Lipkin's Conservation Law in Vacuum Electromagnetic Fields

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Abstract

Lipkin's zilches are a set of little-known conserved quantities in classical electromagnetic theory. Here we report a systematic calculation of the zilches for topologically non-trivial vacuum electromagnetic fields and their interpretation in terms of both the physical and mathematical properties of the fields. Several families of electromagnetic fields have been explored and examined computationally. In these cases it is found that the zilches can be written in terms of more familiar conserved quantities: energy, momentum and angular momentum. Furthermore we demonstrate that the zilches also contain information about the topology of the field lines for the fields we have examined, thus providing a previously unsuspected aspect to their interpretation. We conjecture that these properties generalise to all integrable fields.

INTRODUCTION:

Maxwell's equations in a vacuum can be solved using standard techniques to yield solutions with the properties of monochromatic wave-like electromagnetic radiation[1]. This was one of the great triumphs of 19th century physics. However, such light is not the only solution of these equations and over the last thirty years there has been a great deal of work on fields with exotic properties in electrodynamics, mainly inspired by the influential paper of Ranada[2]. Ranada's work has been shown to be equivalent to earlier solutions[3–5] and has been generalised in a number of directions[6–9], which have led to families of solutions of Maxwell's equations with non-trivial topologies including knotted structures that are stable with time[10, 11].

In 1964, Lipkin[12] introduced a set of new quantities in electromagnetism that form a 3rd rank tensor and which he (unattractively) named zilches. They represent a set of ten (nine independent) conserved quantities that have units of force. As shown by Kibble[13] and others[14–16], for any free field one can find an infinite number of conserved quantities with densities that are bilinear functions of the fields. Kibble[13] was also able to show how the zilches were related to more familiar quantities in electromagnetism including the Electromagnetic Field tensor and the Maxwell stress energy tensor. These authors inferred that the zilches are very unlikely to have any physical significance. A conservation law implies a symmetry of the Lagrangian through Noether's theorem. For a long time the symmetry giving rise to the conservation of zilch remained elusive. However recently it has been identified as a symmetry involving the second derivative of both the magnetic and electric vector potential[17], and of just the magnetic vector potential[18]. For individual zilches this symmetry has been suggested previously[19, 20]. Interpretation of the zilches in terms of physical quantities has been mainly restricted to the Z^{000} zilch, which has been thought of as a measure of optical chirality, and has been used in this capacity to predict and interpret experiment[21, 22]. For monochromatic light, Z^{000} has been shown to be proportional to the helicity of the field, and the 0j0-zilches proportional to the components of spin of the fields[23–25]. As emphasised by Cameron and Barnett[17] there are no such proportionalities in a polychromatic field. Beyond Z^{000} virtually nothing has been said about the physical interpretation of the zilches. Hence the full meaning of these conserved quantities has remained a mystery throughout the fifty years since their discovery, and anything new that can be said about them is of fundamental interest.

Our purpose here is to explore zilch properties in the arena of topologically unusual electromagnetic fields. A profound connection between the zilches and the topology of the fields is demonstrated. We have also found an interpretation of zilches in terms of known conserved physical properties of the fields and hence of the Maxwell stress energy and angular momentum tensors[1] for each of the fields.

In the following section we define and discuss the properties of the zilches and then look at Bateman's method for solving Maxwell's equations[3]. Then we consider how to describe the topology of the fields obtained in this way. Next we discuss three different families of solutions of Maxwell's equations, the topology and physical properties of their zilches, and then finally draw some conclusions from this work.

THEORY:

In this paper, zilch densities will be represented by calligraphic $\mathcal{Z}$, while for the integrated zilches we will use latin letters Z. For an electric field, $\mathbf{E}$, and magnetic field, $\mathbf{B}$, the contravariant components of Lipkin's zilches are given by[26],

$$\mathcal{Z}^{000} = \frac{\epsilon_0}{2}\left(\mathbf{E}\cdot(\nabla\times\mathbf{E}) + c^2\mathbf{B}\cdot(\nabla\times\mathbf{B})\right),$$

$$\mathcal{Z}^{0i0} = \frac{\epsilon_0 c}{2}\left(\mathbf{E}\times(\nabla\times\mathbf{B}) - \mathbf{B}\times(\nabla\times\mathbf{E})\right)_i,$$

$$\mathcal{Z}^{ij0} = \delta_{ij}\mathcal{Z}^{000} - \frac{\epsilon_0}{2}(E_i(\nabla\times\mathbf{E})_j + E_j(\nabla\times\mathbf{E})_i + c^2 B_i(\nabla\times\mathbf{B})_j + c^2 B_j(\nabla\times\mathbf{B})_i),$$

$$\mathcal{Z}^{ijk} = \delta_{ij}\mathcal{Z}^{00k} + \frac{\epsilon_0 c}{2}\left(B_i\frac{\partial E_j}{\partial x_k} + B_j\frac{\partial E_i}{\partial x_k} - E_i\frac{\partial B_j}{\partial x_k} - E_j\frac{\partial B_i}{\partial x_k}\right),$$

$$\mathcal{Z}^{\mu\nu\gamma} = \mathcal{Z}^{\nu\mu\gamma}, \qquad \mathcal{Z}^{00i} = \mathcal{Z}^{0i0}, \qquad \mathcal{Z}^{0ij} = \mathcal{Z}^{ij0},$$

where c is the speed of light. These obey

$$\partial_\gamma \mathcal{Z}^{\mu\nu\gamma} = 0,$$

and hence

$$Z^{\mu\nu 0} = \int\int\int \mathcal{Z}^{\mu\nu 0} d^3\mathbf{r}$$

represents a set of ten conserved quantities.

In a key paper, Kedia *et. al.*[10] have defined knotted solutions of Maxwell's equations in a vacuum using Bateman's construction[3]. A clear discussion and generalisation of this work has been provided by Hoyos *et. al.*[11] and we outline the method below. Henceforth, **B** is multiplied by the speed of light, c, to unify its units with those of **E**. Then Maxwell's equations in a vacuum are

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

which writes them in the most symmetric way possible. If we define the Riemann-Silberstein vector

$$\mathbf{R} = \mathbf{E} + i\mathbf{B},$$

Maxwell's equations become

$$\nabla \cdot \mathbf{R} = 0, \qquad i\frac{\partial \mathbf{R}}{\partial t} = c\nabla \times \mathbf{R}. \tag{1}$$

These are satisfied by complex scalar quantities α and β such that

$$\mathbf{R} = E_0 \nabla\alpha \times \nabla\beta, \tag{2}$$

where E_0 is an arbitrary constant with units of electric field times distance squared. α and β are dimensionless and can be written in terms of the space-time coordinates and a scale factor k which we set equal to one in most of what follows. This definition is automatically consistent with the first of equations (1), and if α and β also satisfy

$$i\left(\frac{\partial\alpha}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\alpha\right) = c\nabla\alpha \times \nabla\beta, \tag{3}$$

then the fields generated represent a full solution to Maxwell's equations. Furthermore, once we have a solution, any well-behaved function of α and β yields another new solution of Maxwell's equations. Then, if the fields are integrable we can invoke the equations from the previous section to calculate numerical values for the zilches. Once we have calculated the fields it is a standard calculation, from Noether's theorem[1] to show that symmetry under translations in time, space, and under rotations leads to the conservation of energy, momentum and angular momentum - which is expressed in the Maxwell stress energy tensor and angular momentum tensor in terms of the fields[1]. Explicitly the respective densities are given by

$$\mathcal{E}_{p,q} = \frac{\epsilon_0}{2}(\mathbf{E}^2 + \mathbf{B}^2), \quad \mathcal{P} = \frac{\epsilon_0}{c}\mathbf{E} \times \mathbf{B}, \quad \mathcal{L} = \mathbf{r} \times \mathcal{P}. \tag{4}$$

We can also calculate the electric and magnetic helicity densities from

$$\mathcal{H}^e = \frac{\epsilon_0}{c}\mathbf{E}.\mathbf{C}, \qquad \mathcal{H}^m = \frac{\epsilon_0}{c}\mathbf{B}.\mathbf{A},$$

where $\mathbf{A}$ is the usual magnetic vector potential, $\mathbf{E} = \nabla \times \mathbf{C}$ and $\mathbf{C}$ is the electric equivalent of $\mathbf{A}$. Numerical values of these observables are found after integrating over all space.

VISUALISATION OF THE ELECTROMAGNETIC FIELDS:

We will generate a number of families of solutions of Maxwell's equations, but before doing so it is important to discuss how to visualise them. An insightful method based on a suggestion of Kedia[10, 11] is to define a scalar field that can be used to generate the electric and magnetic fields and to study the topology of that scalar field. This method requires generalisation to be applicable here. If we define

$$\phi = \alpha\beta = \phi_1 + i\phi_2,$$

and make use of equation (2), it can easily be shown that

$$\mathbf{R}.\nabla\phi = 0. \tag{5}$$

Taking real and imaginary parts

$$\begin{aligned} \mathbf{E}.\nabla\phi_1 - \mathbf{B}.\nabla\phi_2 &= 0, \\ \mathbf{B}.\nabla\phi_1 + \mathbf{E}.\nabla\phi_2 &= 0. \end{aligned} \tag{6}$$

These equations will always be satisfied. If it happens that the first of these are satisfied because $\mathbf{B}.\nabla\phi_1 = \mathbf{E}.\nabla\phi_2 = 0$ then ϕ_1 and ϕ_2 can be said to represent the magnetic and electric fields respectively. This is indeed what happens for our knotted fields below. If the second of equations (6) are satisfied because the two scalar products are zero then ϕ_1 and ϕ_2 can be said to represent the electric and magnetic fields respectively. Then surfaces can be drawn on which ϕ_1 or ϕ_2 have a constant magnitude More usually these two equations will be satisfied because the scalar products cancel. In that case, both ϕ_1 and ϕ_2 are necessary to describe the total electromagnetic field, but the electric and magnetic fields are not described by ϕ_1 and ϕ_2 separately. The fact that we can define ϕ such that ϕ_1 and ϕ_2 appear symmetrically in equation (6) is a reflection of the symmetry seen in Maxwell's equations (1). Thus we can say that ϕ is a complex scalar field which we can use to examine the topology of electromagnetic fields. At each point in space it has a real and an imaginary component. In the following we will plot "equipotential" surfaces of this quantity. We can then investigate the electromagnetic field through the topology of the ϕ_1 and ϕ_2 surfaces. In what follows we will use $N^s_{p,q}$ to denote the number of separate pieces of surface of $\phi_{1(2)}$. This number is the same for both the real ϕ_1 and imaginary ϕ_2 scalar fields.

We note that once we have any pair (α, β) that yield a solution of Maxwell's equations, we can define $(f(\alpha), g(\beta))$, where f and g are differentiable functions of α and β, and these will also be valid solutions of Maxwell's equations[10]. In particular, the choice of raising either or both of them to an integer power p or q respectively, defines a family of solutions. Below we present results for a number of families of solutions of Maxwell's equations, all of which are calculated for the case $E_0 = 1$. In the first section on knotted electromagnetic fields we provide a detailed discussion of the relationships between the familiar conserved observables energy, momentum and angular momentum, and the zilches, and the topology of the zilches. In later sections we treat other exotic families of integrable fields. Much of the implementation is common to all solutions and so we present the principle results with rather less detail for these cases.

KNOTTED ELECTROMAGNETIC FIELDS:

A particularly interesting solution to these equations was originally investigated by Ranada[2], and has also been derived using Bateman's construction[3, 10]. This is the Hopfion solution:

$$A = \frac{1}{2}(k^2x^2 + k^2y^2 + k^2z^2 - k^2c^2t^2 + 1);$$

$$\alpha = \frac{A - 1 + ikz}{A + ikct}; \quad \beta = \frac{kx - iky}{A + ikct}. \tag{7}$$

This definition gives $|\alpha|^2 + |\beta|^2 = 1$, and it has been shown by Ranada[2, 7, 27, 28] that this implies an analogy with the Hopf fibration defined by the Hopf map $S^3 \to S^2$. This Hopfion

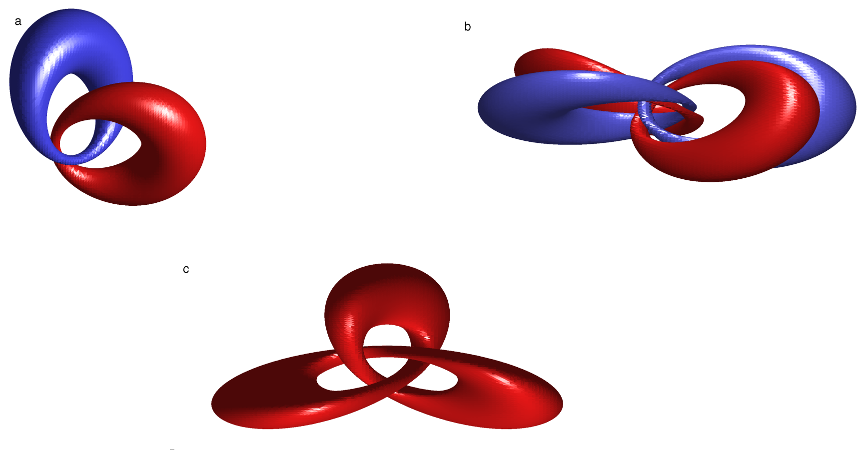

FIG. 1. Knotted ϕ_1 (blue) and ϕ_2 (red) surfaces for: a. $(p, q) = (1, 1), \phi_1 = \phi_2 = 0.45$; b. $(p, q) = (2, 2), \phi_1 = \phi_2 = 0.2$; c. $(p, q) = (2, 3), \phi_2 = 0.1$. All figures are for $k = 1$.

solution was used[10] to generate further solutions. With p and q integer we generate the complete family of toroidal electromagnetic fields. Fig. 1. displays several examples. If p and q are co-prime, $\phi_{1(2)}$ forms a torus knot and if not they form a torus link. For any p and q, each knot and each component of a link is a closed surface of genus 1. The surfaces shown in Figure 1 contain field lines and form tori that are linked through the centre with all other tori. In this case ϕ_1 is a representation of the magnetic field and ϕ_2 represents the electric field directly and Figure 1 demonstrates how the linked tori wrap around each other.

Solutions of Maxwell's equations in a vacuum, such as those above, have been called knotted light. They are a pulse of electromagnetic radiation for which no frequency or wavelength can be defined, which, of course, is true for all pulses. In common with plane waves $\mathbf{E} \cdot \mathbf{B} = 0$ and knotted light has a convergent Fourier transform, it is built from the electromagnetic spectrum. Another analogy with the common view of light has been noted by others[6], at any instant, the field direction rotates as we pass through the knotted structure in certain directions in a manner reminiscent of circularly polarised light. However only a small number of complete rotations can occur and the "wavelength" is not constant in time. A further difference between knotted electromagnetic fields and the electromagnetic spectrum is that the surfaces in Figure 1 are closed and finite for the knotted case, whereas the equivalent surfaces for pure plane waves are open and infinite, as they would be for many forms of light, e.g. a Bessel beam with TE polarisation.

Conserved Quantities:

The expressions for α and β in equation (7) can be substituted into the Bateman construction formulae, with p and q, and expressions for the electric and magnetic fields found as a function of position and time. These can be shown to be solutions of Maxwell's equations explicitly.

A list of conserved quantities, including the non-zero zilches for our hopfion family of solutions to Maxwell's equations, are evaluated and shown in Table I. The results in this

Quantity	Expression
Energy $E_{p,q}$	$2(p+q)k$
Momentum $P_{p,q}$	$(0,0,-2pk/c)$
Angular Momentum $L_{p,q}$	$(0,0,-2q/c)$
Helicity $H^e_{p,q} = H^m_{p,q}$	$2/c$
$Z^{000}_{p,q}$	$(3p+2q)(p+q)k^2$
$Z^{030}_{p,q}$	$(q-3p(p+q))k^2$
$Z^{110}_{p,q} = Z^{220}_{p,q}$	$(q^2+2pq)k^2$
$Z^{330}_{p,q}$	$(3p^2+pq)k^2$

TABLE I. Conserved quantities in electromagnetic knots. Note that there is a sign difference between our angular momentum and that of Hoyos[11] due to a difference in definition. $Z^{\mu,\nu,0}$ are the non-zero zilches for these fields. To obtain the numerical values each of these expressions should be multiplied by $pqp!q!\pi^2\epsilon_0 E_0^2/(p+q)!$

table show that the zilches are not all independent for knotted electromagnetic fields. Simple arithmetic demonstrates that

$$Z^{000}_{p,q} = Z^{110}_{p,q} + Z^{220}_{p,q} + Z^{330}_{p,q}. \tag{8}$$

Further investigation yields that the energies of the various knotted solutions can all be written in terms of the energy of the $(p,q) = (1,1)$ hopfion which in turn can be evaluated exactly as $E_{1,1} = 2\pi^2 k\epsilon_0 E_0^2$.

$$E_{p,q} = \frac{pqp!q!}{(p+q-1)!}E_{1,1} = \frac{2(p+q)pq\pi^2 p!q!\epsilon_0 E_0^2}{(p+q)!}k.$$

This suggests that a profitable way to think about the knotted solutions is not as independent fields, but as a set of excitations of a single field.

Lipkin's Zilches and Knot Properties:

The results in Table 1. allow us to write the zilches entirely in terms of energy, momentum and angular momentum:

$$Z^{000}_{p,q} = -\frac{(p+q)kc}{2}\left(3P^z_{p,q} + 2kL^z_{p,q}\right);$$

$$Z^{030}_{p,q} = -\frac{3(p+q)kc^2 P^z_{p,q}}{2E_{p,q}}\left(P^z_{p,q} + kL^z_{p,q}\right) - \frac{k^2 c}{2}L^z_{p,q};$$

$$Z^{110}_{p,q} = \frac{(p+q)k^2c^2 L^z_{p,q}}{2E_{p,q}}\left(kL^z_{p,q} + 2P^z_{p,q}\right);$$

$$Z^{330}_{p,q} = \frac{(p+q)kc^2 P^z_{p,q}}{2E_{p,q}}\left(3P^z_{p,q} + kL^z_{p,q}\right);$$

and $Z^{220}_{p,q} = Z^{110}_{p,q}$. This shows explicitly that for these torus knotted fields the zilches can be written in terms of other conserved quantities and, thus, are conserved themselves.

In Table II, we show the zilches in a particular set of units. These units are chosen because they are the lowest that leave all zilches as integers. Now we have a prescription.

1. Evaluate $Z^{000}_{p,q}$, $Z^{030}_{p,q}$, $Z^{110}_{p,q}$, and $Z^{330}_{p,q}$.

2. Evaluate $Z^{000}_{q,p}$, $Z^{030}_{q,p}$, $Z^{110}_{q,p}$, and $Z^{330}_{q,p}$.

(p,q)	$(1,5)$	$(2,4)$	$(3,3)$	$(4,2)$	$(5,1)$
$Z^{000}_{p,q}$	78	84	90	96	102
$Z^{030}_{p,q}$	-13	-32	-51	-70	-89
$Z^{110}_{p,q} = Z^{220}_{p,q}$	35	32	27	20	11
$Z^{330}_{p,q}$	8	20	36	56	80
$Y_{p,q}$	1	4	3	2	1
$N^s_{p,q}$	1	2	3	2	1

TABLE II. Table illustrating the link between the zilches and the number of pieces of surface. The first row defines the values of p and q for $p+q=6$. Rows 2 to 5 show the non-zero zilches (divided by $pq\pi^2 p!q!\epsilon_0 E_0^2 k^2/(2(p+q)!)$ to ensure they are dimensionless integers). Row 6 implements the first step of the procedure and we show the gcd of the zilches. The last row is evaluated as the $\gcd(Y_{p,q}, Y_{q,p})$ and is the number of pieces of surface of ϕ_2. The corresponding surfaces are shown in Fig 2. a. - e.

3. Evaluate $Y_{p,q} = \gcd(Z^{000}_{p,q}, Z^{030}_{p,q}, Z^{110}_{p,q}, Z^{330}_{p,q})$ and $Y_{q,p} = \gcd(Z^{000}_{q,p}, Z^{030}_{q,p}, Z^{110}_{q,p}, Z^{330}_{q,p})$.

4. Evaluate $N^s_{p,q} = \gcd(Y_{p,q}, Y_{q,p})$.

$N^s_{p,q}$ is the number of distinct pieces of surface associated with the knotted fields (p,q) and (q,p). The number of surfaces is known as the greatest common divisor (gcd) of p and q from the topology of torus knots/links, but this prescription shows that it is also encoded in the zilches. Let us work through an example. From Table. II, the zilches for $(p,q) = (2,4)$ are 84, -32, 32 and 20, and their gcd is 4. For $(p,q) = (4,2)$ the zilches are 96, -70, 20 and 56, and their gcd is 2. Then to find $N^s_{p,q}$ the gcd of 4 and 2 is 2. This is the number of pieces of surface for $(p,q) = (2,4)$ and $(4,2)$ as shown in Fig. 2b and d. We have verified this prescription explicitly for all knotted electromagnetic fields up to and including $p+q = 15$.

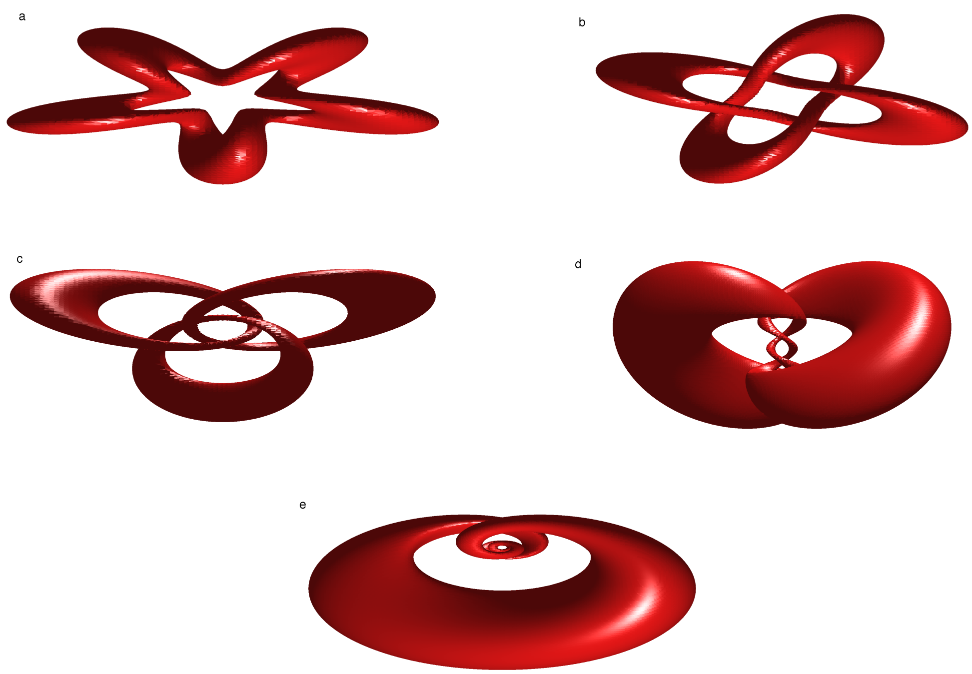

FIG. 2. Knotted surfaces for: a. $(p,q) = (1,5), \phi_E = 0.10$ which has one distinct surface; b. $(p,q) = (2,4), \phi_E = 0.10$ which is composed of two distinct surfaces; c. $(p,q) = (3,3), \phi_E = 0.10$ which is formed from three surfaces; d. $(p,q) = (4,2), \phi_E = 0.10$ which consists of two surfaces, e. $(p,q) = (5,1), \phi_E = 0.20$ which has one distinct surface.

A torus knot, $T(p,q)$, has an unknotting number defined as the number of times the knot must pass through itself to become the unknot. For a torus knot, with p and q not both even, this is given by

$$u(T(p,q)) = \frac{1}{2}(p-1)(q-1).$$

$u(T(p,q))$ can be written in terms of Z^{110} in the units shown in Table 2. For $p+q$ even

$$\begin{aligned} u(T(p,q)) &= \tfrac{1}{2} Z^{110}_{(q-p)/2,p-1}, \qquad q > p, \\ &= \tfrac{1}{2} Z^{110}_{(p-q)/2,q-1}, \qquad q < p, \end{aligned}$$

and for $p+q$ odd

$$u(T(p,q)) = \tfrac{1}{4}\left(Z^{110}_{(q-p-1)/2,p-1} + Z^{110}_{(q-p+1)/2,p-1}\right),\ q>p,$$
$$u(T(p,q)) = \tfrac{1}{4}\left(Z^{110}_{(p-q-1)/2,q-1} + Z^{110}_{(p-q+1)/2,q-1}\right),\ q<p.$$

Similarly the crossing number, $c(T(p,q))$, is the minimum number of crossings that occur in any projection of the knot onto a two-dimensional plane. This number is always at least double the unknotting number. For a torus knot it is given by

$$c(T(p,q)) = \min(p(q-1), q(p-1))$$

The crossing number can also be written in terms of Z^{110}. For $p+q$ odd

$$\begin{aligned} c(T(p,q)) &= Z^{110}_{(q-p+1)/2,p-1}, \qquad q>p,\\ &= Z^{110}_{(p-q+1),q-1}, \qquad q<p, \end{aligned}$$

and for $p+q$ even

$$c(T(p,q)) = \tfrac{1}{2}\left(Z^{110}_{(q-p)/2,p-1} + Z^{110}_{(q-p+2)/2,p-1}\right), \qquad q>p,$$
$$c(T(p,q)) = \tfrac{1}{2}\left(Z^{110}_{(p-q)/2,q-1} + Z^{110}_{(p-q+2)/2,q-1}\right), \qquad q<p.$$

Other simple surface properties can also be obtained. For example, if $q=1$ the surface includes a helix (Fig. 2e for example). The number of turns of the helix is

$$N_t(T(p,1)) = \frac{1}{2}\left(Z^{110}_{p,1} - 3\right).$$

It is now clear that the zilches possess a deep connection to the topology of the knotted electromagnetic fields. We now go on to demonstrate that this is also the case for other families of solutions of Maxwell's equations.

SEGMENTED ELECTROMAGNETIC FIELDS:

A special conformal transformation of the coordinates yields new expressions for α and β that also provide acceptable solutions of Maxwell's equations.

$$A = 1 + 4k^2(x^2+y^2+z^2-c^2t^2);$$

$$\alpha = \frac{2ik(z-ct)-1}{A+4ikct}; \quad \beta = \frac{2k(x-iy)}{A+4ikct}.$$

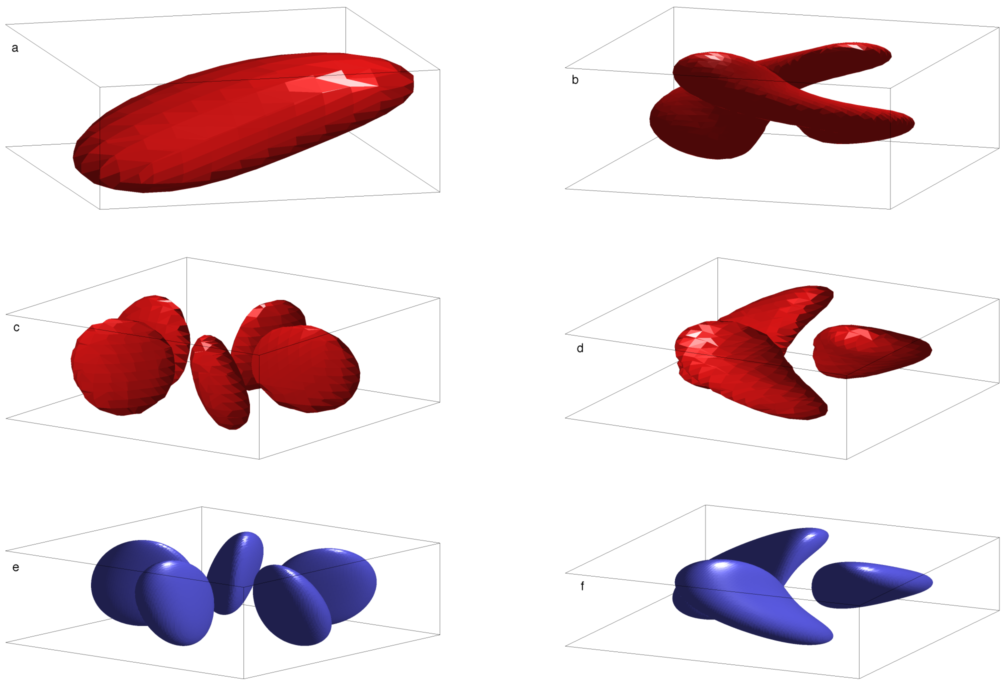

FIG. 3. Segmented surfaces for: a. $(p, q) = (1, 1), \phi_2 = 0.20$ which has one distinct surface; b. $(p, q) = (4, 2), \phi_2 = 0.005$ which is composed of two distinct surfaces; c. $(p, q) = (1, 5), \phi_2 = 0.005$ which is formed from five surfaces; d. $(p, q) = (3, 3), \phi_2 = 0.005$ which consists of three surfaces; e. $(p, q) = (1, 5), \phi_1 = 0.005$ which is formed from five surfaces; f. $(p, q) = (3, 3), \phi_1 = 0.005$ which consists of three surfaces.

In turn these produce a new and very different family of fields. A selection of them, obtained with these expressions with $k = 1$, is shown in Figure 3. The orientation used to display these figures has no meaning and has simply been chosen to display the nature of the surfaces as clearly as possible. In figures 3 a-d we show the surface associated with ϕ_2, the imaginary part of the scalar field ϕ. In figures 3 e-f we show the surface associated with ϕ_1 the real part of ϕ. The values of the p and q and the magnitudes chosen to plot the surface are the same in figure 3 e and f as in c and d respectively. For all of them both the momentum and angular momentum are solely in the z-direction. These representations display a number of properties of these fields. Most obviously the surfaces ϕ_1 and ϕ_2 look very similar and this is usually true. Importantly the number of distinct pieces of field surface is the same for ϕ_1 and ϕ_2, as it is in the knotted case, but this time it is given simply by q, while p is a measure of how much the scalar field "equipotentials" curl around themselves. For these fields there are a number of very simple relations between the zilches and the field properties. Here we again work in units that are the minimum necessary for all the conserved quantities to take on integer values (as described in the caption to Figure 3). Then we obtain the following relations between the standard conserved quantities and the zilches.

$$E_{p,q} = Z^{000}_{p,q}/((2p+2q+1)k),$$

$$P^z_{p,q} = Z^{030}_{p,q}/((2p+2q+1)k),$$

$$L^z_{p,q} = -Z^{110}_{p,q}/((4p+2q)k^2),$$

$$H^e_{p,q} = H^m_{p,q} = -Z^{030}_{p,q}/(2pk^3(2p+2q+1)) = Z^{000}_{p,q}/(2k^3(2(p+q)^2+p+q)).$$

The number of pieces of surface in this case is also easily related to the zilches and to some of the other conserved quantities.

$$N^s_{p,q} = \frac{2(p+q)^2+p+q}{2p+q}\frac{Z^{110}_{p,q}}{Z^{000}_{p,q}} = -\frac{2p^2+2pq+p}{2p+q}\frac{Z^{110}_{p,q}}{Z^{030}_{p,q}} = -\frac{L^z_{p,q}}{kH^e_{p,q}}$$

In table III we write out the values of all conserved quantities for $(p,q) = (2,3)$ as an example and the reader can verify that the above relations hold for this case.

(p,q)	$E_{p,q}$	$P^z_{p,q}$	$L^z_{p,q}$	$H^e_{p,q}$	$Z^{000}_{p,q}$	$Z^{030}_{p,q}$	$Z^{110}_{p,q}$	$Z^{330}_{p,q}$	$N^s_{p,q}$
(2,3)	60	-24	-18	6	660	-264	252	156	3

TABLE III. Table showing values of the conserved quantities for $(p,q) = (2,3)$ where we have used $c = k = 1$. All quantities are determined using the values of α and β in equation (9) and then dividing by $pqp!q!\pi^2\epsilon_0 E_0^2/(2^{2(p+q)-1}(p+q)!$ so that all quantities can be written as integers.

Clearly the number of pieces of surface can easily be found for all members of this family of fields because it is simply equal to q. However again we have seen that it is also encoded in the zilches. It is also clear that the higher p, the higher the circulatory nature of the fields.

"DRIPPING" ELECTROMAGNETIC FIELDS:

A very different family of electromagnetic fields that are solutions of Maxwell's equations in a vacuum is generated by the parameters:

$$A = \frac{1}{2}(k^2x^2 + k^2y^2 + k^2z^2 - k^2c^2t^2 + 1);$$

$$\alpha = \frac{1}{2} - \frac{i(-i + kct + ikx + ky - kz)}{2A + 2ikct}; \quad \beta = \frac{1}{2} - \frac{i(kct - i - kz)}{2A + 2ikct}. \tag{9}$$

Pictures illustrating members of this family of fields are shown in Figure 4. This choice of α and β introduces some overall motion in the y-direction to the field. This means that, not only are P_y and L_y non-zero, but the zilch $Z^{020}_{p,q}$ is also non-zero and now $Z^{110}_{p,q} \neq Z^{220}_{p,q}$ in general. Figure 4 is drawn for fields evaluated at $t = 0$, for later times the topology remains the same although the individual pieces of surface get closer to the lowest piece and tends to flatten out, hence the label "dripping". Clearly there is one major surface and several smaller "droplets" near to it. This is the case in both ϕ_1 and ϕ_2. Figures 4 e and f show ϕ_1 for the same values of p and q as figure 4 a and b show ϕ_2. In a and e the magnitudes differ because the smaller surface is too small to see on this scale if we equalise the magnitudes. The distance between the main sheet and the nearest droplet increases with q for $p + q$ constant.

We have evaluated the constant quantities for the fields generated in this case and find relations between them. This time we divide each calculated zilch by $u = \frac{pqp!q!\pi^2\epsilon_0 E_0^2 k^2}{2^{2(p+q)}(p+q)!}$ making them dimensionless. Then all quantities are integers and we find

$$N^s_{p,q} = gcd(Z^{000}_{p,q}) = gcd(Z^{110}_{p,q} + Z^{220}_{p,q} + Z^{330}_{p,q}) \;\; \forall \;\; p + q = \text{constant}.$$

(p, q)	$(1, 4)$	$(2, 3)$	$(3, 2)$	$(4, 1)$
Z^{000}	70	275	530	500

TABLE IV. Table showing values of the conserved quantities for $(p, q) = (2, 3)$. All quantities are determined using the values of α and β in equation (9) and then dividing by $pqp!q!\pi^2\epsilon_0 E_0^2 k^2/(2^{2(p+q)}(p+q)!)$ so that all quantities can be written as integers.

For example the results for Z^{000}, for all solutions with $p + q = 5$ are shown in Table IV. It is obvious that their greatest common divisor is five and that is indeed the number of

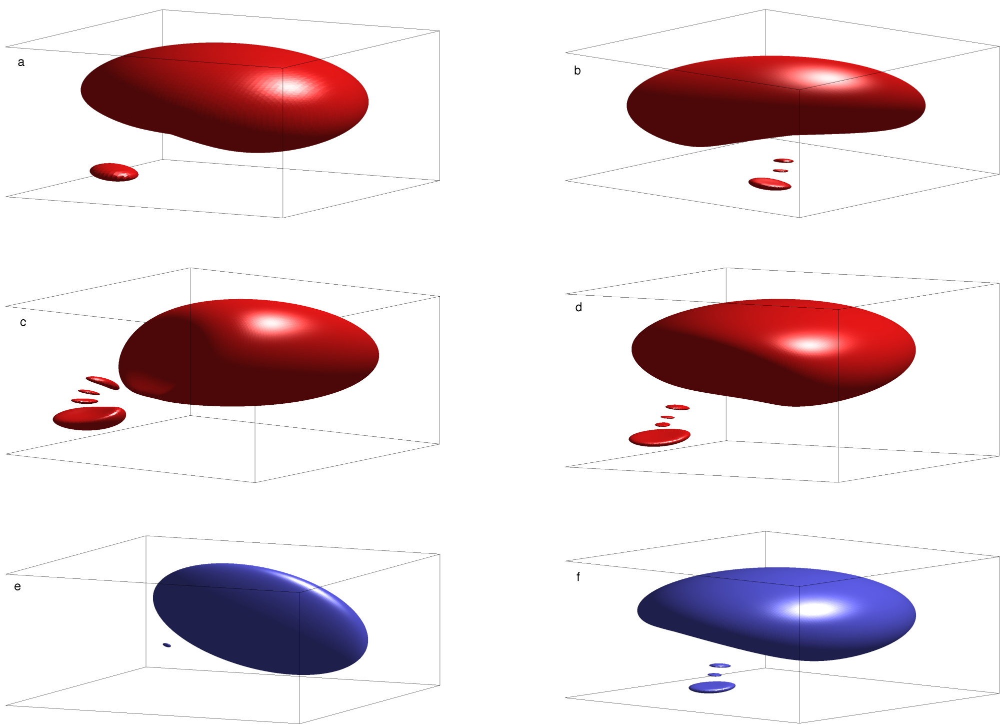

FIG. 4. Surfaces for the "dripping" fields with: a. $(p, q) = (1, 1), \phi_2 = 0.20$ which has two distinct surfaces; b. $(p, q) = (1, 3), \phi_2 = 0.05$ which is composed of four distinct surfaces; c. $(p, q) = (4, 1), \phi_2 = 0.04$ which is formed from five surfaces; d. $(p, q) = (2, 3), \phi_2 = 0.03$ which consists of five surfaces; e. $(p, q) = (1, 1), \phi_1 = 0.30$ which has two distinct surfaces; f. $(p, q) = (1, 3), \phi_1 = 0.05$ which is composed of four distinct surfaces.

surfaces found for all these members of the family described by equations (9). Two examples of the $p + q = 5$ family are shown in Figure 4. For these droplet solutions the zilches are also related to other conserved properties, so for example

$$Z^{000}_{p,q} = -k(p+q)(\frac{3}{2}P^z_{p,q} + kL^z_{p,q})$$

and further relations between the other zilches and observables can be found. Their motion in the y-direction is also encoded in the zilches because

$$P^y_{p,q} = -kL^y_{p,q} = \frac{2}{(3(p+q)+1)k} Z^{020}_{p,q}.$$

Hence we have yet another example where both the observables and the topology of the fields is concealed in the zilches.

Conclusions

Lipkin's zilches have remained enigmatic for around half a century. There have been a number of attempts to interpret and discuss them. In this study we have presented a previously unsuspected view of the zilches which we believe sheds considerable light on their interpretation. We have performed a systematic study of several families of vacuum electromagnetic fields. The fields we have studied produce a convergent value for observables when we integrate their density over all space, and closed surfaces when we plot contours of constant ϕ. In all such cases we have found that:

1. Because we have explicit analytic expressions for the zilches we have shown categorically that for these fields they can be expressed exactly in terms of the energy, momentum, angular momentum and helicity of the fields.

2. The zilches contain information about the topology of the field, particularly the number of distinct sheets of closed surface. In the case of knotted fields we have also been able to determine properties such as the crossing number and unknotting number. It seems that all these quantities can be determined from the number theoretic properties of the zilches. There also appears to be an unquantified correlation between the zilches and the degree of chirality in the field.

On the basis of this work we conjecture that this is always the case for this class of fields. We have not been able to find such a relationship between the zilches and other properties of the electromagnetic fields in cases where the field energy does not converge, such as plane wave and constant field solutions. In these cases when we plot the fields using the methods discussed above, we find the surfaces generated are neither finite nor closed. This further supports our hypothesis that the zilches describe the topology of the fields when their properties are convergent, but cannot do so when the field properties become divergent.

To summarise then, we have found interpretations of the zilches for several examples of this class of fields in terms of both the physical constants associated with the electromagnetic fields and in terms of their topology. We speculate that this can be done for all fields whose properties are convergent. We have certainly not found any examples for which this is not the case. The results of this paper provide a previously unsuspected insight into the properties of the zilches which we expect will give them a more central role in electromagnetic theory in the future.

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