TRANSFER RESULTS FOR FROBENIUS EXTENSIONS

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Abstract. We study Frobenius extensions which are free-filtered by a totally ordered, finitely generated abelian group, and their free-graded counterparts. First we show that the Frobenius property passes up from a free-graded extension to a free-filtered extension, then also from a free-filtered extension to the extension of their Rees algebras. Our main theorem states that, under some natural hypotheses, a free-filtered extension of algebras is Frobenius if and only if the associated graded extension is Frobenius. In the final section we apply this theorem to provide new examples and non-examples of Frobenius extensions.

Keywords. Frobenius extensions; deformation theory; associative algebras.

1. Introduction

Throughout this paper $\Bbbk$ is a field and all algebras are $\Bbbk$-algebras. A finite dimensional algebra $\mathcal{R}$ is called a *classical Frobenius algebra* if the dual of the right regular module is isomorphic to the left regular module $(\mathcal{R}_{\mathcal{R}})^* \cong {}_{\mathcal{R}}\mathcal{R}$. Equivalently $\mathcal{R}$ admits a linear map $\mathcal{R} \to \Bbbk$ whose kernel contains no left or right ideals - we call this the *Frobenius form of* $\mathcal{R}$. The representation theory of classical Frobenius algebras admits extremely nice duality properties. For instance, it is known that the projective and injective modules coincide and, in particular, the left regular module is injective. Three notable families of examples include the group algebras of finite groups, reduced enveloping algebras of restricted Lie algebras and semidirect products $\mathcal{R} \ltimes \mathcal{R}^*$ where $\mathcal{R}$ is any Artinian ring [1, pp. 127], [8, Proposition 1.2].

A natural generalisation of a classical Frobenius algebra is the notion of a *Frobenius extension* (of the first kind), where $\Bbbk$ is replaced by some subring of $\mathcal{R}$, not necessarily central. More precisely we say that a ring extension $\mathcal{S} \subseteq \mathcal{R}$ is a Frobenius extension in case $\mathcal{R}$ is a projective left $\mathcal{S}$-module and $\mathcal{R} \cong \mathrm{Hom}_{\mathcal{S}}({}_{\mathcal{S}}\mathcal{R}, {}_{\mathcal{S}}\mathcal{S})$ as $\mathcal{R}$-$\mathcal{S}$-bimodules. Nakayama and Tsuzuku observed that Frobenius extensions can be characterised by the existence of a $\mathcal{S}$-$\mathcal{S}$-bimodule homomorphism $\Phi : \mathcal{R} \to \mathcal{S}$ generalising the Frobenius form of a classical Frobenius algebra [24], and in this paper we call this map *the Frobenius form of the extension.* Frobenius extensions play an important role in a diverse array of topics, such as link invariants and 2-dimensional TQFT, as well as having many applications in the representation theory of Hopf algebras (see [14] for a survey). The examples which we will be interested in are quantum groups which are free of finite type over their centre, as well as some important new families arising in modular representation theory. For this reason we focus on Frobenius extensions $\mathcal{S} \subseteq \mathcal{R}$ where $\mathcal{S}$ is a subalgebra of the centre of $\mathcal{R}$. It seem plausible that some of our results could be extended to weaken this hypothesis.

Brown–Gordon–Stroppel gave many new examples of Frobenius extensions [3]. Their approach was fairly uniform: in each case they gave an example of a Frobenius form $\Phi : \mathcal{R} \to \mathcal{S}$ and checked the defining property via a single simple hypothesis. In [3, 1.6] they asked whether there exists an axiomatic approach which would apply to all of their examples simultaneously, and it was this question which provided the first motivation for our work. One feature shared by many of their examples, as well as other classical examples, is a filtration by a totally ordered finitely generated abelian group G, and in this paper we

2010 *Mathematics Subject Classification.* 16W70, 16G99.

develop general tools which might help to prove the Frobenius property in the presence of such a filtration. For the rest of the introduction we fix such a group G and we use the words *graded* and *filtered* to mean G-graded and G-filtered.

When dealing with filtrations and gradings it is natural to require that the module structures carry filtrations which are compatible with the actions: such modules are known as *free-filtered* and *free-graded* modules respectively (see §2.4 for an introduction). *Free filtered* and *free-graded* ring extensions are defined in the obvious manner. When $\mathcal{R}$ is a filtered algebra we write $\operatorname{gr}\mathcal{R} := \bigoplus_{g\in G}(\mathcal{R}_g/\sum_{g>h\in G}\mathcal{R}_h)$ for the associated graded algebra.

Now suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a central extension. We say that $\mathcal{S} \subseteq \mathcal{R}$ is a *free-graded Frobenius extension* if $\mathcal{S} \subseteq \mathcal{R}$ is a free-graded extension equipped with a homogeneous Frobenius form $\Phi : \mathcal{R} \to \mathcal{S}$. Similarly we say that $\mathcal{S} \subseteq \mathcal{R}$ is a *free-filtered Frobenius extension* if $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered ring extension and a Frobenius extension.

Main Theorem. *Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered central extension. Then $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered Frobenius extension if and only if $\operatorname{gr}\mathcal{S} \subseteq \operatorname{gr}\mathcal{R}$ is a free-graded Frobenius extension.*

Our method is to show that the Frobenius property passes up from a free-graded central extension to a free-filtered central extension (any choice of filtered lift of the homogeneous Frobenius form will suffice) and then show that the Rees algebra of a free-filtered Frobenius extension is naturally a free-graded Frobenius extension. Finally we observe that the free-graded Frobenius property of $\operatorname{Rees}(\mathcal{S}) \subseteq \operatorname{Rees}(\mathcal{R})$ passes down to the graded quotient $\operatorname{gr}\mathcal{S} \subseteq \operatorname{gr}\mathcal{R}$. This requires us to introduce and define the Rees algebra of an algebra $\mathcal{R}$ filtered by a totally ordered, finitely generated abelian group (§2.6), and observe that $\operatorname{Rees}(\mathcal{R})$ simultaneously deforms $\mathcal{R}$ and $\operatorname{gr}\mathcal{R}$ (Lemma 2.8), an interesting property which generalises the well-known situation where $G = \mathbb{Z}$.

In Section 2 we provide the general background on free-graded (free-filtered) modules and algebras. We also give a brief account of the Rees algebra and prove the deformation property mentioned previously. In Section 3 we give the precise definition of free-graded (free-filtered) Frobenius extensions and recall the definition of the Nakayama automorphism. In Section 4 we state and prove various transfer results and combine them to deduce a proof of the main theorem.

Finally, in §5 we give a few applications of the main theorem, describing new examples and non-examples of Frobenius extensions. For our first new family of examples we consider quantum Schubert cells at an ℓth root of unity, and we show that these are all Frobenius extensions of their ℓ-centre. In Remark 5.3 we observe that the Nakayama automorphism is already well understood. Next we consider modular finite W-algebras, first defined by Premet in [26] and recently studied by Goodwin and the second author in [11]. We show that the modular finite W-algebra is a free-filtered Frobenius extension of its p-centre, with trivial Nakayama automorphism. In the final section we use the transfer result to show that the quantum Grassmanian $\mathcal{O}_q[\operatorname{Gr}(2,4)]$ at an ℓth root of unity actually fails to be a free-Frobenius extension of its ℓ-centre. Our key observation is that if $\mathcal{S} \subseteq \mathcal{R}$ is a filtered Frobenius extension and $\mathcal{R}$ is a free-filtered $\mathcal{S}$-module then the multiset of filtered degrees of the $\mathcal{S}$-basis for $\mathcal{R}$ must exhibit a special symmetry (Lemma 5.5), an observation which we use repeatedly in different guises throughout the paper.

Acknowledgements: We would like to thank the referee for useful comments which helped us to significantly improve the paper. Both authors are grateful for funding received from EPSRC grant EP/N034449/1. Furthermore the second author would like to to acknowledge the support of the European Commission, Seventh Framework Programme, Grant Agreement 600376, as well as grants CPDA125818/12, 60A01-4222/15 and DOR1691049/16 from the University of Padova.

2. Preliminaries on gradings and filtrations

Throughout the paper $\Bbbk$ is a field. Unless otherwise stated every vector space or algebra will be defined over $\Bbbk$. Early in the paper we study the various module structures which can be placed on Hom-spaces between left modules, right modules and bi-modules. For the sake of clarity we will use a very descriptive notation to indicate which structure we are considering. If S is a ring and M, N are S-S-bimodules we write $\mathrm{Hom}_{S-}(M,N)$, $\mathrm{Hom}_{-S}(M,N)$ or $\mathrm{Hom}_{S-S}(M,N)$ to indicate that we are considering homomorphisms of left modules, right modules or bi-modules. When S is commutative we simply use the notation $\mathrm{Hom}_S(M,N)$.

We begin by providing a brief introduction to the theory of algebras and modules endowed with filtrations and gradings by totally ordered groups.

2.1. **Totally ordered groups.** Throughout this paper $(G, \leq)$ shall always be a totally ordered, finitely generated abelian group. Note that such a group is obviously torsion free, hence free. Since we shall need to consider the group algebra $\Bbbk G$, we use multiplicative notation for G, writing 1_G for the identity element. All applications which we have in mind involve finitely many copies of $\mathbb{Z}$ ordered lexicographically.

2.2. **Filtrations by groups.** A G-filtration of a vector space $\mathcal{V}$ is a collection of subspaces $(\mathcal{V}_g : g \in G)$ satisfying $\mathcal{V}_g \subseteq \mathcal{V}_h$ whenever $g \leq h$. Throughout this article we refer to G-filtrations simply as *filtrations* and this shall cause no confusion since G shall be fixed. We say that the filtration is:

- $\Bbbk$-*finite* if $\dim \mathcal{V}_g < \infty$ for all $g \in G$;
- *discrete* if $\mathcal{V}_g = 0$ for some g;
- *non-negative* if $\mathcal{V}_g = 0$ for $g < 1_G$;
- *exhaustive* if $\bigcup_{g \in G} \mathcal{V}_g = \mathcal{V}$;
- *proper* if $\mathcal{V}_g \neq \mathcal{V}$ for all g.

Throughout this paper all filtrations are assumed to be $\Bbbk$-finite, discrete, exhaustive and proper. For $0 \neq v \in \mathcal{V}$ the hypothesis that the filtration is discrete implies that $\{g \in G : v \in \mathcal{V}_g\}$ has a minimum in G, which we define to be the degree $\deg(v)$. The degree of $0 \in \mathcal{V}$ is taken to be $-\infty$ which, by convention, shall satisfy $-\infty < g$ for all $g \in G$. If $\mathcal{S}$ is a $\Bbbk$-algebra then a filtration of $\mathcal{S}$ should satisfy the additional hypothesis that $\mathcal{S}_g \mathcal{S}_h \subseteq \mathcal{S}_{gh}$ for all $g, h \in G$.

2.3. **Gradings by groups.** A G-grading, or just a *grading*, of a vector space is a decomposition $V = \bigoplus_{g \in G} V^g$. All gradings in this article are assumed to be $\Bbbk$-finite, exhaustive, non-negative and proper, and these conditions are defined analogously to those same conditions on filtrations. Every grading $V = \bigoplus_{g \in G} V^g$ induces a filtration $V_g := \bigoplus_{h \leq g} V^h$. If we have a filtered vector space $\mathcal{V}$ then we may define the associated graded space $V = \mathrm{gr}\, \mathcal{V}$ by setting $V^g = \mathcal{V}_g / (\sum_{h<g} \mathcal{V}_h)$ and $V = \bigoplus_{g \in G} V^g$. We define the degree function $\deg : \bigcup_{g \in G} V^g \to G \cup \{-\infty\}$ on homogeneous elements by setting $\deg V^g \setminus \{0\} = g$ and $\deg(0) = -\infty$.

If S is a graded $\Bbbk$-algebra then we insist that gradings satisfy $S^g S^h \subseteq S^{gh}$ for all $g, h \in G$. When $\mathcal{S}$ is a filtered algebra $\mathrm{gr}\, \mathcal{S}$ is a graded algebra in the obvious manner. If M is a graded left S-module and $d \in G$ then we can define the shifted module $M[d]$ by setting $M[d] \cong M$ as S-modules and $M[d]^g := M^{dg}$ as graded vector spaces.

2.4. **Free-filtered and free-graded modules.** If $\mathcal{S}$ is a filtered $\Bbbk$-algebra then a *free-filtered left $\mathcal{S}$-module* is a filtered left module $\mathcal{M}$ that is free on some basis $\{m_i \mid i \in I\}$ for which there exist elements $d_i \in G$ for every $i \in I$ such that

$$\mathcal{M}_g = \bigoplus_{i\in I} \mathcal{S}_{gd_i^{-1}} m_i. \tag{1}$$

We call a basis of a free left module satisfying (1) a *free-filtered basis*. If S is a graded $\Bbbk$-algebra then a *free-graded left S-module* is a graded module M admitting a homogeneous basis $\{m_i \in M \mid i \in I\}$. In other words free-graded left modules are precisely those of the form $\bigoplus_{i\in I} S[d_i]$ for tuples $(d_i \mid i \in I) \in G^I$. As a consequence, for all $g \in G$ we have

$$M^g = \bigoplus_{i\in I} S^{gd_i^{-1}} m_i \tag{2}$$

We define free-filtered and free-graded right $\mathcal{S}$-modules similarly.

Remark 2.1. It is not hard to see that when an algebra $\mathcal{S}$ admits a $\Bbbk$-finite, discrete filtration it satisfies the invariant basis number property on finite rank free-filtered modules. This means that when two finite free-filtered modules are isomorphic they have the same number of independent generators, which we call the *rank* of the module.

Lemma 2.2. *Let $\mathcal{M}$ be a finitely generated filtered left $\mathcal{S}$-module and $M = \operatorname{gr} \mathcal{M}$, $S = \operatorname{gr} \mathcal{S}$. Then $\mathcal{M}$ is free-filtered over $\mathcal{S}$ if and only if M is free-graded over S. In this case they have the same rank.*

Proof. If $\mathcal{M} = \bigoplus_{i\in I} \mathcal{M}_i$ is any direct sum decomposition as $\mathcal{S}$-modules, then $M \cong \bigoplus_{i\in I} \operatorname{gr} \mathcal{M}_i$, and so it suffices to work in the rank one case when proving the 'only if' part. If $\mathcal{M}_g = \mathcal{S}_g m$ for some $m \in \mathcal{M}$ then

$$\operatorname{gr} \mathcal{M} = \bigoplus_{g\in G} (\mathcal{S}_g m / \sum_{h<g} \mathcal{S}_h m) \cong (\operatorname{gr} \mathcal{S})[\deg m]$$

as graded left $\operatorname{gr} \mathcal{S}$-modules. Conversely, if $\{\overline{m}_i \in M \mid i \in I\}$ are homogeneous generators of M then we can choose lifts $\{m_i \in \mathcal{M} \mid i \in I\}$ satisfying $m_i + \sum_{g<d_i} \mathcal{M}_g = \overline{m}_i$ for all i, where $d_i = \deg \overline{m}_i$. The argument of [29, Lemma 4.7(2)] shows that $\mathcal{M}$ is a free left $\mathcal{S}$-module, whilst an easy induction using (2) shows that $\mathcal{M}_g = \bigoplus_{i\in I} \mathcal{S}_{gd_i^{-1}} m_i$, using the fact that the filtration is discrete and $\Bbbk$-finite. □

Lemma 2.3. *The following hold:*

(i) *The multiset of degrees of a free-filtered basis of a finitely generated free-filtered left $\mathcal{S}$-module $\mathcal{M}$ is independent of the choice of free-filtered basis;*
(ii) *The analogue of (i) holds for homogeneous bases of free-graded left modules;*
(iii) *Both (i) and (ii) remain true when left modules are replaced by right modules.*

Proof. Part (i) will follow from a more general claim which we prove in this first paragraph. Let $F := \operatorname{Fun}(G, \mathbb{Z})$ denote the abelian group of all functions $G \to \mathbb{Z}$. The group G acts on F by left translations, and we claim that if $f \in F$, and if there exists some $g' \in G$ such that $f(g') \neq 0$ and $f(g) = 0$ for all $g < g'$ then there are no $\mathbb{Z}$-linear dependences between elements of $G \cdot f$. Suppose for a contradiction that $\sum_{i\in I} a_i(g_i \cdot f) = 0$ for non-zero integers $a_i \in \mathbb{Z}$ and some finite set I, and that g' has the property mentioned previously. Choose $i \in I$ so that g_i is minimal in the ordering. Then by assumption $0 = (\sum_{j\in I} a_j(g_j \cdot f))(g_i g') = \sum_{j\in I} a_j f(g_j^{-1} g_i g') = a_i f(g')$ because $g_i < g_j$ implies $g_j^{-1} g_i g' < g'$ and $f(g_j^{-1} g_i g') = 0$. From $f(g') \neq 0$ we deduce $a_i = 0$, and this contradiction proves the claim.

Now we proceed to deduce part (i). Suppose $\{c_i \mid i \in I'\}$ and $\{d_i \mid i \in I\}$ are both multisets of degrees of the free-filtered left module $\mathcal{M}$. We must show that these multisets are equal. By Remark 2.1 we have $I = I'$. For $i \in I$ write $f_{c_i}(g) := \dim \mathcal{S}_{c_i^{-1}g}$ and similar for f_{d_i}. By equation (1) we know that the function

$$f \ : \ G \longrightarrow \mathbb{Z}_{\geq 0}$$
$$g \longmapsto \dim \mathcal{M}_g$$

is equal to $\sum_{i\in I} f_{c_i} = \sum_{i\in I} f_{d_i}$ and so $\sum_{i\in I}(f_{c_i} - f_{d_i}) = 0$ is a $\mathbb{Z}$-linear dependence between the G-translates of $f \in F$. By the first paragraph the dependence is trivial, which proves (i). Part (ii) is proven similarly using formula (2) in place of (1) and part (iii) follows by duality. □

Before we proceed we shall need a technical lemma.

Lemma 2.4. *Let $\mathcal{S}$ be a filtered $\Bbbk$-algebra, let $\mathcal{M}_1, \mathcal{M}_2$ be finite, free-filtered left $\mathcal{S}$-modules and let D_1, D_2 denote the multisets of degrees of any free-filtered bases for $\mathcal{M}_1$ and $\mathcal{M}_2$. Suppose also that $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ is an injective or surjective filtered linear map of degree d. Then it is bijective if and only if $D_1 d = D_2$. The same holds for right modules*

Proof. Under these hypotheses Ψ is bijective if and only if $\dim(\mathcal{M}_1)_g = \dim(\mathcal{M}_2)_{gd}$ for all $g \in G$. It follows from equation (1) that $\dim(\mathcal{M}_1)_g = \sum_{i\in I} \dim \mathcal{S}_{gd_i^{-1}}$ which is completely determined by the multiset $(d_i \mid i \in I)$ of degrees, and an identical assertion holds for the dimensions $\dim(\mathcal{M}_2)_g$. □

2.5. Hom **spaces between free-filtered modules.** In this subsection we must record some elementary facts about filtered modules and Hom-spaces. Assume throughout §2.5 that $\mathcal{S}$ is a non-negatively filtered commutative $\Bbbk$-algebra and that $\mathcal{M}$ is a finite, free-filtered $\mathcal{S}$-module with basis $m_1, ..., m_n$ in filtered degree $d_1, ..., d_n \in G$ respectively. We are interested in the space $\mathrm{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S})$, which has an $\mathcal{S}$-module structure given by $(s\cdot\phi)(m) := s\phi(m)$ where $s \in \mathcal{S}, \phi \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S})$ and $m \in \mathcal{M}$. It also has a natural filtration given by

$$\mathrm{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S})_g := \{\phi : \mathcal{M} \to \mathcal{S} \mid \phi(\mathcal{M}_h) \subseteq \mathcal{S}_{gh} \text{ for all } h \in G\}. \quad (3)$$

We define special elements $\phi_1, ..., \phi_n \in \mathrm{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S})$ by $\phi_i(sm_j) := s\delta_{i,j}$ for $s \in \mathcal{S}$. The next result is straightforward.

Lemma 2.5. $\mathrm{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S})$ *is a free-filtered $\mathcal{S}$-module with free-filtered basis $\phi_1, ..., \phi_n$ lying in degrees $d_1^{-1}, ..., d_n^{-1}$.* □

Retain the notation so that $\mathcal{M}$ is an $\mathcal{S}$-module, but not necessarily free-filtered. If we write $S = \mathrm{gr}\, \mathcal{S}$ and $M = \mathrm{gr}\, \mathcal{M}$ then S is a commutative $\Bbbk$-algebra, M is an S-module and we can define homogeneous subspace of homomorphisms $\mathrm{Hom}_S(M, S)^d$ of degree d to be those maps taking M^h to S^{dh} for all $h \in G$.

Lemma 2.6. *The following hold:*

(i) *If M is a finitely generated S-module then* $\mathrm{Hom}_S(M, S)$ *is graded: it is a direct sum of the subspaces* $\mathrm{Hom}_S(M, S)^h$ *with $h \in G$.*

(ii) *If $\mathcal{M}$ is finite and free-filtered as an $\mathcal{S}$-module then the natural map*

$$\mathrm{gr}\, \mathrm{Hom}_{\mathcal{S}}(\mathcal{M}, \mathcal{S}) \longrightarrow \mathrm{Hom}_S(M, S) \quad (4)$$

is an isomorphism.

Proof. The first part is standard but we sketch a short proof for the reader's convenience. If $\phi \in \mathrm{Hom}_S(M, S)$ and $M = \sum_{i=1}^n Sm_i$ then the elements ϕ_i defined preceding Lemma 2.5 make sense regardless of whether M is a direct sum of the Sm_i. Now every $\phi \in \mathrm{Hom}_S(M, S)$ is a linear combination of the ϕ_i satisfying relations determined by the relations in M, hence ϕ is a sum of homogeneous parts.

We now prove (ii). The remarks preceding Lemma 2.6 describe explicit bases for $\mathrm{Hom}_S(M,S)$ and $\mathrm{Hom}_{\mathcal{S}}(\mathcal{M},\mathcal{S})$ as a right S-module and a right $\mathcal{S}$-module. Lemma 2.5 shows that the top graded components of the $\mathcal{S}$-basis for $\mathrm{Hom}_{\mathcal{S}}(\mathcal{M},\mathcal{S})$ are mapped to the S-basis for $\mathrm{Hom}_S(M,S)$ under (4) which induces a graded (ie. homogeneous of degree zero) isomorphism $\mathrm{gr}\,\mathrm{Hom}_{\mathcal{S}}(\mathcal{M},\mathcal{S}) \longrightarrow \mathrm{Hom}_S(M,S)$. $\square$

2.6. **The Rees algebra of a filtered algebra.** The Rees algebra $\mathrm{Rees}(\mathcal{S})$ of a $\mathbb{Z}$-filtered algebra $\mathcal{S}$ is a well-known tool from algebraic geometry [7, 6.5]. Since we were unable to find sources in the literature which work at our level of generality, we shall present some of the details here.

We continue to assume that $(G,\le)$ is a totally ordered, finitely generated abelian group. We remind the reader that since we will often consider the group algebra $\Bbbk G$, we prefer to use multiplicative notation in G, writing 1_G for the identity element. More generally, write $\Bbbk P \subseteq \Bbbk G$ for the semigroup algebra of any monoid $P \subseteq G$ and from henceforth fix the notation $P := \{g \in G \mid g \ge 1_G\}$ for the *positive cone of* G.

Let $\mathcal{S}$ be a non-negatively filtered algebra $\mathcal{S} = \bigcup_{g\ge 1_G} \mathcal{S}_g$. The Rees algebra of $\mathcal{S}$ is

$$\mathrm{Rees}(\mathcal{S}) := \bigoplus_{g\in G} \mathcal{S}_g \otimes g \subseteq \mathcal{S} \otimes_{\Bbbk} \Bbbk P.$$

Choose generators $x_1,...,x_n$ for G. After replacing some of these by their inverses we may assume $x_i \ge 1_G$ for all i and so P is the monoid generated by $x_1,...,x_n$. Since $G \cong \mathbb{Z}^n$ for some $n \ge 0$ it follows that $\mathrm{Spec}\,\Bbbk P$ is an n-dimensional affine space over $\Bbbk$.

Lemma 2.7. $\mathrm{Rees}(\mathcal{S})$ *is a free module over* $\Bbbk P$.

Proof. Write $d_g := \dim \mathcal{S}_g$, $L_g := \sum_{h<g} \mathcal{S}_h$ and $l_g := \dim L_g$. Choose a $\Bbbk$-basis for $\mathcal{S}$ inductively: first pick a basis for $\mathcal{S}_{1_G}$ arbitrarily, then for $g > 1_G$ choose a basis $b_1,...,b_{d_g}$ for $\mathcal{S}_g$ which extends the basis for L_g already chosen. We claim that $B := \{b_j \otimes g \mid g \in G, j = l_g+1,...,d_g\}$ is a $\Bbbk P$-basis for $\mathrm{Rees}(\mathcal{S})$. To see the claim observe that there is a bijection

$$\begin{array}{rcl} \{(j,g) \mid g \in G, l_g < d_g, j = l_g+1,...,d_g\} \times P & \to & \{(j,g) \mid g \in G, 0 < d_g, j = 1,...,d_g\} \\ (j,g,h) & \mapsto & (j,gh) \end{array}$$

It is surjective because for every $g \in G$ with $d_g > 0$ and $j = 1,....,d_g$ there is some $g_0 \le g$ such that $l_{g_0} < j \le d_{g_0}$ and so $(j, g_0, g_0^{-1}g) \mapsto (j,g)$. It is injective because if $(j,gh) = (j',g'h')$ then the condition $\{l_g+1,...,d_g\} \ni j = j' \in \{l_{g'}+1,...,d_{g'}\}$ forces $g = g'$ from whence $h = h'$. Now consider the natural map

$$\dot{\bigoplus}_{b\in B} b \otimes \Bbbk P \to \mathrm{Rees}(\mathcal{S}) \tag{5}$$

where $\dot{\oplus}$ denotes the external direct sum. The existence of the previous bijection implies that a basis is mapped to a basis under (5). $\square$

2.7. **Factors of the Rees algebra.** We now consider reductions of $\mathrm{Rees}(\mathcal{S})$ by maximal ideals $\mathfrak{m} \in \mathrm{Spec}\,\Bbbk P$. Consider the following two canonical ideals

$$\mathfrak{m}_0 := (g \in P \mid g \gneq 1_G) \in \mathrm{Spec}\,\Bbbk P;$$
$$\mathfrak{m}_1 := (g - h \mid g,h \in P, g \ge h) \in \mathrm{Spec}\,\Bbbk P.$$

It is not hard to see that $\mathfrak{m}_1 = (x_1 - 1,...,x_n - 1)$.

Lemma 2.8. *Continue to assume that the filtration of* $\mathcal{S}$ *is non-negative. The following hold:*

(i) $\mathrm{Rees}(\mathcal{S})/\mathfrak{m}_0\,\mathrm{Rees}(\mathcal{S}) \cong \mathrm{gr}\,\mathcal{S}$;
(ii) $\mathrm{Rees}(\mathcal{S})/\mathfrak{m}_1\,\mathrm{Rees}(\mathcal{S}) \cong \mathcal{S}$.

Proof. First of all observe that

$$\mathfrak{m}_0 \operatorname{Rees}(\mathcal{S}) = \bigoplus_{g\in G}\Big(\sum_{h>1_G} \mathcal{S}_{gh^{-1}}\Big) \otimes g = \bigoplus_{g\in G}\Big(\sum_{h<g} \mathcal{S}_h\Big) \otimes g.$$

This immediately leads to (i). Now observe that there is a homomorphism

$$\begin{aligned} \operatorname{Rees}(\mathcal{S}) &\longrightarrow \mathcal{S}; \\ \sum_{g\in G} r_g \otimes g &\longmapsto \sum_{g\in G} r_g. \end{aligned}$$

The kernel is generated as a right $\Bbbk P$-module by elements $r \otimes (x_i - 1)$ with $i = 1, ..., n$ and $r \in \mathcal{S}$. This proves (ii). ☐

Remark 2.9. To complete the picture it is worth observing that in general the quotients $\operatorname{Rees}(\mathcal{S})/\mathfrak{m}\operatorname{Rees}(\mathcal{S})$ for $\mathfrak{m} \in \operatorname{Max}\Bbbk P$ are isomorphic to the associated graded algebras with respect to filtrations induced by quotients of G. The picture is also simplified somewhat by taking into account the action of the torus $(\Bbbk^\times)^n$ on $\operatorname{Rees}(\mathcal{S})$ acting on the set of ideals $\{\mathfrak{m}\operatorname{Rees}(\mathcal{S}) \mid \mathfrak{m} \in \operatorname{Max}\Bbbk P\}$ with finitely many orbits.

3. Frobenius extensions and their invariants

3.1. **Free Frobenius extensions.** Throughout this section $\mathcal{S} \subseteq \mathcal{R}$ are $\Bbbk$-algebras, and $\mathcal{S}$ is not necessarily central. The space $\operatorname{Hom}_{\mathcal{S}-}(\mathcal{R}, \mathcal{S})$ carries the structure of an $\mathcal{R}$-$\mathcal{S}$-bimodule by setting

$$(q \cdot f \cdot s)(r) := f(rq)s \tag{6}$$

where $q, r \in \mathcal{R}$, $s \in \mathcal{S}$ and $f \in \operatorname{Hom}_{\mathcal{S}-}(\mathcal{R}, \mathcal{S})$. We say that $\mathcal{S} \subseteq \mathcal{R}$ is a *Frobenius extension* if

(i) $\mathcal{R}$ is a projective left $\mathcal{S}$-module of finite type;
(ii) the $\mathcal{R}$-$\mathcal{S}$-bimodules $\mathcal{R}$ and $\operatorname{Hom}_{\mathcal{S}-}(\mathcal{R}, \mathcal{S})$ are isomorphic.

Cases where $\mathcal{R}$ is a free left $\mathcal{S}$-module are widespread (see [2, III.4] for a survey of some important examples) and these so-called *free Frobenius extensions* shall be our main object of study. There is a right handed definition of free Frobenius extensions, which is equivalent to the definition presented here (see [2, III.4.8, Lemma 1] for example).

These extensions generalise the concept of Frobenius algebras over $\Bbbk$, which are precisely the free Frobenius extensions where $\mathcal{S} = \Bbbk$. The theory was originally motivated by the extremely nice duality properties exhibited in the representation theory of Frobenius algebras.

3.2. **The Frobenius form.** It is well known that classical Frobenius algebras over $\Bbbk$ are characterised by the existence of a 1-form $\mathcal{R} \to \Bbbk$ such that the kernel contains no non-zero proper (left or right) ideals. We shall work with a similar characterisation of free Frobenius extensions which was observed by Nakayama and Tsuzuku.

Lemma 3.1. [24, pp. 11], [25, Proposition 4] *A finite extension $\mathcal{S} \subseteq \mathcal{R}$ such that $\mathcal{R}$ is a free left $\mathcal{S}$-module is Frobenius if and only if there exists $\Phi \in \operatorname{Hom}_{\mathcal{S}-\mathcal{S}}(\mathcal{R}, \mathcal{S})$ such that*

(F1) *the kernel of Φ contains no proper (left or right) ideals;*
(F2) *the assignment*

$$\begin{aligned} \mathcal{R} &\to \operatorname{Hom}_{\mathcal{S}-}(\mathcal{R}, \mathcal{S}) \\ r &\mapsto (q \mapsto \Phi(qr)) \end{aligned}$$

is surjective.

The form Φ shall be called the *Frobenius form* of the extension. The proof of the lemma is easy to describe: supposing $\mathcal{R}$ is isomorphic to $\mathrm{Hom}_{\mathcal{S}-}(\mathcal{R}, \mathcal{S})$ as an $\mathcal{R}$-$\mathcal{S}$-bimodule, the unit $1 \in \mathcal{R}$ corresponds to an element of $\mathrm{Hom}_{\mathcal{S}-\mathcal{S}}(\mathcal{R}, \mathcal{S})$ satisfying (F1) and (F2). Conversely, the assignment described in (F2) is certainly injective whenever (F1) holds.

Example 3.2. Some classical examples of Frobenius extensions are:

- the enveloping algebra of a restricted Lie algebra in characteristic $p > 0$, viewed as an extension of the p-centre [8, Proposition 1.2];
- the quantised enveloping algebra at an ℓth root of unity, viewed as an extension of its ℓ-centre [17];
- more generally, for every PI Hopf triple $Z_0 \subseteq Z \subseteq R$ the extension $Z_0 \subseteq R$ is Frobenius [2, Corollary III.4.7]. Notice that this example subsumes the previous two;
- Numerous examples were discussed by Brown–Gordon–Stroppel in [3]. Constructing machinery which could be applied to large families of examples simultaneously was a motivating goal of the current article.

3.3. **Free-filtered and free-graded Frobenius extensions.** We continue with a totally ordered, finitely generated abelian group $(G, \leq)$. We remind the reader that all gradings and filtrations in this article are $\Bbbk$-finite, proper, discrete and exhaustive. Let $\mathcal{R}$ be a non-negatively filtered $\Bbbk$-algebra and $\mathcal{S} \subseteq \mathcal{R}$ a central subalgebra.

Definition 3.3. We call $\mathcal{S} \subseteq \mathcal{R}$ a *free-filtered Frobenius extension* if $\mathcal{R}$ is a free-filtered $\mathcal{S}$-module and $\mathcal{S} \subseteq \mathcal{R}$ is a Frobenius extension.

Now let $S \subseteq R$ be a central extension of $\Bbbk$-algebras. When R is a finitely generated S-module the space $\mathrm{Hom}_S(R, S)$ is graded (Lemma 2.6, (i)).

Definition 3.4. We say that $S \subseteq R$ is a *free-graded Frobenius extension* if R is a free-graded S-module, $S \subseteq R$ is a free-graded Frobenius extension with homogeneous Frobenius form $\Phi : R \to S$. In other words $\Phi \in \mathrm{Hom}_S(R, S)^d$ for some $d \in G$.

Remark 3.5. (i) In the set up of Definition 3.4, property (F1) is equivalent to

(F1$'$) $\mathrm{Ker}(\Phi)$ contains no proper graded left or right ideals.

(ii) One remarkable corollary of our transfer result is that an extension $\mathcal{S} \subseteq \mathcal{R}$ is a free-graded Frobenius extension if and only if it is a Frobenius extension and $\mathcal{R}$ is a free-graded $\mathcal{S}$-module (Corollary 4.5).

3.4. **Invariants of filtered and graded Frobenius extensions.** Since the Frobenius extensions in this paper are assumed to have $\mathcal{S} \subseteq \mathcal{R}$ a central subalgebra, we assume this is the case throughout this subsection. The *rank of a free-filtered (resp. graded) Frobenius extension* $\mathcal{S} \subseteq \mathcal{R}$ is defined to be the rank of $\mathcal{R}$ as a free $\mathcal{S}$-module (Cf. Remark 2.1). Similarly we define the *degree of the Frobenius extension* to be the filtered (resp. graded) degree of any choice of (homogeneous) Frobenius form $\Phi : \mathcal{R} \to \mathcal{S}$.

Lemma 3.6. *The degree of a free-filtered or free-graded Frobenius extension is a well-defined invariant.*

Proof. The graded claim may be seen as a special case of the filtered claim, and so we prove the latter. Observe that if $\Phi : \mathcal{R} \to \mathcal{S}$ is a homomorphism of $\mathcal{S}$-modules of filtered degree d then the map $\Psi : \mathcal{R} \to \mathrm{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ given by $r \mapsto (q \mapsto \Phi(qr))$ is also of degree d. Thanks to Lemma 2.5 both $\mathcal{R}$ and $\mathrm{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ are free-filtered $\mathcal{S}$-modules and we may write D_1, D_2 for their multisets of filtered degrees (these are well defined thanks to Lemma 2.3). In Lemma 2.4 we observed that $D_1 d = D_2$. Since D_1 and D_2 are bounded this equation holds for at most one $d \in G$, hence $d = \deg(\Phi)$ is uniquely determined. $\square$

3.5. **The Nakayama automorphism.** For simplicity we suppose in this subsection that $\mathcal{S} \subseteq \mathcal{R}$ is a subalgebra of the centre of $\mathcal{R}$. Let $\mathcal{S} \subseteq \mathcal{R}$ be a Frobenius extension with Frobenius form $\Phi : \mathcal{R} \to \mathcal{S}$. Thanks to [25, Proposition 1] we know that $\mathrm{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-$\mathcal{S}$-bimodule and as an $\mathcal{S}$-$\mathcal{R}$-bimodule. In each case, $\mathrm{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ is generated by Φ as a left and right $\mathcal{R}$-module [15, Section 1]. This implies that there exists a bijection $\nu : \mathcal{R} \to \mathcal{R}$ such that $r\Phi = \Phi\nu(r)$ for all $r \in \mathcal{R}$, and it is straightforward to check that this map is actually an algebra automorphism, commonly called *the Nakayama automorphism.* To phrase it another way, ν satisfies

$$\Phi(qr) = \Phi(\nu(r)q) \tag{7}$$

for all $q, r \in \mathcal{R}$. It is clear that ν is only uniquely determined up to inner automorphism and so determines a class in $\mathrm{Out}(\mathcal{R}) = \mathrm{Aut}(\mathcal{R})/\,\mathrm{Inn}(\mathcal{R})$. It is a useful invariant to calculate since the class of ν is trivial if and only if $(q, r) \longmapsto \Phi(qr)$ is a symmetric form $\mathcal{R} \times \mathcal{R} \to \mathcal{S}$ which, in turn, has rather stark consequences for the representation theory of $\mathcal{R}$; see [5, Ch. IX] for example.

3.6. **Example: multiparameter quantum affine space.** We provide one elementary example of a free-graded Frobenius extension which shall be used to deduce some of our later results. Fix $\ell, n \in \mathbb{N}$ and an $n \times n$ matrix $q = (q_{i,j})_{1 \le i,j \le n}$ with entries in $\Bbbk$ satisfying $q_{i,j} = q_{j,i}^{-1}$ whenever $i \neq j$ and $q_{i,i} = 1$. We also suppose $q_{i,j}^{\ell} = 1$ for all i, j. Then $A = \Bbbk_q[\mathbb{A}^n] := \Bbbk_q[x_1, ..., x_n]$ shall denote the n-dimensional multiparameter quantum affine space with generators $x_1, ..., x_n$ satisfying $x_i x_j = q_{i,j} x_j x_i$. Let $(G, \le)$ be any totally ordered abelian group and choose elements $d_1, ..., d_n \in G$. We view A as a G-graded algebra by declaring that each x_i lies in degree d_i. The degree of any homogeneous element $a \in A$ is written $\deg(a)$.

The ℓ-centre of A is the graded unital central subalgebra generated by elements $x_1^{\ell}, ..., x_n^{\ell}$, and is denoted Z_0. For $\underline{a} = (a_1, ..., a_n) \in \mathbb{Z}_{\ge 0}^n$, the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in A shall be denoted $x^{\underline{a}}$. Consider the set of *restricted monomials*

$$\mathcal{B}^{\mathrm{res}} := \{x^{\underline{a}} \mid 0 \le a_i < \ell \text{ for all } i\}$$

and notice that A is a free Z_0-module with basis $\mathcal{B}^{\mathrm{res}}$. The index

$$\underline{\ell - 1} = (\ell - 1, ..., \ell - 1)$$

plays a special role as follows. We have a graded Z_0-module decomposition $A = \bigoplus_{\underline{a} \in \mathcal{B}^{\mathrm{res}}} Z_0 x^{\underline{a}}$ and we let Φ denote the projection onto the factor corresponding to $x^{\underline{\ell-1}}$, which is a homogeneous homomorphism of Z_0-modules.

Proposition 3.7. *The following hold:*

1. *A is a free-graded Frobenius extension of Z_0;*
2. *Φ is a homogeneous Frobenius form;*
3. *the degree of the extension is $\prod d_i^{-(\ell-1)}$;*
4. *the rank of the extension is ℓ^n.*

Proof. It is easy to see that A is a free-graded Z_0-module of rank ℓ^n with basis $\mathcal{B}^{\mathrm{res}}$, and that Φ is a homogeneous Z_0-equivariant map of the requisite degree, so it will suffice to confirm axioms (F1) and (F2) of Lemma 3.1. Suppose that

$$\mathrm{Ker}(\Phi) = \bigoplus_{x^{\underline{a}} \in \mathcal{B}^{\mathrm{res}}; \underline{a} \neq \underline{\ell - 1}} Z_0 x^{\underline{a}}$$

contains a right ideal J. Let $0 \neq y \in J$. We can write $y = \sum z_{\underline{a}} x^{\underline{a}}$ for coefficients $z_{\underline{a}} \in Z_0$, and suppose $z_{\underline{b}} \neq 0$ for some fixed $\underline{b} = (b_1, ..., b_n)$. Right multiplication by any monomial $x^{\underline{c}}$ permutes the summands in the decomposition $A = \bigoplus Z_0 x^{\underline{a}}$ and so if we let

$\underline{c} = (\ell-1-b_1, ..., \ell-1-b_n)$ then $\Phi(yx^{\underline{c}}) \neq 0$, contradicting the fact that $yx^{\underline{c}} \in J \subseteq \operatorname{Ker}(\Phi)$. Similarly, $\operatorname{Ker}(\Phi)$ contains no left ideals, using the decomposition of A as a free right module over Z_0.

In order to see that the assignment $y \mapsto (z \mapsto \Phi(zy))$ is a surjective map from A to $\operatorname{Hom}_{Z_0-}(A, Z_0)$ we show that for each $\underline{b}$ the projection $\sum z_{\underline{a}} x^{\underline{a}} \mapsto z_{\underline{b}}$ lies in the image. The result will follow since $\operatorname{Hom}_{Z_0-}(A, Z_0)$ is a free Z_0-module generated by these projections. Take $\underline{c} := \underline{\ell-1} - \underline{b}$ as above and observe that $y \mapsto \Phi(x^{\underline{c}} y)$ is a non-zero scalar multiple of the projection $\sum z_{\underline{a}} x^{\underline{a}} \mapsto z_{\underline{b}}$, which completes the proof. □

4. Proof of the transfer results

Throughout this section $(G, \leq)$ is a totally ordered, finitely generated abelian group and $\mathcal{R}$ is a non-negatively filtered $\Bbbk$-algebra with subalgebra $\mathcal{S} \subseteq \mathcal{R}$. We write $S := \operatorname{gr} \mathcal{S} \subseteq R := \operatorname{gr} \mathcal{R}$ and we remind the reader that all filtrations in this paper are assumed to be $\Bbbk$-finite, discrete, exhaustive and proper.

4.1. **Passing the Frobenius property down through a quotient.**

Lemma 4.1. *Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a central subalgebra, that $C \subseteq \mathcal{S}$ is a subalgebra of $\mathcal{S}$, $I \trianglelefteq C$ is an ideal (resp. homogeneous ideal) and $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered (resp. free-graded) Frobenius extension. Then $\mathcal{S}/I\mathcal{S} \subseteq \mathcal{R}/I\mathcal{R}$ is a free-filtered (resp. free-graded) Frobenius extension of the same rank and degree.*

Proof. It is easy to check that $\mathcal{R}/I\mathcal{R}$ is a free-filtered left module over $\mathcal{S}/I\mathcal{S}$, with a free-filtered basis given by the image under $\mathcal{R} \twoheadrightarrow \mathcal{R}/I\mathcal{R}$ of a basis for $\mathcal{R}$ over $\mathcal{S}$.

For C-modules $\mathcal{M}$ we may write $\mathcal{M}_I := \mathcal{M}/I\mathcal{M} \cong C/I \otimes_C \mathcal{M}$. We claim that there is a natural surjection

$$\operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})_I \twoheadrightarrow \operatorname{Hom}_{\mathcal{S}_I}(\mathcal{R}_I, \mathcal{S}_I). \tag{8}$$

First of all we have the map $\operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})_I \twoheadrightarrow \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S}_I)$ defined by sending $c \otimes \phi \mapsto (r \mapsto c \otimes \phi(r))$, which is surjective because $\mathcal{R}$ is a projective $\mathcal{S}$-module. Next we observe that, since elements of $\operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ are C-equivariant we have $\operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S}_I) = \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}_I, \mathcal{S}_I)$. Finally, this equals $\operatorname{Hom}_{\mathcal{S}_I}(\mathcal{R}_I, \mathcal{S}_I)$ since the $\mathcal{S}$-action on $\mathcal{R}_I$ and $\mathcal{S}_I$ factors through $\mathcal{S}_I$. This proves that (8) is surjective as required.

Now let $\Phi : \mathcal{R} \to \mathcal{S}$ be a Frobenius form and $\Psi : \mathcal{R} \to \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ the corresponding isomorphism $r \mapsto (q \mapsto \Phi(qr))$ of $\mathcal{R}$-$\mathcal{S}$-bimodules (see Lemma 3.1 and the remarks that follow). We define a form $\Phi_I : \mathcal{R}_I \to \mathcal{S}_I$ by setting $\Phi_I(r + I\mathcal{R}) = \Phi(r) + I\mathcal{S}$. It is well defined since Ψ is C-equivariant and $\Psi_I : \mathcal{R}_I \to \operatorname{Hom}_{\mathcal{S}_I}(\mathcal{R}_I, \mathcal{S}_I)$ is the corresponding map of $\mathcal{R}_I$-$\mathcal{S}_I$-bimodules. It is readily seen that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{R} & \xrightarrow{\Psi} & \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S}) \\
\big\downarrow & & \big\downarrow \\
 & & \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})_I \\
 & & \big\downarrow \\
\mathcal{R}_I & \xrightarrow{\Psi_I} & \operatorname{Hom}_{\mathcal{S}_I}(\mathcal{R}_I, \mathcal{S}_I)
\end{array}$$

Since Ψ is surjective it follows that $\Psi_I \circ (-)_I : \mathcal{R} \to \operatorname{Hom}_{\mathcal{S}_I}(\mathcal{R}_I, \mathcal{S}_I)$ is surjective, and so too is Ψ_I. Using Lemma 2.5 and the remarks at the start of the proof we see that both $\mathcal{R}_I$ and $\operatorname{Hom}_{\mathcal{S}_I}(\mathcal{R}_I, \mathcal{S})_I$ are free-filtered $\mathcal{S}_I$-modules of the same rank (Cf. Remark 2.1). In particular they are isomorphic as $\mathcal{S}$-modules and so by [22, Theorem 2.4] the map Ψ_I is also injective. This complete the proof. □

4.2. **Lifting the Frobenius property through a filtration.** Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a central extension of $\Bbbk$-algebras.

Proposition 4.2. *If $S \subseteq R$ is a free-graded Frobenius extension then $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered Frobenius extension of the same rank and the same degree as $S \subseteq R$.*

Proof. Since $S \subseteq R$ is a free-graded Frobenius extension, we know that R is free-graded as an S-module and there exists a homogeneous map of S-modules $\overline{\Phi} : R \to S$ which satisfies

(F1′) $\operatorname{Ker} \overline{\Phi}$ contains no non-trivial homogeneous left or right ideals (see Remark 3.5);
(F2) $r \mapsto (q \mapsto \overline{\Phi}(qr))$ is surjective $R \to \operatorname{Hom}_S(R, S)$;

Lemma 2.2 implies that $\mathcal{R}$ is free-filtered as an $\mathcal{S}$-module of the same rank, and that the multiset of degrees of the free-filtered generators coincides with that of R over S. By Lemma 2.6 we may find some $\Phi \in \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ with $\overline{\Phi} = \Phi + \sum_{g<\deg(\Phi)} \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})_g$. We claim that Φ satisfies properties (F1) and (F2) from Lemma 3.1.

If $\mathcal{K} \subseteq \mathcal{R}$ is any subspace we identify the associated graded K with a subspace of R in the obvious way. We claim that $\operatorname{gr} \operatorname{Ker} \Phi \subseteq \operatorname{Ker} \overline{\Phi} \subseteq R$. The space $\operatorname{gr} \operatorname{Ker} \Phi$ is spanned by its homogeneous components. Let $\overline{r}$ be one of these homogeneous elements and let $r \in \operatorname{Ker}(\Phi)$ be such that $\overline{r} = r + \sum_{g<\deg(r)} \mathcal{R}_g$. Then

$$\overline{\Phi}(\overline{r}) = \Phi(r) + \sum_{g<\deg(\Phi)+\deg(r)} \mathcal{S}_g = 0.$$

For a general element of $\operatorname{gr} \operatorname{Ker}(\Phi)$ we apply the above reasoning to each of the homogeneous summands, which confirms the claim. Now let us suppose that $I \trianglelefteq \mathcal{R}$ is an ideal contained within $\operatorname{Ker}(\Phi)$. Consider the graded ideal $\operatorname{gr} I \trianglelefteq R$. By the previous observations we have $\operatorname{gr} I \subseteq \operatorname{gr} \operatorname{Ker}(\Phi) \subseteq \operatorname{Ker}(\overline{\Phi})$. By property (F1′) $\operatorname{Ker}(\overline{\Phi})$ does not contain any non-trivial graded ideals, which forces $\operatorname{gr} I = 0$ and $I = 0$, and this verifies (F1).

To finish the proof we must confirm (F2) for Φ. Both R and $\operatorname{Hom}_S(R, S)$ are free-filtered S-modules and we write D_1, D_2 for the multisets of degrees of the free-graded generators. By Lemma 2.4 and Lemma 2.5 we have $D_2 = D_1^{-1}$ and $D_1 d = D_1^{-1}$. By Lemma 2.2, along with Lemma 2.5 the degrees of the free-filtered generators of $\mathcal{R}$ and $\operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$ over $\mathcal{S}$ are D_1 and D_2 respectively. Now we may apply Lemma 2.4 to see that Ψ_I is surjective, which completes the proof. □

4.3. **Passing the Frobenius property to the Rees algebra.** Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a central extension of $\Bbbk$-algebras.

Proposition 4.3. *If $\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered Frobenius extension,* $\operatorname{Rees}(\mathcal{S}) \subseteq \operatorname{Rees}(\mathcal{R})$ *is a free-graded Frobenius extension of the same rank and the same degree as $\mathcal{S} \subseteq \mathcal{R}$.*

Proof. Let $\mathcal{M}$ be a rank one free-filtered left $\mathcal{S}$-module, ie. $\mathcal{M} \cong \mathcal{S}$ as $\mathcal{S}$-modules but the filtration has been shifted by some element $d \in G$. Then we have $\operatorname{Rees}(\mathcal{M}) := \bigoplus_{g\in G} \mathcal{M}_g \otimes g \cong \operatorname{Rees}(\mathcal{S})(1 \otimes d^{-1})$ as $\operatorname{Rees}(\mathcal{S})$-modules, hence $\operatorname{Rees}(\mathcal{M})$ is free-graded $\operatorname{Rees}(\mathcal{S})$-module. Now $\mathcal{R}$ is a finite direct sum of rank one free-filtered modules, hence $\operatorname{Rees}(\mathcal{R})$ is free-graded over $\operatorname{Rees}(\mathcal{S})$.

By Lemma 3.1 there exists an S-module homomorphism $\Phi : R \to S$ which contains no ideals in its kernel, such that $r \mapsto (q \mapsto \Phi(rq))$ is surjective $\mathcal{R} \to \operatorname{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$. We define

$$\begin{aligned} \widetilde{\Phi} \;:\; & \operatorname{Rees}(\mathcal{R}) \longrightarrow \operatorname{Rees}(\mathcal{S}); \\ & \sum_{g\in G} r_g \otimes g \longmapsto \sum_{g\in G} \Phi(r_g) \otimes (gd) \end{aligned}$$

We clearly have $\widetilde{\Phi}(\text{Rees}(\mathcal{R})) \subseteq \mathcal{S} \otimes_{\Bbbk} \Bbbk P$, and since $\Phi(\mathcal{R}_g) \subseteq \mathcal{S}_{gd}$ the image lies in $\text{Rees}(\mathcal{S})$ as claimed. Furthermore $\widetilde{\Phi}$ is evidently homogeneous of degree d, and so it remains to show that $\widetilde{\Phi} : \text{Rees}(\mathcal{R}) \to \text{Rees}(\mathcal{S})$ satisfies properties (F1) and (F2) of Lemma 3.1.

Write $\widetilde{\Psi} : \text{Rees}(\mathcal{R}) \to \text{Hom}_{\text{Rees}(\mathcal{S})}(\text{Rees}(\mathcal{R}), \text{Rees}(\mathcal{S}))$ for the map $r \mapsto (q \mapsto \widetilde{\Phi}(rq))$. In order to show that it is injective it will suffice to check that it is so on the graded components of $\text{Rees}(\mathcal{R})$. Suppose that $a \in \mathcal{R}_g$ and $\widetilde{\Psi}(a \otimes g) = 0$. Then for all $q \in \text{Rees}(\mathcal{R})$ we have $\widetilde{\Phi}((a \otimes g)q) = 0$. In particular we may suppose $b \in \mathcal{R}_h$ and we have $\widetilde{\Phi}((a \otimes g)(b \otimes h)) = \Phi(ab) \otimes gh = 0$ which implies $\Phi(ab) = 0$ for all $b \in \mathcal{R}$. Since Φ contains no non-zero right ideals we have $a\mathcal{R} = 0$ and so $a = 0$. This confirms that axiom (F1) of Lemma 3.1 holds for $\widetilde{\Phi} : \text{Rees}(\mathcal{R}) \to \text{Rees}(\mathcal{S})$. To check axiom (F2) we use the same argument as per the last paragraph of Proposition 4.2, as we now explain. It follows from the remarks at the beginning of this proof that the multiset D_1 of the basis elements of the free-graded basis of $\text{Rees}(\mathcal{R})$ over $\text{Rees}(\mathcal{S})$ are the same as those of $\mathcal{R}$ over $\mathcal{S}$. Similarly, by Lemma 2.5, the multiset of degrees of the basis of $\text{Hom}_{\text{Rees}(\mathcal{S})}(\text{Rees}(\mathcal{R}), \text{Rees}(\mathcal{S}))$ is equal to D_1^{-1}, which equals the multiset of degrees of a basis for $\text{Hom}_{\mathcal{S}}(\mathcal{R}, \mathcal{S})$. The hypotheses of the current Proposition imply that $D_1 d = D_1^{-1}$, and so applying Lemma 2.4 once more we see that $\widetilde{\Psi}$ is an isomorphism.

□

4.4. **The Transfer Theorem.** Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a finite central extension, non-negatively filtered by a totally ordered finitely generated abelian group, so that $S := \text{gr}\,\mathcal{S} \subseteq R := \text{gr}\,\mathcal{R}$ is a finite extension graded by the same group. Once again we recall that all filtrations in this paper are assumed to be $\Bbbk$-finite, discrete, exhaustive and proper.

Theorem 4.4. *$\mathcal{S} \subseteq \mathcal{R}$ is a free-filtered Frobenius extension if and only if $S \subseteq R$ is a free-graded Frobenius extension. In this case the degrees and ranks of the two extensions coincide.*

Proof. The 'if' part follows from Proposition 4.2. Furthermore, Proposition 4.3 tells us that $\text{Rees}(\mathcal{S}) \subseteq \text{Rees}(\mathcal{R})$ is a free-graded Frobenius extension of the same rank and degree as $\mathcal{S} \subseteq \mathcal{R}$. Note that $1 \otimes \mathfrak{m}_0$ is homogeneous ideal of $\Bbbk P$ generated by the generators of P, therefore $1 \otimes \mathfrak{m}_0$ generates a homogeneous ideal of $\text{Rees}(\mathcal{R})$ and of $\text{Rees}(\mathcal{S})$. Applying Lemma 2.8 we see that $\text{Rees}(\mathcal{S})/(1 \otimes \mathfrak{m}_0)\,\text{Rees}(\mathcal{S}) \cong S$ and $\text{Rees}(\mathcal{R})/(1 \otimes \mathfrak{m}_0)\,\text{Rees}(\mathcal{R}) \cong R$ whilst Lemma 4.1 tells us that $S \subseteq R$ is a free-graded Frobenius extension of the same rank and degree as $\mathcal{S} \subseteq \mathcal{R}$, which completes the proof. □

Corollary 4.5. *$S \subseteq R$ is a free-graded Frobenius extension if and only if it is a Frobenius extension and R is a free-graded left S-module.*

Proof. The 'only if' part is obvious so suppose that R a is free-graded left S-module with (not necessarily homogeneous) Frobenius form $\Phi : R \to S$. Viewing $S \subseteq R$ as a filtered extension we may apply the previous result to deduce that $\text{gr}\,S \subseteq \text{gr}\,R$ is a free-graded Frobenius extension. By assumption $\text{gr}\,S \cong S$ and $\text{gr}\,R \cong R$. □

5. Applications: examples and counterexamples

5.1. **Quantum Schubert cells.** Schubert varieties are the closures of the Schubert cells, which provide an affine paving of the flag variety of a reductive algebraic group over $\mathbb{C}$ and they arise in a vast array of contexts in geometric representation theory; see [4, Ch. 6] for example. As a natural step in the program of quantising classical geometric objects, their quantum analogues have been defined and extensively studied (see [6] for example). The quantum coordinate rings on matrices occur as special examples.

Let Φ be a finite, indecomposable, crystallographic root system with associated Weyl group W, let $\mathfrak{g}$ be the corresponding complex, simple Lie algebra and let $U_q(\mathfrak{g})$ denote the Drinfeld-Jimbo quantised enveloping algebra generated by $E_1, ..., E_r, F_1, ..., F_r, K_1^{\pm 1}, ..., K_r^{\pm 1}$ and with relations which may be read in [21, Ch. 3], for example. Lusztig defined an action on $U_q(\mathfrak{g})$ of braid group B associated to the abstract Weyl group of $\mathfrak{g}$. This allows one to construct a (non-minimal) system of generators for $U_q(\mathfrak{g})$ corresponding to the roots of $\mathfrak{g}$ which serve as a system of PBW generators for $U_q(\mathfrak{g})$ (see [2, I.4.6], for example).

Lemma 5.1. *Write $\Phi^+ = \{\alpha_1, ..., \alpha_N\}$ for a choice of positive roots. There exist elements*

$$\{E_\alpha, F_\alpha \in U_q(\mathfrak{g}) \mid \alpha \in \Phi^+\}$$

such that the ordered monomials

$$E_{\alpha_1}^{a_1} \cdots E_{\alpha_N}^{a_N} K_1^{b_1} \cdots K_r^{b_r} F_{\alpha_1}^{c_1} \cdots F_{\alpha_N}^{c_N}$$

(with $a_i, c_i \in \mathbb{Z}_{\geq 0}$ and $b_i \in \mathbb{Z}$) form a $\Bbbk$-basis for $U_q(\mathfrak{g})$.

For each Weyl group element $w \in W$ we can consider the space $\Phi[w] := \Phi^+ \cap w\Phi$ and the *quantum Schubert variety* denoted $U_q[w]$ which is the subalgebra of $U_q(\mathfrak{g})$ generated by $\{E_\alpha \mid \alpha \in \Phi[w]\}$. If we suppose that $q \in \mathbb{C}^\times$ satisfies $q^\ell = 1$ then it is well known that E_α^ℓ is central in $U_q(\mathfrak{g})$. We write $Z_0 \subseteq U_q[w]$ for the subalgebra generated by $\{E_\alpha^\ell \mid \alpha \in \Phi[w]\}$.

As explained in [6, Section 10] there is a natural G-filtration on $U_q(\mathfrak{g})$ where $G = \mathbb{Z}^{2N+1}$ with G ordered lexicographically. We shall not describe this filtration in any detail but we observe that each $U_q[w]$ inherits the subspace filtration. The following is a good illustration of the power of our transfer theorem.

Theorem 5.2. *When $q^\ell = 1$ the quantum Schubert variety $U_q[w]$ is a free-filtered Frobenius extension of Z_0.*

Proof. According to [6, Proposition 10.1] the associated graded algebra $\operatorname{gr} U_q[w]$ is a quantum affine space. The subalgebra Z_0 inherits the filtration and it is clear that $\operatorname{gr} Z_0$ is the subalgebra generated by the ℓth powers of the generators. Now apply Proposition 3.7 and Theorem 4.4. ☐

Remark 5.3.
(i) By a very similar argument the quantum Borel $U_q(\mathfrak{b}) \subseteq U_q(\mathfrak{g})$ which is generated by $\{E_\alpha \mid \alpha \in \Phi^+\} \cup \{K_1^\pm, ..., K_r^\pm\}$ has an associated graded algebra which is a quantum torus. This allows one to recover Theorems 6.5 and 7.2(2) of [3] in a uniform manner.
(ii) The quantum Schubert cells are known to be twisted Calabi–Yau algebras [19], and this implies the existence of a certain automorphism which controls twists appearing in their cohomology theory. The latter is known as the Nakayama automorphism, and in [10] Goodearl and Yakimov gave a precise description in a setting applicable to all quantum Schubert cells. As was observed in [3, 2.5(2)], whenever a Frobenius extension is twisted Calabi–Yau the two different definitions of Nakayama automorphisms actually coincide, and so the Nakayama automorphisms in the sense of §3.5 are well understood for quantum Schubert cells.

5.2. **Modular finite W-algebras.** Finite W-algebras over $\mathbb{C}$ arise via a process of quantum Hamiltonian reduction, and they play a key role in the current developments in the representation theory of complex semisimple Lie algebras (see [20] for a nice survey). The modular analogues of these W-algebras were first studied by Premet in [26] (where their central quotients appeared) and more recently in [27], [29] where the infinite dimensional versions were studied by reduction modulo p. Recently Goodwin and the second author have presented a uniform approach to theory of modular finite W-algebras [11]. In the current section we show that the modular finite W-algebra is a Frobenius extension of the p-centre, with trivial Nakayama automorphism.

Let G be a reductive algebraic group over an algebraically closed field of characteristic $p > 0$ and set $\mathfrak{g} = \mathrm{Lie}(G)$. Recall that the enveloping algebra $U(\mathfrak{g})$ contains a large central subalgebra $Z_p(\mathfrak{g})$ called the p-centre. We also assume the standard hypotheses, which can be read in [11, §2.2] for example, and we let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ denote the non-degenerate G-invariant bilinear form on $\mathfrak{g}$. Pick a nilpotent element $e \in \mathfrak{g}$. According to [13] we can choose an associated cocharacter $\lambda : \Bbbk^\times \to G$ with the property that $\lambda(t)e = t^2 e$ for all $t \in \Bbbk^\times$ and $\lambda(\Bbbk^\times)$ acts rationally on the centraliser $\mathfrak{g}^e$ with non-negative eigenvalues. Write $\mathfrak{g}(i) \subseteq \mathfrak{g}$ for the i-eigenspace of $\lambda(\Bbbk^\times)$.

The bilinear form $\mathfrak{g}(-1) \times \mathfrak{g}(-1) \to \Bbbk$ given by $(x, y) \mapsto \kappa(e, [x, y])$ is non-degenerate and, according to [11, §4.1], we can choose a Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}(-1)$ in such a way that the nilpotent Lie algebra $\mathfrak{m} := \mathfrak{l} \oplus \bigoplus_{i<-1} \mathfrak{g}(i)$ is algebraic, ie. we may suppose $\mathfrak{m} = \mathrm{Lie}(M)$ for some closed connected unipotent subgroup $M \subseteq G$. The linear function $x \mapsto \kappa(e, x)$ defines a character on $\mathfrak{m}$ and we write $\mathfrak{m}_e = \{x - \kappa(e, x) \mid x \in \mathfrak{m}\}$. Now the modular finite W-algebra is defined to be the quantum Hamiltonian reduction $U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_e)^{\mathrm{Ad}(M)}$. The p-centre of the W-algebra is defined to be $Z_p(\mathfrak{g}, e) := (Z_p(\mathfrak{g})/Z_p(\mathfrak{g}) \cap U(\mathfrak{g})\mathfrak{m}_e)^{\mathrm{Ad}(M)}$.

Theorem 5.4. *The W-algebra $U(\mathfrak{g}, e)$ is a free-filtered Frobenius extension of the p-centre with trivial Nakayama automorphism.*

Proof. The PBW theorem for $U(\mathfrak{g}, e)$ states that the associated graded algebra $\mathrm{gr}\, U(\mathfrak{g}, e)$ with respect to the Kazhdan filtration is isomorphic to $\Bbbk[e + \mathfrak{v}]$ where $\mathfrak{v}$ is a homogeneous complement to $T_e \mathrm{Ad}(G)e$ inside $\mathfrak{g}$ [11, Theorem 5.2]. Furthermore, it follows from [11, Lemmas 5.1 & 8.2] that $\mathrm{gr}\, Z_p(\mathfrak{g}, e)$ identifies with $\Bbbk[e+\mathfrak{v}]^p$ as a Kazhdan graded subalgebra of $\Bbbk[e + \mathfrak{v}]$. It follows immediately that $\mathrm{gr}\, U(\mathfrak{g}, e)$ is a free-graded Frobenius extension of $\mathrm{gr}\, Z_p(\mathfrak{g}, e)$ (this is actually a special case of Proposition 3.7 where all commutation parameters are 1). By Theorem 4.4 we see that $U(\mathfrak{g}, e)$ is free-filtered Frobenius extension of $Z_p(\mathfrak{g}, e)$.

The enveloping algebra $U(\mathfrak{g})$ has central reductions $U_\eta(\mathfrak{g}) := U(\mathfrak{g})/\eta U(\mathfrak{g})$ called reduced enveloping algebras parameterised by elements $\eta \in \mathrm{Max}\, Z_p(\mathfrak{g}) \leftrightarrow (\mathfrak{g}^*)^{(1)}$. It is easy to see that the Lie algebras of reductive groups are unimodular, that is to say that they satisfy $\mathrm{tr}(\mathrm{ad}(x)) = 0$ for all $x \in \mathfrak{g}$, and so it follows from [8, Proposition 1.2] that $U_\eta(\mathfrak{g})$ is a symmetric algebra. Now consider the reduced finite W-algebras $U_\eta(\mathfrak{g}, \chi) := U(\mathfrak{g}, \chi)/\eta U(\mathfrak{g}, e)$ where $\eta \in \mathrm{Spec}\, Z_p(\mathfrak{g}, e)$. According to [11, Lemma 8.2] we may identify the maximal spectrum with a subset of the Frobenius twist $(\chi + \mathfrak{m}^\perp)^{(1)} \subseteq (\mathfrak{g}^*)^{(1)}$ and by [11, Remark 9.4] we have an isomorphism $U_\eta(\mathfrak{g}) \cong \mathrm{Mat}_D(U_\eta(\mathfrak{g}, e))$ where $D = p^{\dim \mathfrak{m}}$. It is not hard to see that for A a finite dimensional $\Bbbk$-algebra, A is symmetric if and only if $\mathrm{Mat}_D(A)$ is so. To be precise, if $\mathbb{B} : \mathrm{Mat}_D(A) \times \mathrm{Mat}_D(A) \to \Bbbk$ is a non-degenerate symmetric associative bilinear form and ι is the idempotent corresponding to the $(1, 1)$-entry of $\mathrm{Mat}_D(A)$ then the map $x, y \mapsto \mathbb{B}(\iota x \iota, \iota y \iota)$ is a nondegenerate symmetric associative form on A, which we view as a subalgebra of $\mathrm{Mat}_D(A)$. It transpires that $U_\eta(\mathfrak{g}, e)$ is symmetric for all $\eta \in \mathrm{Spec}\, Z_p(\mathfrak{g}, e)$.

Since $Z_p(\mathfrak{g}, e) \subseteq U(\mathfrak{g}, e)$ is a Frobenius extension we may denote by Φ the Frobenius form. In the notation of Lemma 4.1, taking $C = Z_p(\mathfrak{g}, e)$, we see that Φ_η is a Frobenius form for $U_\eta(\mathfrak{g}, e)$. By the previous paragraph it follows that Φ_η induces a symmetric form on $U_\eta(\mathfrak{g}, e)$.

Now fix $x, y \in U(\mathfrak{g}, e)$ and view $U(\mathfrak{g}, e)$ as a free $Z_p(\mathfrak{g}, e)$-module with basis $b_1, ..., b_r$. If we write

$$\Phi(xy) - \Phi(yx) = \sum_{i=1}^{r} z_i b_i$$

then it follows from the previous paragraph that $z_i \in \eta$ for all maximal ideals $\eta \in \mathrm{Spec}\, Z_p(\mathfrak{g}, e)$. Since $Z_p(\mathfrak{g}, e)$ is a reduced, commutative, affine algebra it is a Jacobson ring and it follows that $\bigcap_{\eta\in \operatorname{Spec} Z_p(\mathfrak{g})} \eta = 0$ and so we conclude that $\Phi(xy) - \Phi(yx) = 0$ for all $x, y \in U(\mathfrak{g}, e)$. We have deduced that Φ induces a symmetric bilinear form from $U(\mathfrak{g}, e)$ to $Z_p(\mathfrak{g}, e)$ which completes the proof. □

5.3. **The Quantum Grassmanian $\mathcal{O}_q[\mathrm{Gr}(n, m)]$.** In this final section we use the theory we have developed thus far to give an example of a very natural quantum algebra at an ℓ^{th} root of unity, which is not a free Frobenius extension of its ℓ-centre. The proof relies on the following elementary lemma which follows directly from Lemmas 2.4 and 2.5.

Lemma 5.5. *Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is a free-graded central Frobenius extension of degree d and D is the multiset of degrees of the basis elements of $\mathcal{R}$ over $\mathcal{S}$. Then*

$$D = D^{-1}d.$$

□

Let $n < m$ be two positive integers and q a nonzero element of the base field $\Bbbk$. We recall that the quantum (n, m)-Grassmanian $\mathcal{O}_q[\mathrm{Gr}(n, m)]$ is the subalgebra of the quantum matrices $\mathcal{O}_q[\mathrm{Mat}(n \times m, \Bbbk)]$ generated by the maximal quantum minors; see [16], for example. Let q be a primitive ℓ^{th} root of unity and consider the subalgebra $Z_0 \subseteq \mathcal{O}_q[\mathrm{Gr}(n, m)]$ which is generated by the ℓ^{th} powers of the maximal quantum minors. The following result is well known, but we could not find it in the literature and so we include a proof for the reader's convenience.

Lemma 5.6. *Z_0 is a central subalgebra of $\mathcal{O}_q[\mathrm{Gr}(n, m)]$, which we call* the ℓ-centre.

Proof. For $I \subseteq \{1, ..., m\}$ we write $[I]$ for the maximal quantum minor in $\mathcal{O}_q[\mathrm{Mat}(n\times m, \Bbbk)]$ with columns indexed by I (and rows $\{1, \dots, n\}$) and by $X_{i,j}$ the canonical generators of $\mathcal{O}_q[\mathrm{Mat}(n \times m, \Bbbk)]$. We prove the stronger result, that $[[I]^\ell, X_{i,j}] = 0$. If $j \in I$ then [9, Lemma 5.2(a)] implies that $[[I], X_{i,j}] = 0$. If $j \notin I$ then [9, Lemma 5.2(b)] implies that $[I]X_{i,j} = qX_{i,j}[I] + (q - q^{-1})Z$ where $Z := \sum_{k>j,k\in I} q^{|I\cap[j,k]|}X_{i,k}[I \setminus \{k\} \cup \{j\}]$. Since Z and $X_{i,j}$ quantum commute (by [9, Lemmas 5.2(a) and 5.3(b)]) an easy induction gives $[I]^tX_{i,j} = q^tX_{i,j}[I]^t + (q^t - q^{-t})Z[I]^{t-1}$ which leads to $[[I]^\ell, X_{i,j}] = 0$ upon setting $t = \ell$. □

The relations between the generators of $\mathcal{O}_q[\mathrm{Gr}(n, m)]$ are called the *quantum Plücker relations* and are quite complicated in general. Building on the work of Lenagan and Rigal [18], Rigal and Zadunaisky have shown that $\mathcal{O}_q[\mathrm{Gr}(n, m)]$ is a symmetric quantum graded algebra with straightening law [28]. One of the many nice consequences of this fact is the existence of a filtration such that the associated graded $\operatorname{gr}\mathcal{O}_q[\mathrm{Gr}(n, m)]$ has a simple presentation. Since we are in survey mode we do not want to describe this filtration in detail, however we shall describe the associated graded algebra when $(n, m) = (2, 4)$.

Lemma 5.7. [28, Theorems 4.9 & 5.1.6] *There is a filtration of $\mathcal{O}_q[\mathrm{Gr}(2, 4)]$ such that $\operatorname{gr}\mathcal{O}_q[\mathrm{Gr}(2, 4)]$ has generators $\{x_1 = [12], x_2 = [13], x_3 = [23], x_4 = [14], x_5 = [24], x_6 = [34]\}$ with relations*

$$x_ix_j = q^{s(i,j)}x_jx_i \text{ for all } 1 \le i, j \le 6 \tag{9}$$

$$x_3x_4 = q^{t(i,j)}x_2x_5. \tag{10}$$

As a consequence, $\operatorname{gr}\mathcal{O}_q[\mathrm{Gr}(2, 4)]$ has a basis given by the standard monomials

$$\{x_1^{k_1}x_2^{k_2}x_i^{k_i}x_5^{k_5}x_6^{k_6} \mid i = 3 \text{ or } 4 \text{ and } k_1, ..., k_6 \ge 0\} \tag{11}$$

□

The following lemma is proven by a simple, but long-winded, combinatorial argument which we sketch in its most basic form.

Lemma 5.8. $\operatorname{gr}\mathcal{O}_q[\mathrm{Gr}(2,4)]$ *is a free module over* $\operatorname{gr} Z_0$ *with basis*

$$\left\{ x_1^{k_1}x_2^{k_2}x_i^{k_i}x_5^{k_5}x_6^{k_6} \mid \begin{array}{c} i=3,4; 0\le k_1,...,k_6<\ell \\ \textit{either } k_2+k_i<\ell \textit{ or } k_i+k_5<\ell \end{array} \right\}$$

Proof. Fix $i = 1,6$. Since the only relations between x_i and the other generators are quantum commutation it suffices to show that the subalgebra generated by x_2, x_3, x_4, x_5 is a free module over the central subalgebra generated by $x_2^\ell, x_3^\ell, x_4^\ell, x_5^\ell$ with basis

$$\mathcal{B} := \{x_2^i x_m^j x_5^k : m = 3,4; 0 \le i,j,k < l; i+j<l \text{ or } j+k<l\}.$$

Let M denote the set of standard monomials in $\langle x_2^\ell, x_3^\ell, x_4^\ell, x_5^\ell\rangle$, so that $M = M_1 \cup M_2$ with

$$M_i = \{x_2^{al}x_i^{bl}x_5^{cl} : a,b,c \ge 0\}.$$

In order to prove the lemma one may show that for every $m \in M$ and $b \in \mathcal{B}$ the product mb is a non-zero scalar multiple of some element of S, and that this assignment induces a bijection between $M \times \mathcal{B}$ and S. This last claim can be proven by explicit calculation, multiplying together various basis elements, however we omit the details since they are long and tedious. ☐

Theorem 5.9. $\mathcal{O}_q[\mathrm{Gr}(2,4)]$ *is not a Frobenius extension of its* ℓ*-centre.*

Proof. We suppose for a contradiction that $\mathcal{O}_q[\mathrm{Gr}(2,4)]$ is a Frobenius extension of Z_0. The filtration defined in [28] assigns a filtered degree to each generator $x_1,...,x_6$ in such a way that $\deg(x_3) + \deg(x_4) = \deg(x_2) + \deg(x_5)$. It is easy to see from Lemma 5.8 that $\operatorname{gr}\mathcal{O}_q[\mathrm{Gr}(2,4)]$ is a free-graded algebra over $\operatorname{gr} Z_0$. It follows from Lemma 2.2 that $Z_0 \subseteq \mathcal{O}_q[\mathrm{Gr}(2,4)]$ is a free-filtered extension and so by Theorem 4.4 we see that $\operatorname{gr} Z_0 \subseteq \operatorname{gr}\mathcal{O}_q[\mathrm{Gr}(2,4)]$ is a free-graded Frobenius extension. In particular, it is a Frobenius extension. Write $\overline{\mathcal{O}}_q := \operatorname{gr}\mathcal{O}_q$ and $\overline{Z}_0 := \operatorname{gr} Z_0$. Now it follows from Corollary 4.5 that $\overline{Z}_0 \subseteq \overline{\mathcal{O}}_q$ is actually a free-graded Frobenius extension with respect to any grading such that

(i) $\overline{Z}_0$ is a graded subalgebra of $\overline{\mathcal{O}}_q$; and
(ii) $\overline{\mathcal{O}}_q$ is a free-graded module over $\overline{Z}_0$.

We consider the grading on $\overline{\mathcal{O}}_q$ defined by setting

$$\begin{aligned} &\deg(x_1) = \deg(x_3) = \deg(x_5) = \deg(x_6) = 2; \\ &\deg(x_2) = \deg(x_4) = 1. \end{aligned}$$

Since $\deg(x_3) + \deg(x_4) = \deg(x_2) + \deg(x_5) = 3$ this really does define a grading on $\overline{\mathcal{O}}_q$ and it is easy to see that $\overline{\mathcal{O}}_q$ is a free-graded module over $\overline{Z}_0$ with basis described in Lemma 5.8. Using Corollary 4.5 again we see that $\overline{Z}_0 \subseteq \overline{\mathcal{O}}_q$ is a free-graded Frobenius extension with respect to this new grading. By inspection the largest degree of any basis element is $\deg(x_1^{\ell-1}x_3^{\ell-1}x_5^{\ell-1}x_6^{\ell-1}) = 8(\ell-1)$. There are precisely 2 basis elements of degree 1 in the basis of Lemma 5.8 and there are no elements of degree $8(\ell-1)-1$. Now Lemma 5.5 states that if D is the multiset of degrees of the free-graded generators of $\overline{\mathcal{O}}_q$ over $\overline{Z}_0$ then $D = D^{-1}d$. Since $D \subseteq \mathbb{Z}$ it is natural to use additive notation here and conclude that the number of basis elements of graded degree 1 is equal to the number of basis element of graded degree $-1 + 8(\ell-1)$. This contradiction confirms completes the proof. ☐

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