$$7.62\times 12+7.03\times 22+10.21\times 32+6.16\times 42+8.87\times 52+9.8\times 62+5.62\times 72+5.71\times 82+ 8.37\times 92+5.81\times 102+6.83\times 112+10.53\times 122+9.83\times 132+16.84\times 142+5.79\times 152 < 4200$$

$$7.62\times 12+7.03\times 22+10.21\times 32+6.16\times 42+8.87\times 52+9.8\times 62+5.62\times 72+5.71\times 82+ 8.37\times 92+5.81\times 102+6.83\times 112+10.53\times 122+9.83\times 132+16.84\times 142+5.79\times 152 > 3800$$

$$7.85\times 13+7.24\times 23+10.52\times 33+6.34\times 43+9.14\times 53+10.1\times 63+5.79\times 73+5.88\times 83+ 8.62\times 93+5.99\times 103+7.04\times 113+10.84\times 123+10.12\times 133+17.35\times 143+5.97\times 153 < 4200$$

$$7.85\times 13+7.24\times 23+10.52\times 33+6.34\times 43+9.14\times 53+10.1\times 63+5.79\times 73+5.88\times 83+ 8.62\times 93+5.99\times 103+7.04\times 113+10.84\times 123+10.12\times 133+17.35\times 143+5.97\times 153 > 3800$$

$$7.35\times 14+7.46\times 24+10.83\times 34+6.53\times 44+9.41\times 54+10.4\times 64+5.97\times 74+6.05\times 84+ 8.88\times 94+6.17\times 104+7.25\times 114+11.17\times 124+10.43\times 134+17.84\times 144+6.15\times 154 < 4200$$

$$7.35\times 14+7.46\times 24+10.83\times 34+6.53\times 44+9.41\times 54+10.4\times 64+5.97\times 74+6.05\times 84+ 8.88\times 94+6.17\times 104+7.25\times 114+11.17\times 124+10.43\times 134+17.84\times 144+6.15\times 154 > 3800$$

$$8.33\times 15+7.68\times 25+11.16\times 35+6.73\times 45+9.7\times 55+10.71\times 65+6.14\times 75+6.24\times 85+ 9.15\times 95+6.35\times 105+7.46\times 115+11.5\times 125+10.74\times 135+18.4\times 145+6.33\times 155 < 4200$$

$$8.33\times 15+7.68\times 25+11.16\times 35+6.73\times 45+9.7\times 55+10.71\times 65+6.14\times 75+6.24\times 85+ 9.15\times 95+6.35\times 105+7.46\times 115+11.5\times 125+10.74\times 135+18.4\times 145+6.33\times 155 > 3800$$

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An Application of Linear Programming for Short-Term Harvest Scheduling

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ABSTRACT. The technique of linear programming (LP) is illustrated by developing a harvest schedule for an industrial forest ownership in northwestern Pennsylvania. The objective was to maximize net present value of the harvest over a five-year planning period. The effect of changes in timber value and growth rate on the optimum schedule was determined. Sensitivity analysis provided additional information the manager could use to make decisions. In order to successfully apply LP, the forester must be able to define the management objective of the harvest schedule and the resource and managerial constraints that will influence its attainment. Data used in the model have to be available and reliable. Many forest enterprises should be in the position to adopt LP since commercial programs for microcomputers are now available for which a high level of computing expertise is not required.

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Linear programming (LP) is a mathematical technique used to allocate limited resources optimally among competing activities to satisfy a given objective. The technique has been widely used in industrial applications to improve resource efficiency. In forestry LP has been applied successfully to maximize allowable timber yields (Leak 1964), timber utilization in sawmills (Fasick and Sampson 1966), and in wildlife habitat management (Mealey and Horn 1981). Forplan is a large LP model, developed by the USDA Forest Service to allocate forestland to general management activities and to schedule treatments and resulting product flows (Field 1984). A comprehensive review of the theory and application of linear programming in natural resource management is given by Dykstra (1984).

Harvest scheduling is well suited to an LP approach since, like many other management problems in forestry, it deals with the optimization of certain measures of economic performance (e.g., maximizing profit or minimizing cost) while having to satisfy management restrictions such as allowable cut and mill requirements. Linear programming is a tool that can be used to cope with the biological, financial, and operational factors inherent in any harvest scheduling problem (Fairweather 1987).

The aim of this paper is to illustrate the use of LP as an aid to harvest scheduling and to show how changes in timber value and growth rates can influence the selection of stands to be harvested. The example problem is relatively simple, and could be solved by hand, but the intention is to show how the problem is described and solved within the framework of linear programming. The example could be easily enlarged to a point where hand solution would be impractical, but linear programming would be quite efficient.

LP—A BRIEF REVIEW

Every linear programming problem consists of three components: decision variables, constraints, and the objective function. The objective function is expressed mathematically as:

$$\text{Max or min } Z = c_1X_1 + c_2X_2 + \ldots + c_nX_n$$

$X_1, X_2, \ldots X_n$ represent the decision variables. Values will be found for the decision variables that maximize (or minimize) Z, the value of the objective function. The value or cost coefficients c_1 through c_n associated with each decision variable represent the contribution one unit of X_i makes to the value of the objective function. The objective function must be linear (no quadratic expressions allowed), and the decision variables must be divisible (can assume noninteger values) and nonnegative. Techniques are available for the solution of problems that do not meet these assumptions, but they are beyond the realm of this discussion.

Constraints should restrict the objective function from reaching zero in a minimization problem or infinity in a maximization problem. They normally represent some finite physical resource or management restriction and take the form:

$$b_1X_1 + b_2X_2 + \ldots + b_nX_n \quad <, >, or = B,$$

where b_1 through b_n are the amounts of total resource B utilized by one unit of the corresponding decision variable. Thus, the LP algorithm finds values for the decision variables that simultaneously satisfy all the constraints and maximize (or minimize) the objective function; these values are known as the optimal solution.

A SIMPLE HARVEST SCHEDULE APPLICATION

In this study, linear programming for harvest scheduling was applied to a forest products company in northwestern Pennsylvania. Its timber inventory is dispersed over 80 properties and consists mainly of black cherry, red and sugar maple, and red oak. The company owns and operates a large sawmill, producing about 15 mmbf annually. About 25% of the mill's timber requirement comes from company land. The company's management objective was to develop a harvest schedule which maximized the net present value (*NPV*) of timber harvested from stands over a 5-year period while satisfying the mill's annual demand for timber. Harvesting is through clearcutting; each acre in a stand is assumed to have the same species mix and volume. Company personnel identified 15 stands that were candidates for cutting sometime in the next 5 years.

Problem Formulation

The complete LP formulation of this problem is in the *Appendix*. The decision variables are designated as X_{ij}, the number of acres to cut in stand i in year j.

The objective function maximizes net present value, and required the calculation of net present value on a per-acre basis for each combination of stand i and year j, or NPV_{ij}. Basic data for these calculations supplied by company personnel were current timber volume and stumpage values, by species, and percent growth and depletion rates, by stand.

The calculation of *NPV* for stand i in year j was as follows:

$$NPV = (\text{net value/ac in yr } j)/(1 + r)^{j-1},$$

where

j = 1 to 5,

r = discount rate, and

net value/ac in yr j = (stumpage value · estimated volume/ac in yr j) − depletion rate/ac.

The depletion rates were \$32.69/mbf for stands bought before 1974, and \$132.02/mbf for stands acquired in or after 1974. These were converted to a per-acre basis for use in the above for-

Table 1. Total stand volume by species.

Stand number:		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Deplt. rate ($/mbf)		32	32	132	32	32	132	132	32	32	132	132	132	32	32	132
Species	Value ($/mbf)	.... mbf														
Ash	250	226	11	647	10	61	26	339	21	631	146	93	47	0	22	167
Aspen	0	0	35	0	0	0	0	467	0	0	0	0	0	0	0	0
Bass/pop	40	60	7	206	0	59	26	333	18	192	0	17	11	0	0	14
Beech	15	588	11	364	0	130	108	628	37	588	8	57	124	230	40	659
Birch	0	77	14	234	0	32	41	184	20	172	0	14	92	284	89	20
B. cherry	300	339	9	456	55	32	14	476	0	699	272	77	19	76	34	3564
Hemlock	10	301	0	134	0	40	38	919	4	0	0	7	0	756	47	894
Red maple	60	322	100	529	15	238	107	628	11	1379	436	23	260	191	161	1254
S. maple	60	498	7	625	144	221	49	1356	48	86	92	76	603	0	8	1279
Red oak	250	202	73	1933	130	47	45	202	3	43	0	9	0	1357	316	0
White oak	100	11	11	786	10	0	4	0	0	0	0	0	0	484	0	0
Mixed	80	88	78	444	31	0	18	0	0	0	26	0	0	1282	100	0
Stand volume		2712	355	6357	394	861	476	5530	162	3790	980	373	1155	4661	818	7851
% of total volume		7.5	1	17	1	2	1.5	15	0.5	10	3	1	3.5	13	2	22

mula. Estimated volume per acre in year *j* was calculated by compounding initial volume per acre by a percent growth rate supplied by the company.

For example, the value of $NPV_{2,3}$ in the *Appendix* is $492.00. This figure was derived as shown, using data from Table 1:

Depletion rate = $32.69/mbf

Growth rate = 3%

Discount rate = 4%

Area = 52 ac

Initial stand volume = 355 mbf

Initial volume/acre = 6.82 mbf

Estimated vol/ac in 3 yrs = $6.82(1.03)^2$
= 7.24 mbf

Depletion rate in 3 yrs = 236.67/ac

Total value in yr 3
= $250 (11 × 1.03 × 1.03)
+ $ 0 (35 × 1.03 × 1.03)
+ $ 40 (7 × 1.03 × 1.03)
:
+ $ 80 (78 × 1.03 × 1.03)
= $40,213

Value/ac in yr 3
= $773.33

Net value/ac in yr 3
= $773.33 − 32.69 (7.24)
= $773.33 − 236.67
= $536.66

Net present value/ac
= $\$536.66/(1.04)^2$
= $496.17 (not equal to $492 due to fractional volumes not shown in Table 1.)

The constraints in the Appendix occur in two major groups. The first group of 15 constraints guarantee that a cutting schedule will not be constructed that cuts more acres than are available in any particular stand. Stand 1 consists of 366.5 ac, stand 2 has 52 ac, and so on.

The second group of constraints guarantee that annual mill requirements of between 3800 and 4200 mbf (from company land) will be met in each year of the planning period. There are 10 constraints, a pair for each year. Each pair specifies the lower and upper limits. The coefficients in each constraint represent the volume per acre in each stand/year combination.

The problem was run using LINDO (Schrage 1986), a commercial LP package on an IBM mainframe computer at The Pennsylvania State University. The following runs were made to study the influence of price and growth rate on the harvest schedule:

1. Run 1 used a growth rate of 3%/yr and a discount rate of 4%. Black cherry was valued at $300/mbf and red oak at $250/mbf (as in Table 1 and the Appendix).
2. Run 2 used the same growth and discount rates as (1), but stand value was derived from different timber values for black cherry and northern red oak. The value of black cherry timber was increased from $300 to $400 per mbf, while red oak was reduced from $250 to $150 per mbf.
3. Run 3 was the same as (1) except stands 9 and 14 were "grown" at a rate of 5%/yr.

RESULTS

The Optimum Schedule

In Run 1 (Fig. 1), the LP model selected stands for harvest which carried a significant proportion of valuable timber species, such as black cherry and red oak, and which were subjected to the lower of the two depletion rates (Table 1). For example, black cherry, red oak, and ash comprise approximately 50% of the total timber volume of stands 4, 9 and 14 and were all harvested in year 1. By comparison less than 20% of timber volume in stands 6 and 7 was attributable to these valuable species and neither stand was selected for harvest. The depletion rate also influences stand selection. Stands 3 and 10 both had a relatively high proportion of good timber but were excluded from the schedule due to the application of the higher depletion rate ($132.02).

Effect of Changing Timber Prices on the Optimum Schedule

The results of Run 2 indicate, not surprisingly, that, due to the increase in black cherry prices and decline in the price of red oak, stands with a relatively high proportion of black cherry and little red oak were favored. (In this context, "favored" means that the stands were selected for harvest earlier in the planning period). For example, stand 15, which has no red oak and a considerable amount of black cherry (Table 1) was completely harvested by year 4 (Fig. 1), while in the previous run, with unchanged prices, the stand was only partially harvested by the end of year 5. Stands 5 and 8 were forced out of the solution and stands 13 and 14 which carry significant proportions of red oak were less favored.

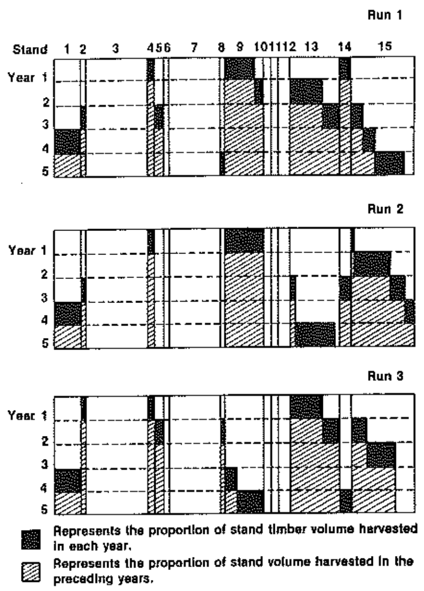

Fig. 1. Optimal harvest schedules for each of the three runs. The area of each stand is drawn to scale to represent the proportion of total forest volume in the stand.

An increase in the value of the harvest occurred, following the changes in timber values, indicating that an upswing in black cherry prices even with a concurrent fall (of the same magnitude), in the value of red oak, is beneficial to the company.

Effect of Changes in Growth Rate on the Optimum Schedule

In Run 3, since the increase in timber volume in stands 9 and 14 was greater than the discount rate, the value of these stands increased over time. As a result, the LP program selected them for harvest toward the end of the five-year period (Fig. 1). Stand 9 was harvested in years 4 and 5 and stand 14 was harvested in year 5. This is of some importance to management since a failure to correctly predict the growth of stands would lead to a nonoptimal harvest schedule. In this case, if stands 9 or 14 were actually growing at a rate of 3%/yr, but were predicted to be growing at 5%, they would be harvested later in the planning period than would be optimal. Unfortunately, the cost of this nonoptimal scheduling cannot be directly determined from the LP solution.

Sensitivity Analysis

Sensitivity analysis shows us by how much the optimal solution will vary with changes in the input data. It is an extremely important aspect of any LP solution. The following summary describes the information provided by sensitivity analysis and its possible implications for management:

1. The *dual price* indicates by how much the optimal value of the objective function will increase (in this problem) following an increase of one unit of any given constraint. Management could use this information to determine, for example, how much to pay for additional acreage of any given stand.
2. The *reduced cost* represented a penalty to the objective function incurred by forcing one unit of a stand not selected for harvest into the harvest schedule. Management may wish to consider this cost if they are forced to cut in stands not selected for harvest (e.g., as a result of mineral extraction).
3. *Slack values* indicate how much of the resource associated with any given constraint was left unused at the optimal solution. For instance, no acres of stand 12 were cut in the optimal solution; the slack value associated with the acreage constraint of stand 12 is, therefore, the full area of the stand (113 ac). From this information management can quickly obtain an impression of the distribution and area of unharvested land.

DISCUSSION

In this study an LP model selected a harvest schedule that maximized net present value within the given constraints. Changes in timber value and growth rate had a strong influence on the stands selected for harvest. The model can, therefore, act in two ways: as a method of determining an optimal harvest schedule, and as a way of predicting the impact of price trends and growth rate on harvesting operations and profitability.

The model could be expanded to incorporate other management considerations such as transportation and harvesting costs. Site restrictions due to seasonal conditions could also be included as a constraint in the model. For example, winter logging may not be possible on some sites due to wet conditions, hence X_{ijw} would be set to 0. That is, the number of acres in stand *i*, which can be logged in year *j* during the winter months, *w*, is zero.

The effectiveness of the LP model in a forest application such as harvest scheduling depends on the availability of reliable data. Inaccurate data will likely result in a less than optimal solution. Linear programming for forest planning can, in fact, be used by management to justify the collection of accurate inventory data, since the formulation and result of an LP problem would indicate which information is of importance in determining an optimum schedule.

Overall, LP appears to be an extremely useful aid to the forest manager. The application of LP does not require a high degree of computer experience, and as more commercial LP software for microcomputers becomes available, its use is likely to spread among smaller forest enterprises. □

APPENDIX

FORMULATION OF HARVEST SCHEDULING PROBLEM

Maximize net present value of harvested acres:

NPV =

$$
\begin{array}{llllll}
522.64(\times 11) & +\ 517.62(\times 12) & +\ 512.64(\times 13) & +\ 508.00(\times 14) & +\ 503.00(\times 15) & + \\
502.00(\times 21) & +\ 497.00(\times 22) & +\ 492.00(\times 23) & +\ 488.00(\times 24) & +\ 483.00(\times 25) & + \\
220.00(\times 31) & +\ 218.00(\times 32) & +\ 216.00(\times 33) & +\ 214.00(\times 34) & +\ 211.00(\times 35) & + \\
781.00(\times 41) & +\ 774.00(\times 42) & +\ 766.00(\times 43) & +\ 759.00(\times 44) & +\ 752.00(\times 45) & + \\
408.00(\times 51) & +\ 404.00(\times 52) & +\ 400.00(\times 53) & +\ 396.00(\times 54) & +\ 392.00(\times 55) & - \\
535.00(\times 61) & -\ 530.00(\times 62) & -\ 525.00(\times 63) & -\ 520.00(\times 64) & -\ 515.00(\times 65) & - \\
297.00(\times 71) & -\ 294.00(\times 72) & -\ 292.00(\times 73) & -\ 289.00(\times 74) & -\ 286.00(\times 75) & + \\
189.00(\times 81) & +\ 187.00(\times 82) & +\ 185.00(\times 83) & +\ 183.00(\times 84) & +\ 182.00(\times 85) & + \\
769.00(\times 91) & +\ 762.00(\times 92) & +\ 754.00(\times 93) & +\ 747.00(\times 94) & +\ 740.00(\times 95) & + \\
130.00(\times 101) & +\ 129.00(\times 102) & +\ 128.00(\times 103) & +\ 127.00(\times 104) & +\ 126.00(\times 105) & + \\
123.00(\times 111) & +\ 122.00(\times 112) & +\ 121.00(\times 113) & +\ 120.00(\times 114) & +\ 119.00(\times 115) & - \\
716.00(\times 121) & -\ 709.00(\times 122) & -\ 702.00(\times 123) & -\ 696.00(\times 124) & -\ 689.00(\times 125) & + \\
784.00(\times 131) & +\ 777.00(\times 132) & +\ 769.00(\times 133) & +\ 762.00(\times 134) & +\ 755.00(\times 135) & + \\
1747(\times 141) & +\ 1730(\times 142) & +\ 1713(\times 143) & +\ 1697(\times 144) & +\ 1680(\times 145) & + \\
176.00(\times 151) & +\ 174.00(\times 152) & +\ 173.00(\times 153) & +\ 171.00(\times 154) & +\ 169.00(\times 155) &
\end{array}
$$

Subject to the following constraints:

1. DON'T CUT MORE ACRES THAN ARE AVAILABLE

$$
\begin{array}{l}
\times 11 + \times 12 + \times 13 + \times 14 + \times 15 < 366.5 \\
\times 21 + \times 22 + \times 23 + \times 24 + \times 25 < 52 \\
\times 31 + \times 32 + \times 33 + \times 34 + \times 35 < 641 \\
\times 41 + \times 42 + \times 43 + \times 44 + \times 45 < 66 \\
\times 51 + \times 52 + \times 53 + \times 54 + \times 55 < 100 \\
\times 61 + \times 62 + \times 63 + \times 64 + \times 65 < 50 \\
\times 71 + \times 72 + \times 73 + \times 74 + \times 75 < 1013 \\
\times 81 + \times 82 + \times 83 + \times 84 + \times 85 < 29.2 \\
\times 91 + \times 92 + \times 93 + \times 94 + \times 95 < 466.3 \\
\times 101 + \times 102 + \times 103 + \times 104 + \times 105 < 173.6 \\
\times 111 + \times 112 + \times 113 + \times 114 + \times 115 < 56.29 \\
\times 121 + \times 122 + \times 123 + \times 124 + \times 125 < 113 \\
\times 131 + \times 132 + \times 133 + \times 134 + \times 135 < 488.5 \\
\times 141 + \times 142 + \times 143 + \times 144 + \times 145 < 50 \\
\times 151 + \times 152 + \times 153 + \times 154 + \times 155 < 1396
\end{array}
$$

2. MUST CUT BETWEEN 3800 AND 4200 MBF EACH YEAR

$$
\begin{array}{l}
7.40\times 11+6.82\times 21+9.92\times 31+5.98\times 41+8.61\times 51+9.51\times 61+5.46\times 71+5.53\times 81+ \\
8.13\times 91+5.64\times 101+6.63\times 111+10.22\times 121+9.54\times 131+16.35\times 141+5.62\times 151 \\
<4200 \\
7.40\times 11+6.82\times 21+9.92\times 31+5.98\times 41+8.61\times 51+9.51\times 61+5.46\times 71+5.53\times 81+ \\
8.13\times 91+5.64\times 101+6.63\times 111+10.22\times 121+9.54\times 131+16.35\times 141+5.62\times 151 \\
>3800
\end{array}
$$