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The rippling effect of nonlinearities

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Abstract Nonlinearities can lead to unexpected dynamic behaviours in supply chain systems that could then either trigger disruptions or make the response and recovery process more difficult. In this chapter we take a control theoretic perspective to discuss the impact of nonlinearities on the ripple effect. This chapter is particularly relevant for researchers wanting to learn more about the different types of nonlinearities that can be found in supply chain systems, the existing analytical methods to deal with each type of nonlinearity and future scope for research based on the current knowledge in this field.

1 Introduction

As a result of globalisation and increasing competitive pressures, modern supply chains have gone through a leaning and lengthening process [1] and now back to reshoring [2]. Managers have attempted to optimise supply chains by reducing holding inventory, outsourcing noncore activities, cutting the number of suppliers and sourcing globally, forgetting that the world market is an erratic and unpredictable place [3]. In addition to this, current trade restrictions as a result of protectionist political environment emerging in North America and Europe introduce additional

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uncertainty and complexity into supply chains, which are more vulnerable to disruptions than ever before [4].

The resulting complex business environment has increased the importance of handling risks which can emerge from the customers, suppliers, manufacturing processes and control systems [5] and of designing ripple effect mitigation strategies through agile and resilient practices [6]. Increased complexity also means that supply chain researchers can no longer disregard capacity limitations, restrictive policies and other system constraints, i.e. that the real world is nonlinear. Nonlinearities can introduce unexpected behaviour in a system causing instability and uncertainty [7, 8], therefore it is important to understand how control systems can be designed to influence dynamic behaviours and how nonlinearities impact the performance of supply chains.

When looking at supply chain problems, researchers have created a number of production and inventory models to represent the flows of information and material between different supply chain players. There are a number of research streams that deal with such problems, such as Markov demand process, Bayesian approach, moving average or ARIMA process [9], mixed-integer programming, stochastic programming, simulation (via system dynamics, agent-based modelling, discrete-event), graph theory [6] and control theory (via feedback control and optimal control mechanism) [6, 9]. The latter approach concerns determining transient responses and systems stability, i.e. understanding and controlling supply chain dynamics. These dynamics are normally driven by the application of different control system policies and can be considered as a source of disruption depending on the control system design [10]. Moreover, a number frameworks exist for tackling the ripple effect in the supply chain dynamics, control and disruption management domain [11].

In this chapter we will discuss the impact of nonlinearities on the ripple effect from a control structure perspective, by revisiting the literature on nonlinear control theory application in supply chain management. As [12] pointed out ‘useful tools for quantitative analysis of control and systems theory for a wide supply chain management research community remain undiscovered’. This chapter reviews new research techniques and recent progress in the analytical understanding of how nonlinearities influence dynamic behaviours and affect the performance of the supply chains. We start by introducing different types of nonlinearities and their typical effect on system transient output response. Then, we suggest existing methods to analyse each type of nonlinearity and detail a selected number of mathematical approaches that can be used alongside simulation methods to explore the hidden dynamics caused by such nonlinearities. Next, we discuss the applications of these methods and compile key findings on the rippling effect of nonlinearities. Finally, the chapter concludes with a future research agenda.

2 Types of nonlinearities

A nonlinear system is one whose performance does not obey the principal of superposition. This means that the output of a nonlinear system is not directly proportional to the input and the variables to be solved cannot be expressed as a linear combination of the independent parts [13]. In supply chain systems, nonlinearities can naturally occur due to the existence of physical and economic constraints, for instance fixed and variable capacity constraints in the manufacturing and shipping processes, resource availability, variable delays and variable control parameters, trade and infrastructure constraints [8].

Since the variety of possible nonlinearities in supply chain systems is extremely wide, it may be worthwhile to classify them into different categories, for which appropriate analytical methods will be suggested. The first research found on categorisation of nonlinearities in business system dynamics research was done by Mohapatra [14] who identified three types of nonlinearities: limiting functions, table functions and product operators. He also recommends some techniques to deal with such properties, including the omission of redundant functions, linearisation through averaging, best-fit line approximations and small perturbation theory. In the control systems literature, nonlinearities are more extensively classified as inherent or intentional, continuous or discontinuous and single- or multiple-valued [15, 16], as in Figure 1.

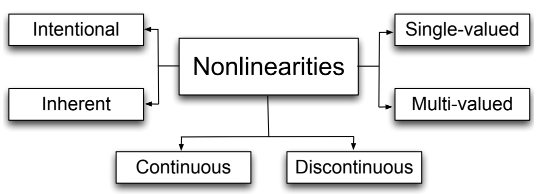


Fig. 1 Types of nonlinearities

Inherent nonlinearities are intrinsic to the nature of the system and arise from the system's hardware and motion. They are normally undesirable and need to be compensated for by the system designer. Intentional nonlinearities are artificial and deliberately introduced by the designer in order to improve system performance [15]. Normally in supply chain systems, nonlinearities are intrinsic to the system due to physical and economic constraints. These nonlinearities may or may not be considered in the system modelling depending on the degree of accuracy and complexity necessary for the supply chain design. On the other hand, supply chain designers may want to include nonlinearities that do not exist in reality for the sake of improving certain performance measures. This type of research has only recently been considered [17] but yet to be duly explored. Other studies have shown that while the presence of nonlinearities may worsen some performance measures, they may improve others. For example, [18] - demand amplification versus service level, [19] - complexity of the production plan versus production cost, [20] - leadtime expectations versus dynamic behaviour in the system.

Continuous and discontinuous nonlinearities are associated with the rate of change in the output in relation to the input. Table 1 contains examples of discontinuous (the first four rows) and continuous (the last two rows) nonlinearities found in production and inventory control models for supply chain management and their block diagram symbol, typical output response given a sinusoidal input and the rate of change between output and input. A feature of the outputs in continuous functions is that they are smooth enough to possess convergent expansions at all points and therefore can be linearised. Examples include any adaptive control system, where certain control

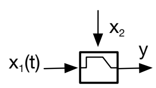
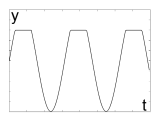
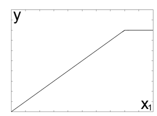
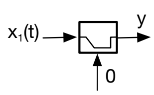
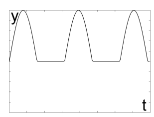
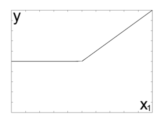
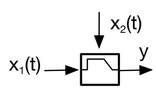
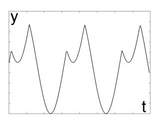
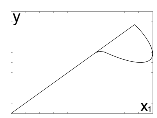
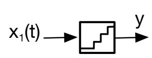
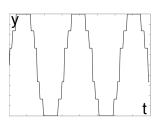
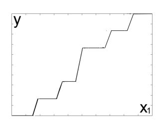
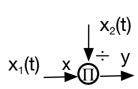
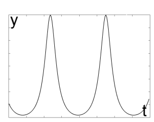
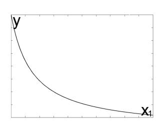
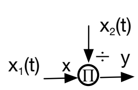
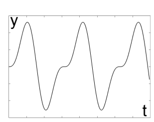
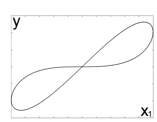
Nonlinearity	Block diagram symbol	Typical Output Response	Input-Output Profile
Fixed Capacity Constraint (discontinuous, single-valued)			
Non-negativity Constraint (discontinuous, single-valued)			
Variable Capacity Constraint (discontinuous, multiple-valued)			
Rounding (discontinuous, single-valued)			
Time-varying parameter (continuous, single-valued)			
Time-varying parameter (continuous, multiple-valued)			

Table 1 Nonlinearities in Production and Inventory Control Systems

parameters, instead of being fixed, vary depending on the state of other variables [16]. Sharp changes in output values or gradients indicate discontinuities. The most common type of discontinuous nonlinearity is the piecewise linear functions, which consist of a set of linear relations for different regions.

In the case of single-valued nonlinearities, the output is a result of the current value of the input, whereas two or more values of output may be possible for the same input value in the case of multiple-valued nonlinearities. The multiple values of the output will depend on the previous history of the input; thus such nonlinearities are said to possess memory. The last column of Table 1 demonstrates the difference between these characteristics. Multiple-valued functions are often used in engineering to model hysteresis of magnetic and elastic materials and mechanical backlash of friction gears [15]. In business studies this kind of nonlinear behaviour has been described in economics [21], for instance between buying/selling states and price [22] and unemployment and economy growth rate [23]. In supply chain management research, multi-valued nonlinearities are not so commonly reported. They have been used to model switching of certain operation strategies depending on cost directions. Examples include investigations on changes in global sourcing [24] and manufacturing strategies [25] depending on foreign exchange rate directions. From a purely production-inventory control system perspective, this kind of effect has been identified in outbound shipments which depend on relational fluctuations between inventory levels and current demand [8]. The normal thinking is that, independent of demand growing direction, the order quantities placed to suppliers or shipped to customers will always match demand. However, when a variable capacity is put in place, these outputs can result in a complex multiple-valued nonlinear behaviour.

3 Methods for the analysis of nonlinearities

When confronted with a nonlinear system, the first approach is to linearise it. The rationale for this is that techniques to analyse linear systems are much more established and understood than nonlinear control theory methods [16]. Linearisation is generally considered an appropriate choice when the solution can be obtained in this manner. While linear system theory is well acknowledged, the literature in nonlinear theory is less conclusive when it comes to generality and applicability [13]. Because of a lack of common terminology and lack of detailed research methods in the nonlinear control systems literature, the complete catalogue of all existing methods and their applicability in the analysis of nonlinear feedback systems is laborious. Table 2 presents a list of the methods that have been sufficiently acknowledged in the literature and whose full details were accessible.

There are a number of methods for system linearisation, such as small perturbation theory, describing function and averaging or best-fit line approximations. The former allows the system with continuous nonlinearities to be analysed through successive approximations in the form of power series around a specific operating point [15]. If

	Method of Analysis	Applications	Considerations
Linearisation methods	Small perturbation theory with Taylor series expansion	Continuous Single-valued	Assumption that the amplitude of the excitation signal is small. Local stability analysis only.
Linearisation methods	Describing function	Continuous, Discontinuous Single-valued, Multi-valued	Less accurate when nonlinearities contain higher harmonics. Analysis of systems with periodic or Gaussian random input only.
Linearisation methods	Small perturbation theory with Volterra/Wiener series expansion	Continuous Multi-valued	Assumption that the amplitude of the excitation signal is small. Difficulty in calculating the kernels and operators of the system, making it impractical for high order systems.
Linearisation methods	Averaging and best-fit line approximations	Continuous, Discontinuous Single-valued, Multi-valued	Gross approximation of real responses. Only when better estimates are not possible.
Graphical and simple methods	Phase plane and graphical solutions	Continuous, Discontinuous Single-valued, Multi-valued	Limited to 1^{st} and 2^{nd} order systems only.
Graphical and simple methods	Point transformation method	Discontinuous Single-valued, Multi-valued	Piecewise linear systems only. For high order systems, automated numerical methods must be employed.
Exact solutions	Direct solution	Continuous Single-valued	Limited to a finite number of equations.
Stability method	Lyapunov-based stability analysis for piecewise-linear systems	Discontinuous Only single-valued examples were found	Piecewise linear systems only. Computation can be complex depending on the system.
Simulation	Numerical and simulation solution	Continuous, Discontinuous Single-valued, Multi-valued	Can be time consuming. Dependent on computer and software calculations capacity.

Table 2 Summary of methods used to analyse nonlinear systems

the system can be represented by the Taylor series or Volterra series, then it can be approximated using perturbation theory [26]. The Volterra series is often compared with Taylor series but it is also suitable to approximate outputs with memory, which means that the Volterra series can mimic systems where the output depends on past inputs so they are suitable for multi-valued nonlinearities [13]. The describing function method is attributed to as a quasi-linearisation, since the approximation process of the nonlinear system is for specific inputs. For instance, sinusoidal inputs are more often used since the frequency response approach is a powerful tool for the analysis and design of systems [27]. Averaging and best-fit line techniques produce rough estimations and can be a simpler alternative for comprehending more complex systems in a qualitative manner [14]. However, whenever precision and reliability are needed these methods should be avoided [16].

Then there are graphical techniques, such as the phase plane analysis. However, this technique is limited to second order systems [16]. The point transformation method allows periodicity and stability of piecewise-linear systems to be investigated by studying the behaviour of trajectories that cross repeatedly from one region to another [15], but it can be complicated for high order systems. There are also exact solutions for a finite number of nonlinear control systems with low order [16], making its application very limited. More complex and sophisticated techniques such as the one developed by [28] are used for stability analysis of piecewise-linear systems combining Lyapunov functions and convex optimisation process.

Finally, there is simulation, which although is a very helpful technique, it should be in principle used as a complementary tool to the above analytical methods. Simulation has many advantages, offering a "middle ground between pure mathematical modelling, empirical observation and experiments for strategic issues in supply chain research" [29]. Because simulation is a numerical technique that allows the analysis of complex models, it does not require specific mathematical forms that are analytically solvable.

In the next subsections, we provide instructions on how to adopt the following linearisation methods: describing functions, small perturbation theory with Taylor and Volterra series expansion. These methods were chosen given their wide applicability, versatility and power in uncovering hidden dynamics caused by different types of nonlinearities and in tracing the transient behaviour, which is necessary to estimate system's performance. In supply chain systems, the understanding of transient responses can elucidate the occurrence of disruptions and how to mitigate its cascading effects on other supply chain members.

3.1 Describing Function

The describing function method is a quasi-linear representation for a nonlinear element subjected to a specific input. This is a method that attempts to estimate the output properties, such as frequency, amplitude and stability, after being affected by a nonlinear component [27]. This method is also used to predict limit cycles or sustained oscillations [30].

The basic idea of the describing function is to express a nonlinear element in the form of a transfer function, or a gain, determined from its effects on a particular input signal. For asymmetric nonlinearities, or symmetric nonlinearities subjected to biased inputs, at least two terms of the describing function are needed: one that expresses the change in the output amplitude (N_A), and another that considers the change in the output mean (N_B). This leads to the so-called dual-input describing function [16, 15]. Another effect caused by this type of nonlinearity is the possible change in phase angle (ϕ) of the output response in relation to its input. Next, we give an example of how sinusoidal describing functions can be determined.

Consider the input to the nonlinearity:

$$x_1(t) = A.cos(\omega t) + B \tag{1}$$

where ω is the angular frequency and $\omega = 2\pi/T$. The output y can be approximated to:

$$y(t) = N_A.A.cos(\omega t + \phi) + N_B.B \tag{2}$$

In order to determine the terms of the describing function (N_A, N_B and ϕ) the series has to be expanded and its first harmonic coefficients must be determined. The Fourier series expansion method is used to represent the output y such as:

$$y(t) \approx b_0 + a_1 cos(\omega t) + b_1 sin(\omega t) + a_2 cos(2\omega t) + b_2 sin(2\omega t) + \cdots$$

$$y(t) \approx b_0 + \sum_{k=1}^{\infty} [a_k cos(k.\omega t) + b_k sin(k.\omega t)] \tag{3}$$

where the Fourier coefficients are given by:

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) cos(k.\omega t) d\omega t \tag{4}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} y(t) sin(k.\omega t) d\omega t \tag{5}$$

$$b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(t) d\omega t \tag{6}$$

The nonlinear function y is then approximated to the first harmonic, resulting in:

$$y(t) = b_0 + a_1 cos(\omega t) + b_1 sin(\omega t) = b_0 + \sqrt{a_1^2 + b_1^2}.cos(\omega t + \phi) \tag{7}$$

where, $\phi = arctan\left(\frac{b_1}{a_1}\right)$

In this way the two terms of the describing function can be determined as:

$$N_A = \frac{\sqrt{a_1^2 + b_1^2}}{A} \tag{8}$$

$$N_B = \frac{b_0}{B} \tag{9}$$

For single-valued nonlinearities the coefficient b_1 will be equal to zero and therefore the phase angle ϕ will be also zero. In case of dynamic multi-valued nonlinearities the describing function will be in the form of $N_A(A, \omega)$. Normally a plot of $N_A(A, \omega)$ versus $\phi(A, \omega)$ for various values of A and ω are used to understand such complex nonlinearities [27].

Examples of supply chain applications of such methods can be found in [8, 30, 31, 32, 33]. By replacing the different describing function values in the system transfer functions, these studies were able to determine the effect of nonlinearities on the system's natural frequency, damping ratio and stability.

3.2 Small Perturbation Theory with Taylor Series Expansion

The Taylor series can be used for approximating the response of a nonlinear system to a given input if the output of this system depends strictly on the input at that particular time.

Given a system with single-valued continuous nonlinearity

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{10}$$

where x is the state vector, $\dot{x}$ is the time derivative of the state vector, y is the output vector and u is the input vector, we can derive an approximate linear system about a nominal operating state space x^* and for a given input u^* by using small perturbation theory with Taylor series expansion. The linearisation process involved in this approach is such that departures from a steady state point are small enough to produce transfer function coefficients. Hence, by assuming a small amplitude of the excitation signal, the nonlinear differential equations are replaced by a set of linearised differential equations with coefficients dependent upon the steady state operating point.

The first order Taylor series approximation of the nonlinear state derivatives leads to the following linearised function:

$$\Delta\dot{x} = A\Delta x + B\Delta u \tag{11}$$

$$\Delta y = C\Delta x + D\Delta u \tag{12}$$

where $\Delta x = x - x^*$, $\Delta\dot{x} = \frac{d\Delta x}{dt}$, $\Delta y = y - y^*$, $\Delta u = u - u^*$ and A, B, C, D can be found through the following partial derivatives:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccc|ccc} \frac{\partial f_1(x^*,u^*)}{\partial x_1} & \cdots & \frac{\partial f_1(x^*,u^*)}{\partial x_n} & \frac{\partial f_1(x^*,u^*)}{\partial u_1} & \cdots & \frac{\partial f_1(x^*,u^*)}{\partial u_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(x^*,u^*)}{\partial x_1} & \cdots & \frac{\partial f_n(x^*,u^*)}{\partial x_n} & \frac{\partial f_n(x^*,u^*)}{\partial u_1} & \cdots & \frac{\partial f_n(x^*,u^*)}{\partial u_k} \\ \hline \frac{\partial h_1(x^*,u^*)}{\partial x_1} & \cdots & \frac{\partial h_1(x^*,u^*)}{\partial x_n} & \frac{\partial h_1(x^*,u^*)}{\partial u_1} & \cdots & \frac{\partial h_1(x^*,u^*)}{\partial u_k} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_m(x^*,u^*)}{\partial x_1} & \cdots & \frac{\partial h_m(x^*,u^*)}{\partial x_n} & \frac{\partial h_m(x^*,u^*)}{\partial u_1} & \cdots & \frac{\partial h_m(x^*,u^*)}{\partial u_k} \end{array}\right) \tag{13}$$

where n is the number of state variables, m is the number of outputs and k is the number of inputs.

The nominal operating point (x^*, u^*) normally corresponds to the equilibrium or resting points where all state derivatives are equal to zero and they can be found by applying the final value theorem:

$$x^* = \lim_{t\to\infty} x(t) = \lim_{s\to 0} s.X(s) \tag{14}$$

for a constant input u.

The reader can refer to the works of [8, 34, 35, 36] for application of this method in supply chain models. After linearisation is performed, it is possible to find the system transfer function and system design will follow linear control system theory.

3.3 Small Perturbation Theory with Volterra-Wiener Series Expansion

The Volterra series has the ability to deal with multi-valued nonlinearities by capturing memory effects. The output from the Volterra series depends on the previous history of the input to the system. Hence, a continuous multi-valued nonlinear output can be approximated to:

$$\begin{aligned} y(t) &= h_0 + \int_{\mathbb{R}} h_1(\tau_1)x(t-\tau_1)d\tau_1 + \int_{\mathbb{R}} h_2(\tau_1,\tau_2)x(t-\tau_1)x(t-\tau_2)d\tau_1 d\tau_2 + \ldots \\ &= h_0 + \sum_{n=1}^{N} \int_{\mathbb{R}} h_n(\tau_1,\ldots,\tau_n) \prod_{j=1}^{N} x(t-\tau_j)d\tau_j \end{aligned} \tag{15}$$

where h_0 is a constant and for n=1, 2...N, the function $h_n(\tau_1, \ldots, \tau_n)$ is referred as n-th order Volterra kernel. Note that when $h_0 = 0$ and $N = 1$, the formula describes the system's impulse response by a convolution of $x(t)$. Volterra extended the linear system representation to nonlinear systems by adding a series of nonlinear integral operators.

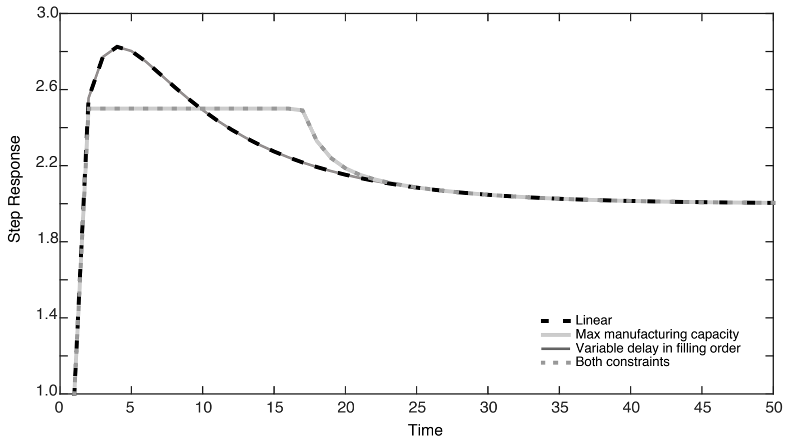
The convergence of the Volterra series is comparable to the convergence of the Taylor series expansion of a function which often allows only for small deviations from the starting point. However, the type of convergence required is very rigorous since not only the error has to approach zero with increasing number of terms, but also the derivatives of the error. Hence, estimation of Volterra coefficients is generally performed by estimating the coefficients of an orthogonalised series, e.g. the Wiener series, and then recomputing the coefficients of the original Volterra series. Readers can refer to [13] for more details. No application of this method was found in the supply chain management literature.

4 The effects of nonlinearities

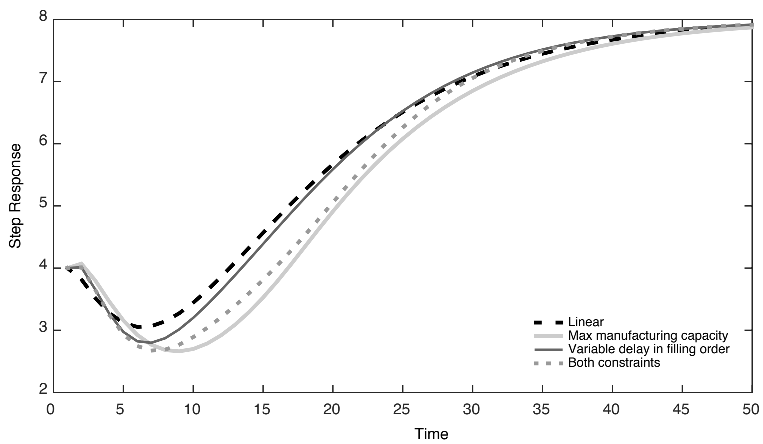
In recent years, researchers have put effort in shaping stability regions of nonlinear supply chain systems and understanding the factors which will lead to chaotic behaviours. These studies made contributions in explaining and tackling uncertainties, dynamics and disruptions in complex supply chain systems, since they elucidated how nonlinearities can change a system's transient response by monitoring the variation in terms of output's amplitude, mean and phase and consequently its repercussion on the system's natural frequency and damping ratio. Figure 2 illustrates a few examples on the effect of some nonlinearities (maximum manufacturing capacity and variable delay in filling order from the model in [8]) on the system's step response. The figure demonstrates how simplistic linear assumptions can be and that indeed nonlinearities can make significant changes to responses' amplitude and settling time. Another observation is that nonlinearities affect different performances in different ways. For instance, while the maximum production capacity seems to diminish inventory levels, it helps to decrease the amplification in manufacturing order rate (bullwhip). The variable delay in filling orders will have more negative impact on the outbound shipment rate than on the inventory response. When both nonlinearities are considered at the same time, outbound shipments' amplitude and recovery time are further worsened. Response and recovery time as well as over/undershoot are good indicative of system's resilience [5] and the diminishment in this performance can affect other players in the supply chain.

Table 3 summarises the current understanding of some types of nonlinearities and their impact on the ripple effect. Discontinuous, single-valued nonlinearities such as maximum capacity constraints, buying quantity constraints and non-negativities have been predominantly studied by describing function methods [8, 30, 31, 32, 33]. This method enabled understanding of the impact of such nonlinearities on system output responses, for example manufacturing and supplier orders and production rates. Although, this nonlinearity does not provoke a shift in the output phase, it will change the output's mean and amplitude. These distortions increase complexity of supply chain dynamics making it difficult for supply chains to respond and recover from disruptions, therefore potentially aggravating the ripple effect. For instance, studies on fixed manufacturing capacity suggest that this nonlinearity decreases the amplification of manufacturing orders, consequently decreasing the Bullwhip effect. However, its impact of the manufacturing output mean can slow down the ripple effect mitigation process. In case of asymmetrical nonlinearities, such as in [33] the output will be relative depending on the relationship between the minimum (non-negative) and maximum capacity constraints. If the mean of the orders received is less than half of the maximum capacity then the non-negative order boundary dominates. This leads to the increase in average inventory level and orders, therefore increasing costs. If the mean of the desired orders exceeds half of the maximum capacity, then the dominant impact on system dynamics will be the capacity constraint rather than the non-negative order low boundary. Under such condition, mean gain will increase with demand amplitude, leading to the decrease in average inventory level and orders, therefore increasing the risk of disruption.

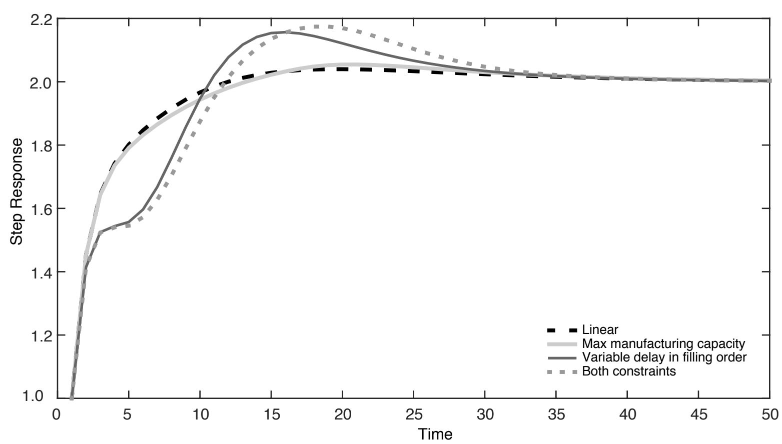
Describing functions have also been used to analyse discontinuous multi-valued nonlinearities, such as variable shipment constraints due to changes in customer orders and inventory levels [8, 30, 32]. For low frequency orders, this dynamic capacity constraint can decrease the output's amplitude and mean and shift the output's phase making the output response lag behind the input. Hence, disruptions are less likely to affect supply chains with high and medium frequency demands. ripple effect mitiga-



(a) Manufacturing order rate



(b) Inventory level



(c) Outbound shipment rate

Fig. 2 Example of nonlinearity effects on transient responses tion strategies would include encouraging high frequency purchasing by developing resilient demand management strategies. In [33, 36], a similar nonlinearity is applied to switch between 'push' and 'pull' production modes, but the authors decided to evaluate both modes separately through transfer function analysis. This analysis was not able to capture the effect of the switch nonlinearity, but the authors were able to conclude that when the upstream operates in make-to-stock mode, other capacity constraints can reduce the bullwhip effect at the expense of increased inventory variability, therefore at the expense of decreased resilience.

Only single-valued continuous nonlinearities have been identified in the production and inventory control literature, therefore only Taylor series expansion has been applied as small perturbation method to predict nonlinear responses of continuous constraints, such as time-varying delays, resource depletion and real lead-time estimation [8, 34, 35, 36]. In [8, 34, 36], the nonlinear differential equation involved more than one variable hence the impact on the output's amplitude and mean will depend on the input and parameter settings. However, linearisation with Taylor series expansion enabled determination of parameter settings that minimise operational disruptions and increase supply chain resilience. In the case of [35], the nonlinear differential equation is used to represent the depletion in resources that will certainly have a negative impact on production rate and therefore on the ripple effect. The analysis of this nonlinearity can shed light on how to best allocate scarce resources to ensure seamless flows of information and material.

It is worth mentioning that other authors have used analytical methods to analyse nonlinearities such as averaging [37, 38] and stability methods [7, 39], which even though were not able to capture the effect of nonlinearities on system's responses, it allowed establishing parameter settings that minimise cost and disruptions and meet stability requirements indispensable for supply chain resilience.

5 Conclusion and future scope

This chapter has revisited the literature of nonlinear control theory application in supply chain management. The chapter instructed readers on the different types of nonlinearities that can commonly appear in supply chain systems and provoke undesirable dynamic behaviours. A number of methods and references to their application have been introduced and key findings on the rippling effect of nonlinearities have been discussed. From the main points discussed in this chapter we outline the following directions for future research:

1. **Supply chain structural development**: There is an opportunity to explore the effect of different capacity constraints to devise tactical and strategic plans regarding potential adjustments in supply chain structure, such as investment in infrastructure and resource efficiency and flexibility. Previous research suggests that adequate capacity levels can help attain desired supply chain performance, therefore a performance-based structure plan can help companies to make right

Source	Nonlinearity	Nonlinearity type	Method Used	Nonlinearity Output	Effect on output			Potential consequences to Ripple effect
					Amp.	Mean	Phase	
[8]	Manufacturing constraint	Discontinuous, single-valued	Describing Function	Manufacturing order	↓	↓	0	Although manufacturing constraints decrease the amplification of manufacturing orders forcing the production manager to prioritise a level production strategy, it substantially decreases the mean of the output, suggesting difficulties in responding to sudden changes in demand and lead-time. Hence, this nonlinearity can cause disruptions that may cascade downstream affecting all other companies in the supply chain.
[8, 32, 30]	Shipment constraint	Discontinuous, multi-valued	Describing Function	Shipment	↓*	↓*	←*	This dynamic capacity constraint in the outbound shipment occurs due to variation in inventory levels and customer orders. A decrease in the output's amplitude and mean is observed and a phase shift makes the output response lag behind the input, only for low frequency orders. Hence, disruptions are less likely to affect supply chains with high and medium frequency demands.
[34, 8]	Time-varying delay in filling orders	Continuous, single-valued	Small perturbation with Taylor series	Shipment	↓↑	↓↑	0	This continuous nonlinearity can have cause different impacts to the system output response depending on the parameter choice. Linearisation with Taylor series expansion enables to determine parameter settings that minimises disruptions.
[32]	Buying quantity constraint	Discontinuous, single valued	Describing Function	Supplier order	↓	↑	0	A combination of non-negativity and batching constraints made this nonlinearity very complex to analyse. Insights obtained in the work suggest that products with the same demand pattern should be grouped together to determine order quantities that minimises disruptions.
[31, 30]	Forbidden return of orders	Discontinuous, single-valued	Describing Function	Supplier order	↓	↑	0	Non-negative constraints in the ordering rule, known as forbidden return constraint, can cause limit cycles, which are oscillations intrinsic to the nonlinear production and inventory control system itself and not imposed by the demand. This problem may be exacerbated as this information signal propagates upstream in the supply chain. Distorted orders sent upstream are more likely to cause disruptions through its backlash downstream.
[37, 38]	Time-varying delay in filling orders	Continuous, single-valued	Averaging technique + simulation	Shipment	Not able to capture			Although method was unable to understand the impact of this nonlinearity on system response, averaging technique enabled researcher to propose a design that minimises disruptions and cost.
[35]	Resource availability	Continuous, single-valued	Small perturbation with Taylor series	Production rate	↓	↓	0	The effective production rate is related to the amount of available resources in the current environment. As available resources decrease, the production rate decreases in a nonlinear fashion, increasing the risk of disruptions.
[7, 39]	Forbidden return of orders	Discontinuous, single-valued	Lyapunov-based stability analysis for piecewise-linear systems	Supplier order	Not able to capture			Although the method was not able to provide understanding on the impact of this nonlinearity on system's responses, it enables to undertake an in-depth stability analysis of the system and recommendations for parameter settings.
[33])	Manufacturing constraint (non-neg + max capacity)	Discontinuous, single-valued	Averaging technique + simulation	Production rate	↓	↓↑ 0**	0	Method enables to compare the results between linear and nonlinear assumptions under the same control policy settings. A slower recovery speed of assembly work-in-process and production rate is observed in the non-linear environment due to the effect of nonlinearity on the output's amplitude.
[33, 36]	Push-Pull decision	Discontinuous, multi-valued	Transfer function analysis of different modes	Assembly rate	Not able to capture			Very complex nonlinearity that was analysed only by transfer function analysis of different operating modes. Insights suggests that the nonlinearity would behave similarly to the shipment constraint in [8, 32, 30] due to limited recovery inventory. However, further research needs to be done. Other insights suggest that when upstream operates in make-to-stock mode, non-linearities can reduce the bullwhip effect at the expense of increased inventory variability.
[36]	Rate change between backlog level and shipment rate	Continuous, single-valued	Small perturbation with Taylor series	Lead-time	↓↑	↓↑	0	This continuous nonlinearity can have cause different impacts to the system output response depending on the parameter choice. Linearisation with Taylor series expansion enables to determine parameter settings that minimises disruptions.

Table 3 Summary of rippling effects of nonlinearities in production and inventory control systems

↓ decrease; ↑ increase; ← lag; * only for low frequencies; ** if symmetrical investments depending on their focus: customer service, cost efficiency, risk management and so on. For ripple effect mitigation, this performance-based structure plan should include all members of the supply chain.

2. **Supply chain design development**: There is an important avenue for future research regarding the deliberate employment of nonlinearities for improved system's design. Computer simulation enabled researchers to increase model accuracy and validation to better represent reality. However, the analytical methods here presented bring us one step forward in unraveling the mechanisms of nonlinear supply chain dynamics. This knowledge can be used to advantage in the improvement of the supply chain performance, from operational, economic and environmental viewpoints.
3. **Continuous, multi-valued nonlinearities**: Despite not being referenced in the supply chain literature, continuous nonlinearities with memory can appear in supply chain models, for example, in circular economy supply chains where there is uncertainty of the volume, timing and quality of both demand and returns, therefore multiple inputs should be considered. Hence, future research can build on previous efforts and discoveries to identify new nonlinearities in supply chain systems, to investigate which limitations and constraints they represent and their effect on system's dynamics.
4. **Discontinuous, multi-valued nonlinearities**: Limited study has explored the multi-valued discontinuous nonlinearities analytically, even though some insights are obtained by the simulation approach. Discontinuous nonlinearities with memory are very common in supply chain systems. For example, the shipment constraint in assemble-to-order systems due to the limited customer order decoupling point (CODP) inventory. Also, the constrained remanufacturing order rate in hybrid manufacturing/remanufacturing systems, driven by the availability of recoverable inventory (limited returned products), is also a multi-valued nonlinearity. Future research should analytically predict its impact on ripple effect and suggest the corresponding control strategies in mitigating the unwanted dynamic behaviour.
5. **Effect of lead-time disturbances**: Most research in the application of both linear and nonlinear control theories in supply chain management focus on understanding the impact of demand uncertainty and on improving demand forecasting methods. Lead-time fluctuations can lead to performance degradation and increased production costs, just as demand uncertainties can [40] and disturbances and uncertainties in production and supply lead-times are reported to be the main sources of supply chain risk [10]. Supply chain control theorists have avoided tackling lead-time disturbance under the assumption that models become nonlinear.

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