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The Planar Multiple Obnoxious Facilities Location Problem: A Voronoi Based Heuristic*

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Abstract

Consider the situation where a given number of facilities are to be located in a convex polygon with the objective of maximizing the minimum distance between facilities and a given set of communities with the important additional condition that the facilities have to be farther than a certain distance from one another. This continuous multiple obnoxious facility location problem, which has two variants, is very complex to solve using commercial nonlinear optimizers. We propose a mathematical formulation and a heuristic approach based on Voronoi diagrams and an optimally solved binary linear program.

As there are no nonlinear optimization solvers that guarantee optimality, we compare our results with a popular multi-start approach using interior point, genetic algorithm (GA), and sparse non-linear optimizer (SNOPT) solvers in Matlab. These are state of the art solvers for dealing with constrained non linear problems. Each instance is solved using 100 randomly generated starting solutions and the overall best is then selected. It was found that the proposed heuristic results are much better and were obtained in a fraction of the computer time required by the other methods.

The multiple obnoxious location problem is a perfect example where all-purpose non-linear non-convex solvers perform poorly and hence the best way forward is to design and analyze heuristics that have the power and the flexibility to deal with such a high level of complexity.

Key Words: Location; Obnoxious Facilities; Continuous Location; Voronoi Diagrams; Matlab; Heuristic; Binary Linear Program.

*This research has been supported in part by the Spanish Ministry of Economy and Competitiveness (MTM2015-70260-P), in part financed by the European Regional Development Fund (ERDF).

25 **1 Introduction**

26 Suppose that 100 communities are located in a 100 by 100 miles square. 20 obnoxious (e.g., noisy
27 or polluting) factories or landfills need to be located in the area. These factories are required to
28 be at least $D = 16$ miles from one another to avoid cumulative nuisance to the communities. The
29 objective is to maximize the minimum distance between facilities and communities.

Figure 1: Configuration of 100 Communities and Some Highest hilltops

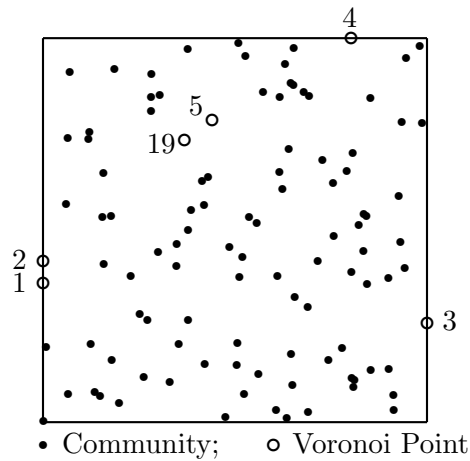
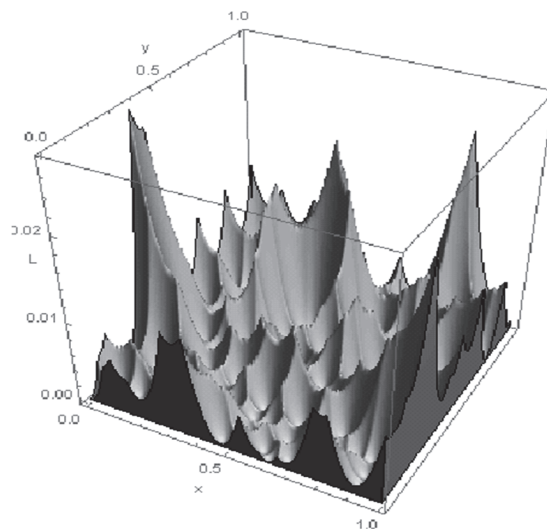


Figure 2: Surface of Distances to the Closest Community



30 To illustrate the problem consider the randomly generated example problem depicted in Fig-
31 ure 1. The surface of the shortest distance to the communities is depicted in Figure 2. There

32 are 202 hilltops. In Figure 1 the five “tallest” hilltops are marked. If a standard non-linear opti-
33 mization method is applied from a random starting solution, the process will likely end on hilltops
34 depending on the starting solution. There are 2×10^{27} possible selections of different 20 hilltops.
35 Intuitively, it is preferred to locate facilities on hilltops as long as the minimum distance of 16
36 miles is maintained. We therefore propose a heuristic that selects the best set of hilltops subject
37 to the distance constraints. Note that if the locations of the facilities are restricted to hilltops, the
38 heuristic solution is optimal. In Section 3.3 we show that the solutions for up to 5 facilities in this
39 example are optimal.

40 1.1 Literature Review

41 Obnoxious location problems involve locating one or more facilities as far as possible from a set
42 of communities. Most papers investigate the problem on networks or in discrete space [1, 4, 5,
43 6, 7, 16, 38]; location in the interior of a network [7, 14]; location on the plane [10, 20, 32, 39];
44 location on the sphere [12]. Applications may include nuisance generated by the facilities such
45 as airports, pollution generating industrial facilities, prisons, and others affecting residents living
46 in a set of communities. Another type of applications assume that the nuisance is generated by
47 the communities and the facilities should be located at locations with minimum nuisance. For
48 example, the location of schools or hospitals which require a low noise level caused by a set of
49 points or locating a telescope as far as possible from light sources.. In most of these applications
50 the nuisance propagates “by air” and not along network links making the use of Euclidean distances
51 appropriate.

52 Such models can be formulated in several ways. The most common way is to maximize the
53 minimum distance between the facilities and the set of communities [4]. Hansen et al. [20] assume
54 that the nuisance caused by communities declines by the square of the distance and suggested to
55 minimize the sum of $\frac{1}{d^2}$ where d is the distance between a community and the facility. Church
56 and Meadows [5] suggested to maximize the sum of distances from communities in a network
57 environment. Colmenar et al. [6] solved the multiple facilities version of this problem.

58 Drezner and Wesolowsky [14] found a location in the interior of a planar network that maximizes
59 the minimum distance between the facility and the links of the network. Drezner et al. [7] found
60 the best location for a facility in the interior of a planar network minimizing the total nuisance
61 generated by the links of the network.

62 The single facility problem is to find a location for one facility that maximizes the minimum
63 distance to a set of n communities. The problem is equivalent to finding the center of the largest
64 possible circle that has no communities in its interior. The facility must be located in a bounded
65 region. Otherwise, the solution will be at infinity. Shamos and Hoey [32] showed how to optimally
66 solve the problem in $O(n \log n)$ time using Voronoi Diagrams [30, 34, 37]. The idea of the Voronoi
67 diagram is to partition the plane into polygons such that all the points inside a polygon are closest
68 to one of the communities. The vertices of these polygons are equally distant to at least three
69 communities (and closest to them) or to at least two communities if the Voronoi vertex is on an
70 edge of the feasible region. The vertices of the feasible region are also Voronoi vertices that are at
71 the minimum distance to at least one community. The circle centered at a Voronoi vertex with a
72 radius equal to the distance to the closest community does not have communities in its interior.
73 Therefore, the best location for the facility is on one of these vertices. Finding all the vertices is
74 done in $O(n \log n)$ time and many computer codes are available for finding all the vertices, which
75 are known as “Voronoi points” [29, 33].

76 The single facility location model suggested by Hansen et al. [20] was optimally solved by
77 the “Big Square Small Square” global optimization method which was introduced in [20]. The
78 problem was also solved by the effective global optimization method known as “Big Triangle Small
79 Triangle” [9]. Another problem that aims to maximize the weighted sum of distances is a special
80 case of minimizing the sum of weighted distances with positive and negative weights [13, 25, 36].
81 It can be efficiently solved by these global optimization methods, for example, [9].

82 Most of the papers mentioned above investigated single facility problems. The only paper
83 that investigated the planar multi-facility obnoxious facility problem using Euclidean distances is
84 [39]. They found the optimal solution by a branch and bound algorithm which can be applied to
85 relatively small problems. They solved problems with up to five facilities and 120 demand points.

86 In this paper we heuristically solve two variants of the multiple obnoxious facility problem. The
87 first variant is maximizing the minimum distance between facilities and communities subject to a
88 required minimum distance between facilities. The second variant is maximizing all the distances
89 between facilities and communities and between facilities. The distances between facilities can be
90 multiplied by a factor to reflect a different weight to the two distance types in the objective function.
91 Suppose that a number of communities are located in an area. A required number of obnoxious
92 facilities (for example, noisy factories, landfills emitting odor) need to be located in the area. The

93 objective is to maximize the minimum distance between the communities and the facilities. The
94 facilities are required to be at least a given distance from one another to avoid cumulative nuisance
95 to the communities. Note that if no separation distance is imposed, the optimal solution is to
96 locate all facilities at the center of the largest circle without any communities. Alternatively, the
97 distance between facilities is required to be at least the minimum distance between communities
98 and facilities multiplied by a given factor.

99 The aim of the study is three fold:

- 100 (i) To heuristically solve the two variants of the multiple obnoxious problem in the plane.
- 101 (ii) To effectively incorporate the power of Voronoi vertices with optimally solving a binary linear
102 program in a recursive manner.
- 103 (iii) To explore the interior point, GA and SNOPT [17] non-linear optimization solvers in Matlab
104 [19] for comparison purposes.

105 The rest of the paper is organized as follows. The continuous multiple obnoxious facility location
106 problem is presented and its formulation is provided in the next section. This is followed in Section
107 3 by our Voronoi-based heuristic using solutions to binary linear programs. The computational
108 results are presented in Section 4. A case study of locating obnoxious facilities in Colorado, U.S.A.
109 is presented and solved in Section 5 and we conclude the paper with a summary of the results.

110 2 The Multiple Obnoxious Facilities Location Problem

111 The multiple obnoxious facility location problem is to locate p obnoxious facilities in a convex
112 polygon among a set of communities [11, 21, 38, 39]. Additional restrictions are required, otherwise,
113 the solution would be to locate all facilities at the optimal single facility location. We wish the
114 facilities not to be close to one another because the facilities may affect each other negatively. For
115 example, when locating schools or hospitals as far as possible from nuisance causing communities,
116 these facilities need to be spread out. When locating airports it does not make sense to locate two
117 airports next to one another. In addition, the location of the facilities must be restricted to a finite
118 area, otherwise, the solution would be to locate all facilities at infinity.

119 Let $A_i = (a_i, b_i)$ for $i = 1, \dots, n$ be the locations of communities, and $X_j = (x_j, y_j)$ for $j =$
120 $1, \dots, p$ be the unknown locations of the p facilities. We assume that the propagation of the

121 nuisance declines as the Euclidean distances increase.

122 Two problems are investigated in this paper:

123 **Maximin1:** Maximize the minimum squared distance between facilities and communities subject
124 to a given minimum distance D between facilities [2]. The non-linear programming formula-
125 tion is:

$$\max\{ L \}$$

Subject to: (1)

$$(x_j - a_i)^2 + (y_j - b_i)^2 \geq L \quad \text{for } i = 1, \dots, n; j = 1, \dots, p$$

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq D^2 \quad \text{for } 1 \leq i < j \leq p$$

126 In addition we need constraints that restrict the facilities' locations to a convex polygon or
127 any region. This formulation has $2p + 1$ variables and $np + \frac{p(p-1)}{2}$ constraints in addition to
128 the constraints restricting the locations to a region in the plane such as a square.

129 Note that it is more convenient to apply squared Euclidean distances in the formulation.

130 **Maximin2:** Maximize the minimum of all distances both between the facilities and communities
131 and between facilities [39]. The distances between facilities are equal to the distances between
132 communities and facilities multiplied by a given factor α (the squared distance is multiplied by
133 α^2). Welch et al. [39] allowed for different weights for different facilities. Such a modification
134 can be easily accommodated. This problem is formulated as

$$\max\{ L \}$$

Subject to: (2)

$$(x_j - a_i)^2 + (y_j - b_i)^2 \geq L \quad \text{for } i = 1, \dots, n; j = 1, \dots, p$$

$$(x_i - x_j)^2 + (y_i - y_j)^2 \geq \alpha^2 L \quad \text{for } 1 \leq i < j \leq p$$

135 As in Maximin1, we also need constraints that restrict the facilities' locations to a convex
136 polygon or any region. The size of this formulation is the same as the one given in (1).

137 The factor α allows flexibility when facilities are more (or less) obnoxious to each other than
138 to communities. A special case of this problem is using $\alpha = 1$ which entails equal importance given
139 to all distances. In Maximin1 the minimum distance between facilities is imposed rather than being
140 dependent on the minimum distance between facilities and communities.

2.1 Relationship to the p -Dispersion Problem

The multiple obnoxious facility problems are related to the p -dispersion problem [3, 15, 22]. In the p -dispersion problem, a set of potential locations for facilities is given and the objective is to select p points out of the potential locations that maximize the minimum distance between facilities. There are no communities in the p -dispersion model.

The p -dispersion location problem in an area is finding locations for p facilities in the area such that the minimum distance between pairs of facilities is maximized. The formulation is similar to (1) without the first type constraints, because there are no communities in the problem formulation, with the objective of maximizing D . The p -dispersion problem in an area [8, 24, 26, 28, 35] is equivalent to packing p circles in an area. **Heuristic solution approaches by solving the non-linear program using all purpose solvers are suggested in [8, 26]. Optimal branch and bound algorithms are proposed in [24, 28, 35].** Most results are for circle packing in a square. The best known solutions for the p -dispersion in a square are given in <http://www.packomania.com/> which reports proven optimal solutions for $p \leq 30$ and $p = 36$. Suppose that the optimal solution to the p -dispersion problem is D^* . For these values of p , there cannot be a feasible solution to problem Maximin1 if D exceeds D^* . Furthermore, if D is close to D^* , there are very few feasible solutions and distances to communities become almost irrelevant. This is not in the spirit of obnoxious facilities applications where distances to communities are the focus of the problem. We are therefore interested in D being considerably smaller than D^* for the problem to have practical applicability.

The value of D^* in a unit square is $\sqrt{2}$ for $p = 2$ and D^* declines to about 0.287 for $p = 20$. It is equal to $\frac{1}{q-1}$ for $p = q^2$ for an integer q up to $p = 36$. We therefore selected $D = \frac{1}{\sqrt{2p}}$ and $D = \frac{1}{\sqrt{p}}$ which is below D^* , see Table 1.

3 A Voronoi Based Heuristic Solution Approach

The problems can be heuristically solved by a multi-start approach solving the non-linear non-convex formulations by an optimization software such as those available in Matlab [19]. However, these problems have numerous local maxima and it is difficult to escape such local maxima. For $p = 20$ facilities and $n = 1000$ communities there are 4×10^{47} local maxima (some of them are infeasible). Even the smallest problem tested in this paper (locating two facilities among 100 communities) has over 20,000 local maxima. Generating a starting solution close to the “correct”

Table 1: p -dispersion Optimal Solutions

p	D^*	$\frac{1}{\sqrt{2p}}$	Ratio	$\frac{1}{\sqrt{p}}$	Ratio
2	1.414214	0.5000	0.3536	0.7071	0.5000
3	1.035276	0.4082	0.3943	0.5774	0.5577
4	1.000000	0.3536	0.3536	0.5000	0.5000
5	0.707107	0.3162	0.4472	0.4472	0.6325
6	0.600925	0.2887	0.4804	0.4082	0.6794
7	0.535898	0.2673	0.4987	0.3780	0.7053
8	0.517638	0.2500	0.4830	0.3536	0.6830
9	0.500000	0.2357	0.4714	0.3333	0.6667
10	0.421280	0.2236	0.5308	0.3162	0.7506
11	0.398207	0.2132	0.5354	0.3015	0.7572
12	0.388730	0.2041	0.5251	0.2887	0.7426
13	0.366096	0.1961	0.5357	0.2774	0.7576
14	0.348915	0.1890	0.5416	0.2673	0.7660
15	0.341081	0.1826	0.5353	0.2582	0.7570
16	0.333333	0.1768	0.5303	0.2500	0.7500
17	0.306154	0.1715	0.5602	0.2425	0.7922
18	0.300463	0.1667	0.5547	0.2357	0.7845
19	0.289542	0.1622	0.5603	0.2294	0.7923
20	0.286612	0.1581	0.5517	0.2236	0.7802

170 local maximum is very unlikely.

171 Maximin1 is equivalent to finding p empty circles such that the distance between any two circles'
 172 centers is at least D with the objective of maximizing the radius of the smallest circle. Maximin2
 173 is similar but the value of D depends on the smallest distance between facilities and communities.
 174 We propose a heuristic approach that found much better results than solving (1) and (2) directly
 175 by a non-linear non-convex available procedure, in a much shorter run time. This approach is based
 176 on selecting p points out of the set of V Voronoi points as potential facilities' locations.

177 It is important to note that even though the heuristic procedures were tested using Euclidean
 178 distances, they can be used for any distance measure once a Voronoi diagram is available for that
 179 distance measure.

180 We first define and prepare the following structure:

- 181 - All V Voronoi points, intersection points with the sides of the convex polygon and its vertices
 182 are generated.
- 183 - The distance to the closest community is calculated for each Voronoi point.
- 184 - The Voronoi points are sorted by decreasing order of these distances.

185 - The sorted list of Voronoi points is $\{V_i\}$ for $i = 1, \dots, V$, with a distance d_i between V_i and
 186 its closest community, such that $d_1 \geq d_2 \geq \dots \geq d_V$.

187 - The distance between Voronoi points i and j is D_{ij} .

188 It is well known (see for example Okabe et al. [30]) that the number of Voronoi points V is
 189 around $2n$.

190 The idea is to find p locations out of the V Voronoi points so that the distances between the
 191 chosen p Voronoi points are feasible, and the minimum distance to communities is maximized.
 192 Suppose that the vector of the V Voronoi points is sorted by the distance to the closest community
 193 (the peaks in Figure 2). Maximizing the shortest distance is equivalent to finding the p feasible
 194 Voronoi points whose p^{th} index in the sorted vector of distances is minimized. Define the optimal
 195 p^{th} index as K^* with the optimal objective function d_{K^*} .

196 Suppose that the first $p \leq K \leq V$ Voronoi points are selected. If $K \geq K^*$ there is a feasible
 197 solution to the problem based on these K Voronoi points. On the other hand, if $K < K^*$ no feasible
 198 solution exists.

199 3.1 Solving Maximin1 Heuristically

200 The first $p \leq K \leq V$ Voronoi points are selected. K binary variables x_i for $i = 1, \dots, K$ are defined
 201 with x_i equals 1 if Voronoi point i is selected and zero otherwise.

202 The following binary linear program solves the problem for a given K . When a feasible solution
 203 for this K exists, the solution is optimal to selecting p out of the V Voronoi points. Otherwise, if
 204 there is no feasible solution, the K need to be increased. Define the constants $\Delta_i = d_1 - d_i$. Note
 205 that $\Delta_i \geq 0$ because d_1 is the maximum distance.

206 Formulation BLP

Maximize $\{L\}$

subject to:

$$\sum_{i=1}^K x_i = p \tag{3}$$

$$x_i + x_j \leq 1 \text{ when } D_{ij} < D \tag{4}$$

$$L + x_i \Delta_i \leq d_1 \text{ for } i = 1, \dots, K \tag{5}$$

$$x_i \in \{0, 1\}$$

207 When Voronoi point i is not selected ($x_i = 0$), the constraint $L + x_i\Delta_i \leq d_1$ is $L \leq d_1$ which
 208 is always satisfied. When Voronoi point i is selected ($x_i = 1$), the constraint reduces to $d_i \geq L$
 209 and maximizing L results in the combination of p Voronoi points whose minimum distance to
 210 communities is maximized.

211 We suggest two approaches that employ BLP solutions. In Algorithm 1 we solve the problem
 212 using $K = V$. In Algorithm 2 we attempt to shorten the run time by solving a sequence of problems
 213 with smaller values of K in order to reduce the number of constraints in the BLP formulation.

214 **Algorithm 1:** Solve the BLP problem using $K = V$.

215 **Algorithm 2:**

- 216 1. Select $K = K_{\min}$.
- 217 2. Solve the BLP problem.
- 218 3. If there is a solution, stop.
- 219 4. If there is no feasible solution, increase K by q and go to Step 2

220 In our implementation we used $K_{\min} = 2p$ and $q = p$.

221 3.2 Solving Maximin2 Heuristically

222 For Maximin2, solving BLP for a given K means replacing D by αd_K in the BLP formulation. We
 223 first present and prove the following two properties.

224 **Property 1:** *If there is a feasible solution for BLP using $K = K_1$, i.e. using $D = \alpha d_{K_1}$, there will*
 225 *be a feasible solution for every $K \geq K_1$.*

226 **Proof:** Since $K \geq K_1$, $\alpha d_K \leq \alpha d_{K_1}$ and the feasible solution using $K = K_1$ is also feasible for
 227 $K \geq K_1$. □

228 **Property 2:** *If there is no feasible solution for BLP using $K = K_1$, there is no feasible solution*
 229 *for every $K \leq K_1$.*

230 **Proof:** If there was a feasible solution for $K \leq K_1$, there would have been a feasible solution for
 231 $K = K_1$ by Property 1. □

232 We conclude, by Properties 1 and 2, that up to a certain value of K there are no feasible
 233 solutions and for all greater values of K there is a feasible solution. The solution is obtained

234 by the solution for the smallest possible K , defined as K^* , that has a feasible solution. The
 235 objective function, which is the minimum distance between facilities and communities, is d_{K^*} and
 236 the distances between facilities are all at least αd_{K^*} .

237 We create a range $K_{\min} \leq K \leq K_{\max}$ such that for $K = K_{\min}$ there is no feasible solution
 238 and for $K = K_{\max}$ there is a feasible solution. The range is decreased every iteration and once
 239 $K_{\max} = K_{\min} + 1$, the feasible solution for K_{\max} is the best (or tied for the best) possible set of p
 240 locations from the set of V Voronoi points.

241 In Algorithm 3 we perform a bisection search on the whole range $p \leq K \leq V$. In algorithm 4 we
 242 attempt to reduce the run times by narrowing the range for the bisection search and avoid solving
 243 unnecessarily BLP problems with large values of K . The power and usefulness of incorporating
 244 neighborhood reduction in the search is shown to be a promising way forward in heuristic search
 245 design in general [31].

246 **Algorithm 3:**

- 247 1. Set $K_{\min} = p - 1$ and $K_{\max} = V$. We select $K_{\min} = p - 1$ because it is possible that for
 248 $K = p$ there is a feasible solution.
- 249 2. Set $K = \frac{1}{2}(K_{\min} + K_{\max})$ rounded down and solve the BLP problem replacing D by
 250 αd_K .
- 251 3. If there is no feasible solution, set $K_{\min} = K$ and go to Step 4. Otherwise,
 252 (a) Save this solution and set $K_{\max} = K$.
 253 (b) Find \bar{K} , the largest index among the Voronoi points in the solution.
 254 (c) If $\bar{K} \leq K_{\min}$ go to Step 4.
 255 (d) If $\bar{K} = K$ go to Step 4. If $\bar{K} < K$ solve the BLP problem for $K = \bar{K}$ replacing D
 256 by $\alpha d_{\bar{K}}$ and go to Step 3.
- 257 4. If $K_{\max} - K_{\min} > 1$ go to Step 2. Otherwise, choose the solution for $K = K_{\max}$ and
 258 stop.

259 Note that Step 3c is valid because there is no feasible solution for \bar{K} by Property 2

260 **Algorithm 4: Tightening Scheme for Initial K_{\min} and K_{\max}**

- 261 1. Select $K = 2p$.

- 262 2. Solve the BLP problem for this K replacing D by αd_K .
- 263 3. If there is no feasible solution, increase K by p and go to Step 2.
- 264 4. Otherwise, set $K_{\max} = K$, $K_{\min} = K - p$ ($K_{\min} = p - 1$ if $K = 2p$), and apply Algorithm 3
- 265 from Step 2 with these values of K_{\min} and K_{\max} .

266 3.3 Properties of the Heuristic Solution

267 This approach is a heuristic because the optimal facilities' locations are not necessarily on hilltops.

268 When there are no distance constraints, then the solution is to locate all the facilities at the top

269 of the tallest hilltop. When D is very small, the solution is a cluster of p facilities near the tallest

270 hilltop as long as the minimum distance to communities does not decrease by much. When D is

271 moderately increased, then it is not possible to locate more than one facility in a disk centered at

272 any hilltop because the hill is not large enough. Even if the hill is large enough, the facilities must

273 be located near the bottom of the hill and the minimum distance to communities (the minimum

274 height of the two facilities' locations) is likely to be too small. It is important to remember that

275 the solution is the height of the lowest facility and facilities that are located at higher locations can

276 be moved without affecting the value of the objective function.

277 It is possible, however, that the optimal solution is not at hilltops. Suppose that two tall

278 hilltops, are slightly closer than D to one another. The heuristic procedure cannot select both

279 hilltops (it may select only one of them). However, it may be possible to move the facilities located

280 at these two hilltops in opposite directions thus attaining a distance of D between them (and

281 not violating the distance constraints with other facilities) while reducing slightly the minimum

282 distance between the translated two facilities and their closest community (sliding downhill but not

283 by much). This may result in a better solution if the minimum distance is still above the heuristic

284 objective function. Such a scenario is possible but not likely. Even if it occurs, the heuristic solution

285 will not deteriorate much. For example, the heuristic solution for locating $p = 20$ facilities includes

286 the 53rd highest hilltop. If the minimum distance is relaxed and we select among the 46 highest

287 hilltops, the objective on hilltops is 2% higher, but the 16 miles distance constraints are violated.

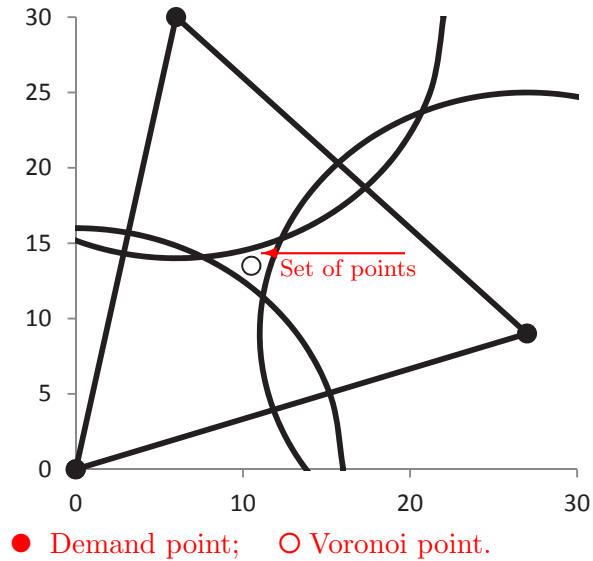
288 If the facilities are moved from the hilltops to accommodate the 16 miles distance requirement, the

289 objective function cannot improve and may well be worse than the heuristic objective function.

290 To illustrate this point we considered a hilltop based on three demand points, see Figure 3. The

291 demand points are located at $(0, 0)$, $(6, 30)$, and $(27, 9)$. The Voronoi point inside the triangle is

Figure 3: Points Near the Top of a Hill Exceeding a Height of 16



292 located at $(10.5, 13.5)$ which is at distance $\sqrt{292.5} = 17.103$ from all three demand points. The
 293 area near the Voronoi point for which the height on the hill is greater than 16 is depicted in Figure
 294 3. Suppose that the distance to another Voronoi point is slightly lower than D and the heuristic
 295 solution is less than 16. It may be possible to move these two points downhill slightly farther from
 296 one another to achieve a distance of D between them and if the points remain in the interior of the
 297 area, the value of the objective function may exceed 16. Such a solution is not on Voronoi points.
 298 Since the “hills” are steep, see Figure 2, the area of exceeding a lower value is usually small as is
 299 observed in Figure 3.

300 Consider the only instance that SNOPT found a better solution than the Voronoi heuristic.
 301 The location of $p = 4$ facilities among $n = 100$ demand points using $D = 0.5$ reported in Table
 302 xx. In Table 3 the distances between the first five Voronoi points are reported. When $D \leq 0.419$
 303 is used, the solution is points 1 (or 2) and points 3,4,5. The objective function for this selection
 304 is 0.150887. However, this solution violates the $D \geq 0.5$ constraints. A better solution was found
 305 by SNOPT by moving slightly Voronoi points 4 and 5 thus obtaining a distance of 0.5 between
 306 them. The objective function was reduced by 17% to 0.124590, but this solution is still better than
 307 using Voronoi point #19 (see Figure 1), which satisfies the $D \geq 0.5$ constraints, with the objective
 308 of 0.114609. This observation may suggest ways to attempt and improve the heuristic approach.
 309 Such options are discussed in the conclusions section as ideas for future research.

310 It may also be possible that a large area contains no communities and there is only one Voronoi
 311 point in the area while two or more facilities that are far enough from one another may be located
 312 there. Note that our algorithm is suited for $p \ll n$ which is the case in most practical applications.
 313 For example, if $p > V$, there must be facilities located at points which are not Voronoi points.
 314 However such instances require a very small value of D which is not practical.

315 Note that in an optimal solution to the original problem a facility cannot be located inside disks
 316 (on hills) whose d_i (height) is smaller than d_{K^*} . If a starting solution for the non-linear optimization
 317 procedure has facilities on such hills, standard non-linear optimization software will not be able to
 318 escape to another hill due to the extreme non-convexity of the surface (see for example Figure 2).
 319 In other words, the procedure has to cross deep “valleys”, which it is not designed to do, and
 320 eventually will result in an inferior value of the objective function.

Table 2: The First Five Voronoi Points

i	x	y	d_i
1	0	0.361453	0.166317
2	0	0.420781	0.158368
3	1	0.257239	0.154282
4	0.802745	1	0.151738
5	0.440903	0.787825	0.150887

Table 3: The Values of D_{ij}

	1	2	3	4	5
1	0.000	0.059	1.005	1.026	0.613
2	0.059	0.000	1.013	0.990	0.574
3	1.005	1.013	0.000	0.769	0.771
4	1.026	0.990	0.769	0.000	0.419
5	0.613	0.574	0.771	0.419	0.000

321 See for example the randomly generated problem with $n = 100$ communities used in the com-
 322 putational experiments. The 100 communities in a square are depicted in Figure 1. We also show
 323 in the same figure the top five Voronoi points which are also given in Table 2. The distances D_{ij}
 324 between the five Voronoi points are given in Table 3. The values of D used for the $p = 2, 3, 4$
 325 instances are $D = \frac{1}{\sqrt{2p}} = 0.5, 0.408, 0.354$, respectively.

326 In Figure 2, the five highest Voronoi points are visible. Compare it also with Figure 1. The

327 first two Voronoi points are on the left wall very close to one another, the third one is on the right
328 (not so clearly visible), the fourth one is in the far right, and the fifth one is close to the middle.

329 Since the distance D_{12} is small, the heuristic solutions for $2 \leq p \leq 4$ include either Voronoi
330 point 1 or Voronoi point 2 but not both. Voronoi point 3 is added to the $p = 2$ heuristic solution,
331 Voronoi points 3 and 4 are added for $p = 3$ and Voronoi points 3,4,5 for the $p = 4$ heuristic solution.
332 The heuristic objectives for these p 's are indeed d_3 , d_4 , and d_5 and K^* are 3,4, and 5 (see Table 4).
333 These solutions are optimal for the original problems because we cannot "separate" Voronoi points
334 1 and 2 far enough so that the distance between them will not be less than D . For example, for the
335 $p = 2$ instance the points in the square that are at least a distance d_3 from all communities, which
336 is the heuristic objective, are Voronoi point 3 and small areas surrounding Voronoi points 1 and 2
337 (intersection of the exteriors of circles centered at communities with a radius d_3). See also Figure
338 3 for an illustration. All other points in the square are closer than d_3 to at least one community. If
339 there is a solution with a distance greater than d_3 , the area where facilities can be located does not
340 include Voronoi point 3 but the facilities must be located in the interior of the small areas around
341 Voronoi points 1 and 2 and none of these points are at least a distance $D = \frac{1}{2}$ from one another.

342 **Theorem 1:** *The heuristic solution based on Voronoi points is a local maximum.*

343 **Proof:** Consider the heuristic solution that includes the Voronoi point V_{K^*} which determines its
344 objective value d_{K^*} . There may be several Voronoi points tied for this distance. By the construction
345 of the Voronoi points, an infinitesimal change in the location of V_{K^*} cannot increase its minimal
346 distance to the closest community. Infinitesimal changes in other Voronoi points which are part of
347 the heuristic solution, V_K for $K \leq K^*$, cannot improve the value of the objective function as well.
348 Therefore, any combination of such infinitesimal changes cannot improve the value of the objective
349 function even if some Voronoi points are at exactly a distance D from one another. \square

350 By Theorem 1 it is clear that if the heuristic solution is used as a starting solution to a non-linear
351 optimization software, such software cannot improve it. For illustration purposes, empirical exper-
352 iments were conducted using Matlab showing that the heuristic solutions could not be improved
353 when they were used as a starting solution, supporting Theorem 1.

354 4 Computational Experiments

355 All experiments were run on a virtual server with 16 vCPUs and 128 GB of vRAM. Algorithms 1-4
356 were implemented with the OPL and run on IBM's CPLEX Optimization Studio 12.4 environment.
357 We used the default CPLEX MIP solver settings for all four algorithms.

358 The non-convex quadratically-constrained (QCP) versions of Maximin1 and Maximin2 problems
359 were implemented in Matlab R2016b and solved using the interior-point method and SNOPT [17]
360 starting from 100 random solutions. The GA method provided similar but poorer results and
361 therefore results using GA are not reported. Unlike in the case of CPLEX, the default settings
362 of QCP interior point solver resulted in poor quality solutions and long processing times, so the
363 following changes were made: (i) analytical gradients and Hessians were specified for the objective
364 function and all non-linear constraints, (ii) scaling was applied to the objective function and all
365 constraints and (iii) the maximum number of function evaluations was increased to 50000. The first
366 two changes significantly improved the quality of the solutions and solver's efficiency (run time)
367 and the last one prevented the solver from exiting prematurely.

368 We experimented with $n = 100$ and 1000 with $p = 2, 3, \dots, 20$ for each problem for a total of 76
369 instances, each solved by two algorithms and QCP for comparison purposes. We generate random
370 locations for communities in a square (for details see the Appendix) that can be easily replicated
371 for future comparisons with other methods. The interior point and SNOPT solvers were applied
372 in a multi-start approach repeating the process from 100 random starting solutions and the best
373 result is reported. We also experimented with $n = 100$ instances and 1000 starting solutions. The
374 results were only slightly better but run times were about 10 times longer and thus these results
375 are not reported. For Maximin1 we applied $D = \frac{1}{\sqrt{2p}}$ and for Maximin2 we applied $\alpha = 2$. The
376 value of V is 202 for $n = 100$ and 2002 for $n = 1000$.

377 In tables 4-7 we report:

- 378 1. for the heuristic algorithms: the value of the objective function which is the minimum distance
379 between facilities and communities,
- 380 2. for Maximin1: the number of pairs of Voronoi points for which $D_{ij} < D$ (constraints of type
381 (4)),
- 382 3. for the heuristic algorithms: the value of K at the optimal solution,

- 383 4. for Maximin2: the number of BLP applications,
384 5. the clock time in seconds,
385 6. for QCP we also report the value of the objective function and the percentage of the QCP
386 objective below the heuristic objective for both the interior point solver and SNOPT.

Table 4: Results for Maximin1 $n = 100$ Instances Using $D = \frac{1}{\sqrt{2p}}$

p	Heuristic		Alg. 1		Alg. 2		Interior Point			SNOPT		
	Objective	†	K	Time (sec.)	K	Time (sec.)	Objective	Time (sec.)	% below Heuristic	Objective	Time (sec.)	% below Heuristic
2	0.154282	9,593	3	1.60	3	0.57	0.111489	23.51	27.7%	0.154282	2.66	0.0%
3	0.151738	6,957	4	1.63	4	0.90	0.110094	23.04	27.4%	0.133824	2.52	11.8%
4	0.150887	5,495	5	2.15	5	1.15	0.108818	26.00	27.9%	0.102189	3.50	32.3%
5	0.111488	4,558	21	2.38	21	2.18	0.092668	31.07	16.9%	0.102189	4.77	8.3%
6	0.111488	3,884	21	2.54	21	2.12	0.092668	37.96	16.9%	0.102189	7.15	8.3%
7	0.110668	3,432	22	2.58	22	2.23	0.095394	56.61	13.8%	0.094258	17.27	14.8%
8	0.108818	3,007	25	2.10	25	2.26	0.081280	48.48	25.3%	0.095395	20.20	12.3%
9	0.106636	2,720	26	2.87	26	2.18	0.081276	54.47	23.8%	0.092658	23.33	13.1%
10	0.102189	2,477	32	2.34	32	1.82	0.081271	67.72	20.5%	0.081767	79.07	20.0%
11	0.101100	2,271	36	2.07	36	2.27	0.081278	68.32	19.6%	0.081281	76.49	19.6%
12	0.100538	2,071	39	2.29	39	2.21	0.075754	76.60	24.7%	0.075618	100.21	24.8%
13	0.100538	1,913	39	1.65	39	1.43	0.081280	84.58	19.2%	0.081407	135.11	19.0%
14	0.096482	1,789	46	2.63	46	2.13	0.055635	93.30	42.3%	0.078265	206.78	18.9%
15	0.096482	1,688	46	3.85	46	2.16	0.058284	112.99	39.6%	0.071777	255.66	25.6%
16	0.096482	1,596	46	2.60	46	1.10	0.027046	135.21	72.0%	0.067402	489.99	30.1%
17	0.096482	1,515	46	2.63	46	1.37	0.050588	208.13	47.6%	0.072201	553.38	25.2%
18	0.095394	1,436	49	3.51	49	1.99	0.027046	199.49	71.6%	0.063215	502.35	33.7%
19	0.094537	1,365	51	2.87	51	1.62	0.027045	209.96	71.4%	0.066939	446.24	29.2%
20	0.094259	1,303	53	3.07	53	1.63	0.050265	212.04	46.7%	0.059051	581.34	37.4%

† Constraints of Type (4) for $K = V$

387 4.1 Maximin1 Results

388 In Table 4 we report results for $n = 100$ and in Table 5 for $n = 1000$ by Algorithms 1 and 2, interior
389 point and SNOPT for $2 \leq p \leq 20$. In two cases the value of K at the heuristic solution is not the
390 same for both algorithms. This is due to ties between some values of d_i resulting in different sorted
391 vectors of d_i . The value of the objective function is the same by applying both algorithms.

392 We note that for $n = 100$, the interior point $p = 2$ best objective is equal to d_{21} , $p = 3$ objective
393 is equal to d_{22} , and the $p = 14, 16, 18, 19, 20$ objectives are equal to d_{197} almost at the bottom of the

Table 5: Results for Maximin1 $n = 1000$ Instances Using $D = \frac{1}{\sqrt{2p}}$

p	Heuristic Objective	†	Alg. 1		Alg. 2		Interior Point			SNOPT		
			K	Time (sec.)	K	Time (sec.)	Objective	Time (sec.)	% below Heuristic	Objective	Time (sec.)	% below Heuristic
2	0.060413	978,875	5	176.37	5	23.44	0.032961	166.01	45.4%	0.043710	20.78	27.6%
3	0.048334	718,783	12	71.32	12	22.21	0.032971	250.88	31.8%	0.040716	43.25	15.8%
4	0.048334	569,500	12	89.24	12	21.24	0.031792	466.65	34.2%	0.038923	79.44	19.5%
5	0.048099	473,078	14	49.04	14	23.74	0.027037	880.24	43.8%	0.035011	924.34	27.2%
6	0.048099	405,490	14	47.72	14	21.15	0.021501	1143.16	55.3%	0.039285	1793.50	18.3%
7	0.044364	355,455	29	74.95	29	23.30	0.013620	1477.08	69.3%	0.030422	1693.56	31.4%
8	0.044364	316,529	29	39.43	29	21.66	0.024378	1869.33	45.0%	0.027604	2736.92	37.8%
9	0.044324	284,873	30	37.19	30	21.77	0.010598	2621.13	76.1%	0.027575	5933.92	37.8%
10	0.043385	259,320	36	50.68	36	20.99	0.019381	2881.38	55.3%	0.026487	4878.93	38.9%
11	0.041560	238,634	44	32.04	43	23.22	0.014727	3535.14	64.6%	0.024715	5576.01	40.5%
12	0.041552	220,422	45	60.26	45	22.40	0.012497	4425.06	69.9%	0.023707	5660.69	42.9%
13	0.041193	204,874	49	29.53	49	22.26	0.010614	5601.43	74.2%	0.025258	8140.24	38.7%
14	0.041193	191,432	49	33.50	49	22.32	0.010578	6416.49	74.3%	0.020032	19105.39	51.4%
15	0.039729	179,592	69	29.95	70	22.02	0.010614	9233.43	73.3%	0.023445	1608.35	41.0%
16	0.039664	169,334	72	62.55	72	22.77	0.003796	10958.50	90.4%	0.020939	5546.08	47.2%
17	0.039647	160,236	74	26.84	74	22.80	0.003797	13542.01	90.4%	0.016033	8355.69	59.6%
18	0.039664	152,157	72	38.14	72	24.07	0.010576	15234.95	73.3%	0.021174	11513.74	46.6%
19	0.039647	144,796	73	30.62	73	22.77	0.003797	16269.15	90.4%	0.020433	11902.36	48.5%
20	0.040239	138,098	65	51.09	65	24.30	0.003796	17831.12	90.6%	0.016277	18924.10	59.5%

† Constraints of Type (4) for $K = V$

394 list of 202 Voronoi points. One of the facilities is located at the top right corner (1,1) (see Figure
395 1). The best QCP solutions have at least one facility at the top of a low height hill (see Figure 2).
396 Note that it is enough that one facility is “stuck” in a “bad” region. All other facilities’ locations
397 become irrelevant to the value of the objective function.

398 We suspect that the interior points and SNOPT are not designed to effectively solve such
399 extreme non-convex problems. To get good solutions one must be “lucky” when selecting the
400 starting solution. In fact, when the heuristic solution was used as a starting solution for QCP, the
401 result remained the same. This is expected in view of Theorem 1. Run times by QCP are much
402 longer. Note that Algorithms 1 and 2 have no random component and replicating them will yield
403 the same solution which is optimal for the BLP.

404 For $n = 100$ run times are comparable for Algorithms 1 and 2. For $n = 1000$, run times required
405 by Algorithm 2 are stable and do not vary much for different values of p . On the other hand, run
406 times by Algorithm 1 decrease as p increases. Since D decreases as p increases, the number of

Table 6: Results for Maximin2 $n = 100$ Instances

p	Heuristic Objective	Algorithm 3			Algorithm 4			QCP (Interior Point)		
		K	BLP Runs	Time (sec.)	K	BLP Runs	Time (sec.)	Objective	Time (sec.)	% below Heuristic
2	0.154282	3	3	1.66	3	4	1.94	0.111489	17.84	27.7%
3	0.151738	4	3	2.36	4	4	1.79	0.108817	19.58	28.3%
4	0.150887	5	3	2.13	5	5	1.75	0.095811	26.21	36.5%
5	0.128668	14	12	9.01	14	9	5.68	0.092664	31.57	28.0%
6	0.111488	21	6	4.14	21	9	3.94	0.091351	35.57	18.1%
7	0.110668	22	4	3.72	22	10	5.42	0.083075	54.85	24.9%
8	0.108818	25	6	4.49	25	10	5.24	0.081279	48.44	25.3%
9	0.106636	26	5	5.07	26	7	4.35	0.081279	51.75	23.8%
10	0.102189	32	9	6.57	32	10	6.30	0.079098	71.50	22.6%
11	0.101100	36	8	9.42	36	11	6.56	0.076545	81.25	24.3%
12	0.100538	39	10	10.98	39	11	9.22	0.079096	77.60	21.3%
13	0.098631	43	13	11.56	43	10	6.77	0.078138	88.80	20.8%
14	0.096482	46	10	10.28	46	11	7.37	0.066097	98.15	31.5%
15	0.095394	49	10	10.17	49	11	6.89	0.066088	107.45	30.7%
16	0.094259	53	11	10.17	53	14	10.07	0.068306	117.49	27.5%
17	0.094259	53	9	9.08	53	11	7.38	0.056740	138.84	39.8%
18	0.094258	54	8	9.97	54	9	8.41	0.056742	146.93	39.8%
19	0.094012	55	8	8.99	55	9	9.95	0.056744	155.11	39.6%
20	0.093847	56	9	10.89	56	10	9.39	0.048573	173.89	48.2%

407 constraints of type (4) decreases as p increases leading to shorter run times. OPL using CPLEX
408 was able to solve such problems with almost a million constraints and two thousand variables in a
409 short run time. Both algorithms required very short run times and Algorithm 2 is clearly preferred
410 for small values of p .

411 The quality of the heuristic solutions is much better than those of the QCP. For $n = 100$, the
412 interior point objectives were below the heuristic objectives by 13%-72% and the SNOPT objective
413 was the same for $p = 2$ but was up to 37% below the heuristic solution for larger values of p . For
414 $n = 1000$, the interior point solutions were 32%-90% below the heuristic objectives and the SNOPT
415 solutions were 16%-59% below the heuristic solutions. In some cases the heuristic objective was
416 more than 10 times better! Run times required by Matlab are much longer. The largest problem
417 was solved heuristically in 24 seconds while it required about five hours by the interior point and
418 SNOPT.

Table 7: Results for Maximin2 $n = 1000$ Instances

p	Heuristic Objective	Algorithm 3			Algorithm 4			QCP (Interior Point)		
		K	BLP Runs	Time (sec.)	K	BLP Runs	Time (sec.)	Objective	Time (sec.)	% below Heuristic
2	0.060413	5	6	15.61	5	6	8.56	0.032961	184.28	45.4%
3	0.052838	7	5	16.92	7	6	10.04	0.032971	320.24	37.6%
4	0.050160	9	5	18.45	9	8	11.74	0.031792	527.52	36.6%
5	0.048652	10	4	16.21	10	6	10.21	0.027037	897.87	44.4%
6	0.048334	12	5	17.86	12	6	11.74	0.021501	1321.51	55.5%
7	0.048099	14	7	17.62	14	6	10.42	0.013620	1656.54	71.7%
8	0.047801	16	7	19.99	16	7	12.00	0.024378	2348.02	49.0%
9	0.044977	24	12	22.96	24	6	11.98	0.010598	2799.99	76.4%
10	0.044977	25	6	18.24	25	8	11.61	0.019381	3371.73	56.9%
11	0.044364	29	12	29.76	29	6	18.15	0.014727	3913.02	66.8%
12	0.044324	30	10	22.64	30	8	11.86	0.012497	4564.89	71.8%
13	0.044324	31	6	27.82	31	9	12.64	0.010614	5467.57	76.1%
14	0.043973	32	10	22.10	32	9	12.36	0.010578	6450.50	75.9%
15	0.043710	33	10	22.40	33	10	13.10	0.010614	7515.70	75.7%
16	0.043710	34	6	18.91	34	10	13.21	0.003796	8120.26	91.3%
17	0.043487	35	10	22.94	35	10	13.52	0.003797	9550.26	91.3%
18	0.043385	36	10	21.63	36	8	13.03	0.010576	10591.68	75.6%
19	0.042742	40	12	24.63	40	10	13.63	0.003797	11294.25	91.1%
20	0.041560	43	12	22.56	43	10	13.68	0.003796	11861.28	90.9%

4.2 Maximin2 Results

In Table 6 we report results for $n = 100$ and in Table 7 for $n = 1000$ by Algorithms 3 and 4 and interior point for $2 \leq p \leq 20$. The SNOPT solver failed to find even one feasible solution in 100 runs for 20 of the 38 instances. We also solved the $n = 100$ instances 10,000 times. There were more feasible solutions and the results are slightly better (much worse than the heuristic results) and run times are 100 times longer. We therefore do not report the SNOPT results for the Maximin2 instances. The performance, except SNOPT, and conclusions are very similar to those obtained for Maximin1.

Both algorithms are very efficient for $n = 100$ and performed about equally well. For $n = 1000$, run times required by Algorithm 4 are generally lower than those required by Algorithm 3.

The quality of the heuristic solutions is much better than those obtained by the interior point solver. For $n = 100$, the interior point objectives were below the heuristic objectives by 18%-48%. For $n = 1000$, they were 37%-91% below the heuristic objectives. Run times of the QCP are much

Table 8: Results for Maximin1 $n = 100$ Instances Using $D = \frac{1}{\sqrt{p}}$

p	Heuristic Objective	Alg. 1		Alg. 2		Interior Point			SNOPT		
		K	Time (sec.)	K	Time (sec.)	Objective	Time (sec.)	% below Heuristic	Objective	Time (sec.)	% below Heuristic
2	0.154282	3	0.59	3	0.18	0.111486	3.81	27.7%	0.154283	3.07	0.0%
3	0.151738	4	0.57	4	0.16	0.108816	3.49	28.3%	0.150887	3.24	0.6%
4	0.114609	19	0.59	19	0.20	0.108818	5.30	5.1%	0.124591	5.13	-8.7%
5	0.111488	21	0.51	21	0.29	0.092665	7.54	16.9%	0.110669	7.39	0.7%
6	0.110668	22	0.47	22	0.19	0.081278	9.72	26.6%	0.095301	9.26	13.9%
7	0.108818	25	0.52	25	0.20	0.073011	23.61	32.9%	0.101620	22.99	6.6%
8	0.102189	32	0.48	32	0.24	0.073038	54.59	28.5%	0.095395	52.58	6.6%
9	0.102189	32	2.18	32	0.32	0.069585	61.80	31.9%	0.086117	59.57	15.7%
10	0.095394	49	0.55	49	0.24	0.055803	81.38	41.5%	0.093217	91.84	2.3%
11	0.095394	49	0.50	49	0.33	0.065434	74.35	31.4%	0.079606	72.08	16.6%
12	0.095169	50	0.50	50	0.35	0.027046	79.35	71.6%	0.079318	77.03	16.7%
13	0.094401	52	0.75	52	0.24	0.059649	151.55	36.8%	0.073015	147.33	22.7%
14	0.081280	78	0.72	78	0.40	0.035561	134.87	56.2%	0.072998	131.88	10.2%
15	0.081407	77	2.78	77	0.45	0.065433	302.97	19.6%	0.070529	294.69	13.4%
16	0.081280	78	0.61	78	0.29	0.055881	237.83	31.2%	0.067402	232.79	17.1%
17	0.075380	91	0.52	91	0.52	0.027045	490.82	64.1%	0.073015	483.18	3.1%
18	0.073080	103	0.53	103	0.52	0.027046	221.06	63.0%	0.069189	245.86	5.3%
19	0.075380	91	0.56	91	0.37	0.070528	268.33	6.4%	0.056615	299.86	24.9%
20	0.076831	88	0.57	88	0.38	0.027046	481.49	64.8%	0.058138	521.96	24.3%

432 longer in some cases by a factor of almost one thousand.

433 5 Case Study: Locating Obnoxious Facilities in Colorado

434 There are 271 municipalities in Colorado and we wish to build p obnoxious facilities such as pollution
435 generating industrial facilities to be as far as possible from these municipalities. The problems were
436 solved by the Voronoi based heuristic algorithms (that need to be solved only once) as well as by
437 Matlab, using interior point method and using SNOPT, reporting the best solution obtained from
438 100 randomly generated starting solutions.

439 The locations for $2 \leq p \leq 20$ by Maximin1 requiring $D = 80$ miles between facilities and
440 Maximin2 with $\alpha = 2$ are depicted in Tables 11 and 12. The results clearly show that the Voronoi
441 based heuristic performed much better than the Matlab procedures on these 38 instances. The best
442 value of the objective function obtained by the Matlab procedures was between 6% and 57% lower
443 than the results obtained by the Voronoi heuristic. Run times by the Voronoi based heuristic are

Table 9: Results for Maximin1 $n = 1000$ Instances Using $D = \frac{1}{\sqrt{p}}$

p	Heuristic Objective	Alg. 1		Alg. 2		Interior Point			SNOPT		
		K	Time (sec.)	K	Time (sec.)	Objective	Time (sec.)	% below Heuristic	Objective	Time (sec.)	% below Heuristic
2	0.060413	5	120.93	5	13.65	0.032968	151.78	45.4%	0.060414	28.85	0.0%
3	0.048099	14	100.05	14	13.89	0.032959	342.97	31.5%	0.038076	70.84	20.8%
4	0.048099	14	90.64	14	13.74	0.029457	523.79	38.8%	0.036250	146.11	24.6%
5	0.044364	29	82.76	29	15.14	0.027045	848.14	39.0%	0.032971	355.05	25.7%
6	0.044364	29	86.52	29	13.44	0.019380	1274.65	56.3%	0.031137	373.78	29.8%
7	0.043710	34	74.77	34	13.73	0.027044	1753.83	38.1%	0.029046	2959.65	33.5%
8	0.041560	43	79.85	43	14.26	0.026407	2518.62	36.5%	0.028746	1745.14	30.8%
9	0.039729	69	84.82	69	14.23	0.019382	2975.67	51.2%	0.027386	1653.56	31.1%
10	0.039729	69	76.05	69	14.15	0.019381	3750.88	51.2%	0.024644	1304.80	38.0%
11	0.038075	115	66.48	115	15.24	0.010609	5058.97	72.1%	0.023472	2954.71	38.4%
12	0.039123	87	68.38	87	14.63	0.009643	7469.20	75.4%	0.024104	3310.78	38.4%
13	0.038075	115	75.59	115	14.91	0.009863	11551.76	74.1%	0.023842	8746.06	37.4%
14	0.038075	115	84.43	115	14.29	0.010608	13240.33	72.1%	0.020034	10845.53	47.4%
15	0.037049	133	67.23	133	14.59	0.010615	19198.95	71.3%	0.020288	6800.51	45.2%
16	0.037049	133	64.99	133	15.31	0.009875	21075.77	73.3%	0.021321	3716.66	42.5%
17	0.035941	147	110.77	147	14.47	0.003795	30142.08	89.4%	0.020564	7650.78	42.8%
18	0.034271	203	82.50	203	17.11	0.003794	34586.89	88.9%	0.016961	4403.85	50.5%
19	0.035288	167	122.84	167	15.06	0.003796	40564.65	89.2%	0.020070	6593.26	43.1%
20	0.033267	222	132.45	222	17.47	0.010612	45396.36	68.1%	0.020019	8934.54	39.8%

444 more than 1,000 times faster for large values of p . Interior point performed better than SNOPT.

445 The solution for locating 20 obnoxious facilities by the maximin1 model is depicted in Fig-
 446 ure 4. The heuristic minimum distance between facilities and communities is about 16.5 miles (see
 447 Table 11). Interior point's best solution is about 15.6 miles while SNOPT's is about 10.8 miles.

448 6 Conclusions

449 We formulated and solved two multiple obnoxious facilities problems. A given number of facilities
 450 are to be located in a convex polygon with the objective of maximizing the minimum distance
 451 between facilities and a given set of communities. The facilities has to be farther than a certain
 452 distance from one another. The proposed heuristic solution approaches are based on generating the
 453 Voronoi points of Voronoi diagrams [30, 34]. A binary linear program (BLP) was constructed and
 454 the solution approaches applied this BLP iteratively. Run times are very short producing excellent
 455 results.

Table 10: Upper Bounds For the Heuristic Results

p	$n = 100$			$n = 1,000$		
	Heuristic	U.B.	% above	Heuristic	U.B.	% above
2	0.154282	0.158368	2.6%	0.060413	0.065301	8.1%
3	0.151738	0.154282	1.7%	0.048099	0.064855	34.8%
4	0.150887	0.151738	0.6%	0.048099	0.063223	31.4%
5	0.111488	0.150887	35.3%	0.044364	0.060413	36.2%
6	0.111488	0.150845	35.3%	0.044364	0.055291	24.6%
7	0.110668	0.148404	34.1%	0.043710	0.052838	20.9%
8	0.108818	0.135640	24.6%	0.041560	0.050315	21.1%
9	0.106636	0.134780	26.4%	0.039729	0.050160	26.3%
10	0.102189	0.134754	31.9%	0.039729	0.048652	22.5%
11	0.101100	0.133824	32.4%	0.038075	0.048627	27.7%
12	0.100538	0.133587	32.9%	0.039123	0.048334	23.5%
13	0.100538	0.132914	32.2%	0.038075	0.048158	26.5%
14	0.096482	0.128668	33.4%	0.038075	0.048099	26.3%
15	0.096482	0.126415	31.0%	0.037049	0.047881	29.2%
16	0.096482	0.124170	28.7%	0.037049	0.047801	29.0%
17	0.096482	0.124036	28.6%	0.035941	0.047774	32.9%
18	0.095394	0.117843	23.5%	0.034271	0.047660	39.1%
19	0.094537	0.114609	21.2%	0.035288	0.046943	33.0%
20	0.094259	0.113482	20.4%	0.033267	0.046500	39.8%

456 For comparison purposes we solved the problem by a multi-start approach applying the non-
457 convex quadratically-constrained (QCP) method in Matlab based on Matlab's default interior point
458 and SNOPT solvers. The best results obtained by Matlab are worse by at least 13% than the
459 heuristic results. In some cases the heuristic results are better by a factor greater than 10. This
460 means that the minimum distance between communities and facilities in the heuristic solution is
461 more than ten times greater than the minimum distance in the best solution found by Matlab!
462 For example, suppose that 1000 communities are located in a 100 by 100 miles square in locations
463 corresponding to our test problem. 20 noisy factories need to be located in the area. These
464 factories are required to be at least 16 miles from one another to avoid cumulative nuisance to
465 the communities. By Matlab using the interior point method the minimum distance between a
466 community and a factory is 0.38 miles (see Table 5). SNOPT found a solution of 1.6 miles. By
467 our heuristic result each community is at least 4 miles away from any factory. When the distances
468 between factories are required to be at least twice the minimum distance to the communities (see
469 Table 7), the minimum distance by the interior point method is the same 0.38 miles, SNOPT failed

Table 11: Results for Colorado Municipalities: Maximin1 Objective

p	Heuristic Objective	Alg. 1		Alg. 2		Interior Point			SNOPT		
		K	Time (sec.)	K	Time (sec.)	Objective	Time (sec.)	% below Heuristic	Objective	Time (sec.)	% below Heuristic
2	39.694395	4	3.53	4	1.06	36.511967	41.96	8.0%	33.728213	10.90	15.0%
3	38.333642	6	3.26	6	1.28	35.181856	54.72	8.2%	33.728213	17.07	12.0%
4	37.388116	8	3.59	8	1.11	29.416538	83.51	21.3%	33.728213	57.46	9.8%
5	35.771728	16	3.25	16	1.34	26.652446	103.13	25.5%	33.728213	492.77	5.7%
6	35.181855	18	3.68	18	1.12	26.600475	143.14	24.4%	31.924244	1169.13	9.3%
7	33.728213	20	3.25	20	1.34	26.600475	171.81	21.1%	25.416189	1957.45	24.6%
8	30.134736	32	3.65	32	1.17	22.520097	210.84	25.3%	22.520097	2322.08	25.3%
9	29.742869	33	3.37	33	1.36	22.520097	290.29	24.3%	20.820193	3175.30	30.0%
10	29.528622	36	3.63	36	1.14	20.126060	335.87	31.8%	22.132283	2938.46	25.0%
11	29.416537	37	3.31	37	1.33	18.718547	452.46	36.4%	19.287761	2987.33	34.4%
12	29.057271	39	3.65	39	1.12	22.113030	561.33	23.9%	17.393139	3999.68	40.1%
13	28.629458	42	3.35	42	1.33	19.079799	689.59	33.4%	16.998044	3119.47	40.6%
14	28.265609	47	3.62	47	1.16	19.575039	873.61	30.7%	15.655950	5069.37	44.6%
15	24.835288	88	3.40	88	1.54	17.559717	1128.02	29.3%	15.255855	5696.77	38.6%
16	22.132283	118	3.79	118	1.64	16.309250	1343.76	26.3%	13.932383	6009.23	37.0%
17	20.949743	133	3.43	133	1.84	18.718547	1736.98	10.7%	13.870493	3498.17	33.8%
18	19.418680	154	3.71	154	1.97	13.261295	2133.28	31.7%	14.485326	4566.82	25.4%
19	19.145613	159	3.46	159	2.23	16.608329	2663.39	13.3%	11.821557	5510.11	38.3%
20	16.535647	215	3.82	215	2.70	15.565481	3009.58	5.9%	10.829746	4801.80	34.5%

470 to find a feasible solution, and our heuristic found a solution with a minimum distance of 4.16
471 miles. Run times required by Matlab employing the interior point method or SNOPT solvers are
472 much longer. The largest problem was solved heuristically in 24 seconds while it required about five
473 hours by Matlab. We do not expect to get much better results by using other non-linear non-convex
474 solvers because there are so many local maxima (4×10^{47} local maxima, some infeasible, for the
475 largest tested problem) and the result depends on the initial random solution because it is unlikely
476 to move from one local maximum to another (see Figure 2).

477 We also solved a case study of locating obnoxious facilities in Colorado among 271 municipalities.
478 The Voronoi heuristic performed much better than Matlab for this case study as well. By inspecting
479 Figures 1 and 4, it seems that solutions tend to be close to the periphery of the convex polygon.
480 It is possible, for example in the Colorado case study, that communities outside the state may be
481 affected and should be considered in the model. In such cases the Voronoi points should be created
482 considering also points outside the convex polygon but restricted to the convex polygon. This can
483 be accomplished by creating a Voronoi diagram based on all points, selecting as Voronoi points the

Table 12: Results for Colorado Municipalities: Maximin2 Objective

p	Heuristic Objective	Alg. 3		Alg. 4		Interior Point			SNOPT		
		K	Time (sec.)	K	Time (sec.)	Objective	Time (sec.)	% below Heuristic	Objective	Time (sec.)	% below Heuristic
2	39.694395	4	1.85	4	1.08	36.511967	44.00	8.0%	33.728213	10.95	15.0%
3	38.333642	6	1.74	6	1.02	35.181856	56.67	8.2%	33.728213	18.43	12.0%
4	37.388116	8	2.08	8	1.04	29.416538	84.32	21.3%	33.728213	62.07	9.8%
5	37.203216	9	2.53	9	1.11	26.652446	108.30	28.4%	33.728213	539.84	9.3%
6	35.181855	18	2.03	18	1.16	26.600475	150.18	24.4%	31.924244	1262.25	9.3%
7	33.728213	20	2.08	20	1.17	26.600475	181.87	21.1%	25.416189	2092.06	24.6%
8	33.119803	22	3.71	22	1.34	22.520097	219.15	32.0%	22.520097	2504.32	32.0%
9	32.110293	24	1.93	24	1.10	22.520097	298.95	29.9%	20.820193	3378.49	35.2%
10	30.134736	32	2.02	32	1.43	20.126060	355.58	33.2%	22.132283	3118.91	26.6%
11	29.416537	37	2.01	37	1.40	18.718547	473.52	36.4%	19.287761	3167.01	34.4%
12	29.057271	39	2.00	39	1.47	22.113030	588.50	23.9%	17.393139	4429.27	40.1%
13	28.902359	40	1.96	40	1.39	19.079799	729.14	34.0%	16.998044	3297.04	41.2%
14	28.629458	42	2.01	42	1.21	19.575039	939.98	31.6%	15.655950	4924.29	45.3%
15	28.359140	46	2.18	46	1.37	17.559717	1134.59	38.1%	15.255855	5216.74	46.2%
16	28.265609	47	2.12	47	1.27	16.309250	1373.46	42.3%	13.932383	5698.57	50.7%
17	27.093737	65	2.40	65	2.04	18.718547	1786.32	30.9%	13.870493	3111.48	48.8%
18	26.388220	72	2.59	72	2.17	13.261295	2193.20	49.7%	14.485326	3921.14	45.1%
19	26.086154	73	2.71	73	2.07	16.608329	2723.78	36.3%	11.821557	4800.02	54.7%
20	25.266566	80	2.37	80	1.65	15.565481	3072.09	38.4%	10.829746	4197.52	57.1%

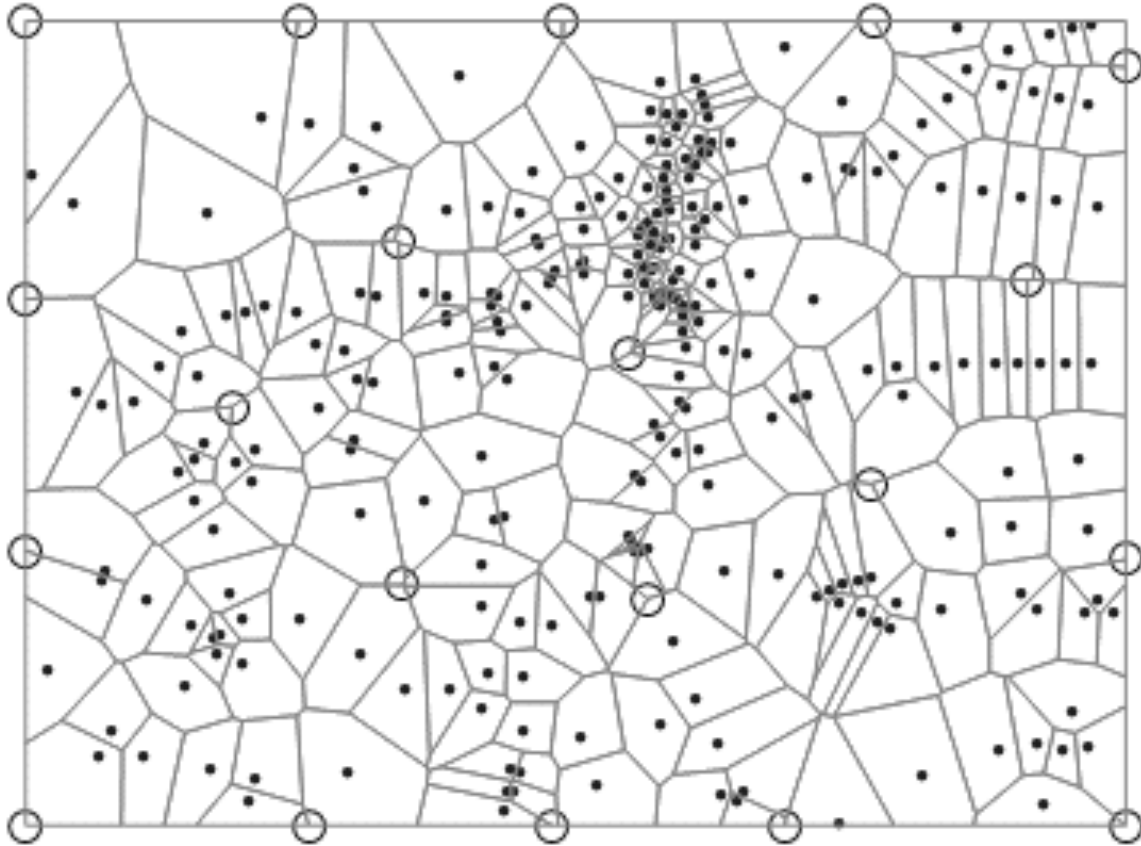
484 Voronoi points in the convex polygon and the intersection points between the Voronoi edges and
485 the boundary of the convex polygon.

486 The problem can also be defined in a cube in three dimensions or on the globe. The heuristic
487 approach requires three-dimensional Voronoi vertices [18] or spherical Voronoi vertices [27]. Non
488 linear optimization procedures such as QCP in Matlab can be implemented in a multi-start approach
489 but from the experience based on the results presented in this paper we do not expect that high
490 quality solutions will be found this way.

491 6.1 Suggestions for Future Research

492 The discussion in Section 3.3 suggests other solution algorithms based on the Voronoi heuristic.
493 There are a few possible approaches. For example, require a lower value of D and apply the
494 Voronoi heuristic. Presumably, some constraints for the original value of D are violated. Apply
495 an optimization procedure from this solution subject to the original D constraints. Some solution
496 points may slide a bit from hilltops and a better solution may possibly be found. Constructing,

Figure 4: Locating 20 Obnoxious Facilities in Colorado



497 analyzing, and testing such approaches will constitute a full fledged new paper.

498 Appendix: Generating Random Configurations

499 We follow the idea presented in [23] for generating random numbers. We generate a sequence
500 of integer numbers in the open range $(0, 100,000)$. A starting seed r_1 , which is the first number in
501 the sequence, is selected. The sequence is generated by the following rule for $k \geq 1$:

- 502 • Set $\theta = 12219r_k$.
- 503 • Set $r_{k+1} = \theta - \lfloor \frac{\theta}{100000} \rfloor \times 100000$, i.e., r_{k+1} is the remainder of dividing θ by 100000. It is
504 also the last five digits of θ .

505 For the x coordinates we used $r_1 = 97$ and for the y -coordinates we used $r_1 = 367$. The first
506 100 points in a square (we divide the coordinates by 100000 so the points are in a unit square) are

507 depicted in Figure 1.

508 These sequences return to the first point after 5000 generations. Note that even though there are
509 99,999 numbers between 1 and 99,999, even numbers and numbers divisible by 5 are not obtained
510 in the sequences. We could get longer sequences if 100,000 or 99,999 were prime numbers. The
511 sequences suggested in [23] exploit the fact that $2^{31} - 1$ is a prime number. Note that the number
512 12,219 can be replaced by many other numbers.

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