A Solution Method for Linear Rational Expectation Models under Imperfect Information

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Proposed Running Head: Solution Method for Imperfect Info Models

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Abstract

This article presents a solution algorithm for linear rational expectation models under imperfect information, in which some decisions are made based on smaller information sets than others. In our solution representation, imperfect information does not affect the coefficients on crawling variables, which implies that, if a perfect information model exhibits saddle path stability, for example, the corresponding imperfect information models also exhibit saddle-path stability. However, imperfect information can significantly alter the quantitative properties of a model. Indeed, this article demonstrates that, with a predetermined wage contract, the standard RBC model remarkably improves the correlation between labour productivity and output.

Keywords: Linear rational expectation models, Imperfect information

JEL Classification Codes: C63, C65, C68

1 Introduction

This article presents a solution algorithm for linear rational expectation models under imperfect information. "Imperfect information" in this article signifies that some decisions are made before observing some shocks, while others are made after observing them. For example, we can consider a variant of the standard RBC model, in which households predetermine wage (and commit themselves to accommodating any labour demand) before observing today's productivity shock. In this variant, the equations that define the equilibrium are the same as in the standard RBC model, except for the information structure; i.e., the first order condition (FOC) with respect to labour supply has an expectation operator.

Imperfect information is an interesting consideration for several reasons. First, imperfect information plays an important role in many important classes of models, such as the sticky information model of Mankiw and Reis (2001). Second, researchers often do not know *a priori* what information is available when each decision is made; hence, they may want to estimate the information structure by parameterising it, or they may want to experiment on a model under several patterns of information structure. It is easy to implement such exercises with our algorithm; once structural equations under the corresponding perfect information are obtained, then the additional input to the algorithm is only the information structure in a model. Third, the obtained numerical result may not be robust for a small change in information structure. Indeed, we show a variant of the RBC model with a predetermined wage contract to demonstrate that changing information structure remarkably improves the model performance in terms of thecorrelation between labour productivity and output.

This article offers a set of easy-to-use Matlab codes to solve a general class of linear models under imperfect information.[1] The solution method is an extension of the QZ method by Sims (2002). The algorithm solves the system of linear difference equations in the following form.

$$0 = \tilde{E}_t[Ay_{t+1} + By_t] + C\xi_t \tag{1}$$

where A, B and C are proper coefficient matrices, and y_t and ξ_t are the vectors of endogenous and exogenous variables, respectively. Expectation operator $\tilde{E}_t[\quad]$ is non-standard because the information set in each equation can differ from each other (imperfect information).

The algorithm provides the solution of a model in the form of

$$\begin{aligned}
\kappa_{t+1} &= H\kappa_t + J\xi^{t,S} \\
\phi_t &= F\kappa_t + G\xi^{t,S} \\
\xi^{t,S} &:= \begin{pmatrix} \xi_t^T & \cdots & \xi_{t-S}^T \end{pmatrix}^T
\end{aligned}$$

where κ_t and ϕ_t are the vectors of crawling and jump variables, respectively,[2] and ξ_{t-s} is the vector of innovations at time $t-s$, for $s = 0, \cdots, S$, where S is such that the minimum information set in the model includes all information up to time $t - S - 1$. The superscript T indicates transposition, and hence $\xi^{\tau,S}$ is the vertical concatenation

[1]The codes and a manual for them are available at:

http://www.kent.ac.uk/economics/papers/papers07.html

[2]Crawling and jump variables are essentially the same concepts as predetermined and non-predetermined variables in the literature. Indeed, they are interchangeable under perfect information, which is a special case of imperfect information. However, the traditional terminologies predetermined/non-predetermined could be misleading, in the sense that typical non-predetermined variables such as consumption and wage can be *already determined before the current period* under imperfect information.

of $\{\xi_{\tau-s}\}_{s=0}^{S}$. H, J, F and G are the solution matrices provided by the algorithm.

It is important to note that the state variables in this solution are κ_t and $\xi^{t,S}$. Imperfect information requires the expansion of the state space, but this can be done either by expanding the innovation vector or by expanding the set of crawling variables; i.e., the representation of state space is not necessarily unique. Our choice of state variables works, intuitively because, if past innovations are recorded, we can recover the past crawling variables and hence recover the information available in past periods.

By keeping the number of crawling variables unchanged, it can be shown that the dynamic parts of the solution (i.e., H and F matrices) are the same as in the corresponding perfect information model.[3] Thus, it is clear that if the corresponding perfect model is saddle-path stable (sunspot, explosive), then an imperfect information model is also saddle-path stable (sunspot, explosive, respectively). That is to say, the information structure does not alter the dynamic stability property. In this sense, we can say that *qualitatively* an imperfect information model inherits key properties of the corresponding perfect information model. However, *quantitatively* imperfect information can have significant effects, as shown in Section 5.

Moreover, invariant H and F matrices imply that the direct effects of imperfect information on impulse response functions (IRFs) last for only S periods after an impulse. In subsequent periods, IRFs follow essentially the same process as those in the perfect information counterpart. More specifically, suppose that an endogenous variable a_t is determined S periods in advance (observing κ_{t-S} and ξ_{t-S}). In this case, the IRFs are directly affected by the information imperfection from time t to $t+S-1$. At $t+S$,

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[3] See Wang and Wen (2006). They point out that the dynamic parts under imperfect information have the same roots as those under perfect information, which is a corollary to our result.

however, the IRFs show sudden jumps because a_{t+S} starts reacting to innovations at t. Let κ_{t+S} be the values of the crawling variables at the beginning of $t+S$. Then, the following IRFs exactly follow the same time path as those under perfect information that starts with κ_{t+S} (without innovations). One such example can be found in Dupor and Tsuruga (2005), who argue that the hump-shaped IRFs found in Mankiw and Reis (2001) critically hinge on the assumption of the Calvo style information updating, in which some agents, though their population decreases over time, cannot renew their information forever. By instead constructing the Taylor style staggered information renewal, Dupor and Tsuruga (2005) show that IRFs jump to zero right after the last cohort renews its information set. We show, however, that such sudden jumps in IRFs are rather common observations in imperfect information models.

There are, at least allegedly, three existing treatments of imperfect information.[4] The

[4]There are three types of methods for perfect information models.

1. King and Watson's method (1998 and 2002)(see also Woodford, undated) implements a two-stage substitution. First, non-dynamic jump variables are substituted out, and then dynamic jump variables are substituted out from the system of equations.

2. In the QZ method by Sims (2002) (see also Klein, 2000), the QZ decomposition is applied to matrices on endogenous variables. Recognising that (1) roots that correspond to non-dynamic jump variables are infinite, and (2) roots that correspond to dynamic jump variables are larger than one in absolute terms, the transversality conditions (TVCs) eliminate both types of jump variables at once.

3. The method of undetermined coefficients by Uhlig (1999) (see also Christiano, 1998) substitutes a guess solution into the given system of equations; the resulting matrix polynomial is solved directly. In principle, this method does not require that given equations are first-order difference equations. Higher order matrix polynomials can be numerically solved (see Appendix).

first remedy for imperfect information is to define the *dummy* variables.[5] For example, consider a variant of the standard RBC model, in which labour supply L_t is determined without observing today's innovations. Then, the optimal labour supply is determined by

$$0 = E_{t-1}\left[\eta L_t + \sigma C_t - W_t\right] \tag{2}$$

where C_t and W_t are consumption and wage at time t, η and σ are parameters provided by the theory, and $E_{t-1}[\,]$ is the expectation operator with all information up to time $t-1$. Define dummy variable L_t^* such that

$$\begin{aligned} 0 &= E_t\left[\eta L_{t+1}^* + \sigma C_{t+1} - W_{t+1}\right] \\ L_{t+1} &= L_t^* \end{aligned}$$

In this method, having additional crawling variable L_t, the set of crawling variables is expanded. The problem with this method is that it cannot solve the model if some endogenous variables are determined before observing some (not all) of today's innovations but after observing the others.

The second method developed by Wang and Wen (2006) is most closely related to our method, in the sense that they also chose to expand expectation error instead of crawling variables. Apart from the difference in the bases of the algorithm (they employ the method of undetermined coefficients, while we use QZ method), however, there are three major differences. First, our algorithm allows more flexible specification; with our method, an endogenous variable can be determined observing some innovations but not observing the others at t, while their method deals with lagged expectations like

[5] See Uhlig (1999) for example.

the dummy variable method mentioned above. Second, our method only requires two indicator matrices (see the next paragraph), which specifies whether each variable is decided with or without observing each innovation, while they require researchers to solve for their Λ_i and Γ_i matrices ($i = 1, \cdots, S$). Third and most importantly, our method reveals sharper analytical results (see footnote 3, for example).

The other possibility is a modification of the method of undetermined coefficients. According to Christiano (1998), his version of method of undetermined coefficients, like ours, can deal with models in which some endogenous variables are determined before observing some (not all) of today's innovations are observed but after observing the others. The most salient difference between his method and ours is in the specification of information structure; Christiano (1998) requires a user to provide only one matrix R that specifies which innovations are to be included in the information set of each expectation operator. Roughly speaking, his R relates equations to observable innovations. In contrast, in the algorithm developed in this paper, a researcher must specify two indicator matrices; one relates innovations to equations (like Christiano, 1998), and the other relates innovations to variables. To understand why the latter matrix is necessary, consider the above example (2). Certainly, it is clear that a researcher must specify the information set of the expectation operator in (2). However, in a given information set, there are generically three possibilities, namely that (a) the representative household fixes labour supply before observing some of today's innovations, (b) it determines wage before innovations (sticky wage), or (c) it decides consumption before innovations. Hence, one more matrix is necessary in our algorithm to specify which of C_t, W_t or H_t is chosen while not having full information. In general, the quantitative behaviour of a model is completely different, depending on which variables are assumed to be decided before observing some information. Indeed, Section 5 shows that the difference between (a) and (b) is very crucial.

The plan of this article is as follows. In Section 2, we define the problem and derive the solution, and show two key observations: (i) if the k-th time t variable $y_{k,t}$ is determined without observing the i-th time $t-s$ innovations $\xi_{i,t-s}$, then $y_{k,t}$ cannot respond to $\xi_{i,t-s}$; and (ii) if the expectation operator in the j-th equation has an information set that includes $\xi_{i,t-s}$, then $\xi_{i,t-s}$ cannot be the source of the expectation error in the j-th equation. It turns out that these two restrictions are enough to determine the unique solution coefficients. In Section 3, we discuss the assumptions that are necessary for guaranteeing the existence of a solution. Each of them has some economic meaning, and the existence condition is slightly tighter under imperfect information than under perfect information. In Section 4, the main features of the solution of imperfect information models are briefly discussed. Most of them are direct consequences of the invariant H and F matrices. In Section 5, we demonstrate the effects of imperfect information on the standard RBC model as an example. The final section concludes the discussion.

2 Derivation of the Solution

Essentially, our algorithm is an extension of the QZ method used in Sims (2002). Our objective is to obtain the state space representation of a solution that satisfies two key zero restrictions. For the details of matrix notation, see Appendix.

2.1 Definition of the Problem

This subsection defines the inputs and outputs of the algorithm.

2.1.1 Given Models

Instead of using expectation operators like (1), following Sims (2002), we formulate the linear rational models with expectation errors as follows.

$$0 = Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \tag{3}$$

where

$$y_t = \begin{pmatrix} \kappa_t \\ \\ \phi_t \end{pmatrix}, \xi^{t,S} = \begin{pmatrix} \xi_t \\ \vdots \\ \xi_{t-S} \end{pmatrix}$$

$$\begin{aligned} E &:= \begin{bmatrix} E_0 & E_1 & \cdots & E_s & \cdots & E_S \end{bmatrix} \\ &:= \begin{bmatrix} E_{0,11} & \cdots & E_{0,1N} & & E_{s,11} & \cdots & E_{s,1N} & & E_{S-1,11} & \cdots & E_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ E_{0,M1} & \cdots & E_{0,MN} & & E_{s,M1} & \cdots & E_{s,MN} & & E_{S-1,M1} & \cdots & E_{S,MN} \end{bmatrix} \end{aligned}$$

y_t is the vector of all endogenous variables, in which κ_t is the vector of crawling variables and ϕ_t is that of jump variables. Stock variables are all recorded at the beginning of each period. M is the number of equations, which is equal to the number of endogenous variables, N is the number of innovations, and S is such that the minimum information set includes ξ_{t-S-1}.

ξ_{t-s} is a column vector of *iid* innovations at time $t-s$. Limiting ξ_t to be *iid* is not restrictive since we can add the law of motions of serially correlated shocks to the system of equations and treat the shocks themselves as crawling variables.[6]

Only two sets of inputs are required: (i) coefficient matrices A, B and C, which are typically the same as in perfect information models; and (ii) indicator matrices $IndE$

[6]See Woodford (undated). This technique simplifies the algebra and computation significantly.

and $IndV$ (their elements are either zero or one).[7] The size of $IndE$ is the same as that of E in (3), and, if the i,j-th element in E is zero, then the i,j-th element $IndE$ is also zero. Essentially, $IndE$ specifies the information set in each equation in (3). The size of $IndV$ is the same as that of the vertical concatenation $[J^T\ G^T]^T$ (see the next subsection), and its zero elements represent variables that do not observe each innovation. The value of the non-zero elements in J, G and E are computed by the algorithm, while (the positions of) their zero elements are provided by a user.

2.1.2 Goal of the Algorithm

Our objective is to obtain the state space representation of (3).

$$\kappa_{t+1} = H\kappa_t + J\xi^{t,S} \tag{4a}$$

$$\phi_t = F\kappa_t + G\xi^{t,S} \tag{4b}$$

[7] See the manual for further details. Note that we do not explicitly mention these two indicator matrix in the rest of this article.

where

$$
\begin{aligned}
J &:= \begin{bmatrix} J_0 & J_1 & \cdots & J_s & \cdots & J_S \end{bmatrix} \\
&:= \begin{bmatrix} J_{0,11} & \cdots & J_{0,1N} & & J_{s,11} & \cdots & J_{s,1N} & & J_{S,11} & \cdots & J_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ J_{0,M_\kappa 1} & \cdots & J_{0,M_\kappa N} & & J_{s,M_\kappa 1} & \cdots & J_{s,M_\kappa N} & & J_{S,M_\kappa 1} & \cdots & J_{S,M_\kappa N} \end{bmatrix} \\
G &:= \begin{bmatrix} G_0 & G_1 & \cdots & G_s & \cdots & G_S \end{bmatrix} \\
&:= \begin{bmatrix} G_{0,11} & \cdots & G_{0,1N} & & G_{s,11} & \cdots & G_{s,1N} & & G_{S,11} & \cdots & G_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ G_{0,M_\phi 1} & \cdots & G_{0,M_\phi N} & & G_{s,M_\phi 1} & \cdots & G_{s,M_\phi N} & & G_{S,M_\phi 1} & \cdots & G_{S,M_\phi N} \end{bmatrix}
\end{aligned}
$$

2.2 Two Key Observations

This subsection shows two zero restrictions. The algorithm seeks the solution that satisfies them.

2.2.1 Repeated Substitutions

To obtain the representation of κ_{t+1} and ϕ_t as functions of κ_{t-S} and $\xi_{t-\tau}$ for $\tau = 0, \cdots, 2S-1$, repeat the substitution of the vertically concatenated guess solution (4) into itself. Defining $\check{H} := [H^T \ F^T]^T$,

$$
\begin{pmatrix} \kappa_{t+1} \\ \phi_t \end{pmatrix} = \check{H}\kappa_t + \tilde{\Gamma}\xi^{t,S} = \check{H} \left(\ H^S \kappa_{t-S} + \sum_{k=1}^{S} H^{k-1} J \xi^{t-k,S} \ \right) + \tilde{\Gamma}\xi^{t,S}
$$

$$
\begin{aligned}
&= \check{H}H^S\kappa_{t-S} + \left(\Gamma_0\xi_{t-0} + \Gamma_1\xi_{t-1} + \cdots + \Gamma_S\xi_{t-S}\right) \\
&\quad + \check{H}\begin{pmatrix} H^0\left(J_0\xi_{t-1} + J_1\xi_{t-2} + \cdots + J_S\xi_{t-1-S}\right) \\ +H^1\left(J_0\xi_{t-2} + J_1\xi_{t-3} + \cdots + J_S\xi_{t-2-S}\right) + \cdots \\ +H^{S-1}\left(J_0\xi_{t-S} + J_1\xi_{t-S-1} + \cdots + J_S\xi_{t-S-S}\right) \end{pmatrix} \\
&= \check{H}H^S\kappa_{t-S} + \Pi_0\xi_t + \Pi_1\xi_{t-1} + \cdots + \Pi_s\xi_{t-s} + \cdots + \Pi_S\xi_{t-S} \\
&\quad + \text{terms with } \xi_{t-\tau} \text{ for } \tau \geq S+1 \qquad (5)
\end{aligned}
$$

where $\tilde{\Gamma} := \begin{bmatrix} \Gamma_0 & \cdots & \Gamma_s & \cdots & \Gamma_S \end{bmatrix}$ with $\Gamma_s := \begin{bmatrix} J_s^T & G_s^T \end{bmatrix}^T$, and

$$
\begin{aligned}
\Pi_0 :=& \quad \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix} \\
\Pi_1 :=& \quad \Gamma_1 + \begin{bmatrix} H \\ F \end{bmatrix} J_0 = \begin{bmatrix} J_1 + HJ_0 \\ G_1 + FJ_0 \end{bmatrix} \\
\Pi_2 :=& \quad \Gamma_2 + \begin{bmatrix} H \\ F \end{bmatrix} \left(J_1 + HJ_0\right) = \begin{bmatrix} J_2 + H\left(J_1 + HJ_0\right) \\ G_2 + F\left(J_1 + HJ_0\right) \end{bmatrix}, \ \cdots \\
\Pi_s :=& \quad \Gamma_s + \begin{bmatrix} H \\ F \end{bmatrix} \left(\sum\nolimits_{k=0}^{s-1} H^{s-1-k}J_k\right) = \begin{bmatrix} J_s + H\sum_{k=0}^{s-1} H^{s-1-k}J_k \\ G_s + F\sum_{k=0}^{s-1} H^{s-1-k}J_k \end{bmatrix}, \ \cdots \\
\Pi_S :=& \quad \Gamma_S + \begin{bmatrix} H \\ F \end{bmatrix} \left(\sum\nolimits_{k=0}^{S-1} H^{S-1-k}J_k\right) = \begin{bmatrix} J_S + H\sum_{k=0}^{S-1} H^{S-1-k}J_k \\ G_S + F\sum_{k=0}^{S-1} H^{S-1-k}J_k \end{bmatrix}
\end{aligned}
$$

In the recursive representation,

$$
\begin{aligned}
\Pi_0 &= \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix} \\
\Pi_s &= \Gamma_s + \tilde{H}\Pi_{s-1} \text{ for } s = 1, \cdots, S \qquad (6)
\end{aligned}
$$

where

$$
\tilde{H} := \begin{bmatrix} H & 0 \\ F & 0 \end{bmatrix}
$$

Intuitively, equation (5) shows that the j,k-th element of Π_s is the effect of $\xi_{k,t-s}$ (the k-th innovation at time $t-s$) on $y_{j,t}$ (the j-th endogenous variable at time t). Thus, given κ_{t-S}, $\Pi_{s,jk}$, which is defined as the j,k-th element of Π_s, is zero if $y_{j,t}$ is determined without observing $\xi_{k,s}$.

In the matrix representation, (6) becomes

$$\Gamma = M_{\Gamma\Pi}\Pi \tag{7}$$

where

$$\Gamma := \begin{bmatrix} \Gamma_0^T & \cdots & \Gamma_s^T & \cdots & \Gamma_S^T \end{bmatrix}^T \tag{8a}$$

$$\Pi := \begin{bmatrix} \Pi_0^T & \cdots & \Pi_s^T & \cdots & \Pi_S^T \end{bmatrix}^T \tag{8b}$$

$$M_{\Gamma\Pi} := \begin{bmatrix} I & & & 0 \\ -\tilde{H} & I & & 0 \\ & \ddots & \ddots & \\ 0 & & -\tilde{H} & I \end{bmatrix} \tag{8c}$$

$M_{\Gamma\Pi}$ is clearly invertible, and plays a key role in the following.

2.2.2 Zero Restrictions

Throughout this paper, we exploit the following two observations.

1. If the k-th set of variables $y_{k,t}$ does not observe the i-th set of time $t-s$ innovations $\xi_{i,t-s}$, then $\partial y_{k,t}/\partial\xi_{t-s} = \Pi_{s,ki} = 0$, given κ_{t-S} and $\xi_{t-\tau}$ for $\tau = s+1, \cdots$. Simply put, no decision can respond to unobserved innovations.

2. If the information set of the expectation operator in the j-th equation includes the i-th time $t-s$ innovation $\xi_{i,t-s}$, then the realization of the j-th equation must hold

for any realisation of the i-th innovation. The expectation error in each expectation operator occurs only due to innovations that are not included in its information sets. Thus, $E_{s,ji} = 0$.

For example, suppose that labour supply L_t (k-th variable, $y_{k,t}$) is decided on before observing today's technology shock (i-th shock, $\xi_{i,t}$), but after today's preference shock (l-th shock, $\xi_{l,t}$), both of which are *iid*. If the FOC with respect to L_t is the j-th equation,

$$\begin{aligned}\Pi_{0,ki} &= 0 \ (\xi_{i,t-0} \text{ does not affect } y_{k,t}) \\ E_{0,jl} &= 0 \ (\xi_{l,t-0} \text{ does not cause expectation error in } j\text{-th eqn})\end{aligned}$$

Roughly speaking, $E_{0,jl} = 0$ means that if the expectation operator of the j-th equation were eliminated from the j-th equation, it would still hold in terms of $\xi_{0,l}$. It is the duty of a user to specify the positions of these zero elements in Π and E (by providing $IndV$ and $IndE$).

2.3 Sketch of Derivation and Key Equations for Computation

The fully detailed derivation is provided in Appendix. This subsection briefly describes the skeleton of the derivation and lists the minimum results necessary for computation.

2.3.1 QZ Decomposition

In order to introduce notations, this subsection briefly reviews the QZ decomposition (or generalised Schur decomposition). For matrices A and B ($\in \mathbb{C}^{n\times n}$), there exist unitary matrices Q and Z such that

$$\begin{aligned}Q^H A Z &= \Omega_A \\ Q^H B Z &= \Omega_B\end{aligned}$$

where Ω_A and Ω_B are both upper triangular matrices, and superscript H indicates a conjugate transpose. Any unitary matrix U satisfies $U^H U = UU^H = I$. Let a_{kk} and b_{kk} be the k-th diagonal elements in Ω_A and Ω_B, respectively. Assuming that a_{kk} and b_{kk} are not zero at the same time, then $\lambda_k := b_{kk}/a_{kk}$ for $k = 1, \cdots, n$ are the generalised eigenvalues of the matrix pencil $B - \lambda_k A$.[8]

The basic idea is that, by applying the QZ decomposition to (3), the algorithm separates unstable roots u_t from stable roots s_t, as in Sims (2002).

$$\begin{aligned}
0 &= Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \\
&= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\
&= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\
&\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S}
\end{aligned}$$

where

$$\begin{pmatrix} s_t \\ u_t \end{pmatrix} := Z^H \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}$$

By using TVCs, the expected values of all unstable roots u_{t+1} are set to be equal to zero.[9]

[8] The generalised eigenvalues have the properties similar to forward operators F; $x_{t+1} = Fx_t$.

[9] Remember that all innovations are assumed to be *iid*. Note also that, if the expectations of u_{t+1} must be zero under perfect information, they must be also zero under imperfect information. This can be shown by simply applying the iterated linear projection. See Appendix for more deliberate discussion.

2.3.2 Notations for the Outputs of QZ Decomposition

For later use, we define submatrices as follows

$$Z^H := \begin{bmatrix} Z^H_{s.} \\ Z^H_{u.} \end{bmatrix} := \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix}, \; Z := \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix}, \; Q^H := \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} \tag{9a}$$

$$\Omega^A := \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix}, \; \Omega^B := \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \tag{9b}$$

where subscripts u and s imply unstable and stable roots, respectively. Note that Ω^A_{ss} and Ω^B_{uu} are both invertible by construction.

Additionally, we define four matrices as

$$\Lambda^A_{s\kappa} := \Omega^A_{ss} Z^H_{s\kappa} + \Omega^A_{su} Z^H_{u\kappa} \tag{10a}$$

$$\Lambda^A_{s\phi} := \Omega^A_{ss} Z^H_{s\phi} + \Omega^A_{su} Z^H_{u\phi} \tag{10b}$$

$$\Lambda^B_{s\kappa} := \Omega^B_{ss} Z^H_{s\kappa} + \Omega^B_{su} Z^H_{u\kappa} \tag{10c}$$

$$\Lambda^B_{s\phi} := \Omega^B_{ss} Z^H_{s\phi} + \Omega^B_{su} Z^H_{u\phi} \tag{10d}$$

Note that all the matrices defined by (10) are obtained from the outputs of the QZ decomposition.

2.3.3 Matrix Subscripts

We introduce the following notation rule for subscripts on matrices. For a matrix A,

- $A_{.x}$ is columns x of a matrix A,
- $A_{x.}$ is rows x of a matrix A,
- $A_{.\neg x}$ is the columns remaining after the elimination of columns x, and
- $A_{\neg x.}$ is the rows remaining after the elimination of rows x,

where x is the name of a set of columns or rows. This notation makes certain matrix operations extremely simple. See Appendix for further details.

2.3.4 Zero Restrictions

As a result of manipulating the matrix equations, it is shown that

$$0 = \Pi + M_{\Pi E}(E + \mathbf{C}) \tag{11}$$

$$M_{\Pi E} := (M_{y\Gamma} M_{\Gamma\Pi}) \backslash \mathbf{Q} \tag{12}$$

where

$$\Gamma := \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_{S-1} \end{pmatrix}, E := \begin{pmatrix} E_0 \\ \vdots \\ E_{S-1} \end{pmatrix}, \mathbf{C} := \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \mathbf{Q} := \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \tag{13a}$$

$$M_{y\Gamma} := \begin{bmatrix} \Phi & \Lambda^{0A} & & 0 \\ & \ddots & & \\ & & \Phi & \Lambda^{0A} \\ 0 & & & \Phi \end{bmatrix}, \quad \begin{array}{l} \Phi := \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \\ \Lambda^{0A} := \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \end{array} \tag{13b}$$

and $X \backslash Y = X^{-1}Y$. Our immediate objective is to find E and Π. Bear in mind that, while $M_{y\Gamma}$ is computable solely from the outputs of the QZ decomposition, we can obtain $M_{\Gamma\Pi}$ only after finding H and F (see equation (8c)).

Given $M_{\Gamma\Pi}$, E and Π are computed column by column (i.e., innovation by innovation) in (11). Because some elements in Π and E are zero due to the two zero restrictions, for the i-th column (or equivalently for the i-th innovation) of (11),

$$0 = \begin{pmatrix} \Pi_{1,i} \\ \vdots \\ \Pi_{k,i} \, (= 0) \\ \vdots \\ \Pi_{M(S+1),i} \end{pmatrix} + M_{\Pi E} \left(\begin{pmatrix} 0 \\ \vdots \\ E_{ji} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} C_{.i} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right) \tag{14}$$

where M in subscripts is the number of equations and hence $M\,(S+1)$ is the number of rows in Π.

From the k-th set of equations in (14)

$$0 = \Big[\, M_{\Pi E} \, \Big]_{kj} E_{ji} + \Big[\, M_{\Pi E} \, \Big]_{kj} \mathbf{C}_{ji} + \Big[\, M_{\Pi E} \, \Big]_{k\neg j} \mathbf{C}_{\neg ji} \tag{15}$$

which gives the values of the non-zero elements of E. From the remaining equations in (14),

$$\begin{aligned} 0 \;&=\; \Pi_{\neg ki} + \Big[\, M_{\Pi E} \, \Big]_{\neg kj} \mathbf{C}_{ji} + \Big[\, M_{\Pi E} \, \Big]_{\neg k\neg j} \mathbf{C}_{\neg ji} \\ &\quad - \Big[\, M_{\Pi E} \, \Big]_{\neg kj} \left(\Big[\, M_{\Pi E} \, \Big]_{kj} \setminus \Big[\, M_{\Pi E} \, \Big]_{k\neg j} \mathbf{C}_{\neg ji} + \mathbf{C}_{ji} \right) \end{aligned} \tag{16}$$

which gives the non-zero elements of Π.

Here we assume that $[M_{\Pi E}]_{kj}$ is invertible, which, however, is not necessarily true in general. The economic meaning of its invertibility is discussed in Section 3.

2.3.5 Solution

The solution algorithm computes key matrices sequentially. The basic structure is as follows:

1. Obtain submatrices form the outputs of the QZ decomposition (9) and (10).

2. Obtain H and F from (17).

3. Obtain $M_{y\Gamma}$, $M_{\Gamma\Pi}$ and $M_{\Pi E}$ from (13b), (8c) and (12), respectively.

4. Obtain E and Π from (18) and (19).

5. Obtain G and J from (20).

H **and** F**:** As in Sims (2002), it turns out that the H and F matrices are derived independently from the G and J matrices, based on the coefficient on κ_{t-S} in (5) (see the Appendix for details). Therefore, they are exactly the same as in perfect information models.

$$F = -Z_{u\phi}^{H} \backslash Z_{u\kappa}^{H} = Z_{\phi s}/Z_{\kappa s} \tag{17a}$$

$$H = -Z_{\kappa s}\left(\Omega_{ss}^{A} \backslash \Omega_{ss}^{B}\right)/Z_{\kappa s} \tag{17b}$$

E **and** Π**:** From (15) and (16), the non-zero elements of E and Π are

$$E_{ji} = -\left[\ M_{\Pi E}\ \right]_{kj} \backslash \left[\ M_{\Pi E}\ \right]_{k\neg j} \mathbf{C}_{\neg ji} - \mathbf{C}_{ji} \tag{18}$$

$$\Pi_{\neg ki} = -\left[\ M_{\Pi E}^{-1}\ \right]_{\neg j \neg k} \backslash \mathbf{C}_{\neg ji} \tag{19}$$

where $M_{\Pi E}$ can be obtained from (8c) and (13) with the solution of H and F. Note that H and F can be computed without referring E, Π or $M_{\Pi E}$. Since $[M_{\Pi E}]_{kj}$ is assumed to be invertible, $\left[M_{\Pi E}^{-1}\right]_{\neg j \neg k}$ is also invertible.

J **and** G**:** From the definition of Γ (8a),

$$\Gamma := \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi}\Pi \tag{20}$$

Note that, with H and F matrices, $M_{\Gamma\Pi}$ are recovered from (8c).

D**:** From a given economic model (3) it is obvious that

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix} \tag{21}$$

3 Assumptions

In this section, we discuss three assumptions. Assumptions 1 and 2 in the following are the same as in the solution method for perfect information models, while Assumption 3 is specific to imperfect information models. This subsection omits discussion about the Blanchard-Kahn condition, which is automatically satisfied by Assumption 1.

3.1 Assumption 1: $Z_{u\phi}^H$ is Invertible

Klein (2000) shows that this assumption is a generalisation of the condition derived in Blanchard and Kahn (1980). Boyd and Dotsey (1990) makes it clear that the Blanchard-Kahn condition, which counts and compares the numbers of unstable roots and jump variables, is a necessary but not sufficient condition for the existence of a unique solution; they provide a counter-example that satisfies the Blanchard-Kahn counting condition but does not have a stable solution. Intuitively, an invertible $Z_{u\phi}^H$ means that we can always find the values of jump variables such that the expectation of u_{t+1} is a zero vector in any states (TVCs). Heuristically, $Z_{u\phi}^H$ maps jump variables ϕ_t to unstable roots u_t, and its inverse maps u_t to ϕ_t. See King and Watson (1998) for an intuitive exposition.

The existence of the right inverse of $Z_{u\phi}^H$ entails the existence of jump variables, while the non-existence of its left inverse implies non-uniqueness of jump variables.[10] Note that typically non-uniqueness causes sunspot equilibria.

3.2 Assumption 2: a_{kk} and b_{kk} are Not Zero at the Same Time

If a_{kk} and b_{kk} are zero at the same time, it implies that there exist row vectors X such that $0 = X\xi$; indeed, X is (a scaler multiple of) the k-th row of Q (see also Sims (2002)). The existence of such row vectors generically implies either of the following:

(a) If $X\xi$ is indeed zero, then some equations are not linearly independent of the others. Essentially, there are fewer equations than endogenous variables. At least one equation can be expressed as a linear combination of others, and such a linear combination is X.

(b) If $X\xi$ is non-zero, clearly there is an internal contradiction. One such example is a two-equation, two-variable non-dynamic model with no state variables:

$$\begin{aligned} \phi_{1,t} &= \alpha\phi_{2,t} + \xi_t \\ \phi_{1,t} &= \alpha\phi_{2,t} + \xi_t + \eta_t \end{aligned}$$

Obviously, both do not hold at the same time for non-zero η_t. Since the QZ decomposition is merely a linear transformation, this implies that there is an internal inconsistency in the original system of equations (3).

[10] See Uhlig (2000) for a treatment of non-uniqueness.

3.3 Assumption 3: $[M_{\Pi E}]_{kj}$ is Invertible

This condition is specific to imperfect information models, though it is analogous to the equation (40) in Sims (2002).[11] Intuitively, if it is not invertible, then the information structure is not consistent. Note that the inverse of $[M_{\Pi E}]_{kj}$, if it exists, maps the j-th set of expectation errors to the k-th set of innovations to which some endogenous variables cannot respond. Hence, if the inverse of $[M_{\Pi E}]_{kj}$ exists, then expectation errors can equate both sides of the equations for any realisation of innovations.

A non-invertible $[M_{\Pi E}]_{kj}$ appears in the following example. Suppose that all production factors and all demand components are decided before observing today's technology shock. In this case, output varies depending on the realisation of technology, while demand cannot respond to it. Thus, the goods market does not clear at any price. One important lesson from this is that a researcher must construct consistent models; an arbitrarily specified information structure may have internal inconsistencies.

4 Properties of the Solution

By construction, of course, any solution generated by the algorithm satisfies the following two solution principles (two zero restrictions): that is, (i) if the k-th time t variable $y_{k,t}$ is determined without observing the i-th time $t-s$ innovations $\xi_{i,t-s}$, then $y_{k,t}$ cannot respond to $\xi_{i,t-s}$ (i.e., $\partial y_{k,t}/\partial \xi_{i,t-s} = 0$ given κ_{t-S}), and (ii) if the expectation operator in the j-th equation has an information set that includes $\xi_{i,t-s}$, $\xi_{i,t-s}$ cannot be the source of the expectation error in the j-th equation. In addition, as mentioned in Introduction,

[11] Note, however, that Sims' condition is related to time $t+1$ expectation errors, while our discussion in the following is related to time τ expectation errors ($\tau < t$).

invariant dynamic parts, H and F matrices, imply that imperfect information models inherit qualitative nature of the corresponding perfect information model: specifically, (a) dynamic stability property is not affected by information structure, and (b) the direct effect of imperfect information on IRFs lasts for only first S periods after an impulse, and then IRFs show sudden jumps.

The rest of this section briefly discusses other interesting features.

4.1 Inference

First, the maximum possible information set at time t (perfect information) is $\{\kappa_{t-j}, \xi_{t-j}\}_{j=0}^{\infty}$ (not includes $\{\phi_{t-j}\}_{j=0}^{\infty}$). Importantly, the algorithm does not allow inference. If the information set of economic agents in a model includes all current and past jump variables $\{\phi_{t-j}\}_{j=0}^{\infty}$, then the economic agents can infer most hidden information, which reduces an imperfect information model to the corresponding perfect information model in most cases. Hence, one natural interpretation of imperfect information is that agents have to make future decisions in the current period, as in sticky price models.

4.2 Noisy Information Models

Second, the algorithm can easily deal with noisy information models. Suppose an AR(1) shock process A_t follows

$$\ln A_{t+1} = \rho \ln A_t + \sqrt{1-\eta}\xi_t^{ob} + \sqrt{\eta}\xi_t^{uo} \qquad (22)$$

where ξ_t^{ob} and ξ_t^{uo} are the observable and unobservable components of innovation, respectively, and $(1-\eta)/\eta$ is the signal to noise ratio. This technique allows us to parameterise the extent of imperfect information.

5 An Example

5.1 Standard RBC Model

To demonstrate the quantitative effects of imperfect information, we consider the standard RBC model under imperfect information, focusing on impulse response functions (IRFs) and second moments.

The main economic motivation is to address an overly high $Corr\left(Y_t - H_t, Y_t\right)$ in the standard RBC model. Under the plausible parameter range, the standard RBC model predicts an almost perfect correlation between labour productivity $Y_t - H_t$ and output Y_t, but the correlation is only slightly positive in the data.

Hence, we modify the standard RBC model by adding imperfect information related to the labour market. The relevant equations are

$$0 = bH_t - W_t - \lambda_t \tag{23a}$$

$$0 = Y_t - H_t - W_t \tag{23b}$$

where Y_t, H_t, W_t, λ_t are output, working hours, wage and the marginal utility of consumption, respectively. All endogenous variables are measured as deviations from their steady-state values in percentage terms. b is a constant, which represents (a multiple of) the elasticity of marginal disutility of labour. The first equation is of the representative household (HH) − the FOC with respect to labour supply −, while the second is of firms − it equates the marginal product of labour $Y_t - H_t$ to wage.[12] The set of state variables

[12]Note that since all endogenous variables are represented as log-deviations from their steady state, $Y_t - H_t$ is the deviation of "output divided by labour hour" (i.e., labour productivity). The Cobb-Douglas production function implies that the marginal product of labour is $(1-\alpha)$ times labour productivity, which means that the percent change of labour productivity is exactly the same as that of the marginal

under perfect information is $\{K_t, A_t, \xi_t\}$, where K_t and A_t are capital and technology at the beginning of time t, respectively, and ξ_t represents the innovation on technology. Note that A_t is regarded as an endogenous crawling variable, and there is only one *iid* exogenous variable ξ_t. That is to say, A_t is treated as a stock variable.

Assuming that today's innovation affects today's output,

$$\begin{aligned} Y_t &= A_{t+1} K_t^{\alpha} H_t^{1-\alpha} \\ \ln A_{t+1} &= \rho \ln A_t + \xi_t \end{aligned}$$

where ρ is a parameter that governs the persistence of technology shock.

5.1.1 Case I: HH Decides Labour Supply before Observing Innovations

In this case, (23a) does not hold. Instead, the labour supply decision is governed by[13]

$$0 = E\left[\ bH_t - W_t - \lambda_t \mid K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}\right]$$

Since H_t cannot react to past innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

[Figure 1 here!]

Figure 1 shows the impulse response functions where $S = 5$, which means that the household decides its labour supply five quarters in advance.

There are several points worth noting here:

---

product of labour. In other words, in the Cobb-Douglas production function, $Y_t - H_t$ represents both the percent deviations of labour productivity and marginal product of labour.

[13]Exactly speaking, information set is $\{K_{t-j}, A_{t-j}, \xi_{t-j}\}_{j=S+1}^{\infty}$, but only $\{K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}\}$ suffices to determine the state of the economy.

- Labour hours do not move for the first S periods. That is, $\partial H_t / \partial \xi_{t-s} = 0$ for $s = 0, 1, \cdots, S$.

- Labour productivity $(Y_t - H_t)$ and investment show unusual movements for the first S periods. However, after $S+1$ periods, all endogenous variables follow (linear combinations of) AR(1) processes. This is one example of the proposition that *the direct effect of imperfect information lasts for only S periods after an impulse.*

- $Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information (around 0.91), but only slightly.

5.1.2 Case II: Firms Decide Labour Demand before Observing Innovations

In this case, (23b) does not hold. Instead, the labour demand decision is governed by

$$0 = E\left[\ Y_t - H_t - W_t \mid K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}\right]$$

Since H_t cannot react to the innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

The results are not very interesting in terms of economics.

- The IRFs are almost the same as in the Case I, except for wage (hence, the figure is omitted).

- $Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information, but only slightly.

However, this experiment demonstrates that, to find a solution, it is not enough to specify which endogenous variables are determined with imperfect information; a researcher must also specify which information sets are imperfect. This is evident in that the results of Cases I and II are not the same.

5.1.3 Case III: HH Decides Wage before Observing Innovations but Accommodates Labour Demand

This case can be regarded as a version of the sticky wage model. The representative household fixes wage before observing innovations, and it commits itself to supplying labour to accommodate labour demand.

In this case, (23a) does not hold. Instead, the labour supply decision is governed by

$$0 = E\left[\ bH_t - W_t - \lambda_t \mid K_{t-S-1}, A_{t-S-1}, \xi_{t-S-1}\right]$$

Since W_t cannot react to the innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial W_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

The results are interesting:

- The volatility of labour is much higher, and $Corr\left(Y_t - H_t, Y_t\right)$ is much lower than under perfect information.
- Given the standard deviation of the innovation, both output and labour are more volatile.
- The variance-covariances of most variables other than labour and labour productivity do not change significantly.

The intuition behind these results is quite simple. Without imperfect information, when there is a positive productivity innovation, wage increases, which discourages firms from hiring more labour. As a result, labour does not increase significantly. Indeed, another failure of the standard RBC model is that it predicts too low labour volatility relative to output volatility. During a boom both Y_t and H_t increase, while $Y_t - H_t$ increases because the increase in H_t is not large enough. Consequently, both Y_t and $Y_t - H_t$ increase in a boom, which is the (one possible) mechanism behind a high $Corr\left(Y_t - H_t, Y_t\right)$ in the standard RBC model.

However, if wage is determined without seeing a positive innovation, it does not change quickly; hence, firms are not discouraged from using more labour. Consequently, in a boom both Y_t and H_t increase, while $Y_t - H_t$ does not increase very much because the increase in H_t is large enough. Hence, the model predicts a low $Corr\left(Y_t - H_t, Y_t\right)$. Indeed, in the otherwise standard RBC model with one-period wage stickiness, the predicted relative volatility of labour almost matches the data. Under the standard parameter set, $Corr\left(Y_t - H_t, Y_t\right)$ is negative for $S \geq 2$.

[Table 1 here!]

Table 1 shows the summary table of the selected second moments for one-period wage stickiness ($S = 1$). One-period wage stickiness significantly improves the labour volatility and correlation between labour productivity and output, while it slightly deteriorates the model performance in terms of the relative volatility of investment.

[Figure 2 here!]

Figure 2 shows the comparison of selected impulse response functions between perfect and imperfect information models. The salient differences appear only in the first period. In the sticky wage model, both labour and output jump in the first period, and the size of the jumps are the same, hence, the labour productivity does not change in the first period. Note that the Cobb-Douglas production function implies that the labour productivity is always equal to wage.

[Figure 3 here!]

Figure 3 shows the relative volatilities and correlations for different degrees of imperfect information (i.e., for different values of S). As S increases, $Corr(Y_t - H_t, H_t)$ decreases.

Case III again reveals one computational requirement; simply specifying the information set in each equation is not enough to find a solution. A researcher must also specify which variables are determined without observing perfect information. This is evident in that the results of Case I and III are not the same.

5.1.4 Conclusion for RBC under Imperfect Information

Adding one-period wage stickiness is quantitatively enough to overcome the two drawbacks of the standard RBC model − where (a) labour volatility is too small and (b) the correlation between labour productivity and output is too high − without deteriorating other dimensions of the model performance. This example shows the possibility that the information structure has significant quantitative effects.

6 Conclusion

This article has developed an algorithm for linear rational models under imperfect information. Imperfect information is important because it includes many interesting classes of models such as sticky information and noisy signal models.

The algorithm exploits two observations: (1) if an endogenous variable $y_{k,t}$ is decided without observing an innovation $\xi_{i,t-s}$, then $y_{k,t}$ is not affected by $\xi_{i,t-s}$ (i.e., $\partial y_{k,t}/\partial \xi_{i,t-s} = 0$ given κ_{t-S}); (2) if the information set in the j-th equation includes $\xi_{i,t-s}$, then $\xi_{i,t-s}$ cannot be the source of expectation error in the j-th equation ($E_{s,ji} = 0$). The solution is defined by these two zero restrictions, and it turns out that they are enough to determine unique solutions.

The state space representation chosen in this algorithm is the set of crawling variables and current and past innovations. This representation reveals that the dynamic parts of the solution (i.e., the H and F matrices) are the same as under the corresponding perfect information models. Invariant H and F matrices imply that (a) the dynamic property, such as sunspot or saddle-path stability is not altered by information structure, and (b) impulse response functions are not (directly) affected by the information structure after the first S periods, where S is such that the minimum information set in a model has all the information up to time S. These findings show that *qualitatively* imperfect information models inherit the properties of their perfect information counterparts.

However, as the RBC example demonstrates, *quantitatively* imperfect information may be important. Hence, it is desirable to check for robustness in terms of the information structure, and our Matlab programme offers an easy way to conduct such experiments. Once structural equations are obtained, then the additional inputs to the algorithm are only two zero-one matrices.

Appendix

A Extension of Uhlig's Theorem 3

Proposition 1 *(Extension of Uhlig's Theorem 3) To find a $m \times m$ matrix X that solves the matrix polynomial*

$$\Theta_n X^n - \Theta_{n-1} X^{n-1} - \cdots - \Theta_1 X - \Theta_0 = 0 \tag{24}$$

Given $m \times m$ coefficient matrices $\{\Theta_{n'}\}_{n'=0}^{n}$, define the $nm \times nm$ matrices Ξ and Δ by

$$\Xi = \begin{bmatrix} \Theta_{n-1} & \cdots & \Theta_1 & \Theta_0 \\ I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I & 0 \end{bmatrix}, \ \Delta = \begin{bmatrix} \Theta_n & 0 & \cdots & 0 \\ 0 & I & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I \end{bmatrix}$$

and obtain the generalized eigenvalues λ and the generalized eigenvector s such that $\lambda \Delta s = \Xi s$. Then, s can be written as

$$s = \begin{pmatrix} \lambda^{n-1} x \\ \vdots \\ \lambda x \\ x \end{pmatrix}$$

for some $x \in \mathbb{R}^m$, and

$$X = \Omega \Lambda \Omega^{-1}$$

where $\Omega = [x_1, \cdots, x_m]$ and $\Lambda = diag(\lambda_1, \cdots, \lambda_m)$.

Proof. Almost identical to Uhlig (1999). ■

B Matrix Operations

To pick up and drop out columns and rows from a matrix, as in the main text, we define (i) $[A]_{.x}$ as columns x of a matrix A, (ii) $[A]_{x.}$ as rows x of a matrix A, (iii) $[A]_{.\neg x}$ as the columns remaining after the elimination of columns x, and (iv) $[A]_{\neg x.}$ as the rows remaining after the elimination of rows x, where x is the name of a set of columns or rows. The brackets are used simply because they often clarify notations, and often can be omitted (i.e., $[B]_{.\neg y} = B_{.\neg y}$). The dot . implies all rows or columns (e.g., $B_{..} = B$). It is quite easy to show the following formulae:

$$
\begin{aligned}
[AB] &= [A]_{.\neg x}[B]_{\neg x.} + [A]_{.x}[B]_{x.} \\
[AB]_{.\neg y} &= [A][B]_{.\neg y} \\
[AB]_{\neg x.} &= [A]_{\neg x.}[B] \\
[AB]_{\neg x \neg y} &= [A]_{\neg x.}[B]_{.\neg y}
\end{aligned}
$$

An example for the first formula is

$$
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}
\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
=
\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}
\begin{bmatrix} b_{11} & b_{12} \end{bmatrix}
+
\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}
\begin{bmatrix} b_{21} & b_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix}
+ \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}
= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}
$$

where $x = 2$.

Note that this notation is consistent with other matrix subscripts; for example, the rows of $Z_{s\kappa}$ are related to stable roots s and its columns are related to crawling variables κ.

C Invertible $Z_{u\phi}^H$ Implies Invertible $Z_{s\kappa}^H$

Proposition 2 *For an invertible matrix Z, which is partitioned as*

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

if Z_{11} is invertible, then $[Z^{-1}]_{22}$ is also invertible.

Proof. Define

$$Z_L := \begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix}$$

$$Z_R := \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix}$$

Note that $Z_L Z Z_R$ has full rank because all of Z_L, Z and Z_R have full rank, and note that

$$\begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \end{bmatrix}$$

Hence, $G := Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}$ must have full rank.

For a full rank matrix with an invertible upper left submatrix, the well-known formula tells us

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Z_{11}^{-1} + Z_{11}^{-1}Z_{12}G^{-1}Z_{21}Z_{11}^{-1} & -Z_{11}^{-1}Z_{12}G^{-1} \\ -G^{-1}Z_{21}Z_{11}^{-1} & G^{-1} \end{bmatrix}$$

Note that the RHS exists since we know that both Z_{11} and G are invertible. Thus, $[Z^{-1}]_{22}$ is invertible. ∎

Since Z is unitary, $Z^{-1} = Z^H$, which implies $G^{-1} = [Z^{-1}]_{22} = Z_{22}^H$. Since Z_{22}^H has full rank, its conjugate transpose Z_{22} $\left(= \left[Z_{22}^H\right]^H\right)$ also has full rank. This proposition is very useful; e.g., some final results in Klein (2000) can be significantly simplified.

D Full Derivation

This section provides the full derivation. For the notation, see the main text.

D.1 QZ Decomposition

Applying the QZ decomposition to (3)

$$
\begin{aligned}
0 &= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\
&= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\
&\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S}
\end{aligned} \tag{25}
$$

where s_t and u_t are stable and unstable roots, respectively, such that

$$
\begin{pmatrix} s_t \\ u_t \end{pmatrix} := \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}
$$

D.1.1 Unstable Roots and Transversality Conditions (TVCs)

Imperfect information requires a slightly careful treatment of TVCs. Focussing on the lower half of (25)

$$
0 = \Omega^A_{uu} u_{t+1} + \Omega^B_{uu} u_t + Q^H_{u.} C\xi_t + Q^H_{u.} D\xi_{t+1} + Q^H_{u.} E\xi^{t,S} \tag{26}
$$

Iterating it forward

$$
\begin{aligned}
&\lim_{l\to\infty} \left\{ \begin{array}{c} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l u_{t+l} \\ + \sum_{s=1}^{l-1} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^s \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) \left(C\xi_{t+s} + D\xi_{t+1+s} + E\tilde{\xi}^{t+s,S} \right) \end{array} \right\} \\
= \; & -u_t - \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) C\xi_t - \sum_{l=0}^{S} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) E\hat{\xi}^{t+l,S}
\end{aligned} \tag{27}
$$

where

$$\xi^{t+l,S} = \begin{pmatrix} \xi_{t+l} \\ \vdots \\ \xi_{t+1} \\ \xi_t \\ \vdots \\ \xi_{t+l-S} \end{pmatrix} = \hat{\xi}^{t+l,S} + \tilde{\xi}^{t+l,S} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_t \\ \vdots \\ \xi_{t+l-S} \end{pmatrix} + \begin{pmatrix} \xi_{t+l} \\ \vdots \\ \xi_{t+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $A\backslash B = A^{-1}B$ and $A/B = AB^{-1}$.

There are many information sets, under each of which TVCs must be satisfied. – that is, TVCs are (seemingly) tighter under imperfect information. However, if the perfect information counterpart satisfies TVCs, corresponding imperfect information models also satisfy them automatically due to the law of iterated linear projection.[14] Thus, the same logic as in the perfect information case holds; because $\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l} \to 0$ as $l \to 0$ by construction, the expected value of u_{t+l} explodes for any non-zero value of the RHS of (27), which contradicts the TVCs. Note that the inside the limit operator in the LHS shows the expected value of u_{t+l} (the realisation of u_{t+l} plus expectation errors) times $\left(-\Omega_{uu}^{B}\backslash\Omega_{uu}^{A}\right)^{l}$. Hence, the RHS of (27) must be zero.

[14]There are two comments. First, (27) must hold for *any* realisation of κ_{t-1} and ξ_{t-s} for $s = 0, 1, \cdots$. Hence, it is *not* possible that TVCs hold under imperfect information but not under perfect information. Second, if an information set does not include, for example, $\xi_{i,t-s}$ then the relevant expected value of u_{t+s} is the RHS with setting $\xi_{i,t-s} = 0$. Hence, if TVCs hold for the full information set, they hold for non-full information sets as well.

Therefore,

$$
\begin{aligned}
-\Omega^B_{uu} u_t &= -\Omega^B_{uu} Z^H_{u\kappa}\kappa_t - \Omega^B_{uu} Z^H_{u\phi}\phi_t \\
&= Q^H_{u.} C\xi_t + \Omega^B_{uu} \sum_{l=0}^{S} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) E\hat{\xi}^{t+l,S} \\
&= Q^H_{u.} C\xi_t + \sum_{l=0}^{S} \left(-\Omega^A_{uu} / \Omega^B_{uu} \right)^l Q^H_{u.} E\hat{\xi}^{t+l,S} \qquad (28)
\end{aligned}
$$

Substituting our "guess solution" (4) into (28),

$$
\begin{aligned}
0 &= \left(\Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F \right) \kappa_t + \Omega^B_{uu} Z^H_{u\phi} G\xi^{t,S} + Q^H_{u.} C\xi_t \\
&\quad + \sum_{l=0}^{S} \left(-\Omega^A_{uu} / \Omega^B_{uu} \right)^l Q^H_{u.} E\hat{\xi}^{t+l,S} \qquad (29)
\end{aligned}
$$

D.1.2 Stable Roots

Similarly, from the upper half,

$$
\begin{aligned}
0 &= \Omega^A_{ss} \left(Z^H_{s\kappa}\kappa_{t+1} + Z^H_{s\phi}\phi_{t+1} \right) + \Omega^A_{su} \left(Z^H_{u\kappa}\kappa_{t+1} + Z^H_{u\phi}\phi_{t+1} \right) \\
&\quad + \Omega^B_{ss} \left(Z^H_{s\kappa}\kappa_t + Z^H_{s\phi}\phi_t \right) + \Omega^B_{su} \left(Z^H_{u\kappa}\kappa_t + Z^H_{u\phi}\phi_t \right) \\
&\quad + Q^H_{s.} C\xi_t + Q^H_{s.} D\xi_{t+1} + Q^H_{s.} E\xi^{t,S} \qquad (30)
\end{aligned}
$$

Again, by substituting (4) into (30), after some manipulation,

$$
\begin{aligned}
0 &= \left(\Lambda^A_{s\phi} FH + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} \right) \kappa_t \\
&\quad + \Lambda^A_{s\phi} G\xi^{t+1,S} + Q^H_{s.} D\xi_{t+1} + Q^H_{s.} C\xi_t \\
&\quad + \left(\Lambda^A_{s\phi} FJ + \Lambda^A_{s\kappa} J + \Lambda^B_{s\phi} G + Q^H_{s.} E \right) \xi^{t,S} \qquad (31)
\end{aligned}
$$

Though the definitions of $\Lambda^A_{s\kappa}$, $\Lambda^A_{s\phi}$, $\Lambda^B_{s\kappa}$ and $\Lambda^B_{s\phi}$ are (10) in the main text, the following definition may be more useful.

$$
\begin{bmatrix} \Lambda^A_{s\kappa} & \Lambda^A_{s\phi} \\ \Lambda^B_{s\kappa} & \Lambda^B_{s\phi} \end{bmatrix} := \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ \Omega^B_{ss} & \Omega^B_{su} \end{bmatrix} \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \qquad (32)
$$

D.2 Expansion of $\xi^{t+1,S}$ and $\xi^{t,S}$

Expanding $\xi^{t+1,S}$ and $\xi^{t,S}$ in (31) and (29),

$$
\begin{aligned}
0 \;=\; & \left(\Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} \right) \kappa_t \\
& + \left(\Lambda^A_{s\phi} G_0 + Q^H_{s.} D \right) \xi_{t+1} \\
& + \left(\Lambda^A_{s\phi} G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_0 + \Lambda^B_{s\phi} G_{0.} + Q^H_{s.} E_{0.} + Q^H_{s.} C \right) \xi_t \\
& + \left(\Lambda^A_{s\phi} G_2 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_1 + \Lambda^B_{s\phi} G_{1.} + Q^H_{s.} E_{1.} \right) \xi_{t-1} + \cdots \\
& + \left(\Lambda^A_{s\phi} G_S + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_{S-1} + \Lambda^B_{s\phi} G_{S-1.} + Q^H_{s.} E_{S-1.} \right) \xi_{t-(S-1)} \\
& + \left(\left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_{S.} + Q^H_{s.} E_{S.} \right) \xi_{t-S}
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(\Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F \right) \kappa_t \\
& + \sum_{s=1}^{S} \left(\Omega^B_{uu} Z^H_{u\phi} G_s + \left(\textstyle\sum_{k=0}^{S-s} \left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^k Q^H_{u.} E_{k+s} \right) \right) \xi_{t-s} \\
& + \left(Q^H_{u.} C + \Omega^B_{uu} Z^H_{u\phi} G_0 + \left(\textstyle\sum_{k=0}^{S} \left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^k Q^H_{u.} E_k \right) \right) \xi_t
\end{aligned}
$$

Since these matrix equations must hold for any realisation of κ_t, $\xi_{t-\tau}$ for $\tau = -1, 0, 1, \cdots, S$,

$$0 \;=\; \Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} \tag{33a}$$

$$0 \;=\; \Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F \tag{33b}$$

$$0 \;=\; \Lambda^A_{s\phi} G_{0.} + Q^H_{s.} D \tag{34a}$$

$$0 \;=\; 0 \tag{34b}$$

$$0 \;=\; \Lambda^A_{s\phi} G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_0 + \Lambda^B_{s\phi} G_0 + Q^H_{s.} E_{S.} + Q^H_{s.} C \tag{35a}$$

$$0 \;=\; \Omega^B_{uu} Z^H_{u\phi} G_0 + \left(\sum_{s=0}^{S} \left(-\Omega^A_{uu}/\Omega^B_{uu}\right)^s Q^H_{u.} E_s \right) + Q^H_{u.} C \tag{35b}$$

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} G_{s+1} + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_s + \Lambda^B_{s\phi} G_s + Q^H_{s.} E_s && \text{(36a)}\\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_s + \left(\sum_{k=0}^{S-s} \left(-\Omega^A_{uu}/\Omega^B_{uu}\right)^k Q^H_{u.} E_{k+s} \right) && \text{(36b)}\\
& \text{for } s = 1, \cdots, S-1
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_S + Q^H_{s.} E_S && \text{(37a)}\\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_S + Q^H_{u.} E_S && \text{(37b)}
\end{aligned}
$$

D.3 Dynamic Parts (H and F)

Since (33a) and (33b) do not include G, J, D, E or Π, these two matrix equations can be solved for H and F independently. Thus, assuming $Z^H_{u\phi}$ has a (right) inverse,[15]

$$
\begin{aligned}
F &= -Z^H_{u\phi} \backslash Z^H_{u\kappa} = Z_{\phi s}/Z_{\kappa s}\\
H &= -Z_{\kappa s} \left(\Omega^A_{ss} \backslash \Omega^B_{ss}\right) / Z_{\kappa s}
\end{aligned}
$$

Note that the H and F matrices are the same as in the corresponding perfect information model.[16]

[15] Remember that an invertible $Z^H_{u\phi}$ implies an invertible $Z^H_{s\kappa}$.

[16] For the F matrix, note

$$
Z^H Z = \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix} = \begin{bmatrix} Z^H_{s\kappa} Z_{\kappa s} + Z^H_{s\phi} Z_{\phi s} & Z^H_{s\kappa} Z_{\kappa u} + Z^H_{s\phi} Z_{\phi u} \\ Z^H_{u\kappa} Z_{\kappa s} + Z^H_{u\phi} Z_{\phi s} & Z^H_{u\kappa} Z_{\kappa u} + Z^H_{u\phi} Z_{\phi u} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
$$

Looking at the lower left element

$$
\begin{aligned}
Z^H_{u\kappa} Z_{\kappa s} + Z^H_{u\phi} Z_{\phi s} &= 0\\
-Z^H_{u\kappa} Z_{\kappa s} &= Z^H_{u\phi} Z_{\phi s}\\
-Z^H_{u\phi} \backslash Z^H_{u\kappa} &= Z_{\phi s}/Z_{\kappa s}
\end{aligned}
$$

Also, remember that

$$
Z^{-1}_{\kappa s} = Z^H_{s\kappa} - Z^H_{s\phi} \left(Z^H_{u\phi} \backslash Z^H_{u\kappa}\right)
$$

D.4 Zero Restrictions on E and Π

Vertically concatenating matrix equations (35a)-(37b) in pairs,

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + \sum_{k=0}^{S} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H \left(E_k + C \right) & \text{(38a)} \\
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + \sum_{k=0}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} & \text{(38b)} \\
&\quad \text{for } s = 1, \cdots, S-1 \\
0 &= \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S & \text{(38c)}
\end{aligned}
$$

Note that

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix} \left(\begin{array}{c} \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+2} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} \\ + \sum_{k=0}^{S-(s+1)} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s+1} \end{array} \right) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \sum_{k=1}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} & \text{(39)}
\end{aligned}
$$

---

and that Ω^A_{ss} is invertible by the reordering of QZ decomposition.

Subtracting (39) from each of (38),[17]

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + Q^H E_k + Q^H C && \text{(40a)} \\
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + Q^H E_{k+s} && \text{(40b)} \\
& \text{for } s = 1, \cdots, S-1 \\
0 &= \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S && \text{(40c)}
\end{aligned}
$$

and again vertically concatenating these equations,

$$
\begin{aligned}
0 &= M_{y\Gamma} \Gamma + \mathbf{Q} \left(E + \mathbf{C} \right) \\
\Gamma &:= \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_S \end{pmatrix}, \; E := \begin{pmatrix} E_0 \\ \vdots \\ E_S \end{pmatrix}, \; \mathbf{C} := \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \; \mathbf{Q} := \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \\
M_{y\Gamma} &:= \begin{bmatrix} \Phi & \Lambda^{0A} & & \\ & \ddots & \ddots & \\ 0 & & \Phi & \Lambda^{0A} \\ & & & \Phi \end{bmatrix}, \; \Phi := \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \; \Lambda^{0A} := \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix}
\end{aligned}
$$

Note that since Φ is invertible, $M_{y\Gamma}$ is also clearly invertible. Hence,

$$
\begin{aligned}
0 &= \Gamma + M_{y\Gamma} \backslash \mathbf{Q} \left(E + \mathbf{C} \right) \\
&= M_{\Gamma\Pi} \Pi + M_{y\Gamma} \backslash \mathbf{Q} \left(E + \mathbf{C} \right)
\end{aligned}
$$

where (7) is used to derive the second line. Hence,

$$
\begin{aligned}
0 &= \Pi + M_{\Pi E} \left(E + \mathbf{C} \right) && \text{(41a)} \\
M_{\Pi E} &:= \left(M_{y\Gamma} M_{\Gamma\Pi} \right) \backslash \mathbf{Q} && \text{(41b)}
\end{aligned}
$$

[17] Though this process is not necessary, it reduces the computational burden.

In the following, we compute E and Π *column by column.*

$$\Pi_{.i} = M_{\Pi E}\left(E_{.i} + \mathbf{C}_{.i}\right)$$

Remember that some elements in $\Pi_{.i}$ are zero due to imperfect information, while some elements in $E_{.i}$ are non-zero. For example,

$$0 = \begin{pmatrix} \Pi_{1,i} \\ \vdots \\ \Pi_{k,i}\,(=0) \\ \vdots \\ \Pi_{M(S+1),i} \end{pmatrix} + M_{\Pi E}\left(\begin{pmatrix} 0 \\ \vdots \\ E_{ji} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} C_{.i} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}\right) \tag{42}$$

D.4.1 E Matrix

From the k-th set of equations in (42)

$$0 = \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{kj} E_{ji} + \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{kj} \mathbf{C}_{ji} + \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{k\neg j} \mathbf{C}_{\neg ji}$$

Hence, assuming $[M_{\Pi E}]_{kj}$ is invertible,

$$E_{ji} = -\left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{kj} \backslash \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{k\neg j} \mathbf{C}_{\neg ji} - \mathbf{C}_{ji}$$

D.4.2 Π matrix

From the other equations in (42), we eliminate the expectation errors E_{ji}.

$$\begin{aligned}
\Pi_{\neg ki} &= \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{\neg kj} \left(\left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{kj} \backslash \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{k\neg j} \mathbf{C}_{\neg ji} + \mathbf{C}_{ji}\right) \\
&- \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{\neg kj} \mathbf{C}_{ji} - \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{\neg k\neg j} \mathbf{C}_{\neg ji} \\
&= \left(\left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{\neg kj} \left(\left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{kj} \backslash \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{k\neg j}\right) - \left[\begin{array}{c} M_{\Pi E} \end{array}\right]_{\neg k\neg j}\right) \mathbf{C}_{\neg ji} \\
&= -\left[\begin{array}{c} M_{\Pi E}^{-1} \end{array}\right]_{\neg j\neg k} \backslash \mathbf{C}_{\neg ji}
\end{aligned}$$

The vector $\Pi_{\neg ki}$ and $\Pi_{ki} = 0$ can be vertically merged to recover $\Pi_{.i}$, and the vectors $\Pi_{.i}$ are horizontally concatenated to recover full Π matrix. Note that an invertible $[M_{\Pi E}]_{kj}$ implies an invertible $\left[M_{\Pi E}^{-1}\right]_{\neg j \neg k}$. Not surprisingly, $\mathbf{C}_{ji}$ does not affect the coefficient matrix $\Pi_{.i}$, because the j-th set of equations does not hold for the i-th innovation in any case; it only affects the expectation error E_{ji}.

D.5 Other Matrices (J, G and D)

D.5.1 J and G Matrices

To obtain the J and G matrices, from (7),

$$\Gamma := \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi}\Pi$$

D.5.2 D Matrix

From the A matrix in a given model (3),

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix}$$

which always satisfies (34a). It can be shown that the j-th rows in D are zeros if the j-th equation does not include $t + 1$ dynamic jump variable (see the next section).

E A Comment on the D Matrix

The direct derivation of the D matrix from (34a) is a bit tricky, and requires careful attention concerning non-square matrices $\Lambda^A_{s\phi}$ and $Q^H_{s.}$. Also, it is perhaps not intuitive. In this article, we exploit an *ex post* relationship (21), and here we show that it always satisfies (34a), which, in turn, reveals an important intuition.

First, we define dynamic and non-dynamic jump variables: $\phi_{t+1} = [(\phi^d_{t+1})^T \ (\phi^n_{t+1})^T]^T$. Note that the coefficients on the non-dynamic jump variables ϕ^n_{t+1} in A matrix must be zero by the definition of "non-dynamic".

$$Ay_{t+1} := \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{pmatrix} \kappa_{t+1} \\ \phi^d_{t+1} \\ \phi^n_{t+1} \end{pmatrix}$$

where ϕ^d_{t+1} is the vector of dynamic variables, such as consumption in the Euler equation. The submatrices in G_0 and Q^H are defined as

$$\tilde{G}_0 := \begin{bmatrix} 0 \\ G_0 \end{bmatrix} := \begin{bmatrix} 0 \\ G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix}$$

$$Q^H := \begin{bmatrix} Q^H_{s.} \\ Q^H_{\phi.} \end{bmatrix}, \ Q^H_{s.} := \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix}, \ Q^H_{u.} := \begin{bmatrix} Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix}$$

where u^f and u^i imply finite and infinite unstable roots, respectively.

Focussing on the second term of (34a)

$$\begin{aligned} Q^H_{s.}D &= Q^H_{s.}A\tilde{G}_0 = \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix} \\ &= \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) G_{0,\phi^d.} \end{aligned} \tag{43}$$

For the first term of (34a) note that $\Lambda^A_{s\phi}$ is the $s\phi$-th elements in $\Omega^A Z^H$, i.e.,

$$\begin{aligned}
\Lambda^A_{s\phi} &= \left[\Omega^A Z^H\right]_{s\phi} = \left[QQ^H\Omega^A Z^H\right]_{s\phi} = [QA]_{s\phi} \\
&= \left[\begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \\ Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix}\begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix}\right]_{s\phi} \\
&= \left[\begin{bmatrix} * & \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d}\right) & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix}\right]_{s\phi} \\
&= \begin{bmatrix} \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d}\right) & 0 \end{bmatrix}
\end{aligned}$$

where $*$ elements are irrelevant for our current interest. Hence,

$$\begin{aligned}
\Lambda^A_{s\phi}G_0 &= \begin{bmatrix} \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d}\right) & 0 \end{bmatrix}\begin{bmatrix} G_{0,\phi^d\cdot} \\ G_{0,\phi^n\cdot} \end{bmatrix} \\
&= \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d}\right)G_{0,\phi^d\cdot} \qquad (44)
\end{aligned}$$

(43) and (44) show that (34a) satisfies (21). The key to the solution is a sort of zero restriction; A matrix has zero columns by the definition of "non-dynamic" variables.

A further question is the consistency of D (i.e. whether the computed D always has zeros at the proper positions?). Specifically, if the j-th equation does not have ϕ^d_{t+1}, it should not have an expectation error due to ξ_{t+1}, and hence the row vector $D_{j\cdot}$ must be zero; this zero restriction on D is analogous to that on E. This is surely satisfied because the rows corresponding to non-dynamic equations in D $(= A\tilde{G}_0)$ is always zero by the construction of A; i.e., the j-th row in A is zero if the j-th equation does not include dynamic jump variables ϕ^d_{t+1}. For example, in the standard RBC model, all but the Euler equation have zero rows in A and hence in D.

What this section discusses is the correspondence between expectation errors and the source of such errors. If, for example, expectation errors with respect to full information up to time t appears in the equations without dynamic jump variables, then it is a logical contradiction (expectation errors without their causes), and hence (34a) is not satisfied. Conceptually, the consistency of the D matrix is parallel to the invertibility of $[M_{\Pi E}]_{kj}$. As mentioned in the main text, the non-invertibility of $[M_{\Pi E}]_{kj}$ implies an incorrect specification of the information structure with respect to $\xi_{t+\tau}$ ($\tau = 0, 1, \cdots, S$). Similarly, an inconsistent D (or the non-existence of a consistent D) implies an incorrect specification of information structure with respect to ξ_{t+1}. Such inconsistency/non-existence happens, for example, if a researcher puts an expectation operator on the evolution of capital, rather than on the consumption Euler equation.

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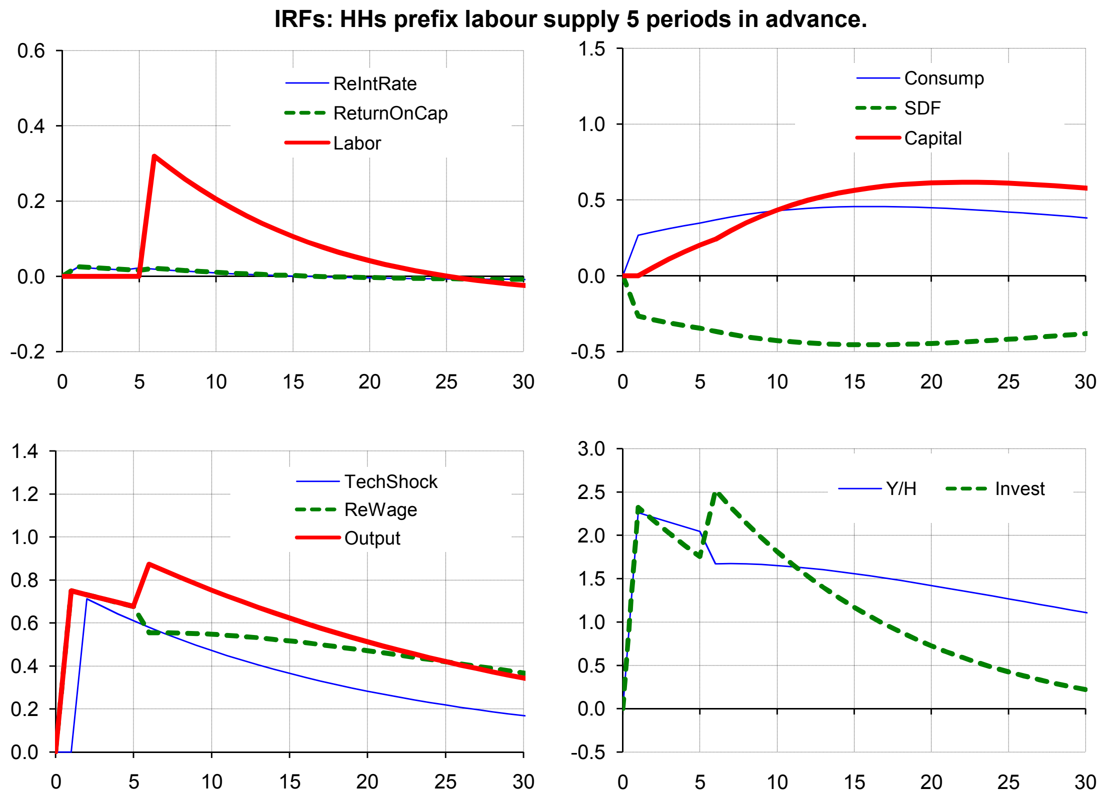

Figure 1: Impulse response functions to a positive technology innovation of the standard RBC model, in which labour supply is determined five periods in advance.

Table 1: Comparison between perfect and imperfect information RBC models.

	Output	Hours	Consumption	Investment	Corr(Output,Outpu/Hours)
			Data		
s.d.	1.72	1.59	0.86	8.24	0.41
relative	1.00	0.92	0.50	4.79	
			Standard RBC		
s.d.	1.35	0.47	0.33	5.95	0.98
relative	1.00	0.35	0.24	4.41	
			Imperfect information (RBC with Prefixed Wage)		
s.d.	2.15	2.10	0.53	7.92	0.25
relative	1.00	0.98	0.25	3.69	

Note: Figures of "Data" and "Standard RBC" are cited from Cooley and Prescott (1995).

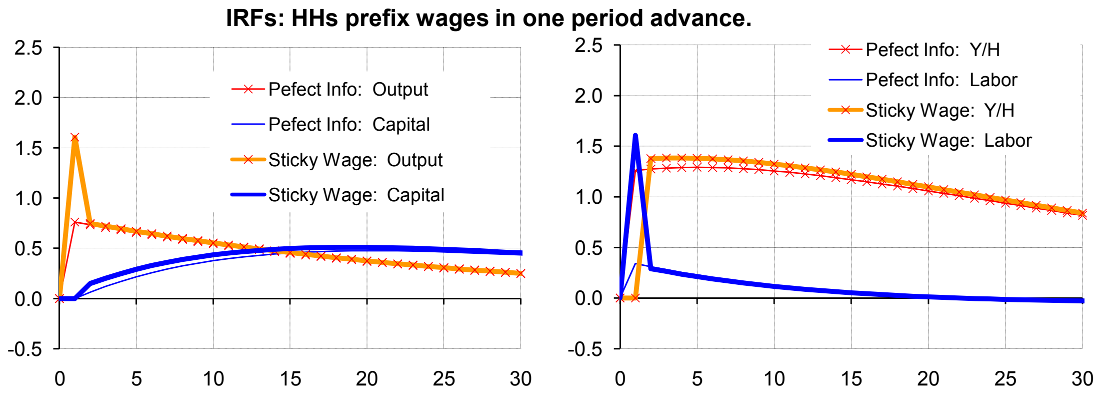

Figure 2: Comparison of selected impulse response functions to a positive technology innovation between standard RBC and RBC with wage stickiness.

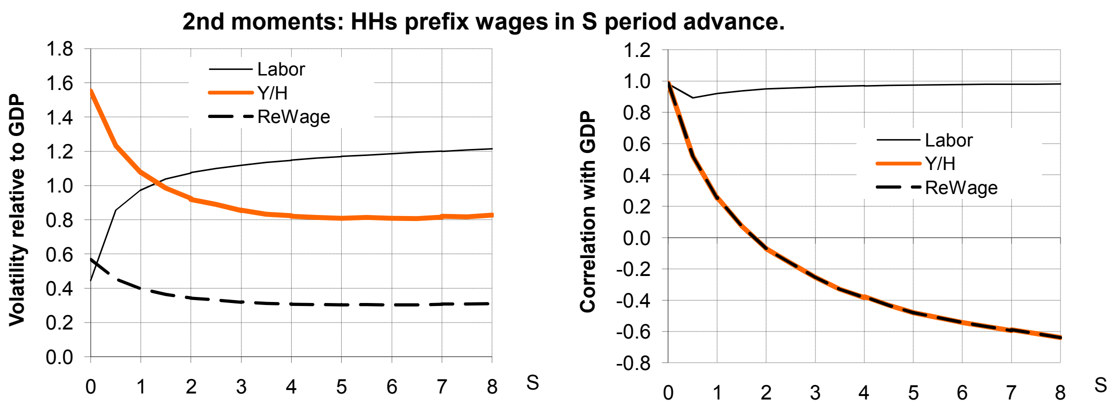

Figure 3: Effect of different degrees of imperfect information on selected second moments.