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Research Article

Time-Consistent Strategies for Multi-Period Portfolio Optimization with/without the Risk-Free Asset

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The pre-commitment and time-consistent strategies are the two most representative investment strategies for the classic multi-period mean-variance portfolio selection problem. In this paper, we revisit the case in which there exists one risk-free asset in the market and prove that the time-consistent solution is equivalent to the optimal open-loop solution for the classic multi-period mean-variance model. Then, we further derive the explicit time-consistent solution for the classic multi-period mean-variance model only with risky assets, by constructing a novel Lagrange function and using backward induction. Also, we prove that the Sharpe ratio with both risky and risk-free assets strictly dominates that of only with risky assets under the time-consistent strategy setting. After the theoretical investigation, we perform extensive numerical simulations and out-of-sample tests to compare the performance of pre-commitment and time-consistent strategies. The empirical studies shed light on the important question: what is the primary motivation of using the time-consistent investment strategy.

1. Introduction

Currently, the portfolio selection problem is one of the most popular topics in financial economics, and it has played a central role in modern financial studies since the publication of the work on the static mean-variance portfolio theory introduced by Markowitz [1], which primarily examined the optimal allocation of an investor's wealth to a basket of assets. After the publication of Markowitz's pioneering work, multi-period portfolio selection problems have been discussed by many researchers. Many of the previous studies have followed the approach of maximizing the expected utility functions of the terminal wealth or consumption, such as Mossin [2], Dumas and Luciano [3], etc. Nevertheless, the explicit expression of the optimal strategy has not been derived under the mean-variance criterion for a long time, in part because the variance is not separable, and thus the multi-period mean-variance problem cannot be directly solved by applying dynamic programming. In the study by Li and Ng [4], the authors first derived the analytical optimal solution and efficient frontier for the classic mean-variance model using a novel approach: the embedding scheme. In the same year, Zhou and Li [5] also derived the explicit solution for the mean-variance formulation in continuous-time setting by using the same method.

Since then, many researchers have followed the work by Li and Ng [4] and further studied the dynamic mean-variance portfolio optimization problem. Zhou and Yin [6] considered a continuous-time mean-variance portfolio selection in a regime-switching capital market, while Yin and Zhou [7] studied a discrete-time version. Leippold et al. [8] presented a geometric approach to solve the multi-period asset-liability management mean-variance portfolio selection problem. In addition, Zhu et al. [9] discussed a generalized multi-period mean-variance model that provides useful advice to help investors not only achieve an optimal expected return but also have a good risk control over bankruptcy. As far as we are aware, the studies above always assumed a predetermined time horizon. However, in the real world, the investor could not determine the time horizon at the beginning of the investment in some cases. In other words, some investors might be forced to abandon the original investment plan as a consequence of some exogenous or endogenous factors. Wu and Li [10] investigated an uncertain exit time horizon multi-period mean-variance portfolio in the regime-switching capital market. Yao et al. [11] further studied the work of Wu and Li [10] and considered an uncertain exit time multi-period mean-variance portfolio selection problem with endogenous liabilities in a regime-switching capital market. In fact, there is a growing literature that addresses dynamic mean-variance portfolio selection problems. For more detailed discussions on the above subject, see Cui et al. [12], Li and Li [13], Yao et al. [14], etc.

By examining the above literatures, it is not difficult to determine that all of the optimal strategies are made at an initial date (at the time that the investor has just joined the capital market), and they not only depend on the current wealth but also rely on the initial capital and the statistic characteristics of the assets for different periods of time. Many of the researchers designated these optimal investment strategies derived in the above mean-variance models as the pre-commitment strategy, which has been criticized for lacking rationality since the above optimal investment strategies do not satisfy the Bellman principle or time-consistency. It is easy to find that the cause of this time inconsistency in the above optimal investment strategies is that the variance in optimization objective functions is not separable. For more details of the subject of time-consistent strategy, readers may refer to see Bjork and Murgoci [15] and Basak and Chabakauri [16].

To the best of our knowledge, there exist three approaches to address the time-inconsistent mean-variance problem in the current literature. In addition to the method of Bjork and Murgoci [15], Cui et al. [17] presented a weak time consistency and also derived the corresponding weak time-consistent strategy. On the other hand, Chen et al. [18] proposed the notation of an expected conditional mapping and then constructed a time-consistent multi-period mean-variance portfolio optimization model. Chen et al. [19] generalized the work of Chen et al. [18] to a regime-switching capital market. However, compared with the first approach, the last two approaches have not been used to derive the explicit time-consistent solutions directly for the classical model. Therefore, the approach that is first used by Bjork and Murgoci [15] is by far the most widely used. Zeng and Li [20] applied this game approach to study the time-consistent investment and reinsurance strategies for mean-variance insurers. Czichowsky [21] proved that the continuous-time mean-variance portfolio selection formulation is coincident with the continuous-time limit of the discrete-time formulation under the time-consistent setting. Furthermore, Hu et al. [22] and Björk et al. [23] also discussed a more realistic mean-variance model whereby the risk aversion depends on the current total wealth. Bensoussan et al. [24] extended the work of Björk et al. [23] to the portfolio selection with no short-selling constraints.

Along the above research line, we discuss the relationship between the time-consistent and optimal open-loop strategies when there exists one risk-free asset in the classic multi-period mean-variance model. Most of the existing studies on the time-consistent solutions of the mean-variance portfolio selection problem are only concerned with the market with both risky assets and one risk-free asset. In real applications, it is easy to find the case in which the investor only invests on risky assets. Note that the time-consistent strategy shown in Bjork and Murgoci [15] with a non-feedback form might not be true for the case without risk-free assets. In this paper, we derive the explicit time-consistent solution for the multi-period mean-variance model in the case that there only exist risky assets by using a novel approach. What is more, we prove that the best Sharpe ratio generated by both risky and risk-free assets strictly dominates that of only risky assets under the time-consistent strategy setting. In addition, we discuss the time-consistent solution of the adjustment model, and prove the time-consistent strategy is coincident with that of the classic multi-period mean variance model. According to the results of Bjork and Murgoci [15], the pre-commitment strategy might produce more expected return than the time-consistent strategy in terms of theory. However, in real investment, the time-consistent strategy may be a better choice for many investors. Thus, motivated by the work of DeMiguel et al. [25], we use Sharpe ratio, Portfolio turnover and Maximum drawdown to compare the pre-commitment and time-consistent investment strategies presented in this paper.

The remainder of this paper is organized as follows. In Section 2, we introduce the classic multi-period mean-variance portfolio selection model. In Section 3, we discuss the time-consistent strategies and optimal open-loop strategies for the classic model; in addition, we also present the relationship between the best Sharpe ratio generated by both risky and risk-free assets and that generated by only risky assets under the time-consistent strategy setting. In Section 4, we further discuss the above two strategies for the adjustment model in Calafiore [26]. Next, some numerical simulations and empirical studies are given in Section 5 to illustrate the conclusions for different models or optimal strategies. The concluding remarks are presented in the end.

2. Problem Formulations

In this paper, we assume that the investor joins the capital market at time 0 with an initial known wealth w_0 and wishes to make a plan for allocating all of his/her wealth among the n risky assets and one risk-free asset within a time horizon of T periods. In addition, the investor can rebalance his/her portfolio at the beginning of each of the subsequent periods. Let $e_t = [e_t^1, \ldots, e_t^n]'$ be the return at time period t, where e_t^i denotes the return of the ith risky asset at time period t, $i = 1, \ldots, n$, $t = 0, 1, \ldots, T-1$. We assume that the risk-free asset with a deterministic rate of return s_t and the risky return e_t, $t = 0, 1, \ldots, T-1$ are statistically independent and have a known mean $E(e_t) = [E(e_t^1), \ldots, E(e_t^n)]'$ and a known covariance matrix

$$\Omega_t = \begin{bmatrix} \operatorname{cov}\left(e_t^1, e_t^1\right) & \operatorname{cov}\left(e_t^1, e_t^2\right) & \cdots & \operatorname{cov}\left(e_t^1, e_t^n\right) \\ \operatorname{cov}\left(e_t^2, e_t^1\right) & \operatorname{cov}\left(e_t^2, e_t^2\right) & \cdots & \operatorname{cov}\left(e_t^2, e_t^n\right) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{cov}\left(e_t^n, e_t^1\right) & \operatorname{cov}\left(e_t^n, e_t^2\right) & \cdots & \operatorname{cov}\left(e_t^n, e_t^n\right) \end{bmatrix}. \quad (1)$$

Let w_t denote the wealth variable for the investment process at time t and the investment portfolio $u_t = (u_t^0, u_t^1, \dots, u_t^n)'$ represent the amounts that have been invested in the n risky assets at the beginning of the tth time period, then the amount invested in the risk-free asset is equal to $w_t - \sum_{i=1}^n u_t^i$ under the self-financing condition, where $t = 0, 1, \dots, T-1$. Thus, the dynamic wealth process can be described as

$$w_{t+1} = s_t w_t + P_t' u_t, \quad t = 0, 1, \dots, T-1. \tag{2}$$

where $P_t = [(e_t^1 - s_t), (e_t^2 - s_t), \dots, (e_t^n - s_t)]'$.

Therefore, the classic multi-period mean-variance portfolio selection can be expressed as

$$\begin{aligned} \max \quad & E(w_T) - \omega \operatorname{var}(w_T) \\ s.t. \quad & w_{t+1} = s_t w_t + P_t' u_t, \quad t = 0, 1, \dots, T-1. \end{aligned} \tag{3}$$

where ω is a risk aversion parameter that reflects the investor's attitude toward risk.

3. Solutions of the Classic Multi-Period Mean-Variance Portfolio Optimization Model

Because the optimal solution of the classic multi-period mean-variance model (henceforth, the pre-commitment strategy) is time-inconsistent, in the literature, many authors have developed time-consistent solutions for it in the case in which there is a risk-free asset, such as Basak and Chabakauri [16], Bjork and Murgoci [15] and Bensoussan et al. [24], etc. In Section 3.1, we revisit this model due to its importance and then present the solution obtained by Bjork and Murgoci [15] which is coincident with the optimal open-loop solution of this model. In addition, since the pioneering work of Bjork and Murgoci [15] only considers both risk-free asset and risky assets as investment target, it is necessary to discuss the time-consistent strategy for Model (3) when the capital pool is only with risky assets (note that if investor does not invest risk-free assets, in this case, we only need to add the extra condition $w_t = \sum_{i=1}^n u_t^i$, $t = 0, 1, \dots, T-1$ into Model (3)). When the capital pool is only with risky assets, along with the work of Bjork and Murgoci [15], we derive the time-consistent solution of multi-period mean-variance model by applying the backward induction method in Section 3.2. It is interesting that the solution obtained is a feedback, while the solution obtained by Bjork and Murgoci [15] is a deterministic solution (an open-loop solution).

3.1. Definition of Time-Consistent Strategy and Solution Methodology. To this end, it is necessary to introduce the definition of time-consistent solutions for the above optimization models. Let the different control strategies for the above models be represented by $u = (u_0, u_1, \dots, u_{T-1})$, and $E_k(w_T)$, $\operatorname{var}_k(w_T)$, $k = 0, 1, \dots T-1$ denote the conditional expectation and variance based on the information of the time period k, respectively.

Let

$$J(w_k, u) = E_k(w_T) - \omega \operatorname{var}_k(w_T), \qquad k = 0, 1, \dots T-1. \tag{4}$$

According to definition 2.2 in Bjork and Murgoci [15], the time-consistent strategy for Model (4) can be defined as follows.

Definition 1. Consider a given control law $\hat{u} = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{T-1})$. For $k = 0, 1, \dots T-1$, we let

$$\begin{aligned} u(k) &= (u_k, \hat{u}_{k+1}, \dots, \hat{u}_{T-1}), \\ \hat{u}(k) &= (\hat{u}_k, \hat{u}_{k+1}, \dots, \hat{u}_{T-1}), \end{aligned} \tag{5}$$

where u_k is an arbitrary control variable. Then, $\hat{u}$ is said to be a time-consistent strategy if, for all $k = 0, 1, \cdots T-1$, the following conditions hold:

$$\max_{u_k} J(w_k, u(k)) = J(w_k, \hat{u}(k)). \tag{6}$$

To make the above definition easier to understand, it is necessary to introduce detailed procedures to derive the above time-consistent solution, which is presented as follows:

(1) For the given initial time $k = T-1$, as w_{T-1} is known, the above Model (4) degenerates into a single period mean-variance portfolio problem. By maximizing the following objective function $E_{T-1}(w_T) - \omega \operatorname{var}_{T-1}(w_T)$, we can find the time-consistent strategy $\hat{u}_{T-1}$, the optimal value functions $E_{T-1}(\hat{w}_T)$, and $\operatorname{var}_{T-1}(\hat{w}_T)$, which are the function of w_{T-1}. And the corresponding value function can be expressed as

$$V_{T-1}(w_{T-1}) = E_{T-1}(\hat{w}_T) - \omega \operatorname{var}_{T-1}(\hat{w}_T). \tag{7}$$

(2) By applying the backward induction method, the law of iterated expectations and the law of total variance, given the initial time $k = T-2$ and fixed wealth w_{T-2}, the time-consistent strategy $\hat{u}_{T-2}$ satisfies the following condition:

$$\begin{aligned} \hat{u}_{T-2} &= \arg\max_{u_{T-2}} \left([E_{T-2}(w_T) - \omega \operatorname{var}_{T-2}(w_T)] \mid \hat{u}_{T-1} \right) \\ &= \arg\max_{u_{T-2}} \left(E_{T-2}[E_{T-1}(\hat{w}_T) - \omega \operatorname{var}_{T-1}(\hat{w}_T)] \right. \\ &\quad \left. - \omega \operatorname{var}_{T-2}[E_{T-1}(\hat{w}_T)] \right) \\ &= \arg\max_{u_{T-2}} \left(E_{T-2}[V_{T-1}(w_{T-1})] \right. \\ &\quad \left. - \omega \operatorname{var}_{T-2}[E_{T-1}(\hat{w}_T)] \right). \end{aligned} \tag{8}$$

Additionally, the value function can be derived by

$$V_{T-2}(w_{T-2}) = E_{T-2}(\hat{w}_T) - \omega \operatorname{var}_{T-2}(\hat{w}_T). \tag{9}$$

(3) Repeating the above steps, we can find the time-consistent investment policy for all the time periods $u = (\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{T-1})$.

Bjork and Murgoci [15] showed that the so-called time-consistent strategy means that optimal strategy obtained at time period k_1 agrees with that derived at time period k_2

where $k_1 < k_2$. Compared the above solution methodology with the embedding scheme presented in Li and Ng [4], we can conclude that the former adopts a novel approach to deal with this time-inconsistent mean-variance problem, which directly forces the proposed solution methodology to satisfy the Bellman principle or time-consistency, while the latter mainly uses an indirect approach where a time-consistent auxiliary problem $\max_u \lambda E(w_T) - \omega E(w_T^2)$ is solved; after that, the gap between the original problem and auxiliary problem can be bridged by using the relationship among them. However, the embedding approach cannot assure that the derived investment strategy satisfies time-consistency.

3.2. Revisiting the Time-Consistent Control Policy with Both Risky Assets and Risk-Free Assets. Here, we first describe the concept of the open-loop strategy. An open-loop strategy is also called a non-feedback strategy. That is, it only uses the information at current state and does not need feedback information to determine the investment strategy. According to definition 2.1 in Fershtman [27], the open-loop strategy for Model (4) can be defined as follows.

Definition 2. Consider a given control law $u^{OL} = (u_0^{OL}, u_1^{OL}, \ldots, u_{T-1}^{OL})$. For $k = 0, 1, \ldots T - 1$, if u_k^{OL} is only a time path $u_k^{OL}(w_0)$ such that, given the initial state variable w_0, it assigns for every k a control in the set of admissible control, then u^{OL} is called as open-loop strategy. Furthermore, $\widehat{u}^{OL} = (\widehat{u}_0^{OL}, \widehat{u}_1^{OL}, \ldots, \widehat{u}_{T-1}^{OL})$ is said to be the optimal open-loop strategy, when $\widehat{u}^{OL}$ satisfies the following conditions:

$$\widehat{u}^{OL} = \arg\max_{u^{OL}} \left(\left[E_0(w_T) - \omega \mathrm{var}_0(w_T) \right] \Big|_{w_0} \right). \tag{10}$$

It is not difficult to find that the optimal open-loop strategy takes the viewpoint that the decision maker wants to compute and freeze the whole sequence of control strategies $\widehat{u}^{OL} = (\widehat{u}_0^{OL}, \widehat{u}_1^{OL}, \ldots, \widehat{u}_{T-1}^{OL})$ at the initial time $k = 0$; see Calafiore [26, 28] for the detailed discussion. In this case, the multi-period portfolio selection problem does not need to use dynamic programming approach because it is a deterministic optimization problem, which can be solved by some existing optimization algorithms. Compared to the optimal feedback strategy, the performance of the optimal open-loop strategy would generally be worse in theory; for this curious phenomenon, Calafiore [26] gave an interpretation that the optimal open-loop strategy $\widehat{u}^{OL}$ is “here and now”, while the feedback strategy is “wait and see”. However, the open-loop strategy is much easier to compute in general.

According to the definition of the time-consistent strategy, one can derive the general time-consistent strategy and efficient frontier by using backward induction, and the main results are as follows.

Lemma 3. *The time-consistent strategy and efficient frontier for Model (3) can be expressed as*

$$\widehat{u}_t = \frac{\Omega_t^{-1} E(P_t)}{2\omega \prod_{i=t+1}^{T-1} s_i}, \quad t = 0, 1, \ldots, T-1 \tag{11}$$

and

$$\mathrm{var}_0(w_T) = \frac{\left(E_0(w_T) - w_0 \prod_{k=0}^{T-1} s_k \right)^2}{\sum_{k=0}^{T-1} E(P_k') \Omega_k^{-1} E(P_k)} \tag{12}$$

where $E(P_t)$ is assumed to be not identically equal to zero vector.

Proof. The details of the proof can be found in Proposition 6.1 provided by Bjork and Murgoci [15], Proposition 9.1 provided by Bjork and Murgoci [29], and Theorem 2 provided by Wu [30], and then we omit the proof here. □

Additionally, Lemma 3 also indicates that when there is a risk-free asset, the time-consistent solution of Model (11) is demonstrated to be determinist. The question is whether it is an optimal open-loop strategy for this case. In general, an optimal open-loop strategy is not necessarily a time-consistent one. In this section, we prove that it is indeed the case.

Theorem 4. *The time-consistent strategy (11) is also the optimal open-loop strategy for Model (3) with risky assets and one risk-free asset. In addition, the efficient frontiers are the same under the above two investment strategies.*

Proof. According to the conclusions of Lemma 3, to prove Theorem 4, we only need to derive the optimal open-loop strategy for Model (3) and check whether they are equivalent. Based on the dynamic wealth process, we have

$$\begin{aligned} w_t &= P_{t-1}' u_{t-1} + \left(\prod_{i=t-1}^{t-1} s_i \right) P_{t-2}' u_{t-2} + \cdots \\ &\quad + \left(\prod_{i=1}^{t-1} s_i \right) P_0' u_0 + \left(\prod_{i=0}^{t-1} s_i \right) w_0 \\ &= \left(\prod_{i=0}^{t-1} s_i \right) w_0 + \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} s_i \right) P_k' u_k. \end{aligned} \tag{13}$$

From the definition of the open-loop strategy, we can obtain the explicit expressions for the expectation and variance of the terminal of total wealth.

$$E_0(w_t) = \left(\prod_{i=0}^{t-1} s_i \right) w_0 + \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} s_i \right) E(P_k') u_k, \tag{14}$$

and

$$\begin{aligned} \mathrm{var}_0(w_t) &= \mathrm{var}_0 \left(\sum_{k=0}^{t-1} \left[\prod_{i=k+1}^{t-1} s_i \right] E(P_k') u_k \right) \\ &= \sum_{k=0}^{t-1} \left(\prod_{i=k+1}^{t-1} s_i \right)^2 u_k' \Omega_k u_k. \end{aligned} \tag{15}$$

Therefore, the explicit expression for the open-loop model can be expressed as

$$\begin{aligned} \max \quad & E_0(w_T) - \omega \mathrm{var}_0(w_T) \\ s.t. \quad & E_0(w_T) \\ & = \left(\prod_{i=0}^{T-1} s_i\right) w_0 + \sum_{k=0}^{T-1} \left(\prod_{i=k+1}^{T-1} s_i\right) E(P_k') u_k \\ & \mathrm{var}_0(w_T) = \sum_{k=0}^{T-1} \left(\prod_{i=k+1}^{T-1} s_i\right)^2 u_k' \Omega_k u_k. \end{aligned} \tag{16}$$

Note that problem (16) is a convex optimization problem with respect to the open-loop control variable u_k. By the first-order necessary optimality condition, the optimal open-loop strategy can be derived:

$$\hat{u}_t^{OL} = \frac{\Omega_t^{-1} E(P_t)}{2\omega \prod_{i=t+1}^{T-1} s_i}, \quad t = 0, 1, \ldots, T-1. \tag{17}$$

Substituting (17) into (14) and (15), the efficient frontier under the optimal open-loop strategy (17) can be derived:

$$\mathrm{var}_0(w_T) = \frac{\left(E_0(w_T) - w_0 \prod_{k=0}^{T-1} s_k\right)^2}{\sum_{k=0}^{T-1} E(P_k') \Omega_k^{-1} E(P_k)}. \tag{18}$$

Compared with Lemma 3 and the above results, it is not difficult to find that the optimal open-loop strategy (17) and corresponding efficient frontier (18) are the same as the time-consistent strategy (11) and time-consistent efficient frontier (12), respectively. Therefore, Theorem 4 is derived. □

In addition, according to the formulation of the time-consistent strategy presented in Lemma 3, we can find that it is not dependent on the information in the future investment, which has the same form of the following myopic investment strategies: the myopic investment strategies by optimizing a one-period investment problem at each period time t, where $0 \le t \le T-1$.

$$\begin{aligned} \max \quad & E_t(w_{t+1}) - \omega \mathrm{var}_t(w_{t+1}) \\ \text{s.t.} \quad & w_{t+1} = s_t w_t + P_t' u_t, \quad t = 0, 1, \ldots, T-1. \end{aligned} \tag{19}$$

Obviously, we can derive the myopic investment strategies at each time t for Model (19), and the detailed results can be expressed as follows:

$$u_t^{MP} = \frac{\Omega_t^{-1} E(P_t)}{2\omega}, \quad t = 0, 1, \ldots, T-1. \tag{20}$$

Compared the time-consistent strategy (11) with the myopic strategy (20), we can find that the time-consistent solution of Model (11) at each time period t is similar to the solution of the common single-period optimization problem (19). The only difference is that the risk aversion for the investor is varying at different period t. Thus the time-consistent strategy for Model (11) can also be derived by optimizing the single-period problem (19) with time-vary risk aversion coefficient $\omega_t = \omega \prod_{i=t+1}^{T-1} s_i$. In addition, suppose that the length of each rebalancing period is very short (high frequency trading), the investor might modify the investment strategy after one minute or one hour, then the return of the risk asset becomes very small, and it almost can be ignored in this case ($s_t \longrightarrow 1$). As a result, we can obtain some interesting finding that the time-consistent strategy (11) is almost equivalent to the myopic strategy (20) in this case.

3.3. Time-Consistent Control Policy without Risk-Free Asset. If we further consider the investor allocate all his/her wealth on the above n risky assets, the dynamic wealth process (2) can be rewritten as

$$\begin{aligned} w_{t+1} &= e_t' u_t, \quad t = 0, 1, \ldots, T-1, \\ w_t &= \mathbf{I}' u_t \quad t = 0, 1, \ldots, T-1, \end{aligned} \tag{21}$$

where $\mathbf{I} = (1, \ldots, 1)'$ denotes $(n+1) \times 1$ vector.

Based on the above wealth processes, the classic multi-period mean-variance portfolio optimization model reads as follows:

$$\begin{aligned} \max \quad & E(w_T) - \omega \,\mathrm{var}(w_T) \\ s.t. \quad & w_{t+1} = e_t' u_t, \quad t = 0, 1, \ldots, T-1, \\ & w_t = \mathbf{I}' u_t, \quad t = 0, 1, \ldots, T-1. \end{aligned} \tag{22}$$

In this section, we will develop the time-consistent solution of Model (22) for the case in which there is no risk-free asset by directly applying the backward induction method. We assume that the covariance matrix Ω_t is positively definite. For notational simplicity, we first put together all the notations that will appear hereafter.

$$\begin{aligned} A_t &= \mathbf{I}' \widehat{\Omega}_t^{-1} \mathbf{I}, \\ B_t &= \mathbf{I}' \widehat{\Omega}_t^{-1} E(e_t), \end{aligned} \tag{23}$$

$$\begin{aligned} C_t &= E(e_t') \widehat{\Omega}_t^{-1} E(e_t), \\ D_t &= A_t C_t - B_t^2, \end{aligned} \tag{24}$$

$$\begin{aligned} \alpha_t &= \frac{1}{A_t}, \\ m_t &= \prod_{k=t}^{T-1} \frac{B_k}{A_k}, \end{aligned} \tag{25}$$

$$\widehat{\Omega}_t = \begin{cases} \Omega_{T-1}, & t = T-1 \\ \alpha_{t+1} E(e_t e_t') + m_{t+1}^2 \Omega_t, & t < T-1, \end{cases} \tag{26}$$

where $t = 0, 1, \ldots, T-1$. From the definitions of A_t and m_t, it is obvious that $\alpha_t > 0$ and $m_t^2 > 0$ for $t = 0, 1, \ldots, T-1$. In this section, we also assume that $E(e_t e_t')$ is positive definite matrix. Since $\widehat{\Omega}_t$ is a linear combination of Ω_t and $E(e_t e_t')$ and the combination coefficients are all positive, it is easy to see that $\widehat{\Omega}_t$ is positive definite matrix. Then, the time-consistent strategy and the optimal value function can be derived by using the backward induction method and constructing the Lagrange function for the investor who only invests in risky assets.

From the work of Bjork and Murgoci [15], one can see that they first conjectured that the optimal value function was a linear function of current total wealth, and they subsequently were able to find the explicit formulation for the solution. However, it will be observed later that this conjecture is untrue for the capital market with only risky assets. The main results of this section can be expressed as the following theorem.

Theorem 5. *When there are no risk-free assets, the time-consistent strategy, and the corresponding conditional expectation and variance of terminal wealth for the Model (22) can be expressed as follows:*

$$\widehat{u}_t = a_t w_t + b_t, \tag{27}$$

$$E_t\left(\widehat{w}_T\right) = m_t w_t + n_t, \tag{28}$$

$$\operatorname{var}_t\left(\widehat{w}_T\right) = \alpha_t w_t^2 + \beta_t w_t + \gamma_t, \tag{29}$$

where $t = 0, 1, \ldots, T-1$, and the above parameters satisfy the following relations:

$$\begin{aligned} a_\tau &= \frac{\widehat{\Omega}_\tau^{-1}\mathbf{I}}{A_\tau}, \\ b_\tau &= \frac{m_{\tau+1}}{2\omega}\widehat{\Omega}_\tau^{-1}\left(E\left(e_\tau\right) - \frac{B_\tau \mathbf{I}}{A_\tau}\right), \end{aligned} \tag{30}$$

$$\begin{aligned} m_\tau &= \prod_{k=\tau}^{T-1}\frac{B_k}{A_k}, \\ n_\tau &= \frac{m_{\tau+1}^2 D_\tau}{2\omega A_\tau} + n_{\tau+1}, \end{aligned} \tag{31}$$

$$\begin{aligned} \alpha_\tau &= \frac{1}{A_\tau}, \\ \beta_\tau &= 0, \\ \gamma_\tau &= \gamma_{\tau+1} + \frac{m_{\tau+1}^2 D_\tau}{4\omega^2 A_\tau}, \end{aligned} \tag{32}$$

where $\tau = 0, 1, \ldots, T-2$, as well as the terminal conditions:

$$\begin{aligned} a_{T-1} &= \frac{\Omega_{T-1}^{-1}\mathbf{I}}{A_{T-1}}, \\ b_{T-1} &= \frac{1}{2\omega}\Omega_{T-1}^{-1}\left(E\left(e_{T-1}\right) - \frac{B_{T-1}\mathbf{I}}{A_{T-1}}\right), \\ m_{T-1} &= \frac{B_{T-1}}{A_{T-1}}, \\ n_{T-1} &= \frac{D_{T-1}}{2\omega A_{T-1}}, \\ \alpha_{T-1} &= \frac{1}{A_{T-1}}, \\ \beta_{T-1} &= 0, \\ \gamma_{T-1} &= \frac{D_{T-1}}{4\omega^2 A_{T-1}}, \end{aligned} \tag{33}$$

Proof. The proof of Theorem 5 is shown in Appendix A. □

According to (28) and (29), the efficient frontier under the time-consistent strategy (27) can be derived; for the main results, see Corollary 6.

Corollary 6. *The efficient frontier under the time-consistent strategy (27) for the mean-variance model (22) is given by*

$$\operatorname{var}_0\left(w_T\right) = \frac{\left[E_0\left(w_T\right) - w_0\prod_{i=0}^{T-1}\left(B_i/A_i\right)\right]^2}{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(B_i/A_i\right)^2\right]\left(D_k/A_k\right)} + \frac{w_0^2}{A_0}, \tag{34}$$

where $\prod_{i=k+1}^{T-1}(B_i/A_i)$ in (34) is defined to be unity when $k = T-1$.

Proof. From Theorem 5, it is easy to derive the following results:

$$\begin{aligned} n_t &= \frac{1}{2\omega}\sum_{k=t}^{T-1}\left(\prod_{i=k+1}^{T-1}\frac{B_i}{A_i}\right)\frac{D_k}{A_k}, \\ \gamma_t &= \frac{1}{4\omega^2}\sum_{k=t}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(\frac{B_i}{A_i}\right)^2\right]\frac{D_k}{A_k}, \end{aligned} \tag{35}$$

For the given $t = 0$ and the initial wealth w_0, (28) and (29) can be rewritten as

$$E_0\left(w_T\right) = \prod_{i=0}^{T-1}\frac{B_i}{A_i}w_0 + \frac{1}{2\omega}\sum_{k=0}^{T-1}\left(\prod_{i=k+1}^{T-1}\frac{B_i}{A_i}\right)\frac{D_k}{A_k}, \tag{36}$$

$$\operatorname{var}_0\left(w_T\right) = \frac{w_0^2}{A_0} + \frac{1}{4\omega^2}\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(\frac{B_i}{A_i}\right)^2\right]\frac{D_k}{A_k}. \tag{37}$$

Thus, the following efficient frontier can be obtained:

$$\operatorname{var}_0\left(w_T\right) = \frac{\left[E_0\left(w_T\right) - w_0\prod_{i=0}^{T-1}\left(B_i/A_i\right)\right]^2}{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(B_i/A_i\right)^2\right]\left(D_k/A_k\right)} + \frac{w_0^2}{A_0}. \tag{38}$$

Then, Corollary 6 is proved. □

Suppose that the risk aversion ω in Model (22) tends to infinity, then the investor only considers the risk in the investment process and the expected return is ignored in this case, which is also known as the global minimum variance portfolio optimization problem in literature. According to Theorem 4, we can directly derive the time-consistent strategy and the corresponding optimal expected and variance values of the terminal wealth; for the detailed results see the following corollary.

Corollary 7. *Suppose that there are no risk-free assets, the time-consistent control policy, and the corresponding expectation and variance of terminal wealth for the global minimum variance model can be expressed as follows:*

$$\widehat{u}_t = \frac{\widehat{\Omega}_t^{-1}\mathbf{I}}{A_t} w_t, \quad t = 0, 1, \ldots, T-1 \tag{39}$$

$$E_t\left(\widehat{w}_T\right) = \prod_{k=t}^{T-1} \frac{B_k}{A_k} w_t, \quad t = 0, 1, \ldots, T-1 \tag{40}$$

$$\operatorname{var}_t\left(\widehat{w}_T\right) = \frac{1}{A_t} w_t^2, \quad t = 0, 1, \ldots, T-1 \tag{41}$$

Proof. From Theorem 5 and Corollary 6, we can obtain that b_t, n_t, and γ_t trend to zero when $\omega \longrightarrow \infty$. Therefore, Corollary 7 is proved. □

Notice that the solution obtained is a feedback or close-loop solution, while the solution obtained by Bjork and Murgoci [15, 29] for the case with a risk-free asset is a deterministic solution (or an open-loop solution).

3.4. The Premium of Dynamic Trading under the Time-Consistent Strategies Setting. In the previous section, we mainly have discussed the time-consistent investment strategies for the classic model with and without risk-free assets. However, the premiums of dynamic trading for the above two investment strategies are also concerned by most investors, such as the Sharpe ratios of the portfolios under different strategies. To the best of our knowledge, Chiu and Zhou [31] showed that the Sharpe ratio of this portfolio included a risk-free asset that strictly dominates that of this portfolio without including a risk-free asset under continuous-time setting. Additionally, Yao et al. [32] extended the work of Chiu and Zhou [31] to a discrete-time setting. However, the above literatures are limited to the conclusions under pre-commitment investment strategy framework. In the following, we will further discuss the premium of dynamic trading under the time-consistent strategy setting.

We consider the Sharpe ratio defined as

$$SR\left(E_0\left(w_T\right)\right) = \frac{E_0\left(w_T\right) - w_0 \prod_{t=0}^{T-1} s_t}{\sqrt{\operatorname{var}_0\left(w_T\right)}}. \tag{42}$$

For convenience, let BSR_{rf} and BSR denote the best Sharpe ratios for the portfolio with and without risk-free assets, respectively. According to the efficient frontier (12), the best Sharpe ratio that with both risky assets and risk-free asset can be expressed as

$$BSR_{rf} = \sqrt{\sum_{k=0}^{T-1} E\left(P_k'\right) \Omega_k^{-1} E\left(P_k\right)}. \tag{43}$$

Similarly, we can also derive the best Sharpe ratio that of only risky assets. From the efficient frontier (34), we have the following Sharpe ratio:

$$SR\left(E_0\left(w_T\right)\right) = \frac{E_0\left(w_T\right) - w_0 \prod_{t=0}^{T-1} s_t}{\sqrt{\left(1/\varphi_0\right)\left[E_0\left(w_T\right) - w_0 \prod_{i=0}^{T-1}\left(B_i/A_i\right)\right]^2 + w_0^2/A_0}}, \tag{44}$$

where $\varphi_0 = \sum_{k=0}^{T-1}[\prod_{i=k+1}^{T-1}(B_i/A_i)^2](D_k/A_k)$.

By the first-order derivative of SR, we have

$$\frac{dSR\left(E_0\left(w_T\right)\right)}{dE_0\left(w_T\right)} = \frac{w_0^2/A_0 + \left(1/\varphi_0\right)\left[E_0\left(w_T\right) - w_0 \prod_{t=0}^{T-1} s_t\right]\left[\prod_{t=0}^{T-1} s_t - \prod_{i=0}^{T-1}\left(B_i/A_i\right)\right] w_0}{\left(\left(1/\varphi_0\right)\left[E_0\left(w_T\right) - w_0 \prod_{i=0}^{T-1}\left(B_i/A_i\right)\right]^2 + w_0^2/A_0\right)^{3/2}}. \tag{45}$$

Due to the fact that $E_0(w_T) - w_0\prod_{t=0}^{T-1} s_t \geq 0$, if $\prod_{t=0}^{T-1} s_t - \prod_{i=0}^{T-1}(B_i/A_i) \geq 0$, we can easily find that SR is an increasing function about $E_0(w_T)$ on its domain $[w_0\prod_{i=0}^{T-1}(B_i/A_i), \infty)$. In this case, we can derive the supremum of the Sharpe ratio (44), which can be referred to as the best Sharpe ratio.

$$BSR = \sqrt{\varphi_0} = \sqrt{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(\frac{B_i}{A_i}\right)^2\right] \frac{D_k}{A_k}}. \tag{46}$$

On the other hand, for $\prod_{t=0}^{T-1} s_t - \prod_{i=0}^{T-1}(B_i/A_i) < 0$, we rewrite the efficient frontiers (12) and (34) as follows, respectively.

$$\begin{aligned}\sigma^r\left(w_T\right) &= \sqrt{\operatorname{var}_0\left(w_T\right)} \\ &= \sqrt{\frac{\left[E_0\left(w_T\right) - w_0 \prod_{i=0}^{T-1}\left(B_i/A_i\right)\right]^2}{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(B_i/A_i\right)^2\right]\left(D_k/A_k\right)} + \frac{w_0^2}{A_0}}\end{aligned} \tag{47}$$

and

$$\sigma^{rf}\left(w_T\right) = \sqrt{\operatorname{var}_0\left(w_T\right)} = \frac{\left(E_0\left(w_T\right) - w_0 \prod_{k=0}^{T-1} s_k\right)}{\sqrt{\sum_{k=0}^{T-1} E\left(P_k'\right) \Omega_k^{-1} E\left(P_k\right)}}. \tag{48}$$

It is easy to find that the tangent line for the efficient frontier (47) passes through the point $(w_0\prod_{k=0}^{T-1} s_k, 0)$, and its slope is the best Sharpe ratio generated by only risky assets. Therefore, the best Sharpe ratio can be expressed as

$$BSR = \sqrt{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(\frac{B_i}{A_i}\right)^2\right] \frac{D_k}{A_k} + A_0\left[\prod_{i=0}^{T-1} \frac{B_i}{A_i} - \prod_{t=0}^{T-1} s_t\right]^2} \tag{49}$$

Similar to Chiu and Zhou [31] and Yao et al. [32], we can conclude the following theorem.

Theorem 8. *If $T \geq 2$, under the time-consistent strategy framework, we have the following conclusions.*

(a) The efficient frontier (12) with both risky and risk-free assets strictly dominates the efficient frontier (34) only with risky assets.

(b) The best Sharpe ratio with both risky and risk-free assets is strictly greater than that of only with risky assets, that is, $BSR_{rf} > BSR$.

Proof. The proof of Theorem 8 is shown in Appendix B. □

As shown in Theorem 8 and its proof, we can find, under the time-consistent strategy framework, that the probability of any efficient portfolio involving a risk-free asset at any time is strictly greater than zero in the multi-period mean-variance model with a risk-free asset. This indicates that the efficient frontier with a risk-free asset is strictly above that with only risky assets in the multi-period setting. In other words, the multi-period mean-variance model with a risk-free asset can achieve a strictly better Sharpe ratio than a model with only risky assets, which is different from the case of the classical static model (the efficient frontier in the single-period case is tangent to the risky region). Theorem 8 additionally shows that the availability of a risk-free asset can increase the Sharpe ratio of portfolios derived by the multi-period time-consistent strategy.

4. Further Discussions on the Time-Consistent Solutions for the Adjustment Model

In many applications, such as those with transaction costs, it is more convenient to take the control variable in Model (3) in the form of adjustments of investments. These types of models have been studied in Calafiore [26, 28].

We denote v_t^i as the value of the portion of the investor's total wealth invested in risky asset i at the beginning of time t. The vector of portfolio with component v_t^i can be expressed $v_t = [v_t^1, \ldots, v_t^n]'$, where $i = 1, \ldots, n$, $t = 0, 1, \ldots, T-1$. At the end of each period, the investor has the opportunity to adjust his investment by rebalancing the portfolio composition v_t. Let Δv_t^i be the adjustment value of ith risky asset and $\Delta v_t = [\Delta v_t^1, \ldots, \Delta v_t^n]'$ denote the control variables. Based on the self-financing condition, the adjustment value of risk-free asset at the time t can be denoted as $\Delta v_t^0 = -\mathbf{I}'\Delta v_t$. Then, we can construct the following dynamic wealth process under this control variable of adjustment value.

$$w_{t+1} = s_t w_t + P_t' \left(v_t + \Delta v_t\right), \quad t = 0, 1, \ldots, T-1 \tag{50}$$

where $P_t = [(e_t^1 - s_t), (e_t^2 - s_t), \ldots, (e_t^n - s_t)]'$.

According to (50), the following multi-period mean-variance portfolio adjustment model can be presented as

$$\begin{aligned} \max \quad & E\left(w_T\right) - \omega \operatorname{var}\left(w_T\right) \\ \text{s.t.} \quad & w_{t+1} = s_t w_t + P_t'\left(v_t + \Delta v_t\right), \\ & \qquad t = 0, 1, \ldots, T-1. \end{aligned} \tag{51}$$

Similarly, if we assume that the investment is only on the risky assets, we need add the condition $w_t = \mathbf{I}'(v_t + \Delta v_t)$ into Model (51).

Because $u_t = v_t + \Delta v_t$, where u_t is the control variable in Model (3) and Model (22). Thus, it is clear that the dynamic wealth processes for Model (3), Model (22), and the above adjustment model can be transformed by each other, respectively.

In the following, we will also discuss the time-consistent strategy for the adjustment model. According to the relationship among Model (3), Model (22), and the above adjustment model, we can readily derive the time-consistent strategy and efficient frontier, and the main conclusions are summarized as follows.

Theorem 9. *When a capital market has risky assets and one risk-free asset, the time-consistent strategy and efficient frontier for the adjustment model (51) can be expressed as*

$$\Delta \widehat{v}_t = \frac{\Omega_t^{-1} E\left(P_t\right)}{2\omega \prod_{i=t+1}^{T-1} s_i} - \widehat{v}_t, \quad t = 0, 1, \ldots, T-1 \tag{52}$$

where $\Delta \widehat{v}_t = [\Delta v_t^1, \ldots, \Delta v_t^n]'$, and

$$\operatorname{var}_0\left(w_T\right) = \frac{\left(E_0\left(w_T\right) - w_0 \prod_{k=0}^{T-1} s_k\right)^2}{\sum_{k=0}^{T-1} E\left(P_k'\right) \Omega_k^{-1} E\left(P_k\right)}. \tag{53}$$

Theorem 10. *The time-consistent strategy and efficient frontier for the problem (51) can be expressed as the following conclusions when a capital market has no risk-free asset.*

$$\begin{aligned} \Delta v_t = {} & \left(\frac{\widehat{\Omega}_t^{-1} \mathbf{I}\mathbf{I}'}{A_t} - \Sigma\right) v_t \\ & + \frac{1}{2\omega} \prod_{i=t+1}^{T-1} \frac{B_i}{A_i} \widehat{\Omega}_t^{-1} \left(E\left(e_t\right) - \frac{B_t \mathbf{I}}{A_t}\right), \\ & \qquad t = 0, 1, \ldots, T-1 \end{aligned} \tag{54}$$

and

$$\begin{aligned} & \operatorname{var}_0\left(w_T\right) \\ & = \frac{\left[E_0\left(w_T\right) - w_0 \prod_{i=0}^{T-1}\left(B_i/A_i\right)\right]^2}{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(B_i/A_i\right)^2\right]\left(D_k/A_k\right)} \left[E_0\left(w_T\right)\right. \\ & \quad \left. - \mathbf{I}' v_0 \prod_{i=0}^{T-1} \frac{B_i}{A_i}\right]^2 + \frac{v_0' \mathbf{I}\mathbf{I}' v_0}{A_0}, \end{aligned} \tag{55}$$

where Σ denotes $n \times n$ identity matrix, and the notions of $\widehat{\Omega}_t$, A_t, B_t, D_t are the same as that in Section 3.2.

5. Numerical Simulation and Empirical Analysis

For the purpose of comparing the relative performance of the above two models under the different investment strategies,

Table 1: Sharpe ratios for portfolios with one risk-free asset.

Periods	Sharpe ratios for different risk aversion parameters					
	risk aversion parameter $\omega = 0.1$		risk aversion parameter $\omega = 0.5$		risk aversion parameter $\omega = 2.5$	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
1	1.2091	1.2091	1.2091	1.2091	1.2091	1.2091
2	2.2497	1.7099	2.2497	1.7099	2.2497	1.7099
3	3.7313	2.0942	3.7313	2.0942	3.7313	2.0942
4	5.9781	2.4182	5.9781	2.4182	5.9781	2.4182
5	9.4576	2.7037	9.4576	2.7037	9.4576	2.7037
6	14.8888	2.9617	14.8888	2.9617	14.8888	2.9617
7	23.3926	3.1990	23.3926	3.1990	23.3926	3.1990
8	36.7243	3.4199	36.7243	3.4199	36.7243	3.4199
9	57.6353	3.6273	57.6353	3.6273	57.6353	3.6273
10	90.4412	3.8235	90.4412	3.8235	90.4412	3.8235

we mainly compare their Sharpe ratios and efficient frontiers. We assume that the investor has one unit of initial wealth ($w_0 = 1$). In this section, our simulation and empirical analysis have twofold contributions. The first contribution is to provide some simulations to check our conclusions presented in Section 3 and to compare efficient frontiers and Sharpe ratios of the pre-commitment and time-consistent strategies. Second, we provide an empirical analysis to compare the performances of portfolios under pre-commitment and time-consistent strategies, respectively. The results show that the time-consistent strategies almost have better performance than pre-commitment strategies in out-of-sample test.

5.1. Comparison of Pre-Commitment and Time-Consistent Strategies. In this section, we mainly aim to check the theoretical findings presented in Section 3, and quantify the difference between pre-commitment and time-consistent strategies. To make it easier to compare our results with those in Li and Ng [4], we adopt the data in examples 1 and 2 from Li and Ng [4]. Although there are some existing researches regarding the difference in efficient frontiers between the time-consistent and pre-commitment strategies, such as Lioui [33] and Wu [30], they did not quantify the differences. In this section, we will give some examples to illustrate the results using both efficient frontiers and Sharpe ratio. For a given risk aversion parameter ω, we can obtain the expectation and variance of the terminal wealth under different investment strategies. Based on the Sharpe ratio presented in Section 3.4, we have

$$\theta^{sharpe}(\omega) = \frac{E(w_T) - w_0 (r_f)^T}{\sqrt{\operatorname{var}(w_T)}}, \tag{56}$$

where r_f denotes the risk-free rate for each time period with a given return rate of 1.04.

According to the definition of Sharpe ratio, we can conclude that the portfolio with a large Sharpe ratio is more effective than the one with a small Sharpe ratio. In the following, we will consider both the capital market with and without the risk-free asset. In the first case, in addition to the following three risky assets, suppose there exists a risk-free asset with a given annual return rate of 1.04, where the expected annual return vectors and covariance matrices at different periods are given as

$$E(e_t) = [1.162, 1.246, 1.228], \quad t = 0, 1, \ldots, T-1,$$
$$\Omega_t = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}, \tag{57}$$
$$t = 0, 1, \ldots, T-1.$$

In the second case, we consider the market only with the above three risky assets.

According to the efficient frontiers shown in Section 3 and Li and Ng [4], we can compare the efficient frontiers under pre-commitment and time-consistent strategies, as shown in Figure 1. In addition, from the definition of the Sharpe ratio, we can obtain the Sharpe ratios for the different risk aversion parameters and investment strategies. The main results are displayed in Tables 1 and 2.

Figure 1 reports the efficient frontiers with different investment strategies. It is clear that the efficient frontiers under pre-commitment strategies are always higher than those of time-consistent strategies, no matter risk-free asset exists in capital pool or not. In addition, when the capital pool with one risk-free asset and risky assets, the gap between the efficient frontiers under pre-commitment strategy and time-consistent strategy is quite large even if the investment period are very small ($T = 2$), while the gap in the efficient frontiers of the pre-commitment strategy and time-consistent strategy is quite close when there is not a risk-free asset in capital pool, comparatively speaking. More importantly, we can find that the efficient frontiers with a risk-free asset are always higher than the efficient frontiers only with risky assets, no matter that the investor chooses the pre-commitment strategies or time-consistent strategies, which also check the conclusions shown in Section 3 and Yao et al. [32].

Table 2: Sharpe ratios for portfolios without risk-free assets.

Periods	risk aversion parameter $\omega = 0.1$		risk aversion parameter $\omega = 0.5$		risk aversion parameter $\omega = 2.5$	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
1	0.7748	0.7748	0.8863	0.8863	1.1771	1.1771
2	1.2205	1.0941	1.3671	1.2580	1.7512	1.6121
3	1.6684	1.3379	1.8304	1.5446	2.2484	1.8941
4	2.1470	1.5425	2.3095	1.7851	2.7094	2.0795
5	2.6596	1.7215	2.8091	1.9932	3.1401	2.1927
6	3.1932	1.8820	3.3176	2.1749	3.5318	2.2492
7	3.7215	2.0280	3.8104	2.3321	3.8677	2.2607
8	4.2112	2.1618	4.2576	2.4655	4.1310	2.2370
9	4.6329	2.2849	4.6335	2.5743	4.3122	2.1862
10	4.9703	2.3982	4.9256	2.6579	4.4145	2.1147

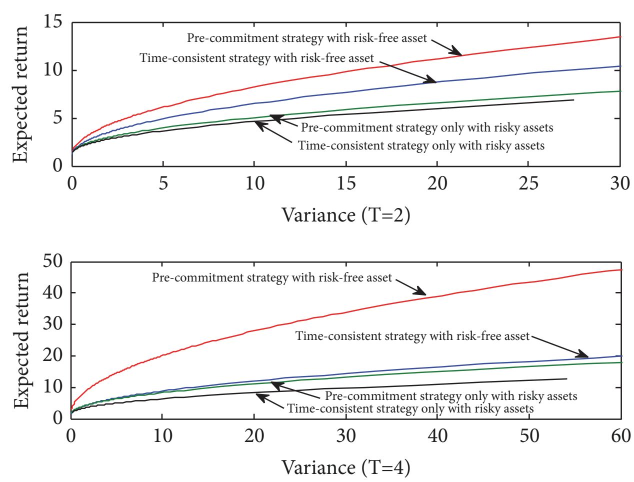

Figure 1: Comparison of frontiers under pre-commitment and time-consistent strategies.

As indicated in Tables 1 and 2, the Sharpe ratios under the time pre-commitment strategies are always higher than those of the time-consistent strategies for $T > 1$. This is because that the pre-commitment strategy achieves the global optimality from the initial point 0, while the time-consistent strategies only obtains sub-optimality since it insures the investment strategies with time consistency at the expense of global interests. For a fixed investment horizon, the Sharpe ratios are the same for different risk aversion parameters when the market with one risk-free asset exists (this conclusion can be directly derived by applying the efficient frontiers for the different investment strategies). In addition, for a fixed risk aversion parameter ω, we can find that the Sharpe ratios with both risk-free asset and risky assets are always larger than those only with risky assets, no matter for the pre-commitment strategies or time-consistent strategies. The simulation results in Tables 1 and 2 are also coincident with the theoretical findings in Section 3.

Although the pre-commitment strategies always perform better than the time-consistent strategies from a theoretical point of view, the differences of the above two investment strategies in real investment process are not clear. More importantly, for a real investor, he/she would like to use the re-estimation approach to update new information at each time period and then to reestimate the future return distributions. Based on the new information, then the investor can re-calculate the investment strategies in the rest of the investment periods. According to the re-estimation approach, we can find that the investor only adopt the previously calculated investment strategy to invest in the first stage; after the first stage, the investor will recalculate the investment policies according to the changes. In this case, the above two investment strategies are called as recalculated pre-commitment strategies and recalculated time-consistent strategies, respectively. In the following, we will compare the pre-commitment strategy (the recalculated pre-commitment strategy) and the time-consistent strategy (the recalculated time-consistent strategy) by using Out-of-sample test, and the main results are shown in Sections 5.2 and 5.3, respectively.

TABLE 3: Sharpe ratios of portfolios under different strategies (T=2, M=120).

Sharpe ratios for different risk aversion parameters ω				
ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	0.1053	0.2810	0.1243	0.1465
0.5	0.1053	0.2810	0.1302	0.1524
1.0	0.1053	0.2810	0.1308	0.1530
1.5	0.1053	0.2810	0.1310	0.1531
2.0	0.1053	0.2810	0.1311	0.1532

TABLE 4: Sharpe ratios of portfolios under different strategies (T=4, M=120).

Sharpe ratios for different risk aversion parameters ω				
ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	0.0000	0.4183	0.1226	0.1451
0.5	0.0000	0.4183	0.1221	0.1448
1.0	0.0000	0.4183	0.1221	0.1447
1.5	0.0000	0.4183	0.1221	0.1447
2.0	0.0000	0.4183	0.1221	0.1447

TABLE 5: Sharpe ratios of portfolios under different strategies (T=8, M=120).

Sharpe ratios for different risk aversion parameters ω				
ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	-0.0712	0.5542	-0.1034	-0.1581
0.5	-0.0712	0.5542	-0.1034	-0.1581
1.0	-0.0712	0.5542	-0.1034	-0.1581
1.5	-0.0712	0.5542	-0.1034	-0.1581
2.0	-0.0712	0.5542	-0.1034	-0.1581

5.2. Out-of-Sample Evaluation of the Pre-Commitment and Time-Consistent Strategies. In the section, we assume that the investor will strictly adhere to the pre-committed strategy and time-consistent strategy without recalculating the investment strategies. In this case, the investor does not need to update the market parameters, and only need to estimate the expected return vectors and covariance matrices of risky assets at the initial time. To further discuss the difference between the pre-commitment strategies and time-consistent strategies, we assume the expected return vectors and covariance matrices at different periods with the same values. We randomly choose 12 industry portfolios in United States from July 1976 to September 2016, which is downloaded from Kenneth French's web site (http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html). The 12 industries considered are Food, Beer, Smoke, Games, Books, Steel, Carry, Coal, Oil, Paper, Finance, and Other.

In addition, we arbitrarily regard the monthly return of risk-free asset as $s_t = 1.0003$. Motivated by Lan [34], we assume that there is a group of investors who have the same risk preference except that they start their investments at different periods. In the section, we assume that the length of the sample returns is N. Also, we suppose that in the investment period T and in each of the T months, there is one investor who starts a T-period investment. Similar to DeMiguel et al. [25], we adopt "rolling-sample" approach to evaluate the above two strategies and choose an estimation window of length M. In the following, we provide some portfolio performance indexes to evaluate the investment strategies presented in this paper, such as Sharpe ratio, Portfolio turnover, and Maximum drawdown.

5.2.1. Sharpe Ratios. In this section, we will assess the performance of the pre-commitment strategy and time-consistent strategy by using Sharpe ratio index. Table 3 contain the results for Sharpe ratios of portfolios under different strategies when $T = 2$, $\omega = 0.1, 0.5, 1.0, 1.5, 2.0$ and $M = 120$. It is fact that the time-consistent strategies almost have higher Sharpe ratios than pre-commitment strategies in out-of-sample test, no matter that the risk-free assets are considered into capital pool or not. In this case, the time-consistent strategy might be a good choice for the investor in the real financial market.

Additionally, we will test the robustness for our conclusions when T take different values. Let $T = 4, 8$, then the empirical results can be derived similarly. The detailed results are shown in Tables 4 and 5.

Table 6: Portfolio turnovers of portfolios under different strategies (T=2, M=120).

ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	3.1412	1.3826	8.5531	7.4077
0.5	0.6282	0.2765	8.4973	7.3446
1.0	0.3141	0.1383	8.5051	7.3579
1.5	0.2094	0.0922	8.5083	7.3635
2.0	0.1571	0.0691	8.5101	7.3665

Portfolio turnovers for different risk aversion parameters ω

Table 7: Portfolio turnovers of portfolios under different strategies (T=4, M=120).

ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	9.3703	1.3880	116.5914	99.1490
0.5	1.8741	0.2776	116.5625	99.2885
1.0	0.9370	0.1388	116.5610	99.3106
1.5	0.6247	0.0925	116.5607	99.3183
2.0	0.4685	0.0694	116.5605	99.3222

Portfolio turnovers for different risk aversion parameters ω

Table 8: Portfolio turnovers of portfolios under different strategies (T=8, M=120).

ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	70.4774	1.3730	3.8704 e+04	3.6401 e+04
0.5	14.0955	0.2746	3.8703 e+04	3.6401 e+04
1.0	7.0477	0.1373	3.8703 e+04	3.6401 e+04
1.5	4.6985	0.0915	3.8703 e+04	3.6401 e+04
2.0	3.5239	0.0687	3.8703 e+04	3.6401 e+04

Portfolio turnovers for different risk aversion parameters ω

As shown in Tables 4 and 5, the Sharpe ratios under the time-consistent strategy are almost better than that of the pre-commitment strategy. Also, it is coincident with the conclusion shown in Tables 3–6. This indicates that the time-consistent strategy might be a more suitable investment strategy for investors under Sharpe ratio evaluation framework.

5.2.2. Portfolio Turnovers. In this section, we will compare the performance of the above two investment strategies under Portfolio turnover framework. Let $M = 120$, $T = 2, 4, 8$ and $\omega = 0.1, 0.5, 1.0, 1.5, 2.0$, according to the definition of Portfolio turnover shown in DeMiguel et al. [25], then the empirical results can be obtained similarly. These detailed results are shown in Tables 6–8.

As shown in Tables 6–8, the Portfolio turnovers under the time-consistent strategy are almost smaller than that of the pre-commitment strategy. As we all know, more turnover mean the investor should pay more transaction cost in the actual investment process. This indicates that the time-consistent strategy might be a more perfect investment strategy for investors under Portfolio turnover framework.

5.2.3. Maximum Drawdowns. In this section, we will compare the performance of the above two investment strategies under Maximum drawdown framework. Maximum drawdown up to time T is the maximum of the drawdown over the history of the variable. According to the definition of Maximum drawdown, we can find that the larger Maximum drawdown mean the investor will face higher risk in the actual investment process. Let M = 120, T = 2, 4, 8 and $\omega = 0.1, 0.5, 1.0, 1.5, 2.0$, then the empirical results can be obtained similarly. These detailed results are shown in Tables 9–11.

As shown in Tables 9–11, the Maximum drawdowns under the time-consistent strategy are almost smaller than that of the pre-commitment strategy. It means that the investor will undertake less loss by using time-consistent strategy compared with pre-commitment strategy. This also indicates that the time-consistent strategy might be a more suitable investment strategy for investors under Maximum drawdowns evaluation framework.

5.3. Out-of-Sample Evaluation of the Recalculated Pre-Commitment and Recalculated Time-Consistent Strategies. In this

Table 9: Maximum drawdowns of portfolios under different strategies (T=2, M=120).

Maximum drawdowns for different risk aversion parameters ω				
ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	70.6723	30.6784	179.6937	165.7188
0.5	14.1345	6.1357	172.0585	158.6346
1.0	7.0672	3.0678	171.2166	157.7491
1.5	4.7115	2.0452	171.1129	157.4540
2.0	3.5336	1.5339	171.0611	157.3064

Table 10: Maximum drawdowns of portfolios under different strategies (T=4, M=120).

Maximum drawdowns for different risk aversion parameters ω				
ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	410.2703	30.5253	7.8974 e+03	7.0880 e+03
0.5	82.0541	6.1051	7.8906 e+03	7.0876 e+03
1.0	41.0270	3.0525	7.8898 e+03	7.0875 e+03
1.5	27.3514	2.0350	7.8895 e+03	7.0875 e+03
2.0	20.5135	1.5263	7.8893 e+03	7.0875 e+03

Table 11: Maximum drawdowns of portfolios under different strategies (T=8, M=120).

Maximum drawdowns for different risk aversion parameters ω				
ω	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Pre-commitment	Time-consistent	Pre-commitment	Time-consistent
0.1	1.6876 e+03	34.3380	1.7671 e+06	2.5050 e+06
0.5	0.3375 e+03	6.8676	1.7671 e+06	2.5050 e+06
1.0	0.1688 e+03	3.4338	1.7671 e+06	2.5050 e+06
1.5	0.1125 e+03	2.2892	1.7671 e+06	2.5050 e+06
2.0	0.0844 e+03	1.7169	1.7671 e+06	2.5050 e+06

section, we assume that these investors can use the re-estimation approach to update new information that appears over the investment horizon and then to recalculate the corresponding investment strategies. Similar to Section 5.2, we also adopt "rolling-sample" approach to evaluate the above two strategies and choose an estimation window of length M.

In fact, before carrying out the T-period investment, these investors can only re-estimate the expected return and covariance matrix for the first period investment by "rolling-sample" approach. Based on the information at time $t = M$, these investors still need to forecast the expected return and covariance matrix of the return of risky assets at the later period time. In this section, we adopt a simulation-based approach to forecast the expected return and covariance matrix at the second period. The main procedures are as follows:

(1) Starting from $t = M$, the return data in the previous M months are used to estimate the parameters for the first period, such as expected return and covariance matrix of risky assets;

(2) Following, we should estimate the expected return and covariance matrix of risky assets at the later investment periods. When these investors stand the first investment period time, they can not obverse the realized return of the risky assets at this time. However, if we want to estimate the expected return and covariance matrix of risky assets at the second investment period time, the return of the risky assets at the first period time should be forecasted. Based on the expected return and covariance matrix estimated in (1), we can simulate the return of risky assets will appear in first investment period by using Monte Carlo approach, we add the above simulated return data into the previous M data and drop the earliest return in the dataset, and then we can forecast the parameters at the later investment periods.

(3) When finishing the first period investment, we can reestimate the expected return and covariance matrix of risky assets for the second period by using the updated information, and then recalculate the investment strategies for the second investment.

(4) Similarly, when $t = M + 2$, repeating step (3), then we can obtain a T-month investment path. This process is continued by adding the return of the next period and dropping the earliest return in the dataset, until $t = N - 1$.

Table 12: Mean of Sharpe ratios of portfolios under different strategies (T=2, M=120).

ω	Mean of Sharpe ratios for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	0.2311	0.2314	0.1336	0.1339
0.5	0.2311	0.2314	0.1159	0.1165
1.0	0.2311	0.2314	0.1136	0.1142
1.5	0.2311	0.2314	0.1128	0.1135
2.0	0.2311	0.2314	0.1125	0.1131

Table 13: Mean of Sharpe ratios of portfolios under different strategies (T=4, M=120).

ω	Mean of Sharpe ratios for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	0.3953	0.4346	0.1865	0.1867
0.5	0.3953	0.4346	0.1847	0.1848
1.0	0.3953	0.4346	0.1844	0.1846
1.5	0.3953	0.4346	0.1843	0.1846
2.0	0.3953	0.4346	0.1843	0.1846

Table 14: Mean of Sharpe ratios of portfolios under different strategies (T=8, M=120).

ω	Mean of Sharpe ratios for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	0.3579	0.6138	0.0569	0.0570
0.5	0.3579	0.6138	0.0568	0.0568
1.0	0.3579	0.6138	0.0568	0.0568
1.5	0.3579	0.6138	0.0568	0.0568
2.0	0.3578	0.6138	0.0569	0.0570

Table 15: Mean of Portfolio turnovers of portfolios under different strategies (T=2, M=120).

ω	Mean of Portfolio turnovers for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	1.4693	1.1340	5.3115	5.3079
0.5	0.2939	0.2268	5.1604	5.1539
1.0	0.1469	0.1134	5.1523	5.1457
1.5	0.0980	0.0756	5.1505	5.1439
2.0	0.0735	0.0567	5.1499	5.1432

(5) Based on the above T-month investment paths, then we calculate the Sharpe ratios for the different investment strategies.

(6) Repeating the above procedures H times, we can calculate the mean of the above H Sharpe ratios. It is defined as the out-of-sample mean of Sharpe ratios here.

From the above real investment paths, we can obtain the out-of-sample mean of Sharpe ratios, Portfolio turnovers and Maximum drawdowns of different investment strategies under the different risk aversion parameters ω and estimation window of length M. The main results are presented in the following Tables 12–20.

5.3.1. Sharpe Ratios. Assuming that $H = 1000$, $M = 120$, $T = 2, 4, 8$ and $\omega = 0.1, 0.5, 1.0, 1.5, 2.0$, we can obtain the empirical results under Sharpe ratio evaluation framework. Tables 12–14 report the empirical results when investors adopt recalculation approach to invest under different situations.

As shown in Tables 12–14, the Sharpe ratios under the recalculated time-consistent strategy are almost larger than that of the recalculated pre-commitment strategy. This indicates that the recalculated time-consistent strategy might be a more suitable investment strategy for investors under Sharpe ratios evaluation framework.

Table 16: Mean of Portfolio turnovers of portfolios under different strategies (T=4, M=120).

ω	Mean of Portfolio turnovers for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	4.2591	1.8665	90.8734	90.7543
0.5	0.8518	0.3733	90.7378	90.6176
1.0	0.4259	0.1866	90.7278	90.6074
1.5	0.2839	0.1244	90.7251	90.6054
2.0	0.2130	0.0933	90.7231	90.6027

Table 17: Mean of Portfolio turnovers of portfolios under different strategies (T=8, M=120).

ω	Mean of Portfolio turnovers for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	13.4218	2.4709	2.1103 e+04	2.1029e+04
0.5	2.6839	0.4942	2.1103 e+04	2.1028e+04
1.0	1.3421	0.2471	2.1103 e+04	2.1028e+04
1.5	0.8949	0.1647	2.1103 e+04	2.1028e+04
2.0	0.6710	0.1235	2.1103 e+04	2.1028e+04

Table 18: Mean of Maximum drawdowns of portfolios under different strategies (T=2, M=120).

ω	Mean of Maximum drawdowns for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	22.2764	20.6553	101.2131	101.1972
0.5	4.4553	4.1311	98.1210	98.0921
1.0	2.2277	2.0655	97.7348	97.7046
1.5	1.4851	1.3770	97.6057	97.5745
2.0	1.1138	1.0328	97.5413	97.5102

Table 19: Mean of Maximum drawdowns of portfolios under different strategies (T=4, M=120).

ω	Mean of Maximum drawdowns for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	48.3886	25.2976	3.2589 e+03	3.2748 e+03
0.5	9.6794	5.0595	3.2597 e+03	3.2756 e+03
1.0	4.8412	2.5298	3.2601 e+03	3.2754 e+03
1.5	3.2260	1.6865	3.2600 e+03	3.2757 e+03
2.0	2.4189	1.2649	3.2600 e+03	3.2756 e+03

5.3.2. Portfolio Turnovers. Similarly, assuming that $H = 1000$, $M = 120$, $\mathrm{T} = 2, 4, 8$ and $\omega = 0.1, 0.5, 1.0, 1.5, 2.0$, we can also obtain the empirical results under Portfolio turnover evaluation framework. Tables 15–17 report the empirical results when investors adopt recalculation approach to invest under different situations.

As shown in Tables 15–17, the Portfolio turnovers under the recalculated time-consistent strategy are almost smaller than that of the recalculated pre-commitment strategy. It is interesting that the above conclusions derived in this situation are consistent with the findings of Tables 12–14, even then the recalculated time-consistent strategies are not always better than recalculated pre-commitment strategies. This indicates that the time-consistent strategy might be a more suitable investment strategy for investors under Portfolio turnovers framework.

5.3.3. Maximum Drawdowns. Assuming that $H = 1000$, $M = 120$, $T = 2, 4, 8$ and $\omega = 0.1, 0.5, 1.0, 1.5, 2.0$, we can similarly obtain the empirical results. Tables 18–20 report the empirical results when investors adopt recalculation approach to invest under different situations.

Table 20: Mean of Maximum drawdowns of portfolios under different strategies (T=8, M=120).

ω	Mean of Maximum drawdowns for different risk aversion parameters ω			
	Portfolio with risk-free assets		Portfolio without risk-free assets	
	Recalculated pre-commitment	Recalculated time-consistent	Recalculated pre-commitment	Recalculated time-consistent
0.1	164.1320	36.0936	6.8385 e+05	6.8196 e+05
0.5	32.8023	7.2187	6.8384 e+05	6.8194 e+05
1.0	16.4079	3.6094	6.8383 e+05	6.8192 e+05
1.5	10.9409	2.4062	6.8383 e+05	6.8192 e+05
2.0	8.2069	1.8047	6.8383 e+05	6.8193 e+05

Tables 18–20 report the empirical results when investors adopt recalculation approach to invest under different situations. It is interesting that the recalculated time-consistent strategies always have higher mean of Maximum drawdowns than recalculated pre-commitment strategies in out-of-sample test, no matter that the risk-free assets are considered into capital pool or not. In other words, the recalculated time-consistent strategy is more dominant than the recalculated pre-commitment strategy. This indicates that the recalculated time-consistent strategy might be a good choice for the investor in the real financial market.

6. Conclusion

To answer the important question: what are the main practical gains of using the time-consistent investment strategy and when would it be advantageous to use it instead of the pre-commitment strategy? To achieve this target, we prove that the time-consistent strategy is also the optimal open-loop strategy for the classic model when there is one risk-free asset. We first derive the explicit time-consistent solution for the classic model in the case only with risky assets. In addition, we prove that the Sharpe ratio with both risky and risk-free assets strictly dominates that of only risky assets under the time-consistent strategy setting. Finally, extensive numerical simulations and empirical studies are given to compare the pre-commitment and time consistent strategies. This indicates that the investor should adopt the time-consistent strategies, since the investor needs to recalculate the investment strategy based on the updated market information in actual applications.

In this paper, we only limit to discuss the performance of pre-commitment and time-consistent strategies, aiming to answer the question that for the pre-commitment and time-consistent strategies which one is better for a real investor, although there are some other simple dynamic investment strategies in the literatures. For future research, considering the market is always with perturbation, we can generalize our results under the robust optimization framework and consider more existing dynamic investment strategies for comparison, for example, the 1/N strategy and the myopic strategy which is derived by optimizing a one-period investment problem at each rebalancing period. In addition, we assume the investor with a constant risk aversion parameter in our model. In order to have a more realistic model we can investigate the case when the risk aversion depends dynamically on current wealth, then a state-dependent time-consistent strategy will be provided for real investor.

Appendix

A. The proof of Theorem 5

For the given $t = T-1$ and a fixed wealth w_{T-1}, we will obtain the optimal strategy $\widehat{u}_{T-1}$ by maximizing the objection function $E_{T-1}(w_T) - \omega \mathrm{var}_{T-1}(w_T)$. Based on the wealth process $w_T = e'_{T-1}u_{T-1}$ and the self-financing condition $\mathbf{I}'u_{T-1} = w_{T-1}$, we can construct the following Lagrange function:

$$\begin{aligned} L_{T-1}(u_{T-1}) = {} & E\left(e'_{T-1}\right)u_{T-1} - \omega u'_{T-1}\Omega_{T-1}u_{T-1} \\ & - \lambda_{T-1}\left(\mathbf{I}'u_{T-1} - w_{T-1}\right), \end{aligned} \tag{A.1}$$

where λ_{T-1} is the Lagrange multiplier. By the first-order necessary optimality condition, we have the following equations:

$$\begin{aligned} E\left(e_{T-1}\right) - 2\omega\Omega_{T-1}u_{T-1} - \lambda_{T-1}\mathbf{I} &= 0 \\ \mathbf{I}'u_{T-1} &= w_{T-1}. \end{aligned} \tag{A.2}$$

Then, we have

$$\lambda_{T-1} = \frac{B_{T-1} - 2\omega w_{T-1}}{A_{T-1}}, \tag{A.3}$$

$$\widehat{u}_{T-1} = a_{T-1}w_{T-1} + b_{T-1}, \tag{A.4}$$

where

$$\begin{aligned} a_{T-1} &= \frac{\Omega_{T-1}^{-1}\mathbf{I}}{A_{T-1}}, \\ b_{T-1} &= \frac{1}{2\omega}\Omega_{T-1}^{-1}\left(E\left(e_{T-1}\right) - \frac{B_{T-1}\mathbf{I}}{A_{T-1}}\right). \end{aligned} \tag{A.5}$$

Based on the optimal decision at the beginning of the $(T-1)$th period, we can obtain the conditional expectation and variance of the terminal wealth, and the main results are the following:

$$\begin{aligned} E_{T-1}(\widehat{w}_T) &= E(e'_{T-1})\widehat{u}_{T-1} \\ &= E(e'_{T-1})a_{T-1}w_{T-1} + E(e'_{T-1})b_{T-1} \\ &= m_{T-1}w_{T-1} + n_{T-1} \end{aligned} \quad (A.6)$$

$$\begin{aligned} \mathrm{var}_{T-1}(\widehat{w}_T) &= \widehat{u}'_{T-1}\Omega_{T-1}\widehat{u}_{T-1} \\ &= a'_{T-1}\Omega_{T-1}a_{T-1}w^2_{T-1} \\ &\quad + 2a'_{T-1}\Omega_{T-1}b_{T-1}w_{T-1} \\ &\quad + b'_{T-1}\Omega_{T-1}b_{T-1} \\ &= \alpha_{T-1}w^2_{T-1} + \beta_{T-1}w_{T-1} + \gamma_{T-1} \end{aligned} \quad (A.7)$$

where the boundary conditions are as follows:

$$\begin{aligned} m_{T-1} &= \frac{B_{T-1}}{A_{T-1}}, \\ n_{T-1} &= \frac{D_{T-1}}{2\omega A_{T-1}}, \end{aligned} \quad (A.8)$$

$$\begin{aligned} \alpha_{T-1} &= \frac{1}{A_{T-1}}, \\ \beta_{T-1} &= 0, \\ \gamma_{T-1} &= \frac{D_{T-1}}{4\omega^2 A_{T-1}}. \end{aligned} \quad (A.9)$$

Apparently, Theorem 5 holds for $t = T - 1$.

Now, we will prove that Theorem 5 is true for $t = 0, 1, \ldots, T-2$ by using the mathematical induction approach.

In the following, we assume that Theorem 5 holds for $t = k + 1$, where $k < T - 2$. For the given $t = k$, initial wealth w_k and the time-consistent strategy $\widehat{u}_i$, $i = k + 1, \ldots, T - 1$. Then, we can derive the time-consistent investment strategy $\widehat{u}_k$ by maximizing the objective function $E_k(w_T) - \omega\mathrm{var}_k(w_T) = E_k[V_{k+1}(w_{k+1})] - \omega\mathrm{var}_k[E_{k+1}(\widehat{w}_T)]$. Similarly, we can construct the following Lagrange function:

$$\begin{aligned} L_k(u_k) &= E_k[V_{k+1}(w_{k+1})] - \omega\mathrm{var}_k[E_{k+1}(\widehat{w}_T)] \\ &\quad - \lambda_k(\mathbf{I}'u_k - w_k) = m_{k+1}E_k(w_{k+1}) + n_{k+1} \\ &\quad - \omega[\alpha_{k+1}E_k(w^2_{k+1}) + \beta_{k+1}E_k(w_{k+1}) + \gamma_{k+1} \\ &\quad + m^2_{k+1}\mathrm{var}_k(w_{k+1})] - \lambda_k(\mathbf{I}'u_k - w_k) \\ &= m_{k+1}u'_kE(e_k) + n_{k+1} - \omega[\alpha_{k+1}u'_kE(e_ke'_k)u_k \\ &\quad + \beta_{k+1}u_kE(e_k) + \gamma_{k+1} + m^2_{k+1}u'_k\Omega_{T-2}u_k] \\ &\quad - \lambda_k(\mathbf{I}'u_k - w_k), \end{aligned} \quad (A.10)$$

Where λ_k is the Lagrange multiplier at time period $t = k$. Finally, we can obtain the following:

$$\widehat{u}_k = a_kw_k + b_k, \quad (A.11)$$

$$E_k(\widehat{w}_T) = m_kw_k + n_k, \quad (A.12)$$

$$\mathrm{var}_k(\widehat{w}_T) = \alpha_kw^2_k + \beta_kw_k + \gamma_k, \quad (A.13)$$

where

$$\begin{aligned} a_k &= \frac{\widehat{\Omega}^{-1}_k\mathbf{I}}{A_k}, \\ b_k &= \frac{m_{k+1}}{2\omega}\widehat{\Omega}^{-1}_k\left(E(e_k) - \frac{B_k\mathbf{I}}{A_k}\right), \end{aligned} \quad (A.14)$$

$$\begin{aligned} m_k &= \prod_{i=k}^{T-1}\frac{B_i}{A_i}, \\ n_k &= \frac{m^2_{k+1}D_k}{2\omega A_k} + n_{k+1}, \end{aligned} \quad (A.15)$$

$$\begin{aligned} \alpha_k &= \frac{1}{A_k}, \\ \beta_k &= 0, \\ \gamma_k &= \gamma_{k+1} + \frac{m^2_{k+1}D_k}{4\omega^2A_k}. \end{aligned} \quad (A.16)$$

Therefore, Theorem 5 holds for $t = k$. From the above discussion, we know that the time-consistent strategy, conditional expectation and variance of terminal wealth are given by (27), (28) and (29), respectively.

B. The proof of Theorem 8

In the following, we will prove Theorem 8 from the following two cases.

For **Case 1**: $T \geq 2$, $\prod_{t=0}^{T-1}s_t - \prod_{i=0}^{T-1}(B_i/A_i) \geq 0$.

For convenience, let $\breve{A}_t = \mathbf{I}'\Omega^{-1}_t\mathbf{I}$, $\breve{B}_t = \mathbf{I}'\Omega^{-1}_tE(e_t)$, $\breve{C}_t = E(e'_t)\Omega^{-1}_tE(e_t)$, $\breve{D}_t = \breve{A}_t\breve{C}_t - \breve{B}^2_t$ and $l_{t+1} = \alpha_{t+1}/m^2_{t+1}$ hereafter, where $t = 0, 1, \ldots, T-2$. Similar to Yao et al. [32], we can easily prove that $\breve{A}_t > 0$, $\breve{C}_t > 0$ and $\breve{D}_t > 0$. In addition, let $l_T = 0$, then we have

$$\begin{aligned} \varphi_0 &= \sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(\frac{B_i}{A_i}\right)^2\right]\frac{D_k}{A_k} = \sum_{k=0}^{T-1}\frac{m^2_{k+1}D_k}{A_k} \\ &= \sum_{k=0}^{T-1}\frac{\breve{D}_k}{\breve{A}_k + l_{k+1}(\breve{A}_k + \breve{D}_k)} > 0. \end{aligned} \quad (B.1)$$

Due to the fact that

$$\sum_{k=0}^{T-1} E\left(P_k'\right)\Omega_k^{-1}E\left(P_k\right) = \sum_{k=0}^{T-1}\left(\breve{C}_k + \breve{A}_k r_k^2 - 2r_k\breve{B}_k\right) = \sum_{k=0}^{T-1}\frac{\breve{D}_k + \left(\breve{A}_k r_k - \breve{B}_k\right)^2}{\breve{A}_k} \geq \sum_{k=0}^{T-1}\frac{\breve{D}_k}{\breve{A}_k} > 0 \quad \text{(B.2)}$$

and $l_{t+1} = \alpha_{t+1}/m_{t+1}^2 > 0$ for $t = 0, 1, \ldots, T-2$, then we have

$$\begin{aligned}\left(BSR_{rf}\right)^2 - (BSR)^2 &= \sum_{k=0}^{T-1}\frac{\breve{D}_k + \left(\breve{A}_k r_k - \breve{B}_k\right)^2}{\breve{A}_k} - \sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}\left(\frac{B_i}{A_i}\right)^2\right]\frac{D_k}{A_k} \\ &= \sum_{k=0}^{T-1}\frac{\breve{D}_k + \left(\breve{A}_k r_k - \breve{B}_k\right)^2}{\breve{A}_k} - \sum_{k=0}^{T-1}\frac{\breve{D}_k}{\breve{A}_k + l_{k+1}\left(\breve{A}_k + \breve{D}_k\right)} \\ &> \sum_{k=0}^{T-1}\frac{\breve{D}_k + \left(\breve{A}_k r_k - \breve{B}_k\right)^2}{\breve{A}_k} - \sum_{k=0}^{T-1}\frac{\breve{D}_k}{\breve{A}_k} = \sum_{k=0}^{T-1}\frac{\left(\breve{A}_k r_k - \breve{B}_k\right)^2}{\breve{A}_k} \\ &\geq 0.\end{aligned} \quad \text{(B.3)}$$

Thus, we can obtain that $BSR_{rf} > BSR$ in this case.

In addition, we denote $\operatorname{var}_0^{rf}(w_T)$ and $\operatorname{var}_0^{r}(w_T)$ as the variances of terminal wealth for the portfolio with and without risk-free asset under the assumption that $E_0(w_T) \geq \prod_{k=0}^{T-1} s_k$, respectively. Then we can conclude that

$$\begin{aligned}\operatorname{var}_0^{rf}(w_T) - \operatorname{var}_0^{r}(w_T) &= \frac{\left(E_0(w_T) - w_0\prod_{k=0}^{T-1} r_k^0\right)^2}{\sum_{k=0}^{T-1} E\left(P_k'\right)\Omega_k^{-1}E\left(P_k\right)} - \frac{\left[E_0(w_T) - w_0\prod_{i=0}^{T-1}(B_i/A_i)\right]^2}{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}(B_i/A_i)^2\right](D_k/A_k)} - \frac{w_0^2}{A_0} \\ &= \frac{\left(2E_0(w_T) - w_0\prod_{k=0}^{T-1} r_k^0 - w_0\prod_{i=0}^{T-1}(B_i/A_i)\right)\left(w_0\prod_{i=0}^{T-1}(B_i/A_i) - w_0\prod_{k=0}^{T-1} r_k^0\right)}{\sum_{k=0}^{T-1}\left[\prod_{i=k+1}^{T-1}(B_i/A_i)^2\right](D_k/A_k)} - \frac{w_0^2}{A_0} < 0.\end{aligned} \quad \text{(B.4)}$$

According to (B.4), we can easily derive that the efficient frontier (12) with both risky and risk-free assets strictly dominates the efficient frontier (34) only with risky assets. Therefore, Theorem 8 is proved in this case.

For **Case 2**: $T \geq 2$, $\prod_{t=0}^{T-1} s_t - \prod_{i=0}^{T-1}(B_i/A_i) < 0$.

Since the parameters in efficient frontier (34) are semi-analytical formulas, it is difficult to directly compare the differences between the above two efficient frontiers in this case. Motivated by Theorem 4.5 in Chiu and Zhou [31], to prove the conclusion (a) in Theorem 8, we only need to prove that the following conclusion (B.5) holds for the time-consistent strategy $\widehat{u}_t$ derived by Model (3) when the terminal wealth satisfies $E_0(w_T) \geq w_0\prod_{t=0}^{T-1} s_t$ setting.

$$P\left\{\widehat{u}_t^0 \neq 0\right\} > 0, \quad \text{holds for } \forall t \in \{1, \ldots, T-1\}, \quad \text{(B.5)}$$

where $\widehat{u}_t^0$ denotes the amount invested in risk-free asset at period t ($t = 0, 1, \ldots, T$).

If $E_0(w_T) = w_0\prod_{t=0}^{T-1} s_t$, then (B.5) is obviously true. In the following, we only need to prove that (B.5) holds for the assumption $E_0(w_T) > w_0\prod_{t=0}^{T-1} s_t$. According to Lemma 3, for each investment period time $t \in \{1, \ldots, T-1\}$, the time-consistent strategy $\widehat{u}_t$ can be expressed as $\widehat{u}_t = \Omega_t^{-1}E(P_t)/(2\omega\prod_{i=t+1}^{T-1} s_i)$. Note that we will prove (B.5) by using the proof by contradiction.

Assume that (B.6) is not true, then there exists $\tau \in \{1, \ldots, T-1\}$, such that $P\{\widehat{u}_\tau^0 = 0\} = 1$. Then we have that $w_\tau = \sum_{i=1}^n \widehat{u}_\tau^i = I'\Omega_\tau^{-1}E(P_\tau)/(2\omega\prod_{i=\tau+1}^{T-1} s_i)$, where w_τ is a deterministic value in this case. According to (2), we also have that $w_\tau = (\prod_{i=0}^{\tau-1} s_i)w_0 + \sum_{k=0}^{\tau-1}(\prod_{i=k+1}^{\tau-1} s_i)P_k'\widehat{u}_k$, and the following equation can be derived.

$$\left(\prod_{i=0}^{\tau-1} s_i\right)w_0 + \sum_{k=0}^{\tau-1}\left(\prod_{i=k+1}^{\tau-1} s_i\right)P_k'\widehat{u}_k = \frac{I'\Omega_\tau^{-1}E\left(P_\tau\right)}{\left(2\omega\prod_{i=\tau+1}^{T-1} s_i\right)}. \quad \text{(B.6)}$$

As shown in (B.6), we can easily find that the left of (B.6) is a random variable on the excess returns, while the right of (B.6) is a deterministic value. So if (B.6) is true, we can obtain that $\widehat{u}_k = 0$ a.s. for $k = 0, 1, \tau - 1$, which contradicts with the time-consistent strategy $\widehat{u}_k$ shown in Lemma 3. Therefore, (B.5) is proved, and the conclusion (a) in Theorem 8 is true in this case.

Due to the fact that the efficient frontier (12) strictly dominates the efficient frontier (34), for the given expected return level $E_0(w_T)$, we have

$$\begin{aligned}\operatorname{var}_0^{rf}(w_T) - \operatorname{var}_0^{r}(w_T) &= \frac{\left(E_0(w_T) - w_0\prod_{k=0}^{T-1} s_k\right)^2}{\sum_{k=0}^{T-1} E(P_k')\,\Omega_k^{-1} E(P_k)} \\ &\quad - \frac{\left[E_0(w_T) - w_0\prod_{i=0}^{T-1}(B_i/A_i)\right]^2}{\varphi_0} - \frac{w_0^2}{A_0} < 0.\end{aligned} \tag{B.7}$$

Then, we obtain that

$$\begin{aligned}&\varphi_0 A_0\left(E(w_T) - w_0\prod_{k=0}^{T-1} s_k\right)^2 \\ &\quad - \left[\sum_{k=0}^{T-1} E(P_k')\,\Omega_k^{-1}E(P_k)\right] A_0 \left[E(w_T) - w_0\prod_{i=0}^{T-1}\frac{B_i}{A_i}\right]^2 \\ &\quad - w_0^2\left[\sum_{k=0}^{T-1} E(P_k')\,\Omega_k^{-1}E(P_k)\right]\varphi_0 < 0 \Longrightarrow \\ &\frac{\left(E(w_T) - w_0\prod_{k=0}^{T-1}s_k\right)^2}{\operatorname{var}_0^r(w_T)} < \frac{w_0^2\sum_{k=0}^{T-1}E(P_k')\,\Omega_k^{-1}E(P_k)}{A_0\operatorname{var}_0^r(w_T)} \\ &\quad + \frac{\left[\sum_{k=0}^{T-1}E(P_k')\,\Omega_k^{-1}E(P_k)\right]\left[E(w_T) - w_0\prod_{i=0}^{T-1}(B_i/A_i)\right]^2}{\varphi_0\operatorname{var}_0^r(w_T)} \Longrightarrow \\ &\frac{\left(E(w_T) - w_0\prod_{k=0}^{T-1}s_k\right)^2}{\operatorname{var}_0^r(w_T)} < \frac{w_0^2\sum_{k=0}^{T-1}E(P_k')\,\Omega_k^{-1}E(P_k)}{A_0\operatorname{var}_0^r(w_T)} \\ &\quad + \left[\sum_{k=0}^{T-1}E(P_k')\,\Omega_k^{-1}E(P_k)\right]\left[1 - \frac{w_0^2}{A_0\operatorname{var}_0^r(w_T)}\right] \Longrightarrow \\ &\frac{\left(E(w_T) - w_0\prod_{k=0}^{T-1}s_k\right)^2}{\operatorname{var}_0^r(w_T)} < \left[\sum_{k=0}^{T-1}E(P_k')\,\Omega_k^{-1}E(P_k)\right] \Longrightarrow \\ &(SR)^2 < \left(BSR_{rf}\right)^2.\end{aligned} \tag{B.8}$$

Since (B.8) holds for the entire portfolio only with risk assets, and then we can prove that $(BSR)^2 < (BSR_{rf})^2$ or $BSR < BSR_{rf}$ is also true in this case. Finally, we can conclude that Theorem 8 is proved.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Supplementary Materials

In addition, we also provide the results of several additional analysis that we have undertaken to test the robustness of our findings. The results, reported in Tables A1–A54 of the Online Appendix, show that the time-consistent strategy (the recalculated time-consistent strategy) almost outperform the pre-commitment strategy (the recalculated pre-commitment strategy) under the different evaluation criterions. Obviously, this conclusion is coincident with that of Sections 5.2 and 5.3. For more detailed results, see the Online Appendix. *(Supplementary Materials)*

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