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ON THE NUMBER OF PAIRWISE TOUCHING SIMPLICES

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ABSTRACT. In this note it is shown that the maximum number of pairwise touching translates of an n-simplex is at least $n+3$ for $n=7$, and for all $n \geq 5$ such that $n \equiv 1 \bmod 4$. The current best known lower bound for general n is $n+2$. For $n = 2^k - 1$ and $k \geq 2$, we will also present an alternative construction to give $n+2$ touching simplices using Hadamard matrices.

1. INTRODUCTION

A classic problem in discrete geometry is to determine for a given convex body K in $\mathbb{R}^n$ the maximum number of pairwise touching translates of K. This number is called the *touching number of* K and is denoted by $t(K)$. It is well-known that for any convex body K in $\mathbb{R}^n$,

$$t(K) \leq 2^n,$$

and equality holds if, and only if, K is a parallelotope, see [3, 9, 10]. On the other hand, it is unknown if for each convex body K in $\mathbb{R}^n$ the inequality $t(K) \geq n+1$ holds when $n \geq 4$, see [4, Section 2.3].

This paper concerns the touching number of n-dimensional simplices, Δ_n. This number was studied by Koolen, Laurent and Schrijver in [7]. They showed, among other things, that $t(\Delta_n) \geq n+2$ for all $n \geq 3$ and $t(\Delta_3) = 5$, see Figure 1. In [8] the first author gave examples that showed that $t(\Delta_4) \geq 7$ and $t(\Delta_5) \geq 9$.

The main goal of this short note is to present a construction that gives the following small improvement of the lower bound for $t(\Delta_n)$.

Theorem 1.1. *For $n = 7$ and $n \equiv 1 \bmod 4$, with $n \geq 5$, we have that*

$$t(\Delta_n) \geq n+3.$$

The problem of determining $t(\Delta_n)$ is known [7, 8] to be equivalent to finding the maximum size of ℓ_1-norm equilateral sets in a hyperplane. We will discuss the equivalence between these two problems in the next section.

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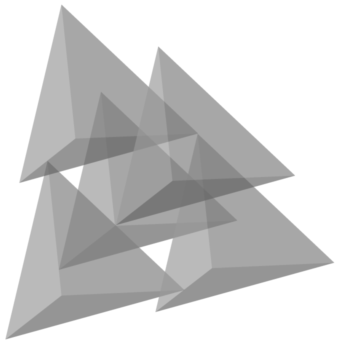

Figure 1. Five pairwise touching tetrahedra

2. Equilateral sets

A convex body K in $\mathbb{R}^n$ which is centrally symmetric, i.e., $x \in K$ if and only if $-x \in K$, is the unit ball of a norm $\|\cdot\|_K$ on $\mathbb{R}^n$. Indeed, for $x \in \mathbb{R}^n$ we can define the norm by

$$\|x\|_K = \inf\{\lambda > 0 \colon x \in \lambda K\}.$$

A set S in a normed space $(\mathbb{R}^n, \|\cdot\|)$ is called an *equilateral set* if there exists a constant $\delta > 0$ such that

$$\|s - t\| = \delta \quad \text{for all } s \neq t \text{ in } S.$$

The maximum size of an equilateral set in $(\mathbb{R}^n, \|\cdot\|)$ is the *equilateral dimension of* $(\mathbb{R}^n, \|\cdot\|)$, and is denoted by $e(\mathbb{R}^n, \|\cdot\|)$. Note that the constant $\delta > 0$ does not play a role, as we can always scale the equilateral set. Clearly, if K is a centrally symmetric body in $\mathbb{R}^n$, then $S = \{s_1, \ldots, s_p\}$ is an equilateral set in $(\mathbb{R}^n, \|\cdot\|_K)$ with pairwise distance 2 if, and only if, the set of unit balls with centers $s_1, \ldots, s_p$ is a configuration of p pairwise touching translates of K.

The equilateral dimension has been studied for many normed spaces, see for example [1, 11, 12]. Particular attention has been given to so called ℓ_p-norms which are defined as follows. For $1 \leq p < \infty$, the *ℓ_p-norm* on $\mathbb{R}^n$ is given by $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$. For the ℓ_1-norm it has been conjectured by Kusner [6] that $e(\mathbb{R}^n, \|\cdot\|_1) = 2n$, but at present this has only been confirmed for $1 \leq n \leq 4$, see [2, 7]. Obviously, $2n$ is a lower bound for $e(\mathbb{R}^n, \|\cdot\|_1)$, as the set of standard basis vectors and their opposites form an equilateral set. The best known upper bound is $Cn \log n$, where $C > 0$ is a constant, which was obtained using probabilistic methods by Alon and Pudlak [1].

The touching number for the n-dimensional simplex turns out to be equivalent to determining the maximum size of an ℓ_1-norm equilateral set contained in a hyperplane. More precisely, if $h(n)$ is the maximum size of an

ℓ_1-norm equilateral set in $H_\alpha = \{x \in \mathbb{R}^n \colon \sum_i x_i = \alpha\}$ for some $\alpha \in \mathbb{R}$, then

$$t(\Delta_n) = h(n+1) \quad \text{for all } n \geq 1, \tag{2.1}$$

see [7, 8]. For example, the ℓ_1-norm equilateral set

$$S = \{(2,0,1,1),(0,2,1,1),(1,1,2,0),(1,1,0,2),(2,2,0,0)\} \tag{2.2}$$

in the hyperplane $H_4 \subseteq \mathbb{R}^4$ corresponds to the configuration of 5 pairwise touching translates of a tetrahedron depicted in Figure 1. The examples of equilateral sets in Table 1 were found with the aid of a computer. In particular, we see that $t(\Delta_7) \geq 10$, which settles the $n = 7$ case in Theorem 1.1.

Table 1. Equilateral sets

$n=5$	$n=6$	$n=8$
$(4,0,1,1,2)$	$(4,0,1,1,1,1)$	$(0,4,2,2,0,4,2,2)$
$(0,4,1,1,2)$	$(0,4,1,1,1,1)$	$(4,0,2,2,4,0,2,2)$
$(1,1,4,0,2)$	$(1,1,4,0,1,1)$	$(2,2,0,4,2,2,0,4)$
$(1,1,0,4,2)$	$(1,1,0,4,1,1)$	$(2,2,4,0,2,2,4,0)$
$(2,2,0,0,4)$	$(1,1,1,1,4,0)$	$(8,2,1,1,0,2,1,1)$
$(0,0,2,2,4)$	$(1,1,1,1,0,4)$	$(4,4,4,4,0,0,0,0)$
$(2,2,2,2,0)$	$(2,2,2,2,0,0)$	$(4,4,0,0,4,4,0,0)$
	$(2,2,0,0,2,2)$	$(4,4,0,0,0,0,4,4)$
	$(0,0,2,2,2,2)$	$(4,0,4,0,0,4,0,4)$
		$(4,0,0,4,0,4,4,0)$

It is interesting to note that in these examples all the nonzero coordinates are powers of 2. We have looked into those type of examples in more detail, which let to the construction in Proposition 4.1. At present, however, we have no clear understanding of why these coordinate values generate large examples.

Before we prove Theorem 1.1, we mention that the inequalities

$$h(n) \leq e(\mathbb{R}^n, \|\cdot\|_1) \leq h(2n-1)$$

are known [7, 8] to hold for all $n \geq 1$, Thus, $e(\mathbb{R}^n, \|\cdot\|_1)$ grows linearly in n if, and only if, $h(n)$ does.

3. Proof of Theorem 1.1

For each $n \equiv 2 \bmod 4$ with $n \geq 6$ we shall construct an ℓ_1-norm equilateral set in $H_\alpha = \{x \in \mathbb{R}^n \colon \sum_i x_i = \alpha\}$ of size $n+2$, where $\alpha = (n-2)^2/2$. The result then follows from equation (2.1). So let $n \equiv 2 \bmod 4$ with $n \geq 6$.

Define

$$\begin{aligned} v^1 &= (b,0,a,a,\dots,a,a),\\ v^2 &= (0,b,a,a,\dots,a,a),\\ v^3 &= (a,a,b,0,\dots,a,a),\\ v^4 &= (a,a,0,b,\dots,a,a),\\ &\vdots\\ v^{n-1} &= (a,a,a,a,\dots,b,0),\\ v^n &= (a,a,a,a,\dots,0,b), \end{aligned}$$

in $\mathbb{R}^n$, where $a=(n-4)/2$ and $b=n-2$. Furthermore let

$$v^{n+1} = (\overbrace{y,y,\dots,y}^{k},\overbrace{z,z,\dots,z}^{n-k}) \quad \text{and} \quad v^{n+2} = (\overbrace{z,z,\dots,z}^{n-k},\overbrace{y,y,\dots,y}^{k})$$

in $\mathbb{R}^n$. We now show that if we take

$$k=(n-2)/2, \quad y=(n-6)/2, \quad \text{and} \quad z=(n-2)/2,$$

then $V=\{v^1,\dots,v^{n+2}\}$ is an ℓ_1-norm equilateral set in H_α, where $\alpha=(n-2)^2/2$ and the distance is $2(n-2)$.

To verify this we note first that $b\geq z\geq a\geq y\geq 0$. For $i=1,\dots,n$ the coefficient sum of v^i is given by

$$b+(n-2)a=(n-2)+(n-2)(n-4)/2=(n-2)^2/2.$$

Similarly the coefficient sum for the vectors v^{n+1} and v^{n+2} is equal to

$$(n-k)z+ky=(n+2)(n-2)/4+(n-2)(n-6)/4=(n-2)^2/2.$$

Let $1\leq i\neq j\leq n$. For $i=2k-1$ and $j=2k$, the distance between v^i and v^j is given by

$$\|v^i-v^j\|_1=|b-0|+|0-b|=2(n-2),$$

and for all other $i\neq j$,

$$\|v^i-v^j\|_1=|b-a|+|0-a|+|a-b|+|a-0|=2(b-a)+2a=2(n-2).$$

Also

$$\|v^{n+1}-v^{n+2}\|_1=k|z-y|+k|y-z|=(n-2)\left(\tfrac{n-2}{2}-\tfrac{n-6}{2}\right)=2(n-2).$$

Finally the distance between any of the first n vectors and the last two is calculated as in either the case of v^1 and v^{n+1},

$$\begin{aligned} \|v^1-v^{n+1}\|_1 &= |b-y|+|0-y|+(k-2)|a-y|+(n-k)|a-z|\\ &=(n-2)+(n-6)/2+(n+2)/2\\ &=2(n-2), \end{aligned}$$

or, as in the case of v^1 and v^{n+2},

$$\begin{aligned}\|v^1 - v^{n+2}\|_1 &= |b-z| + |0-z| + (n-k-2)|a-z| + k|a-y| \\ &= (n-2) + (n-2)/2 + (n-2)/2 \\ &= 2(n-2).\end{aligned}$$

Thus, V is an ℓ_1-norm equilateral set in H_α of size $n+2$. Table 2 shows examples in dimensions $n = 6$, 10 and 14.

TABLE 2. Equilateral sets of size $n+2$

$n=6$	$n=10$	$n=14$
(4,0,1,1,1,1)	(8,0,3,3,3,3,3,3,3,3)	(12, 0, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(0,4,1,1,1,1)	(0,8,3,3,3,3,3,3,3,3)	(0,12, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1,1,4,0,1,1)	(3,3,8,0,3,3,3,3,3,3)	(5, 5,12, 0, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1,1,0,4,1,1)	(3,3,0,8,3,3,3,3,3,3)	(5, 5, 0,12, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1,1,1,1,4,0)	(3,3,3,3,8,0,3,3,3,3)	(5, 5, 5, 5,12, 0, 5, 5, 5, 5, 5, 5, 5, 5)
(1,1,1,1,0,4)	(3,3,3,3,0,8,3,3,3,3)	(5, 5, 5, 5, 0,12, 5, 5, 5, 5, 5, 5, 5, 5)
(2,2,2,2,0,0)	(3,3,3,3,3,3,8,0,3,3)	(5, 5, 5, 5, 5, 5,12, 0, 5, 5, 5, 5, 5, 5)
(0,0,2,2,2,2)	(3,3,3,3,3,3,0,8,3,3)	(5, 5, 5, 5, 5, 5, 0,12, 5, 5, 5, 5, 5, 5)
	(3,3,3,3,3,3,3,3,8,0)	(5, 5, 5, 5, 5, 5, 5, 5,12, 0, 5, 5, 5, 5)
	(3,3,3,3,3,3,3,3,0,8)	(5, 5, 5, 5, 5, 5, 5, 5, 0,12, 5, 5, 5, 5)
	(4,4,4,4,4,4,2,2,2,2)	(5, 5, 5, 5, 5, 5, 5, 5, 5, 5,12, 0, 5, 5)
	(2,2,2,2,4,4,4,4,4,4)	(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 0,12, 5, 5)
		(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5,12, 0)
		(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 0,12)
		(6, 6, 6, 6, 6, 6, 6, 6, 4, 4, 4, 4, 4, 4)
		(4, 4, 4, 4, 4, 4, 6, 6, 6, 6, 6, 6, 6, 6)

4. HADAMARD MATRICES

In this section we will give an alternative construction that shows that $t(\Delta_n) \geq n+2$ for all $n = 2^k - 1$ with $k \geq 2$ using ℓ_1-norm equilateral sets and Hadamard matrices. Recall that an $n \times n$ matrix $H = [h_{ij}]$ with entries $h_{ij} \in \{-1, 1\}$ for all i and j, is called a *Hadamard matrix* if $HH^T = nI$. There exists a simple well-known construction of Hadamard matrices of size 2^k. Define $H_1 = [1]$ and

$$H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix}$$

for all $k \geq 1$. So,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \ldots .$$

Now suppose $k \geq 2$. Let $v^1, \ldots, v^{2^k} \in \mathbb{R}^{2^k}$ denote the rows of the Hadamard matrix H_{2^k}, and define the set

$$V_k = \{v^3\} \cup \{v^i : i = 5, \ldots, 2^k\}.$$

Furthermore let $W_k = \{w^1, w^2, w^3, w^4\} \in \mathbb{R}^{2^k}$ be given by

$$\begin{aligned} w^1 &= (1, -1, 0, 0, 1, -1, 0, 0, \ldots, 1, -1, 0, 0), \\ w^2 &= (-1, 1, 0, 0, -1, 1, 0, 0, \ldots, -1, 1, 0, 0), \\ w^3 &= (0, 0, 1, -1, 0, 0, 1, -1, \ldots, 0, 0, 1, -1), \\ w^4 &= (0, 0, -1, 1, 0, 0, -1, 1, \ldots, 0, 0, -1, 1). \end{aligned}$$

Proposition 4.1. *For each $k \geq 2$ the set $V_k \cup W_k$ is an ℓ_1-norm equilateral set of size $2^k + 1$ in $H_0 = \{x \in \mathbb{R}^{2^k} : \sum_i x_i = 0\}$.*

Proof. Let $k \geq 2$. It is easy to show that each $u \in V_k \cup W_k$ lies in H_0. Also note that any two distinct points v^i and v^j in V_k satisfy

$$\|v^i - v^j\|_1 = 2^k,$$

as the rows in H_{2^k} differ in exactly 2^{k-1} places. The reader can check that $\|w^i - w^j\|_1 = 2^k$ for all $1 \leq i \neq j \leq 4$.

So, it remains to show that

$$\|v^i - w^j\|_1 = 2^k \quad \text{for all } v^i \in V_k \text{ and } w^j \in W_k. \tag{4.1}$$

We use induction on k. Note that if $k = 2$, we have that

$$V_2 \cup W_2 = \{(1, 1, -1, -1), (1, -1, 0, 0), (-1, 1, 0, 0), (0, 0, 1, -1), (0, 0, -1, 1)\},$$

which is an ℓ_1-norm equilateral set with distance 4. Now suppose that (4.1) holds for k. Denote the points in V_{k+1} by $\bar{v}^i$ and the points in W_{k+1} by $\bar{w}^j$. Note that for $j = 1, \ldots, 4$ we have $\bar{w}^j = (w^j, w^j)$, where $w^j \in W_k$. Also observe that for $i = 3, 5 \ldots, 2^k$ we have $\bar{v}^i = (v^i, v^i)$, and for $i = 2^k + 1, \ldots, 2^{k+1}$ we have $\bar{v}^i = (v^{i-2^k}, -v^{i-2^k})$, where $v^i \in V_k$.

So, for $i = 3, 5, \ldots, 2^k$ and $j = 1, \ldots, 4$, we have that

$$\|\bar{v}^i - \bar{w}^j\|_1 = \sum_{l=1}^{2^{k+1}} |\bar{v}^i_l - \bar{w}^j_l| = 2 \sum_{l=1}^{2^k} |v^i_l - w^j_l| = 2 \cdot 2^k = 2^{k+1}$$

by the induction hypothesis. Also for $i = 2^k + 1, \ldots, 2^{k+1}$ and $j = 1, \ldots, 4$, we have that

$$\|\bar{v}^i - \bar{w}^j\|_1 = \sum_{l=1}^{2^k} (|v^{i-2^k}_l - w^j_l| + |v^{i-2^k}_l + w^j_l|) = \sum_{l=1}^{2^k} (1 - w^j_l + 1 + w^j_l) = 2^{k+1},$$

as $v^i_l \in \{-1, 1\}$ and $-1 \leq w^j_l \leq 1$ for all l. $\square$

The reader should note that the equilateral set $V_k \cup W_k$ can be seen as a generalization of the equilateral set S in (2.2), as $V_2 \cup W_2 = S - (1,1,1,1)$. Furthermore, the example in Table 1 with $n = 8$ is also of this type, if one ignores the point $(8,2,1,1,0,2,1,1)$.

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