The Fourth Painlevé Transcendent

P. A. CLARKSON,
Institute of Mathematics, Statistics and Actuarial Science,
University of Kent, Canterbury, CT2 7NF, UK
E-mail: P.A.Clarkson@kent.ac.uk

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Abstract

The six Painlevé equations (P_{I}–P_{VI}) were first discovered about a hundred years ago by Painlevé and his colleagues in an investigation of nonlinear second-order ordinary differential equations. During the past 30 years there has been considerable interest in the Painlevé equations primarily due to the fact that they arise as symmetry reductions of the soliton equations which are solvable by inverse scattering. Although first discovered from pure mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications.

The Painlevé equations may be thought of as nonlinear analogues of the classical special functions. They have a Hamiltonian structure and associated isomonodromy problems, which express the Painlevé equations as the compatibility condition of two linear systems. The Painlevé equations also admit symmetries under affine Weyl groups which are related to the associated Bäcklund transformations. These can be used to generate hierarchies of rational solutions and one-parameter families of solutions expressible in terms of the classical special functions, for special values of the parameters. Further solutions of the Painlevé equations have some interesting asymptotics which are use in applications. In this paper I discuss some of the remarkable properties which the Painlevé equations possess using the fourth Painlevé equation (P_{IV}) as an illustrative example.

1 Introduction

The six Painlevé equations (P_{I}–P_{VI}) are the nonlinear ordinary differential equations

$$w'' = 6w^2 + z, \tag{1.1}$$

$$w'' = 2w^3 + zw + \alpha, \tag{1.2}$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{1.3}$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{1.4}$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{1.5}$$

$$\begin{aligned} w'' = {} & \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}, \end{aligned} \tag{1.6}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$ and α, β, γ and δ are arbitrary constants, whose solutions are called the *Painlevé transcendents*. The Painlevé equations P_{I}–P_{VI} were discovered about a hundred years ago by Painlevé, Gambier and their colleagues whilst studying which second order ordinary differential equations of the form

$$w'' = F(z; w, w'), \tag{1.7}$$

where F is rational in w' and w and analytic in z, have the property that the solutions have no movable branch points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property*. Painlevé, Gambier *et al.*

showed that there were fifty canonical equations of the form (1.7) with this property, up to a Möbius (bilinear rational) transformation

$$W(\zeta)=\frac{a(z)w+b(z)}{c(z)w+d(z)},\qquad \zeta=\phi(z), \tag{1.8}$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ and $\phi(z)$ are locally analytic functions. Further they showed that of these fifty equations, forty-four are either integrable in terms of previously known functions (such as elliptic functions or are equivalent to linear equations) or reducible to one of six new nonlinear ordinary differential equations, which define new transcendental functions (see Ince (48)). The Painlevé equations can be thought of as nonlinear analogues of the classical special functions (see Refs. (23; 30; 52; 100)). Indeed Iwasaki, Kimura, Shimomura and Yoshida (52) characterize the Painlevé equations as "the most important nonlinear ordinary differential equations" and state that "many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions". Further Umemura (100) states that "Kazuo Okamoto and his circle predict that in the 21st century a new chapter on Painlevé equations will be added to Whittaker and Watson's book".

Although first discovered from mathematical considerations, the Painlevé equations have subsequently arisen in a variety of applications including statistical mechanics (correlation functions of the XY model and the Ising model), random matrix theory, topological field theory, plasma physics, nonlinear waves (resonant oscillations in shallow water, convective flows with viscous dissipation, Görtler vortices in boundary layers and Hele-shaw problems), quantum gravity, quantum field theory, general relativity, nonlinear and fiber optics, polyelectrolytes, Bose-Einstein condensation and stimulated Raman scattering. Further the Painlevé equations have attracted much interest since they also arise as reductions of the soliton equations, such as the Korteweg-de Vries, Boussinesq and nonlinear Schrödinger equations, which are solvable by inverse scattering (see Ablowitz and Clarkson (2) and references therein).

The general solutions of the Painlevé equations are transcendental, i.e. irreducible in the sense that they cannot be expressed in terms of previously known functions, such as rational functions, elliptic functions or the classical special functions (see Umemura (100)). Essentially, generic solutions of the $\mathrm{P_I}$–$\mathrm{P_V}$ are meromorphic functions in the complex plane (see Hinkkanen and Laine (44–46) and Steinmetz (93), also Gromak, Laine and Shimomura (42)); solutions of $\mathrm{P_{VI}}$ are rather different since the equation has three singular points (see Hinkkanen and Laine (47)).

The Painlevé equations, in common with other integrable equations such as soliton equations, have a plethora of fascinating properties, some of which we discuss in the following sections, using $\mathrm{P_{IV}}$ (1.4) as an illustrative example.

- The equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be written as a (non-autonomous) *Hamiltonian system*, which are discussed in equation (2.
- The equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be expressed as the compatibility condition of a linear system, called an *isomonodromy problem* or *Lax pair*, which are discussed in §3.
- The equations $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess *Bäcklund transformations* which relate one solution to another solution either of the same equation, with different values of the parameters, or another equation, which are discussed in §4; these transformations play an important role in the theory of integrable systems.
- The equations $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess many *rational solutions*, *algebraic solutions* and solutions expressible in terms of the *classical special functions* for certain values of the parameters, which are discussed in Secs. 5–6; these solutions of the Painlevé equations are called "classical solutions", see Umemura(100).
- Since the solutions of the equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ are transcendental, then a study of the *asymptotic behaviour* of these solutions, together with the associated *connection formulae* which relate the asymptotic behaviours of the solutions as $z\to\pm\infty$, play an important role in the application of the equations. These are discussed in §7.

A representation of $\mathrm{P_{IV}}$ which has attracted much recent interest is the *symmetric* $\mathrm{P_{IV}}$ ($\mathrm{sP_{IV}}$) system

$$\varphi_1'+\varphi_1(\varphi_2-\varphi_3)+2\mu_1=0, \tag{1.9a}$$
$$\varphi_2'+\varphi_2(\varphi_3-\varphi_1)+2\mu_2=0, \tag{1.9b}$$
$$\varphi_3'+\varphi_3(\varphi_1-\varphi_2)+2\mu_3=0, \tag{1.9c}$$

where μ_1, μ_2 and μ_3 are constants, with the constraints

$$\mu_1+\mu_2+\mu_3=1,\qquad \varphi_1+\varphi_2+\varphi_3=-2z. \tag{1.9d}$$

Eliminating φ_2 and φ_3, then $w=\varphi_1$ satisfies $\mathrm{P_{IV}}$ (1.4) with $\alpha=\mu_3-\mu_2$ and $\beta=-2\mu_1^2$. The system $\mathrm{sP_{IV}}$ (1.9) was derived by Bureau (15; 16) and later by Adler (5). The $\mathrm{sP_{IV}}$ system (1.9) is extensively discussed by Noumi (79) and arises in applications such as random matrices (see Forrester and Witte (35; 36)); other studies of $\mathrm{sP_{IV}}$ include Refs. (55; 72; 80; 83; 90–92; 94; 99; 106; 109).

2 Hamiltonian Structure

The Hamiltonian structure associated with the Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ is $\mathcal{H}_\mathrm{J} = (q, p, \mathrm{H_J}, z)$, where $\mathrm{H_J}$, the Hamiltonian function, is a polynomial in q, p and rational in z. Each of the Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be written as a Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathrm{H_J}}{\partial p}, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathrm{H_J}}{\partial q}, \tag{2.1}$$

for a suitable Hamiltonian function $\mathrm{H_J}(q, p, z)$ (see Okamoto (88)). Further the function $\sigma(z) \equiv \mathrm{H_J}(q, p, z)$ satisfies a second-order, second-degree ordinary differential equation, whose solution is expressible in terms of the solution of the associated Painlevé equation.

Example 2.1. The Hamiltonian for $\mathrm{P_{IV}}$ (1.4) is

$$\mathrm{H_{IV}}(q, p, z) = 2qp^2 - (q^2 + 2zq + 2\kappa_0)p + \kappa_\infty q, \tag{2.2}$$

with κ_0 and κ_∞ parameters, and so

$$q' = 4qp - q^2 - 2zq - 2\kappa_0, \tag{2.3a}$$

$$p' = -2p^2 + 2pq + 2zp - \kappa_\infty. \tag{2.3b}$$

Eliminating p then $w = q$ satisfies $\mathrm{P_{IV}}$ (1.4) with $\alpha = 1 - \kappa_0 + 2\kappa_\infty$ and $\beta = -2\kappa_0^2$, whilst eliminating q then $w = -2p$ satisfies $\mathrm{P_{IV}}$ (1.4) with $\alpha = 2\kappa_0 - \kappa_\infty - 1$ and $\beta = -2\kappa_\infty^2$. The function $\sigma = \mathrm{H_{IV}}(q, p, z)$, defined by (2.2) satisfies the second-order second-degree ordinary differential equation

$$(\sigma'')^2 - 4(z\sigma' - \sigma)^2 + 4\sigma'(\sigma' + 2\kappa_0)(\sigma' + 2\kappa_\infty) = 0 \tag{2.4}$$

(see Jimbo and Miwa (53) and Okamoto (84)). Equation (2.4) is equivalent to equation SD-I.c in the classification of second order, second degree ordinary differential equations with the Painlevé property due to Cosgrove and Scoufis (27), an equation first derived and solved by Chazy (17) and rederived by Bureau (13; 14). Equation (2.4 also arises in various applications including random matrix theory, see, for example, Refs. (35; 36; 57; 98). Conversely, if σ is a solution of (2.4), then solutions of (2.3) are given by

$$q = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\kappa_\infty)}, \qquad p = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\kappa_0)}. \tag{2.5}$$

We remark for $\mathrm{sP_{IV}}$ (1.9), the Hamiltonian is

$$\mathrm{H_{IV}}(\varphi_1, \varphi_2, \varphi_3, z) = -\tfrac{1}{2}\varphi_1\varphi_2\varphi_3 + \mu_1\varphi_2 - \mu_2\varphi_1, \tag{2.6}$$

since $q = \varphi_1$, $p = -\frac{1}{2}\varphi_2$, $\kappa_0 = \mu_1$ and $\kappa_\infty = -\mu_2$.

3 Isomonodromy Problems

Each of the Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be expressed as the compatibility condition of the linear system

$$\frac{\partial \boldsymbol{\Psi}}{\partial \lambda} = \mathbf{A}(z;\lambda)\boldsymbol{\Psi}, \qquad \frac{\partial \boldsymbol{\Psi}}{\partial z} = \mathbf{B}(z;\lambda)\boldsymbol{\Psi}, \tag{3.1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices whose entries depend on the solution $w(z)$ of the Painlevé equation. The equation

$$\frac{\partial^2 \boldsymbol{\Psi}}{\partial z \partial \lambda} = \frac{\partial^2 \boldsymbol{\Psi}}{\partial \lambda \partial z},$$

is satisfied provided that

$$\frac{\partial \mathbf{A}}{\partial z} - \frac{\partial \mathbf{B}}{\partial \lambda} + \mathbf{AB} - \mathbf{BA} = 0, \tag{3.2}$$

which is the compatibility condition of (3.1). Matrices $\mathbf{A}$ and $\mathbf{B}$ for $\mathrm{P_I}$–$\mathrm{P_{VI}}$ satisfying (3.2) are given by Jimbo and Miwa (54) (see also Refs. (30) and (51)), though these are not unique, as illustrated in the following exmaples.

Example 3.1. Jimbo and Miwa (54) show that the compatibility condition of the linear system (3.1) with

$$\mathbf{A}(z;\lambda) = \begin{pmatrix} \lambda + z + \dfrac{\theta_0 - 2v}{2\lambda} & u - \dfrac{uw}{2\lambda} \\ \dfrac{2v - \theta_0 - \theta_\infty}{u} + \dfrac{2v(v-\theta_0)}{\lambda uw} & -\lambda - z + \dfrac{2v - \theta_0}{2\lambda} \end{pmatrix}, \tag{3.3a}$$

$$\mathbf{B}(z;\lambda) = \begin{pmatrix} \lambda & u \\ \dfrac{2v - \theta_0 - \theta_\infty}{u} & \lambda \end{pmatrix}, \tag{3.3b}$$

with θ_0 and θ_∞ parameters, is

$$u' = -u(w + 2z), \tag{3.4a}$$

$$v' = \frac{2\theta_0 - 2v^2}{w} + \tfrac{1}{2}(\theta_0 + \theta_\infty - 2v)w, \tag{3.4b}$$

$$w' = -4v + w^2 + 2zw + 2\theta_0. \tag{3.4c}$$

Eliminating v then w satisfies $\mathrm{P_{IV}}$ (1.4) with $\alpha = \theta_\infty - 1$ and $\beta = -2\theta_0^2$.

Example 3.2. As remarked above, the isomonodromy problem is not unique. Kitaev (60) and Milne, Clarkson and Bassom (74) show that $\mathrm{P_{IV}}$ (1.4) also arises as the compatibility condition of the linear system (3.1) with

$$\mathbf{A}(z;\lambda) = \begin{pmatrix} \tfrac{1}{2}\lambda^3 + (z + uv) + \dfrac{\Theta_0}{\lambda} & \mathrm{i}(\lambda^2 + 2zu + u') \\ \mathrm{i}(\lambda^2 + 2zv - v') & -\tfrac{1}{2}\lambda^3 - (z + uv) - \dfrac{\Theta_0}{\lambda} \end{pmatrix}, \tag{3.5a}$$

$$\mathbf{B}(z;\lambda) = \begin{pmatrix} \tfrac{1}{2}\lambda^2 + uv & \mathrm{i}\lambda u \\ \mathrm{i}\lambda v & -\tfrac{1}{2}\lambda^2 - uv \end{pmatrix}. \tag{3.5b}$$

with Θ_0 a constant, which comes from a scaling reduction of the Lax pair for the modified nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}(|u|^2 u)_x, \tag{3.6}$$

due to Kaup and Newell (59). Substituting (3.5) into the compatibility condition (3.2) yields

$$u'' + 2zu' + u + 2\Theta_0 u - 4zu^2 v - 2uvu' = 0, \tag{3.7a}$$

$$v'' - 2zv' - v + 2\Theta_0 v - 4zuv^2 + 2uvv' = 0, \tag{3.7b}$$

which implies that

$$u'v - uv' + 2zuv - u^2v^2 = 2\Theta_1, \tag{3.8}$$

with Θ_1 a constant. Then $w = uv$ satisfies $\mathrm{P_{IV}}$ (1.4) with $\alpha = 2\Theta_0 - \Theta_1$ and $\beta = -2\Theta_1^2$.

4 Bäcklund and Schlesinger Transformations

4.1 Bäcklund Transformations

The Painlevé equations $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess *Bäcklund transformations* which relate one solution to another solution either of the same equation, with different values of the parameters, or another equation (see Refs. (23; 28; 42) and the references therein). An important application of the Bäcklund transformations is that they generate hierarchies of classical solutions of the Painlevé equations, which are discussed in §5 and §6.

Theorem 4.1. *Let $w_0 = w(z;\alpha_0,\beta_0)$ and $w_j^{\pm} = w(z;\alpha_j^{\pm},\beta_j^{\pm})$, $j = 1,2,3,4$ be solutions of* $\mathrm{P_{IV}}$ *(1.4) with*

$$\alpha_1^{\pm} = \tfrac{1}{4}(2 - 2\alpha_0 \pm 3\sqrt{-2\beta_0}), \qquad \beta_1^{\pm} = -\tfrac{1}{2}(1 + \alpha_0 \pm \tfrac{1}{2}\sqrt{-2\beta_0})^2, \tag{4.1a}$$

$$\alpha_2^{\pm} = -\tfrac{1}{4}(2 + 2\alpha_0 \pm 3\sqrt{-2\beta_0}), \qquad \beta_2^{\pm} = -\tfrac{1}{2}(1 - \alpha_0 \pm \tfrac{1}{2}\sqrt{-2\beta_0})^2, \tag{4.1b}$$

$$\alpha_3^{\pm} = \tfrac{3}{2} - \tfrac{1}{2}\alpha_0 \mp \tfrac{3}{4}\sqrt{-2\beta_0}, \qquad \beta_3^{\pm} = -\tfrac{1}{2}(1 - \alpha_0 \pm \tfrac{1}{2}\sqrt{-2\beta_0})^2, \tag{4.1c}$$

$$\alpha_4^{\pm} = -\tfrac{3}{2} - \tfrac{1}{2}\alpha_0 \mp \tfrac{3}{4}\sqrt{-2\beta_0}, \qquad \beta_4^{\pm} = -\tfrac{1}{2}(-1 - \alpha_0 \pm \tfrac{1}{2}\sqrt{-2\beta_0})^2. \tag{4.1d}$$

Table 4.1: The effect of the Bäcklund transformations of $\mathrm{sP_{IV}}$

	μ_1	μ_2	μ_3	φ_1	φ_2	φ_3
$\mathcal{S}_1$	$-\mu_1$	$\mu_1+\mu_2$	$\mu_1+\mu_3$	φ_1	$\varphi_2-\dfrac{2\mu_1}{\varphi_1}$	$\varphi_3-\dfrac{2\mu_1}{\varphi_1}$
$\mathcal{S}_2$	$\mu_1+\mu_2$	$-\mu_2$	$\mu_2+\mu_3$	$\varphi_1-\dfrac{2\mu_2}{\varphi_2}$	φ_2	$\varphi_3+\dfrac{2\mu_2}{\varphi_2}$
$\mathcal{S}_3$	$\mu_1+\mu_3$	$\mu_2+\mu_3$	$-\mu_3$	$\varphi_1+\dfrac{2\mu_3}{\varphi_3}$	$\varphi_2-\dfrac{2\mu_3}{\varphi_3}$	φ_3
$\boldsymbol{\pi}$	μ_2	μ_3	μ_1	φ_2	φ_3	φ_1

Then

$$\mathcal{T}_1^{\pm}: \qquad w_1^{\pm} = \frac{w_0' - w_0^2 - 2zw_0 \mp \sqrt{-2\beta_0}}{2w_0}, \tag{4.2a}$$

$$\mathcal{T}_2^{\pm}: \qquad w_2^{\pm} = -\frac{w_0' + w_0^2 + 2zw_0 \mp \sqrt{-2\beta_0}}{2w_0}, \tag{4.2b}$$

$$\mathcal{T}_3^{\pm}: \qquad w_3^{\pm} = w_0 + \frac{2\left(1-\alpha_0 \mp \frac{1}{2}\sqrt{-2\beta_0}\,\right) w_0}{w_0' \pm \sqrt{-2\beta_0} + 2zw_0 + w_0^2}, \tag{4.2c}$$

$$\mathcal{T}_4^{\pm}: \qquad w_4^{\pm} = w_0 + \frac{2\left(1+\alpha_0 \pm \frac{1}{2}\sqrt{-2\beta_0}\,\right) w_0}{w_0' \mp \sqrt{-2\beta_0} - 2zw_0 - w_0^2}, \tag{4.2d}$$

valid when the denominators are non-zero, and where the upper signs or the lower signs are taken throughout each transformation.

Proof. See Lukashevich (64), Gromak (40; 41); also Bassom, Clarkson and Hicks (8), Gromak, Laine and Shimomura (42) and Murata (77). □

The parameter space of $\mathrm{P_{IV}}$ can be identified with the Cartan subalgebra of a simple Lie algebra and the corresponding *affine Weyl group* $\widetilde{A}_2$ acts on $\mathrm{P_{IV}}$ as a group of Bäcklund transformations (for further details see Refs. (80; 83; 84; 101)). An affine Weyl group is essentially a group of translations and reflections on a lattice, which for $\mathrm{P_{IV}}$ is in the parameter space. The affine Weyl group $\widetilde{A}_2 = \langle \mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \boldsymbol{\pi}\rangle$ with fundamental relations

$$\mathcal{S}_j^2 = \mathcal{I}, \qquad (\mathcal{S}_j\mathcal{S}_{j+1})^3 = \mathcal{I}, \qquad \boldsymbol{\pi}^3 = \mathcal{I}, \qquad \boldsymbol{\pi}\mathcal{S}_j = \mathcal{S}_{j+1}\boldsymbol{\pi}, \tag{4.3}$$

for $j = 1, 2, 3$, with $\mathcal{S}_{j+3} = \mathcal{S}_j$ for $j \in \mathbb{Z}$.

Theorem 4.2. *The Bäcklund transformations of* $\mathrm{sP_{IV}}$ *(1.9) are defined by the fundamental relations (4.3) and realized as a group of automorphisms of the field of rational functions* $\mathbb{C}(\mu_j, \varphi_j)$*, for* $j = 1, 2, 3$*, as*

$$\mathcal{S}_j(\mu_j) = -\mu_j, \quad \mathcal{S}_j(\mu_{j\pm1}) = \mu_{j\pm1} + \mu_j, \tag{4.4a}$$

$$\mathcal{S}_j(\varphi_j) = \varphi_j, \qquad \mathcal{S}_j(\varphi_{j\pm1}) = \varphi_{j\pm1} \pm \frac{\mu_j}{\varphi_j}, \tag{4.4b}$$

$$\boldsymbol{\pi}(\mu_j) = \mu_{j+1}, \quad \boldsymbol{\pi}(\varphi_j) = \varphi_{j+1}. \tag{4.4c}$$

Proof. See Noumi and Yamada (80; 82); see also Noumi (79). The effect of these Bäcklund transformations is summarized in Table 4.1. □

Discrete Painlevé equations, which are discrete equations (difference equations) that have Painlevé equations as their continuum limits, arising from the Bäcklund transformations of $\mathrm{P_{IV}}$ (1.4) given by (4.2) are discussed in Refs. (26; 29; 38), as illustrated in the following example.

Example 4.3. Consider the Bäcklund transformations $\mathcal{T}_1^+$ and $\mathcal{T}_2^-$. Setting $a_{n+1} = \alpha_1^+$, $a_n = \alpha_0$, $a_{n-1} = \alpha_2^-$, $c_{n+1} = \sqrt{-2(\beta_1^+)^2}$, $c_n = \sqrt{-2\beta_0^2}$ and $c_{n-1} = \sqrt{-2(\beta_2^-)^2}$ yields the difference equations

$$\begin{aligned} a_{n+1} &= \tfrac{1}{2} - \tfrac{1}{2}a_n + \tfrac{3}{4}c_n, \qquad & a_{n-1} &= -\tfrac{1}{2} + \tfrac{1}{2}a_n - \tfrac{3}{4}c_n,\\ c_{n+1} &= 1 + a_n + \tfrac{1}{2}c_n, & c_{n-1} &= -1 + a_n + \tfrac{1}{2}c_n, \end{aligned}$$

which have the solution

$$a_n = \tfrac{1}{2}\kappa - \tfrac{3}{2}\mu(-1)^n + \tfrac{1}{2}n, \qquad c_n = \kappa + \mu(-1)^n + n,$$

with κ and μ arbitrary constants. Eliminating w_0' between the Bäcklund transformations $\mathcal{T}_1^+$ and $\mathcal{T}_2^-$, then setting $x_n = w_0(z; a_n, c_n)$, $x_{n+1} = w_1^+(z; a_{n+1}, c_{n+1})$ and $x_{n-1} = w_2^-(z; a_{n-1}, c_{n-1})$, with $c_m = \sqrt{-2b_m}$, gives

$$(x_{n+1} + x_n + x_{n-1})x_n = -2zx_n - n + \kappa(-1)^n + \mu, \tag{4.5}$$

which is the discrete Painlevé equation $\mathrm{dP_I}$ (29). The relationship between solutions of $\mathrm{P_{IV}}$ (1.4) and solutions of dPI (4.5) was first noted by Fokas, Its and Kapaev (31; 32). Subsequently this relationship was studied by Bassom, Clarkson and Hicks (8) who derived hierarchies of simultaneous solutions of $\mathrm{P_{IV}}$ (1.4) and $\mathrm{dP_I}$ (4.5) in terms of parabolic cylinder functions (the *half-integer hierarchy* discussed in §6.3); see also Ref. (38). The relationship between solutions of $\mathrm{P_{IV}}$ (1.4) and $\mathrm{dP_I}$ (4.5) is reflected in the similarity of the results for $\mathrm{dP_I}$ in Ref. (38) to those for $\mathrm{P_{IV}}$ in Refs. (8; 77; 84; 103).

4.2 Schlesinger transformations

A class of Bäcklund transformations for the Painlevé equations is generated by so-called *Schlesinger transformations* of the associated isomondromy problems (see Refs. (33) and (76)).

Fokas, Mugan and Ablowitz (33), using a Schlesinger transformation formulation associated with the isomonodromy formulation of $\mathrm{P_{IV}}$, deduced the following transformations $\mathcal{R}_1$–$\mathcal{R}_4$ for $\mathrm{P_{IV}}$.

$$\mathcal{R}_1 : w_1 = \frac{\left(w' + \sqrt{-2\beta}\right)^2 + \left(4\alpha + 4 - 2\sqrt{-2\beta}\right) w^2 - w^2(w+2z)^2}{2w\left(w^2 + 2zw - w' - \sqrt{-2\beta}\right)}, \tag{4.6a}$$

$$\mathcal{R}_2 : w_2 = \frac{\left(w' - \sqrt{-2\beta}\right)^2 + \left(4\alpha - 4 - 2\sqrt{-2\beta}\right) w^2 - w^2(w+2z)^2}{2w\left(w^2 + 2zw + w' - \sqrt{-2\beta}\right)}, \tag{4.6b}$$

$$\mathcal{R}_3 : w_3 = \frac{\left(w' - \sqrt{-2\beta}\right)^2 - \left(4\alpha + 4 + 2\sqrt{-2\beta}\right) w^2 - w^2(w+2z)^2}{2w\left(w^2 + 2zw - w' + \sqrt{-2\beta}\right)}, \tag{4.6c}$$

$$\mathcal{R}_4 : w_4 = \frac{\left(w' + \sqrt{-2\beta}\right)^2 + \left(4\alpha - 4 + 2\sqrt{-2\beta}\right) w^2 - w^2(w+2z)^2}{2w\left(w^2 + 2zw + w' + \sqrt{-2\beta}\right)}, \tag{4.6d}$$

where $w \equiv w(z; \alpha, \beta)$, $w_j \equiv w(z; \alpha_j, \beta_j)$. Fokas, Mugan and Ablowitz (33) also defined the composite transformations $\mathcal{R}_5 = \mathcal{R}_1\mathcal{R}_3$, $\mathcal{R}_6^+ \equiv \mathcal{R}_2\mathcal{R}_3$, $\mathcal{R}_6^- \equiv \mathcal{R}_1\mathcal{R}_4$ and $\mathcal{R}_7 = \mathcal{R}_2\mathcal{R}_4$ given by

$$\mathcal{R}_5 : w_5 = \frac{\left(w' - w^2 - 2zw\right)^2 + 2\beta}{2w\left\{w' - w^2 - 2zw + 2(\alpha+1)\right\}}, \tag{4.7a}$$

$$\begin{aligned}\mathcal{R}_6^{\pm} : w_6 = w &+ \frac{\left(2\alpha - 2 \mp \sqrt{-2\beta}\right) w M^{\pm}(w,w',z)}{w(4 \pm 2\sqrt{-2\beta}) - M^{\pm}(w,w',z)\left(w' - 2zw - w^2 \mp \sqrt{-2\beta}\right)} \\ &+ \frac{\left(2 + 2\alpha \pm \sqrt{-2\beta}\right) w}{w' - 2zw - w^2 \mp \sqrt{-2\beta}} + \frac{2 \pm \sqrt{-2\beta}}{M^{\pm}(w,w',z)},\end{aligned} \tag{4.7b}$$

$$\mathcal{R}_7 : w_7 = \frac{\left(w' + w^2 + 2zw\right)^2 + 2\beta}{2w\left\{w' + w^2 + 2zw - 2(\alpha-1)\right\}}, \tag{4.7c}$$

respectively, where

$$M^{\pm}(w,w',z) = \tfrac{1}{2}w + z + \frac{\left(2 + 2\alpha \pm \sqrt{-2\beta}\right) w}{w' - 2zw - w^2 \mp \sqrt{-2\beta}} + \frac{w' \mp \sqrt{-2\beta}}{2w}. \tag{4.7d}$$

We remark that $\mathcal{R}_5$ and $\mathcal{R}_7$ are the transformations $\mathcal{T}_+$ and $\mathcal{T}_-$, respectively, given by Murata (77) and a special case of $\mathcal{R}_6^{\pm}$ is given by Boiti and Pempinelli (11). Further note that w_1 and w_2 in (4.6a,b) are respectively $x_{n\pm 2}$ in the sense of (4.5). The effect of these Schlesinger transformations on the $\mathrm{P_{IV}}$ parameters α and β, the isomonodromy parameters θ_0 and θ_∞, and the $\mathrm{sP_{IV}}$ parameters μ_1, μ_2 and μ_3 is given in Table 4.2.

The Schlesinger transformations $\mathcal{R}_1$–$\mathcal{R}_5$, $\mathcal{R}_6^{\pm}$ and $\mathcal{R}_7$ are Bäcklund transformations of $\mathrm{P_{IV}}$ which involve $(w')^2$. The Bäcklund transformations for $\mathrm{P_{IV}}$ given by $\mathcal{T}_1^{\pm}$–$\mathcal{T}_4^{\pm}$ (4.2), have the form

$$\widehat{w}(z; \widehat{\alpha}, \widehat{\beta}) = \frac{A(w,z)w' + B(w,z)}{C(w,z)w' + D(w,z)}, \tag{4.8}$$

Table 4.2: The effect of the Schlesinger transformations $\mathcal{R}_1$–$\mathcal{R}_5$, $\mathcal{R}_6^{\pm}$ and $\mathcal{R}_7$ on the parameters.

	α	β	θ_0	θ_∞	μ_1	μ_2	μ_3
$\mathcal{R}_1$	$\alpha+1$	$-\frac{1}{2}\left(2-\sqrt{-2\beta}\right)^2$	θ_0-1	$\theta_\infty+1$	μ_1-1	μ_2	μ_3+1
$\mathcal{R}_2$	$\alpha-1$	$-\frac{1}{2}\left(2+\sqrt{-2\beta}\right)^2$	θ_0+1	$\theta_\infty-1$	μ_1+1	μ_2	μ_3-1
$\mathcal{R}_3$	$\alpha+1$	$-\frac{1}{2}\left(2+\sqrt{-2\beta}\right)^2$	θ_0+1	$\theta_\infty+1$	μ_1+1	μ_2-1	μ_3
$\mathcal{R}_4$	$\alpha-1$	$-\frac{1}{2}\left(2-\sqrt{-2\beta}\right)^2$	θ_0-1	$\theta_\infty-1$	μ_1-1	μ_2+1	μ_3
$\mathcal{R}_5\equiv\mathcal{R}_1\mathcal{R}_3$	$\alpha+2$	β	θ_0	$\theta_\infty+2$	μ_1	μ_2-1	μ_3+1
$\mathcal{R}_6^+\equiv\mathcal{R}_2\mathcal{R}_3$	α	$-\frac{1}{2}\left(4+\sqrt{-2\beta}\right)^2$	θ_0+2	θ_∞	μ_1+2	μ_2-1	μ_3-1
$\mathcal{R}_6^-\equiv\mathcal{R}_1\mathcal{R}_4$	α	$-\frac{1}{2}\left(4-\sqrt{-2\beta}\right)^2$	θ_0-2	θ_∞	μ_1-2	μ_2+1	μ_3+1
$\mathcal{R}_7\equiv\mathcal{R}_2\mathcal{R}_4$	$\alpha-2$	β	θ_0	$\theta_\infty-2$	μ_1	μ_2+1	μ_3-1

where $A(w,z)$, $B(w,z)$, $C(w,z)$ and $D(w,z)$ are polynomials in w and locally analytic in z, with $AD-BC\neq 0$; for further details see Refs. (41) and (64). Bassom, Clarkson and Hicks (8) show that the Schlesinger transformations $\mathcal{R}_1$–$\mathcal{R}_5$, $\mathcal{R}_6^{\pm}$ and $\mathcal{R}_7$ can be written in terms of the Bäcklund transformations $\mathcal{T}_1^{\pm}$–$\mathcal{T}_4^{\pm}$ (4.2). Second order difference equations which are linear the highest coefficient, in particular the discrete Painlevé equations, can be derived from Bäcklund transformations of the form (4.8), cf. Ref. (29), whereas Bäcklund transformations which involve $(w')^2$ usually give rise to difference equations that are of second degree in the highest coefficient, cf. Ref. (26).

5 Rational solutions of $\mathrm{P_{IV}}$

The Painlevé equations $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess sets of rational solutions, often called hierarchies, for special values of the parameters (see Refs. (4; 23; 42) and the references therein). These hierarchies can be generated from "seed solutions", such as the simple solution for $\mathrm{P_{IV}}$ (1.4) given in (5.1) using the Bäcklund transformations discussed in §4 above and frequently are expressed in the form of determinants.

Simple rational solutions of $\mathrm{P_{IV}}$ (1.4) are

$$w_1(z;\pm2,-2)=\pm1/z,\quad w_2(z;0,-2)=-2z,\quad w_3(z;0,-\tfrac{2}{9})=-\tfrac{2}{3}z. \tag{5.1}$$

It is known that there are three sets of rational solutions of $\mathrm{P_{IV}}$, which have the solutions (5.1) as the simplest members. These sets are known as the "$-1/z$ hierarchy", the "$-2z$ hierarchy" and the "$-\frac{2}{3}z$ hierarchy", respectively (see Bassom, Clarkson and Hicks (8)). The "$-1/z$ hierarchy" and the "$-2z$ hierarchy" form the set of rational solutions of $\mathrm{P_{IV}}$ (1.4) with parameters given by (5.2) and the "$-\frac{2}{3}z$ hierarchy" forms the set with parameters given by (5.3). The rational solutions of $\mathrm{P_{IV}}$ (1.4) with parameters given by (5.2) lie at the vertexes of the "Weyl chambers" and those with parameters given by (5.3) lie at the centres of the "Weyl chamber", see Umemura and Watanabe (103). The corresponding simple rational solutions of $\mathrm{sP_{IV}}$ (1.9) are

$$\varphi_1=-1/z,\quad \varphi_2=1/z,\quad \varphi_3=-2z,$$

with parameters $\mu_1=\mu_2=1$ and $\mu_3=-1$ and

$$\varphi_1=\varphi_2=\varphi_3=-\tfrac{2}{3}z,$$

with parameters $\mu_1=\mu_2=\mu_3=\frac{1}{3}$.

Theorem 5.1. $\mathrm{P_{IV}}$ *(1.4) has rational solutions if and only if the parameters α and β are given by either*

$$\alpha=m,\qquad \beta=-2(2n-m+1)^2, \tag{5.2}$$

or

$$\alpha=m,\qquad \beta=-2(2n-m+\tfrac{1}{3})^2, \tag{5.3}$$

with $m,n\in\mathbb{Z}$. Further the rational solutions for these parameters are unique.

Proof. See Lukashevich (64), Gromak (41) and Murata (77); also Bassom, Clarkson and Hicks (8), Gromak, Laine and Shimomura (42), Umemura and Watanabe (103). □

In a comprehensive study of properties of solutions of $\mathrm{P_{IV}}$, Okamoto (84) introduced two sets of polynomials associated with rational solutions of $\mathrm{P_{IV}}$, analogous to the *Yablonskii–Vorob'ev polynomials*, which are special polynomials associated with rational solutions of $\mathrm{P_{II}}$ (for further details see Refs. (25; 107; 110)). Noumi and Yamada (83) generalized Okamoto's results and introduced the *generalized Hermite polynomials* $H_{m,n}$, defined in Theorem 5.2, and the *generalized Okamoto polynomials* $Q_{m,n}$, defined in Theorem 5.4 (see also Clarkson (20)). Noumi and Yamada (83) expressed the generalized Hermite and generalized Okamoto polynomials in terms of Schur functions related to the modified Kadomtsev-Petviashvili hierarchy. Kajiwara and Ohta (56) also expressed rational solutions of $\mathrm{P_{IV}}$ in terms of Schur functions by expressing the solutions in the form of determinants, see equation (5.8.

5.1 Generalized Hermite polynomials

Here we consider the generalized Hermite polynomials $H_{m,n}$ which are defined in the following theorem.

Theorem 5.2. *Suppose $H_{m,n}$ satisfies the recurrence relations*

$$2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 + 2mH^2_{m,n}, \tag{5.4a}$$

$$2nH_{m,n+1}H_{m,n-1} = -H_{m,n}H''_{m,n} + \left(H'_{m,n}\right)^2 + 2nH^2_{m,n}, \tag{5.4b}$$

with $H_{0,0} = H_{1,0} = H_{0,1} = 1$ and $H_{1,1} = 2z$, then solutions of $\mathrm{P_{IV}}$ *are*

$$w^{[1]}_{m,n} = w(z; 2m+n+1, -2n^2) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{H_{m+1,n}}{H_{m,n}}\right), \tag{5.5a}$$

$$w^{[2]}_{m,n} = w(z; -(m+2n+1), -2m^2) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{H_{m,n}}{H_{m,n+1}}\right), \tag{5.5b}$$

$$w^{[3]}_{m,n} = w(z; n-m, -2(m+n+1)^2) = -2z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{H_{m,n+1}}{H_{m+1,n}}\right). \tag{5.5c}$$

Proof. See Theorem 4.4 in Noumi and Yamada (83); also Theorem 3.1 in Clarkson (20). In terms of the rational solutions (5.5), the corresponding rational solutions of $\mathrm{sP_{IV}}$ (1.9) are given by $\varphi_j = w^{[j]}_{m,n}$, for $j = 1, 2, 3$, with parameters $\mu_1 = -n$, $\mu_2 = -m$ and $\mu_3 = m + n + 1$. □

The polynomials $H_{m,n}$ defined by (5.4) are called the *generalized Hermite polynomials* since $H_{m,1}(z) = H_m(z)$ and $H_{1,m}(z) = \mathrm{i}^{-m}H_m(\mathrm{i}z)$, where $H_m(z)$ is the standard Hermite polynomial defined by

$$H_m(z) = (-1)^m \exp(z^2)\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{\exp(-z^2)\right\} \tag{5.6}$$

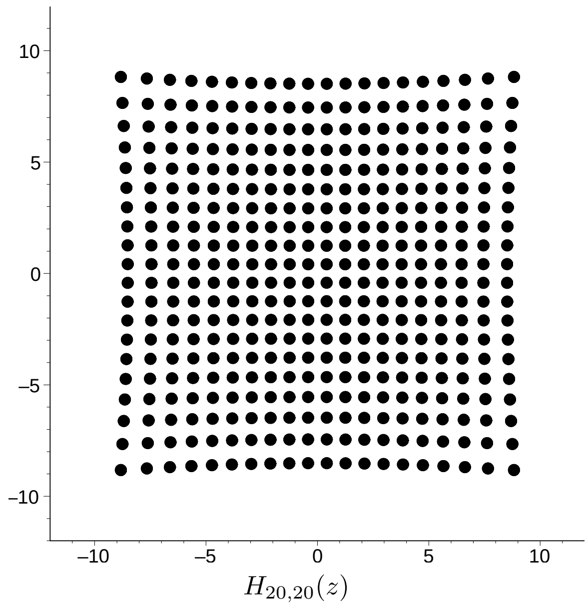
or alternatively through the generating function

$$\sum_{m=0}^{\infty} \frac{H_m(z)\,\lambda^m}{m!} = \exp(2\lambda z - \lambda^2) \tag{5.7}$$

(cf. Refs. (3; 6; 97)). The rational solutions of $\mathrm{P_{IV}}$ defined by (5.5) include all solutions in the "$-1/z$" and "$-2z$" hierarchies, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (5.2), and can be expressed in terms of determinants whose entries are Hermite polynomials; see equation (5.8. These rational solutions of $\mathrm{P_{IV}}$ are special cases of the special function solutions which are expressible in terms of parabolic cylinder functions $D_\nu(\xi)$, as shown in §6.2. The polynomial $H_{m,n}$ has degree mn with integer coefficients, in fact $H_{m,n}(\tfrac{1}{2}\zeta)$ is a monic polynomial in ζ with integer coefficients (for further details see Refs. (20; 83)). Further $H_{m,n}$ possesses the symmetry $H_{n,m}(z) = \mathrm{i}^{-mn}H_{m,n}(\mathrm{i}z)$.

Plots of the complex roots of $H_{20,20}(z)$ and $H_{21,19}(z)$ are given in Figure 5.1. These plots, which are invariant under reflections in the real and imaginary z-axes, take the form of $m \times n$ "rectangles", though these are only approximate rectangles since the roots lie on arcs rather than straight lines as can be seen by looking at the actual values of the roots.

Remark 5.3.



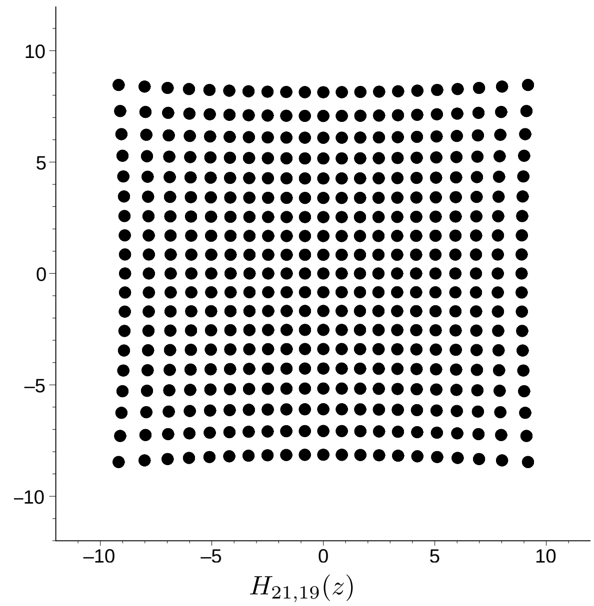


Figure 5.1: Roots of the generalized Hermite polynomials $H_{20,20}(z)$, $H_{21,19}(z)$

1. The generalized Hermite polynomials $H_{m,n}$ can be expressed in determinantal form as follows

$$H_{m,n}(z) = c_{m,n}\mathcal{W}(H_m, H_{m+1}, \ldots, H_{m+n-1}), \tag{5.8}$$

where $H_n(z)$ is the nth Hermite polynomial, $c_{m,n}$ is a constant and $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the usual Wronskian. The generalized Hermite polynomials $H_{m,n}$ can also be expressed in terms of Schur polynomials; for further details see Kajiwara and Ohta (56) and Noumi and Yamada(83).

2. Using the Hamiltonian formalism for $\mathrm{P_{IV}}$ discussed in §2, it can be shown that the generalized Hermite polynomials $H_{m,n}$, which are defined by the differential-difference equations (5.4), also satisfy fourth order bilinear ordinary differential equation

$$\begin{aligned} H_{m,n}H_{m,n}'''' - 4H_{m,n}'H_{m,n}''' + 3\left(H_{m,n}''\right)^2 + 4zH_{m,n}H_{m,n}' - 8mnH_{m,n}^2 \\ - 4(z^2+2n-2m)\left\{H_{m,n}H_{m,n}'' - \left(H_{m,n}'\right)^2\right\} = 0, \end{aligned} \tag{5.9}$$

and homogeneous difference equations; for further details see Clarkson(21; 22).

3. The polynomial $H_{m,n}(z)$ can be expressed as the multiple integral

$$\begin{aligned} H_{m,n}(z) = \frac{\pi^{m/2}\prod_{k=1}^{m} k!}{2^{m(m+2n-1)/2}} \int_{-\infty}^{\infty} \overset{n}{\cdots} \int_{-\infty}^{\infty} \prod_{i=1}^{n}\prod_{j=i+1}^{n} (x_i - x_j)^2 \\ \times \prod_{k=1}^{n} (z - x_k)^m \exp\left(-x_k^2\right) \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n, \end{aligned} \tag{5.10}$$

which arises in random matrix theory; for further details see Brézin and Hikami (12), Forrester and Witte (35).

4. The orthogonal polynomials on the real line with respect to the weight $w(x;z,m) = (x-z)^m \exp(-x^2)$, satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + a_n(z;m)p_n(x) + b_n(z;m)p_{n-1}(x), \tag{5.11a}$$

where

$$a_n(z;m) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{n+1,m}}{H_{n,m}}, \quad b_n(z;m) = \frac{nH_{n+1,m}H_{n-1,m}}{2H_{n,m}^2}, \tag{5.11b}$$

for further details see Chen and Feigin (18).

5.2 Generalized Okamoto polynomials

Here we consider the generalized Okamoto polynomials $Q_{m,n}$ which were introduced by Noumi and Yamada (83) and are defined in following Theorem.

Theorem 5.4. *Suppose $Q_{m,n}$ satisfies the recurrence relations*

$$Q_{m+1,n}Q_{m-1,n} = \tfrac{9}{2}\left\{Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right\} + \left\{2z^2 + 3(2m+n-1)\right\}Q^2_{m,n}, \tag{5.12a}$$

$$Q_{m,n+1}Q_{m,n-1} = \tfrac{9}{2}\left\{Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right\} + \left\{2z^2 + 3(1-m-2n)\right\}Q^2_{m,n}, \tag{5.12b}$$

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}\,z$, then solutions of $\mathrm{P_{IV}}$ *are*

$$\widetilde{w}^{[1]}_{m,n} = w(z; 2m+n, -2(n-\tfrac{1}{3})^2) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{Q_{m+1,n}}{Q_{m,n}}\right), \tag{5.13a}$$

$$\widetilde{w}^{[2]}_{m,n} = w(z; -m-2n, -2(m-\tfrac{1}{3})^2) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{Q_{m,n}}{Q_{m,n+1}}\right), \tag{5.13b}$$

$$\widetilde{w}^{[3]}_{m,n} = w(z; n-m, -2(m+n+\tfrac{1}{3})^2) = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{Q_{m,n+1}}{Q_{m+1,n}}\right). \tag{5.13c}$$

Proof. See Theorem 4.3 in Noumi and Yamada (83); also Theorem 4.1 in Clarkson (20). In terms of the rational solutions (5.13), the corresponding rational solutions of $\mathrm{sP_{IV}}$ (1.9) are given by $\varphi_j = \widetilde{w}^{[j]}_{m,n}$, for $j = 1,2,3$, with parameters $\mu_1 = -n + \tfrac{1}{3}$, $\mu_2 = -m + \tfrac{1}{3}$ and $\mu_3 = m + n + \tfrac{1}{3}$. □

The polynomials $Q_{m,n}$ defined by (5.12) are called the *generalized Okamoto polynomials* since Okamoto (84) defined the polynomials in the cases when $n = 0$ and $n = 1$. The rational solutions of $\mathrm{P_{IV}}$ defined by (5.13) include all solutions in the "$-\tfrac{2}{3}z$" hierarchy, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (5.3), which also can be expressed in the form of determinants; see equation (5.15. The polynomial $Q_{m,n}$ has degree $d_{m,n} = m^2 + n^2 + mn - m - n$ with integer coefficients (83); in fact $Q_{m,n}(\tfrac{1}{2}\sqrt{2}\,\zeta)$ is a monic polynomial in ζ with integer coefficients. Further $Q_{m,n}$ possesses the symmetries

$$Q_{n,m}(z) = \exp(-\tfrac{1}{2}\pi\mathrm{i}d_{m,n})\,Q_{m,n}(\mathrm{i}z), \tag{5.14a}$$

$$Q_{1-m-n,n}(z) = \exp(-\tfrac{1}{2}\pi\mathrm{i}d_{m,n})\,Q_{m,n}(\mathrm{i}z). \tag{5.14b}$$

Note that $d_{m,n} = m^2 + n^2 + mn - m - n$ satisfies $d_{m,n} = d_{n,m} = d_{1-m-n,n}$.

Plots of the complex roots of $Q_{10,10}(z)$ and $Q_{-9,-7}(z)$ are given in Figure 5.2. The roots of the polynomial $Q_{m,n}$, with $m,n \geq 1$, take the form of $m \times n$ "rectangle" with an "equilateral triangle", which have either $m-1$ or $n-1$ roots, on each of its sides. The roots of the polynomial $Q_{-m,-n}$, with $m,n \geq 1$, take the form of $m \times n$ "rectangle" with an "equilateral triangle", which now have either m or n roots, on each of its sides. These are only approximate rectangles and equilateral triangles as can be seen by looking at the actual values of the roots. The plots are invariant under reflections in the real and imaginary z-axes.

Due to the symmetries (5.14), the roots of the polynomials $Q_{-m,n}$ and $Q_{m,-n}$, with $m,n \geq 1$ take similar forms as these polynomials they can be expressed in terms of $Q_{M,N}$ and $Q_{-M,-N}$ for suitable $M,N \geq 1$. Specifically, the roots of the polynomial $Q_{-m,n}$, with $m \geq n \geq 1$, has the form of a $n \times (m-n+1)$ "rectangle" with an "equilateral triangle", which have either $n-1$ or $n-m-1$ roots, on each of its sides. Also the roots of the polynomial $Q_{-m,n}$ with $n > m \geq 1$, has the form of a $m \times (n-m-1)$ "rectangle" with an "equilateral triangle", which have either m or $n-m-1$ roots, on each of its sides. Further, we note that $Q_{-m,m} = Q_{m,1}$ and $Q_{1-m,m} = Q_{m,0}$, for all $m \in \mathbb{Z}$, where $Q_{m,0}$ and $Q_{m,0}$ are the original polynomials introduced by Okamoto (84). Analogous results hold for $Q_{m,-n}$, with $m,n \geq 1$.

Remark 5.5.

1. The generalized Okamoto polynomials $Q_{m,n}$ can be expressed in determinantal form as follows

$$Q_{m,n} = c_{m,n}\mathcal{W}\left(\psi_1, \psi_4, \ldots, \psi_{3m+3n-5}, \psi_2, \psi_5, \ldots, \psi_{3n-4}\right), \tag{5.15a}$$

$$Q_{-m,-n} = \widetilde{c}_{m,n}\mathcal{W}\left(\psi_1, \psi_4, \ldots, \psi_{3n-2}, \psi_2, \psi_5, \ldots, \psi_{3m+3n-1}\right), \tag{5.15b}$$

for $m,n \geq 0$, with $c_{m,n}$ and $\widetilde{c}_{m,n}$ constants, $\mathcal{W}(\psi_1, \psi_2, \ldots, \psi_n)$ the Wronskian, and $\psi_n(z)$ given by

$$\sum_{n=0}^{\infty}\frac{\psi_n(z)\,\zeta^n}{n!} = \exp\left(2z\zeta + 3\zeta^2\right), \tag{5.16}$$

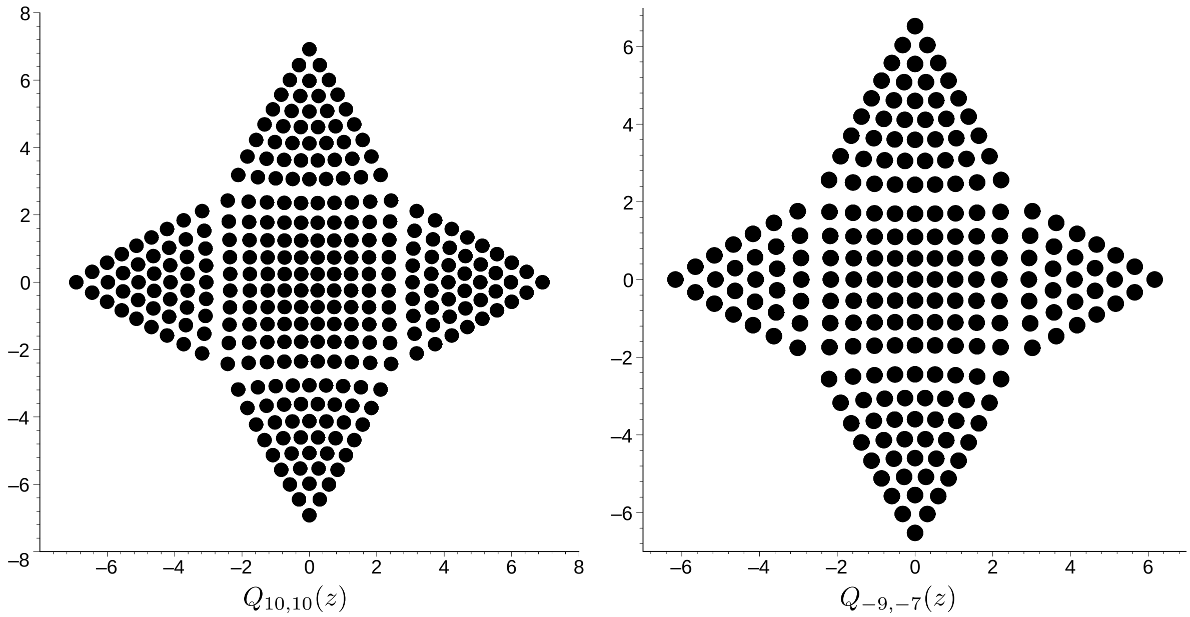


Figure 5.2: Roots of the generalized Okamoto polynomials $Q_{10,10}(z)$ and $Q_{-9,-7}(z)$

so $\psi_n(z) = (-3)^{n/2} H_n\left(\frac{1}{3}\sqrt{3}\,\mathrm{i}z\right)$. This follows from equation (20) in Kajiwara and Ohta (56), though it is not given there explicitly (see also Ref. (24)). The generalized Okamoto polynomials $Q_{m,n}$ can also be expressed in terms of Schur polynomials (for further details see Refs. (24; 56; 83)).

2. As for the generalized Hermite polynomials,using the Hamiltonian formalism for $\mathrm{P_{IV}}$ discussed in §2, it can be shown that the generalized Okamoto polynomials $Q_{m,n}$, which are defined by the differential-difference equations (5.12), also satisfy the fourth order bilinear ordinary differential equation

$$\begin{aligned} Q_{m,n}Q''''_{m,n} - 4Q'_{m,n}Q'''_{m,n} + 3\left(Q''_{m,n}\right)^2 + \tfrac{4}{3}z^2\left\{Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right\} \\ + 4zQ_{m,n}Q'_{m,n} - \tfrac{8}{3}(m^2+n^2+mn-m-n)Q^2_{m,n} = 0, \end{aligned} \tag{5.17}$$

and homogeneous difference equations; for details see Refs. (21) and (22).

6 Special Function Solutions

The Painlevé equations $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess hierarchies of solutions expressible in terms of classical special functions, for special values of the parameters through an associated Riccati equation,

$$w' = p_2(z)w^2 + p_1(z)w + p_0(z), \tag{6.1}$$

where $p_2(z)$, $p_1(z)$ and $p_0(z)$ are rational functions. Hierarchies of solutions, which are often referred to as "one-parameter solutions" (since they have one arbitrary constant), are generated from "seed solutions" derived from the Riccati equation using the Bäcklund transformations given in §4. Furthermore, as for the rational solutions, these special function solutions are often expressed in the form of determinants. See Sachdev (89) for details of the derivation of the associated Riccati equations for $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$.

Special function solutions of $\mathrm{P_{II}}$ are expressed in terms of Airy functions $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$(4; 37; 40; 84), of $\mathrm{P_{III}}$ are expressed in terms of Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ (65; 75; 78; 87; 104), of $\mathrm{P_{IV}}$ in terms of Weber-Hermite (parabolic cylinder) functions $D_\nu(z)$ (8; 41; 64; 77; 84; 103), of $\mathrm{P_V}$ in terms of Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$, or equivalently confluent hypergeometric functions ${}_1F_1(a;c;z)$ (39; 66; 86; 108) and of $\mathrm{P_{VI}}$ in terms of hypergeometric functions $F(a,b;c;z)$ (34; 68; 85); see also Refs. (23; 42; 70; 95). Some classical orthogonal polynomials arise as particular cases of these special function solutions and thus yield rational solutions of the associated Painlevé equations. For $\mathrm{P_{III}}$ and $\mathrm{P_V}$ these are in terms of associated Laguerre polynomials $L_k^{(m)}(z)$ (71; 81), for $\mathrm{P_{IV}}$ in terms of Hermite polynomials $H_n(z)$ (8; 56; 77; 84) and for for $\mathrm{P_{VI}}$ in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ (69; 96). In fact all rational solutions of $\mathrm{P_{VI}}$ arise as particular cases of the special solutions given in terms of hypergeometric functions (73).

6.1 Weber-Hermite function solutions of $\mathrm{P_{IV}}$

Theorem 6.1. $\mathrm{P_{IV}}$ *(1.4) has solutions expressible in terms of parabolic cylinder functions if and only if either*

$$\beta = -2(2n+1+\varepsilon\alpha)^2, \tag{6.2}$$

or

$$\beta = -2n^2, \tag{6.3}$$

with $n \in \mathbb{Z}$ and $\varepsilon = \pm 1$.

Proof. See Gambier (37), Gromak (41), Gromak and Lukashevich (43), Lukashevich (63; 64); also Gromak, Laine and Shimomura (42). □

In the case when $n = 0$ in (6.2), then the associated Riccati equation is

$$w' = \varepsilon(w^2 + 2zw) + 2\nu, \qquad \varepsilon^2 = 1, \tag{6.4}$$

with $\mathrm{P_{IV}}$ parameters $\alpha = -\varepsilon(1+\nu)$ and $\beta = -2\nu^2$. Letting $w(z) = -\varepsilon\psi_\nu'(z;\varepsilon)/\psi_\nu(z;\varepsilon)$ in this equation yields the Weber-Hermite equation

$$\psi_\nu'' - 2\varepsilon z\psi_\nu' + 2\varepsilon\nu\psi_\nu = 0, \tag{6.5}$$

which, provided that $\nu \notin \mathbb{Z}$, has general solution

$$\psi_\nu(z;\varepsilon) = \left\{C_1 D_\nu(\sqrt{2\varepsilon}\, z) + C_2 D_\nu(-\sqrt{2\varepsilon}\, z)\right\} \exp\left(\tfrac{1}{2}\varepsilon z^2\right), \tag{6.6}$$

where C_1 and C_2 are arbitrary constants and $D_\nu(\zeta)$ is the parabolic cylinder function which satisfies

$$\frac{\mathrm{d}^2 D_\nu}{\mathrm{d}\zeta^2} = (\tfrac{1}{4}\zeta^2 - \nu - \tfrac{1}{2})D_\nu, \tag{6.7}$$

with boundary condition

$$D_\nu(\zeta) \sim \zeta^\nu \exp\left(-\tfrac{1}{4}\zeta^2\right), \qquad \text{as} \quad \zeta \to +\infty. \tag{6.8}$$

Equivalently

$$\psi_\nu(z;\varepsilon) = \left\{\widetilde{C}_1 M_{\frac{1}{2}\nu+\frac{1}{4},\frac{1}{4}}(\varepsilon z^2) + \widetilde{C}_2 W_{\frac{1}{2}\nu+\frac{1}{4},\frac{1}{4}}(\varepsilon z^2)\right\} \frac{\exp\left(\frac{1}{2}\varepsilon z^2\right)}{z^{1/2}}, \tag{6.9}$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants, and $M_{\kappa,\mu}(\xi)$ and $W_{\kappa,\mu}(\xi)$ are the Whittaker functions which satisfy

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \left(\frac{\frac{1}{4} - \mu^2}{\xi^2} + \frac{\kappa}{\xi} - \frac{1}{4}\right) u = 0. \tag{6.10}$$

The seed solutions which generate the special function solutions of $\mathrm{P_{IV}}$ are

$$w(z, -\varepsilon(\nu+1), -2\nu^2) = -\varepsilon \frac{\mathrm{d}}{\mathrm{d}z} \ln \psi_\nu(z;\varepsilon), \tag{6.11a}$$

$$w(z, -\varepsilon\nu, -2(\nu+1)^2) = -2z + \varepsilon \frac{\mathrm{d}}{\mathrm{d}z} \ln \psi_\nu(z;\varepsilon), \tag{6.11b}$$

where $\psi_\nu(z;\varepsilon)$ is given by (6.6) or (6.9). The corresponding special function solutions of $\mathrm{sP_{IV}}$ (1.9) are given by, for $\varepsilon = 1$

$$\varphi_1 = -\frac{\mathrm{d}}{\mathrm{d}z} \ln \psi_\nu(z;1), \qquad \varphi_2 = -2z + \frac{\mathrm{d}}{\mathrm{d}z} \ln \psi_\nu(z;1), \qquad \varphi_3 = 0,$$

with parameters $\mu_1 = -\nu$, $\mu_2 = \nu + 1$ and $\mu_3 = 0$ and for $\varepsilon = -1$

$$\varphi_1 = \frac{\mathrm{d}}{\mathrm{d}z} \ln \psi_\nu(z;-1), \qquad \varphi_2 = 0, \qquad \varphi_3 = -2z - \frac{\mathrm{d}}{\mathrm{d}z} \ln \psi_\nu(z;-1),$$

with parameters $\mu_1 = -\nu$, $\mu_2 = 0$ and $\mu_3 = \nu + 1$.

Determinantal representations of special function solutions of $\mathrm{P_{IV}}$ are discussed by Okamoto (84); see also Forrester and Witte (35). The following theorem is a generalization of Theorem 5.2.

Theorem 6.2. *Suppose $\tau_{\nu,n}(z;\varepsilon)$ is given by*

$$\tau_{\nu,n}(z;\varepsilon)=\mathcal{W}\left(\psi_\nu(z;\varepsilon),\psi_{\nu+1}(z;\varepsilon),\ldots,\psi_{\nu+n-1}(z;\varepsilon)\right), \tag{6.12}$$

with $\psi_\nu(z;\varepsilon)$ given by (6.6) and $\mathcal{W}(\psi_\nu,\psi_{\nu+1},\ldots,\psi_{\nu+n-1})$ the usual Wronskian, then solutions of $\mathrm{P_{IV}}$ *are given by*

$$w(z;\varepsilon(2\nu+n+1),-2n^2)=\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}\right), \tag{6.13a}$$

$$w(z;-\varepsilon(\nu+2n+1),-2\nu^2)=\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_{\nu,n}(z;\varepsilon)}{\tau_{\nu,n+1}(z;\varepsilon)}\right), \tag{6.13b}$$

$$w(z;\varepsilon(n-\nu),-2(\nu+n+1)^2)=-2z+\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}\right). \tag{6.13c}$$

In the case when $\varepsilon=1$, the corresponding solutions of $\mathrm{sP_{IV}}$ (1.9) are given by

$$\varphi_1=\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_{\nu+1,n}(z;1)}{\tau_{\nu,n}(z;1)}\right),\qquad \varphi_2=\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_{\nu,n}(z;1)}{\tau_{\nu,n+1}(z;1)}\right),\qquad \varphi_3=-2z+\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{\tau_{\nu,n+1}(z;1)}{\tau_{\nu+1,n}(z;1)}\right),$$

with parameters $\mu_1=-n$, $\mu_2=-\nu$ and $\mu_3=\nu+n+1$. There are analogous solutions of $\mathrm{sP_{IV}}$ in the case when $\varepsilon=-1$.

We shall now discuss some special cases which are of particular interest.

6.2 Rational solutions

If $\alpha=n\in\mathbb{Z}$, then for $\nu=n\in\mathbb{Z}^+$ the parabolic cylinder function is given by $D_n(\zeta)=2^{-n/2}H_n(\zeta/\sqrt{2})\exp(-\tfrac14\zeta^2)$, with $H_n(z)$ the Hermite polynomial. Consequently, $\mathrm{P_{IV}}$ (1.4) has the solutions

$$w(z;-\varepsilon(n+1),-2n^2)=-\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln H_n(\sqrt{\varepsilon}\,z), \tag{6.14a}$$

$$w(z;-\varepsilon n,-2(n+1)^2)=-2z+\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln H_n(\sqrt{\varepsilon}\,z), \tag{6.14b}$$

which are special cases of the rational solutions of $\mathrm{P_{IV}}$ that are expressed in terms of the generalized Hermite polynomials discussed in §5.1. In the case when $\varepsilon=1$, the corresponding rational solutions of $\mathrm{sP_{IV}}$ (1.9) are

$$\varphi_1=-\frac{\mathrm{d}}{\mathrm{d}z}\ln H_n(z),\qquad \varphi_2=-2z+\frac{\mathrm{d}}{\mathrm{d}z}\ln H_n(z),\qquad \varphi_3=0,$$

with parameters $\mu_1=-n$, $\mu_2=n+1$ and $\mu_3=0$.

6.3 Half-integer hierarchy

If $\nu=-\frac12$, then equation (6.5 has solution

$$\psi_{-1/2}(z;\varepsilon)=\left\{C_1D_{-1/2}(\sqrt{2}\,z)+C_2D_{-1/2}(-\sqrt{2}\,z)\right\}\exp\left(\tfrac12\varepsilon z^2\right), \tag{6.15}$$

with C_1 and C_2 arbitrary constants, and so $\mathrm{P_{IV}}$ (1.4) has the solutions

$$w(z;\tfrac12,-\tfrac12)=-\frac{\sqrt{2}\left\{C_1D_{1/2}(\sqrt{2}\,z)-C_2D_{1/2}(-\sqrt{2}\,z)\right\}}{C_1D_{-1/2}(\sqrt{2}\,z)+C_2D_{-1/2}(-\sqrt{2}\,z)}, \tag{6.16a}$$

$$w(z;-\tfrac12,-\tfrac12)=-2z+\frac{\sqrt{2}\left\{C_1D_{1/2}(\sqrt{2}\,z)-C_2D_{1/2}(-\sqrt{2}\,z)\right\}}{C_1D_{-1/2}(\sqrt{2}\,z)+C_2D_{-1/2}(-\sqrt{2}\,z)}. \tag{6.16b}$$

The corresponding solutions of $\mathrm{sP_{IV}}$ (1.9) are

$$\varphi_1=-\frac{\sqrt{2}\left\{C_1D_{1/2}(\sqrt{2}\,z)-C_2D_{1/2}(-\sqrt{2}\,z)\right\}}{C_1D_{-1/2}(\sqrt{2}\,z)+C_2D_{-1/2}(-\sqrt{2}\,z)},\qquad \varphi_2=0,$$

$$\varphi_3=-2z+\frac{\sqrt{2}\left\{C_1D_{1/2}(\sqrt{2}\,z)-C_2D_{1/2}(-\sqrt{2}\,z)\right\}}{C_1D_{-1/2}(\sqrt{2}\,z)+C_2D_{-1/2}(-\sqrt{2}\,z)},$$

with parameters $\mu_1 = \frac{1}{2}$, $\mu_2 = 0$ and $\mu_3 = \frac{1}{2}$.

Using (6.16) as seed solutions generates a hierarchy, the *half-integer hierarchy*, of solutions of $\mathrm{P_{IV}}$ (1.4) expressed in terms of $D_{\pm1/2}(\zeta)$, which is discussed in detail by Bassom, Clarkson and Hicks (8) who plot some of the solutions. We remark that the special solutions of $\mathrm{P_{IV}}$ (1.4) when $\alpha = \frac{1}{2}n$ and $\beta = -\frac{1}{2}n^2$ are also solutions of $\mathrm{dP_I}$ (4.5) and arise in quantum gravity (see Fokas, Its and Kitaev (31; 32)).

6.4 Complementary error function hierarchy

If $\nu = 0$, then equation (6.5 has solution

$$\psi_0(z;\varepsilon) = C_1 + C_2\,\mathrm{erfc}\left(\sqrt{-\varepsilon}\,z\right), \tag{6.17}$$

where C_1 and C_2 are arbitrary constants and $\mathrm{erfc}(z)$ is the complementary error function defined by

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-t^2)\,\mathrm{d}t, \tag{6.18}$$

and so $\mathrm{P_{IV}}$ (1.4) has the solutions

$$w(z;1,0) = -\frac{2C_2\exp(-z^2)}{\sqrt{\pi}\,\{C_1 + C_2\,\mathrm{erfc}(z)\}}, \tag{6.19a}$$

$$w(z;-1,0) = \frac{2\mathrm{i}C_2\exp(z^2)}{\sqrt{\pi}\,\{C_1 + C_2\,\mathrm{erfc}(\mathrm{i}z)\}} \tag{6.19b}$$

(see Bassom, Clarkson and Hicks (8), Gromak and Lukashevich (43)). In the case when $\varepsilon = 1$, the corresponding error function solutions of $\mathrm{sP_{IV}}$ (1.9) are

$$\varphi_1 = -\frac{2C_2\exp(-z^2)}{\sqrt{\pi}\,\{C_1 + C_2\,\mathrm{erfc}(z)\}}, \qquad \varphi_2 = 0,$$
$$\varphi_3 = -2z + \frac{2C_2\exp(-z^2)}{\sqrt{\pi}\,\{C_1 + C_2\,\mathrm{erfc}(z)\}},$$

with parameters $\mu_1 = \mu_2 = 0$ and $\mu_3 = 1$.

The seed solutions (6.19) generate a hierarchy, the *complementary error function hierarchy*, of solutions of $\mathrm{P_{IV}}$ (1.4) which have the form

$$w(z) = \frac{P(z,\Psi)}{Q(z,\Psi)}, \qquad \Psi(z;\xi) = \frac{2\xi\exp(-z^2)}{\sqrt{\pi}[1 - \xi\,\mathrm{erfc}(z)]}, \tag{6.20}$$

where $P(z,\Psi)$ and $Q(z,\Psi)$ are polynomials in z and Ψ (see Bassom, Clarkson and Hicks (8)). Solutions of the form (6.20) are particular cases of the special function solutions associated with the Weber-Hermite equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2} + 2z\frac{\mathrm{d}\psi}{\mathrm{d}z} - 2m\psi = 0, \qquad m\in\mathbb{Z}, \tag{6.21}$$

If $\xi = 0$ then $\Psi \equiv 0$, and the solutions (6.20) then reduce to rational solutions that are expressed in terms of the generalized Hermite polynomials discussed in §5.1.

A special case of the complementary error function hierarchy hierarchy occurs when $\alpha = 2n+1$ and $\beta = 0$ which gives *nonlinear bound state solutions* which exponential decay as $z \to \pm\infty$ and so are nonlinear analogues of bound states for the linear harmonic oscillator; for details see (8; 10), also Eqs. (7.10) and (7.11). These bound state solutions arise in the theory of (i), orthogonal polynomials with the discontinuous Hermite weight

$$\omega(x;z,\mu) = \exp(-x^2)\left\{1 - \mu + 2\mu\mathcal{H}(x-z)\right\}, \tag{6.22}$$

with $\mathcal{H}(\zeta)$ the Heaviside function and μ a parameter (see Chen and Pruessner (19)), and (ii), GUE random matrices which are expressed as Hankel determinants of the function

$$\psi_m(z;\xi) = \left(\int_{-\infty}^\infty - \xi\int_z^\infty\right)(x-z)^m\exp(-x^2)\,\mathrm{d}x, \tag{6.23}$$

with ξ a parameter (see Forrester and Witte (36)). We remark that $\psi_m(z;\xi)$ given by (6.23) is the general solution of equation (6.21.

7 Asymptotics and Connection Formulae

In this section we are concerned with the special case of $\mathrm{P_{IV}}$ (1.4) with $w(z) = 2\sqrt{2}\,y_k^2(x;\nu)$, $z = \frac{1}{2}\sqrt{2}\,x$, $\alpha = 2\nu+1 (\in \mathbb{R})$ and $\beta = 0$, so that

$$\frac{\mathrm{d}^2 y_k}{\mathrm{d}x^2} = 3y_k^5 + 2xy_k^3 + (\tfrac{1}{4}x^2 - \nu - \tfrac{1}{2})y_k, \tag{7.1}$$

together with boundary condition

$$y_k(x;\nu) \to 0, \qquad \text{as} \quad x \to +\infty. \tag{7.2}$$

Theorem 7.1. *Any nontrivial solution of (7.1) that satisfies (7.2) is asymptotic to $kD_\nu(x)$ as $x \to +\infty$, where $k(\neq 0)$ is a constant. Conversely, for any $k(\neq 0)$, there is a unique solution $y_k(x;\nu)$ of (7.1) such that*

$$y_k(x;\nu) \sim kD_\nu(x), \qquad \textit{as} \quad x \to +\infty, \tag{7.3}$$

with $D_\nu(x)$ the parabolic cylinder function. Now suppose $x \to -\infty$.

1. *If $0 \le k < k_*$, where*

$$k_*^2 = \frac{1}{2\sqrt{2\pi}\,\Gamma(\nu+1)}, \tag{7.4}$$

then this solution exists for all real x as $x \to -\infty$.

   (a) If $\nu = n \in \mathbb{N}$

$$y_k(x;n) \sim \frac{kD_n(x)}{\sqrt{1 - 2\sqrt{2\pi}\,n!\,k^2}}, \qquad \textit{as} \quad x \to -\infty. \tag{7.5}$$

   (b) If $\nu \notin \mathbb{Z}$, then for some d and $\theta_0 \in \mathbb{R}$,

$$y_k(x;\nu) = (-1)^{[\nu+1]}\left(-\tfrac{1}{6}x\right)^{1/2} + d|x|^{-1/2}\sin\varphi(x) + \mathcal{O}\left(|x|^{-3/2}\right), \tag{7.6a}$$

   as $x \to -\infty$, where

$$\varphi(x) = \tfrac{1}{6}\sqrt{3}\,x^2 - \tfrac{4}{3}\sqrt{3}\,d^2\,\ln|x| - \theta_0. \tag{7.6b}$$

2. *If $k = k_*$, then*

$$y_k(x;\nu) \sim \mathrm{sgn}(k)\left(-\tfrac{1}{2}x\right)^{1/2}, \qquad \textit{as} \quad x \to -\infty. \tag{7.7}$$

3. *If $k > k_*$ then $y_k(x;\nu)$ has a pole at a finite x_0 depending on k, so*

$$y_k(x;\nu) \sim \mathrm{sgn}(k)(x - x_0)^{-1/2}, \qquad \textit{as} \quad x \downarrow x_0. \tag{7.8}$$

Proof. See Bassom, Clarkson, Hicks and McLeod (10); these asymptotics of $\mathrm{P_{IV}}$ are also discussed by Abdullayev (1) and Lu (61; 62). □

Plots of $y_k(x;\nu)$ for $\nu = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ for values of k which are just below and just above the critical value of k given by (7.4) and the curves $y^2 + \frac{1}{3}x \pm \sqrt{x^2 + 12\nu + 6} = 0$ are given in Figure 7.1.

The connection formulae relating the asymptotics of $y_k(x;\nu)$ as $x \to \pm\infty$ given by (7.3) and (7.6) are discussed in the following theorem.

Theorem 7.2. *Connection formulas for d and θ_0 are given by*

$$d^2 = -\tfrac{1}{4}\sqrt{3}\,\pi^{-1}\ln(1 - |\mu|^2), \tag{7.9a}$$

$$\theta_0 = \tfrac{1}{3}d^2\sqrt{3}\,\ln 3 + \tfrac{2}{3}\pi\nu + \tfrac{7}{12}\pi + \arg\mu + \arg\Gamma\left(-\tfrac{2}{3}\mathrm{i}\sqrt{3}\,d^2\right), \tag{7.9b}$$

where $\mu = 1 + 2\mathrm{i}k\pi^{3/2}\exp(-\mathrm{i}\pi\nu)/\Gamma(-\nu)$.

Proof. See Its and Kapaev (49), who use the linear system (3.1) with matrices $\mathbf{A}$ and $\mathbf{B}$ given by (3.5). □

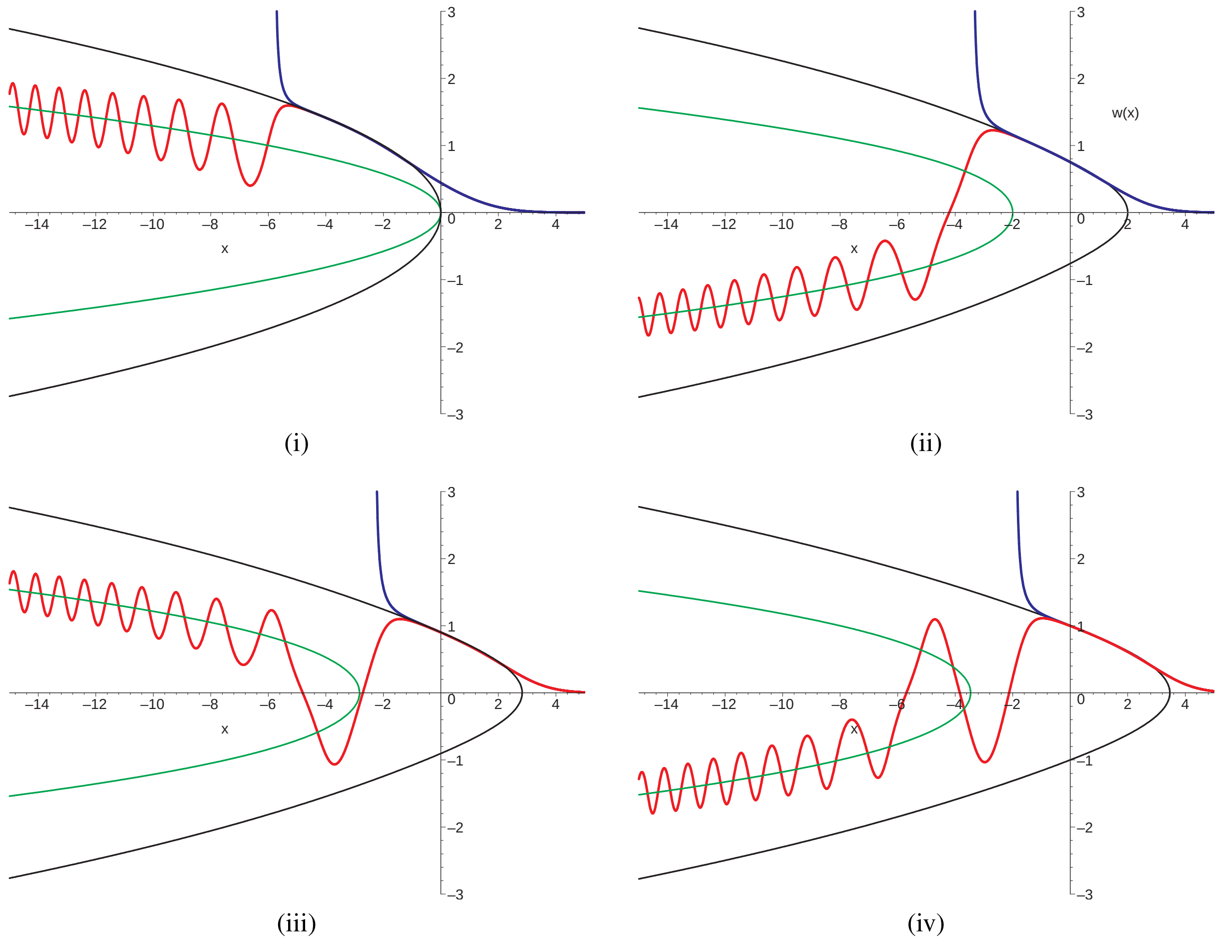


Figure 7.1: (i), $y_k(x;-\frac{1}{2})$ with $k=0.33554691$ (red), 0.33554692 (blue), and the parabolas $y^2+\frac{1}{2}x=0$ (green), $y^2+\frac{1}{6}x=0$ (black). (ii), $y_k(x;\frac{1}{2})$ with $k=0.47442$ (red), 0.47443 (blue), and the curves $y^2+\frac{1}{3}x+\frac{1}{6}\sqrt{x^2+12}=0$ (green) and $y^2+\frac{1}{3}x-\frac{1}{6}\sqrt{x^2+12}=0$ (black). (iii), $y_k(x;\frac{3}{2})$ with $k=0.38736$ (red), 0.38737 (blue), and the curves $y^2+\frac{1}{3}x+\frac{1}{6}\sqrt{x^2+24}=0$ (green) and $y^2+\frac{1}{3}x-\frac{1}{6}\sqrt{x^2+24}=0$ (black). (iv) $y_k(x;\frac{5}{2})$ with $k=0.244992$ (red), 0.244993 (blue), and the curves $y^2+\frac{1}{3}x+\frac{1}{6}\sqrt{x^2+36}=0$ (green) and $y^2+\frac{1}{3}x-\frac{1}{6}\sqrt{x^2+36}=0$ (black).

For $n\in\mathbb{Z}^+$, $y_k(x;n)$ exists for all x provided that $k^2<1/(2\sqrt{2\pi}\,n!)$, has n distinct zeros and decays exponentially to zero as $x\to\pm\infty$ with asymptotic behaviour

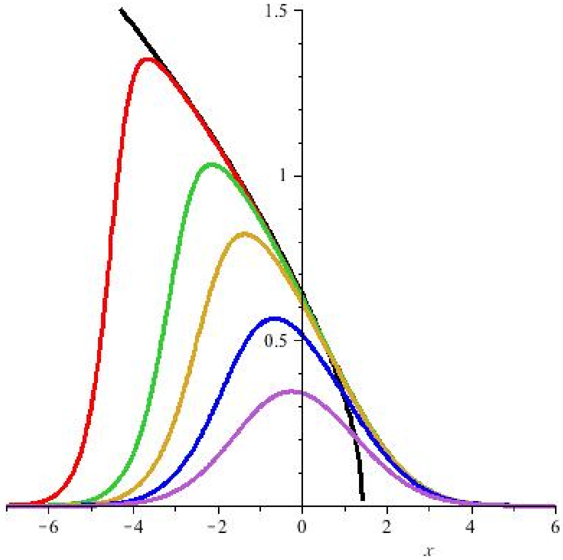
$$y_k(x;n)\sim\begin{cases} k\,x^n\exp(-\frac{1}{4}x^2), & \text{as}\quad x\to\infty,\\ \dfrac{k\,x^n\exp(-\frac{1}{4}x^2)}{\sqrt{1-2\sqrt{2\pi}\,n!\,k^2}}, & \text{as}\quad x\to-\infty.\end{cases} \tag{7.10}$$

There solutions are nonlinear analogues of the bound state solutions for the linear harmonic oscillator and so may be thought of as *nonlinear bound state solutions*. The first two nonlinear bound state solutions are

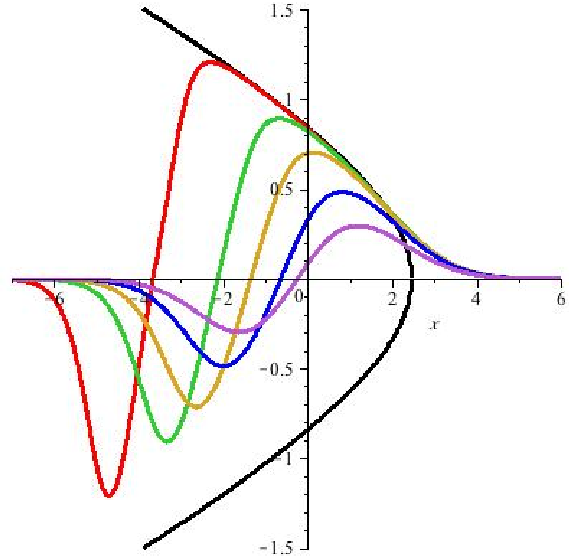
$$y_k(x;0)=\frac{k\exp(-\frac{1}{4}x^2)}{\sqrt{1-k^2\sqrt{2\pi}\ \mathrm{erfc}\left(\frac{1}{2}\sqrt{2}\,x\right)}}\equiv\Psi_k(x), \tag{7.11a}$$

$$y_k(x;1)=\frac{\Psi_k(x)\left\{2\Psi_k^2(x)+x\right\}}{\sqrt{1-2x\Psi_k^2(x)-4\Psi_k^4(x)}}. \tag{7.11b}$$

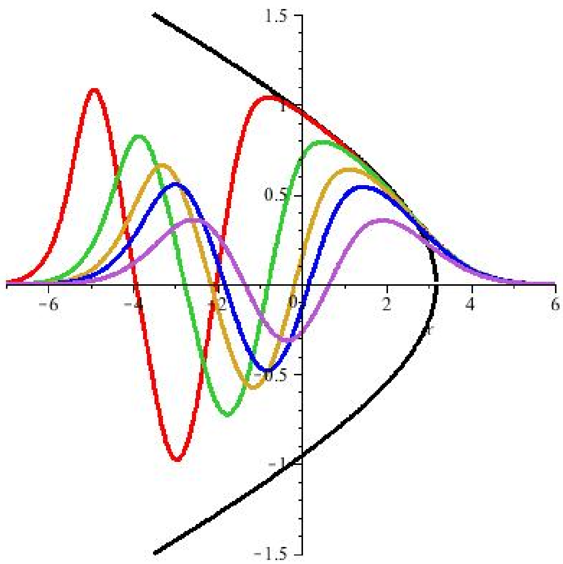
Plots of $y_k(x;n)$, for $n=0,1,2,3$, for several values of k are given in Figure 7.2.



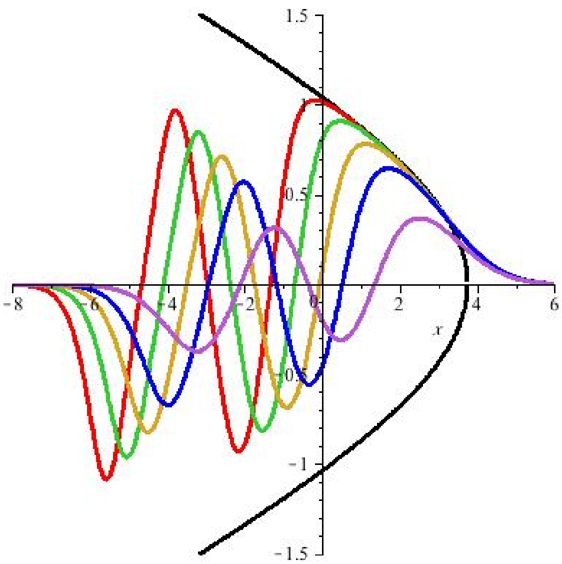
$y_k(x;0)$
$k = 0.3, 0.4, 0.44, 0.446, 0.44662$



$y_k(x;1)$
$k = 0.3, 0.4, 0.44, 0.446, 0.44662$



$y_k(x;2)$
$k = 0.25, 0.3, 0.31, 0.315, 0.3158$



$y_k(x;3)$
$k = 0.15, 0.18, 0.182, 0.1823, 0.18233$

Figure 7.2: Plots of $y_k(x;n)$, for $n = 0, 1, 2, 3$, for several values of k.

8 Discussion

This paper gives an introduction to some of the fascinating properties which the Painlevé equations possess including Hamiltonian structure, isomonodromy problems, Bäcklund transformations, hierarchies of exact solutions, asymptotics and connection formulae. These properties show that the Painlevé equations may be thought of as nonlinear analogues of the classical special functions.

There are still several very important open problems relating to the following three major areas of modern theory of Painlevé equations.

(i) Bäcklund transformations and exact solutions of Painlevé equations; a summary of many of the currently known results are given in Refs. (23) and (42).

(ii) The relationship between affine Weyl groups, Painlevé equations, Bäcklund transformations and discrete equations; see Ref. (79), for an introduction to this topic.

(iii) Asymptotics and connection formulae for the Painlevé equations using the isomonodromy method, for example the construction of uniform asymptotics around a nonlinear Stokes ray; see Refs. (30; 50; 58).

The ultimate objective is to provide a complete classification and unified structure for the exact solutions and Bäcklund transformations for the Painlevé equations (and the discrete Painlevé equations) — the presently known results are rather fragmentary and non-systematic.

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