developed theoretical results, it is assumed that $(l - b(t)) = \theta_1(t) = \theta_2(t)$ is an unknown time varying parameter for $i = 1, 2$ where l is the natural length of spring and $b(t)$ is the distance between the two pendulum hinges.

In order to avoid system states going to infinity, and for simulation purposes, the following feedback transformation is introduced

$$\begin{aligned} u_i &= -k_i x_i + v_i, \qquad i = 1,2 && (5.33)\\ k_1 &= \begin{bmatrix} 10 & 15 \end{bmatrix} && (5.34)\\ k_2 &= \begin{bmatrix} 8 & 12 \end{bmatrix} && (5.35) \end{aligned}$$

Then with the given parameters above, the system (5.29)-(5.32) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A_1} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0.06215\sin(x_{11}) + \frac{1}{5}v_1 \end{bmatrix}}_{f_1(x_1,u_1)} + \underbrace{\begin{bmatrix} 0 \\ 2.25 \end{bmatrix}}_{B_1} \underbrace{(l - b(t))}_{\theta_1(t)} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ 0.2813\sin(x_{21}) \end{bmatrix}}_{H_{12}(x_2)} && (5.36)\\ y_1 &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C_1} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} && (5.37)\\ \dot{x}_2 &= \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A_2} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0.01632\sin(x_{21}) + \frac{1}{4}v_2 \end{bmatrix}}_{f_2(x_2,u_2)} + \underbrace{\begin{bmatrix} 0 \\ 2.8125 \end{bmatrix}}_{B_2} \underbrace{(l - b(t))}_{\theta_2(t)} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ 0.352\sin(x_{11}) \end{bmatrix}}_{H_{21}(x_1)} && (5.38)\\ y_2 &= \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C_2} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} && (5.39) \end{aligned}$$

Choose

$$L_i = [0 \;\; 0] \qquad \text{and} \qquad Q_i = 4I$$

for $i = 1, 2$. It follows that the Lyapunov equations (5.3) have unique solutions:

$$P_i = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}, \qquad i = 1,2 \tag{5.40}$$

satisfying the condition (5.4) with

$$F_1 = 2.25, \qquad \text{and} \qquad F_2 = 2.8125$$

For simplicity, it is assumed that

$$\xi_i(t) = 1, \qquad \epsilon_{0_i} = 1 \text{ and } \delta_i = 2$$

for $i = 1, 2$.

By direct computation, it follows that the matrix $W^T + W$ is positive definite. Thus, all the conditions of Theorem 6.1 are satisfied. This implies that the following dynamical systems are the asymptotic observer of the nonlinear interconnected system (5.36)–(5.39):

$$\begin{aligned} \dot{\hat{x}}_1 &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.06215\sin(\hat{x}_{11}) + \frac{1}{5}v_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2.25 \end{bmatrix} \hat{\theta}_1(t) \\ &\quad - \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} \frac{(\hat{y}_1 - y_1)}{\|\hat{y}_1 - y_1\|} - \begin{bmatrix} 0 \\ 2.25 \end{bmatrix} \hat{\epsilon}_1(t) + \begin{bmatrix} 0 \\ 0.2813\sin(\hat{x}_{21}) \end{bmatrix} && (5.41) \\ \hat{y}_1 &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} && (5.42) \\ \dot{\hat{x}}_2 &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.01632\sin(\hat{x}_{21}) + \frac{1}{4}v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2.8125 \end{bmatrix} \hat{\theta}_2(t) \\ &\quad - \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \frac{(\hat{y}_2 - y_2)}{\|\hat{y}_2 - y_2\|} - \begin{bmatrix} 0 \\ 2.8125 \end{bmatrix} \hat{\epsilon}_2(t) + \begin{bmatrix} 0 \\ 0.352\sin(\hat{x}_{11}) \end{bmatrix} && (5.43) \\ \hat{y}_2 &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix} && (5.44) \end{aligned}$$

The designed adaptive laws are given by

$$\dot{\hat{\theta}}_1(t) = -4(2.25(\hat{y}_1 - y_1))^T \qquad (5.45)$$

$$\dot{\hat{\theta}}_2(t) = -4(2.8125(\hat{y}_2 - y_2))^T \qquad (5.46)$$

For simulation purpose, the unknown parameters θ_{0_i} and $\theta_i(t)$ are chosen as $\theta_{0_i} = 0$ and $\theta_i(t) = 0.6\sin t$ for $i = 1, 2$. Simulation in Figures 5.2 and 5.3 shows that the estimation error between the states of the system (5.29)-(5.32) and the states of the observer (5.41)-(5.44) converges to zero asymptotically. Figure 5.4 shows that the estimation of the parameters is uniformly ultimately bounded with satisfactory accuracy.

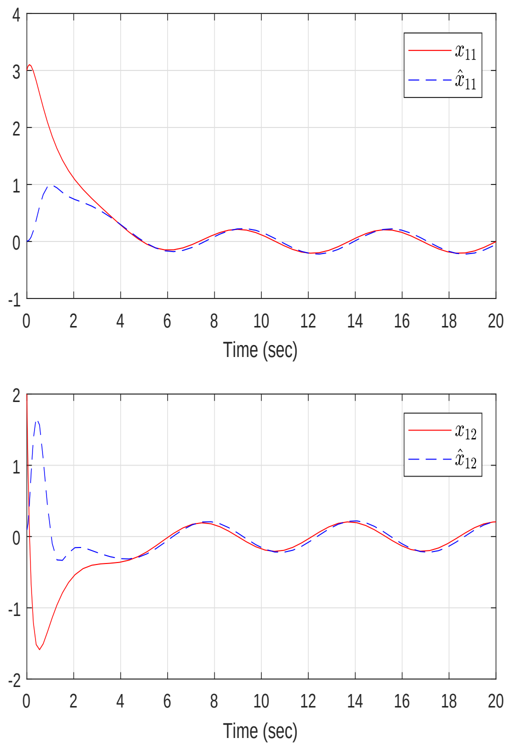

Figure 5.2: The time responses of the 1st subsystem states $x_1 = \text{col}\,(x_{11}, x_{12})$ and their estimation $\hat{x}_1 = \text{col}\,(\hat{x}_{11}, \hat{x}_{12})$.

5.4. Case study 2: A Quarter-Car Suspension

Consider a vehicle (car, bus etc.) divided into 4 parts in case of 4 wheels, each part is a composite mechanical spring-damper system consisting of the quarter part of the mass of the body (together with passengers) and the mass of the wheel. The vertical positions are described by upward directed x_1 and x_2, see Figure 5.5. The distance between the road surface and the wheel's contact point is the disturbance w varying together with the road surface. The suspension is active which means that the actuator produces the force F (control signal).

A good suspension system should have satisfactory road holding stability, while providing good traveling comfort when riding over bumps and holes in the roads.

Denote m_1 and m_2 the mass of the quarter body and the wheel respectively, while

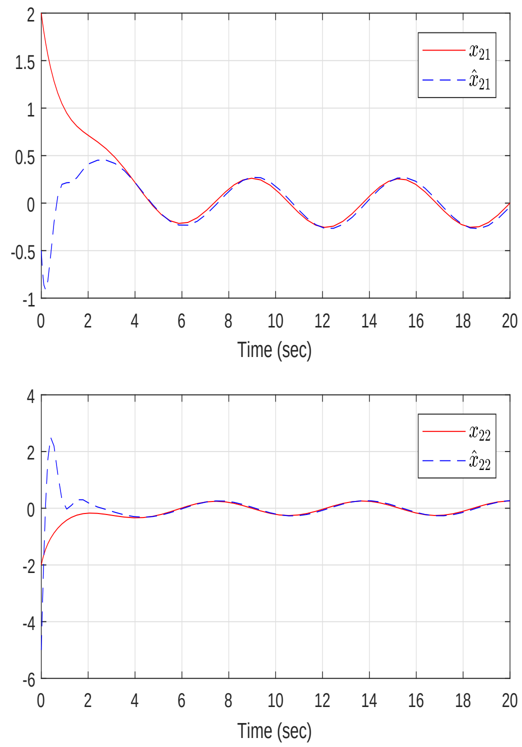

Figure 5.3: The time responses of the 2nd subsystem states $x_2 = \text{col}\,(x_{21}, x_{22})$ and their estimation $\hat{x}_2 = \text{col}\,(\hat{x}_{21}, \hat{x}_{22})$.

the flexible connections are described by the viscous damping factor b_1 and the spring constants (k_1, k_2). The motion equations can be described as follows (see e.g. [1]):

$$m_1\ddot{x}_1 = F - b_1(\dot{x}_1 - \dot{x}_2) - k_1(x_1 - x_2) \tag{5.47}$$

$$m_2\ddot{x}_2 = -F + b_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) - k_2(x_2 - w) \tag{5.48}$$

The system can be described in a state space model

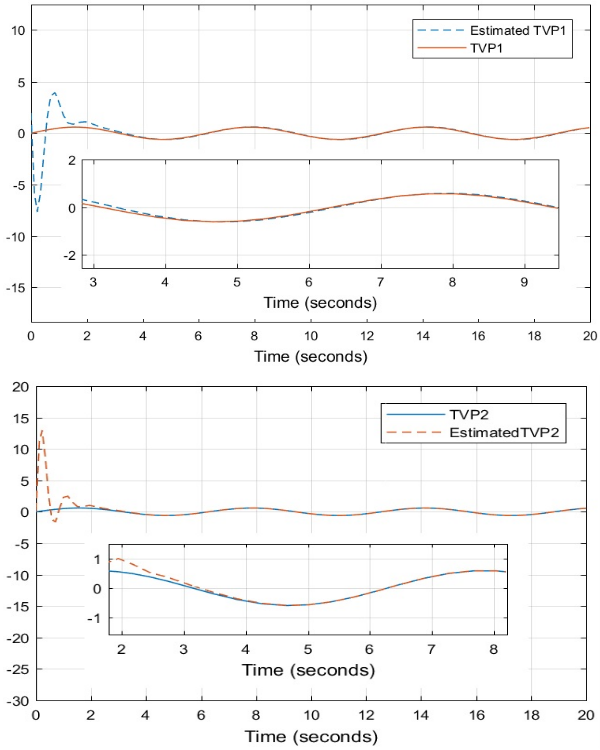

Figure 5.4: Upper: the time responses of $\hat{\theta}_1(t)$ (dashed line) and $\theta_1(t)$ (solid line); Bottom: the time responses of $\hat{\theta}_2(t)$ (dashed line) and $\theta_2(t)$ (solid line).

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ \frac{-k_1}{m_1} & \frac{-b_1}{m_1} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1}F \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1}(k_1 x_{21} + b_1 x_{22}) \end{bmatrix} \tag{5.49}$$

$$y_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \tag{5.50}$$

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ \frac{-k_2}{m_2} & \frac{-b_1}{m_2} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-k_1}{m_2}x_{21} - \frac{1}{m_2}F \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_2}{m_2} \end{bmatrix} w + \begin{bmatrix} 0 \\ \frac{1}{m_2}(k_1 x_{11} + b_1 x_{12}) \end{bmatrix} \tag{5.51}$$

$$y_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \tag{5.52}$$

where $m_1 = 500\ kg,\ m_2 = 300\ kg,\ b_1 = 900\ N/m/s,\ k_1 = 900\ N/m$ and $k_2 = 600\ N/m$.

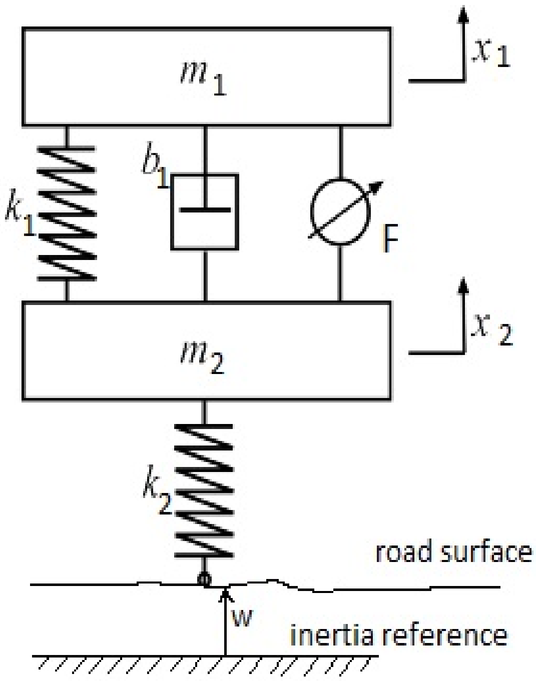

Figure 5.5: A quarter-car suspension

In order to avoid system states going to infinity, and for simulation purposes, the following feedback transformation is introduced

$$\begin{aligned} u_i &= -k_i x_i, && (5.53)\\ k_i &= \begin{bmatrix} 3.18 & 4.18 \end{bmatrix}, \qquad i = 1,2 && (5.54)\end{aligned}$$

Then with the given parameters, the system (5.49)-(5.52) can be rewritten as

$$\dot{x}_1 = \underbrace{\begin{bmatrix} 0 & 1 \\ -1.8 & -1.8 \end{bmatrix}}_{A_1} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0.002\, u \end{bmatrix}}_{f_1(x_1,u_1)} + \underbrace{\begin{bmatrix} 0 \\ 1.8(x_{21}+x_{22}) \end{bmatrix}}_{H_{12}(x_2)} \quad (5.55)$$

$$y_1 = \underbrace{\begin{bmatrix} 3 & 3 \end{bmatrix}}_{C_1} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \quad (5.56)$$

$$\begin{aligned}\dot{x}_2 &= \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A_2} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -3x_{21} - 0.0033\,u \end{bmatrix}}_{f_2(x_2,u_2)} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_{B_2} \underbrace{w}_{\theta(t)} \\ &\quad + \underbrace{\begin{bmatrix} 0 \\ 3(x_{11} + x_{12}) \end{bmatrix}}_{H_{21}(x_1)} \end{aligned} \tag{5.57}$$

$$y_2 = \underbrace{\begin{bmatrix} 1.8 & 1.8 \end{bmatrix}}_{C_2} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \tag{5.58}$$

In order to illustrate the developed theoretical results, it is assumed that all the system states are available, and the aim is to estimate the distance between the road surface and the wheel's contact point w which is varying together with the road surface.

Choose

$$L = [0 \;\; 0] \qquad \text{and} \qquad Q = 4I$$

. It follows that the Lyapunov equations (5.3) have unique solutions:

$$P = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}, \tag{5.59}$$

satisfying the condition (5.4) with

$$F = 0.667$$

For simplicity, it is assumed that

$$\xi(t) = 1, \qquad \epsilon_0 = 1 \;\text{ and }\; \delta = 5$$

The following dynamical systems are the observer of the nonlinear interconnected system (5.55)–(5.58):

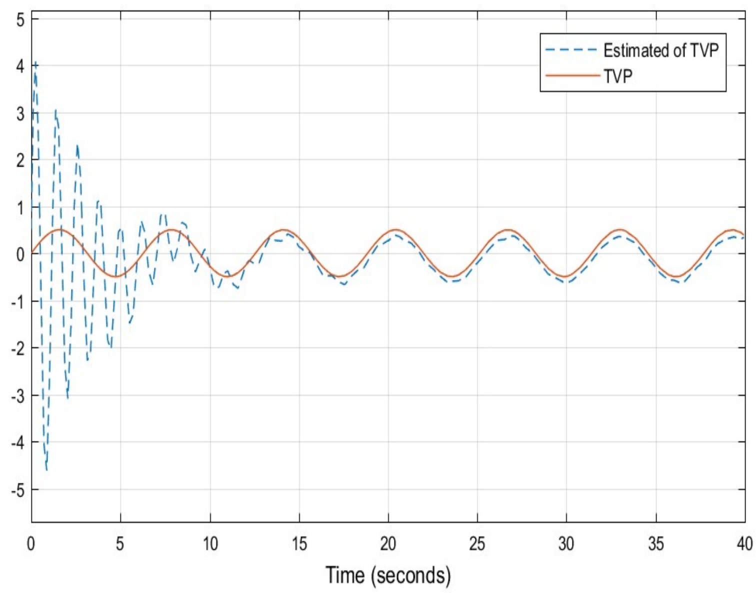

Figure 5.6: The time responses of $\hat{\theta}(t)$ (dashed line) and $\theta(t)$ (solid line).

$$\dot{\hat{x}}_1 = \begin{bmatrix} 0 & 1 \\ -1.8 & -1.8 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.002\, u \end{bmatrix} + \begin{bmatrix} 0 \\ 1.8(\hat{x}_{21} + \hat{x}_{22}) \end{bmatrix} \tag{5.60}$$

$$\hat{y}_1 = \begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_{11} \\ \hat{x}_{12} \end{bmatrix} \tag{5.61}$$

$$\begin{aligned} \dot{\hat{x}}_2 &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ -3\hat{x}_{21} - 0.0033\, u \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \hat{\theta}(t) - \begin{bmatrix} 0 \\ 1.2 \end{bmatrix} \frac{(\hat{y}_2 - y_2)}{\|\hat{y}_2 - y_2\|} \\ &\quad - \begin{bmatrix} 0 \\ 2 \end{bmatrix} \hat{\epsilon}_2(t) + \begin{bmatrix} 0 \\ 3(\hat{x}_{11} + \hat{x}_{12}) \end{bmatrix} \end{aligned} \tag{5.62}$$

$$\hat{y}_2 = \begin{bmatrix} 1.8 & 1.8 \end{bmatrix} \begin{bmatrix} \hat{x}_{21} \\ \hat{x}_{22} \end{bmatrix} \tag{5.63}$$

The designed adaptive law is given by

$$\dot{\hat{\theta}}(t) = -10(0.667(\hat{y}_2 - y_2))^T \tag{5.64}$$

For simulation purpose, the unknown parameter θ_0 and $\theta(t)$ are chosen as 0 and $0.5 \sin t$ respectively. Figure 5.6 shows that the estimation of the parameter is uniformly ultimately bounded with satisfactory accuracy.

Remark 6.7. For a real system, the positions and/or the velocities are usually chosen as system output. However, some times, the linear combination of the position and velocity are taken as system output. Physically, such an aggregation of the output might arise in some real systems [31, 144], for example, certain remote-control applications where the number of transmission and receive lines/frequencies are limited [31].

5.5. CONCLUSION

In this chapter, an adaptive observer design for a class of nonlinear large scale interconnected systems with unknown time varying parameters has been proposed based on Lyapunov direct method. The unknown parameters vary within a given range. A set of sufficient conditions has been developed to guarantee that the observation error system with the proposed adaptive laws is globally uniformly bounded. The states of the designed observer are asymptotically convergent to the original system states. Therefore, from the state estimation point of view, the designed observers are asymptotic observers. Case study examples on a coupled inverted pendulum system and a quarter-car suspension show the practicability of the developed observer scheme for nonlinear interconnected systems.

CHAPTER. 6

ADAPTIVE SLIDING MODE OBSERVER FOR NONLINEAR INTERCONNECTED SYSTEMS WITH TIME VARYING PARAMETERS

In this chapter, a class of nonlinear interconnected systems with uncertain time varying parameters (TVPs) is considered. Both the interconnections and the isolated subsystems are nonlinear. Sliding mode control method and adaptive techniques are employed together to design an observer to estimate the state variables of the systems in presence of unknown TVPs. The Lyapunov direct method is used to analysis the stability of the sliding motion and it is not required to solve the so-called constrained Lyapunov problem (CLP). A set of conditions is developed under which the augmented systems formed by the error dynamical systems and the designed adaptive laws, are globally uniformly ultimately bounded. A simulation example is presented and the results show that the method proposed in this chapter is effective.

6.1. System Description and Preliminaries

Consider a nonlinear interconnected system composed of N subsystems as follows

$$\dot{x}_i = A_i x_i + g_i(x_i, u_i) + \phi_i(y_i, u_i)\Theta_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}(x_j) \tag{6.1}$$

$$y_i = C_i x_i \tag{6.2}$$

where $x_i \in \Omega_i \subset R^{n_i}$ (Ω_i are neighborhoods of the origin), $u_i \in U_i \in R^{m_i}$ (U_i are the admissible control sets) and $y_i \in R^{p_i}$ with $m_i \leq p_i \leq n_i$ are the state variables, inputs and outputs of the i-th subsystem respectively, $g_i(x_i, u_i) \in R^{n_i}$ are nonlinear known functions, $\phi_i(y_i, u_i) \in R^{n_i}$ are known functions and $\Theta_i(t) \in R$ are unknown TVPs. The matrix triples (A_i, C_i) are constant with appropriate dimensions and C_i are full rank. The terms $\sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}(x_j)$ are the known interconnections for $i = 1, \cdots, N$.

Since the C_i are full row rank, there exist nonsingular matrices T_{c_i} such that

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i1} & \bar{A}_{i2} \\ \bar{A}_{i3} & \bar{A}_{i4} \end{bmatrix} := T_{c_i} A_i T_{c_i}^{-1}, \tag{6.3}$$

$$\bar{C}_i = \begin{bmatrix} 0 & I_{p_i} \end{bmatrix} := C_i T_{c_i}^{-1} \tag{6.4}$$

where $\bar{A}_{i1} \in R^{(n_i-p_i)\times(n_i-p_i)}$ for $i = 1, \cdots, N$. Then in the new coordinates $\bar{x}_i$ defined by

$$\bar{x}_i = T_{c_i} x_i \tag{6.5}$$

The system (6.1)-(6.2) can be rewritten as

$$\begin{aligned}\dot{\bar{x}}_{i1} = {} & \bar{A}_{i1}\bar{x}_{i1} + \bar{A}_{i2}\bar{x}_{i2} + \bar{g}_{i1}(\bar{x}_i, u_i) \\ & + \bar{\phi}_{i1}(y_i, u_i)\Theta_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}^{a}(\bar{x}_j)\end{aligned} \tag{6.6}$$

$$\begin{aligned}\dot{\bar{x}}_{i2} = {} & \bar{A}_{i3}\bar{x}_{i1} + \bar{A}_{i4}\bar{x}_{i2} + \bar{g}_{i2}(\bar{x}_i, u_i) \\ & + \bar{\phi}_{i2}(y_i, u_i)\Theta_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}^{b}(\bar{x}_j)\end{aligned} \tag{6.7}$$

$$y_i = \bar{x}_{i2} \tag{6.8}$$

where $\bar{x} = \text{col}(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_N)$, $\bar{x}_i = \text{col}(\bar{x}_{i1}, \bar{x}_{i2})$, $\bar{x}_{i1} \in R^{n_i - p_i}$, $\bar{x}_{i2} \in R^{p_i}$, and

$$\begin{bmatrix} \bar{g}_{i1}(\bar{x}_i, u_i) \\ \bar{g}_{i2}(\bar{x}_i, u_i) \end{bmatrix} := \bar{g}_i(\bar{x}_i, u_i) = T_{c_i}[g_i(x_i, u_i)]_{x_i = T_{c_i}^{-1}\bar{x}_i} \tag{6.9}$$

$$\begin{bmatrix} \bar{\phi}_{i1}(y_i, u_i) \\ \bar{\phi}_{i2}(y_i, u_i) \end{bmatrix} := T_{c_i}\phi_i(y_i, u_i), \tag{6.10}$$

$$\begin{bmatrix} H_{ij}^a(x_j) \\ H_{ij}^b(x_j) \end{bmatrix} := T_{c_i}[H_{ij}(x_j)]_{x_j = T_{c_j}^{-1}\bar{x}_j} \tag{6.11}$$

Assumption 7.1. The uncertain TVPs $\Theta_i(t)$ satisfy

$$|\dot{\Theta}_i(t)| \leq \mu_i \tag{6.12}$$

where μ_i are known constants and $\mu_i > 0$.

Assumption 7.1 means the bounds on the unknown TVPs are not required, but the rate of changes of these parameters are required to be bounded.

Assumption 7.2. The matrix pairs $(\bar{A}_i, \bar{C}_i)$ in (6.3)-(6.4) are observable for $i = 1, 2, \cdots, N$.

Under Assumption 7.2, there exist matrices L_i such that $\bar{A}_i - L_i\bar{C}_i$ are stable, and thus for any $Q_i > 0$ the Lyapunov equations

$$(\bar{A}_i - L_i\bar{C}_i)^T P_i + P_i(\bar{A}_i - L_i\bar{C}_i) = -Q_i \tag{6.13}$$

have unique solutions $P_i > 0$ for $i = 1, 2, \cdots, N$.

For further analysis, introduce partitions of P_i and Q_i which are conformable with the decomposition in (6.6)-(6.8) as follows

$$P_i = \begin{bmatrix} P_{i1} & P_{i2} \\ P_{i2}^T & P_{i3} \end{bmatrix}, \qquad Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix} \tag{6.14}$$

where $P_{i1} \in R^{(n_i - p_i)\times(n_i - p_i)}$, $Q_{i1} \in R^{(n_i - p_i)\times(n_i - p_i)}$. Then, from $P_i > 0$ and $Q_i > 0$, it follows that $P_{i1} > 0$, $P_{i3} > 0$, $Q_{i1} > 0$ and $Q_{i3} > 0$. The following result is required for further analysis.

Lemma 7.1. The matrices $\bar{A}_{i1} + P_{i1}^{-1}P_{i2}\bar{A}_{i3}$ are Hurwitz stable, where P_{i1} and P_{i2} are defined in (6.14) and $\bar{A}_{i1}$ and $\bar{A}_{i3}$ are defined in (6.3), if the Lyapunov equations (6.13) are satisfied.

Proof. See Lemma 2.1 in [126].

Assumption 7.3. The functions $\bar{g}_i(\bar{x}_i, u_i)$ defined in (6.9) satisfy the Lipschitz condition with respect to $\bar{x}_i \in R^{n_i}$ and uniformly for $u_i \in U_i \in R^{m_i}$ for $i = 1, 2, \cdots, N$.

Assumption 7.3 implies that there exist nonnegative functions $\ell_{\bar{g}_{i1}}$ and $\ell_{\bar{g}_{i2}}$ such that

$$\|\bar{g}_{i1}(\bar{x}_i, u_i) - \bar{g}_{i1}(\hat{\bar{x}}_i, u_i)\| \le \ell_{\bar{g}_{i1}}(u_i)\|\bar{x}_i - \hat{\bar{x}}_i\| \tag{6.15}$$

$$\|\bar{g}_{i2}(\bar{x}_i, u_i) - \bar{g}_{i2}(\hat{\bar{x}}_i, u_i)\| \le \ell_{\bar{g}_{i2}}(u_i)\|\bar{x}_i - \hat{\bar{x}}_i\| \tag{6.16}$$

for $i = 1, 2, \cdots, N$.

Remark 7.1. Assumption 7.3 shows that the functions $\bar{g}_i(\bar{x}_i, u_i)$ defined in (6.9) satisfy the Lipschitz condition with respect to only $\bar{x}_i$ instead of $(\bar{x}_i, u_i)$. Such an Assumption is reasonable because control inputs u_i are usually known in observer design, and may relax the limitation to the functions $\bar{g}_i(\bar{x}_i, u_i)$.

6.2. ADAPTIVE SLIDING MODE OBSERVER DESIGN

Consider the system in (6.6)-(6.8). Introduce a linear coordinate transformation

$$z_i = \underbrace{\begin{bmatrix} I_{n_i - p_i} & K_i \\ 0 & I_{p_i} \end{bmatrix}}_{T_i} \bar{x}_i \tag{6.17}$$

where $K_i = P_{i1}^{-1} P_{i2}$. In the new coordinate system z_i, system (6.6)-(6.8) has the following form

$$\begin{aligned}
\dot{z}_{i1} &= (\bar{A}_{i1} + K_i\bar{A}_{i3})z_{i1} + (\bar{A}_{i2} - \bar{A}_{i1}K_i + K_i(\bar{A}_{i4} - \bar{A}_{i3}K_i)z_{i2} + \bar{g}_{i1}(T_i^{-1}z_i, u_i) \\
&\quad + \bar{\phi}_{i1}(y_i, u_i)\Theta_i(t) + K_i\bar{g}_{i2}(T_i^{-1}z_i, u_i) + \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}^a(T^{-1}z) \\
&\quad + K_i\bar{\phi}_{i2}(y_i, u_i)\Theta_i(t) + K_i\sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}^b(T^{-1}z) \qquad (6.18) \\
\dot{z}_{i2} &= \bar{A}_{i3}z_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}K_i)z_{i2} + \bar{g}_{i2}(T_i^{-1}z_i, u_i) + \bar{\phi}_{i2}(y_i, u_i)\Theta_i(t) \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}^b(T^{-1}z) \qquad (6.19) \\
y_i &= z_{i2} \qquad (6.20)
\end{aligned}$$

where $z_i = \text{col}(z_{i1}, z_{i2})$ with $z_{i1} \in R^{n_i - p_i}$.

For system (6.18)-(6.20), consider a dynamical system

$$\begin{aligned}
\dot{\hat{z}}_{i1} &= (\bar{A}_{i1} + K_i\bar{A}_{i3})\hat{z}_{i1} + (\bar{A}_{i2} - \bar{A}_{i1}K_i + K_i(\bar{A}_{i4} - \bar{A}_{i3}K_i))y_i + \bar{g}_{i1}(T_i^{-1}\hat{z}_i, u_i) \\
&\quad + \bar{\phi}_{i1}(y_i, u_i)\hat{\Theta}_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}^a(T^{-1}\hat{z}) + K_i\bar{g}_{i2}(T_i^{-1}\hat{z}_i, u_i) \\
&\quad + K_i\bar{\phi}_{i2}(y_i, u_i)\hat{\Theta}_i(t) + K_i\sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}^b(T^{-1}\hat{z}) \qquad (6.21) \\
\dot{\hat{z}}_{i2} &= \bar{A}_{i3}\hat{z}_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}K_i)y_i + \bar{g}_{i2}(T_i^{-1}\hat{z}_i, u_i) + \bar{\phi}_{i2}(y_i, u_i)\hat{\Theta}_i(t) \\
&\quad + \sum_{\substack{j=1 \\ j\neq i}}^{N} H_{ij}^b(T^{-1}\hat{z}) + d_i(\cdot) \qquad (6.22) \\
\hat{y}_i &= \hat{z}_{i2} \qquad (6.23)
\end{aligned}$$

where $\hat{z} = \text{col}(\hat{z}_1, y)$, and the injection term $d_i(\cdot)$ is defined by

$$d_i(\cdot) = \rho_i \text{sgn}(y_i - \hat{y}_i) \qquad (6.24)$$

where ρ_i are positive constants for $i = 1, 2, \cdots, N$, with adaptive laws

$$\begin{aligned}
\dot{\Gamma}_i &= -\sigma_i[\dot{\hat{y}}_i - d_i(\cdot)] \qquad (6.25) \\
\hat{\Theta}_i(t) &= \Gamma_i + \sigma_i y_i \qquad (6.26)
\end{aligned}$$

where $d_i(\cdot)$ is given in (6.24), and σ_i are positive constants.

Let $e_{i1} = z_{i1} - \hat{z}_{i1}$, $e_{y_i} = y_i - \hat{y}_i$ and $e_{\Theta_i} = \Theta_i(t) - \hat{\Theta}_i(t)$. Then from (6.18)-(6.20) and (6.21)-(6.23), the error dynamics are described by

$$\begin{aligned}
\dot{e}_{i1} &= (\bar{A}_{i1} + K_i\bar{A}_{i3})e_{i1} + [\bar{g}_{i1}(\cdot) - \bar{g}_{i1}(\hat{\cdot})] + \bar{\phi}_{i1}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^a(\cdot) - H_{ij}^a(\hat{\cdot})] + K_i[\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] + K_i\bar{\phi}_{i2}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] \\
&\quad + K_i\sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})] \qquad (6.27)\\
\dot{e}_{y_i} &= \bar{A}_{i3}e_{i1} + [\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] + \bar{\phi}_{i2}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] \\
&\quad + \sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})] - d_i(\cdot) \qquad (6.28)
\end{aligned}$$

where $d_i(\cdot)$ is given in (6.24) for $i = 1, 2, \cdots, N$, and

$$\begin{aligned}
\bar{g}_{i1}(T_i^{-1}z_i, u_i) &= \bar{g}_{i1}(\cdot),\ \ \bar{g}_{i1}(T_i^{-1}\hat{z}_i, u_i) = \bar{g}_{i1}(\hat{\cdot}) \\
\bar{g}_{i2}(T_i^{-1}z_i, u_i) &= \bar{g}_{i2}(\cdot),\ \ \bar{g}_{i2}(T_i^{-1}\hat{z}_i, u_i) = \bar{g}_{i2}(\hat{\cdot}) \\
H_{ij}^a(T^{-1}z) &= H_{ij}^a(\cdot),\ \ H_{ij}^a(T^{-1}\hat{z}) = H_{ij}^a(\hat{\cdot}) \\
H_{ij}^b(T^{-1}z) &= H_{ij}^b(\cdot),\ \ H_{ij}^b(T^{-1}\hat{z}) = H_{ij}^b(\hat{\cdot})
\end{aligned}$$

From (6.25) and (6.26)

$$\begin{aligned}
\dot{e}_{\Theta_i} &= \dot{\Theta}_i(t) - \dot{\hat{\Theta}}_i(t) \\
&= \dot{\Theta}_i(t) - \{\dot{\Gamma}_i + \sigma_i\dot{y}_i\} \\
&= \dot{\Theta}_i(t) - \big\{\{-\sigma_i[\bar{A}_{i3}\hat{z}_{i1} + (\bar{A}_{i4} - \bar{A}_{i3}K_i)y_i + \bar{g}_{i2}(\hat{\cdot}) \\
&\quad + \bar{\phi}_{i2}(y_i, u_i)\hat{\Theta}_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}^b(\hat{\cdot}) + d_i(\cdot) - d_i(\cdot)]\} + \{\sigma_i[\bar{A}_{i3}z_{i1} \\
&\quad + (\bar{A}_{i4} - \bar{A}_{i3}K_i)z_{i2} + \bar{g}_{i2}(\cdot) + \bar{\phi}_{i2}(y_i, u_i)\Theta_i(t) + \sum_{\substack{j=1\\ j\neq i}}^{N} H_{ij}^b(\cdot)]\}\big\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & -\sigma_i \bar{A}_{i3} e_{i1} - \sigma_i [\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] - \sigma_i \bar{\phi}_{i2}(y_i, u_i) e_{\Theta_i} - \sum_{\substack{j=1 \\ j \neq i}}^{N} \sigma_i [H_{ij}^b(\cdot) \\
& -H_{ij}^b(\hat{\cdot})] + \dot{\Theta}_i(t) \qquad (6.29)
\end{aligned}
$$

From the structure of the transformation matrix T_i in (6.17) and the fact that $\hat{z}_i = \text{col}(\hat{z}_{i1}, y_i)$, it follows that

$$
\begin{aligned}
\|T_i^{-1} z_i - T_i^{-1} \hat{z}_i\| &= \|T_i^{-1}(z_i - \hat{z}_i)\| \\
&= \left\| T_i^{-1} \begin{bmatrix} e_{i1} \\ 0 \end{bmatrix} \right\| = \|e_{i1}\| \qquad (6.30)
\end{aligned}
$$

From the analysis above, it is straightforward to see

$$
\|T^{-1} z - T^{-1} \hat{z}\| = \|e_1\| \qquad (6.31)
$$

where

$$
e_1 := \text{col}(e_{11}, e_{21}, \cdots, e_{N1}) \qquad (6.32)
$$

Therefore, from (6.15), (6.16), (6.30) and (6.31)

$$
\begin{aligned}
\|\bar{g}_{i1}(T_i^{-1} z_i, u_i) - \bar{g}_{i1}(T_i^{-1} \hat{z}_i, u_i)\| &\leq \ell_{\bar{g}_{i1}}(u_i) \|e_{i1}\| \qquad (6.33) \\
\|\bar{g}_{i2}(T_i^{-1} z_i, u_i) - \bar{g}_{i2}(T_i^{-1} \hat{z}_i, u_i)\| &\leq \ell_{\bar{g}_{i2}}(u_i) \|e_{i1}\| \qquad (6.34) \\
\|H_{ij}^a(T^{-1} z) - H_{ij}^a(T^{-1} \hat{z})\| &\leq \ell_{H^a} \|e_1\| \qquad (6.35) \\
\|H_{ij}^b(T^{-1} z) - H_{ij}^b(T^{-1} \hat{z})\| &\leq \ell_{H^b} \|e_1\| \qquad (6.36)
\end{aligned}
$$

where $\ell_{\bar{g}_{i1}}(u_i)$ and $\ell_{\bar{g}_{i2}}(u_i)$ are nonnegative functions, and ℓ_{H^a} and ℓ_{H^b} are constants.

Remark 7.2. It is well known that sliding mode is a reduced order system. In this chapter, the sliding motion governs by the error dynamical systems (6.27) with adaptive laws (6.25) - (6.26) while the error dynamical systems (6.28) does not affect the sliding motion, which makes the obtained results less conservative.

6.3. Stability of the Error Dynamical Systems

The following result is ready to be presented:

Theorem 7.1. Under Assumptions 7.1-7.3, the error dynamical systems (6.27) with adaptive laws (6.25) - (6.26) are globally uniformly ultimately bounded if the matrix $W^T + W$ is positive definite, where

$$W = \begin{bmatrix} w^a & w^b \\ w^c & w^d \end{bmatrix}_{2N\times 2N} \tag{6.37}$$

where $w^a = (w^a_{ij})_{N\times N}$, $w^b = (w^b_{ij})_{N\times N}$, $w^c = (w^c_{ij})_{N\times N}$, $w^d = (w^d_{ij})_{N\times N}$, and

$$w^a_{ij} = \begin{cases} \{\lambda_{min}(Q_{i1}) - 2\|P_{i1}\|[\ell_{g_{i1}} + \|K_i\|\ell_{g_{i2}}] \\ -2\|P_{i1}\|[\ell_{H^a} + \|K_i\|\ell_{H^b}]\}, & i = j \\ \\ -\|P_{i1}\|[\ell_{H^a} + \|K_i\|\ell_{H^b}], & i \neq j \end{cases}$$

$$w^b_{ij} = w^c_{ij} = \begin{cases} -\{\|P_{i1}\|\alpha_{i1} + \sigma_i\|\bar{A}_{i3}\| \\ +\sigma_i\ell_{g_{i2}} + \sigma_i\ell_{H^b}\}, & i = j \\ \\ \sigma_i\ell_{H^b}, & i \neq j \end{cases}$$

$$w^d_{ij} = \begin{cases} 2\sigma_i\alpha_{i2}, & i = j \\ \\ 0, & i \neq j \end{cases}$$

where P_{i1} and Q_{i1} are given in (6.14), and

$$\|\bar{\phi}_{i1}(\cdot) + K_i\bar{\phi}_{i2}(\cdot)\| \leq \alpha_{i1} \tag{6.38}$$

$$\|\bar{\phi}_{i2}(\cdot)\| \leq \alpha_{i2} \tag{6.39}$$

for $i, j = 1, 2, \cdots, N$.

Proof. For systems (6.27) and (6.29), consider the candidate Lyapunov function

$$V = \sum_{i=1}^{N} e_{i1}^T P_{i1} e_{i1} + \sum_{i=1}^{N} e_{\Theta_i}^T e_{\Theta_i} \tag{6.40}$$

The time derivative of $V(\cdot)$ along the trajectories of system (6.27) and (6.29) is given by

$$\begin{aligned}
\dot{V} \;&= \sum_{i=1}^{N}[\dot{e}_{i1}^T P_{i1} e_{i1} + e_{i1}^T P_{i1}\dot{e}_{i1} + \dot{e}_{\Theta_i}^T e_{\Theta_i} + e_{\Theta_i}^T \dot{e}_{\Theta_i}] \\
&= \sum_{i=1}^{N}\Big\{\big\{(\bar{A}_{i1} + K_i\bar{A}_{i3})e_{i1} + [\bar{g}_{i1}(\cdot) - \bar{g}_{i1}(\hat{\cdot})] \\
&\quad +\bar{\phi}_{i1}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] + \sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^a(\cdot) - H_{ij}^a(\hat{\cdot})] \\
&\quad +K_i[\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] + K_i\bar{\phi}_{i2}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] \\
&\quad +K_i\sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^b(\cdot) - H_{ij}^b(\cdot)]\big\}^T P_{i1}e_{i1} \\
&\quad +e_{i1}^T P_{i1}\big\{(\bar{A}_{i1} + K_i\bar{A}_{i3})e_{i1} + [\bar{g}_{i1}(\cdot) - \bar{g}_{i1}(\hat{\cdot})] \\
&\quad +\bar{\phi}_{i1}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] + \sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^a(\cdot) - H_{ij}^a(\hat{\cdot})] \\
&\quad +K_i[\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] + K_i\bar{\phi}_{i2}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] \\
&\quad +K_i\sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})]\big\} \\
&\quad +\big\{-\sigma_i\bar{A}_{i3}e_{i1} - \sigma_i[\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] \\
&\quad -\sigma_i\bar{\phi}_{i2}(\cdot)e_{\Theta_i} + \sum_{\substack{j=1\\ j\neq i}}^{N}\sigma_i[H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})] \\
&\quad +\dot{\Theta}_i(t)\big\}^T e_{\Theta_i} + e_{\Theta_i}^T\big\{-\sigma_i\bar{A}_{i3}e_{i1} \\
&\quad -\sigma_i[\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] \\
&\quad -\sigma_i\bar{\phi}_{i2}(\cdot)e_{\Theta_i} + \sum_{\substack{j=1\\ j\neq i}}^{N}\sigma_i[H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})] + \dot{\Theta}_i(t)\big\}\Big\} \\
&= \sum_{i=1}^{N}\Big\{e_{i1}^T[(\bar{A}_{i1} + K_i\bar{A}_{i3})^T P_{i1} + P_{i1}(\bar{A}_{i1} \\
&\quad +K_i\bar{A}_{i3})]e_{i1} + 2e_{i1}^T P_{i1}[\bar{g}_{i1}(\cdot) - \bar{g}_{i1}(\hat{\cdot})] \\
&\quad +2e_{i1}^T P_{i1}\bar{\phi}_{i1}(\cdot)e_{\Theta_i} + 2e_{i1}^T P_{i1}\sum_{\substack{j=1\\ j\neq i}}^{N}[H_{ij}^a(\cdot) - H_{ij}^a(\hat{\cdot})] \\
&\quad +2e_{i1}^T P_{i1}K_i[\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] + 2e_{i1}^T P_{i1}K_i\bar{\phi}_{i2}(\cdot)e_{\Theta_i}
\end{aligned}$$

$$
\begin{aligned}
&+2e_{i1}^T P_{i1} K_i \sum_{\substack{j=1\\j\neq i}}^{N} [H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})] \\
&-2e_{\Theta_i}^T \sigma_i \bar{A}_{i3} e_{i1} - 2e_{\Theta_i}^T \sigma_i [\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] \\
&-2e_{\Theta_i}^T \sigma_i \sum_{\substack{j=1\\j\neq i}}^{N} [H_{ij}^b(\cdot) - H_{ij}^b(\hat{\cdot})] \\
&-2e_{\Theta_i}^T \sigma_i \bar{\phi}_{i2}(\cdot) e_{\Theta_i} + 2e_{\Theta_i}^T \dot{\Theta}_i(t) \Big\}
\end{aligned}
$$

From (6.33)-(6.36),

$$
\begin{aligned}
\dot{V} \;\leq\; & \sum_{i=1}^{N} \Big\{ - e_{i1}^T Q_{i1} e_{i1} \\
& +2\|P_{i1}\| [\ell_{g_{i1}} + \|K_i\| \ell_{g_{i2}}] \|e_{i1}\|^2 + 2\|P_{i1}\| \\
& \times \|[\bar{\phi}_{i1}(\cdot) + K_i \bar{\phi}_{i2}(\cdot)]\| \|e_{i1}\| \|e_{\Theta_i}\| \\
& +2\|e_{i1}\| \|P_{i1}\| [\ell_{H^a} + \|K_i\| \ell_{H^b}] \|e_1\| \\
& -2\|e_{\Theta_i}\| \sigma_i \|\bar{A}_{i3}\| \|e_{i1}\| - 2\|e_{\Theta_i}\| \sigma_i \ell_{g_{i2}} \|e_{i1}\| \\
& -2\|e_{\Theta_i}\| \sigma_i \ell_{H^b} \|e_1\| - 2\|e_{\Theta_i}\| \sigma_i \|\bar{\phi}_{i2}(\cdot)\| \|e_{\Theta_i}\| \\
& +2\|e_{\Theta_i}\| \|\dot{\Theta}_i(t)\| \Big\}
\end{aligned}
\tag{6.41}
$$

From the definition of e_1 in (6.32)

$$
\|e_1\| \leq \sum_{j=1}^{N} \|e_{j1}\| = \|e_{i1}\| + \sum_{\substack{j=1\\j\neq i}}^{N} \|e_{j1}\| \tag{6.42}
$$

Then, from (6.41) and (6.42)

$$
\begin{aligned}
\dot{V} \;\leq\; & \sum_{i=1}^{N} \Big\{ - e_{i1}^T Q_{i1} e_{i1} \\
& +2\|P_{i1}\| [\ell_{g_{i1}} + \|K_i\| \ell_{g_{i2}}] \|e_{i1}\|^2 \\
& +2\|P_{i1}\| \|[\bar{\phi}_{i1}(\cdot) + K_i \bar{\phi}_{i2}(\cdot)]\| \|e_{i1}\| \|e_{\Theta_i}\| \\
& +2\|e_{i1}\| \|P_{i1}\| [\ell_{H^a} + \|K_i\| \ell_{H^b}] [\|e_{i1}\| \\
& + \sum_{\substack{j=1\\j\neq i}}^{N} \|e_{j1}\|] - 2\|e_{\Theta_i}\| \sigma_i \|\bar{A}_{i3}\| \|e_{i1}\| \\
& -2\|e_{\Theta_i}\| \sigma_i \ell_{g_{i2}} \|e_{i1}\| - 2\|e_{\Theta_i}\| \sigma_i \ell_{H^b} [\|e_{i1}\|
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{\substack{j=1\\ j\neq i}}^{N}\|e_{j1}\|] - 2\sigma_i\|\bar{\phi}_{i2}(\cdot)\|\|e_{\Theta_i}\|^2 \\
& +2\|e_{\Theta_i}\|\|\dot{\Theta}_i(t)\|\Big\} \\
\leq\ & \sum_{i=1}^{N}\Big\{-e_{i1}^T Q_{i1} e_{i1} \\
& +2\|P_{i1}\|[\ell_{g_{i1}} + \|K_i\|\ell_{g_{i2}}]\|e_{i1}\|^2 \\
& +2\|P_{i1}\|\|[\bar{\phi}_{i1}(\cdot) + K_i\bar{\phi}_{i2}(\cdot)]\|\|e_{i1}\|\|e_{\Theta_i}\| \\
& +2\|P_{i1}\|[\ell_{H^a} + \|K_i\|\ell_{H^b}]\|e_{i1}\|^2 \\
& +\sum_{\substack{j=1\\ j\neq i}}^{N}[2\|P_{i1}\|[\ell_{H^a} + \|K_i\|\ell_{H^b}]\|e_{i1}\|\|e_{j1}\|] \\
& -2\sigma_i\|\bar{A}_{i3}\|\|e_{i1}\|\|e_{\Theta_i}\| - 2\sigma_i\ell_{g_{i2}}\|e_{i1}\|\|e_{\Theta_i}\| \\
& -2\sigma_i\ell_{H^b}\|e_{i1}\|\|e_{\Theta_i}\| - \sum_{\substack{j=1\\ j\neq i}}^{N} 2\sigma_i\ell_{H^b}\|e_{\Theta_i}\|\|e_{j1}\| \\
& -2\sigma_i\|\bar{\phi}_{i2}(\cdot)\|\|e_{\Theta_i}\|^2 + 2\|e_{\Theta_i}\|\|\dot{\Theta}_i(t)\|\Big\} \qquad (6.43)
\end{aligned}
$$

From Assumption 1, (6.38) and (6.39)

$$
\begin{aligned}
\dot{V}\ \leq\ & -\sum_{i=1}^{N}\Big\{\{\lambda_{min}(Q_{i1}) - 2\|P_{i1}\|[\ell_{g_{i1}} + \|K_i\|\ell_{g_{i2}}] \\
& -2\|P_{i1}\|[\ell_{H^a} + \|K_i\|\ell_{H^b}]\}\|e_{i1}\|^2 - \{2\|P_{i1}\|\alpha_{i1} \\
& +2\sigma_i\|\bar{A}_{i3}\| + 2\sigma_i\ell_{g_{i2}} + 2\sigma_i\ell_{H^b}\}\|e_{i1}\|\|e_{\Theta_i}\| \\
& -\sum_{\substack{j=1\\ j\neq i}}^{N}[2\|P_{i1}\|[\ell_{H^a} + \|K_i\|\ell_{H^b}]\|e_{i1}\|\|e_{j1}\|] \\
& +\sum_{\substack{j=1\\ j\neq i}}^{N} 2\sigma_i\ell_{H^b}\|e_{\Theta_i}\|\|e_{j1}\| + 2\sigma_i\alpha_{i2}\|e_{\Theta_i}\|^2\Big\} \\
& +\sum_{i=1}^{N} 2\mu_i\|e_{\Theta_i}\| \qquad (6.44)
\end{aligned}
$$

Then, from the definition of the matrix W in Theorem 7.1 and the inequality above, it follows that

$$
\begin{aligned}
\dot{V} &\leq -\frac{1}{2}X^T[W^T + W]X + \gamma\|X\| \\
&\leq -\frac{1}{2}\lambda_{min}(W^T + W)\|X\|^2 + \gamma\|X\| \\
&\leq -\left(\frac{1}{2}\lambda_{min}(W^T + W)\|X\| - \gamma\right)\|X\|
\end{aligned} \tag{6.45}
$$

where $\gamma = 2\mu_i$ and $X = [\|e_{11}\|, \|e_{21}\|, \cdots, \|e_{N1}\|, \|e_{\Theta_1}\|, \|e_{\Theta_2}\|, \cdots, \|e_{\Theta_N}\|]^T$.

From the definition of Lyapunov function in (6.40), it is straightforward to see that

$$
\lambda_{min}(P_{i1})\|X\|^2 \quad \leq \quad V \leq \lambda_{max}(P_{i1})\|X\|^2
$$

where $X = [\|e_{11}\|, \|e_{21}\|, \cdots\cdots, \|e_{N1}\|, \|e_{\Theta_1}\|, \|e_{\Theta_2}\|, \cdots, \|e_{\Theta_N}\|]^T$, for all $X \in R^n$. It can been seen clearly that $\lambda_{min}(P_{i1})\|X\|^2$ belongs to class K_∞.

Therefore, from the condition that $W^T + W$ is positive definite, system (6.27) is globally uniformly ultimately bounded. Hence the result follows. △

Remark 3. From Theorem 1, it follows that e_1 and e_{Θ_i} are bounded and thus there exist constants $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$
\|e_1\| \leq \beta_1, \qquad \|e_{\Theta_i}\| \leq \beta_2 \tag{6.46}
$$

where β_1 can be estimated using the approach given in [126] by slightly modification.

For system (6.27)-(6.28), consider a sliding surface

$$
S_i \quad = \quad \{(e_{i1}, e_{y_i}, e_{\Theta_i})\big| e_{y_i} = 0\} \tag{6.47}
$$

From the structure of the error dynamical system (6.27)-(6.28), it follows that the sliding mode of the error system (6.27)-(6.28) with respect to the sliding surface (6.47) is the system (6.27) when limited to the sliding surface (6.47). All that remains is to determine the gains ρ_i in (6.24) such that the system (6.27)-(6.28) can be driven to the sliding surface S_i in finite time and a sliding motion maintained thereafter.

Theorem 7.2. Under Assumptions 7.1-7.3 and the inequality (6.39), system (6.27)-(6.28) is driven to the sliding surface (6.47) in finite time and remains on it thereafter if

$$
\rho_i \geq (\|\bar{A}_{i3}\| + \ell_{\bar{g}_{i2}} + \ell_{H^b} + \alpha_{i2}\beta_2)\beta_1 + \eta \tag{6.48}
$$

where $\eta > 0$ is constant, β_1 and β_2 satisfy (6.46).

Proof. From (6.28)

$$\begin{aligned}
\sum_{i=1}^{N} e_{y_i}^T \dot{e}_{y_i} &= \sum_{i=1}^{N} e_{y_i}^T \Big\{ \bar{A}_{i3} e_{i1} + [\bar{g}_{i2}(\cdot) - \bar{g}_{i2}(\hat{\cdot})] \\
&\quad + \bar{\phi}_{i2}(\cdot)[\Theta_i(t) - \hat{\Theta}_i(t)] \\
&\quad + \sum_{\substack{j=1 \\ j \neq i}}^{N} [H_{ij}^b(\cdot) - H_{ij}^b(\cdot)] - d_i(\cdot) \Big\} \\
&\leq \sum_{i=1}^{N} \Big\{ \|\bar{A}_{i3}\| \|e_{i1}\| \|e_{y_i}\| + \ell_{g_{i2}} \|e_{i1}\| \|e_{y_i}\| \\
&\quad + \ell_{H^b} \|e_1\| \|e_{y_i}\| + \|\bar{\phi}_{i2}(\cdot)\| \|e_{\Theta_i}\| \|e_{y_i}\| \\
&\quad - \rho_i \mathrm{sgn}(e_{y_i}) \Big\} \\
&\leq \sum_{i=1}^{N} \Big\{ \big\{ (\|\bar{A}_{i3}\| + \ell_{\bar{g}_{i2}} + \ell_{H^b} + \alpha_{i2}\beta_2)\beta_1 \\
&\quad - \rho_i \big\} \|e_{y_i}\| \Big\}
\end{aligned} \tag{6.49}$$

Applying (6.48) into (6.49)

$$\sum_{i=1}^{N} e_{y_i}^T \dot{e}_{y_i} = -\eta \sum_{i=1}^{N} \|e_{y_i}\| \tag{6.50}$$

which implies that

$$e_y^T \dot{e}_y \leq -\eta \|e_y\|$$

where $e_y = \mathrm{col}(e_{y_1}, e_{y_2}, \cdots, e_{y_N})$ and the inequality $\|e_y\| \leq \sum_{i=1}^{N} \|e_{y_i}\|$ is applied to obtain the inequality above. This shows that the reachability condition is satisfied. Hence the conclusion follows. △

Remark 7.4. From sliding mode theory, Theorems 7.1 and 7.2 show that system (6.21)-(6.23) is an approximate observer for the system (6.18)-(6.20) and the estimation error enters a bounded domain in finite time.

Remark 7.5. It should be noted that Theorem 7.1 holds no matter whether the system is on the sliding surface or not. It guarantees that the inequalities in (6.46) always hold. Therefore, (6.46) can be used to obtain the results in Theorem 7.2.

6.4. SIMULATION EXAMPLE

Consider a nonlinear interconnected system as follows:

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} + \begin{bmatrix} u_1 \\ \sin x_{12} \end{bmatrix} + \begin{bmatrix} x_{11} \\ 0 \end{bmatrix} \theta_1(t) + \begin{bmatrix} 0 \\ 0.1x_{21}^2 \end{bmatrix} \quad (6.51)$$

$$y_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \quad (6.52)$$

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} + \begin{bmatrix} u_2 \\ 0.7\cos x_{22} \end{bmatrix} + \begin{bmatrix} x_{21} \\ 0 \end{bmatrix} \theta_2(t) + \begin{bmatrix} 0 \\ 0.7\sin x_{11} \end{bmatrix} \quad (6.53)$$

$$y_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \quad (6.54)$$

where col(x_1, x_2) are the system states, y_1 and y_2 are the system outputs.

Let

$$T_{c_i} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad i = 1, 2 \quad (6.55)$$

From Section II, the system (6.51)-(6.54) can be transformed to

$$\dot{\bar{x}}_1 = \underbrace{\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}}_{\bar{A}_1} \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} + \underbrace{\begin{bmatrix} \sin \bar{x}_{12} \\ u_1 \end{bmatrix}}_{\bar{g}_1(\cdot)} + \underbrace{\begin{bmatrix} 0 \\ \bar{x}_{11} \end{bmatrix}}_{\bar{\phi}_1(\cdot)} \theta_1(t) + \underbrace{\begin{bmatrix} 0.1\bar{x}_{21}^2 \\ 0 \end{bmatrix}}_{H_{12}(\bar{x}_2)} \quad (6.56)$$

$$y_1 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\bar{C}_1} \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} \quad (6.57)$$

$$\dot{\bar{x}}_2 = \underbrace{\begin{bmatrix} -3 & 2 \\ 1 & 0 \end{bmatrix}}_{\bar{A}_2} \begin{bmatrix} \bar{x}_{21} \\ \bar{x}_{22} \end{bmatrix} + \underbrace{\begin{bmatrix} 0.7\cos \bar{x}_{22} \\ u_2 \end{bmatrix}}_{\bar{g}_2(\cdot)} + \underbrace{\begin{bmatrix} 0 \\ \bar{x}_{21} \end{bmatrix}}_{\bar{\phi}_2(\cdot)} \theta_2(t) + \underbrace{\begin{bmatrix} 0.7\sin \bar{x}_{11} \\ 0 \end{bmatrix}}_{H_{21}(\bar{x}_1)} \quad (6.58)$$

$$y_2 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\bar{C}_2} \begin{bmatrix} \bar{x}_{21} \\ \bar{x}_{22} \end{bmatrix} \quad (6.59)$$

Choose

$$L_i = [1 \;\; 1] \qquad \text{and} \qquad Q_i = 8I$$

for $i = 1, 2$. It follows that the Lyapunov equation (6.13) has unique solution:

$$P_i = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 4.2 \end{bmatrix}, \qquad i = 1, 2 \tag{6.60}$$

and the matrix W defined in Theorem 7.1 is

$$W = \begin{bmatrix} 6.8 & -0.1 & -2.2 & 0 \\ -0.1 & 6.8 & 0 & -2.2 \\ -2.2 & 0 & 2 & 0 \\ 0 & -2.2 & 0 & 2 \end{bmatrix} \tag{6.61}$$

Therefore, under the transformation $x_i = (T_i T_{c_i})^{-1} z_i$ with T_{c_i} defined in (6.55) and T_i given by

$$T_i = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \qquad i = 1, 2 \tag{6.62}$$

The system can be described in z coordinates as follows

$$\dot{z}_{11} = -2.8z_{11} + 2.2z_{12} + \sin z_{12} + 0.1(z_{21} - 0.2z_22)^2 \tag{6.63}$$

$$\dot{z}_{12} = z_{11} - 0.2z_{12} + (z_{11} - 0.2z_{12})\theta_1(t) + u_1 \tag{6.64}$$

$$y_1 = z_{12} \tag{6.65}$$

$$\dot{z}_{21} = -2.8z_{21} + 2.2z_{22} + 0.7\cos z_{22} + 0.7\sin(z_{11} - 0.2z_{12}) \tag{6.66}$$

$$\dot{z}_{22} = z_{21} - 0.2z_{22} + (z_{21} - 0.2z_{22})\theta_2(t) + u_2 \tag{6.67}$$

$$y_2 = z_{22} \tag{6.68}$$

For simulation purposes, the controllers are chosen as:

$$u_i = -k_i x_i, \qquad i = 1, 2 \tag{6.69}$$

$$k_i = \begin{bmatrix} 8 & 2 \end{bmatrix} \tag{6.70}$$

By direct computation, it follows that the matrix $W^T + W$ is positive definite. Thus, all the conditions of Theorem 7.1 are satisfied. Therefore the following dynamical system is an asymptotic observer of the system $(6.63) - (6.68)$

$$\dot{\hat{z}}_{11} = -2.8\hat{z}_{11} + 2.2y_1 + \sin \hat{z}_{12} + 0.1(\hat{z}_{21} - 0.2\hat{z}_{22})^2 \tag{6.71}$$

$$\dot{\hat{z}}_{12} = \hat{z}_{11} - 0.2\hat{z}_{12} + (\hat{z}_{11} - 0.2\hat{z}_{12})\hat{\theta}_1(t) + u_1 + d_1(\cdot) \tag{6.72}$$

$$\hat{y}_1 = \hat{z}_{12} \tag{6.73}$$

$$\dot{\hat{z}}_{21} = -2.8\hat{z}_{21} + 2.2y_2 + 0.7\cos \hat{z}_{22} + 0.7\sin(\hat{z}_{11} - 0.2\hat{z}_{12}) \tag{6.74}$$

$$\dot{\hat{z}}_{22} = \hat{z}_{21} - 0.2\hat{z}_{22} + (\hat{z}_{21} - 0.2\hat{z}_{22})\hat{\theta}_2(t) + u_2 + d_2(\cdot) \tag{6.75}$$

$$\hat{y}_2 = \hat{z}_{22} \tag{6.76}$$

where the terms $d_1(\cdot)$ and $d_2(\cdot)$ are defined by

$$d_1(\cdot) = 9 \text{ sgn}(y_1 - \hat{y}_1) \tag{6.77}$$

$$d_2(\cdot) = 9 \text{ sgn}(y_2 - \hat{y}_2) \tag{6.78}$$

where ρ_1 and ρ_2 are defined in (6.48) and given by

$$\rho_1 \geq \beta_1 + \eta \tag{6.79}$$

$$\rho_2 \geq \beta_1 + \eta \tag{6.80}$$

where $\beta_1 = 6.5$, and the design parameter η is chosen as $\eta = 2.5$. From (6.25) and (6.26) with $\sigma_1 = \sigma_2 = 1$, the designed adaptive laws are given by

$$\dot{\Gamma}_1 = -[\dot{\hat{y}}_1 - d_1(\cdot)] \tag{6.81}$$

$$\hat{\Theta}_1(t) = \Gamma_1 + y_1 \tag{6.82}$$

$$\dot{\Gamma}_2 = -[\dot{\hat{y}}_2 - d_2(\cdot)] \tag{6.83}$$

$$\hat{\Theta}_2(t) = \Gamma_2 + y_2 \tag{6.84}$$

Simulation in Figures 6.1-6.4 shows the system state variables and their estimations in presence of unknown time varying parameters $\Theta_1(t) = \Theta_2(t) = 0.3t$, and simulation in Figures 6.5-6.6 shows that the system state variables and their estimations in presence of unknown time varying parameters $\Theta_1(t) = \Theta_2(t) = 0.6t$. The estimation error between the states of the system (6.63)-(6.68) and the states of the observer (6.71)-(6.76) converges to zero globally ultimately bounded. Therefore, $\hat{z}_i = \text{col}(\hat{z}_{i1}, \hat{z}_{i2})$ in (6.71)-(6.76) is an asymptotic estimation of $z_i = \text{col}(z_{i1}, z_{i2})$ in (6.63)-(6.68).

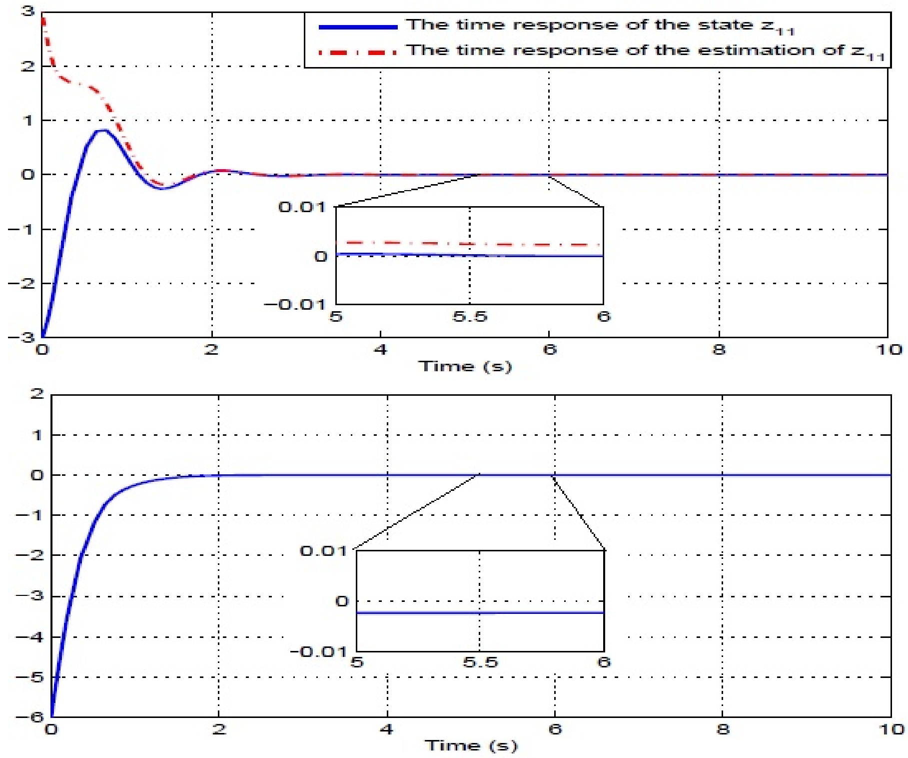

Figure 6.1: Upper: the time response of the z_{11} (solid line) and the time response of the estimated $\hat{z}_{11}$ (dashed line); Bottom: the time response of the error system $z_{11} - \hat{z}_{11}$.

Remark 7.6. It should be noted that the states $\hat{z}_i = \text{col}(\hat{z}_{i1}, \hat{z}_{i2})$ in (6.71)-(6.76) give estimations of the variable $z_i = \text{col}(z_{i1}, z_{i2})$ in (6.63)-(6.68) for $i = 1, 2$. From the analysis in Sections II and III, it is straightforward to see that $\hat{x}_i = (T_i T_{c_i})^{-1}\hat{z}_i$ are estimations of the states $x_i = [x_{i1} \;\; x_{i2}]^T$ of the system (6.51)-(6.54) where T_{c_i} and T_i are defined in (6.55) and (6.62) respectively for $i = 1, 2$.

Moreover, to validate the obtained results and to show that these results are applicable to wide range of systems, $A_i, i = 1, 2$ have been changed.

Let

$$A_i = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \qquad \text{Then} \qquad \bar{A} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}, \qquad i = 1, 2 \tag{6.85}$$

Choose

$$L_i = [15 \;\; 9] \qquad \text{and} \qquad Q_i = 6I$$

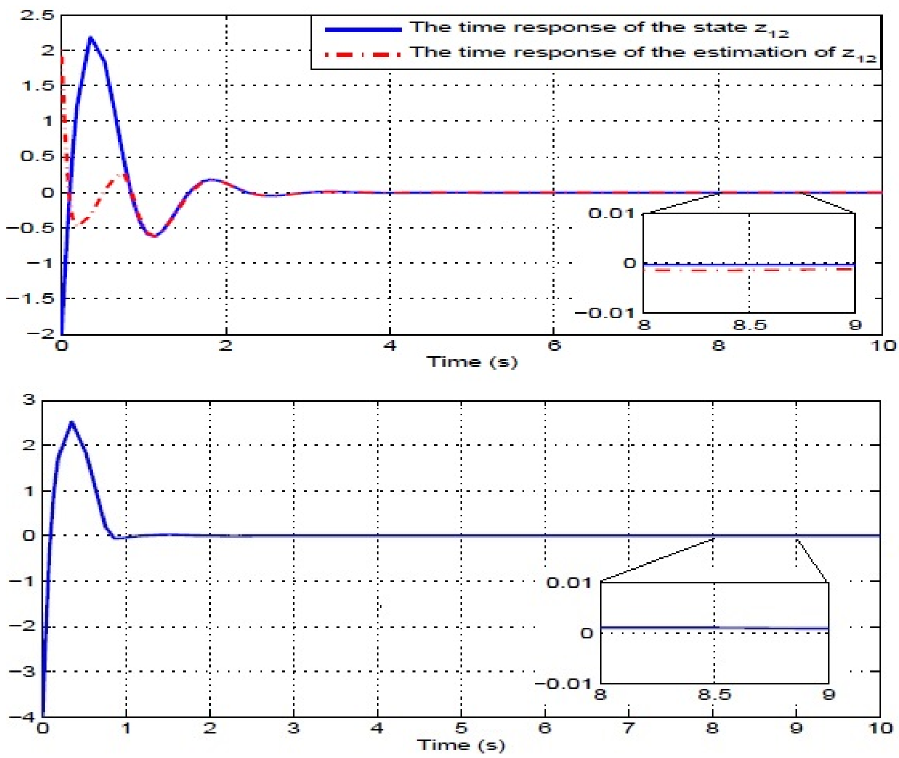

Figure 6.2: Upper: the time response of the z_{12} (solid line) and the time response of the estimated $\hat{z}_{12}$ (dashed line); Bottom: the time response of the error system $z_{12} - \hat{z}_{12}$.

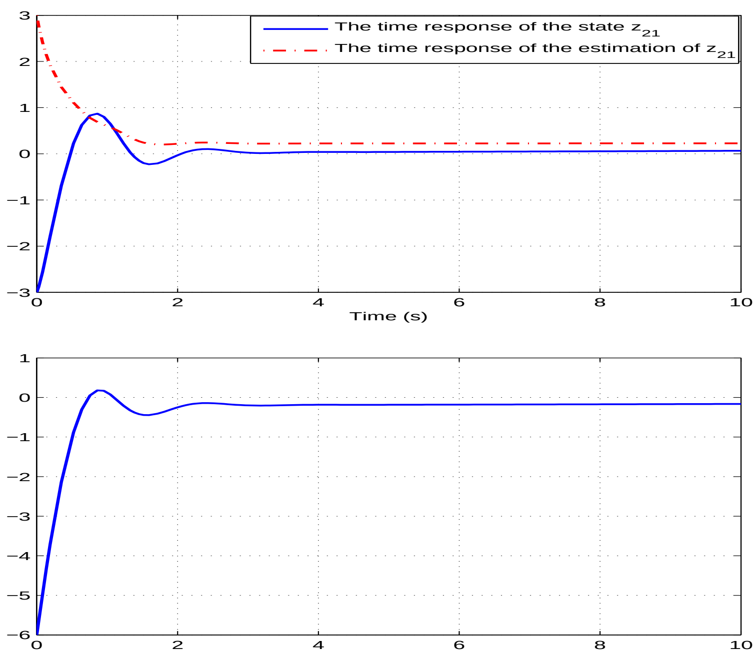

Figure 6.3: Upper: the time response of the z_{21} (solid line) and the time response of the estimated $\hat{z}_{21}$ (dashed line); Bottom: the time response of the error system $z_{21} - \hat{z}_{21}$.

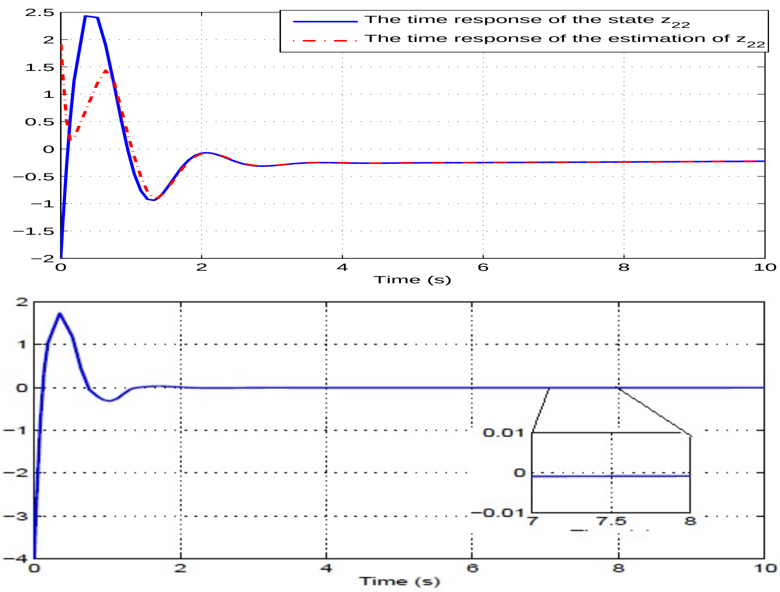

Figure 6.4: Upper: the time response of the z_{22} (solid line) and the time response of the estimated $\hat{z}_{22}$ (dashed line); Bottom: the time response of the error system $z_{22} - \hat{z}_{22}$.

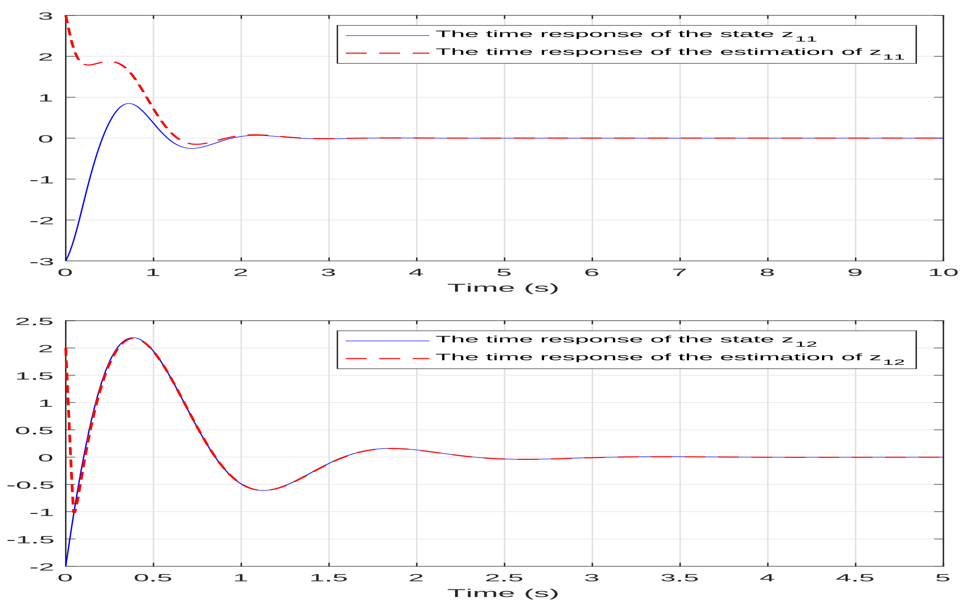

Figure 6.5: Upper: the time response of the z_{11} (solid line) and the time response of the estimated $\hat{z}_{11}$ (dashed line); Bottom: the time response of the z_{12} (solid line) and the time response of the estimated $\hat{z}_{12}$.

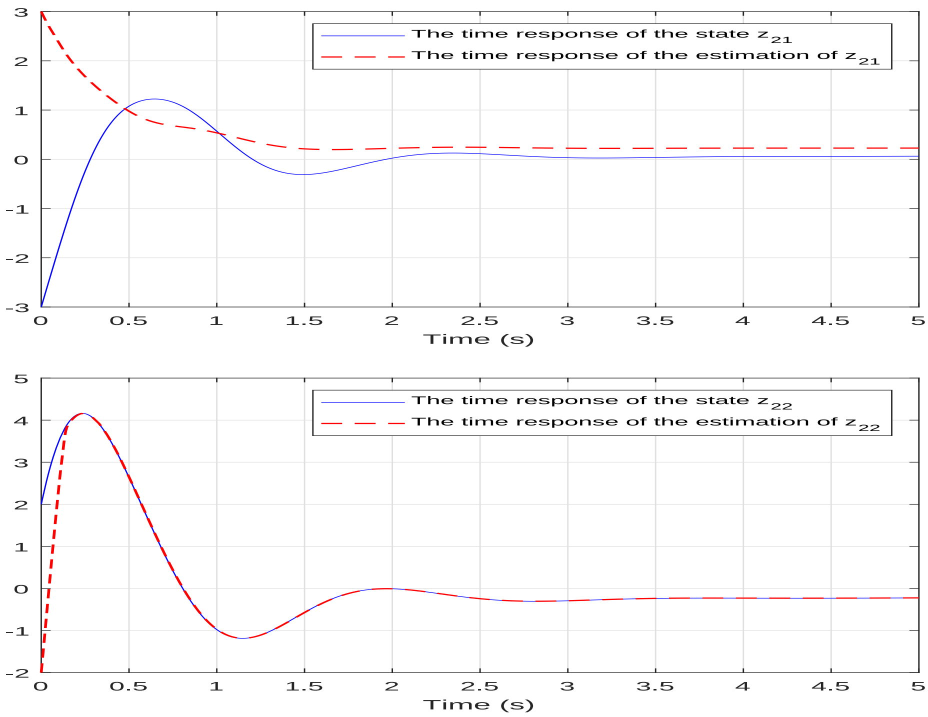

Figure 6.6: Upper: the time response of the z_{21} (solid line) and the time response of the estimated $\hat{z}_{21}$ (dashed line); Bottom: the time response of the z_{22} (solid line) and the time response of the estimated $\hat{z}_{22}$ (dashed line)

for $i = 1, 2$. It follows that the Lyapunov equation (6.13) has unique solution:

$$P_i = \begin{bmatrix} 0.8647 & -0.6529 \\ -0.6529 & 0.8412 \end{bmatrix}, \qquad i = 1, 2 \tag{6.86}$$

and the matrix W defined in Theorem 7.1 is

$$W = \begin{bmatrix} 4.9624 & -0.0865 & -2.6529 & 0 \\ -0.0865 & 4.9624 & 0 & -2.6529 \\ -2.6529 & 0 & 2 & 0 \\ 0 & -2.6529 & 0 & 2 \end{bmatrix} \tag{6.87}$$

Therefore, under the transformation $x_i = (T_i T_{c_i})^{-1} z_i$ with T_{c_i} defined in (6.55) and T_i given by

$$T_i = \begin{bmatrix} 1 & -0.76 \\ 0 & 1 \end{bmatrix}, \qquad i = 1, 2 \tag{6.88}$$

The system can be described in z coordinates as follows

$$
\begin{aligned}
\dot{z}_{11} &= -1.1z_{11} - 1.47z_{12} + \sin z_{12} + 0.1(z_{21} + 0.76z_22)^2 && (6.89)\\
\dot{z}_{12} &= 3z_{11} + 3.7z_{12} + (z_{11} + 0.76z_{12})\theta_1(t) + u_1 && (6.90)\\
y_1 &= z_{12} && (6.91)\\
\dot{z}_{21} &= -1.1z_{21} - 1.47z_{22} + 0.7\cos z_{22} + 0.7\sin(z_{11} + 0.76z_{12}) && (6.92)\\
\dot{z}_{22} &= 3z_{21} + 3.7z_{22} + (z_{21} + 0.7z_{22})\theta_2(t) + u_2 && (6.93)\\
y_2 &= z_{22} && (6.94)
\end{aligned}
$$

For simulation purposes, the controllers are chosen as:

$$
\begin{aligned}
u_i &= -k_i x_i, \qquad i = 1, 2 && (6.95)\\
k_i &= \begin{bmatrix} 9 & 21 \end{bmatrix} && (6.96)
\end{aligned}
$$

By direct computation, it follows that the matrix $W^T + W$ is positive definite. Thus, all the conditions of Theorem 7.1 are satisfied. Therefore the following dynamical system is an asymptotic observer of the system $(6.89) - (6.94)$

$$
\begin{aligned}
\dot{\hat{z}}_{11} &= -1.1\hat{z}_{11} - 1.47y_1 + \sin \hat{z}_{12} + 0.1(\hat{z}_{21} + 0.76\hat{z}_{22})^2 && (6.97)\\
\dot{\hat{z}}_{12} &= 3\hat{z}_{11} + 3.7\hat{z}_{12} + (\hat{z}_{11} + 0.76\hat{z}_{12})\hat{\theta}_1(t) + u_1 + d_1(\cdot) && (6.98)\\
\hat{y}_1 &= \hat{z}_{12} && (6.99)\\
\dot{\hat{z}}_{21} &= -1.1\hat{z}_{21} - 1.47y_2 + 0.7\cos \hat{z}_{22} + 0.7\sin(\hat{z}_{11} + 0.76\hat{z}_{12}) && (6.100)\\
\dot{\hat{z}}_{22} &= 3\hat{z}_{21}3.7\hat{z}_{22} + (\hat{z}_{21} + 0.76\hat{z}_{22})\hat{\theta}_2(t) + u_2 + d_2(\cdot) && (6.101)\\
\hat{y}_2 &= \hat{z}_{22} && (6.102)
\end{aligned}
$$

where the terms $d_1(\cdot)$ and $d_2(\cdot)$ are defined by

$$
\begin{aligned}
d_1(\cdot) &= 9\,\text{sgn}(y_1 - \hat{y}_1) && (6.103)\\
d_2(\cdot) &= 9\,\text{sgn}(y_2 - \hat{y}_2) && (6.104)
\end{aligned}
$$

where ρ_1 and ρ_2 are defined in (6.48) and given by

$$
\rho_1 \geq \beta_1 + \eta \qquad (6.105)
$$

$$
\rho_2 \geq \beta_1 + \eta \qquad (6.106)
$$

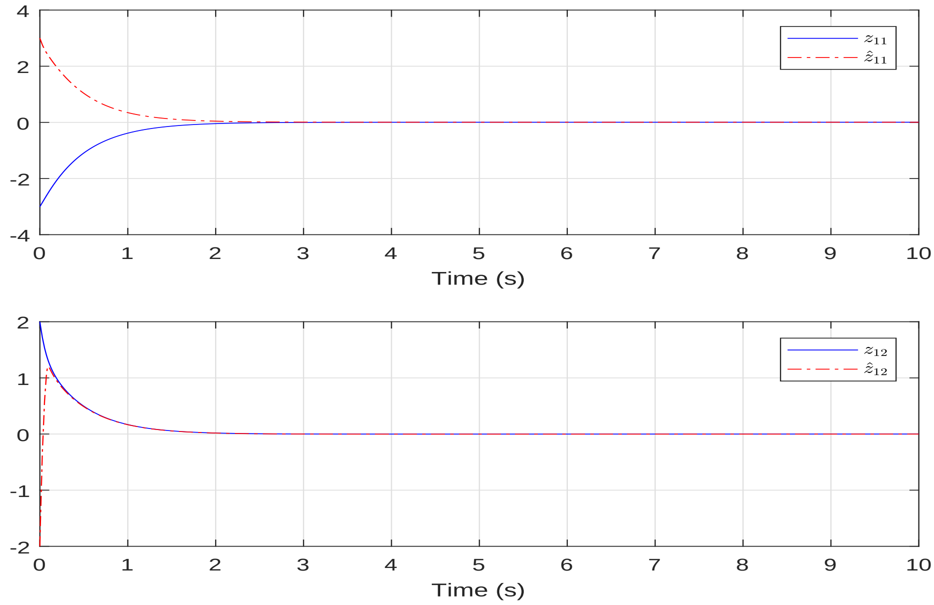

Figure 6.7: Upper: the time response of the z_{11} (solid line) and the time response of the estimated $\hat{z}_{11}$ (dashed line); Bottom: the time response of the z_{12} (solid line) and the time response of the estimated $\hat{z}_{12}$.

where $\beta_1 = 6.5$, and the design parameter η is chosen as $\eta = 2.5$. From (6.25) and (6.26) with $\sigma_1 = \sigma_2 = 1$, the designed adaptive laws are given by

$$\dot{\Gamma}_1 = -[\dot{\hat{y}}_1 - d_1(\cdot)] \tag{6.107}$$

$$\hat{\Theta}_1(t) = \Gamma_1 + y_1 \tag{6.108}$$

$$\dot{\Gamma}_2 = -[\dot{\hat{y}}_2 - d_2(\cdot)] \tag{6.109}$$

$$\hat{\Theta}_2(t) = \Gamma_2 + y_2 \tag{6.110}$$

Simulation in Figures 6.7-6.8 shows the system state variables and their estimations in presence of unknown time varying parameters $\Theta_1(t) = \Theta_2(t) = 0.6t$. The estimation error between the states of the system (6.89)-(6.94 and the states of the observer (6.97)-(6.102) converges to zero globally ultimately bounded. Therefore, $\hat{z}_i = \mathrm{col}(\hat{z}_{i1}, \hat{z}_{i2})$ in (6.97)-(6.102) is an asymptotic estimation of $z_i = \mathrm{col}(z_{i1}, z_{i2})$ in (6.89)-(6.94).

Remark 7.7. It should be noted that the states $\hat{z}_i = \mathrm{col}(\hat{z}_{i1}, \hat{z}_{i2})$ in (6.97)-(6.102) give estimations of the variable $z_i = \mathrm{col}(z_{i1}, z_{i2})$ in (6.89)-(6.94) for $i = 1, 2$. From the analysis in Sections II and III, it is straightforward to see that $\hat{x}_i = (T_i T_{c_i})^{-1} \hat{z}_i$ are estimations of

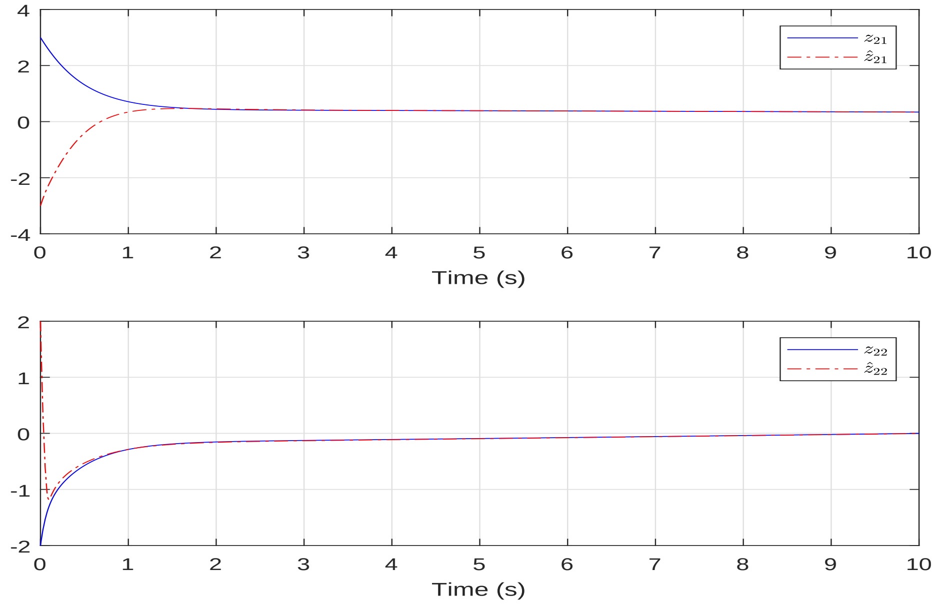

Figure 6.8: Upper: the time response of the z_{21} (solid line) and the time response of the estimated $\hat{z}_{21}$ (dashed line); Bottom: the time response of the z_{22} (solid line) and the time response of the estimated $\hat{z}_{22}$ (dashed line)

the states $x_i = [x_{i1} \;\; x_{i2}]^T$ of the system (6.51)-(6.54) with a new matrix A_i for $i = 1, 2$ defined in (6.85) where T_{c_i} and T_i are defined in (6.55) and (6.88) respectively for $i = 1, 2$.

6.5. Conclusion

In this chapter, an adaptive sliding mode observer for a class of nonlinear large scale interconnected systems with unknown TVPs has been proposed based on Lyapunov direct method. Although bounds on the unknown TVPs are not required, the rate of changes of these parameters are bounded. The technique that used in this chapter is combined with sliding mode and adaptive techniques, which is developed to estimate the states of the system in presence of unknown TVPs and to guarantee the ultimate boundedness of the estimation error of the designed observer. Two simulation examples with different parameters are presented to show the method used is effective and to show that the obtained results are applicable to wide range of systems.

It should be noted that there are some differences between the work in chapter 5 and the work in chapter 6 in: the system structure, the technique used to design the observers, and the dynamics of unknown TVPs.

The system structure, the term of the unknown TVPs considered in the chapter 5 is presented along side with known distribution matrix, while in chpter 6 is considered along side with known function.

The technique used, the technique that used in chapter 6 is Sliding Mode Control technique. This technique needs to design a sliding surface where the sliding motion will take place. However, this is not the case with Variable Structure Control technique where no sliding motion is needed.

The dynamics of unknown TVPs, It is assumed in chapter 5 that the unknown TVPs vary within a specific range, and the nominal value is not required to be known, while in the chapter 6, it is assumed that the unknown TVPs change with time, but the rate of change is required to be bonded.

CHAPTER. 7

CONCLUSIONS AND FUTURE WORK

7.1. SUMMARY AND CONCLUSIONS

In this thesis, a survey of the background and the history of observer and adaptive observer for nonlinear interconnected systems have been presented. System structures have been considered to reduce the conservatism and to guarantee the obtained results to be applicable to a wide range of real physical systems. Novel approaches have been developed to estimate the system states, and to estimate state variables and unknown time varying parameters of the system.

The fundamental knowledge and the basic concept of Observability, Luenburger Observer, Sliding Mode Observer and Adaptive tools have been given in Chapter 3 after outline of mathematical preliminaries and basic stability theories in Chapter 2.

In Chapter 4, a novel approach to design an observer for a class of nonlinear large scale interconnected systems with uniform relative degree has been developed. An asymptotic observer has been proposed for nonlinear interconnected systems with uncertainties using the Lyapunov approach together with a geometric transformation. It is not required that either the isolated nominal subsystems or the interconnections are linearisable. Robustness to uncertainties is enhanced by using the system structure and the structure of the uncertainties within the design framework. The developed results are applicable to a wide class of nonlinear interconnected systems. A simulation example and a case study have been given to show the effectiveness of the obtained results. A robust sliding mode observer has been designed for a class of nonlinear interconnected systems with uncertainties in Chapter 5. The known nonlinear interconnections and uncertain nonlinear interconnections have been dealt with separately to reduce the effects of the interconnections without introducing unnecessary conservatism. A set of sufficient conditions has been provided such that the error dynamics are asymptotically stable if the structure of the uncertainties is known. All the bounds on the uncertainties involved are nonlinear and are employed in the observer design to reject/reduce the effect of uncertainties. An ultimately bounded sliding mode observer is proposed to estimate the states of the interconnected system if the structure of the uncertainties is not available. A case study relating to a multimachine power system has been presented to demonstrate the proposed approach.

Chapters 6-7 focus on interconnected systems that have unknown time varying parameters as well as inaccessible system states. More specifically, in Chapter 6, an adaptive observer design scheme has been proposed for a class of nonlinear large scale interconnected systems with unknown time varying parameters. The unknown parameters vary within a given range. A set of sufficient conditions has been developed to guarantee that the observation error system with the proposed adaptive law is uniformly ultimately bounded. The states of the designed observer are asymptotically convergent to the original system states. Therefore, from the state estimation point of view, the designed observers are asymptotic observers. Case study on a coupled inverted pendulum system shows the practicability of the developed observer scheme for nonlinear interconnected systems. In Chapter 7, an adaptive sliding mode observer for a class of nonlinear large scale interconnected systems with unknown time varying parameters has been proposed based on Lyapunov direct method. Although bounds on the unknown time varying parameters are not required, the rate of changes of these parameters are required to be bounded. The technique used in this chapter is a harmonious combination of sliding mode techniques and adaptive techniques, which has been developed to estimate the system states in presence of unknown time varying parameters and to guarantee the ultimate boundedness of the estimation error of the designed observer. Simulation examples with different parameters are presented to show that the developed method is effective and also to show that the obtained results are applicable to a wide range of interconnected systems.

7.2. Ideas for Future Research

There are some possible interesting ideas for further research.

Although it has been shown that the observers effectively used to estimate the inaccessible states for nonlinear interconnected systems, how to design observers for systems with unknown nonlinear interconnection functions are still challenging. As a possible solution, it might use bounds on these unknown nonlinear functions and if the bounds are known then based on Lipschitz condition, observers may be designed. However, to design an asymptotic observer is a great challenge in this case. Moreover, many practical systems include time varying parameters, it is still an open problem to design adaptive observers for interconnected systems that have unknown time varying parameters if bounds on the variation of the parameters are not available.

Nowadays, the fault diagnosis area of research including fault detection, fault isolation, and fault identification, is a very interesting topic due to its importance for system safety and reliability. It becomes interesting to use sliding mode and adaptive techniques in fault detection for interconnected systems. In addition, it is well known that Supervisory Control and Data Acquisition (SCADA) systems play a curial role in modern industrial control systems. SCADA systems collect data from remote terminal units installed in an industrial field and send them to the central control room. SCADA systems are vulnerable to cyber attacks and as a result of these attacks, very serious results are expected on the whole control systems. Recently, this issue attracts a huge amount of attention and the main focus is to secure critical infrastructures that use SCADA systems in face of cyber and cyber-physical attacks by design fault detection and reconstruction schemes. This is a very interesting area to research in the future.

Appendices

APPENDIX. A

MATHEMATICAL PRELIMINARIES

In this chapter, some fundamental mathematical definitions and lemmas are given to outline the mathematical terms and results that will be used for analysis and design in the thesis.

A.1. MATHEMATICAL REVIEW

A.1.1. EUCLIDEAN SPACE

The set of all n-dimensional vectors $x = [x_1, \cdots, x_n]^T$, where $x_1, \cdots, x_n$ are real numbers, defines the n-dimensional Euclidean space denoted by R^n. The one-dimensional Euclidean space consists of all real numbers and is denoted by R. Vectors in R^n can be added by adding their corresponding component by the scalar. They can be multiplied by a scalar by multiplying each component by the scalar. The inner product of two vectors x and y is defined by $x^T y = \sum_{i=1}^{n} x_i y_i$, where $x = [x_1, \cdots, x_n]^T$ and $y = [y_1, \cdots, y_n]^T$.

A.1.2. Vector and Matrix Norms

The norm $\|x\|$ of a vector x is a real valued function with the following properties

- $\|x\| \geq 0$ for all $x \in R^n$, with $\|x\| = 0$ if and only if $x = 0$.
- $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in R^n$.
- $\|\alpha x\| = |\alpha| \; \|x\|$, for all $\alpha \in R$ and $x \in R^n$.

For a vector $x = [x_1, \cdots, x_n]^T$, its p-norm can be defined by

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \qquad 1 \leq p \leq \infty \tag{A.1}$$

and

$$\|x\|_\infty = \text{max}_i |x_i| \tag{A.2}$$

The Euclidean norm is defined by

$$\|x\|_2 = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}} = (x^T x)^{\frac{1}{2}} \tag{A.3}$$

A.1.3. Differentiable Functions

A function $f : R \longrightarrow R$ is said to be differentiable at $x \in R$ if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \tag{A.4}$$

exists. The limit $f'(x)$ is called the derivative of f at x. A function $f : R^n \longrightarrow R^m$ is said to be *continuously differentiable* at a point x_0 if the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and continuous at $x_0 \in R^n$ for $1 \leq i \leq m, 1 \leq j \leq n$. A function f is continuously differentiable on a set S if it is continuously differentiable at every point of S.

A.2. Lyapunov Stability

Throughout the development of entire system theory, stability theory plays a significant role in analysis and design. In this thesis, the stability of equilibrium points, which is usually characterised in the sense of Lyapunov named after a Russian mathematician and engineer who laid the foundation of the enter theory [5], is recurrently discussed. Therefore, some main results of stability of equilibrium points are provided in this section.

Consider the autonomous system

$$\dot{x} = f(x) \tag{A.5}$$

where $x \in D \subset R^n$, and $f : D \longrightarrow R^n$ is a locally Lipschitz map in domain D.

Definition 2.2.1 (Stability and Asymptotically Stability [5]). *An equilibrium point $x = 0$ of system (A.5) is said to be*

- *stable if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that*

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon, \quad \forall\, t \geq 0 \tag{A.6}$$

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| < \delta \Longrightarrow \lim_{x \to \infty} x(t) = 0 \tag{A.7}$$

For non-autonomous systems

$$\dot{x} = f(t, x) \tag{A.8}$$

firstly it is required to introduce some fundamental definitions. A system (A.8) satisfying the inequality

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \tag{A.9}$$

for all (t, x) and (t, y) in some neighbourhood of (t_0, x_0), is said to be *Lipschitz* in x, and the positive L is called a *Lipschitz constant*. Also the words *locally Lipschitz* and *globally Lipschitz* have been widely used to indicate the domain over which the Lipschitz condition holds. A function $f(x)$ is said to be *locally Lipschitz* on a domain (open and connected set) $D \subset R^n$ if each point in D has a neighbourhood D_0 such that f satisfies Lipschitz condition (A.9) for all points in D_0 with some Lipschitz constant L_0. We say that f is Lipschitz on a set W if it satisfies (A.9) for all points in W, with the same Lipschitz constant L. A function $f(x)$ is said to be *globally Lipschitz* if it is Lipschitz on R^n.

Now consider the system in (A.8), where $f\,:\,[0,\infty)\times D\longrightarrow R^n$ is piecewise continuous in t and locally Lipschitz in x on $f\,:\,[0,\infty)\times D\subset R^n$ is a domain that contains the origin $x=0$ is an equilibrium point for (A.8) at $t=0$ if

$$f(t,0)=0,\quad \forall\, t\geq 0 \tag{A.10}$$

Definition 2.2.2 (Stability and Asymptotically Stability [5]). *An equilibrium point $x=0$ of system (A.8) is said to be*

- *stable if, for each $\epsilon>0$, there exists $\delta=\delta(\epsilon,t_0)>0$ such that*

$$\|x(t_0)\|<\delta\Longrightarrow\|x(t)\|<\epsilon,\quad\forall\, t\geq t_0\geq 0 \tag{A.11}$$

- *uniformly stable if, for each $\epsilon>0$, there is $\delta=\delta(\epsilon)>0$, independent of t_0 such that (A.11) is satisfied .*

- *unstable if it is not stable.*

- *asymptotically stable if it is stable and there is a positive constant $c=c(t_0)$ such that $x(t)\longrightarrow 0$ as $t\longrightarrow\infty$, for all $\|x(t_0)\|<c$.*

- *uniformly asymptotically stable if it is uniformly stable and there is a positive constant c independent of t_0 such that for all $\|x(t_0)\|<c, x(t)\longrightarrow 0$ as $t\longrightarrow\infty$, uniformly in t_0 ; that is, for each $\eta>0$, there is $T=T(\eta)$ such that*

$$\|x(t)\|<\eta,\;\;\forall\, t\geq t_0+T(\eta),\;\;\forall\,\|x(t)\|<c \tag{A.12}$$

- *globally uniformly asymptotically stable if it is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty}\delta(\epsilon)=\infty$, and for each pair of positive numbers η and c, there is $T=T(\eta,c)>0$ such that*

$$\|x(t)\|<\eta,\;\;\forall\, t\geq t_0+T(\eta,c),\;\;\forall\,\|x(t)\|<c \tag{A.13}$$

Then, without loss of generality, consider a linear autonomous system

$$\dot{x} = Ax \tag{A.14}$$

where $x \in R^n$.

Lemma 2.2.1 (Stability of Linear System [5]). *An equilibrium point x_0 of system (A.14) is said to be stable if and only if all eigenvalues λ_i of A satisfy Re$\lambda_i \leq 0$ and for every eigenvalue with Re$\lambda_i = 0$ and algebraic multiplicity $q_i \geq 0$, rank$(A - \lambda_i I) = n - q_i$, where n is the dimension of x. The equilibrium point is said to be asymptotically stable if and only if all eigenvalues of A satisfy Re$\lambda_i < 0$.*

Definition 2.2.3 (Hurwitz Matrix and Lyapunov Equation [5]). A matrix $A \in R^{n\times n}$ of system (A.14) is said to be *Hurwitz* if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P satisfies

$$PA + A^{\mathrm{T}}P = -Q \tag{A.15}$$

Moreover, if A is Hurwitz, then P is the unique solution of (A.15) which is called the *Lyapunov equation.*

For a nonlinear system (A.5), the following theorem is usually used to determine the stability of the system.

Lemma 2.2.2 (Lyapunov Stability Theorem [5]). *Let $x = 0$ be an equilibrium point for system (A.5) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : D \longrightarrow R$ be a continuously differentiable function such that*

$$\begin{aligned} V(0) &= 0 \text{ and } V(x) > 0 \;\; in \;\; D - 0 && \text{(A.16)} \\ \dot{V}(x) &\leq 0 \text{ in } D && \text{(A.17)} \end{aligned}$$

then, $x = 0$ is stable. Moreover, if

$$\begin{aligned} V(0) &= 0 \text{ and } V(x) > 0 \;\; in \;\; D - 0 && \text{(A.18)} \\ \dot{V}(x) &< 0 \text{ in } D && \text{(A.19)} \end{aligned}$$

then, $x = 0$ is asymptotically stable.

A function $V(x)$ satisfying condition (A.16), that is $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ is said to be *positive definite* . If $V(x) \geq 0$ for $x \neq 0$ is said to be *positive* *semidefinite*. With this terminology, roughly speaking, *the origin is stable if there is a continuously differentiable positive definite function $V(x)$ such that $\dot{V}(x)$ is negative semidefinite, and it is asymptotically stable if $\dot{V}(x)$ is negative definite.*

Whether a function is positive or negative can be easily checked if the function is in a quadratic form as

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j \tag{A.20}$$

where $P = (p_{ij})_{n\times n}$ is a real symmetric matrix. In this case, $V(x)$ is positive definite (positive semidefinite) if and only if all the eigenvalues of P are positive (nonnegative).

Definition 2.2.4 (Class $\mathcal{K}$ functions [5]). A continuous function $\alpha : [0, a) \longrightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \longrightarrow \infty$ as $r \longrightarrow \infty$.

Definition 2.2.5 (Class $\mathcal{KL}$ functions [5]). A continuous function $\beta : [0, a) \times [0, \infty) \longrightarrow [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed $s \in [0, \infty)$, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respective to s and $\beta(r, s) \longrightarrow \infty$ as $s \longrightarrow \infty$.

Lemma 2.2.3 (Expanded Lyapunov Asymptotically Stability Theorem [5]). *Let $x = 0$ be an equilibrium point of system (A.8) and $D \subset R^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \longrightarrow R$ be a continuously differentiable function such that*

$$W_1(x) \le V(t, x) \le W_2(x) \tag{A.21}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x) \tag{A.22}$$

$\forall t \ge 0$ and $\forall x \in D$, where $W_i(x)$ for $i = 1, 2, 3$ are continuous positive definite functions on D. Then, $x = 0$ is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \le r\} \subset D$ and $c < min_{\|x\|=r} W_1(x)$, then every trajectory staring in $\{x \in B_r | W_2(x) \le c\}$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall t \ge t_0 \ge 0 \tag{A.23}$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then $x = 0$ is globally uniformly asymptoticly stable.

Lemma 2.2.4 (Converse Lyapunov Function [5]). *Let $x = 0$ be an equilibrium point of system*

$$\dot{x} = f(t, x) \tag{A.24}$$

where $f : [0, \infty) \times D \longrightarrow R^n$ is continuously differentiable, $D = \{x \in R^n | \|x\| < r\}$, and the Jacobian matrix $[\partial f/\partial x]$ is bounded on D, uniformly in t. Let k, λ and r_0 be postive constants with $r_0 < r/k$. Let $D_0 = \{x \in R^n | \|x\| < r_0\}$. Assume that the trajectories of the system satisfy

$$\|x(t)\| \leq k\|x(t_0)\| \exp^{-\lambda(t-t_0)}, \quad \forall x(t_0) \in D, \quad \forall t \geq \ t_0 \geq 0 \tag{A.25}$$

Then, there exists a function $V : [0, \infty) \times D_0 \longrightarrow R$ such that

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{A.26}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial t} f(t, x) \leq -\alpha_3(\|x\|) \tag{A.27}$$

$$\|\frac{\partial V}{\partial x}\| \leq \alpha_4(\|x\|) \tag{A.28}$$

where $\alpha_i(\cdot)$ for $i = 1, 2, 3, 4$ are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, the function V can be chosen independent of t.

Lemma 2.2.5 (Lyapunov-Like Theorem [5]). *Let $D \subset R^n$ be a domain that contains the origin and $V : [0, \infty) \times D \longrightarrow R$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \ \leq \ V(t, x) \leq \alpha_2(\|x\|) \tag{A.29}$$

$$\frac{\partial V}{\partial t} \ + \ \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \qquad \forall \|x\| \geq \mu \geq 0 \tag{A.30}$$

$\forall\, t \geq 0$ and $\forall\, x \in D$, where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Take $r > 0$ such that $B_r \subset D$ and suppose that

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \tag{A.31}$$

Then, there exists a class $\mathcal{KL}$ function β and for every initial state $x(t_0)$, satisfying $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T \geq 0$ (dependent on $x(t_0)$ and μ) such that the solution of $\dot{x} = f(t, x)$ satisfies

$$\|x(t)\| \ \leq \ \beta(\|x(t_0)\|, t - t_0), \qquad \forall\, t_0 \leq t \leq t_0 + T \tag{A.32}$$

$$\|x(t)\| \ \leq \ \alpha_2^{-1}(\alpha_1(\mu)), \qquad \forall\, t \geq t_0 + T \tag{A.33}$$

Moreover, if $D = R^n$ and α_1 belongs to class $\mathcal{K}_\infty$, then (A.32) and (A.33) hold for any initial state $x(t_0)$, with no restriction on how large μ is.

Inequalities (A.32) and (A.33) show that $x(t)$ is uniformly bounded for all $t \geq t_0$ and uniformly ultimately bounded with the ultimate bound $\alpha_2^{-1}(\alpha_1(\mu))$. The ultimate bound is a class $\mathcal{K}$ function of μ ; hence, from the fact that the functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathcal{K}$ functions, it follows that the smaller the value of μ, the smaller the ultimate bound. As $\mu \longrightarrow 0$, the ultimate bound approaches zero.

Lemma 2.2.6 (LaSalle's Theorem [5]). *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to (A.5). Let $V : D \to R$ be a continuously differentiable function such that $\dot{V} \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V} = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

Proof: Let $x(t)$ be a solution of (A.5) starting in Ω. Since $\dot{V} \leq 0$ in Ω, $V(x(t))$ is a decreasing function of t. Since $V(x)$ is continuous on the compact set Ω, it is bounded from below on Ω. Therefore, $V(x(t))$ has a limit a as $t \to \infty$, where a is a positive value. Note also that the positive limit set L^+ is in Ω because Ω is a closed set. For any $p \in L^+$, there is a seqence t_n with $t_n \to \infty$ and $x(t_n) \to p$ as $n \to \infty$. By continuity of $V(x), V(p) = \lim_{n\to\infty} V(x(t_n)) = a$. Hence, $V(x) = a$ on L^+. Since (by Lemma 4.1 in [5]) $L^+$is an invariant set, $\dot{V}(x = 0)$ on L^+. Thus,

$$L^+ \subset M \subset E \subset \Omega \tag{A.34}$$

Since $x(t)$ is bounded, $x(t)$ approaches L^+ as $t \to \infty$ (by Lemma 4.1 in [5]). Hence, $x(t)$ approaches M as $t \to \infty$.

Unlike the Lyapunov's theorem, LaSalle's theorem does not require the function $V(x)$ to be positive definite.

A.3. ELEMENTARY THEORY OF NONLINEAR SINGLE INPUT SINGLE OUTPUT SYSTEMS

Definition 2.3.1 (Relative Degree [66]).

The single input single output nonlinear system

$$\dot{x} = f(x) + g(x)u \tag{A.35}$$

$$y = h(x) \tag{A.36}$$

is said to have *relative degree* r at a ponit x° if

- $L_g L_f^k h(x) = 0$ for all x in a neighborhood of x° and all $k < r - 1$.
- $L_g L_f^{r-1} h(x^\circ) = 0$

Assume that $x(t^\circ) = x^\circ$ and suppose we wish to calculate the value of the output $y(t)$ and its k-th derivatives $y^{(k)}(t)$ with respect to time , for $k = 1, 2, \cdots$ at $t = t^\circ$. The process can be explained as follows:

$$y(t^\circ) = h(x(t^\circ)) = h(x^\circ) \tag{A.37}$$

$$y^{(1)}(t) = \frac{\partial h}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}(f(x) + g(x)u(t) \tag{A.38}$$

$$= L_f h(x(t)) + L_g h(x(t))u(t) \tag{A.39}$$

If the relative degree r is larger than 1, for all t such that $x(t)$ is near x°, i.e. for all t near t°, we have $L_g h(x(t)) = 0$ and therefore

$$y^{(1)}(t) = L_f h(x(t)) \tag{A.40}$$

This yields

$$y^{(2)}(t) = \frac{\partial L_f h}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial L_f h}{\partial x}(f(x) + g(x)u(t)) \tag{A.41}$$

$$= L_f^2 h(x(t)) + L_g L_f h(x(t))u(t) \tag{A.42}$$

Again, if the relative degree r is larger than 2, for all t near t°, then $L_g L_f h(x(t)) = 0$ and

$$y^{(2)}(t) = L_f^2 h(x(t)) \tag{A.43}$$

Continuing in this way, it follows that

$$y^{(k)}(t) = L_f^k h(x(t)), \quad \text{for all } k < r \text{ and all } t \text{ near } t^\circ \tag{A.44}$$

$$y^{(r)}(t) = L_f^r h(x^\circ) + L_g L_f^{r-1} h(x^\circ)u(t^\circ) \tag{A.45}$$

Thus, *the relative degree* r is exactly equal to the number of times one has to differentiate the output $y(t)$ at time $t = t^\circ$ in order to have the value $u(t^\circ)$ of the input explicitly appearing.

Proposition 2.3.1 *Suppose the system has relative degree* r *at* x° *(*x° *is a point where* $L_g L_f^{r-1} h(x^\circ) \neq 0$*). Then* $r \leq n$*. Set*

$$\begin{aligned}
\phi_1(x) &= h(x) \\
\phi_2(x) &= L_f h(x) \\
&\vdots \\
\phi_r(x) &= L_f^{r-1} h(x)
\end{aligned}$$

If r *is strictly less than* n*, it is always possible to find* $n-r$ *more functions* $\phi_{r+1}(x), \cdots, \phi_n(x)$ *such that the mapping*

$$\phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_n(x) \end{pmatrix} \tag{A.46}$$

has the Jacobian matrix which is nonsingular at x° *and therefore qualifies as a local coordinates transformation in a neighbourhood of* x°*. The value at* x° *of these additional functions can be find in different ways. However, it is always possible to choose* $\phi_{r+1}(x), \cdots, \phi_n(x)$ *in such a way that*

$$L_g \phi_i(x) = 0, \;\; \textit{for all } r+1 \leq i \leq n, \textit{ and all } x \textit{ around } x^\circ \tag{A.47}$$

Proof. By definition of relative degree, the vector $g(x^\circ)$ is nonzero, and thus the distribution $G = span\{g\}$ is nonsingular around x°. Being 1-dimensional this distribution is also involutive. Therefore, by the Frobenius' Theorem, we deduce the existence of $n-1$ real-valued functions, $\lambda_1(x), \cdots, \lambda_{n-1}(x)$, defined in a neighbourhood of x°, such that

$$\text{span}\{d\lambda_1, \cdots, d\lambda_{n-1}\} = G^\perp \tag{A.48}$$

It is easy to show that

$$\dim(G^\perp + \text{span}\{dh, dL_f h, \cdots, dL_f^{r-1} h\}) = n \tag{A.49}$$

at x°. Suppose this is false. Then

$$G(x^\circ) \cap (\text{span}\{dh, dL_f h, \cdots, dL_f^{r-1}h\})^\perp(x^\circ) \neq \{0\} \tag{A.50}$$

i.e. the vector $g(x^\circ)$ annihilates all the covectors in

$$\text{span}\{dh, dL_f h, \cdots, dL_f^{r-1}h\}(x^\circ) \tag{A.51}$$

But this is a contradiction, because by definition $\langle dL_f^{r-1}h(x^\circ), g(x^\circ)\rangle$ is nonzero.

From (A.48), (A.49) and from the fact that span$\{dh, dL_f h, \cdots, dL_f^{r-1}h\}$ has dimension r, by Lemma 4.1.1 in [66], it is concluded that in the set $\{\lambda_1, \cdots, \lambda_{n-1}\}$ it is possible to find $n-r$ functions: without loss of generality, $\lambda_1, \cdots, \lambda_{n-1}$ with the property that the n differentials $dh, dL_f h, \cdots, dL_f^{r-1}h, d\lambda_1, \cdots, d\lambda_{n-r}$ are linearly independent at x°. Since by construction the functions $\lambda_1, \cdots, \lambda_{n-r}$ are such that

$$\langle dL_f^{r-1}h(x), g(x)\rangle = L_g\lambda_i(x) = 0 \quad \text{for all } x \text{ near } x^\circ \text{and all } 1 \leq i \leq n-r \tag{A.52}$$

This establishes the required result. Note that any other set of functions of the form $\lambda'_i(x) = \lambda_i(x) + c_i$ where c_i is a constant, satisfies the same conditions, thus showing that the value of these functions at the point x° can be chosen arbitrarily.

The description of the system in the new coordinates $z_i = \phi_i(x), 1 \leq i \leq n$, is found very easily. Looking at the calculations already carried out at the begining, it follows that for $z_1, \cdots, z_r$

$$\begin{aligned}
\frac{dz_1}{dt} &= \frac{\partial \phi_1}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} = L_f h(x(t)) = \phi_2(x(t)) = z_2(t) \\
&\vdots \\
\frac{dz_{r-1}}{dt} &= \frac{\partial \phi_{r-1}}{\partial x}\frac{dx}{dt} = \frac{\partial (L_f^{r-2}h)}{\partial x}\frac{dx}{dt} = L_f^{r-1}h(x(t)) = \phi_r(x(t)) = z_r(t)
\end{aligned}$$

For z_r

$$\frac{dz_r}{dt} = L_f^r h(x(t)) + L_g L_f^{r-1}h(x(t))u(t) \tag{A.53}$$

On the right hand side of this equation, it needs to replace $x(t)$ with its expression as a function of $z(t)$ i.e. $x(t) = \phi^{-1}(z(t))$. Thus, setting

$$\begin{aligned}
a(z) &= L_g L_f^{r-1}h(\phi^{-1}(z)) \\
b(z) &= L_f^r h(\phi^{-1}(z))
\end{aligned}$$

The equation can be rewritten as

$$\frac{dz_r}{dt} = b(z(t)) + a(z(t))u(t)$$

Note that at the point $z^\circ = \phi(x^\circ), a(z^\circ) \neq 0$ by definition. Thus, the coefficient $a(z)$ is nonzero for all z in a neighborhood of z°.

As far as the other new coordinates are concerned, any special structure can not be expected for the corresponding equations, if nothing else has been specified. However, if $\phi_{r+1}(x), \cdots, \phi_n(x)$ have been chosen in such a way that $L_g\phi_i(x) = 0$, then

$$\frac{dz_i}{dt} = \frac{\partial \phi_i}{\partial x}(f(x(t)) + g(x(t))u(t)) = L_f\phi_i(x(t)) + L_g\phi_i(x(t))u(t) = L_f\phi_i(x(t))$$

Setting

$$q_i(z) = L_f\phi_i(\phi^{-1}(z)) \quad \text{for all} r+1 \leq i \leq n$$

The latter can be rewritten as

$$\frac{dz_i}{dt} = q_i(z(t))$$

Thus, in summary, the state-space description of the system in the new coordinates is given as follows

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{r-1} &= z_r \\
\dot{z}_r &= b(z) + a(z)u \\
\dot{z}_{r+1} &= q_{r+1}(z) \\
&\vdots \\
\dot{z}_n &= q_n(z)
\end{aligned} \tag{A.54}$$

In addition to these equations one has to specify how the output of the system is related to the new state variables. From $y = h(x)$, it is immediately seen that

$$y = z_1 \tag{A.55}$$

The following example shows how to find a coordinate transformation $z = T(x)$ such that the system considered in new coordinates can be transformed to system (A.54)

Example: Consider the system

$$\dot{x} = \begin{bmatrix} -x_1 \\ x_1 x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \text{exp}(x_2) \\ 1 \\ 0 \end{bmatrix} u \tag{A.56}$$

$$y = h(x) = x_3 \tag{A.57}$$

By direct calculation,

$$\begin{aligned} \frac{\partial h}{\partial x} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \ L_g h(x) = 0, \ L_f h(x) = x_2 \\ \frac{\partial (L_f h)}{\partial x} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \ L_g L_f h(x) = 1 \end{aligned} \tag{A.58}$$

In order to find the normal form, let

$$\begin{aligned} z_1 &= \phi_1(x) = h(x) = x_3 \\ z_2 &= \phi_2(x) = L_f h(x) = x_2 \end{aligned} \tag{A.59}$$

Now, it is required to find a function $\phi_3(x)$ such that

$$\frac{\partial \phi_3}{\partial x} g(x) = \frac{\partial \phi_3}{\partial x_1} \text{exp}(x_2) + \frac{\partial \phi_3}{\partial x_2} = 0 \tag{A.60}$$

It is easily to see that the function

$$\phi_3(x) = 1 + x_1 - \text{exp}(x_2) \tag{A.61}$$

satisfies condition (A.60). This and the previous two functions in (A.59) define a transformation $z = \phi$ whose Jacobian matrix

$$\frac{\partial \phi}{\partial x} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -\text{exp}(x_2) & 0 \end{pmatrix} \tag{A.62}$$

is nonsingular for all x. The inverse transformation is given by

$$\begin{aligned} x_1 &= -1 + z_3 + \text{exp}(z_2) \\ x_2 &= z_2 \\ x_3 &= z_1 \end{aligned}$$

Note also that $\phi(0) = 0$. In the new coordinates the system is described by

$$\begin{aligned}\dot{z}_1 &= z_2 \\ \dot{z}_2 &= (-1 + z_3 + \exp(z_2))z_2 + u \\ \dot{z}_3 &= (1 - z_3 - \exp(z_2))(1 + z_2\exp(z_2))\end{aligned}$$

These equations are globally valid because the transformation we considered was a global coordinates transformation.

APPENDIX. B

APPENDIX: EXPLANATION EXAMPLE OF SOFTWARE USED IN THE THESIS

MATLAB/SIMULINK software are mainly used to the design observers/adaptive observers for nonlinear interconnected systems which are considered in this thesis. To illustrate how the simulation was carried out, the example which is proposed in Chapter 7 is used to show the Simulink blocks as an example of all case studies and numerical examples that presented in the thesis. The figure B.1 shows the structure of the whole system which consists of the actual system and the observer designed. More details in the figure B.2 shows the subsystems and their observers. More detailed information about the Simulink block of the whole system is shown in the figure B.3.

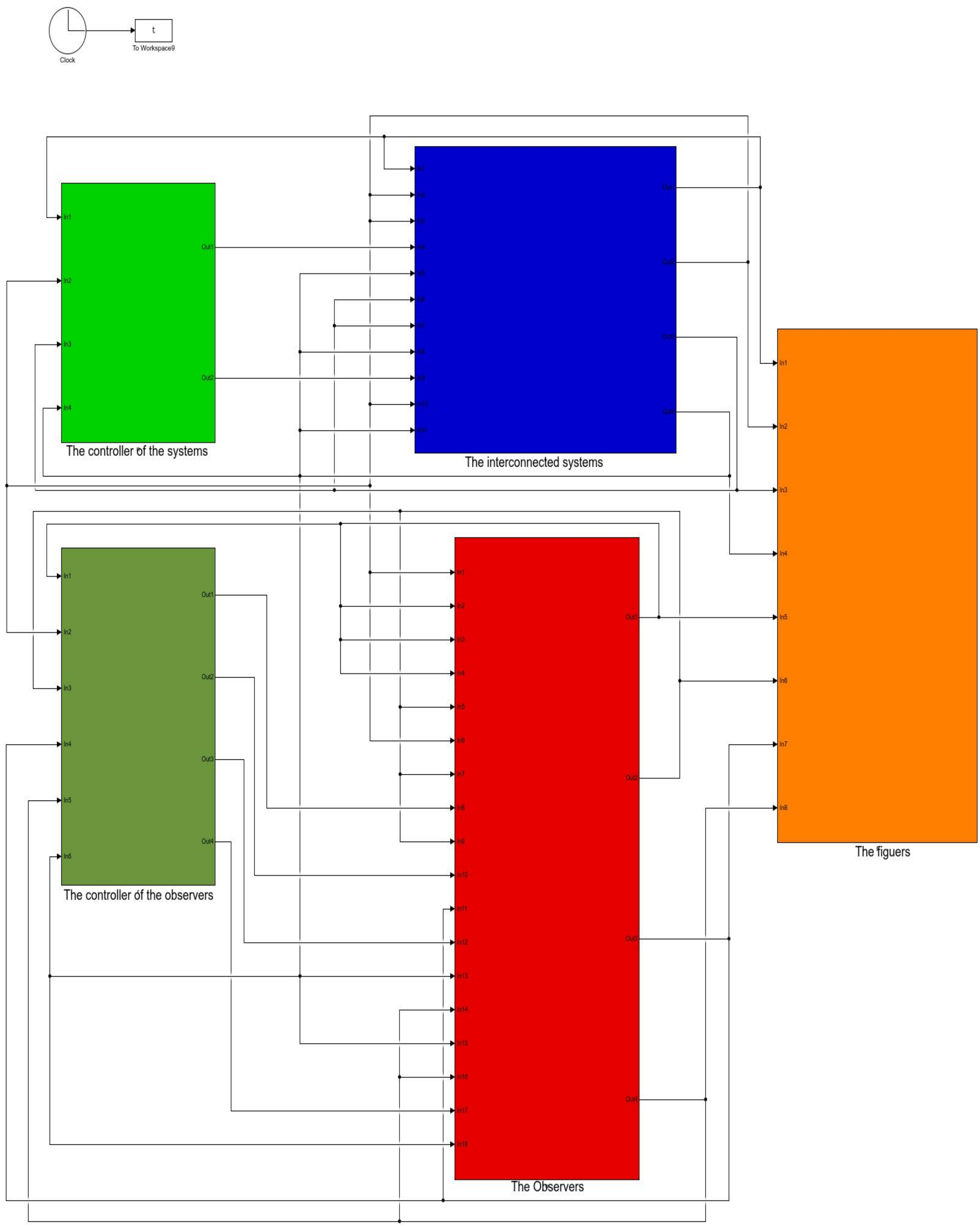

Figure B.1: The whole system representing the general interconnected system and it's observer

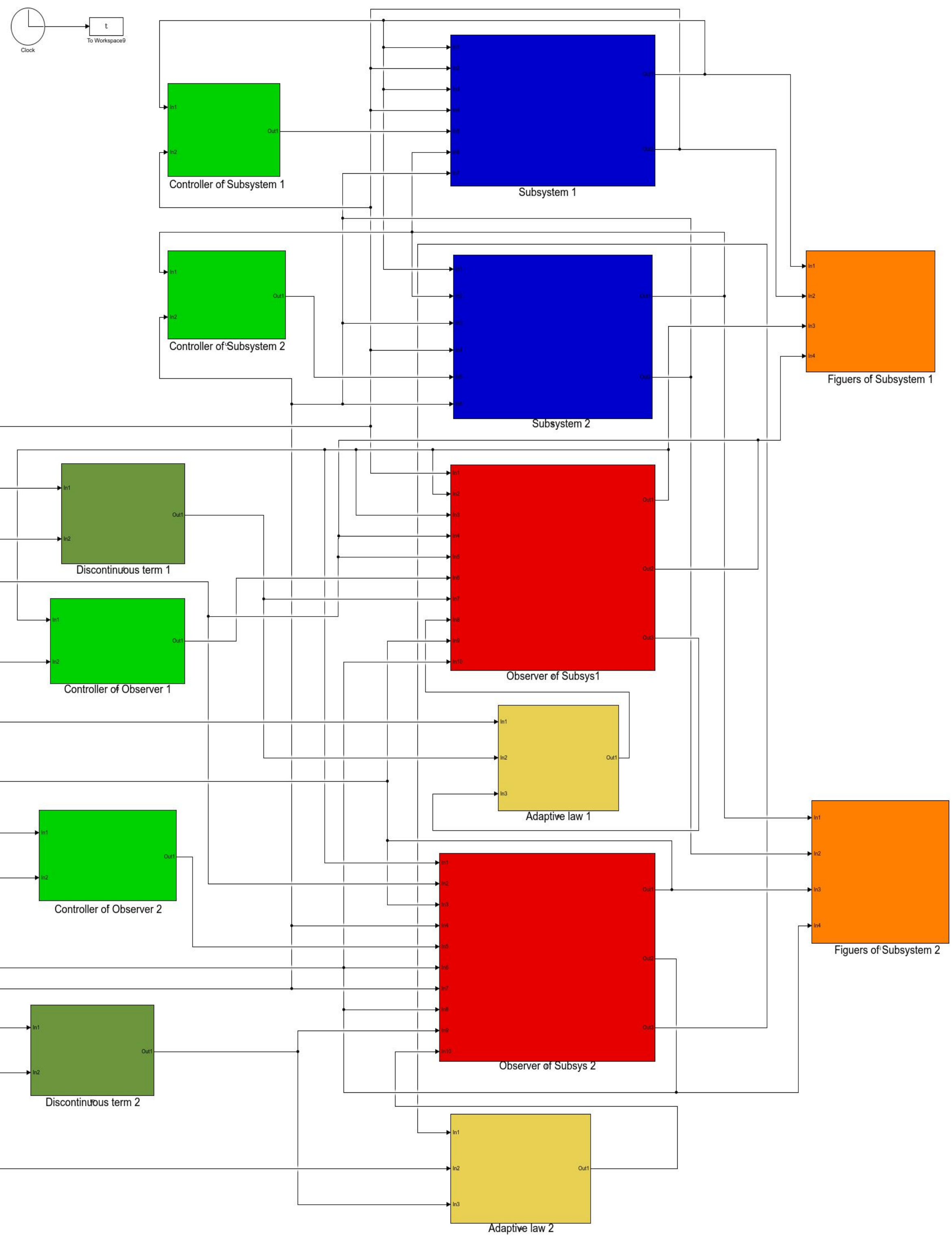

Figure B.2: The whole system representing subsystems and their observers

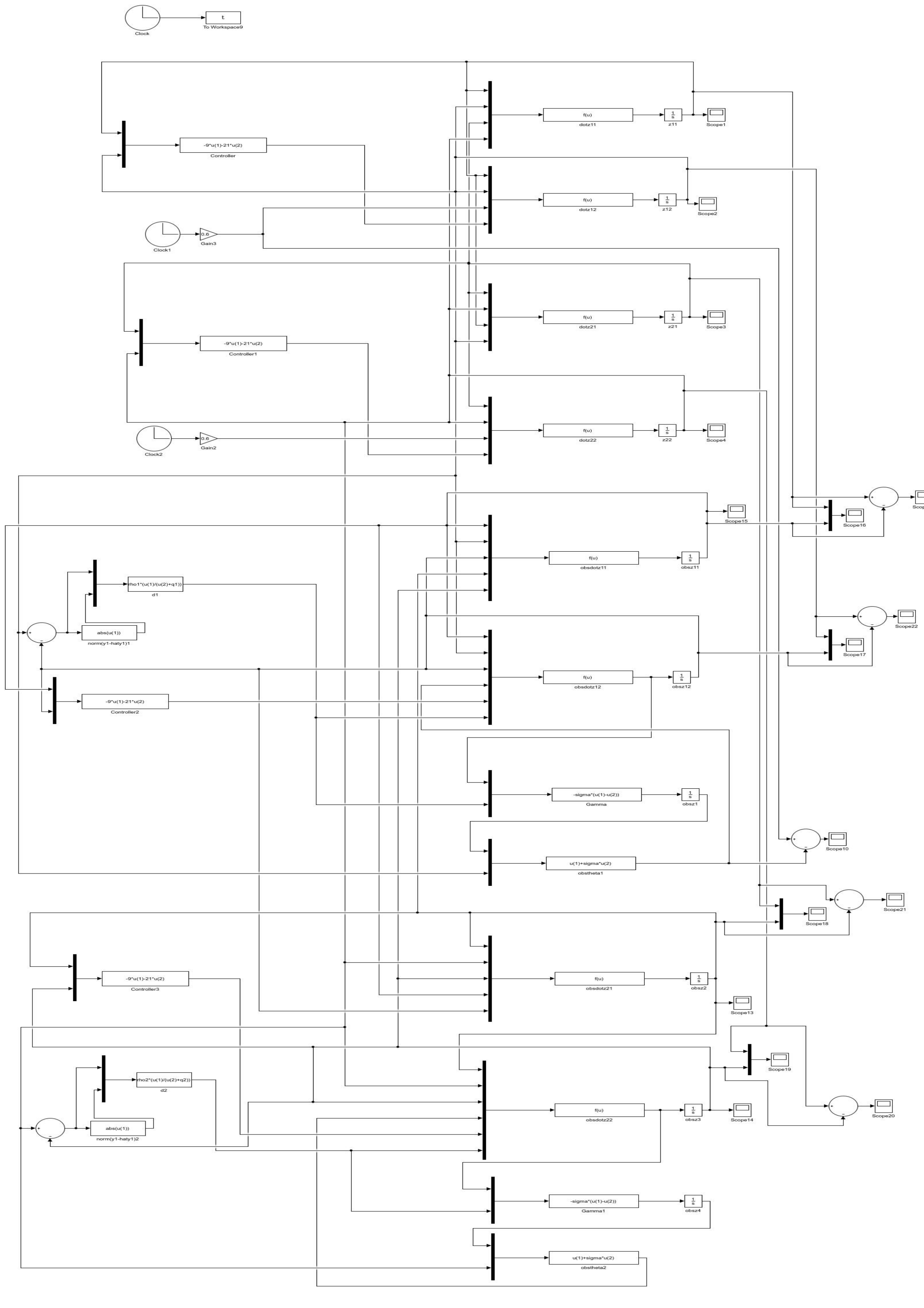

Figure B.3: The interconnected systems in Simulink block diagram

APPENDIX. C

LIST OF PUBLICATION

JOURNAL PAPERS

[1] M. Mohamed, X. G. Yan, S. K. Spurgeon, and B. Jiang, "Robust sliding-mode observers for large-scale systems with application to a multimachine power system," *IET Control Theory & Applications,*, 11(8), 1307-1315, 2016.

[2] M. Mohamed, X. G. Yan, S. K. Spurgeon, and Z. Mao, "Robust Variable Structure Observer Design for Nonlinear Large-Scale Systems with Nonlinear Interconnections," *IMA Journal of Mathematical Control and Information,* Volume 35, Issue 2, 20 , Pages 535-553, 2018.

[3] M. Mohamed, X. G. Yan, Z. Mao, and B. Jiang, "Adaptive Sliding Mode Observer for Nonlinear Interconnected Systems with Time Varying Parameters," *Asian Journal of Control*, 2018. (Accepted)

[4] M. Mohamed, X. G. Yan, Z. Mao, and B. Jiang, "Adaptive Observer Design for Nonlinear Interconnected Systems by the Application of LaSalle's Theorem," *International Journal of Adaptive Control and Signal Processing*, (Submitted)

BOOK CHAPTER

[1] M. Mohamed, X. G. Yan, S. K. Spurgeon and Z. Mao, *Variable structure observers for nonlinear interconnected systems*. In Advances in Variable Structure Systems and Sliding Mode Control Theory and Applications (pp. 195-221), Springer, 2018.

[2] M. Mohamed, X. G. Yan, Z. Mao, B. Jiang and A. Sherif, *Adaptive Observer Design for Nonlinear Interconnected Systems with Applications*. In New Trends in Observer-based Control - Design and Applications, Elsevier (Accepted).

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[1] M. Mohamed, X. G. Yan, S. K. Spurgeon and Z. Mao, "Variable structure observer for a class of nonlinear large-scale interconnected systems with uncertainties," in *Variable Structure Systems (VSS), IEEE 14th International Workshop on,* pp. 385-390, 2016.

[2] M. Mohamed, X. G. Yan, S. K. Spurgeon and B. Jiang, "Robust sliding mode observer design for interconnected systems," in *Decision and Control (CDC), IEEE 55th Conference on*, pp. 6246-6251, 2016.

[3] M. Mohamed, X. G. Yan, Z. Mao, and B. Jiang, "Adaptive Observer Design for a Class of Nonlinear Interconnected Systems with Uncertain Time Varying Parameters", in *the 20th World Congress of the International Federation of Automatic Control*, 9-14, 50(1), 1421-1426, 2017.

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