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Modular Representations and Invariants of Elementary Abelian p-Groups

by

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Declaration

I hereby declare that the content of this dissertation is my own and where work from outside sources has been used it has been properly and accurately cited.

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6th August, 2018

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Abstract

The purpose of this thesis is to develop tools to more easily classify the modular representations of elementary abelian p-groups and better understand their invariant rings. Since these groups almost always have wild representation type complete classification of the indecomposables is considered impossible and as such an alternative perspective is required.

We reformulate the representation classification in the perspective of classifying maximal abelian subgroups of unipotent groups. Thence we express the problem as determining finitely many 'covering' homomorphisms of the form $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$ whose images collectively contain the images of all such representations up to equivalence. To aid in this we attach a combinatorial equivalence invariant object to modular p-group representations thereby allowing us to segment the problem and more easily distinguish between inequivalent families.

Using these tools we build upon the work of [11] and develop a full set of covering homomorphisms for all modular elementary abelian p-groups in $GL_4(\mathbb{F}), GL_5(\mathbb{F})$ and $GL_6(\mathbb{F})$. In doing so we also provide covering homomorphisms for select families in arbitrary dimension with specific patterns in their combinatorial invariant. By way of example we use these to provide an explicit construction for the Sylow p-subgroups of the finite orthogonal groups.

Thereafter our focus switches to invariant rings. Given a matrix group in the image of a homomorphism $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$ we explore methods of recovering the $W \leq (\mathbb{F}^d, +)$ used to generate the group purely through its action on specific elements in the symmetric algebra of the dual, properties of which are indicative of properties of the invariant ring. Using this we provide an alternative explicit construction for the invariant rings implicitly generated in [8]. Further we generalise a long-exploited technique for inductively defining invariants from those of maximal subgroups. After classifying the invariant rings and fields of several hitherto classified families we focus on the four-dimensional modular elementary abelian p-groups with rank 2, providing their invariant rings if Cohen-Macaulay and algorithmic methods to procure it otherwise.

Contents

Chapter 1

Introduction

Overview

Whilst the worlds of representation theory and invariant theory both flourish in recent years there seems to exist a dichotomy between the two. Whilst the representation theoretic world begins to gravitate towards more fruitful modern techniques – representations of algebras, category theory, sheaves and cohomology – invariant theory remains caught, relying on the classical focus they have left behind. Frequently it is left to invariant theorists to develop their own tools to construct and study their initial groups.

The modular case, when the field characteristic $p > 0$ divides the group's order, causes complications. Many of the grand structural questions of non-modular representation and invariant theory have long been answered. In the case of wild representation type – a case in which almost all p-groups sit in the modular case – it is generally considered to be impossible to obtain a full classification of decomposable modules. Hence the representation theorists divert attention.

However the deriving a modular representation's invariants from those of its Sylow p-subgroups enjoys many of the luxuries of the non-modular case. It is thus a desire to understand the modular representations of p-groups to better understand their invariants. This is the baton we grasp in our study of the modular representations and invariants of the elementary abelian p-groups $\mathbb{Z}_p^r$.

This document may be considered a spiritual, if unlicensed, successor to the paper [11]. Therein the authors classified all three-dimensional modular $\mathbb{Z}_p^r$-representations and thence determined their invariants when $r \leq 3$. Here we mimic and generalise many of their techniques, alongside developing original structural tools, to many more dimensions and ranks.

We divide this document fairly neatly into two parts, the representation theory and the invariant theory, flanked by this introduction and their conclusions. These sections are as self contained as possible, making explicit references between them as necessary. If the reader is particularly invariant-inclined one may begin there to give motivation to the section prior.

We proffer much of the preliminary definition from the literature in Section 1.1, all required and well-worn throughout this document, including the heavily leaned-upon 'socle-type'. Section 1.2 recaps the work of [11] in dimension 3 and provides motivation and philosophy going forward. We provide a rough guideline to the document's structure as follows.

Representation Theory

Excluding the cyclic groups and $\mathbb{Z}_2^2$ the representation type of an elementary abelian p-group is wild and thus the problem of classifying the indecomposable modules is generally looked upon as folly. However this does not diminish our interest and so we develop techniques of our own.

We divide the problem of classification into two components: We deal with parameterisation in Section 2.1 by developing *covering homomorphisms* $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$ from which we may construct representations we desire. Section 2.2 then divides representations into smaller families by attaching a combinatorial equivalence invariant, the *socle tabloid*, to them. These sections are distinct, the latter applying to any general p-group. We combine them in Section 2.3 to demonstrate how these techniques allow us to classify infinitely many families of representations from select small-dimensional examples.

Hereafter we begin concrete formulation, classifying those $\mathbb{Z}_p^r$-representations with longest possible socle-series in Section 2.4 and using iterative techniques extend to many more families in Section 2.5. On the opposite end of the spectrum Section 2.6 deals with $\mathbb{Z}_p^r$-representations whose socle series has length 3, the smallest interesting case, reformulating the problem of classification into one of invariants of algebraic groups. For flavour this section includes a direct application by providing an explicit construction for the Sylow p-subgroups of the finite orthogonal groups.

We continue the thread of [11] in their classification of all 3-dimensional $\mathbb{Z}_p^r$-representations and use Sections 2.7, 2.8 and 2.9 to classify all modular $\mathbb{Z}_p^r$-representations in dimensions 4, 5 and 6 respectively.

Invariant Theory

Having concluded constructing $\mathbb{Z}_p^r$-representations we go about the business of examining their invariants. Unless otherwise stated this part of the document is self-contained. Having recapped the notation and useful results from the literature in Section 3.1, we develop two tools for invariant construction in situ.

Section 3.2 develops the matrix minor method of constructing invariants from *recovery functions* which allows us to directly transfer information about the group's generation into an invariant theoretic context. We demonstrate by explicitly constructing a generating set for the invariant ring of the hyperplane groups, an implicit construction having been given in [8].

We take this into Section 3.3 applying representations with socle-type $(m, 1, \ldots, 1)$ – as constructed in Section 2.5 – yielding their invariant fields and thence a method of constructing a basis for their invariant rings.

Section 3.4 diverts course to develop a technique of iteratively constructing invariants from those of maximal subgroups. We demonstrate their effectiveness upon representations with socle-length 2.

Combining all prior sections together we spend Section 3.5 studying and constructing the invariant fields of a family of socle-length 3 representations whose images consist wholly of bireflections.

We conclude with Section 3.6 in which we examine the invariants of four-dimensional $\mathbb{Z}_p^r$-representations. Therein we consider all rank 2 representations, providing either generating sets for the invariant ring when the ring is Cohen-Macaulay, or otherwise the invariant field with an algorithm to construct the ring thereafter.

1.1 Preliminaries

In this section we shall introduce the preliminary definitions and results which provide impetus for all questions that follow.

1.1.1 Basic Definitions

The notation we introduce here shall be consistent throughout the remainder of this document.

We denote by $\mathbb{F}$ a field of prime characteristic $\mathrm{char}(\mathbb{F}) = p > 0$ and $\overline{\mathbb{F}}$ its algebraic closure. By $GL_n(\mathbb{F})$ we denote the general linear group over $\mathbb{F}$ of dimension n. For V an n-dimensional $\mathbb{F}$-vector space we write $GL(V) \cong GL_n(\mathbb{F})$ as specifically the general linear group acting on V. When we wish to view $\mathbb{F}^n$ as an additive group we shall denote it $(\mathbb{F}^n, +)$ and its elements $\underline{c} := (c_1, \ldots, c_n) \in (\mathbb{F}^n, +)$.

We take $\mathbb{N}$ to be the natural numbers *without* zero and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n, m \in \mathbb{N}$ we write $M_{n,m}(\mathbb{F})$ for the set of all $n \times m$ matrices over $\mathbb{F}$. For shorthand we write $M_n(\mathbb{F}) := M_{n,n}(\mathbb{F})$. We denote by $A[i,j]$ the (i,j)th entry of $A \in M_{n,m}(\mathbb{F})$. As the title of this document may suggest we desire abelian representations and as such we shall make heavy use of the matrix commutator

$$[A, B] := AB - BA.$$

Given $\underline{v} \in \mathbb{N}_0^r$ with $\sum v_i = s$ we use the multinomial short-hand

$$\binom{s}{\underline{v}} := \binom{s}{v_1, \ldots, v_r} = \frac{s!}{v_1! \cdots v_r!}.$$

We write $U_n(\mathbb{F})$ for the set of n-dimensional upper-triangular unipotent matrices over $\mathbb{F}$. More specifically for $(m_1, \ldots, m_k) \in \mathbb{N}^k$ with $\sum_i m_i = n$ we denote

$$U_{(m_1,\ldots,m_k)}(\mathbb{F}) := \left\{ \begin{bmatrix} I_{m_1} & \Gamma_{12} & \cdots & \Gamma_{1k} \\ 0 & I_{m_2} & \cdots & \Gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_k} \end{bmatrix} \;\middle|\; \Gamma_{i,j} \in M_{m_i,m_j}(\mathbb{F}) \right\}.$$

As multiplicative groups we note that $U_{(m_1,\ldots,m_k)}(\mathbb{F}) \leq U_n(\mathbb{F}) \leq GL_n(\mathbb{F})$.

Let p be a prime number and denote by $\mathbb{Z}_p$ the group of integers modulo p. We write

$$\mathbb{Z}_p^r := \underbrace{\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p}_{r}$$

to denote an elementary abelian p-group of rank r.

For G a (finite) group we let V be a finite-dimensional left $\mathbb{F}G$-module. By choosing a basis $\mathcal{B} = \{v_1, \ldots, v_n\} \subset V$ the linear (left) action of $g \in G$ may be presented by left multiplication of a matrix on column vectors in V.

By a *representation of* G we shall often hop between referring to an explicit $\mathbb{F}G$-module V, a homomorphism $G \to GL(V)$ and the image of such a homomorphism. Which of these we use at any given time we hope to make clear in situ. We consider representations as *modular* when char($\mathbb{F}$) divides $|G|$.

We often use the short-hand $\mathbb{Z}_p^r \cong G \leq GL(V)$ to refer to an $\mathbb{F}\mathbb{Z}_p^r$-module V with a given basis upon which the action of the group induces the matrix group $G \leq GL_n(\mathbb{F})$.

Given $\mathbb{Z}_p^r \cong G \leq GL(V)$ we observe the dual module $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ of V with dual basis $\mathcal{B} = \{v_1^*, \ldots, v_n^*\}$. Then V^* is a right $\mathbb{F}\mathbb{Z}_p^r$-module upon which $\sigma \in \mathbb{Z}_p^r$ acts by $(x \cdot \sigma)(v) = x(\sigma \cdot v)$. This action is represented by G acting upon row vectors in V^* by right multiplication.

We remark that there is a one-to-one correspondence between quotient modules of V and submodules of its dual V^* and vice-versa induced by duality, as seen in citeLandrock Lemma 6.5 for instance. We shall often make use of this correspondence.

Two modules $V, V' \in \mathbb{F}G$-mod are *equivalent* if there exists a linear, invertible $T : V \to V'$ such that $g \cdot T(v) = T(g \cdot v)$ for all $g \in G$. Two representation homomorphisms $\sigma, \sigma' : G \to GL_n(\mathbb{F})$ are said to be *equivalent* if there exists an $A \in GL_n(\mathbb{F})$ such that $\sigma(g) = A\sigma'(g)A^{-1}$ for all $g \in G$. The modules induced by such homomorphisms are equivalent if and only if the homomorphisms themselves are equivalent, and so we use these terms interchangeably.

Given an $\mathbb{F}G$-module V we denote by $\mathbb{F}[V] := S(V^*)$ the symmetric algebra of the dual space. We extend the action of G on V^* to an action on $\mathbb{F}[V]$ by extending multiplicatively. A prime focus of the latter part of this document will be the invariant ring

$$\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] \mid f \cdot g = f, \ \forall g \in G\}.$$

1.1.2 Socle Series and Socle-Type

A significant effort in this paper is to classify all modular representations of $\mathbb{Z}_p^r$ in a given dimension up to equivalence. In order to better distinguish between these cases we utilise more representation-specific properties which are invariant under equivalence. To this end we note the following lemma which may be found as Lemma 4.0.1 in [12], among others.

Lemma 1.1.1. *Let P be a p-group, $\mathbb{F}$ a field of characteristic p and V a positive-dimensional $\mathbb{F}P$-module. Then*

$$V^P := \{v \in V \mid \sigma \cdot v = v, \ \forall \sigma \in P\} \neq \{0\}.$$

In particular given a representation $\mathbb{Z}_p^r \cong G \leq GL(V)$ the value of $\dim(V^G) > 0$ is invariant under equivalence. We also have that V^G is the socle of the representation, i.e. the sum of the irreducible submodules. This is because the only irreducible submodule is the one-dimensional trivial module. Hence $\mathrm{Soc}(V) = V^G$.

Using this construct inductively $\mathrm{Soc}_1(V) := \mathrm{Soc}(V)$ and

$$\mathrm{Soc}_{i+1}(V)/\mathrm{Soc}_i(V) := \mathrm{Soc}\big(V/\mathrm{Soc}_i(V)\big).$$

Since at each stage the socle of $V/\mathrm{Soc}_i(V)$ is simply the fixed point subspace, which we have ascertained is nonzero, this sequence can be easily calculated when the representation is explicit and terminates at $\mathrm{Soc}_k(V) = V$. Hence we acquire the *socle series*

$$0 < \mathrm{Soc}_1(V) < \mathrm{Soc}_2(V) < \cdots < \mathrm{Soc}_k(V) = V.$$

Naturally the same construction can be performed on V^* to acquire

$$0 < \mathrm{Soc}_1(V^*) < \mathrm{Soc}_2(V^*) < \cdots < \mathrm{Soc}_\ell(V^*) = V^*.$$

It is known – see, for instance, [22] Lemma 8.2 – that the socle-series has the same length as the Loewy series defined by

$$\begin{aligned} &\mathrm{Rad}_1(V) := \mathrm{Rad}(V) = \bigcap \{W \mid W \text{ a maximal submodule of } V\}, \\ &\mathrm{Rad}_{i+1} := \mathrm{Rad}(\mathrm{Rad}_i(V)). \end{aligned}$$

This coupled with the fact that

$$\mathrm{Rad}_i(V) = \{v \in V \mid x(v) = 0 \ \ \forall x \in \mathrm{Soc}_i(V^*)\}$$

as in [22] Lemma 8.4, implies that the socle-series' of V and V^* have the same length. We refer to this integer $k = \ell$ as the *socle-length* of V (or G). The following definition is inspired by the work of [11].

Definition 1.1.2. Given $\mathbb{Z}_p^r \cong G \leq GL(V)$ define the positive integers

$$m_1 := \dim\big(\mathrm{Soc}_1(V)\big), \quad m_i := \dim\big(\mathrm{Soc}_i(V)/\mathrm{Soc}_{i-1}(V)\big),$$

$$n_1 := \dim\big(\mathrm{Soc}_1(V^*)\big), \quad n_i := \dim\big(\mathrm{Soc}_i(V^*)/\mathrm{Soc}_{i-1}(V^*)\big),$$

for $i = 2, \ldots, k$ where k is the socle-length of V. Then we say V and G have *socle-type* $(m_1, \ldots, m_k)$ and *dual socle-type*, or simply *dual-type*, $(n_1, \ldots, n_k)$.

Suppose G acts upon the basis $\{v_1, \ldots, v_n\}$ for V such that $\{v_1, \ldots, v_{m_1+\cdots+m_i}\}$ forms a basis for $\mathrm{Soc}_i(V)$ for each $i \in [\![1, k]\!] := \{1, 2, \ldots, k\}$. We call such a basis *socle-conforming* and so $G \subset U_{(m_1,\ldots,m_k)}(\mathbb{F})$.

We define these for the purpose of classification. The socle- and dual-type of a representation are invariant under equivalence, and socle-conforming bases provide a pseudo-canonical form which makes the socle-type visible at a glance. We divide our classifications into distinct inequivalent families parameterised by such invariants.

1.2 Three-Dimensional Representations

Here we summarise the work performed in [11] in their classification of all three-dimensional $\mathbb{Z}_p^r$-representations. They do so by dividing representations into families by what we've come to call their socle-type. Honourable mention goes to the socle-type (3) representation which is, by construction, trivial.

1.2.1 Socle-Types $(2, 1)$ and $(1, 2)$

Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(2, 1)$, that is $V \in \mathbb{F}\mathbb{Z}_p^r$-mod with socle-type $(2, 1)$ induces the action of the matrix group $G \leq GL_n(\mathbb{F})$. By choosing a socle-conforming basis for V we may ensure that each element $g \in G$ is of the form

$$\sigma_{2,1}(c_1, c_2) := \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix} \in U_{(2,1)}(\mathbb{F})$$

for some $c_1, c_2 \in \mathbb{F}$. Note then that $\sigma_{2,1}(\underline{c})\sigma_{2,1}(\underline{d}) = \sigma_{2,1}(\underline{c}+\underline{d})$, that is

$$\sigma_{2,1} : (\mathbb{F}^2, +) \to U_{(2,1)}(\mathbb{F})$$

is a group homomorphism. Hence G is conjugate to $\sigma_{2,1}(W)$ for some $W \leq (\mathbb{F}^2, +)$. Thus by choosing an appropriate injection $\iota : \mathbb{Z}_p^r \to W$ we see that the representation induced by V is equivalent to $\sigma_{2,1} \circ \iota$, that is a representation *induced* by $\sigma_{2,1}$.

Similarly should V instead have socle-type $(1,2)$ then given a socle-conforming basis $g \in G$ adopts the form

$$\sigma_{1,2}(c_1, c_2) := \begin{bmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in U_{(1,2)}(\mathbb{F})$$

where $\sigma_{1,2}(\underline{c})\sigma_{1,2}(\underline{d}) = \sigma_{1,2}(\underline{c}+\underline{d})$. Thus V is then equivalent to a representation induced by $\sigma_{1,2}$.

Fortuitously $U_{(2,1)}(\mathbb{F})$ and $U_{(1,2)}(\mathbb{F})$ are both already abelian groups. Indeed this extends to all

$$U_{(m_1,m_2)}(\mathbb{F}) = \left\{ \begin{bmatrix} I_{m_1} & \Gamma \\ 0 & I_{m_2} \end{bmatrix} \,\middle|\, \Gamma \in M_{m_1,m_2}(\mathbb{F}) \right\} \leq U_{m_1+m_2}(\mathbb{F}).$$

There thus exist natural homomorphisms between the rank r subgroups of $(\mathbb{F}^{m_1 m_2}, +)$ and the images of $\mathbb{Z}_p^r$-representations with socle-type (m_1, m_2) up to equivalence, depending on how one wishes to unravel the corner matrices.

Remark. Note that not all representations induced from such homomorphisms may have the desired socle-type. For instance if for some $\alpha \in \mathbb{F}$ we have $c_2 = \alpha c_1$ for all $\underline{c} \in W \leq (\mathbb{F}^2, +)$ then a representation induced from $\sigma_{1,2}(W)$ shall have socle-type $(2,1)$. In a sense these additional cases are 'degenerate'. Hence we often demand that the coefficients used to construct the homomorphism be, in some precise sense, sufficiently independent throughout the chosen subgroup of $(\mathbb{F}^d, +)$ in order to acquire the desired form.

For longer socle-types we are required to seek maximal abelian subgroups of $U_M(\mathbb{F})$ as we shall see in the only remaining dimension 3 case.

1.2.2 Socle-Type $(1, 1, 1)$

Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(1, 1, 1)$. By choosing a socle-conforming basis $\{v_1, v_2, v_3\}$ for V we ensure $G \leq U_{(1,1,1)}(\mathbb{F}) = U_3(\mathbb{F})$. Since $U_3(\mathbb{F})$ is non-abelian we seek abelian subgroups therein.

Suppose we wish to find all abelian $G \leq U_{(1,1,1)}(\mathbb{F})$ with socle-type $(1, 1, 1)$. These must contain at least one element

$$C = \begin{bmatrix} 1 & c_{1,2} & c_{1,3} \\ 0 & 1 & c_{2,3} \\ 0 & 0 & 1 \end{bmatrix}$$

for which $c_{2,3} \neq 0$, lest we degrade to socle-type $(1, 2)$. For any other D in our abelian subgroup we have

$$[C, D][1, 3] = c_{1,2}d_{2,3} - d_{1,2}c_{2,3} = 0 \tag{1.1}$$

and so $d_{1,2} = \frac{c_{1,2}}{c_{2,3}} d_{2,3}$. Since the $[1, 2]$ entry of some element must also not vanish, lest we degrade into socle-type $(2, 1)$, it follows that $c_{1,2} \neq 0$. In particular $(C - I_2)^2 \neq 0$ and so we acquire the following.

Corollary 1.2.1. *There exist no modular $\mathbb{Z}_2^r$-representations with socle-type* $(1, 1, 1)$.

It is an ongoing question to determine which socle-types have no associated representations in fields of a given characteristic. Sections 2.3.2 and 2.6.3 examine this problem in more detail.

Once we find an element with a large Jordan normal form, such as the C given above, we wish to view these elements as acting on $V \cong \mathbb{F}^3$ by left multiplication and fix a basis on which it acts conveniently. Suppose C acts as above on the basis $\{v_1', v_2', v_3'\}$ we upper-triangularly transform this into

$$\{v_1, v_2, v_3\} := \{(C - I_3)^2 \cdot v_3', (C - I_3) \cdot v_3', v_3'\}.$$

As an upper-triangular change of basis this leaves the action of $U_3(\mathbb{F})$ unchanged as a whole. However C acts on this basis in its Jordan normal form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We find it computationally convenient to adopt the following: Since $p > 2$ we alter the dual basis to become $\{w_1^*, w_2^*, w_3^*\} := \{2v_1^* + 3v_2^* + v_3^*, v_2^* + v_3^*, v_3^*\}$ on which C acts by

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Remark. We shall often find ourselves possessing an element J which acts as a single Jordan block on a basis $\{v_1, \ldots, v_n\}$, so that $J \cdot v_{i+1} = v_{i+1} + v_i$, and wish to convert this into a binomial form by an upper-triangular basis change. This may be achieved when $p \geq n$ by choosing the dual basis $\{w_1^*, \ldots, w_n^*\}$ where

$$w_{n+1-m}^* := \sum_{i=1}^{m} (i-1)! S_2(m,i) v_{n+1-i}^* \tag{1.2}$$

for $S_2(i,j)$ the Stirling numbers of the second kind. By utilising well-known relations we see that

$$\begin{aligned}
w_{n+1-m}^* \cdot J &= \sum_{i=1}^{m} (i-1)! S_2(m,i) v_{n+1-i}^* + \sum_{i=2}^{m} (i-1)! S_2(m,i) v_{n+2-i}^* \\
&= \sum_{i=1}^{m} (i-1)! v_{n+1-i}^* \left[S_2(m,i) + i S_2(m,i+1) \right] \\
&= \sum_{i=1}^{m} (i-1)! v_{n+1-i}^* \left[S_2(m+1,i+1) - S_2(m,i+1) \right] \\
&= \sum_{i=1}^{m} (i-1)! v_{n+1-i}^* \left[\sum_{j=i}^{m} \binom{m}{j} S_2(j,i) - \sum_{j=i}^{m-1} \binom{m-1}{j} S_2(j,i) \right] \\
&= \sum_{i=1}^{m} (i-1)! v_{n+1-i}^* \sum_{j=i}^{m} \binom{m-1}{j-1} S_2(j,i) \\
&= \sum_{j=1}^{m} \binom{m-1}{j-1} \sum_{i=1}^{j} (i-1)! S_2(j,i) v_{n+1-i}^* \\
&= \sum_{j=1}^{m} \binom{m-1}{j-1} w_{n+1-j}^*.
\end{aligned}$$

Thus the action of J on the resulting basis shall render J into the form $J = [\binom{n-i}{n-j}]_{i,j}$.

Returning to our example it follows from (1.1) that $d_{2,3} = 2d_{1,2}$ for all other elements D of our subgroup. Thus all modular representations of $\mathbb{Z}_p^r$ with socle-type $(1,1,1)$ have images conjugate to a subgroup of

$$\widetilde{U_3(\mathbb{F})} := \left\{ \begin{bmatrix} 1 & 2c_{2,3} & c_{1,3} \\ 0 & 1 & c_{2,3} \\ 0 & 0 & 1 \end{bmatrix} \mid c_{2,3}, c_{1,3} \in \mathbb{F} \right\} \leq U_3(\mathbb{F}). \tag{1.3}$$

Theoretically our task is complete. However easily procuring subgroups therein is not a simple affair since the multiplication in this group is tangled. We thus follow the advice of our predecessors and re-parameterise our matrix entries.

Proposition 1.2.2 ([11], Proposition 5.1)**.** *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(1,1,1)$. Then G is conjugate to a subgroup in the image of the homomorphism*

$$\sigma_{111} : (\mathbb{F}^2, +) \to U_3(\mathbb{F}), \quad \sigma_{111}(\underline{c}) := \begin{bmatrix} 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus V is equivalent to a representation induced by σ_{111}, that is a representation of the form $\sigma_{111} \circ \iota$ for some injection $\iota : \mathbb{Z}_p^r \to W \leq (\mathbb{F}^2, +)$.

Proof. The set $\sigma_{111}(\mathbb{F}^2)$ is equal to the set $\widetilde{U_3(\mathbb{F})}$ procured in (1.3) and thus contains the images of all such representations up to equivalence. Thereafter we verify that $\sigma_{111} : (\mathbb{F}^2, +) \to \widetilde{U_3(\mathbb{F})}$ is a homomorphism and that conjugacy of images implies equivalence of representations up to only precomposition with an automorphism of $\mathbb{Z}_p^r$. The freedom of choice in ι then yields the result. $\square$

Classification of the modular $\mathbb{Z}_p^r$-representations with socle-type $(1,1,1)$ required more effort than with socle-length 2. We first had to find the maximal abelian subgroups of $U_{(1,1,1)}(\mathbb{F})$ and then reparameterise their entries to acquire a homomorphism from $(\mathbb{F}^2, +)$ into such groups.

This process shall act as a prototype for our general classification of $\mathbb{Z}_p^r$ representations explored in generality in Section 2.1.

Conclusion

Here we recapped the work of [11] in which the authors determine the three-dimensional modular representations of $\mathbb{Z}_p^r$ up to equivalence.

We slightly reworked their original construction to demonstrate the techniques we generalise in chapter 2. We shall thence apply these to our classification of all modular $\mathbb{Z}_p^r$-representations in dimensions $n = 4, 5, 6$, as well as certain general-dimensional families which conveniently extend from their smaller brethren.

Chapter 2

Representation Theory

The modular representation theory of p-groups suffers many natural knock-backs. In particular since most have *wild* representation type the quest to fully acquire the indecomposables has been dubbed unsolvable [5]. In lieu of such conveniences we develop methods to aid the classification of modular $\mathbb{Z}_p^r$-representations, with focus on dividing the problem into smaller manageable families and thence the management itself. Our methods do not classify on indecomposable modules alone but we shall briefly touch upon how to distinguish them when possible.

Section 2.1 extends the methods of Section 1.2 associating homomorphisms from groups of the form $(\mathbb{F}^d, +)$ onto the images of our families. We acquire through this a *covering homomorphism* for the family.

Separately Section 2.2 refines the socle-type further into the *socle tabloid*, and further still into deconstruction data, to separate representations into finitely many inequivalent, parameterisable families. This section, unlike the remainder of the thesis, is valid for any given p-group.

Section 2.3 combines the prior sections and illustrates how this combination extends previously known representations into infinite families with relatively little effort.

Using this cut-and-cover system we classify all representations with socle-type $(1, \ldots, 1)$ in Section 2.4. Thence we extend to socle-type $(m, 1, \ldots, 1)$ and $(1, m, 1, \ldots, 1)$ in Section 2.5.

In Section 2.6 we take time to study the representations with socle-length 3. Therein we discover a reformulation of the classification problem as an invariant-theoretic problem for algebraic groups. We conclude by demonstrating a method for constructing the Sylow p-subgroup of finite orthogonal groups.

Collecting prior general results and examples together we explicitly provide the reader all modular $\mathbb{Z}_p^r$-representations in dimensions 4, 5 and 6 in sections 2.7, 2.8 and 2.9 respectively.

2.1 Covering Homomorphisms

We remarked in Section 1.2.1 that since the group

$$U_{(m_1,m_2)}(\mathbb{F}) := \left\{ \begin{bmatrix} I_{m_1} & \Gamma_{12} \\ 0 & I_{m_2} \end{bmatrix} \mid \Gamma_{12} \in M_{m_1,m_2}(\mathbb{F}) \right\}$$

is abelian we may write the image of any $\mathbb{Z}_p^r$-representation with socle-type (m_1, m_2) as a subgroup of $U_{(m_1,m_2)}(\mathbb{F})$ up to equivalence. Thus the group isomorphism $\sigma : (M_{m_1,m_2}(\mathbb{F}), +) \to U_{(m_1,m_2)}(\mathbb{F})$ yields a relation between those representations and certain additive subgroups of $M_{m_1,m_2}(\mathbb{F}) \cong (\mathbb{F}^{m_1 m_2}, +)$. These homomorphisms often yield more 'degenerate' representations than those we covet.

This shall act as a prototype for a general construction introduced in this chapter. We determine a method of constructing homomorphisms from additive groups of vectors into maximal abelian subgroups of $U_M(\mathbb{F})$ which collectively contain the images all of the representations with socle-type $M = (m_1, \ldots, m_k)$ up to equivalence within their image.

Here we demand that $\mathrm{char}(\mathbb{F}) = p \geq k$, however Section 2.1.4 illustrates how this may be bypassed in practical constructions.

2.1.1 Defining Covering Homomorphisms

Throughout this document our attempts to classify modular $\mathbb{Z}_p^r$-representations rest heavily on the following definition.

Definition 2.1.1. Given a (group) homomorphism $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$ by choosing an injection $\iota : \mathbb{Z}_p^r \to W \leq (\mathbb{F}^d, +)$ we construct the $\mathbb{Z}_p^r$-representation $\sigma \circ \iota$. Such representations are said to be *induced* from σ.

Fix a family of elementary abelian p-group representations in $GL_n(\mathbb{F})$, not necessarily all of the same rank. Define *covering homomorphisms* for this family as a sequence of (group) homomorphisms $\sigma_i : (\mathbb{F}^{d_i}, +) \to U_n(\mathbb{F})$ such that each representation in the family is equivalent to a representation induced from some σ_i.

We remark that we since our focus is primarily upon faithful representations, their images being the prime target of acquisition, we would prefer our covering homomorphisms to be injective. Whilst all examples given in this document are injective we cannot yet preclude the necessity for a non-injective covering homomorphism in more advanced cases.

By reformulating the work of [11] in Section 1.2, we have seen that separating the 3-dimensional representations by their socle-type is sufficient to procure a single covering homomorphism per family.

Remark. Recall that two representations $\rho_1, \rho_2 : \mathbb{Z}_p^r \to GL_n(\mathbb{F})$ have conjugate images if and only if they are equivalent up to precomposing one with an automorphism. Thus in order to verify whether a homomorphism is a covering homomorphism for a family of representations we need only check that their images are conjugate to a subgroup of the image of the homomorphism. The additional stipulation of precomposing with an automorphism is automatically absorbed into the choice of injection to construct the induced representation.

Much of the first part of this document is attempting to determine when such a technique is possible and how to construct these homomorphisms. Often we non-uniquely identify the group $\mathbb{Z}_p^r \cong \sigma(W)$ for some $W \leq (\mathbb{F}^d, +)$ with a matrix $A \in M_{d,r}(\mathbb{F})$ whose columns serve as generators of W. Then we say A *generates* the representation (with respect to σ).

Warning. The antagonist for the first part of this document is the seemingly innocuous word *wild*: The representation type of an elementary abelian p-group is, in general, wild. This means (see for instance [5] Section 4.4) that the problem of classifying the indecomposable $\mathbb{F}\mathbb{Z}_p^r$-modules is at least as complex as classifying the indecomposable modules for a free $\mathbb{F}$-algebra over two variables. Since the latter has been considered 'undecidable' – that is no Turing machine algorithm can verify the validity of any given sentence in the language of these modules – the same extends to $\mathbb{F}\mathbb{Z}_p^r$. Thus the modular indecomposable representations for elementary abelian p-groups has been dubbed 'unclassifiable'.

As the document progresses and our techniques for separating representations into manageable families with covering homomorphisms become more acute, one might believe that they extend forth to ever-higher dimensional representations thus rendering the whole problem more manageable. To believe so is to vastly underestimate the power of the word 'wild' and one will soon find oneself retreating to the drawing board, eraser in the dominant hand, hubris in the other.

2.1.2 The Matrix Exponential and Logarithm

In order to further study these covering homomorphisms it is worth first convincing ourselves that in any given situation their existence and construction are guaranteed. We do so by exploiting the matrix exponential, defined in characteristic zero as

$$\exp(\Gamma) := \sum_{i=0}^{\infty} \frac{1}{i!}(\Gamma)^i \quad \text{where} \quad \Gamma^0 := I_n.$$

Given an $M \in \mathbb{N}^k$ and a matrix $\Gamma \in U_M(\mathbb{F})$ define $\Gamma_0 := \Gamma - I_n$. Thence we consider the additive group

$$U_{M,0}(\mathbb{F}) := \{\Gamma_0 \mid \Gamma \in U_M(\mathbb{F})\}.$$

Any $\Gamma_0 \in U_{M,0}(\mathbb{F})$ is nilpotent of degree at most k. Hence over fields of characteristic $p \geq k$ we may define the matrix exponential $\exp : U_{M,0}(\mathbb{F}) \to U_M(\mathbb{F})$ by the truncated sum

$$\exp(\Gamma_0) := \sum_{i=0}^{k-1} \frac{1}{i!}(\Gamma_0)^i.$$

We would be surprised were the following not to exist elsewhere, but we include it here for completion.

Proposition 2.1.2. *The truncated matrix exponential* $\exp : U_{M,0}(\mathbb{F}) \to U_M(\mathbb{F})$ *is a bijection. The inverse map* $\log : U_M(\mathbb{F}) \to U_{M,0}(\mathbb{F})$ *is given by*

$$\log(\Gamma) := \sum_{i=1}^{k-1} \frac{(-1)^i}{i}\Gamma_0^i.$$

Proof. To see that exp is a bijection is clear from the image: Each term Γ_0^i has the first i diagonals (i.e. the blocks $\Gamma_{\ell,\ell+j}$ for $j = 0, \ldots, i-1$) vanish. As such each block-entry of $\exp(\Gamma_0)$ is of the form $\exp(\Gamma_0)_{ij} = \Gamma_{ij} + L.O.T.S$ where the remaining terms are monomials in the $\Gamma_{i',j'}$ for $j' - i' < j - i$.

However we show that exp and log are inverses by direct calculation. We observe the following:

$$\log(\exp(\Gamma_0)) = \sum_{i=1}^{k-1} \frac{(-1)^i}{i} \left(\sum_{j=1}^{k-1} \frac{\Gamma_0^j}{j!} \right)^i = \sum_{i=1}^{k-1} \frac{(-1)^i}{i} \sum_{j_1=1}^{k-1} \cdots \sum_{j_i=1}^{k-1} \frac{\Gamma_0^{j_1+\cdots+j_i}}{j_1! \cdots j_i!}.$$

Recalling that $\Gamma_0^s = 0$ for any $s \geq k$ we may rewrite this as

$$\begin{aligned}
\log(\exp(\Gamma_0)) &= \sum_{i=1}^{k-1} \frac{(-1)^i}{i} \sum_{s=i}^{k-1} \sum_{\substack{\underline{\alpha} \in \mathbb{N}^i \\ \sum \alpha_j = s}} \frac{\Gamma_0^s}{\alpha_1! \cdots \alpha_i!} \\
&= \sum_{s=1}^{k-1} \frac{\Gamma_0^s}{s!} \sum_{i=1}^{s} (-1)^i (i-1)! \sum_{\substack{\underline{\alpha} \in \mathbb{N}^i \\ \sum \alpha_j = s}} \frac{1}{i!} \binom{s}{\underline{\alpha}}.
\end{aligned}$$

Note that $\sum_{\substack{\underline{\alpha} \in \mathbb{N}^i \\ \sum \alpha_j = s}} \frac{1}{i!} \binom{s}{\underline{\alpha}}$ is the Sterling number of the second kind $S_2(s,i)$. It is also well known that $\sum_{i=1}^{s} (-1)^i (i-1)! S_2(s,i) = 0$ whenever $s > 1$. Thus we acquire

$$\log(\exp(\Gamma_0)) = \sum_{s=1}^{k-1} \frac{1}{s!} \Gamma_0^s \sum_{i=1}^{s} (-1)^i (i-1)! S_2(s,i) = \Gamma_0.$$

Since exp is a bijection the result then follows. □

The purpose of this section is the proof of following result which is indicative of our ultimate goal.

Theorem 2.1.3. *Let $\mathbb{F}$ have characteristic $p > 0$ and $M \in \mathbb{N}^k$ for $k \leq p$. For any $\Gamma_0, \Delta_0 \in U_{M,0}(\mathbb{F})$,*

$$[\exp(\Gamma_0), \exp(\Delta_0)] = 0 \implies \exp(\Gamma_0)\exp(\Delta_0) = \exp(\Gamma_0 + \Delta_0).$$

Any reader familiar with the scalar exponential will be unsurprised by this result. However it is not immediate when extended to the matrix case, and thus such a statement must be treated with care.

The immediate consequence of this is the following: Given a commutative multiplicative group $G \leq U_M(\mathbb{F})$ we construct the additive group $\log(G) := \{\log(\Gamma) \mid \Gamma \in G\} \leq U_{M,0}(\mathbb{F})$ and thus $\exp : \log(G) \to G$ becomes a group homomorphism indicative of our covering homomorphisms.

Furthermore since $C^{-1}\exp(\Gamma_0)C = \exp(C^{-1}\Gamma_0 C)$ for $C \in GL_n(\mathbb{F})$ the conjugacy class of G in $GL_n(\mathbb{F})$ corresponds to the conjugacy class of $\log(G)$.

It is well known that if two arbitrary matrices $M_1, M_2 \in M_n(\mathbb{F})$ commute then $\exp(M_1)\exp(M_2) = \exp(M_1 + M_2)$. Hence Theorem 2.1.3 is effectively a corollary of the following.

Proposition 2.1.4. *Let $M \in \mathbb{N}^k$ and $\mathbb{F}$ have characteristic $p \geq k$. Given two $\Gamma_0, \Delta_0 \in U_{M,0}(\mathbb{F})$*

$$[\exp(\Gamma_0), \exp(\Delta_0)] = 0 \iff [\Gamma_0, \Delta_0] = 0.$$

Proof. If $[\Gamma_0, \Delta_0] = 0$ then

$$\exp(\Gamma_0)\exp(\Delta_0) = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\frac{1}{i!j!}\Gamma_0^i\Delta_0^j = \sum_{j=0}^{k-1}\sum_{i=0}^{k-1}\frac{1}{j!i!}\Delta_0^j\Gamma_0^i = \exp(\Delta_0)\exp(\Gamma_0).$$

Suppose instead that $[\exp(\Gamma_0), \exp(\Delta_0)] = 0$. Recalling that any monomial in Γ_0 and Δ_0 of degree k or more vanishes, we see that

$$\begin{aligned}
[\exp(\Gamma_0), \exp(\Delta_0)] &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-i-1}\frac{1}{i!j!}(\Gamma_0^i\Delta_0^j - \Delta_0^j\Gamma_0^i) \\
&= \sum_{i=1}^{k-1}\sum_{j=i+1}^{k-1}\frac{1}{i!(j-i)!}(\Gamma_0^i\Delta_0^{j-i} - \Delta_0^{j-i}\Gamma_0^i) \\
&= \sum_{j=2}^{k-1}\frac{1}{j!}\sum_{i=1}^{j-1}\binom{j}{i}(\Gamma_0^i\Delta_0^{j-i} - \Delta_0^{j-i}\Gamma_0^i) = 0. \qquad (2.1)
\end{aligned}$$

Define $D_j := \sum_{i=1}^{j-1}\binom{j}{i}(\Gamma_0^i\Delta_0^{j-i} - \Delta_0^{j-i}\Gamma_0^i)$ so that

$$[\exp(\Gamma_0), \exp(\Delta_0)] = \sum_{j=2}^{k-1}\frac{1}{j!}D_j = 0.$$

For a matrix

$$A = \begin{bmatrix} I_{m_1} & A_{1,2} & \cdots & A_{1,k} \\ 0 & I_{m_2} & \cdots & A_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_k} \end{bmatrix} \in U_M(\mathbb{F})$$

and similarly for $A_0 \in U_{M,0}(\mathbb{F})$, we refer to the ith diagonal as the blocks of the form $A_{\ell,\ell+i}$, thus referring to the regular block diagonal of the matrix a s the 0th diagonal. Any product of Γ_0 and Δ_0 of degree j vanishes along the diagonals $0, \ldots, j-1$. Hence the 2nd diagonal is the first in (2.1) which isn't immediately zero. The only terms which contribute to this diagonal are the summands of

$$D_2 = \Gamma_0\Delta_0 - \Delta_0\Gamma_0.$$

Showing that $D_2 = 0$ shall conclude the proof.

Since the 2nd diagonal of D_2 must vanish it follows that $\Gamma_{\ell,\ell+1}\Delta_{\ell+1,\ell+2} = \Delta_{\ell,\ell+1}\Gamma_{\ell+1,\ell+2}$. We propagate this effect by observing that we may write all other D_j involving D_2. Since

$$\begin{aligned}
\Gamma_0\Delta_0^n &= D_2\Delta_0^{n-1} + \Delta_0\Gamma_0\Delta_0^{n-1} \\
&= D_2\Delta_0^{n-1} + \Delta_0 D_2\Delta_0^{n-2} + \Delta_0^2\Gamma_0\Delta_0^{n-2} \\
&= \cdots \\
&= D_2\Delta_0^{n-1} + \Delta_0 D_2\Delta_0^{n-2} + \cdots + \Delta_0^{n-1}D_2 + \Delta_0^n\Gamma_0
\end{aligned}$$

It then follows that $\Gamma_0\Delta_0^n - \Delta_0^n\Gamma_0 = \sum_{i=0}^{n-1}\Delta_0^i D_2\Delta_0^{n-1-i}$. Repeating this inductively for higher powers of Γ_0 we acquire

$$D_j = \sum_{i=1}^{j-1}\binom{j}{i}\left[\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{j-i-1}\Gamma_0^\alpha\Delta_0^\beta D_2\Delta_0^{j-i-\beta-1}\Gamma_0^{i-\alpha-1}\right]. \tag{2.2}$$

More pertinently one can observe that if the ℓth diagonal of D_2 vanishes then so must the $\ell + j - 2$th diagonal of D_j. Furthermore the mth diagonal of $[\exp(\Gamma_0), \exp(\Delta_0)]$ relies only on the D_j from $j = 2, \ldots, m$. Thus we have ourselves an inductive argument.

Since the 2nd diagonal of D_2 must vanish so too must the jth diagonal of each D_j for $j = 3, \ldots, k-1$ by (2.2).

The third diagonal of $[\exp(\Gamma_0), \exp(\Delta_0)]$ relies only on D_2 and D_3. However by our prior step the 3rd diagonal of D_3 already vanishes, and so we conclude that so must the 3rd diagonal of D_2. It then follows that the $j+1$st diagonal of each D_j must vanish for each $j = 3, \ldots, k-2$.

We repeat this inductively to show

- The ℓth diagonal of $[\exp(\Gamma_0), \exp(\Delta_0)]$ relies solely on $D_2, \ldots, D_\ell$;

- The ℓth diagonal of each $D_3, \dots, D_\ell$ already vanish by induction, thus so must the ℓth diagonal of D_2;
- The $j + \ell - 2$th diagonals of D_j for $j = 3, \dots, k + 1 - \ell$ vanish, since they all depend on the ℓth diagonal of D_2.

We repeat this process until we show that the $k - 1$st diagonal of D_2 vanishes, thus vanishing all of D_2 thereby concluding the proof. □

2.1.3 The Construction of Covering Homomorphisms

Having ascertained a correspondence between additive subgroups of $U_{M,0}(\mathbb{F})$ and images of modular representations of $\mathbb{Z}_p^r$ (for large enough p) using the truncated matrix exponential/logarithm, our purpose is to generate such representations in a predictable and manipulable way. Currently the construction of the representation relies on acquiring the additive subgroup, itself currently constructed from the representation thus rendering the process fruitless.

Here we demonstrate that the relationship between the additive group and representation is closer than one may presume at first glance. We determine a (group) isomorphism between some $(\mathbb{F}^d, +)$ and each maximal abelian subgroup of $U_M(\mathbb{F})$ in a method indicative of the work of Section 1.2.

In order to acquire this we use the following method, demanding that our maximal abelian subgroups be described as subgroups of $U_M(\mathbb{F})$ of elements whose entries satisfy certain *linear* relations. We have yet to see if this process is immune to generalisation, but it suffices for all cases we consider in this document.

Definition 2.1.5. Choose a field $\mathbb{F}$ of characteristic $p > 0$ and a nominal socle-type $M \in \mathbb{N}^k$ for $k \leq p$.

- We call maximal abelian subgroups $G_i \leq U_M(\mathbb{F})$ with socle-type M **unrefined groups**.

  To classify these we construct a collection of sets of matrices such that at least one exists, up to conjugacy, inside each maximal abelian subgroup of $U_M(\mathbb{F})$ with socle-type M. Such a collection must necessarily exist, although choosing it to be minimal is often a challenge.

  Every other element in the subgroup must commute with each element of this subset. Applying these commutativity criteria to a general element of $U_M(\mathbb{F})$ yields linear relations between its entries. All matrices satisfying these linear relations form a group, the maximal abelian subgroups of which yield the groups we desire.

Any maximal abelian subgroups without socle-type M are unimportant to the classification and thus ignored.

A maximal sequence of non-conjugate unrefined groups $G_1, \ldots, G_s \leq U_M(\mathbb{F})$ collectively contain the images of all $\mathbb{Z}_p^r$-representations (for arbitrary r) over $\mathbb{F}$ up to equivalence.

- Since the relations defining each G_i are linear, each

$$G_{i,0} := \{\Gamma_0 \mid \Gamma \in G_i\} \subset U_{M,0}(\mathbb{F})$$

is an $\mathbb{F}$-vector space. Furthermore since for any two $\Gamma_0, \Delta_0 \in G_{i,0}$ we have

$$\Gamma_0 \Delta_0 = \Gamma\Delta - \Gamma - \Delta + I_n = (\Gamma\Delta)_0 - \Gamma_0 - \Delta_0 \in G_{i,0}$$

it is also multiplicatively closed and commutative. We call these **unrefined algebras**, viewed as an analogue to Lie algebras with commutator $[-,-]$ acting as the bracket. Since we chose our G_i to be non-conjugate so too are the $G_{i,0}$.

- We then construct the **refined groups** $\exp(G_{i,0})$.

Theorem 2.1.6. *Let $\mathbb{F}$ have characteristic $p > 0$ and $M \in \mathbb{N}^k$ be such that $p \geq k$. Suppose $G_1, \ldots, G_s \leq U_M(\mathbb{F})$ form a maximal sequence of pairwise non-conjugate unrefined groups. Then $\exp(G_{1,0}), \ldots, \exp(G_{s,0}) \leq U_M(\mathbb{F})$ form a maximal sequence of pairwise non-conjugate refined groups.*

Hence there exist covering homomorphisms $\sigma_i : (\mathbb{F}^{d_i}, +) \to \exp(G_{i,0})$ for $d_i \in \mathbb{N}$ and all $i = 1, \ldots, s$ covering all modular $\mathbb{Z}_p^r$-representations with socle-type M.

Proof. It is clear from the definition of exp that $\exp(G_{i,0})$ are subgroups of $U_M(\mathbb{F})$. Since the G_i were chosen to be non-conjugate then the $G_{i,0}$ are non-conjugate. Since the conjugacy action and exponentiation commute it follows that each $\exp(G_{i,0})$ are all non-conjugate.

Since $U_M(\mathbb{F}) \xrightarrow{-I_n} U_{M,0}(\mathbb{F}) \xrightarrow{\exp} U_M(\mathbb{F})$ is formed of bijections it follows that the $\exp(G_{i,0})$ are also maximal. As conjugacy is preserved under this map and the unrefined groups are pairwise non-conjugate so too must be the refined groups.

Further since the G_i form a maximal sequence of maximal abelian subgroups up to conjugacy each $\exp(G_{i,0})$, as an abelian subgroup, is conjugate to a subgroup of one of the G_i. However since they are maximal they must be conjugate to the entire G_i. Hence there is a one-to-one correspondence between the unrefined and refined groups each conjugate to their paired kith. Thus the refined groups also form a maximal sequence of non-conjugate maximal abelian subgroups.

The choice of bijections allow us to construct our covering homomorphisms from the following diagram:

$$\begin{tikzcd}
U_M(\mathbb{F}) \arrow[d, "\substack{\text{linear}\\ \text{commutativity}\\ \text{relations}}"] & (\mathbb{F}^{d_i}, +) \arrow[d, "\tau_i"] \arrow[dr, dashed, "\sigma_i"] & \\
\underset{\left(\substack{\text{unrefined}\\ \text{groups}}\right)}{G_i} \arrow[r, "-I_n"] & \underset{\left(\substack{\text{unrefined}\\ \text{algebras}}\right)}{G_{i,0}} \arrow[r, "\exp"] & \underset{\left(\substack{\text{refined}\\ \text{groups}}\right)}{\exp(G_{i,0})}
\end{tikzcd}$$

The τ_i are given by any natural isomorphism between $(\mathbb{F}^{d_i}, +)$ and $G_{i,0}$ since the latter are $\mathbb{F}$-vector spaces whose entries are linear and thus easy to unravel. Thence the $\sigma_i := \exp \circ \tau_i$ for $i = 1, \ldots, s$ collectively form a full set of covering homomorphisms for socle-type M, thus concluding the proof. $\square$

Naturally our focus in this document then is to acquire a maximal sequence of inequivalent unrefined groups for each socle-type and the remainder of the classification follows from Theorem 2.1.6. We hope the reader will garner a better understanding of this process as more examples are considered.

Example 2.1.7. Consider a field $\mathbb{F} = \overline{\mathbb{F}}$ of characteristic $p \geq 3$. We wish to classify representations with both socle-type and dual-type $(1, 2, 1)$. We thus focus our attentions upon the maximal abelian subgroups of

$$U_{(1,2,1)}(\mathbb{F}) = \left\{ \begin{bmatrix} 1 & c_{12} & c_{13} & c_{14} \\ 0 & 1 & 0 & c_{24} \\ 0 & 0 & 1 & c_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; c_{12}, c_{13}, c_{14}, c_{24}, c_{34} \in \mathbb{F} \right\} \leq U_4(\mathbb{F}).$$

Suppose for such a subgroup that any two elements C, D satisfy $\begin{vmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{vmatrix} = 0$. Then, up to reordering of basis, we may write $e_{24} = \alpha e_{34}$ for all elements E in the group for some $\alpha \in \mathbb{F}$. However it then follows that $[0, 0, 0, 1]$ and $[0, -1, \alpha, 0]$ are both fixed by the right action of these elements. Then the resulting group cannot have dual-type $(1, 2, 1)$ providing a contradiction.

Thus we take C, D in our group to satisfy $\begin{vmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{vmatrix} \neq 0$. Given another element E in our proposed subgroup we demand

$$\big[[C, E][1, 4]\ [D, E][1, 4]\big] = [e_{24}\ e_{34}] \begin{bmatrix} c_{12} & d_{12} \\ c_{13} & d_{13} \end{bmatrix} - [e_{12}\ e_{13}] \begin{bmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{bmatrix} = 0$$

and so

$$[e_{12}\ e_{13}] = [e_{24}\ e_{34}] \begin{bmatrix} c_{12} & d_{12} \\ c_{13} & d_{13} \end{bmatrix} \begin{bmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{bmatrix}^{-1} =: \underline{e}A \tag{2.3}$$

where $A \in M_2(\mathbb{F})$. Applying the same arguments as before to the fact that we require socle-type $(1, 2, 1)$ representations, the $[1, 2]$ and $[1, 3]$ entries of our elements cannot have the same ratio throughout the group. Thus it follows from (2.3) that $\det(A) \neq 0$.

Specifically applying the commutativity criteria to both C and D themselves we acquire

$$\begin{aligned}
\begin{bmatrix} [C, C][1, 4] & [D, C][1, 4] \\ [C, D]\,[1, 4] & [D, D][1, 4] \end{bmatrix} &= \begin{bmatrix} c_{12} & c_{13} \\ d_{12} & d_{13} \end{bmatrix} \begin{bmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{bmatrix} - \begin{bmatrix} c_{24} & c_{34} \\ d_{24} & d_{34} \end{bmatrix} \begin{bmatrix} c_{12} & d_{12} \\ c_{13} & d_{13} \end{bmatrix} \\
&= \begin{bmatrix} c_{12} & d_{12} \\ c_{13} & d_{13} \end{bmatrix}^T \begin{bmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{bmatrix} - \left(\begin{bmatrix} c_{12} & d_{12} \\ c_{13} & d_{13} \end{bmatrix}^T \begin{bmatrix} c_{24} & d_{24} \\ c_{34} & d_{34} \end{bmatrix} \right)^T \\
&= 0
\end{aligned}$$

It thus follows that A is symmetric. Furthermore $[E, E'][1, 4] = \underline{e}(A^T - A)\underline{e'}^T = 0$ and so any group of elements satisfying the linear relation (2.3) for some symmetric $A \in GL_2(\mathbb{F})$ is abelian and thus an unrefined group. We need only now determine which choices of A yield conjugate groups.

Suppose two such groups G, G' with associated symmetric matrices $A, B \in$

$GL_2(\mathbb{F})$ are conjugate with respect to the relations

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \begin{bmatrix} 1 & \underline{e}A & e_{14} \\ 0 & I_2 & \underline{e}^T \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & \underline{e}'B & e'_{14} \\ 0 & I_2 & \underline{e}'^T \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ 0 & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & X_{11}\underline{e}A - \underline{e}'BX_{22} & X_{11}e_{14} + X_{12}\underline{e}^T - \underline{e}'BX_{23} - e'_{14}X_{33} \\ 0 & 0 & X_{22}\underline{e}^T - \underline{e}'^T X_{33} \\ 0 & 0 & 0 \end{bmatrix} = 0$$

for all appropriately paired $E \in G, E' \in G'$. Thus we observe that $\underline{e}' = X_{33}^{-1}\underline{e}X_{22}^T$ for $X_{33} \in \mathbb{F}^*$ and $X_{22} \in GL_2(\mathbb{F})$. Therefore

$$X_{11}\underline{e}A - \underline{e}'BX_{22} = \underline{e}\big(X_{11}A - X_{33}^{-1}X_{22}^T BX_{22}\big) = 0.$$

Due to the independence of the $\underline{e}$ in the group it follows that $B = M^T AM$ for some $M \in GL_2(\mathbb{F})$. Hence via conjugation we may choose our symmetric matrix up to congruence. By leaning on our assumption that $\mathbb{F}$ is algebraically closed all such matrices are congruent and thus we choose $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus up to conjugation we acquire the sole abelian unrefined group

$$G_1 := \left\{ \tau(c_1, c_2, c_3) + I_n := \begin{bmatrix} 1 & c_1 & c_2 & c_3 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid (c_1, c_2, c_3) \in \mathbb{F}^3 \right\}$$

with homomorphism $\tau : (\mathbb{F}^3, +) \to U_{(1,2,1),0}(\mathbb{F})$, within which the image of every socle- and dual-type $(1, 2, 1)$ $\mathbb{Z}_p^r$-representation over $\mathbb{F}$ sits up to equivalence. Concatenating this with exp we obtain our covering homomorphism $\sigma : (\mathbb{F}^3, +) \to GL_4(\mathbb{F})$

$$\sigma(\underline{c}) = \exp(\tau(c_1, c_2, c_3)) = \begin{bmatrix} 1 & c_1 & c_2 & c_3 + c_1c_2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We can see that $\exp(G_{1,0}) = \sigma(\mathbb{F}^3) = G_1$ and thus σ is a covering homomorphism for the $\mathbb{Z}_p^r$-representations over $\mathbb{F}$ with socle- and dual-type $(1, 2, 1)$.

2.1.4 Small Primes

Since any p-group can be chosen up to equivalence to belong to some $U_M(\mathbb{F})$ the following is indicative that the relationship between the length of the socle-series k and the field characteristic p is important.

Corollary 2.1.8. *Let $\mathbb{F}$ have characteristic $p > 0$, $M = (m_1, \ldots, m_k) \in \mathbb{N}^k$ for $k \leq p$ and $P \leq U_M(\mathbb{F})$ be a group. If P is abelian then P is elementary abelian.*

Proof. This follows directly from Theorem 2.1.3 via the observation that

$$\exp(\Gamma_0)^p = \exp(p\Gamma_0) = \exp(0) = I_n.$$

$\square$

Non-elementary abelian p-groups begin to obfuscate our approach when our prime is relatively small. Our methods for constructing unrefined groups remain intact but since the refinement process involves exp, a function undefined if $p < k$, the reader may be forgiven for abandoning this approach altogether. However by temporarily abusing notation we may still ensure a definition for exp.

If $p < k$ then the first term of $\exp(\Gamma_0)$ to be undefined is $\frac{1}{p!}\Gamma_0^p$. However since we seek elementary abelian p-groups we expect $\Gamma_0^p = 0$. There are occasions where we may reasonably define $\frac{1}{p!}\Gamma_0^p$ outside of the characteristic before re-entering. To do so we must explicitly ensure each element has the requisite order, a fact we often take for granted when $p \geq k$.

We may view the covering homomorphisms as a reparameterisation of the entries of the unrefined groups since we no longer have access to exp. In practice we do this without cause to mention, however we illustrate here with an example.

Example 2.1.9. Recall from Example 2.1.7 that for $p \geq 3$ the images all representations with socle- and dual-type $(1, 2, 1)$ exist up to conjugacy in the image of the homomorphism $\sigma : (\mathbb{F}^3, +) \to U_4(\mathbb{F})$ given by

$$\sigma(\underline{c}) := \exp\left(\begin{bmatrix} 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & c_1 & c_2 & c_3 + c_1c_2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We remarked that $\sigma(\mathbb{F}^3)$ was equal to the unrefined group. However the unrefined group was still well-defined over $p = 2$. For $\mathrm{char}(\mathbb{F}) = p = 2$ the final form of

$\sigma(\underline{c})$ is also well-defined and thus equals the unrefined group. Hence σ still acts as a covering homomorphism for the char($\mathbb{F}$) $= 2$ case whether its method of construction is valid or not.

We may envisage this process as follows: By temporarily working over $\mathbb{Z}$ we observe that

$$\tau(c_1, c_2, c_3)^2 = \begin{bmatrix} 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 & 2c_1c_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which then naturally vanishes in characteristic 2. However by abusing notation we may define the violating exp term as

$$\frac{1}{2!}\begin{bmatrix} 0 & c_1 & c_2 & c_3 \\ 0 & 0 & 0 & c_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 := \begin{bmatrix} 0 & 0 & 0 & c_1c_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

regardless of characteristic. Thus we may include the $p = 2$ case into this construction by temporarily abusing notation.

If this unsettles the reader simply verify that $\sigma : (\mathbb{F}^3, +) \to GL_4(\mathbb{F})$ is still a homomorphism of groups and that the image equals the unrefined group. Since procuring the unrefined group did not involve the characteristic of the field σ still acts as a covering homomorphism.

2.1.5 Equivalence within Covering Homomorphisms

Given a sequence of covering homomorphisms $\sigma_i(\mathbb{F}^{d_i}, +) \to GL_n(\mathbb{F})$ for a family of $\mathbb{Z}_p^r$-representations we acquire the images of all representations in the family up to equivalence within the homomorphisms' images. It would be beneficial to know which subgroups in the image were equivalent to one another.

Proposition 2.1.10. *Let $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$ be a covering homomorphism for a family of $\mathbb{Z}_p^r$-representations as constructed in Section 2.1.3. Then the conjugacy action of the stabiliser of the image of σ corresponds to a linear group action on the subgroups of $(\mathbb{F}^d, +)$.*

Proof. Let A be an element of the conjugacy stabiliser of $\sigma(\mathbb{F}^d)$ and suppose $W, W' \leq (\mathbb{F}^d, +)$ are such that $A^{-1}\sigma(W)A = \sigma(W')$. Then for any two $\underline{c}, \underline{d} \in W$

acquire their conjugate partners $\underline{c}', \underline{d}' \in W'$, that is

$$\sigma(\underline{c}') = A^{-1}\sigma(\underline{c})A \qquad \text{and} \qquad \sigma(\underline{d}') = A^{-1}\sigma(\underline{d})A.$$

Then we see from Theorem 2.1.3 that

$$\sigma(\underline{c}' + \underline{d}') = \sigma(\underline{c}')\sigma(\underline{d}') = A^{-1}\sigma(\underline{c})AA^{-1}\sigma(\underline{d})A = A^{-1}\sigma(\underline{c})\sigma(\underline{d})A = A^{-1}\sigma(\underline{c} + \underline{d})A$$

Thus $c' + d' \in W'$ is the conjugate partner of $c + d \in W$ and thus conjugation of groups in the image of σ corresponds to a linear action on the subgroups of $(\mathbb{F}^{d_i}, +)$ as required. $\square$

We note that some subgroups in the image of a homomorphism may be conjugate by matrices not within the image's stabiliser. However it is often the case that such groups are, in some sense, degenerate. A covering homomorphism often induces many representations outside of the family it was originally designed to cover, often less complex in structure than those desired. Since the structure of the homomorphism was explicitly defined by the structure of the desired representation family any conjugacy among those is reflected by the stabiliser.

In all examples hence the stabiliser will be sufficient for our understanding of representation equivalence. Thus we shall often reformulate the problem of equivalence as a linear group action on the underlying vector space. Thereafter the problem of determining inequivalent representations is a matter determine the orbits of this action and thus (often algebraic) invariant theory.

Conclusion

By acquiring a maximal sequence of inequivalent, maximal, abelian subgroups (unrefined groups) in $U_M(\mathbb{F})$ and applying the refinement process in definition 2.1.5 we may parameterise all such representations via covering homomorphisms, as in Theorem 2.1.3.

It still remains to be seen how best to construct the maximal sequence of unrefined groups. Due to the wildness of the representation type it is unlikely this process shall find a generalisation-immune formulation. However we can make the process much simpler by dividing $U_M(\mathbb{F})$ into further inequivalent parts, where $U_M(\mathbb{F}) \leq U_n(\mathbb{F})$ acts as a first step.

2.2 Socle Tabloids

In $GL_3(\mathbb{F})$ dividing the $\mathbb{Z}_p^r$-representations into distinct families by their socle-type was a sufficient enough distinction to allow their parameterisation. In higher dimensional examples it is prudent to enter their study with more refined equipment.

We examine in more detail the socle- and dual socle-series to find that their interaction encodes into a more refined combinatorial invariant than their types alone, thereby allowing us to further separate inequivalent representations. Unless otherwise stated the results in this section apply to any modular p-group representation.

2.2.1 Defining Socle Tabloids

Let $P \leq GL(V)$ be a modular p-group representation with socle-type $M = (m_1, \ldots, m_k)$. Recall that an ordered basis for V is called *socle-conforming* if for each $i \in [\![1, k]\!]$ the first $m_1 + \cdots + m_i$ elements of the basis form a basis for $\mathrm{Soc}_i(V)$. We can, however, be more specific.

Definition 2.2.1. We call an ordered basis for an $\mathbb{F}P$-module V *doubly-conforming* if it is socle-conforming and the associated dual basis is also socle-conforming, i.e. conforms to the dual socle series $(\mathrm{Soc}_i(V^*))_i$, up to some permutation of the elements.

Coercing representations into doubly-conforming form presents their properties in as upfront a fashion as possible. The following is of minor assistance to this end.

Theorem 2.2.2. *Let V be an $\mathbb{F}P$-module. Then V has a doubly-conforming basis.*

Proof. The proof we provide is constructive. We remark that applying an upper-triangular change to a socle-conforming basis, one for which each v_i is replaced with some element of the form $\sum_{j=1}^{i} \alpha_j v_j$, yields another socle-conforming basis. Furthermore an upper-triangular change of basis for V corresponds to a lower-triangular change of basis for V^*, one for which v_i^* is replaced with $\sum_{j=i}^{n} \beta_j v_j^*$.

We show that a socle-conforming basis whose dual contains bases for $\mathrm{Soc}_i(V^*)$ for $1 \leq i \leq t$ may be upper-triangularly transformed into one whose dual contains bases for $\mathrm{Soc}_i(V^*)$ for $1 \leq i \leq t+1$. The result then follows from recursive application.

Denote by $(n_1, \ldots, n_k)$ the dual-type of V and by $d_i := \dim(\mathrm{Soc}_i(V^*))$. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a socle-conforming basis for V whose dual $\mathcal{B}^* = \{v_1^*, \ldots, v_n^*\}$ contains bases for $\mathrm{Soc}_i(V^*)$ for $i \leq t$.

In the case $t = 0$ we simply begin with any socle-conforming basis. If $t > 0$ let $\tau \in S_{d_t}$ be a permutation such that $\{v^*_{\tau(1)}, \ldots, v^*_{\tau(d_t)}\}$ is a basis for $\mathrm{Soc}_t(V^*)$.

Construct elements $w_1^*, \ldots, w^*_{n_{t+1}} \in V^*$ such that $\{v^*_{\tau(1)}, \ldots, v^*_{\tau(d_t)}, w_1^*, \ldots, w^*_{n_{t+1}}\}$ is a basis for $\mathrm{Soc}_{t+1}(V^*)$. Since $\mathcal{B}^*$ is a basis for V^* write

$$w_i^* = \sum_{j=\ell_i}^{n} \alpha_{i,j} v_j^*$$

where $\alpha_{i,\ell_i} \neq 0$ for all $i = 1, \ldots, n_{t+1}$. We construct the V^* basis $\widehat{\mathcal{B}}^*$ from $\mathcal{B}^*$ by replacing $v^*_{\ell_i}$ with w_i^* for all $1 \leq i \leq m_{t+1}$, subject to the following stipulations:

By taking linear combinations of the w_i^* and relabelling we may ensure that $\ell_1 < \cdots < \ell_{m_{t+1}}$. Thus the change of basis given shall be lower-triangular. Furthermore if $t > 0$ by adding linear combinations of the $v^*_{\tau(i)} \in \mathrm{Soc}_t(V^*)$ to the w_i^* we also ensure that $\ell_i \neq \tau(j)$ for any $j \in [\![1, d_t]\!], i \in [\![1, m_{t+1}]\!]$. Thus the bases for $\mathrm{Soc}_i(V^*)$ for $i \leq t$ remain intact.

Thence construct $\widehat{\mathcal{B}}$ by dualising $\widehat{\mathcal{B}}^*$. By construction the corresponding change of basis to V will be upper-triangular. Hence $\widehat{\mathcal{B}}$ is a socle-conforming basis for V whose dual contains a bases for $\mathrm{Soc}_i(V^*)$ for all $1 \leq i \leq t+1$. The result then follows. □

With a doubly conforming basis comes information beyond the socle-type and dual-type alone. There may be many permutations which place the dual basis into a socle-conforming order. The orbit of such a permutation under the subgroup of S_n which keep the basis elements of each $\mathrm{Soc}_i(V^*)$ in place all act as candidates. We refer to this as the *class* of permutations associated to the doubly-conforming basis.

Different permutation classes for different representations shall remain distinct under equivalence. This gives us the incentive to introduce the notion of a tabloid associated to such a representation which encodes this information.

A *tabloid* is an equivalence class of tableaux of left-justified boxes containing entries in $\mathbb{N}$ such that two tableaux are considered equivalent if their ith rows from the top contain the same number of $\boxed{j}$ for all $i, j \in \mathbb{N}$. That is two tableaux are equivalent if one can acquire the second from the first only by reordering the boxes in each row. In this work we identify the equivalence class with the representative whose row elements are *non-increasing*. We also index the rows of tabloids in $\mathbb{N}$ from top to bottom, unless otherwise stated. The following are examples of tabloids:

$$\begin{array}{lll} \boxed{3}\boxed{3}\boxed{1} \\ \boxed{2} \\ \boxed{1} \end{array} \qquad \begin{array}{llll} \boxed{4} \\ \boxed{3}\boxed{2}\boxed{1}\boxed{1} \\ \boxed{2} \\ \boxed{1}\boxed{1} \end{array} \qquad \begin{array}{ll} \boxed{1} \\ \boxed{2} \\ \boxed{3}\boxed{1} \end{array}.$$

We are now equipped to define the focus of this section.

Definition 2.2.3. Let $P \leq GL(V)$ be a p-group representation with doubly-conforming basis $\mathcal{B}$. We define the *socle tabloid* of P (or V) to be the tabloid δ such that there is a one-to-one correspondence between the elements of $\mathcal{B}$ and the boxes of the tabloids, where each $v \in \mathcal{B}$ with

$$v \in \mathrm{Soc}_i(V) \setminus \mathrm{Soc}_{i-1}(V), \quad v^* \in \mathrm{Soc}_j(V^*) \setminus \mathrm{Soc}_{j-1}(V^*)$$

corresponds to a $\boxed{j}$ in row i, with no other boxes present. We write $P \sim \delta$ (or $V \sim \delta$) to denote that δ is the socle tabloid of P (or V).

The socle tabloid of a representation encodes three pieces of information: The socle-type is given by the row box counts, the dual-type is given by counting the frequency of each number's appearance in the tabloid, and the class of permutations required to ensure the dual basis conforms is encoded in the arrangement of the numbers in the tabloid.

Strictly our definition ought to make reference to the chosen doubly-conforming basis used to construct the tabloid. The efforts which follow show that this is unnecessary.

Effectively the first i rows of the socle tabloid tell the reader where the duals of each basis element of $\mathrm{Soc}_i(V)$ sit in the dual-socle series. This naturally leads us to the following property.

Lemma 2.2.4. *Let $P \leq GL(V)$ be a modular p-group representation with doubly-conforming basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. Let δ be the associated socle tabloid. Then the set of coset representatives $\mathcal{B}' = \{v_{\dim(\mathrm{Soc}_i(V))+1}, \ldots, v_n\}$ form a doubly conforming basis for $V/\mathrm{Soc}_i(V)$ with socle tabloid δ' acquired by deleting the first i rows from δ.*

Proof. The shape of δ' is acquired from the last $k-i$ rows of δ given that the box count of each row is given by the socle series.

Since our action is upper-triangular with respect to the socle-conforming basis $\mathcal{B}$ taking this quotient does not alter the action on the duals of the remaining coset representatives. Hence the positions of the basis elements' duals in the dual socle series remains unchanged. Consequently $\mathcal{B}'$ is doubly conforming and the socle tabloid information remains excepting the deletion of the first i rows. $\square$

This allows us the following luxury.

Proposition 2.2.5. *Let V be an $\mathbb{F}P$-module. Let $\mathcal{B}_1, \mathcal{B}_2$ be doubly conforming bases which yield socle tabloids δ_1, δ_2 for V respectively. Then $\delta_1 = \delta_2$.*

Proof. Suppose for the sake of contradiction that δ_1 and δ_2 first disagree in row i, say δ_1 has more $\boxed{j}$ in this row than δ_2.

Construct the corresponding tabloids δ_1' and δ_2' for $V_{(i)} := V/\mathrm{Soc}_i(V)$, as in Lemma 2.2.4, by deleting the first i rows from δ_1 and δ_2 respectively. Then since δ_1 and δ_2 had exactly $\dim(\mathrm{Soc}_j(V^*)/\mathrm{Soc}_{j-1}(V^*))$ many $\boxed{j}$, δ_1' now has fewer $\boxed{j}$ overall than δ_2'. However both should contain exactly $\dim(\mathrm{Soc}_j(V_{(i)}^*)/\mathrm{Soc}_{j-1}(V_{(i)}^*))$ many $\boxed{j}$, thus yielding a contradiction. $\square$

The socle tabloid is thus an invariant under equivalence, since it remains static for any choice of doubly-conforming basis. It can thus serve as a visual replacement for the socle- and dual-type. The socle tabloid is indeed more specific since, for example, the tabloids

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

both correspond to socle-type $(2,2,1)$ (the row box count) and dual-type $(2,2,1)$ (two $\boxed{1}$'s, two $\boxed{2}$'s and one $\boxed{3}$) but to a different family of permutations required to conform the associated dual basis.

We illustrate the construction of a socle tabloid in the following example.

Example 2.2.6. Consider a representation V of $\mathbb{Z}_3^2$ with image generated over

$\mathbb{F}_3$ by the elements

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denote by $\mathcal{B} := \{v_1, v_2, v_3, v_4, v_5\}$ the corresponding basis for V upon which these elements act by left multiplication. Thence denote by $\mathcal{B}^* := \{v_1^*, v_2^*, v_3^*, v_4^*, v_5^*\}$ the corresponding dual basis for V^* upon which these elements act by right multiplication. One can check that $\mathcal{B}$ is already doubly-conforming and as such we can see that the socle-type is $(2, 2, 1)$. Thus we begin with the empty tabloid

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1} \end{array}$$

Since the action of these elements is on a doubly-conforming basis, a basis for $\mathrm{Soc}(V^*)$ exists in $\mathcal{B}'$. Only rows 2 and 5 of each generator agree with the identity and so $\mathrm{Soc}(V^*) = \langle v_2^*, v_5^* \rangle$. Using this we begin filling in the tabloid by

$$\begin{aligned} v_2 \in \mathrm{Soc}_1(V), \quad v_2^* \in \mathrm{Soc}_1(V^*) &\implies \text{row 1 contains a } \boxed{1}, \\ v_5 \in \mathrm{Soc}_3(V), \quad v_5^* \in \mathrm{Soc}_1(V^*) &\implies \text{row 3 contains a } \boxed{1}. \end{aligned}$$

Considering the submodule generated by v_1, v_3 and v_4 we acquire the generator actions

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We see that the images of the dual elements v_3^* and v_4^* are acted upon trivially in the quotient and so they belong to $\mathrm{Soc}_2(V^*)$. Since both v_3 and v_4 belong to $\mathrm{Soc}_2(V)$ and their duals belong to $\mathrm{Soc}_2(V^*)$ it follows that row 2 of our tabloid contains a pair of $\boxed{2}$s.

It is then clear that $v_1^* \in \mathrm{Soc}_3(V^*)$ and so we place a $\boxed{3}$ in the remaining box of row 1 to acquire the socle tabloid

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}.$$

From this example we see the construction of the socle tabloid as a matter of record-keeping while constructing the socle- and dual-types

2.2.2 Manipulating Socle Tabloids

Whilst acquiring socle tabloids aids in distinguishing between inequivalent families of representations we would like to know more and find, if possible, practical applications. The following provides a certain incentive for our election of a non-increasing order on tabloid rows.

Lemma 2.2.7. *Let $P \leq GL(V)$ be a p-group representation with socle tabloid δ. Then the ith row from the bottom of δ contains at least one $\boxed{i}$ and no entries which exceed i.*

Proof. This is a translated property of the socle series: For a representation V with socle-length k any $v \in \mathrm{Soc}_{k+1-i}(V) \setminus \mathrm{Soc}_{k-i}(V)$ has dual $v^* \in \mathrm{Soc}_i(V^*)$ with at least one element attaining $v^* \in \mathrm{Soc}_i(V^*) \setminus \mathrm{Soc}_{i-1}(V^*)$.

The quotient module $V/\mathrm{Soc}_{k-i}(V)$ has, by construction, socle-length i. Thus so shall its dual, a submodule of V^*. Hence every element within this submodule cannot sit further in the socle series of V^* than $\mathrm{Soc}_i(V^*)$. Furthermore there must exist at least one element of this submodule sitting within $\mathrm{Soc}_i(V^*) \setminus \mathrm{Soc}_{i-1}(V^*)$ lest the socle length of the submodule fall short of i. $\square$

Since the ith row from the bottom must contain an $\boxed{i}$ and nothing greater we may ensure that the left-most column of our tabloid is a strictly ascending sequence of integers $1, 2, \ldots, k$ from bottom to top. The reader, of course, is welcome to elect their own ordering.

We saw in Lemma 2.2.4 that taking the quotient of part of the socle series yielded a predictable manipulation of the socle tabloid. To the end of determining further predictable manipulations we posit the following library.

Definition 2.2.8. Let γ, δ be tabloids. Then we define the following.

- We define $\gamma \oplus \delta$ to be the tabloid whose ith row is constructed from all elements of the ith rows of δ and γ.
- Define $\delta^{(i)}$ to be the tabloid acquired from δ by removing its first i rows.
- Define $\delta_{(i)}$ to be the tabloid acquired from δ by removing all boxes containing entries less than or equal to i and rescaling all remaining boxes by subtracting i.
- Define δ^* as containing a $\boxed{i}$ in row j for every $\boxed{j}$ in row i of δ for all $i, j \in [\![1, k]\!]$.

Example 2.2.9. Consider the tabloids

$$\gamma = \begin{ytableau} 2 & 2 \\ 1 \end{ytableau} \qquad \delta = \begin{ytableau} 3 & 1 \\ 2 & 1 \\ 1 \end{ytableau}.$$

Then

$$\gamma \oplus \delta = \begin{ytableau} 3 & 1 \\ 2 & 1 \\ 1 \end{ytableau} \oplus \begin{ytableau} 2 & 2 \\ 1 \end{ytableau} = \begin{ytableau} 3 & 2 & 2 & 1 \\ 2 & 1 & 1 \\ 1 \end{ytableau} \qquad \delta^{(1)} = \begin{ytableau} *(red) 3 & *(red) 1 \\ 2 & 1 \\ 1 \end{ytableau} = \begin{ytableau} 2 & 1 \\ 1 \end{ytableau}$$

$$\delta_{(1)} = \begin{ytableau} 3-1 & *(red) 1 \\ 2-1 & *(red) 1 \\ *(red) 1 \end{ytableau} = \begin{ytableau} 2 \\ 1 \end{ytableau} \qquad \delta^* = \begin{ytableau} 3 & 2 & 1 \\ 2 \\ 1 \end{ytableau}$$

Theorem 2.2.10. *Let V, W be $\mathbb{F}P$-modules. Suppose $V \sim \delta$ and $W \sim \gamma$. Then the following hold:*

i) $V \oplus W \sim \delta \oplus \gamma$,

ii) $V^{(i)} := V/\mathrm{Soc}_i(V) \sim \delta^{(i)}$,

iii) $V_{(i)} := \big(V^*/\mathrm{Soc}_i(V^*)\big)^* \sim \delta_{(i)}$,

iv) $V^* \sim \delta^*$.

Proof. Claim $i)$ is a direct consequence of the definition of the socle tabloid and the fact $\mathrm{Soc}_i(V_1 \oplus V_2) = \mathrm{Soc}_i(V_1) \oplus \mathrm{Soc}_i(V_2)$.

Claim $ii)$ is a reiteration of Lemma 2.2.4.

Claim *iv*) follows directly from the definition of δ.

Claim *iii*) is a corollary of *ii*) and *iv*) observing that $\delta_{(i)} = \left((\delta^*)^{(i)}\right)^*$. $\square$

We might ask whether a similar formulation is possible when taking other combinations of submodules and quotients, or indeed whether in some sense we may extend the module with socle tabloid predictability. We begin by examining the smallest possible submodules.

Lemma 2.2.11. *Let the $\mathbb{F}P$ module $V \sim \delta$ have socle length k. Suppose $v \in \mathrm{Soc}_1(V)$ belongs to a doubly-conforming basis and $v^* \in \mathrm{Soc}_i(V^*) \setminus \mathrm{Soc}_{i-1}(V^*)$, i.e. there exists an $\boxed{i}$ in row 1 of δ. Suppose further that no box in row 2 of δ contains an entry less than i. Then the tabloid of $V/\langle v\rangle$ is acquired from δ by removing an $\boxed{i}$ from the first row of δ.*

Proof. Order the doubly-conforming basis of V such that v is first. Then our group elements act on V in the form

$$\left[\begin{array}{c|cccc|c} & 0 & 0 & \cdots & 0 & \\ & * & * & \cdots & * & \\ I_{m_1} & \vdots & \vdots & \ddots & \vdots & \cdots \\ & * & * & \cdots & * & \\ \hline 0 & & I_{m_2} & & & \cdots \\ \hline \vdots & & \vdots & & & \ddots \end{array}\right].$$

This is because the image of the action on v^* can only include elements in $\mathrm{Soc}_{i-1}(V^*)$ and v^* itself. By assumption the elements dual to the basis of $\mathrm{Soc}_2(V)$ are not included in this. Hence under the quotient it follows that no alteration is made to either the socle-type, the dual-type or their basis interactions besides the removal of the element v. The result then follows. $\square$

For this lemma to be applicable to a representation V it demands that our proposed element v not exist in the action of any element in $\mathrm{Soc}_2(V)$. The following is the case where there is so little freedom of movement that such requirements are unnecessary.

Lemma 2.2.12. *Let $P \leq GL(V)$ have socle tabloid δ with $\dim(\mathrm{Soc}\, V) > 1$ and second row consisting of a single $\boxed{k-1}$. Let $v \in \mathrm{Soc}(V)$ be part of a doubly conforming basis for V with $v^* \in \mathrm{Soc}_i(V) \setminus \mathrm{Soc}_{i-1}(V)$. Then the representation*

$V/\langle v\rangle$ has socle tabloid δ' acquired by deleting a single $\boxed{i}$ from row 1 of δ if and only if this removal leaves at least one $\boxed{k}$ in the first row.

Proof. If the dual of v is belongs to $\mathrm{Soc}_{k-1}(V^*)$ (with respect to the aforementioned doubly-conforming basis) then the result follows from Lemma 2.2.11.

Suppose instead that v is the sole element corresponding to a $\boxed{k}$ in the first row – and thus, by Lemma 2.2.7, the entire tabloid. Then, by definition the dual of $V/\langle v\rangle$ is $\mathrm{Soc}_{k-1}(V^*)$. Thus the resulting quotient has socle-length $k-1$ and so the resulting socle tabloid will possess $k-1$ rows, thus bearing little resemblance to δ.

Suppose instead that v is not the sole element corresponding to a $\boxed{k}$ in the first row. Order our doubly-conforming basis $\{v_1,\ldots,v_n\}$ such that $v_1,\ldots,v_\alpha$ are precisely those elements with duals not in $\mathrm{Soc}_{k-1}(V^*)$, so that $\alpha > 1$ is the number of k-blocks in δ. Then the action of an element of the representation dons the form

$$\begin{bmatrix} & c_{1,m_1+1} & \\ & \vdots & \\ I_{m_1} & c_{\alpha,m_1+1} & * \\ & 0 & \\ & \vdots & \\ & 0 & \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix}$$

Since our basis is doubly-conforming no non-trivial linear combination of $v_1^*,\ldots,v_\alpha^*$ lies in $\mathrm{Soc}_{k-1}(V^*)$. Hence there exists no relation between the $c_{1,m_1+1},\ldots,c_{\alpha,m_1+1}$ which holds across all elements of the representation. Thus taking the quotient of V with some v_i for $1\le i\le\alpha$ shall not affect where in the dual socle series the dual of any element in $\mathrm{Soc}(V)$ sits. Furthermore since $\alpha>1$ by assumption no element of $\mathrm{Soc}_2(V)$ shall be repositioned in the quotient's socle series. Thus the tabloid remains the same, excepting for the removal of a single $\boxed{k}$. $\square$

This lemma, unlike its predecessor, poses a restriction on the dimension of $\mathrm{Soc}_2(V)/\mathrm{Soc}_1(V)$. Whilst this might seem less than optimal we shall see in Section 2.3.2 that its application is less for the observation of representations and more for their outright refusal. The following generalisation, however, shall become pivotal for constructing families of representations.

Lemma 2.2.13. *Let $P \leq GL(V)$ be a p-group representation with doubly-conforming basis $\mathcal{B}$. Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_{m'}$ denote all of the basis elements with the strict inclusions*

$$v_i \in \mathrm{Soc}_1(V), v_i^* \in \mathrm{Soc}_2(V^*) \quad \textit{and} \quad w_i \in \mathrm{Soc}_2(V), w_i^* \in \mathrm{Soc}_1(V^*)$$

i.e. the elements of $\mathcal{B}$ corresponding to the $\boxed{2}$ in the first row and the $\boxed{1}$ in the second row of δ respectively. Then the representation resulting from

$$\langle \mathcal{B} \setminus \{w_1, \ldots, w_{m'}\} \rangle \,/\, \langle v_1, \ldots, v_m \rangle$$

has tabloid obtained by deleting the aforementioned boxes from δ.

Proof. Taking the quotient of a v_i will affect where in the socle series some w in the basis appears only if $w \in \mathrm{Soc}_2(V)$. Since this corresponds to taking a submodule in V^* the dual action remains unaffected.

Dualising the argument says that taking the submodule without w_i will affect where in the dual socle series some v^* appears only if $v^* \in \mathrm{Soc}_2(V^*)$.

Hence taking the quotient by all v_i and the submodule without the w_i will not affect where in either socle series any remaining element of the given sits. Furthermore the dimensions of the series remain the same, modulo removing the v_i and w_i. Hence the tabloid remains untouched bar the removal of the boxes corresponding with the v_i and w_i. □

We shall only see the true power of this result come Theorem 2.3.3.

Having ascertained that the direct sum of two $\mathbb{F}P$-modules simply concatenates their socle tabloids and that the $\delta^{(i)}_{(j)}$ are socle tabloids of sub / quotient modules it behoves us to consider how these modules might decompose simply by observing their tabloids. We shall find this useful for further refining our representation management.

Definition 2.2.14. Given an $\mathbb{F}P$-module V with socle tabloid δ we define the *deconstruction* of δ with respect to V to be the following data: For each $0 \leq i, j \leq k$ we take

$$\delta^{(i)}_{(j)} = \delta_{i,j,1} \oplus \cdots \oplus \delta_{i,j,s_{i,j}}$$

where

$$V^{(i)}_{(j)} = V_{i,j,1} \oplus \cdots \oplus V_{i,j,s_{i,j}}$$

is a decomposition by indecomposables with $V_{i,j,\ell} \sim \delta_{i,j,\ell}$.

Example 2.2.15. Let V be an $\mathbb{F}P$-module with socle tabloid $\delta = \begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}$. The possible deconstructions of δ are

- $\delta = \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \oplus \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array} \quad \Longrightarrow \quad \delta^{(1)} = \delta_{(1)} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \oplus \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$;
- $\delta = \begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}$, $\delta^{(1)} = \delta_{(1)} = \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}$;
- $\delta = \begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}$, $\delta^{(1)} = \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}$, $\delta_{(1)} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \oplus \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$;
- $\delta = \begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}$, $\delta^{(1)} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \oplus \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$, $\delta_{(1)} = \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}$;
- $\delta = \begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1&1\\\hline\end{array}$, $\delta^{(1)} = \delta_{(1)} = \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \oplus \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$;

It follows that any two representations with different deconstructions cannot be equivalent. We shall use this to our advantage when determining the maximal abelian subgroups with a given tabloid δ.

Conclusion

However convenient the socle tabloid may seem in its translation of key properties its introduction means nought if it cannot yield information unlikely to have been noticed without it. We have yet to consider how this impacts the work of Section 2.1. Hereafter we specify to $\mathbb{F}\mathbb{Z}_p^r$-modules and back into the purview of covering homomorphisms.

2.3 Iterating Covering Homomorphisms

In Section 2.1 we developed the notion of covering all equivalence classes of $\mathbb{Z}_p^r$-representations with socle-type M using homomorphisms from $\mathbb{F}$-vector spaces. Quite how many homomorphisms are required depends on the length of the largest sequence of maximal abelian subgroups of $U_M(\mathbb{F})$.

With the introduction of socle tabloids in Section 2.2 we may further divide our representations into more manageable families, and further still with deconstructions. We wish to see how this refinement aids our classification of covering homomorphisms and maximal abelian subgroups.

This extends beyond a simple refinement of families and allows us to iteratively construct covering homomorphisms for infinite families of socle tabloids given only the lowest-dimensional case.

2.3.1 Iteration

We take our first step towards an inductive argument with the following.

Definition 2.3.1. Let $\sigma : (\mathbb{F}^d, +) \to GL(V)$ be a homomorphism of the form

$$\sigma(\underline{c}) = \begin{bmatrix} I_{m_1} & \sigma_{1,2}(\underline{c}) & \cdots & \sigma_{1,k}(\underline{c}) \\ 0 & I_{m_2} & \cdots & \sigma_{2,k}(\underline{c}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_k} \end{bmatrix}$$

such that the image has socle tabloid δ and is already given acting on a doubly conforming basis $\{v_1, \dots, v_n\}$ for V. Let the corresponding socle-type be $(m_1, \dots, m_k)$ and dual type $(n_1, \dots, n_k)$.

For $i \in \mathbb{N}_0$ define the homomorphism $\sigma^{+(i,0)} : (\mathbb{F}^{d+in_1}, +) \to GL_{n+i}(\mathbb{F}) = GL(V')$ by

$$\sigma^{+(i,0)}(\underline{c}) := \left[\begin{array}{c:cccc} I_{m_1+i} & \begin{matrix}\sigma_{1,2}(\underline{c}) \\ M_2(\underline{c})\end{matrix} & \cdots & \begin{matrix}\sigma_{1,k}(\underline{c}) \\ M_k(\underline{c})\end{matrix} \\ \hdashline 0 & I_{m_2} & \cdots & \sigma_{2,k}(\underline{c}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{m_k} \end{array}\right]$$

where the entries of $M_j(\underline{c})$ are defined as follows: In those columns corresponding to the action on v_ℓ for which $v_\ell^* \in \mathrm{Soc}_1(V^*)$ we fill the entries with the independent variables $c_{d+1}, \dots, c_{d+in_1}$ in some specified order. In all remaining columns we place zero entries.

Hence we augment V to a larger-dimensional module V' by introducing i many $v \in \mathrm{Soc}_1(V), v^* \in \mathrm{Soc}_2(V^*)$ in the most generic manner possible. Thus $\sigma^{+(i,0)}(\mathbb{F}^{d+in_1})$ has tabloid acquired by adding i many $\boxed{2}$ to the first row of δ.

Dually for a given $j \in \mathbb{N}_0$ define the homomorphism $\sigma^{+(0,j)} : (\mathbb{F}^{d+jm_1}, +) \to$

$GL_{n+j}(\mathbb{F})$ by

$$\sigma^{+(0,j)}(\underline{c}) := \left[\begin{array}{c|cc|ccc} I_{m_1} & \sigma_{1,2}(\underline{c}) & [c_{d+(\iota-1)j+\kappa}]_{\iota,\kappa} & \sigma_{1,3}(\underline{c}) & \cdots & \sigma_{1,k}(\underline{c}) \\ \hline 0 & \multicolumn{2}{c|}{I_{m_2+j}} & \begin{matrix}\sigma_{2,3}(\underline{c})\\ 0\end{matrix} & \cdots & \begin{matrix}\sigma_{2,k}(\underline{c})\\ 0\end{matrix} \\ 0 & \multicolumn{2}{c|}{0} & I_{m_3} & \cdots & \sigma_{3,k}(\underline{c}) \\ \vdots & \multicolumn{2}{c|}{\vdots} & \vdots & \ddots & \vdots \\ 0 & \multicolumn{2}{c|}{0} & 0 & \cdots & I_{m_k} \end{array}\right].$$

Thus we augment V with j many $v \in \mathrm{Soc}_2(V), v^* \in \mathrm{Soc}_1(V^*)$ as generically as possible. Hence $\sigma^{+(0,j)}(\mathbb{F}^{d+jm_1})$ has tabloid acquired by adding j many $\boxed{1}$ to the second row of δ.

Thence we may define $\sigma^{+(i,j)} : (\mathbb{F}^{d+(i+m_1)(j+n_1)-m_1n_1}, +) \to GL_{n+i+j}(\mathbb{F})$ by

$$\sigma^{+(i,j)} := (\sigma^{+(i,0)})^{+(0,j)}.$$

Note that up to reordering the parameters of $\mathbb{F}^{d+(i+m_1)(j+n_1)-m_1n_1}$ the images of $(\sigma^{+(i,0)})^{+(0,j)}$ and $(\sigma^{+(0,j)})^{+(i,0)}$ are equal. We often abuse this by relabelling the parameters more so for aesthetic purposes than following rigorous definition.

Before we prove results regarding these homomorphism we demonstrate their construction by way of example.

Example 2.3.2. It has been shown by [11] (and replicated in Section 1.2.2) that all modular $\mathbb{Z}_p^r$-representations over a field $\mathbb{F}$ with socle-type $(1,1,1)$ are covered by the homomorphism

$$\sigma_{111} : (\mathbb{F}^2, +) \to GL_3(\mathbb{F}), \quad \sigma_{111}(c_1, c_2) = \begin{bmatrix} 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}.$$

We thus endeavour to acquire a homomorphism containing representations with socle tabloid

$$\begin{array}{|c|c|c|}\hline 3 & 2 & 2 \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

by constructing $\sigma_{111}^{+(2,1)}$ as follows:

$$\sigma_{111}(c_1,c_2) \to \sigma_{111}^{+(2,0)}(c_1,\ldots,c_4) := \begin{bmatrix} 1 & 0 & 0 & 2c_1 & c_1^2+c_2 \\ 0 & 1 & 0 & 0 & c_4 \\ 0 & 0 & 1 & 0 & c_3 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \sigma_{111}^{+(2,1)}(c_1,\ldots,c_7) := \begin{bmatrix} 1 & 0 & 0 & 2c_1 & c_7 & c_1^2+c_2 \\ 0 & 1 & 0 & 0 & c_6 & c_4 \\ 0 & 0 & 1 & 0 & c_5 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The following theorem acts as a powerful tool demonstrating that this construction not only yields higher dimensional representations from smaller ones but is arguably exhaustive in doing so.

Theorem 2.3.3. *Suppose $\sigma_\ell : (\mathbb{F}^{d_\ell},+) \to GL_n(\mathbb{F})$ for $\ell = 1,\ldots,s$ are a full set of covering homomorphisms for the modular $\mathbb{Z}_p^r$-representations with a given socle tabloid δ, socle-type $(m_1,\ldots,m_k)$ and dual-type $(n_1,\ldots,n_k)$.*

Construct δ' by adding i many $\boxed{2}$ to the first row and j many $\boxed{1}$ to the second row of δ. Then

$$\sigma_\ell^{+(i,j)}(\mathbb{F}^{d_\ell+(i+m_1)(j+n_1)-m_1n_1},+) \to GL_{n+i+j}(\mathbb{F}) \quad \textit{for } \ell = 1,\ldots,s$$

collectively form a full set covering homomorphisms for all $\mathbb{Z}_p^r$-representations with socle tabloid δ'.

Proof. For convenience we may assume that the images of each σ_ℓ are already acting on a doubly-conforming basis. Let $\mathbb{Z}_p^r \cong G' \leq GL(V')$ have socle tabloid δ' and doubly-conforming basis $\mathcal{B}'$. Denote by $v_1,\ldots,v_i,w_1,\ldots,w_j \in \mathcal{B}'$ basis elements of V' such that

$$v_\ell \in \mathrm{Soc}_1(V'), v_\ell^* \in \mathrm{Soc}_2((V')^*) \quad \text{and} \quad w_\ell \in \mathrm{Soc}_2(V'), w_\ell^* \in \mathrm{Soc}_1((V')^*).$$

Then denote by $V := \langle \mathcal{B}' \setminus \{w_1,\ldots,w_j\}\rangle \,/\, \langle v_1,\ldots,v_i\rangle$ another $\mathbb{F}\mathbb{Z}_p^r$-module inducing the action $G \leq GL(V)$.

By Lemma 2.2.13 the socle tabloid of G (resp. V) is δ. Since this was the only alteration to the socle tabloid the quotient representatives of the remaining elements of $\mathcal{B}'$ form a doubly-conforming basis for V. We may alter this basis such that G exists in the image of one of the homomorphisms σ_ℓ by assumption. Applying this same basis change to the corresponding elements of $\mathcal{B}'$ we acquire a new doubly-conforming basis for V' containing the v_i and w_i such that the induced G has image inside σ_ℓ.

Since $v_s^* \in \mathrm{Soc}_2((V')^*)$ the dual action on v_s may only involve elements of $\mathrm{Soc}((V')^*)$. Since $w_s \in \mathrm{Soc}_2(V')$ the action on w_s may only involve elements in $\mathrm{Soc}(V')$. The resulting form must then exist within the image of $\sigma_\ell^{+(i,j)}$ by design. Hence up to equivalence all such representations' images exist within the image of some $\sigma_\ell^{+(i,j)}$ as required. $\square$

It follows from this that the homomorphism acquired at the conclusion of exercise 2.3.2 is in fact a covering homomorphism for all $\mathbb{Z}_p^r$-representations with socle tabloid $\begin{array}{|c|c|c|}\hline 3&2&2\\ \hline 2&1 \\ \cline{1-2} 1 \\ \cline{1-1}\end{array}$.

If one considers a tabloid with a $\boxed{1}$ in its first row then both the basis element to which it corresponds and its dual shall be fixed under the action of the group. Given that our search is often to find covering homomorphisms for representations of a given degree Theorem 2.3.3 yields a corollary ensuring we need not induce these representations separately.

Corollary 2.3.4. *Suppose for $\ell = 1, \dots, s$ that $\sigma_\ell : (\mathbb{F}^{d_\ell}, +) \to GL_n(\mathbb{F})$ collectively cover the $\mathbb{Z}_p^r$-representations with socle tabloid δ. Then the $\sigma_\ell^{+(1,0)}$ for $\ell = 1, \dots, s$ collectively cover all $\mathbb{Z}_p^r$-representations with socle tabloid δ' acquired from δ by adding a $\boxed{1}$ to its first row.*

Proof. Let $\mathbb{Z}_p^r \cong G' \leq GL(V')$ have socle tabloid δ'. Fix a doubly-conforming basis with element $v \in \mathrm{Soc}(V')$ such that $v^* \in \mathrm{Soc}((V')^*)$. Since $\langle v \rangle$ is a trivial free summand $V := V'/\langle v \rangle$ has socle tabloid δ and thus is covered by some σ_ℓ.

By choosing the appropriate doubly-conforming basis the elements of G' may be written as elements in the image of σ_ℓ augmented with a trivial row and column. This can be achieved by constructing $\sigma_\ell^{+(1,0)} : (\mathbb{F}^{d_\ell + \dim(\mathrm{Soc}\ V^*)}, +) \to GL_n(\mathbb{F})$ and choosing the appropriate subspace $W \leq (\mathbb{F}^{d_\ell + \dim(\mathrm{Soc}\ V^*)}, +)$ for which the newly introduced variables $c_{d_\ell+1} = \cdots = c_{d_\ell + \dim(\mathrm{Soc}\ V^*)} = 0$ for all $\underline{c} \in W$. $\square$

The upside of these results is the following: In classifying all $\mathbb{Z}_p^r$-representations of a given degree, any representation whose socle tabloid contains a $\boxed{1}$ or $\boxed{2}$ in its first row or a $\boxed{1}$ in its second can be easily induced from covering homomorphisms which collectively cover the tabloid with these boxes removed altogether.

Example 2.3.5. In Example 2.3.2 we showed that the homomorphism

$$\sigma' : (\mathbb{F}^7, +) \to GL_6(\mathbb{F}), \quad \sigma'(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & 2c_1 & c_7 & c_1^2 + c_2 \\ 0 & 1 & 0 & 0 & c_6 & c_4 \\ 0 & 0 & 1 & 0 & c_5 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

covers the representations with socle tabloid $\begin{array}{|c|c|c|}\hline 3&2&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$. Since, up to ordering the parameters, $\sigma' = (\sigma^{+(0,1)})^{+(2,0)}$ Corollary 2.3.4 tells us that σ' also covers the representations with socle tabloid

$$\delta_1 = \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array} \quad \text{and} \quad \delta_2 = \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}.$$

For δ_1 we, for instance, may assume that for each element $\underline{c} \in \mathbb{F}^7$ in our vector group satisfies $c_3 = c_5 = 0$. This ensures that our third basis element is fixed in the dual action and thus we acquire a trivial free summand.

For δ_2 we ensure that all generating vectors satisfy $c_3 = c_4 = c_5 = c_6 = 0$ for a similar effect.

Indeed since $\sigma' = (\sigma^{+(2,0)})^{+(0,1)}$ we may further dualise the result and cover all representations with socle tabloid $\begin{array}{|c|c|c|c|}\hline 3&2&2&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}$. These exist in σ' by setting $c_5 = c_6 = c_7 = 0$ for all generating vectors. This is the dual of Corollary 2.3.4 because exchanging a $\boxed{2}$ in the first row for a $\boxed{1}$ is the equivalent in the dual diagram of moving a $\boxed{1}$ from the second row to the first:

$$\begin{array}{|c|c|c|}\hline 3&2&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}^{*} = \begin{array}{|c|c|c|}\hline 3&2\\\hline 2&1&1\\\hline 1\\\cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 3&1\\\hline 2&1&1\\\hline 1\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 3&2&2&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}^{*}.$$

Hence we conclude that σ' in fact covers all $\mathbb{Z}_p^r$-representations with any of the following tabloids:

$$\begin{array}{|c|c|c|}\hline 3&2&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 3&2&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|}\hline 3&1&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 3&2&2&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 3&2&1&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}, \quad \begin{array}{|c|c|c|c|}\hline 3&1&1&1\\\hline 2\\\cline{1-1} 1\\\cline{1-1}\end{array}.$$

Recall that we further our classification refinement by specifying the deconstruction of a tabloid, encoding the way in which the sub/quotient modules $V_{(i)}^{(j)}$ decompose. Thus if some $V_{(i)}$ decomposes we may wish to ask how the quotient of V by these submodules affects the socle tabloid. This process is often less predictable, but worth investigation.

Example 2.3.6. Suppose $\mathbb{Z}_p^r \cong G \leq GL(V)$ has socle tabloid $\begin{array}{|c|c|}\hline 3&3\\\hline 2&2\\\hline 1\\\cline{1-1}\end{array}$. Suppose further that its deconstruction specifies $V_{(1)} = V_1 \oplus V_2 \sim \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array} \oplus \begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ so that we may write the elements in G in the form

$$\begin{bmatrix} 1 & 0 & c_{13} & 0 & c_{15} \\ 0 & 1 & 0 & c_{24} & c_{25} \\ 0 & 0 & 1 & 0 & c_{35} \\ 0 & 0 & 0 & 1 & c_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to preserve the socle tabloid neither the entries c_{35} nor c_{45} are identically zero across all elements, and thus both V/V_1 and V/V_2 have socle tabloid $\begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$. Since we know that representations with this socle tabloid are covered by the homomorphism

$$\sigma_{111}(\underline{c}) = \begin{bmatrix} 1 & 2c_1 & c_2 + c_1^2 \\ 0 & 1 & c_1 \\ 0 & 0 & 1 \end{bmatrix}$$

it follows that we may write our original elements in the form

$$\begin{bmatrix} 1 & 0 & 2c_{35} & 0 & c_{15} + c_{35}^2 \\ 0 & 1 & 0 & 2c_{45} & c_{25} + c_{45}^2 \\ 0 & 0 & 1 & 0 & c_{35} \\ 0 & 0 & 0 & 1 & c_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

One may thence construct a covering homomorphism from this, however we shall give an alternate construction in Example 2.6.5 equal to this but in a more generalised context.

Thus far our attempts to augment socle tabloids with additional boxes and examine the fallout has been fruitful. Naturally, however, no experienced mathematician ought to take an example as proof of a claim beyond the example's scope. For instance is there a method to augment representations with a tabloid

δ to those of δ' acquired by appending a $\boxed{3}$ to δ's first row? We examine this trail of thought in the following section, the title and introduction of which may possibly go some way to spoiling the conclusion.

2.3.2 Vanishing Tabloids

In all results and examples given prior we have constructed representations from others by either enlarging or reducing the dimension in a manner such that the associated tabloid is affected predictably. This all lies under the assumption that for a given tabloid there exists a representation associated to it.

Proposition 2.3.7. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(m_1, \ldots, m_k)$. If $m_j = 1$ for some $1 < j < k$ then $m_j = m_{j+1} = \cdots = m_k = 1$.*

Proof. By taking the quotient of $\mathrm{Soc}_{j-2}(V)$ we need only prove by contradiction the result in the case $j = 2$ and subsequently extend the non-existence upwards.

Assume that $m_2 = 1 < m_3$. Adopting a socle-conforming basis we can write elements in G in the form

$$C = \begin{bmatrix} I_{m_1} & C_{12} & C_{13} & \cdots \\ 0 & 1 & C_{23} & \cdots \\ 0 & 0 & I_{m_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $C_{12} = [c_{1,m_1+1}, \ldots, c_{m_1,m_1+1}]^T$. Choosing this basis appropriately we ensure that $c_{1,m_1+1} \neq 0$ for some element C. Given a general $D \in G$ of a similar form to above we see that $[C, D] = 0$ implies that $C_{12}D_{23} = D_{12}C_{23}$. This is a matrix equation and so, in particular, we must have equality of the first rows:

$$c_{1,m_1+1}D_{23} = d_{1,m_1+1}C_{23} \qquad \Longrightarrow \qquad D_{23} = \frac{d_{1,m_1+1}}{c_{1,m_1+1}}C_{23}.$$

Hence the $[2, 3]$ blocks of all elements are scalar multiples of C_{23}. This means that, for example,

$$[1, 0, \ldots, 0]^T, [0, c_{m_1+1,m_1+3}, -c_{m_1+1,m_1+2}, 0, \ldots, 0]^T \in (V^{(1)})^G$$

as indeed does any augmented nonzero element from the kernel of C_{23}. However we must have $\dim((V^{(1)})^G) = m_2 = 1$. This contradiction yields the result. $\square$

This is the first instance, since the impositions from the socle series, of tabloids with no valid $\mathbb{Z}_p^r$-representations. They are accompanied by the following.

Corollary 2.3.8. *For all $k, \ell \geq 3$, there exist no modular abelian p-group representations associated to the tabloids*

$$\begin{array}{|c|c|}\hline k & \ell \\ \hline k-1 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad . \tag{2.4}$$

Proof. The result follows from Proposition 2.3.7, noting that the dual-type associated to each of these tabloids is $(1, \ldots, 1, \underbrace{2}_{\ell}, 1, \ldots, 1)$. $\square$

This puts pay to the notion that we might arbitrarily extend the dimension of the socle of our representation and expect to acquire valid representations with predictable tabloids. Indeed the following is a generalisation of only the $k = \ell = 3$ case of Corollary 2.3.8 acquired by applying most of the results appearing in Section 2.2.2.

Proposition 2.3.9. *Let δ be a tabloid with k rows. Suppose that for some $i > 1$ the ith row from the top contains exactly one $\boxed{k-i+1}$ and the $(i-1)$th row contains at least two $\boxed{k-i+2}$. Then there are no abelian p-group representations with socle tabloid δ.*

Proof. Suppose an abelian p-group representation over V exists with socle tabloid δ. Then by Theorem 2.2.10 the representation $V_{(k-i-1)}^{(i-2)}$ has socle-length 3 and has socle-tabloid whose first row contains at least two $\boxed{3}$'s and whose second row contains exactly one $\boxed{2}$, that is $\delta_{(k-i-1)}^{(i-2)} = \begin{array}{|c|c|c|c|}\hline 3 & 3 & \star & \cdots \\ \hline 2 & 1 & \cdots & \multicolumn{1}{c}{} \\ \cline{1-3} 1 & \cdots & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$.

Using Lemma 2.2.13 we may take the quotient of all $v \in \mathrm{Soc}_1(V_{(k-i-1)}^{(i-2)})$ and submodule without $w \in \mathrm{Soc}_2(V_{(k-i-1)}^{(i-2)})$ such that $v^* \in \mathrm{Soc}_2\left((V_{(k-i-1)}^{(i-2)})^*\right)$ and $w \in \mathrm{Soc}_1\left((V_{(k-i-1)}^{(i-2)})^*\right)$ to acquire a representation

$$V' \sim \begin{array}{|c|c|c|c|}\hline 3 & 3 & \cdots & 3 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c|}{\cdots} & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array} \; .$$

The socle-type of V' is then $(m_1, 1, m_3)$ for $m_1 > 1$ equal to the number of $(k-i+2)$-boxes in row $(i-1)$ of δ. This then contradicts Proposition 2.3.7 thus yielding the result.

$\square$

This result aims to save us a certain amount of time. For instance were we to wish to classify all modular $\mathbb{Z}_p^r$-representations of dimension 5 (see Section 2.8) by separating them into socle tabloid families then we would not need to consider any of the tabloids

$$\begin{ytableau} 3 & 3 & 3 \\ 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 3 \\ 2 \\ 1 & 1 & 1 \end{ytableau} \quad \begin{ytableau} 3 & 3 & 2 \\ 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 3 \\ 2 & 1 \\ 1 & 1 \end{ytableau} \quad \begin{ytableau} 3 & 3 & 1 \\ 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 3 & 1 \\ 2 \\ 1 & 1 \end{ytableau}$$

$$\begin{ytableau} 3 & 3 \\ 2 & 1 \\ 1 \end{ytableau} \quad \begin{ytableau} 3 & 2 \\ 2 \\ 1 & 1 \end{ytableau} \quad \begin{ytableau} 3 & 3 \\ 2 \\ 1 & 1 \end{ytableau} \quad \begin{ytableau} 4 & 4 \\ 3 \\ 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 4 \\ 3 \\ 2 \\ 1 & 1 \end{ytableau} \quad \begin{ytableau} 4 \\ 3 & 3 \\ 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 4 \\ 3 \\ 2 & 2 \\ 1 \end{ytableau} \quad \begin{ytableau} 4 & 3 \\ 3 \\ 2 \\ 1 \end{ytableau}$$

since either they, their dual or some sub/quotient form of either satisfy the conditions of Proposition 2.3.7, Corollary 2.3.8 or Proposition 2.3.9.

Since we are no longer guaranteed that a given tabloid has associated representations we consider whether certain tabloids do have representations but only for specific deconstructions.

Example 2.3.10. Consider a representation $P \leq GL(V)$ of an abelian p-group with socle tabloid $\delta = \begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{ytableau}$. In Example 2.2.15 we detailed the possible deconstructions of δ corresponding to the decompositions of the sub/quotient modules of V.

Suppose we are in the case with $V_{(1)} = V_1 \oplus V_2 \sim \begin{ytableau} 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} 2 \\ 1 \end{ytableau}$ and with $V^{(1)} \sim \begin{ytableau} 2 & 2 \\ 1 & 1 \end{ytableau}$ indecomposable. Then by choosing the appropriate basis $\{v_1, \ldots, v_6\}$ we may write the elements of P in the form

$$\begin{bmatrix} 1 & 0 & c_{13} & 0 & c_{15} & c_{16} \\ 0 & 1 & 0 & c_{24} & c_{25} & c_{26} \\ 0 & 0 & 1 & 0 & c_{35} & c_{36} \\ 0 & 0 & 0 & 1 & c_{45} & c_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $V_{(1)}$ is a submodule of V, up to choice of V_i we have the action of V/V_1

given by

$$\begin{bmatrix} 1 & c_{24} & c_{25} & c_{26} \\ 0 & 1 & c_{45} & c_{46} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \delta = \begin{array}{|c|l} \cline{1-1} 3 & \\ \cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array}.$$

However we have seen that such representations cannot exist. Hence we have shown that only certain deconstructions of δ have associated representations. We summarise this in a handy-to-reference result for ease of later use.

Corollary 2.3.11. *Let $V \in \mathbb{F}\mathbb{Z}_p^r$-mod have socle tabloid $\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$. Then $V_{(1)}$ is decomposable if and only if $V^{(1)}$ is decomposable.*

Proof. If $V_{(1)}$ is decomposable then the proof is given in example (2.3.10). If $V^{(1)}$ is decomposable then apply the same argument to V^*. □

Conclusion

We have now introduced the notion of iterating covering homomorphisms to arbitrary dimension by socle tabloid-predictable manipulations. This assists in the construction and negation of proposed representations from existing ones. This, however, is of little use were we not to have a solid foundation of existing representations from which to work.

2.4 Socle-Type $(1, \ldots, 1)$ Representations

Hitherto our studies of the modular $\mathbb{Z}_p^r$-representations have been presented in as general a fashion as possible. However in order to apply these to acquire specific representations we require foundations.

We consider the case where the representation dimension equals the socle length. We ascertain up to equivalence all socle-type $(1, \ldots, 1)$ representations of $\mathbb{Z}_p^r$ by way of a covering homomorphism. To this end we employ well known combinatorial objects and, in the process, learn more about their properties.

We then describe the equivalence of such representations in terms of a linear action which permutes these additive preimages.

2.4.1 Prime Restriction

For a representation $\mathbb{Z}_p^r \cong G \leq GL(V)$ to have socle-type $1^n := (1, \ldots, 1)$ is, by a certain measure, either an upper limit of our representations or lower base for induction. It sits as a low base since the associated socle tabloid

$$G \sim \begin{array}{|c|}\hline n \\ \hline \vdots \\ \hline 2 \\ \hline 1 \\ \hline \end{array}$$

is prime foundation material for our inductive constructions. Yet it is an upper limit in the sense that we begin with $U_{1^n}(\mathbb{F}) = U_n(\mathbb{F})$ and can restrict no further without losing the desired socle-type. Fortunately we shall see that this inability to restrict affords us greater control.

Recall from Section 2.1 that the process of constructing covering homomorphisms becomes obfuscated when the prime characteristic of our field sits below the socle length. This is a situation we need not concern ourselves with.

Lemma 2.4.1. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type 1^n. Then there exists an element $\Gamma \in G$ for which $(\Gamma - I_n)^{n-1} \neq 0$. In particular $n \leq p$.*

Proof. Suppose G acts upon a socle-conforming basis for V and, for a contradiction, assume that $(\Gamma - I_n)^{n-1} = 0$ for all elements $\Gamma = [\gamma_{i,j}] \in G$. This is equivalent to ensuring $\prod_{i=1}^{n-1} \gamma_{i,i+1} = 0$.

Since the representation has socle-type 1^n, for all $k \in [\![1, n-1]\!]$ there must exist elements $\Gamma^{(k)} = [\gamma_{i,j}^{(k)}]$ for which $\gamma_{k,k+1}^{(k)} \neq 0$.

Let $i \in [\![2, n-1]\!]$ be the smallest such integer for which $\gamma_{i,i+1}^{(1)} = 0$, which must exist by assumption. Since the group is abelian we acquire

$$[\gamma^{(1)}, \gamma^{(i)}][i-1, i+1] = \gamma_{i-1,i}^{(1)}\gamma_{i,i+1}^{(i)} - \gamma_{i-1,i}^{(i)}\gamma_{i,i+1}^{(1)} = \gamma_{i-1,i}^{(1)}\gamma_{i,i+1}^{(i)} = 0.$$

Since i was minimal we have $\gamma_{i-1,i}^{(1)}\gamma_{i,i+1}^{(i)} \neq 0$. This yields the desired contradiction. $\square$

Since there only exist $\mathbb{Z}_p^r$-representations with socle-type 1^n over fields of characteristic $p \geq n$ we can use the techniques of Section 2.1.3 to classify them. We determine the commutativity criteria to acquire our maximal abelian (unrefined) groups as follows.

Let $J \in G$ be a representation element with $(J - I_n)^{n-1} \neq 0$ already acting on a socle-conforming basis $\{v'_1, \dots, v'_n\}$. Upper-triangularly altering our basis to $\{v_1, \dots, v_n\} := \{(J - I)^{n-1}v'_n, \dots, (J - I)v'_n, v'_n\}$ for V we may ensure that this element is in Jordan form whilst retaining the socle-conforming property.

In keeping with previous conventions set in [11] we use the fact that $p \geq n$ to lower-triangularly alter the dual of this basis to

$$w^*_{n+1-m} := \sum_{i=0}^{m-1} i! S_2(m, i+1) v^*_{n-i}$$

for $S_2(m, i)$ the Stirling numbers of the second kind. Then J acts upon this basis by

$$w^*_{n+1-m} \cdot J = \sum_{i=1}^{m} \binom{m-1}{i-1} w^*_{n+1-i} \quad \text{and so} \quad J = \left[\binom{n-i}{n-j}\right]_{i,j} \in U_n(\mathbb{F}) \tag{2.5}$$

as shown in Section 1.2.2. We determine our linear commutativity criteria by coercing a general element in $U_n(\mathbb{F})$ to commute with this fixed form J.

Lemma 2.4.2. *Suppose an $M \in U_n(\mathbb{F})$ commutes with J as given as in* (2.5)*. Then for some $m_i \in \mathbb{F}$*

$$\begin{aligned} M &= \tau_{1^n}(m_{n-1}, \dots, m_1) + I_n \\ &:= \begin{bmatrix} 1 & \binom{n-1}{n-2}m_{n-1} & \binom{n-1}{n-3}m_{n-2} & \cdots & \binom{n-1}{1}m_2 & m_1 \\ 0 & 1 & \binom{n-2}{n-3}m_{n-1} & \cdots & \binom{n-2}{1}m_3 & m_2 \\ 0 & 0 & 1 & \cdots & \binom{n-3}{1}m_4 & m_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & m_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \end{aligned} \tag{2.6}$$

Proof. We may state the result as any $M = [m_{i,j}] \in U_n(\mathbb{F})$ which commutes with $J = [\binom{n-i}{n-j}]$ satisfies $m_{i,i+k} = \binom{n-i}{n-i-k} m_{n-k,n}$ for all $1 \leq i < n-k$ and $1 \leq k < n$. We prove this by induction on k.

For $k = 1$ we consider for each $i \in [\![1, n-2]\!]$

$$[J, M][i, i+2] = \binom{n-i}{n-i-1} m_{i+1,i+2} - \binom{n-i-1}{n-i-2} m_{i,i+1} = 0.$$

Thus we acquire our base of induction as

$$\begin{aligned} m_{i,i+1} &= \binom{n-i}{n-i-1}\binom{n-i-1}{n-i-2}^{-1} m_{i+1,i+2} = \binom{n-i}{n-i-1}\binom{n-i-2}{n-i-3}^{-1} m_{i+2,i+3} = \dots \\ &= \binom{n-i}{n-i-1}\binom{1}{0}^{-1} m_{n-1,n}. \end{aligned}$$

Now suppose $m_{i,i+j} = \binom{n-i}{n-i-j} m_{n-j,n}$ for all $j < k \leq n-1$. Then for $i \leq n-k-1$ we apply the induction hypothesis to

$$\begin{aligned} (JM)[i,i+k+1] &= m_{i,i+k+1} + \binom{n-i}{n-i-1} m_{i+1,i+k+1} + \binom{n-i}{n-i-2} m_{i+2,i+k+1} + \dots \\ &\quad + \binom{n-i}{n-i-k} m_{i+k,i+k+1} + \binom{n-i}{n-i-k-1} \\ &= m_{i,i+k+1} + \binom{n-i}{n-i-1} m_{i+1,i+k+1} + \binom{n-i}{n-i-2}\binom{n-i-2}{n-i-k-1} m_{n-k+1,n} + \dots \\ &\quad + \binom{n-i}{n-i-k}\binom{n-i-k}{n-i-k-1} m_{n-1,n} + \binom{n-i}{n-i-k-1} \\ (MJ)[i,i+k+1] &= \binom{n-i}{n-i-k-1} + \binom{n-i-1}{n-i-k-1} m_{i,i+1} + \dots \\ &\quad + \binom{n-i-k+1}{n-i-k-1} m_{i,i+k-1} + \binom{n-i-k}{n-i-k-1} m_{i,i+k} + m_{i,i+k+1} \\ &= \binom{n-i}{n-i-k-1} + \binom{n-i-1}{n-i-k-1}\binom{n-i}{n-i-1} m_{n-1,n} + \dots \\ &\quad + \binom{n-i-k+1}{n-i-k-1}\binom{n-i}{n-i-k+1} m_{n-k+1,n} + \binom{n-i-k}{n-i-k-1} m_{i,i+k} + m_{i,i+k+1} \end{aligned}$$

Comparing these and using the identity $\binom{a}{a-b}\binom{a-b}{a-b-c} = \binom{a}{a-c}\binom{a-c}{a-b-c}$ it follows that $[J,M][i,i+k+1] = \binom{n-i}{n-i-1} m_{i+1,i+k+1} - \binom{n-i-k}{n-i-k-1} m_{i,i+k}$. Applying this recursively we acquire

$$\begin{aligned} m_{i,i+k} &= \binom{n-i}{n-i-1}\binom{n-i-k}{n-i-k-1}^{-1} m_{i+1,i+k+1} \\ &= \binom{n-i}{n-i-1}\binom{n-i-k}{n-i-k-1}^{-1}\binom{n-i-1}{n-i-2}\binom{n-i-k-1}{n-i-k-2}^{-1} m_{i+2,i+k+2} \\ &= \cdots \\ &= \binom{n-i}{n-i-1}\cdots\binom{k+1}{k}\binom{n-i-k}{n-i-k-1}^{-1}\cdots\binom{1}{0}^{-1} m_{n-k,n} \\ &= \frac{(n-i)!}{k!}\frac{1}{(n-i-k)!} m_{n-k,n} = \binom{n-i}{n-i-k} m_{n-k,n} \end{aligned}$$

from which the result follows by induction. □

One may verify that any two elements of the form $\tau_{1^n}(\underline{c}) + I_n$ commute. Thus $\{\tau_{1^n}(\underline{c}) + I_n \bigm| \underline{c} \in \mathbb{F}^{n-1}\}$ is a commutative, multiplicative group and serves as our lone unrefined group. Thence $\tau_{1^n}(\mathbb{F}^{n-1})$ becomes our unrefined algebra. In order to easily describe the exponentiated refined group we borrow some information from the common library of combinatorics.

2.4.2 Parameterising Socle-Type 1^n Representations and Bell Polynomials

We shall hereafter make use of the **complete exponential Bell polynomials** from [3], defined here by

$$B_m(c_1, \dots, c_m) = \sum_{\substack{\underline{v} \in \mathbb{N}_0^m \\ \mathrm{wt}(\underline{v})=m}} \frac{m!}{\prod_{i=1}^m v_i!(i!)^{v_i}} \, c_1^{v_1} \cdots c_m^{v_m}$$

where $\mathrm{wt}(\underline{v}) := \sum_{i=1}^m i v_i$ for all $\underline{v} \in \mathbb{N}_0^m$. We occasionally abuse notation and define for $n > m$

$$B_m(c_1, \dots, c_n) := B_m(c_1, \dots, c_m).$$

Theorem 2.4.3. *The homomorphism*

$$\sigma_{1^n} : (\mathbb{F}^{n-1}, +) \to GL_n(\mathbb{F}), \qquad \sigma_{1^n}(\underline{\gamma})[i,j] = \tbinom{n-i}{n-j} B_{j-i}(\underline{\gamma}),$$

where B_m is the mth complete exponential Bell polynomial, is a covering homomorphism for all modular $\mathbb{Z}_p^r$-representations over $\mathbb{F}$ with socle-type $1^n = (1, \dots, 1)$.

Proof. Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type 1^n. By Lemma 2.4.1 we choose the socle-conforming basis for V upon which G acts such that some $J \in G$ is in the form (2.5). Then by Lemma 2.4.2 any element of $U_n(\mathbb{F})$ which commutes with J is of the form $\tau_{1^n}(\underline{c}) + I_n$. Thus $\tau_{1^n}(\mathbb{F}^{n-1})$ is our unrefined algebra. We need only show that $\sigma_{1^n} = \exp \circ \tau_{1^n}$, that is

$$\exp(\tau_{1^n}(\underline{c}))[i,j] = \tbinom{n-i}{n-j} B_{j-i}(\underline{c}).$$

For any given $\underline{v} \in \mathbb{N}^\alpha$ denote $s(\underline{v}, \beta) := \sum_{i=1}^{\beta} v_i$ with special cases $s(\underline{v}) := s(\underline{v}, \alpha)$ and $s(\underline{v}, 0) = 0$. By straight-forward calculation

$$[\tau_{1^n}(\underline{c})^\alpha][i,j] = \sum_{\substack{\underline{v} \in \mathbb{N}^\alpha \\ s(\underline{v})=j-i}} \prod_{\beta=1}^{\alpha} \binom{n-(i+s(\underline{v},\beta-1))}{n-(i+s(\underline{v},\beta))} c_{v_\beta} = \binom{n-i}{n-j} \sum_{\substack{\underline{v} \in \mathbb{N}^\alpha \\ s(\underline{v})=j-i}} \binom{j-i}{\underline{v}} \prod_{\beta=1}^{\alpha} c_{v_\beta}$$

There is a correspondence between the sets

$$\{\underline{v} \in \mathbb{N}^{\alpha} \mid s(\underline{v}) = j-i\} \to \{\underline{w} \in \mathbb{N}_0^{j-i} \mid \mathrm{wt}(\underline{w}) = j-i, s(\underline{w}) = \alpha\}.$$

where to each such $\underline{v}$ we associate $\underline{w}$ where w_ℓ is the number of entries of $\underline{v}$ equal to ℓ. To each $\underline{w}$ there are naturally $\binom{\alpha}{\underline{w}}$ associated $\underline{v}$ acquired from any given example by permuting the entries. We may thus rewrite our expression as follows:

$$[\tau_{1^n}(\underline{c})^{\alpha}][i,j] = \binom{n-i}{n-j} \sum_{\substack{\underline{w} \in \mathbb{N}_0^{j-i} \\ s(\underline{w})=\alpha \\ \mathrm{wt}(\underline{w})=j-i}} \binom{\alpha}{\underline{w}} \frac{(j-i)!}{[1!]^{w_1}[2!]^{w_2}\cdots[(j-i)!]^{w_{j-i}}} \prod_{\beta=1}^{j-i} c_\beta^{w_\beta}.$$

Thus we obtain

$$\begin{aligned}
\exp(\tau_{1^n}(\underline{c}))[i,j] &= \sum_{\alpha=0}^{n-1} \frac{1}{\alpha!}[\tau_{1^n}(\underline{c})^{\alpha}][i,j] \\
&= \sum_{\alpha=0}^{n-1} \frac{1}{\alpha!}\binom{n-i}{n-j} \sum_{\substack{\underline{w} \in \mathbb{N}_0^{j-i} \\ s(\underline{w})=\alpha \\ \mathrm{wt}(\underline{w})=j-i}} \binom{\alpha}{\underline{w}} \frac{(j-i)!}{[1!]^{w_\ell}[2!]^{w_2}\cdots[(j-i)!]^{w_{j-i}}} \prod_{\beta=1}^{j-i} c_\beta^{w_\beta} \\
&= \binom{n-i}{n-j} \sum_{\substack{\underline{w} \in \mathbb{N}_0^{j-i} \\ \mathrm{wt}(\underline{w})=j-i}} \frac{(j-i)!}{\prod_{\ell=1}^{j-i}[\ell!]^{w_1} w_\ell!} \prod_{\beta=1}^{j-i} c_\beta^{w_\beta} = \binom{n-i}{n-j} B_{j-i}(\underline{c})
\end{aligned}$$

as required. □

We conclude with a few well known properties of the Bell polynomials.

Lemma 2.4.4 ([3] (4.9))**.** *For all $\underline{c}, \underline{\delta} \in \mathbb{F}^k$ and $k \geq 0$ we have*

$$B_m(\underline{c} + \underline{d}) = \sum_{i=0}^{m} \binom{m}{i} B_i(\underline{c}) B_{m-i}(\underline{d}).$$

Note that this result is readable from Theorem 2.4.3 were one to examine the $[n-m, n]$ entry of the equation

$$\left[\tbinom{n-i}{n-j} B_{j-i}(\underline{c})\right]\left[\tbinom{n-i}{n-j} B_{j-i}(\underline{d})\right] = \left[\tbinom{n-i}{n-j} B_{j-i}(\underline{c}+\underline{d})\right].$$

Lemma 2.4.5 ([3] (4.2))**.** *For all $m \geq 0$*

$$B_{m+1}(\underline{c}) = \sum_{i=0}^{m} \binom{m}{i} c_{i+1} B_{m-i}(\underline{c}).$$

2.4.3 Equivalence of Socle-Type 1^n Representations

Theorem 2.4.3 tells us that any representation with socle-type 1^n is induced by the covering homomorphism $\sigma_{1^n} : (\mathbb{F}^{n-1}, +) \to GL_n(\mathbb{F})$ up to equivalence. Recall that Proposition 2.1.10 suggests a connection between equivalence of representations induced by the same covering homomorphism and some linear action upon the underlying vector group.

Here we construct an explicit linear action on $\mathbb{F}^{n-1}$ whose orbits directly connect the images of equivalent representations induced by σ_{1^n}. To do so we require the following.

Definition 2.4.6. For $1 \leq k \leq m$ the *incomplete exponential Bell Polynomials* are given by

$$B_{m,k}(\underline{c}) := \sum_{\substack{\underline{v} \in \mathbb{N}_0^m \\ s(\underline{v},m)=k \\ \mathrm{wt}(\underline{v})=m}} \frac{m!}{\prod_{i=1}^{m} v_i!(i!)^{v_i}} \, c_1^{v_1} \cdots c_m^{v_m}.$$

The incomplete Bell polynomials, as with their complete kin $B_m(\underline{c}) = \sum_{k=1}^{m} B_{m,k}(\underline{c})$, are well-studied.

Lemma 2.4.7 ([13] (1.4))**.** *For $0 \leq s \leq j$,*

$$\binom{j}{s} B_{i,j}(\underline{\alpha}) = \sum_{k=j-s}^{i-s} \binom{i}{k} B_{i-k,s}(\underline{\alpha}) B_{k,j-s}(\underline{\alpha}).$$

Using this we may determine the equivalence action as follows.

Proposition 2.4.8. *Two $W, W' \leq (\mathbb{F}^{n-1}, +)$ that the groups $\sigma_{1^n}(W), \sigma_{1^n}(W')$ both induce socle-type 1^n representations. Then these representations are equivalent if and only if there exists an $\underline{\alpha} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$ such that*

$$W' = [B_{i,j}(\underline{\alpha})]\, W.$$

Proof. Assume that the representations are equivalent, that is there exists an $E \leq GL_n(\mathbb{F})$ such that $\sigma_{1^n}(W) = E^{-1}\sigma_{1^n}(W')E$. Since both representations act on socle-conforming bases we have $e_{i,j} = 0$ when $i > j$.

The claim can be restated as

$$E^{-1}\sigma_{1^n}(\underline{d})E = \sigma_{1^n}(\underline{c}) \implies d_m = \sum_{i=1}^{m} B_{m,i}(\underline{\alpha})c_i$$

for all $m = 1, \dots, n-1$ and all $\underline{c} \in W$ with corresponding $\underline{d} \in W'$. We prove by induction on m.

Observe that

$$(E\sigma_{1^n}(\underline{c}) - \sigma_{1^n}(\underline{d})E)[i, i+1] = (n-i)(e_{i,i}c_1 - e_{i+1,i+1}d_1) = 0$$

for all $i = 1, \dots, n-1$. Thus we require $d_1 = \frac{e_{i+1,i+1}}{e_{i,i}}c_1$ for all $i = 1, \dots, n-1$, thus yielding the constant

$$\alpha_1 := \frac{e_{2,2}}{e_{1,1}} = \frac{e_{3,3}}{e_{2,2}} = \cdots = \frac{e_{n,n}}{e_{n-1,n-1}}.$$

It follows then that $e_{i,i} = \alpha_1 e_{i+1,i+1} = \cdots = \alpha_1^{n-i}e_{n,n}$. The further stipulation that our representations have socle-type 1^n demands that there exist elements of both W and W' for which the first coordinate is nonzero. Thus $\alpha_1 \neq 0$. This case acts as the base of induction.

Suppose now that for all $m = 0, \dots, t-2 < n-1$ we have

$$d_{m+1} = \sum_{\ell=1}^{m+1} B_{m+1,\ell}(\underline{\alpha})c_\ell,$$

$$e_{i,i+m} = \sum_{\ell=0}^{m} \binom{n-i}{\ell} B_{n-(i+\ell),n-(i+m)}(\underline{\alpha})e_{n-\ell,n}$$

for some $\underline{\alpha} \in \mathbb{F}^* \times \mathbb{F}^{t-2}$. We prove that this then holds for $m = t-1$ as follows. Without loss of generality we preempt success by writing

$$d_t = \sum_{\ell=1}^{t} B_{t,\ell}(\underline{\alpha})c_\ell + d_t'$$

$$e_{i,i+(t-1)} = \sum_{\ell=0}^{t-1} \binom{n-i}{\ell} B_{n-(i+\ell),n-(i+t-1)}(\underline{\alpha}) e_{n-\ell,n} + e'_{i,i+(t-1)} \tag{2.7}$$

for $i = 1, \dots, n-t$ for some indeterminate $d'_t, e'_{i,i+(t-1)}$ and $\alpha_t \in \mathbb{F}$. Our aim is to show that by choosing α_t appropriately we have $d'_t = e'_{i,i+(t-1)} = 0$ for all $i = 1, \dots, n-t$.

Recall that $\sigma_{1^n}(\underline{c}) = \exp\left(\left[\binom{n-i}{n-j} c_{j-i}\right]_{ij}\right) =: \exp(C)$ and similarly for $\sigma_{1^n}(\underline{d}) =: \exp(D)$. Thus $E\sigma_{1^n}(\underline{c})E^{-1} = \exp\left(ECE^{-1}\right)$ and so $EC - DE = 0$.

Thus using the induction hypothesis for $i < n-t$

$$\begin{aligned}
(DE)_{i,i+t} &= \sum_{j=1}^{t} \tbinom{n-i}{j} e_{i+j,i+t} d_j \\
&= \sum_{j=1}^{t} \tbinom{n-i}{j} \left[\sum_{\ell=0}^{t-j} \tbinom{n-(i+j)}{\ell} B_{n-(i+j+\ell),n-(i+t)}(\underline{\alpha}) e_{n-\ell,n}\right] \left[\sum_{k=1}^{j} B_{j,k}(\underline{\alpha}) c_k\right] \\
&\qquad + \tbinom{n-i}{1} e'_{i+1,i+t}[B_{1,1}(\underline{\alpha})c_1] + \tbinom{n-i}{t}[B_{n-i,n-i}(\underline{\alpha})e_{n,n}]d'_t \\
&= \sum_{j=1}^{t} \sum_{k=1}^{j} \sum_{\ell=0}^{t-j} \tbinom{n-i}{j} \tbinom{n-(i+j)}{\ell} B_{n-(i+j+\ell),n-(i+t)}(\underline{\alpha}) B_{j,k}(\underline{\alpha}) c_k e_{n-\ell,n} \\
&\qquad + \tbinom{n-i}{1} B_{1,1}(\underline{\alpha}) c_1 e'_{i+1,i+t} + \tbinom{n-i}{t} B_{n-i,n-i}(\underline{\alpha}) d'_t e_{n,n}.
\end{aligned}$$

By observing that $\sum_{j=1}^{t}\sum_{k=1}^{j}\sum_{\ell=0}^{t-j} = \sum_{k=1}^{t}\sum_{\ell=0}^{t-k}\sum_{j=k}^{t-\ell}$ and applying Lemma 2.4.7 we acquire

$$\begin{aligned}
(DE)_{i,i+t} &= \sum_{k=1}^{t} c_k \sum_{\ell=0}^{t-k} \tbinom{n-i}{\ell} e_{n-\ell,n} \left[\sum_{j=k}^{t-\ell} \tbinom{n-(i+\ell)}{j} B_{n-(i+j+\ell),n-(i+t)}(\underline{\alpha}) B_{j,k}(\underline{\alpha})\right] \\
&\qquad + \tbinom{n-i}{1} B_{1,1}(\underline{\alpha}) c_1 e'_{i+1,i+t} + \tbinom{n-i}{t} B_{n-i,n-i}(\underline{\alpha}) d'_t e_{n,n} \\
&= \sum_{k=1}^{t} c_k \sum_{\ell=0}^{t-k} \tbinom{n-i}{\ell} e_{n-\ell,n} \left[\tbinom{k+n-i-t}{k} B_{n-i-\ell,k+n-i-t}(\underline{\alpha})\right] \\
&\qquad + \tbinom{n-i}{1} B_{1,1}(\underline{\alpha}) c_1 e'_{i+1,i+t} + \tbinom{n-i}{t} B_{n-i,n-i}(\underline{\alpha}) d'_t e_{n,n} \\
&= \sum_{k=1}^{t} \tbinom{n-(i+t)+k}{k} c_k \sum_{\ell=0}^{t-k} \tbinom{n-i}{\ell} B_{n-(i+\ell),n-(i+t)+k}(\underline{\alpha}) e_{n-\ell,n} \\
&\qquad + \tbinom{n-i}{1} B_{1,1}(\underline{\alpha}) c_1 e'_{i+1,i+t} + \tbinom{n-i}{t} B_{n-i,n-i}(\underline{\alpha}) d'_t e_{n,n}.
\end{aligned}$$

On the other hand for all i we observe that

$$\begin{aligned}(EC)_{i,i+t} &= \sum_{j=1}^{t} \tbinom{n-(i+t)+j}{j} e_{i,i+t-j} c_j \\ &= \sum_{j=1}^{t} \tbinom{n-(i+t)+j}{j} c_j \sum_{\ell=0}^{t-j} \tbinom{n-i}{\ell} B_{n-(i+\ell),n-(i+t)+j}(\underline{\alpha}) e_{n-\ell,n} \\ &\quad + \tbinom{n-(i+t)+1}{1} c_1 e'_{i,i+(t-1)}\end{aligned}$$

In taking the difference we cancel their terms to acquire

$$\begin{aligned}0 &= (EC - DE)_{i,i+t} \\ &= \left[\tbinom{n-(i+t)+1}{1} e'_{i,i+(t-1)} - \tbinom{n-i}{1} B_{1,1}(\underline{\alpha}) e'_{i+1,i+t}\right] c_1 - \tbinom{n-i}{t} B_{n-i,n-i}(\underline{\alpha}) d'_t e_{n,n}\end{aligned}$$

for all $i < n-t$. Hence

$$d'_t = \frac{\tbinom{n-(i+t)+1}{1} e'_{i,i+(t-1)} - \tbinom{n-i}{1} B_{1,1}(\underline{\alpha}) e'_{i+1,i+t}}{\tbinom{n-i}{t} B_{n-i,n-i}(\underline{\alpha})} c_1.$$

However by altering our choice of α_t we see

$$d_t = \sum_{\ell=1}^{t} B_{t,\ell}(\alpha_1,\ldots,\alpha_{t-1},\alpha_t+\beta) c_\ell + d'_t = \sum_{\ell=1}^{t} B_{t,\ell}(\alpha_1,\ldots,\alpha_{t-1},\alpha_t) c_\ell + \beta c_1 + d'_t$$

and thus we may choose α_t so that d'_t contains no multiple of c_1, and thus $d'_t = 0$. Thus we have the recurrence relation

$$e'_{i,i+(t-1)} = \frac{n-i}{n-(i+t)+1} B_{1,1}(\underline{\alpha}) e'_{i+1,i+t}. \tag{2.8}$$

To solve this recurrence relation we rely on the one element of the t'th diagonal of $EC - DE$ we have yet to consider. For $i = n-t$ we have

$$\begin{aligned}(EC - DE)_{n-t,n} &= \sum_{j=1}^{t} \sum_{\ell=0}^{t-j} \binom{t}{\ell} B_{t-\ell,j}(\underline{\alpha}) c_j e_{n-\ell,n} + e'_{n-t,n-1} c_1 \\ &\quad - \sum_{j=1}^{t} \sum_{\ell=1}^{j} \binom{t}{j} B_{j,\ell}(\underline{\alpha}) c_\ell e_{n-t+j,n} - e_{n,n} d'_t\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{t}\sum_{\ell=0}^{t-j}\binom{t}{\ell}B_{t-\ell,j}(\underline{\alpha})c_j e_{n-\ell,n} + e'_{n-t,n-1}c_1 \\
&\qquad - \sum_{\ell=1}^{t}\sum_{j=0}^{t-\ell}\binom{t}{j}B_{t-j,\ell}(\underline{\alpha})c_\ell e_{n-j,n} - e_{n,n}d'_t \\
&= e'_{n-t,n-1}c_1 = 0.
\end{aligned}
$$

Thus since there exists some $c_1 \neq 0$ in W we have $e'_{n-t,n-1} = 0$. The relation (2.8) then yields $e'_{i,i+(t-1)} = 0$ for all $i = 1, \dots, n-t$. Since the $e_{i,i+(t-1)}$ and d_t have the desired form the implication follows by induction.

The reverse implication simply requires reversing the above argument. Supposing instead we have two $W, W' \leq (\mathbb{F}^{n-1}, +)$ and a $\underline{\alpha} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$ such that $W' = [B_{i,j}(\underline{\alpha})]W$. Then by constructing the matrix $E \in GL_n(\mathbb{F})$ which satisfies (2.7) for any choice of $e_{i,n} \in \mathbb{F}$ and $e_{n,n} \in \mathbb{F}^*$ one can see that $E^{-1}\sigma_{1^n}(W')E = \sigma_{1^n}(W)$ as desired. $\square$

From this we garner an immediate corollary for the Bell polynomials.

Corollary 2.4.9. *Let $\underline{\alpha}, \underline{\beta} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$. Define $\underline{\alpha} \circ \underline{\beta} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$ by*

$$(\underline{\alpha} \circ \underline{\beta})_i := \sum_{\ell=1}^{i} B_{i,\ell}(\underline{\alpha})\beta_\ell.$$

Then

$$[B_{i,j}(\underline{\alpha})][B_{i,j}(\underline{\beta})] = [B_{i,j}(\underline{\alpha} \circ \underline{\beta})].$$

Proof. Proposition 2.4.8 shows that conjugate subgroups in the image of σ_{1^n} yield linear actions by matrices $[B_{i,j}(\underline{\alpha})]$ upon $\mathbb{F}^{n-1}$. However taking a relation $[B_{i,j}(\underline{\alpha})]W = W'$ for some $\underline{\alpha} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$ and $W, W' \leq (\mathbb{F}^{n-2}, +)$ we can construct a conjugation matrix E such that $E^{-1}\sigma_{1^n}(W')E = \sigma_{1^n}(W)$ according to the relations given in (2.7).

Thus since equivalence is transitive it follows that the product of two of these linear equivalence matrices is another linear equivalence matrix. Thus for any $\underline{\alpha}, \underline{\beta} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$ we have $[B_{i,j}(\underline{\alpha})][B_{i,j}(\underline{\beta})] = [B_{i,j}(\underline{\gamma})]$ for some $\underline{\gamma} \in \mathbb{F}^* \times \mathbb{F}^{n-2}$. Then observing that $B_{i,1}(\underline{\gamma}) = \gamma_i$ it follows that

$$\gamma_i = \sum_{\ell=1}^{i} B_{i,\ell}(\underline{\alpha})\beta_\ell = (\underline{\alpha} \circ \underline{\beta})_i.$$

□

Conclusion

Using the covering homomorphisms $\sigma_{1^n} : (\mathbb{F}^{n-1}, +) \to GL_n(\mathbb{F})$ we have successfully determined all equivalence classes of $\mathbb{Z}_p^r$-representations with socle-type 1^n. Whilst precise representatives of these classes remain unclassified we have presented the equivalence action as a linear action upon the subgroups of $(\mathbb{F}^{n-1}, +)$.

These representations are, of course, far from all we wish to consider. At the outset we introduced these as a basis for further induction to acquire yet larger families of representations. It is there we travel next.

2.5 Extensions of Socle-Type $(1, \ldots, 1)$

In Section 2.3 we developed techniques for iteratively constructing families of representations from small examples. Whilst specific examples were given alongside these formulations we have yet to see them in full effect. Having spent the previous section classifying all representations with socle-type $1^n := (1, \ldots, 1)$ we now possess a firm foundation onto which to erect this architecture.

Here we shall construct covering homomorphisms for representations with socle-type of the form $(1, \ldots, 1, m, 1, \ldots, 1)$ for arbitrary $1 < m \in \mathbb{N}$. Recall, however, from Proposition 2.3.7 that the only cases with valid abelian representations are $(m, 1, \ldots, 1)$ and $(1, m, 1, \ldots, 1)$.

2.5.1 Socle-Type $(m, 1, \ldots, 1)$ Representations

The classification of representations with socle-type $(m, 1, \ldots, 1)$ may almost be considered a corollary of our work thus far.

Theorem 2.5.1. *For $1 < m \in \mathbb{N}$ the homomorphism*

$$\sigma_{m,1,\ldots,1} : (\mathbb{F}^{n-1}, +) \to GL_n(\mathbb{F}), \qquad \sigma_{m,1,\ldots,1}(\underline{c}) = \left[\begin{array}{c|cccc} & 0 & \cdots & 0 & c_{n-1} \\ & 0 & \cdots & 0 & c_{n-2} \\ I_{m-1} & \vdots & \vdots & \vdots & \vdots \\ & 0 & \cdots & 0 & c_{n-m+1} \\ \hline 0 & \multicolumn{4}{c}{\sigma_{1^{n-m+1}}(c_1, \ldots, c_{n-m})} \end{array}\right],$$

where $\sigma_{1^k}(\underline{c}) = \left[\binom{k-i}{k-j} B_{j-i}(\underline{c})\right]$ *is given in Proposition 2.4.3, is a covering homomorphism for all modular* $\mathbb{Z}_p^r$*-representations with socle-type* $(m, 1, \ldots, 1)$.

Proof. By definition V has socle tabloid of the form

$$\begin{array}{|c|c|c|c|}\hline k & \delta_{1,2} & \cdots & \delta_{1,m} \\ \hline k-1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$$

for some $\delta_{1,i} \in [\![1, n-1]\!]$. By Lemma 2.2.12 and Corollary 2.3.8 if any $\delta_{1,i} \geq 3$ then no such representation with this tabloid exists. Hence $\delta_{1,i} \in [\![1, 2]\!]$.

By Corollary 2.3.4 a full set of covering homomorphisms for the representations with this tabloid

$$\begin{array}{|c|c|c|c|}\hline k & 2 & \cdots & 2 \\ \hline k-1 \\ \cline{1-1} \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array} \quad . \tag{2.9}$$

automatically cover the cases for which some $\delta_{1,i} = 1$. The result then follows immediately by applying the inductive methods of Theorem 2.3.3 to the covering homomorphism $\sigma_{1^{n-m+1}}$ as given in Theorem 2.4.3. □

Equivalence of Socle-Type $(m, 1, \ldots, 1)$ Representations

Proposition 2.4.8 translates the equivalence of representations induced by the homomorphism σ_{1^n} into a linear group action upon the subgroups of the preimage $(\mathbb{F}^{n-1}, +)$. Here we determine a similar linear action for the covering homomorphism given in Theorem 2.5.1.

Recall from Section 2.4.3 that two representations with images $\sigma_{1^n}(W), \sigma_{1^n}(W')$ for some $W, W' \in \mathbb{F}^{(n-1)\times r}$ are equivalent if and only if there exists an $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{F}^* \times \mathbb{F}^{n-2}$ such that

$$W' = [B_{i,j}(\underline{\alpha})]W$$

for the partial Bell polynomials $B_{i,j}(\underline{\alpha})$. We posit the analogue here.

Proposition 2.5.2. *Let $W, W' \leq (\mathbb{F}^{(n-1)\times r}, +)$ and suppose $\sigma_{m,1,\ldots,1}(W)$ and $\sigma_{m,1,\ldots,1}(W')$ are conjugate. Then there exist constants $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{F}^* \times \mathbb{F}^{n-2}$, $\beta_1, \ldots, \beta_{m-1} \in \mathbb{F}$ and $[\gamma_{ij}] \in GL_{m-1}(\mathbb{F})$ such that*

$$W' = \left[\begin{array}{cccc|ccc} & & & & 0 & \cdots & 0 \\ & [B_{i,j}(\underline{\alpha})]_{i=1,\ldots,n-m}^{j=1,\ldots,n-m} & & & \vdots & & \vdots \\ & & & & 0 & \cdots & 0 \\ & & & & \beta_1 & \cdots & \beta_{m-1} \\ \hline \alpha_{n-m+1} & 0 & \cdots & 0 & & & \\ \vdots & \vdots & & \vdots & & [\gamma_{ij}]_{i=1,\ldots,m-1}^{j=1,\ldots,m-1} & \\ \alpha_{n-1} & 0 & \cdots & 0 & & & \end{array}\right] W.$$

Proof. By assumption $E\sigma_{m,1,\ldots,1}(W)E^{-1} = \sigma_{m,1,\ldots,1}(W')$ for some $E = [e_{i,j}] \in GL_n(\mathbb{F})$. Recall from Section 2.1 that we may reformulate the problem, noting that

$$\sigma_{m,1,\ldots,1}(\underline{c}) = \left[\begin{array}{c:cc} & & c_{n-1} \\ I_{m-1} & 0 & \vdots \\ & & c_{n-m+1} \\ \hdashline 0 & & \sigma_{1^{n-m+1}}(\underline{c}) \end{array}\right] = \exp \left[\begin{array}{c:cc} & & c_{n-1} \\ 0 & 0 & \vdots \\ & & c_{n-m+1} \\ \hdashline 0 & & \tau_{1^{n-m+1}}(\underline{c}) \end{array}\right] =: \exp(\tau_{m,1,\ldots,1}(\underline{c}))$$

where $\tau_{1^k}(\underline{c}) = \left[\binom{k-i}{k-j} c_{j-i}\right]$ taking $c_0 = c_{-1} = \cdots = 0$. Then since $E\exp(M)E^{-1} = \exp(EME^{-1})$ our problem is equivalent to ensuring the additive groups $\tau_{m,1,\ldots,1}(W)$ and $\tau_{m,1,\ldots,1}(W)$ exist in the same conjugacy class.

Given the socle-type $M = (m_1, \ldots, m_k) = (m, 1, \ldots, 1)$ the matrix E must be block upper-triangular with arbitrary diagonal blocks in $M_{m_i}(\mathbb{F})$. Furthermore since the socle-tabloid of the representation must be as in (2.9) there must exist in any given doubly-conforming basis a single element $v \in \mathrm{Soc}(V)$ whose dual under this basis sits in $V^* \setminus \mathrm{Soc}_{k-1}(V)$. The remaining basis elements in $\mathrm{Soc}(V)$ have duals in $\mathrm{Soc}_2(V^*)$. As such we cannot introduce a multiple of v into any of these under any basis change lest the resulting basis fail to be doubly conforming. The effect of this upon E is that $e_{1,m} = \cdots = e_{m-1,m} = 0$.

By defining

$$E_1 := \begin{bmatrix} e_{1,1} & \cdots & e_{1,m-1} \\ \vdots & \ddots & \vdots \\ e_{m-1,1} & \cdots & e_{m-1,m-1} \end{bmatrix} \in GL_{m-1}(\mathbb{F}), \quad E_2 := \begin{bmatrix} e_{1,m+1} & \cdots & e_{1,n} \\ \vdots & \ddots & \vdots \\ e_{m-1,m+1} & \cdots & e_{m-1,n} \end{bmatrix}$$

$$E_3 := [e_{m,1}, \ldots, e_{m,m-1}] \qquad E_4 := \begin{bmatrix} e_{m,m} & e_{m,m+1} & \cdots & e_{m,n} \\ 0 & e_{m+1,m+1} & \cdots & e_{m+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{n,n} \end{bmatrix} \in GL_{n-m+1}(\mathbb{F})$$

we may write our equivalence matrix

$$E = \left[\begin{array}{c:c} E_1 & 0 \quad E_2 \\ \hdashline E_3 & \\ 0 & E_4 \end{array}\right]. \tag{2.10}$$

If the images of $\underline{c} \in W$ and $\underline{d} \in W'$ are paired under this equivalence then

$$\Delta_{\underline{c},\underline{d}} := E\tau_{m,1,\ldots,1}(\underline{c}) - \tau_{m,1,\ldots,1}(\underline{d})E$$
$$= \left[\begin{array}{c:c} 0 & E_1 \begin{bmatrix} & c_{n-1} \\ 0 & \vdots \\ & c_{n-m+1} \end{bmatrix} + [0 \ \ E_2]\tau_{1^{n-m+1}}(\underline{c}) - \begin{bmatrix} & d_{n-1} \\ 0 & \vdots \\ & d_{n-m+1} \end{bmatrix} E_4 \\ \hdashline 0 & \begin{bmatrix} E_3 \\ 0 \end{bmatrix} \begin{bmatrix} & c_{n-1} \\ 0 & \vdots \\ & c_{n-m+1} \end{bmatrix} - (E_4\tau_{1^{n-m+1}}(\underline{c}) - \tau_{1^{n-m+1}}(\underline{c})E_4) \end{array}\right] = 0.$$

In the top-right block of $\Delta_{\underline{c},\underline{d}}$ the only term contributing to the first $n-m$ columns is $[0 \ \ E_2]\tau_{1^{n-m+1}}(\underline{c})$. Since these must vanish, and $\tau_{1^{n-m+1}}(\underline{c})$ is upper-triangular with zero diagonal it follows that $e_{i,j} = 0$ for all $(i,j) \in [\![1, m-1]\!] \times [\![m+1, n-2]\!]$. Thus the final column of the upper-right component of $\Delta_{\underline{c},\underline{d}}$ equals

$$E_1 \begin{bmatrix} c_{n-1} \\ \vdots \\ c_{n-m+1} \end{bmatrix} + c_1 \begin{bmatrix} e_{1,n-1} \\ \vdots \\ e_{m-1,n-1} \end{bmatrix} - e_{n,n} \begin{bmatrix} d_{n-1} \\ \vdots \\ d_{n-m+1} \end{bmatrix}.$$

Thus defining $(\gamma_{i,j}) \in GL_{m-1}(\mathbb{F})$ and $\alpha_{n-m+1}, \ldots, \alpha_{n-1} \in \mathbb{F}$ appropriately the vanishing of this block yields

$$\begin{bmatrix} d_{n-m+1} \\ \vdots \\ d_{n-1} \end{bmatrix} = (\gamma_{i,j}) \begin{bmatrix} c_{n-m+1} \\ \vdots \\ c_{n-1} \end{bmatrix} + c_1 \begin{bmatrix} \alpha_{n-m+1} \\ \vdots \\ \alpha_{n-1} \end{bmatrix}.$$

This leaves the lower-right block of $\Delta_{\underline{c},\underline{d}}$ which greatly resembles the conjugacy of groups in the image of $\tau_{1^{n-m+1}}$. Since

$$\begin{bmatrix} E_3 \\ 0 \end{bmatrix} \begin{bmatrix} & c_{n-1} \\ 0 & \vdots \\ & c_{n-m+1} \end{bmatrix} = \begin{bmatrix} 0 & \sum_{i=1}^{m-1} e_{m,i}c_{n-i} \\ 0 & 0 \end{bmatrix}$$

only the very upper-right entry of this block differs from the calculations given in the proof of Proposition 2.4.8. We may apply the same arguments to conclude that $d_i = \sum_{j=1}^{i} B_{i,j}(\underline{\alpha})c_j$ for some $\underline{\alpha} = (\alpha_1, \dots, \alpha_{n-m-1}) \in \mathbb{F}^* \times \mathbb{F}^{n-m-2}$ for all $i = 1, \dots, n-m-1$.

Furthermore the entries of E_4 satisfy the form of the equivalence matrices given in the proof of Proposition 2.4.8. The only unconsidered entry of $\Delta_{\underline{c},\underline{d}}$ is

$$(\Delta_{\underline{c},\underline{d}})_{m,n} = \sum_{j=1}^{m-1} e_{m,j}c_{n-j} + (E_4\tau_{1^{n-m+1}}(\underline{c}) - \tau_{1^{n-m+1}}(\underline{c})E_4)_{1,n-m+1} = 0.$$

Defining $d_{n-m} := \sum_{j=1}^{n-m} B_{n-m,j}(\underline{\alpha}, \alpha_{n-m})c_j + d'_{n-m}$ for some $\alpha_{n-m}, d'_{n-m} \in \mathbb{F}$ we may once again use the same arguments as in the proof of Proposition 2.4.8: By choosing α_{n-m} appropriately we ensure that

$$(\Delta_{\underline{c},\underline{d}})_{m,n} = \sum_{j=1}^{m-1} e_{m,j}c_{n-j} - e_{n,n}d'_{n-m} = 0$$

and thus by defining $\beta_1, \dots, \beta_{m-1}$ appropriately we acquire

$$d_{n-m} = \sum_{j=1}^{n-m} B_{n-m,j}(\underline{\alpha}, \alpha_{n-m})c_j + \sum_{j=1}^{m-1} \beta_j c_{n-m+j}$$

thus concluding the proof. □

2.5.2 Socle-Type $(1, m, 1, \dots, 1)$ Representations

Thus far our techniques for augmenting existing representations into higher dimensions center around expanding either the first or second part of the socle series. When considering representations with socle-type $(1, m, 1, \dots, 1)$ however we must take a different approach.

Naturally a representation $V \in \mathbb{F}\mathbb{Z}_p^r$-mod with socle-type $(1, m, 1, \dots, 1)$ induces the representation $V/\mathrm{Soc}(V)$ with socle-type $(m, 1, \dots, 1)$ and as such is covered by the covering homomorphism $\sigma_{m,1,\dots,1}$ given in Theorem 2.5.1. Thus the socle tabloid of V must be of the form

$$\begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} k \\ \hline k-1 & 2 & \cdots & 2 & 1 & \cdots & 1 \\ \hline \vdots \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}.$$

In this instance we must take care: There exist representations for which our chosen prime p does not exceed the socle series length, a situation we have yet to find ourselves in. Fortunately we shall that this is not a concern when $p > 2$.

Case 1: Characteristic $p > 2$

Whilst the characteristic of the representation avoids $p = 2$ we may employ the services of covering homomorphisms as designed to acquire the following.

Theorem 2.5.3. *Let $\mathbb{F} = \overline{\mathbb{F}}$ have characteristic $p > 2$ and fix $1 < m \in \mathbb{N}$. Then the homomorphism*

$$\sigma^{(\mu)}_{1m1\dots1} : (\mathbb{F}^{n-1}, +) \to GL(V)$$

where $\sigma^{(\mu)}_{1m1\dots1}(\underline{c})$ takes the form

$$\left[\begin{array}{ccc:cccc} 1 & 2\underline{v}_c & 2\underline{v}'_c & \binom{n-m}{n-m-1}B_1(\underline{c}) & \cdots & \binom{n-m}{1}B_{n-m-1}(\underline{c}) & B_{n-m}(\underline{c}) + \underline{v}_c\underline{v}_c^T \\ 0 & I_{m-1-\mu} & 0 & 0 & \cdots & 0 & \underline{v}_c^T \\ 0 & 0 & I_\mu & 0 & \cdots & 0 & 0 \\ \hdashline \multicolumn{3}{c:}{0} & \multicolumn{4}{c}{\sigma_{1^{n-m}}(\underline{c})} \end{array}\right]$$

with $\underline{v}_c := [c_{n-1}, \dots, c_{n-m+\mu+1}]$, $\underline{v}'_c = [c_{n-m+\mu}, \dots, c_{n-m+1}]$, is a covering homomorphism for all modular $\mathbb{Z}_p^r$-representations with socle-type $(1, m, 1, \dots, 1)$ with $\dim(\mathrm{Soc}\, V^*) = \mu + 1$.

We prove Theorem 2.5.3 via sequential lemmas which, in turn, enforce the prime restriction, construct the homomorphism and finally determine homomorphism uniqueness up to equivalence.

Consider $\mathbb{Z}_p^r \cong G \leq GL(V) = GL_n(\mathbb{F})$ with socle-type $(1, m, 1, \dots, 1)$ and $\dim(\mathrm{Soc}\, V^*) = \mu + 1$. Since $V/\mathrm{Soc}(V)$ has socle-type $(m, 1, \dots, 1)$ we may use

Theorem 2.5.1 to choose a doubly-conforming basis and place the image in the image of $\sigma_{m,1,\ldots,1}$. Augmenting this to a doubly-conforming basis for V an arbitrary element $C \in G$ then dons form

$$C = \begin{bmatrix} 1 & \underline{w}_c & \underline{w_c}' & c_{n-m} \\ 0 & I_{m-1} & 0 & (\underline{w}_c'')^T \\ 0 & 0 & \sigma_{1^{n-m}}(c_1, \ldots, c_{n-m-1}) & \end{bmatrix}$$

where $\underline{w}_c = [c_n, \ldots, c_{n+m-2}]$, $\underline{w}_c' = [c_{n+m-1}, \ldots, c_{2n-3}]$ and $\underline{w}_c'' = [c_{n-1}, \ldots, c_{n-m+1}]$.

Lemma 2.5.4. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ be as above. Then $p > n-m$ and there exists an element $J \in G$ for which $(J - I_n)^{n-m} \neq 0$.*

Proof. Let $m \neq n-2$ and the representation acts on a doubly-conforming basis. It is a simple exercise to show that $(C - I_n)^{n-m}[1, n] = (n-m-1)! c_1^{n-m-1} c_{n+m-1}$ and all other entries immediately vanish. Recall that $p \geq n-m$ due to the form of $V/\mathrm{Soc}(V)$.

Assume $(C - I_n)^{n-m}[1, n] = 0$ for all elements $C \in G$. By the socle-tabloid structure there must exist $C, D \in G$ for whom $c_1, d_{n+m-1} \neq 0$. Thence by our assumption $c_{n+m-1} = d_1 = 0$ and so

$$\begin{aligned} (CD - I_n)^{n-m}[1, n] &= (n-m-1)!(c_1 + d_1)^{n-m-1}(c_{n+m-1} + d_{n+m-1}) \\ &= (n-m-1)! c_1^{n-m-1} d_{n+m-1} \neq 0. \end{aligned}$$

This is a contradiction and so such an element J with $(J - I_n)^{n-m}[1, n] \neq 0$ exists. Since our group is elementary abelian the prime restriction then follows.

Suppose instead that $m = n-2 > 1$ and thus our elements are of the form

$$C = \begin{bmatrix} 1 & v_{\underline{c}} & c_{n-m} \\ 0 & I_{n-2} & w_{\underline{c}}^T \\ 0 & 0 & 1 \end{bmatrix}.$$

By assumption we have $p > 2$ and we see that $(C - I_n)^2[1, n] = v_{\underline{c}} w_{\underline{c}}^T$. Suppose this vanishes for all elements. Given two $C, D \in G$ which are not $\mathbb{F}_p$-multiples of one another we have

$$[C, D][1, n] = v_{\underline{c}} w_{\underline{d}}^T - v_{\underline{d}} w_{\underline{c}}^T = 0.$$

$$(CD - I_n)^2[1, n] = (v_{\underline{c}} + v_{\underline{d}})(w_{\underline{c}} + w_{\underline{d}})^T = v_{\underline{c}} w_{\underline{d}}^T + v_{\underline{d}} w_{\underline{c}}^T = 0.$$

The fact that $p > 2$ then gives a contradiction and thus the result. $\square$

Given $J \in G$ with $(J - I_n)^{n-m} \neq 0$ we alter our chosen doubly-conforming basis $\{v_1, \dots, v_n\}$ to better fix J as follows.

We begin by ensuring that $v_2, \dots, v_m$ are the basis elements of $\mathrm{Soc}_2(V)$ whose duals, under this basis, sit in $\mathrm{Soc}_2(V^*)$. Let $v_i^* \cdot (J - I_n) = j_{i,n} v_n^*$ for each $i = 2, \dots, m$ and $i = n - 1$. Then since $j_{n-1,n} \neq 0$ by assumption we replace v_i^* for $i = 2, \dots, m$ with $(v_i^*)' := v_i^* - \frac{j_{i,n}}{j_{n-1,n}} v_{n-1}^*$ in the dual basis to ensure that $(v_i^*)'$ are acted upon trivially. Dualising back we acquire a new doubly-conforming basis for V.

For the remaining elements we replace $v_1, v_{m+1}, v_{m+2}, \dots, v_n$ with

$$v_n \cdot (J - I_n)^{n-m}, \quad v_n \cdot (J - I_n)^{n-m-1}, \quad \dots, \quad v_n \cdot (J - I_n), \quad v_n$$

so that now J acts in Jordan normal form. Transforming these elements as per the methods of Section 1.2.2 and (1.2) we acquire the form

$$J = \left[\begin{array}{cc|cccc} 1 & 0 & \binom{n-m}{n-m-1} & \cdots & \binom{n-m}{1} & 1 \\ 0 & I_{m-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & & & & \\ \vdots & \vdots & & \sigma_{1^{n-m}}(1,0,\dots,0) & & \\ 0 & 0 & & & & \end{array}\right] = \sigma_{1m1\dots1}(1, 0, \dots, 0).$$

Naturally we wish to ascertain how other elements $C \in G$ commute with this, given that

$$\begin{aligned} [C, J][1, m + \ell] &= \sum_{i=1}^{m+\ell} C_{1,i} J_{i,m+\ell} - \sum_{i=1}^{m+\ell} J_{1,i} C_{i,m+\ell} \\ &= \binom{n-m}{\ell} + \sum_{i=1}^{\ell} \binom{n-m-i}{n-m-\ell} (\underline{w}'_c)_i - C_{1,m+\ell} - \sum_{i=1}^{\ell} \binom{n-m}{i} \sigma_{1^{n-m}}(\underline{c})_{i,\ell} \\ &= \sum_{i=1}^{\ell-1} \binom{n-m-i}{n-m-\ell} \left[c_{n+m-2+i} - \binom{n-m}{i} B_{\ell-i}(\underline{c})\right] = 0 \end{aligned} \tag{2.11}$$

for $\ell = 1, \dots, n - m - 1$. This leads to the following.

Lemma 2.5.5. *Suppose $\mathbb{Z}_p^r \cong G \leq GL(V)$ contains J as above. Then for all other elements $C = [c_{i,j}] \in G$ there exists a $\underline{c} \in \mathbb{F}^{n-1}$ such that $c_{n+m-2-i} = \binom{n-m}{n-m-i} B_i(\underline{c})$ for all $i \in [\![1, n - m - 1]\!]$.*

Proof. We prove by induction on i. For $i = 1$ we take $\ell = 2$ in (2.11) to acquire

$$\binom{n-m-1}{n-m-2}^{-1}\left[c_{n+m-1} - \binom{n-m}{n-m-1}B_1(\underline{c})\right] = 0$$

which yields the base. Now assume the result holds for $i = 1, \dots, t-1 < n-m-1$. Then by (2.11) for $\ell = t+1$ we have

$$\begin{aligned}
c_{n+m-2+t} &= \binom{n-m-t}{1}^{-1}\left[\sum_{i=1}^{t}\binom{n-m-i}{n-m-t-1}\binom{n-m}{n-m-i}B_{t-i+1}(\underline{c}) - \sum_{i=1}^{t-1}\binom{n-m-i}{n-m-t-1}c_{n+m-2+i}\right]\\
&= \binom{n-m-t}{1}^{-1}\left[\sum_{j=1}^{t}\binom{n-m-(t+1-j)}{n-m-t-1}\binom{n-m}{n-m-(t+1-j)}B_j(\underline{c}) - \sum_{i=1}^{t-1}\binom{n-m-i}{n-m-t-1}\binom{n-m}{n-m-i}B_i(\underline{c})\right]\\
&= \binom{n-m-t}{1}^{-1}\binom{n-m}{t+1}\left[\sum_{j=1}^{t}\binom{t+1}{j}B_j(\underline{c}) - \sum_{i=1}^{t-1}\binom{t+1}{i}B_i(\underline{c})\right]\\
&= \binom{n-m-t}{1}^{-1}\binom{n-m}{t+1}\binom{t+1}{t}B_t(\underline{c}) = \binom{n-m}{n-m-t}B_t(\underline{c}).
\end{aligned}$$

The result follows by induction. $\square$

Proof of Theorem 2.5.3. By Lemma 2.5.4 we may assume $J := \sigma^{(\mu)}_{1m1\dots1}(1, 0, \dots, 0)$ lies in the image of our representation. All other elements of the representation must then satisfy Lemma 2.5.5. Using this and the Bell polynomial property as in Lemma 2.4.4 we see that for two such elements

$$(CD)[1, m+j] = \binom{n-m}{n-m-j}B_j(\underline{c} + \underline{d})$$

for $j = 1, \dots, n-m-1$ as we would desire. The only entry not clearly fulfilling the form of our proposed homomorphisms is the $[1, n]$ entry. By observing that

$$(CD)[1, n] = d_{n-m} + c_{n-m} + \underline{w}_c(\underline{w}''_d)^T + B_{n-m}(\underline{c} + \underline{d}) - B_{n-m}(\underline{c}) - B_{n-m}(\underline{d})$$

we demand $\underline{w}_c(\underline{w}''_d)^T = \underline{w}_d(\underline{w}''_c)^T$ in order for these two elements C, D to commute.

By the restriction $\dim(\mathrm{Soc}\, V) = 1$ there exist distinct elements $C^{(1)}, \dots, C^{(m-1)}$ such that the matrix $[\underline{w}^T_{c^{(1)}} \cdots \underline{w}^T_{c^{(m-1)}}]$ is invertible. Then by observing the commutators $[C^{(i)}, D]$ for all of these elements we acquire

$$(\underline{w}''_d)^T = \begin{bmatrix} \underline{w}_{c^{(1)}} \\ \vdots \\ \underline{w}_{c^{(m-1)}} \end{bmatrix}^{-1} \begin{bmatrix} \underline{w}''_{c^{(1)}} \\ \vdots \\ \underline{w}''_{c^{(m-1)}} \end{bmatrix} \underline{w}_d^T = \frac{1}{2} A \underline{w}_d^T$$

for the appropriate matrix $A \in \mathbb{F}^{(m-1)\times(m-1)}$. Then for general elements C, D we have

$$\begin{aligned} 2(\underline{w}_c(\underline{w_d}'')^T - \underline{w}_d(\underline{w}''_c)^T) &= \underline{w}_c A \underline{w}_d^T - \underline{w}_d A \underline{w}_c^T \\ &= \underline{w}_d A^T \underline{w}_c^T - \underline{w}_d A \underline{w}_c^T \\ &= \underline{w}_d (A^T - A) \underline{w}_c^T = 0. \end{aligned}$$

Since we can find $m-1$ linearly independent $\underline{w}_c$ throughout the representation it follows that A is symmetric.

Without loss of generality relabel $c_{n-m} \mapsto B_{n-m}(\underline{c}) + \underline{v}_c A \underline{v}_c^T$ so that

$$(CD)[1,n] = B_{n-m}(\underline{c} + \underline{d}) + (\underline{w}_c + \underline{w}_d) A (\underline{w}_c + \underline{w}_d)^T$$

recalling that $B_{n-m}(\underline{c} + \underline{d}) = \sum_{i=0}^{n-m} \binom{n-m}{i} B_i(\underline{c}) B_{n-m-i}(\underline{d})$.

Note that the image of $\sigma^{(\mu)}_{1m1\ldots1}(\underline{c})$ differs from our interim construction

$$C = \left[\begin{array}{cc:ccccc} 1 & 2\underline{w}_c & \binom{n-m}{n-m-1} B_1(\underline{c}) & \cdots & \binom{n-m}{1} B_{n-m-1}(\underline{c}) & B_{n-m}(\underline{c}) + \underline{w}_c A \underline{w}_c^T \\ 0 & I_{m-1} & 0 & \cdots & 0 & A\underline{w}_c^T \\ \hdashline \multicolumn{2}{c:}{0} & \multicolumn{4}{c}{\sigma_{1^{n-m}}(\underline{c})} \end{array}\right]$$

only by virtue of specifying that $A = \begin{bmatrix} I_{m-1-\mu} & 0 \\ 0 & 0 \end{bmatrix}$.

Suppose $\{v_1, \ldots, v_n\}$ is the doubly-conforming basis for V under which our elements act in the form above for some symmetric A. Since $\dim(\mathrm{Soc}\, V^*) = \mu + 1$ we see that A has rank $m-1-\mu$. By exchanging the basis elements $v_2, \ldots, v_m$ with

$$\begin{bmatrix} \widetilde{v_2} \\ \vdots \\ \widetilde{v_m} \end{bmatrix} = E \begin{bmatrix} v_2 \\ \vdots \\ v_m \end{bmatrix}$$

for some $E \in GL_{m-1}(\mathbb{F})$ we acquire a similar action, altering our symmetric matrix by $A \mapsto EAE^T$. Since any two symmetric matrices A, B over an algebraically closed field with characteristic $p \neq 2$ are congruent – that is there exists an invertible E such that $B = EAE^T$ – if and only if they have the same rank, the result then follows. $\square$

Whilst in the socle-type $(m, 1, \ldots, 1)$ cases had no small-prime instances, we cannot presume their non-existence for all cases. We shall now see such caution pay off.

Characteristic $p = 2$

Let $G \leq GL(V)$ be a modular $\mathbb{Z}_2^r$-representation with socle-type $(1, m, 1, \ldots, 1)$. Since V/V^G has socle-type $(m, 1, \ldots, 1)$ this demands that $p = 2 \geq n - m$ and thus $m = n - 2$. The socle tabloid of V then takes the form

$$\begin{array}{|c|cccccc}
\hline
3 & & & & & & \\
\hline
2 & \multicolumn{1}{|c|}{\cdots} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{1} & . \\
\hline
1 & & & & & & \\
\hline
\end{array}$$

Recall that if there are any 1-boxes in the second row of the tableau, then one may construct homomorphisms covering these from lower-dimensional representations by utilising Theorem 2.3.3.

Since $\dim(V^G) = 1$ we may choose a doubly-conforming basis for V such that G contains elements $C^{(k)}$ for all $k \in [\![1, n-2]\!]$ satisfying

$$C^{(k)}_{1,k+1} = 1, \quad C^{(k)}_{1,j} = 0 \quad \forall j \in [\![2, n]\!] \setminus \{k+1\}.$$

Since G is commutative we have

$$[C^{(i)}, C^{(j)}][1, n] = C^{(j)}_{i+1,n} - C^{(i)}_{j+1,n} = 0$$

and so $C^{(j)}_{i+1,n} = C^{(i)}_{j+1,n}$ for all $1 \leq i, j \leq n-2$. Furthermore since G is an elementary abelian 2-group $(C^{(i)})^2[1, n] = C^{(i)}_{i+1,n} = 0$ for all $1 \leq i \leq n-2$.

Let A be the $n-2 \times n-2$ matrix with entries $A[i, j] := C^{(i)}_{j+1,n}$. By the above arguments A is alternate — that is (skew-)symmetric with identically zero diagonal. Then for any other given element M in the representation

$$[C^{(i)}, M][1, n] = M_{i+1,n} - \sum_{j=2}^{n-1} M_{1,j} C^{(i)}_{j,n} = 0$$

for all $1 \le i \le n-2$, and so all elements adhere to the form

$$M_A(\underline{c}, c_{1,n}) := \begin{bmatrix} 1 & \underline{c} & c_{1,n} \\ 0 & I_{n-2} & A\underline{c}^T \\ 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

where $\underline{c} = [c_{1,2}, \ldots, c_{1,n-1}]$. To aid in our manipulations and understanding of A we provide the following result, a combination of [23] Lemma 3.3 and [1] Theorem 4.

Lemma 2.5.6 ([23] Lemma 3.3, [1] Theorem 4)**.** *Suppose A is an $n \times n$ alternate matrix with entries over a field $\mathbb{F}$ of arbitrary characteristic. Then* $\operatorname{rank}(A) = 2t$ *for some $t \in \mathbb{N}_0$ and A is congruent in $\mathbb{F}$ to*

$$\begin{bmatrix} 0 & -I_t & 0 \\ I_t & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This finally moves us along to the following.

Theorem 2.5.7. *Let $\mathbb{F}$ have characteristic 2. For $t \in \mathbb{N}$ define the homomorphism*

$$\sigma_t(\underline{c}) = \begin{bmatrix} 1 & v_1(\underline{c}) & v_2(\underline{c}) & v_3(\underline{c}) & c_{n-1} + v_1(\underline{c})v_2(\underline{c})^T \\ 0 & I_t & 0 & 0 & v_2(\underline{c})^T \\ 0 & 0 & I_t & 0 & v_1(\underline{c})^T \\ 0 & 0 & 0 & I_{n-2t-2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $v_1(\underline{c}) = [c_1, \ldots, c_t], v_2(\underline{c}) = [c_{t+1}, \ldots, c_{2t}]$ and $v_3(\underline{c}) = [c_{2t+1}, \ldots, c_{n-2}]$. Then σ_t is a covering homomorphism for all modular $\mathbb{Z}_2^r$ representations with socle tabloid

$$\begin{array}{l} \boxed{3} \\ \boxed{2}\boxed{\cdots}\boxed{2}\boxed{1}\boxed{\cdots}\boxed{1} \\ \boxed{1} \end{array}$$

where the number of $\boxed{2}$ equals $2t$. Furthermore no such representation exist with an odd number of $\boxed{2}$ in the tabloid.

Proof. Using the above arguments we may choose a basis for such a representation such that all elements act in the form $M_A(\underline{c}, c_{1,n})$ as given in (2.12) for some alternate $A \in M_{n-2}(\mathbb{F})$. Taking an $E \in GL_{n-2}(\mathbb{F})$ and constructing $E' = \mathrm{diag}(1, E, 1)$ we observe that

$$E' M_A(\underline{c}, c_{1,n})(E')^{-1} = M_{EAE^T}(\underline{c}E^{-1}, c_{1,n}).$$

Hence we may choose our basis such that A is given as in the statement Lemma 2.5.6. Each matrix of the form $M_A(\underline{c}, c_{1,n})$ then exists in the image of σ_t. The only difference between their forms is the $[1, n]$ entry, chosen in σ_t to ensure that

$$\begin{aligned}(\sigma_t(\underline{c})\sigma_t(\underline{d}))_{1,n} &= c_{n-1} + d_{n-1} + v_1(\underline{c})v_2(\underline{c})^T + v_1(\underline{c})v_2(\underline{d})^T + v_2(\underline{c})v_1(\underline{d})^T + v_1(\underline{d})v_2(\underline{d})^T \\ &= (c_{n-1} + d_{n-1}) + (v_1(\underline{c}) + v_1(\underline{d}))(v_2(\underline{c}) + v_2(\underline{d}))^T \\ &= (\underline{c} + \underline{d})_{n-1} + v_1(\underline{c} + \underline{d})v_2(\underline{c} + \underline{d})^T = \sigma_t(\underline{c} + \underline{d})_{1,n}\end{aligned}$$

and thus that it is a homomorphism. Thus σ_t acts as a covering homomorphism which yields the proof. □

Recall from the proof of Theorem 2.5.3 that the $p > 2$ case yielded a choice in covering homomorphisms up to congruence of a symmetric matrix. One may easily choose this symmetric matrix to be precisely of the form given in 2.5.6 when its rank is even. This would then yield consistence between the $p = 2$ and $p > 2$ cases. However we choose the forms as given for ease of later application.

Conclusion

This and the prior sections' work particularly focus on the cases with long socle series with low-dimensional factors. We would be remiss not to at least consider the opposite end of this extreme.

2.6 Socle-Length 3 Representations

Having dealt with representations with long socle series we now turn our attention to more minimalist constructions. As our aim is to classify low-dimension representations first our attentions ought to be drawn here eventually.

During our classification of representations with socle-type $(1, m, 1, \ldots, 1)$ in Section 2.5.2 we provided a covering homomorphism for these up to a choice of symmetric or alternate matrix, all choices yielding equivalent images under algebraically closed fields. These represent the hummock of quite a considerable iceberg, the interconnected nature of classifying such representations and differentiating vector spaces of symmetric matrices. By the climax of the section it shall be clear to the reader that representatives of such orbits immediately yield for us the covering homomorphisms we so covet.

2.6.1 Symmetric Matrices, Vector Spaces and Orbits

For this section we take $\mathbb{F} = \overline{\mathbb{F}}$ an algebraically closed field of positive characteristic $p > 0$.

Notation 2.6.1. Denote by $\mathrm{Sym}_k(\mathbb{F}) \subset GL_k(\mathbb{F})$ the $\mathbb{F}$-vector space of $k \times k$ symmetric matrices and by $\mathrm{Sym}_k^0(\mathbb{F}) \leq \mathrm{Sym}_k(\mathbb{F})$ the subspace of symmetric matrices with zeroes on the diagonal. We observe that $\mathrm{Sym}_k^0(\mathbb{F}) \leq \mathrm{Sym}_k(\mathbb{F})$ have dimension $\frac{k(k-1)}{2} \leq \frac{k(k+1)}{2}$ as $\mathbb{F}$-vector spaces.

Let

$$\mathcal{S}_k := \begin{cases} \mathrm{Sym}_k(\mathbb{F}) & p > 2 \\ \mathrm{Sym}_k^0(\mathbb{F}) & p = 2. \end{cases}$$

Then we consider the (right) action of $GL_k(\mathbb{F})$ on $\mathcal{S}_k$ given by the congruence action $M \cdot A = A^T M A \in \mathcal{S}_k$ for $M \in \mathcal{S}_k$ and $A \in GL_k(\mathbb{F})$.

Consider a subspace $S \leq \mathcal{S}_k$ of dimension $d \leq \frac{k(k\pm1)}{2}$. The action of $GL_k(\mathbb{F})$ on the elements of S naturally yields an action $S \cdot A := \{M \cdot A \mid M \in S\}$ on the set of all subspaces of $\mathcal{S}_k$ of dimension d.

We say two subspaces $S_1, S_2 \leq \mathcal{S}_k$ are *congruent* if there exists $A \in GL_k(\mathbb{F})$ such that $S_1 \cdot A = S_2$ and *incongruent* otherwise.

We guide the use of these subspaces in our representation theoretic direction using the following.

Definition 2.6.2. For a given subspace $S \leq \mathcal{S}_k$ with basis $\underline{A} = \{A_1, \ldots, A_d\}$ we define the homomorphism $\sigma_{\underline{A}} : (\mathbb{F}^{n-1}, +) \to GL_n(\mathbb{F})$ by

$$\sigma_{\underline{A}}(\underline{c}) := \begin{bmatrix} & w_{\underline{c}} A_1 & c_{n-1} + w_{\underline{c}} A_1 w_{\underline{c}}^T \\ I_d & \vdots & \vdots \\ & w_{\underline{c}} A_d & c_{k+1} + w_{\underline{c}} A_d w_{\underline{c}}^T \\ 0 & I_k & w_{\underline{c}}^T \\ 0 & 0 & 1 \end{bmatrix}$$

for $w_{\underline{c}} = [c_k, \ldots, c_1]$.

With a bit of effort the reader may discern that the images of such homomorphisms contain the images of representations with dual-type $(1, k, d)$. In fact we may be stronger.

Theorem 2.6.3. *Let $S_1, \ldots, S_m \leq \mathcal{S}_s$ be a sequence of incongruent d-dimensional subspaces for which m is maximal. Then for any choice of bases $\underline{A}_i \subset S_i$ for $i = 1, \ldots, s$ the homomorphisms $\sigma_{\underline{A}_1}, \ldots, \sigma_{\underline{A}_m}$ collectively form a complete set of covering homomorphisms for all modular $\mathbb{Z}_p^r$-representations with dual-type $(1, k, d)$ up to equivalence .*

Proof. We proceed using the notation of the result adding that our representation is of dimension $n := k + d + 1$.

Consider $\mathbb{Z}_p^r \cong G \leq GL(V)$ with dual-type $(1, k, d)$. By choosing a doubly conforming basis we may write the elements of G in the form

$$M := \begin{bmatrix} & v_{1,M} & \\ I_d & \vdots & M' \\ & v_{d,M} & \\ 0 & I_k & w_M^T \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the dual-type is $(1, k, d)$ there exist no $\mathbb{F}$-relations between the $v_{i,M}$ which hold for every element and there exist $M_1, \ldots, M_k$ such that the matrix $[w_{M_1}^T, \ldots, w_{M_k}^T]$ is invertible.

Remark. One might be convinced that such matrices would induce a socle-type $(d, k, 1)$ representation. However we do not make any further assumptions on the entries of the $v_{i,M}$ and as such the dimension of V^G may exceed d. All we stipulate is that the dual socle-type is $(1, k, d)$.

Given another element N we observe for each $i = 1, \ldots, k$ that

$$[M_i, N] = \begin{bmatrix} 0 & 0 & \begin{bmatrix} v_{1,M_i} \\ \vdots \\ v_{d,M_i} \end{bmatrix} w_N^T - \begin{bmatrix} v_{1,N} \\ \vdots \\ v_{d,N} \end{bmatrix} w_{M_i}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Thus since $v_{j,N}w_{M_i}^T = v_{j,M_i}w_N^T = w_N v_{j,M_i}^T$ for all $i = 1, \dots, k$, $j = 1, \dots, d$ it then follows that

$$v_{j,N}[w_{M_1}^T, \dots, w_{M_k}^T] = w_N[v_{j,M_1}^T, \dots, v_{j,M_k}^T]$$

and thus

$$v_{j,N} = w_N[v_{j,M_1}^T, \dots, v_{j,M_k}^T][w_{M_1}^T, \dots, w_{M_k}^T]^{-1}. \tag{2.13}$$

Hence every $v_{j,N}$ is a (right) matrix multiple of w_N. Moreover by altering the basis of V upon which these act such that $[w_{M_1}^T, \dots, w_{M_k}^T] = I_k$ we see that

$$\begin{aligned}
[M_i, M_j] &= \begin{bmatrix} 0 & 0 & \begin{bmatrix} v_{1,M_i} \\ \vdots \\ v_{d,M_i} \end{bmatrix} w_{M_j}^T - \begin{bmatrix} v_{1,M_j} \\ \vdots \\ v_{d,M_j} \end{bmatrix} w_{M_i}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & \begin{bmatrix} (v_{1,M_i})_j \\ \vdots \\ (v_{d,M_i})_j \end{bmatrix} - \begin{bmatrix} (v_{1,M_j})_i \\ \vdots \\ (v_{d,M_j})_i \end{bmatrix} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.
\end{aligned}$$

Thus $[v_{j,M_1}^T, \dots, v_{j,M_k}^T]$ is symmetric for each $j = 1, \dots, d$. Therefore by (2.13) each $v_{j,N}$ is a acquired from w_N by (right) multiplication by a *symmetric* matrix.

Noting that

$$M_i^p = \begin{bmatrix} I_d & p\begin{bmatrix} v_{1,M_i} \\ \vdots \\ v_{d,M_i} \end{bmatrix} & pM_i' + \binom{p}{2}\begin{bmatrix} v_{1,M_i} \\ \vdots \\ v_{d,M} \end{bmatrix} w_{M_i}^T \\ 0 & I_k & pw_{M_i}^T \\ 0 & 0 & 1 \end{bmatrix}$$

each element of this form has order p except when $p = 2$ where we demand $w_{M_i}[v_{1,M_i}^T, \dots, v_{d,M_i}^T] = [(v_{1,M_i})_i, \dots, (v_{d,M_i})_i] = 0$. Then the matrices $[v_{j,M_1}^T, \dots, v_{j,M_k}^T]$ have zero diagonals and so when $p = 2$ each $v_{j,N}$ is a acquired from w_N by (right) multiplication by an element of $\mathrm{Sym}_k^0(\mathbb{F})$.

By the above arguments any representation with dual-type $(1, k, d)$ are equivalent to one induced by a homomorphism of the form $\sigma_{\underline{A}}$ for $\underline{A} = \{A_1, \dots, A_d\} \subset \mathcal{S}_k$. All which remains it to determine which of these homomorphisms induce equivalent representations.

For some $\Gamma \in GL_n(\mathbb{F})$ and $W, W' \leq \mathbb{F}^{n-1}$ suppose $\Gamma\sigma_{\underline{A}}(W) = \sigma_{\underline{B}}(W')\Gamma$, that is $\sigma_{\underline{A}}$ and $\sigma_{\underline{B}}$ yield conjugate images for $A_1, \ldots, A_d, B_1, \ldots, B_d \in \mathrm{Sym}_k(\mathbb{F})$. Then by denoting

$$\Gamma = \begin{bmatrix} \gamma_{1,1} & \cdots & \gamma_{1,d} & \underline{\gamma}_1 & \gamma_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \gamma_{d,1} & \cdots & \gamma_{d,d} & \underline{\gamma}_d & \gamma_{d,n} \\ & 0 & & \Gamma' & \underline{\gamma}' \\ & 0 & & 0 & \gamma_{n,n} \end{bmatrix}$$

we have

$$\begin{aligned} 0 &= \Gamma\sigma_{\underline{A}}(\underline{c}) - \sigma_{\underline{B}}(\underline{b})\Gamma \\ &= \begin{bmatrix} & w_{\underline{c}}\sum_{i=1}^{d}\gamma_{1,i}A_i - w_{\underline{b}}B_1\Gamma' & \sum_{i=1}^{d}\gamma_{1,i}(c_{n-i} + w_{\underline{c}}A_i w_{\underline{c}}^T) - w_{\underline{b}}B_1\underline{\gamma}' - (b_{n-1} + w_{\underline{b}}B_1 w_{\underline{b}}^T)\gamma_{n,n} \\ 0 & \vdots & \vdots \\ & w_{\underline{c}}\sum_{i=1}^{d}\gamma_{d,i}A_i - w_{\underline{b}}B_d\Gamma' & \sum_{i=1}^{d}\gamma_{d,i}(c_{n-i} + w_{\underline{c}}A_i w_{\underline{c}}^T) - w_{\underline{b}}B_d\underline{\gamma}' - (b_{n-d} + w_{\underline{b}}B_d w_{\underline{b}}^T)\gamma_{n,n} \\ 0 & 0 & \Gamma' w_{\underline{c}}^T - \gamma_{n,n}w_{\underline{b}}^T \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

In particular $w_{\underline{b}}^T = \gamma_{n,n}^{-1}\Gamma' w_{\underline{c}}^T$. Thence we acquire the relations

$$w_{\underline{c}}\sum_{i=1}^{d}\gamma_{j,i}A_i - \gamma_{n,n}^{-1}w_{\underline{c}}\Gamma'^T B_j\Gamma' = w_{\underline{c}}\left[\sum_{i=1}^{d}\gamma_{j,i}A_i - \gamma_{n,n}^{-1}\Gamma'^T B_j\Gamma'\right] = 0$$

for each $j = 1, \ldots, d$. Since there must be k linearly independent $w_{\underline{c}}$ it follows that

$$B_j = ((\Gamma')^{-1})^T\gamma_{n,n}\left[\sum_{i=1}^{d}\gamma_{j,i}A_i\right](\Gamma')^{-1}$$

for all $j = 1, \ldots, d$. The remaining relations simply yield expressions defining $b_{k+1}, \ldots, b_n$. Thus two homomorphisms $\sigma_{\underline{A}}$ and $\sigma_{\underline{B}}$ corresponding to the sequences $A_1, \ldots, A_d$ and $B_1, \ldots, B_d$ have conjugate images if and only if the $\mathbb{F}$-linear span of the A_i is congruent to the $\mathbb{F}$-linear span of the B_i under $GL_k(\mathbb{F})$.

Hence two homomorphisms of the form $\sigma_{\underline{A}}$ have non-conjugate images if and only if their sequence of symmetric/alternate matrices span incongruent vector spaces. The result then follows. $\square$

2.6.2 Degeneracy

Having determined the usefulness of the congruent subspaces of $\mathcal{S}_k$ in classifying modular $\mathbb{Z}_p^r$-representations with dual-type $(1, k, d)$ we aim to understand these orbits and the representations which their resulting homomorphisms cover. We are focusing on representations with dual-type $(1, k, d)$, but there are many socle tabloids with this information, all of the form

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
3 & 3 & \cdots & 3 & 2 & \cdots & 2 \\
\hline
2 & \cdots & 2 & \multicolumn{4}{c}{} \\
\cline{1-3}
1 & \multicolumn{6}{c}{} \\
\cline{1-1}
\end{array}
\qquad \text{with } d \text{ total } \boxed{3},\ k \text{ total } \boxed{2} \tag{2.14}$$

To better manipulate these orbits we divide them into smaller families with respect to their corresponding tabloids.

Definition 2.6.4. For a given subspace $S \leq \mathrm{Sym}_k(\mathbb{F})$ define $\ker(S) := \cap_{M \in S} \ker(M)$. We call S *degenerate* if $\dim(\ker S) > 0$ and *non-degenerate* otherwise.

If S is degenerate then we choose a congruence matrix in $GL_k(\mathbb{F})$ to incorporate a basis of $\ker(S)$. Thus we can choose a representative of its congruence orbit whose elements are of the form

$$M = \begin{bmatrix} M' & 0 \\ 0 & 0 \end{bmatrix}$$

for $M' \in \mathrm{Sym}_{k-\dim(\ker S)}(\mathbb{F})$. Hence the problem of determining orbits of degenerate $S \leq \mathrm{Sym}_k(\mathbb{F})$ is equivalent to determining the orbits of non-degenerate $S' \leq \mathrm{Sym}_{k-\dim(\ker S)}(\mathbb{F})$.

Remark. If S is degenerate then the socle tabloid of the representations induced by image of σ_S is of the form (2.14) with $\dim(\ker S)$ many $\boxed{2}$ in the first row. We saw in Theorem 2.3.3 that covering homomorphisms for these are constructed from those covering the tabloid with these boxes removed. This corroborates the correspondence between the degenerate cases and the lower-dimensional non-degenerate cases.

We focus our efforts on constructing orbit representatives of non-degenerate subspaces for increasing k. The following case is useful in the document's future.

Example 2.6.5. Let $\mathbb{F} = \overline{\mathbb{F}}$ have characteristic $p > 2$ and consider 2-dimensional subspaces $S \leq \mathrm{Sym}_2(\mathbb{F})$. The orbits are divided into the non-degenerate cases and cases equivalent to the non-degenerate cases in $\mathrm{Sym}_1(\mathbb{F})$. However $\mathrm{Sym}_1(\mathbb{F}) \cong \mathbb{F}$ has no dimension 2 subspaces and so the latter is trivial.

If $S \leq \mathrm{Sym}_2(\mathbb{F})$ is non-degenerate and all nonzero elements in S have rank 2 then up to congruence we choose $S = \mathrm{span}_{\mathbb{F}}\{I_2, A\}$. However

$$\det\left(A - \left[\mathrm{tr}(A) - \sqrt{\mathrm{tr}(A)^2 \pm 4\det(A)}\right] I_2\right) = 0.$$

Thus there must exist an element in S of rank 1 and so we may ensure that our vector space basis contains

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 0 & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

up to congruence. We then split into two cases. If $a_{22} \neq 0$ then

$$\begin{bmatrix} 1 & 0 \\ -\frac{a_{12}}{a_{22}} & \frac{1}{\sqrt{a_{22}}} \end{bmatrix}^T \begin{bmatrix} 0 & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{a_{12}}{a_{22}} & \frac{1}{\sqrt{a_{22}}} \end{bmatrix} = \begin{bmatrix} \frac{-a_{12}^2}{a_{22}} & 0 \\ 0 & 1 \end{bmatrix}.$$

If $a_{22} = 0$ then by non-degeneracy a_{12} is nonzero. Then instead we may take

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix}^T \begin{bmatrix} 0 & a_{12} \\ a_{12} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{a_{12}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since neither transformation alters J it follows that up to congruence we have two distinct orbit representatives

$$S_1 := \mathrm{span}_{\mathbb{F}}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}, \quad \text{and} \quad S_2 := \mathrm{span}_{\mathbb{F}}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}.$$

This process was ultimately to classify representations with dual-type $(1,2,2)$. Here both cases correspond to the socle tabloid $\delta = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$. The covering homomorphisms these yield are $\sigma_{S_1}, \sigma_{S_2} : (\mathbb{F}^4, +) \to GL_5(\mathbb{F})$ given by

$$\sigma_{S_1}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_2 & 0 & c_4 + c_2^2 \\ 0 & 1 & 0 & 2c_1 & c_3 + c_1^2 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \sigma_{S_2}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_2 & 0 & c_4 + c_2^2 \\ 0 & 1 & c_1 & c_2 & c_3 + c_1c_2 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

These collectively cover the representations with socle tabloid $\begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\\ \boxed{1}\end{array}$. Specifically σ_{S_1} covers representations V for which $V_{(1)} := (V^*/\mathrm{Soc}(V^*))^*$ is decomposable – so that $\delta_{(1)} = \begin{array}{l}\boxed{2}\\ \boxed{1}\end{array} \oplus \begin{array}{l}\boxed{2}\\ \boxed{1}\end{array}$ – and σ_{S_2} covers those for which $V_{(1)}$ is indecomposable.

If we consider the same case for $p = 2$ we observe that since $\mathrm{Sym}_2^0(\mathbb{F}) \cong \mathbb{F}$ there are no two-dimensional subspaces. Hence there are no representations with dual-type $(1, 2, 2)$ for $p = 2$.

2.6.3 More Vanishing Tabloids

To classify the representations with socle tabloid $\begin{array}{l}\boxed{3}\boxed{3}\boxed{\cdots}\boxed{3}\\ \boxed{2}\boxed{\cdots}\boxed{2}\\ \boxed{1}\end{array}$ we consider the nondegenerate subspaces of $\mathcal{S}_k(\mathbb{F}) := \mathrm{Sym}_k(\mathbb{F})$ (or $\mathcal{S}_k(\mathbb{F}) := \mathrm{Sym}_k^0(\mathbb{F})$ for $\mathrm{char}(\mathbb{F}) = 2$) of dimension d. However $\mathcal{S}_k(\mathbb{F})$ is an $\mathbb{F}$-vector space of dimension $\frac{k(k\pm 1)}{2}$. Thus if d exceeds this bound then no such subspaces can exist. From this we achieve the following.

Proposition 2.6.6. *Let* $\delta := \begin{array}{l}\boxed{3}\boxed{3}\boxed{\cdots}\boxed{3}\\ \boxed{2}\boxed{\cdots}\boxed{2}\\ \boxed{1}\end{array}$ *have d many $\boxed{3}$ and k many $\boxed{2}$. Then if $d > \frac{k(k+1)}{2}$ there are no modular $\mathbb{Z}_p^r$-representations with socle tabloid δ. Further no modular $\mathbb{Z}_2^r$-representations exist with socle tabloid δ if $d > \frac{k(k-1)}{2}$.*

Section 2.3.2 introduced tabloids with no associated $\mathbb{Z}_p^r$-representations such as $\begin{array}{l}\boxed{3}\boxed{3}\\ \boxed{2}\\ \boxed{1}\end{array}$. This corresponds to $\mathrm{Sym}_1(\mathbb{F}) \cong \mathbb{F}$ having no 2-dimensional subspaces. However this result provides additional examples. For instance the tabloids

$$\begin{array}{l}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\\ \boxed{1}\end{array}, \quad \begin{array}{l}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\end{array}, \quad \begin{array}{l}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\end{array}$$

are now known not to have representations associated. One may also augment these with $\boxed{1}$ and $\boxed{2}$ in the first row and $\boxed{1}$ in the second to acquire more vanishing examples.

2.6.4 The Orthogonal Groups

We provide here an application of our work on those socle-type $(1, n - 2, 1)$ $\mathbb{Z}_p^r$-representations to the orthogonal groups. An excellent resource for the background of these can be found in [28] whose content we summarise here.

Let $p > 2$ be a prime and $q = p^r$ for some $r \in \mathbb{N}$. For $V = \mathbb{F}_q^n$ define a non-degenerate quadratic form $Q : V \to \mathbb{F}_q$ and its polar form $\beta : V^2 \to \mathbb{F}$ by

$$\beta(v, u) := Q(v + u) - Q(v) - Q(u).$$

From this we then construct the orthogonal group

$$O(V, Q) := \{f \in GL(V) \mid Q(f(v)) = Q(v),\ \forall v \in V\}. \tag{2.15}$$

An alternate, convenient definition of the orthogonal groups is as follows: Let $M \in GL(V)$ and define

$$O(V, M) := \{N \in GL(V) \mid N^T M N = M\}.$$

We note that given Q as in (2.15) and its corresponding β we may take $M = (\beta(e_i, e_j))_{i,j}$ where $\{e_1, \ldots, e_n\}$ is a basis for V and then $O(V, M) = O(V, Q)$.

Consider a $\mathbb{Z}_p^r$-representation $G \leq GL(V)$ over $\mathbb{F}_q$ with socle tabloid $\begin{array}{|c|c|c|}\hline 3 \\ \hline 2 & \cdots & 2 \\ \hline 1 \\ \hline\end{array}$ induced by $\sigma^A(W)$ for some $W \leq (\mathbb{F}^{n-2}, +)$ where

$$\sigma^A(\underline{c}) := \sigma_{1,n-2,1}(\underline{c}, 0) = \begin{bmatrix} 1 & 2\underline{c} & \underline{c}A\underline{c}^T \\ 0 & I_{n-2} & A\underline{c}^T \\ 0 & 0 & 1 \end{bmatrix} \tag{2.16}$$

for an invertible $A \in \mathrm{Sym}_{n-2}(\mathbb{F})$. Let $\{v_1, \ldots, v_n\}$ denote the given basis of V where G acts on $\sum \alpha_i v_i$ by left multiplication on $[\alpha_1, \ldots, \alpha_n]^T$. Dually let $\{x_1, \ldots, x_n\}$ be the associated dual basis for V^* where G acts on $\sum \alpha_i x_i$ by right multiplication on $[\alpha_n, \ldots, \alpha_1]$.

Lemma 2.6.7. *Let G be as above. Define $\underline{x} := [x_{n-1}, \ldots, x_2]$. Then*

$$\delta := x_1 x_n - \underline{x} A^{-1} \underline{x}^T \in \mathbb{F}[V]^G.$$

Proof. The proof is straight-forward substitution as follows:

$$\delta \cdot \sigma^A(\underline{c}) = x_1 \left(x_n + \sum_{i=2}^{n-1} 2c_{i-1} x_i + (\underline{c}A\underline{c}^T) x_1 \right) - (\underline{x} + (A\underline{c}^T)^T x_1) A^{-1} (\underline{x} + (A\underline{c}^T)^T x_1)^T$$

$$
\begin{aligned}
&= x_1x_n + 2x_1\sum_{i=2}^{n-1} c_{i-1}x_i + (\underline{c}A\underline{c}^T)x_1^2 \\
&\qquad\qquad - \underline{x}A^{-1}\underline{x}^T - \underline{x}A^{-1}A\underline{c}^T x_1 - \underline{c}AA^{-1}\underline{x}^T x_1 - \underline{c}AA^{-1}A\underline{c}^T x_1^2 \\
&= x_1x_n + 2\underline{x}\,\underline{c}^T x_1 + (\underline{c}A\underline{c}^T)x_1^2 - \underline{x}A^{-1}\underline{x}^T - 2\underline{x}\,\underline{c}^T x_1 - (\underline{c}A\underline{c}^T)x_1^2 \\
&= x_1x_n - \underline{x}A^{-1}\underline{x}^T = \delta.
\end{aligned}
$$

□

An important observation from an invariant-theoretic perspective is the fact that these group consist wholly of *bireflections*, that is elements $g \in G$ for which $\dim(V^g) \geq \dim(V)-2$. Further we acquire from this result an 'integral' invariant, an element of $\mathbb{F}[V]^G$ whose structure does not rely on the prime characteristic. More immediately however the result tells us that such representations are subgroups of an orthogonal group with δ as the given quadratic form.

Proposition 2.6.8. *Let $\mathbb{F}_q = \langle \alpha_1, \dots, \alpha_r \rangle_{\mathbb{F}_p}$ and define*

$$W = \langle (\alpha_i, 0, \dots, 0), (0, \alpha_i, \dots, 0), \dots, (0, 0, \dots, \alpha_i) \mid i \in [\![1, r]\!] \rangle \leq (\mathbb{F}^{n-2}, +)$$

For some invertible $A \in \mathrm{Sym}_{n-2}(\mathbb{F})$ construct the group

$$E(n, A, q) := \left\langle \sigma_c := \begin{bmatrix} 1 & 2\underline{c} & \underline{c}A\underline{c}^T \\ 0 & I_{n-2} & A\underline{c}^T \\ 0 & 0 & 1 \end{bmatrix} \mid \underline{c} \in W \right\rangle.$$

Then $|E(n, A, q)| = q^{n-2}$ and $E(n, A, q) \leq O(V, \delta)$ for $\delta = x_1x_n - \underline{x}A^{-1}\underline{x}^T$ where $\underline{x} := [x_{n-1}, \dots, x_2]$.

Proof. Lemma 2.6.7 yields $E(n, A, q) \leq O(V, Q)$. It is then easy to see from the fact that $W \to E(n, A, q)$ is a bijection that G has order $p^{r(n-2)} = q^{n-2}$. □

Having shown that $E(n, A, q) \cong \mathbb{Z}_p^{r(n-2)}$ is a subgroup of the Sylow p-subgroup of $O(V, \delta)$ we go on to consider elements of $GL(V)$ of the form

$$M' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose $M' \in O(V, \delta)$, that is

$$\delta \cdot M = x_1 x_n - (\underline{x} M^T) A^{-1} (\underline{x} M^T)^T = x_1 x_n - \underline{x} (M^T A^{-1} M) \underline{x}^T = x_1 x_n - \underline{x} A^{-1} \underline{x}^T = \delta.$$

Then $M \in O(V', \underline{x} A^{-1} \underline{x}^T) = O(V', A^{-1})$ for $V' := \langle v_1, \ldots, v_{n-1} \rangle / \langle v_1 \rangle$. Hence we can augment $E(n, A, q)$ by adding elements M' constructed from $M \in O(V', A^{-1})$ further into the orthogonal group $O(V, \delta)$. We take this to its ultimate conclusion.

Theorem 2.6.9. *Let V be an n-dimensional $\mathbb{F}_q$-vector space with basis $v_1, \ldots, v_n$, dual basis $x_n, \ldots, x_1$ and $n \geq 3$. Define $V' := V / \langle v_1, v_n \rangle$ and $\delta := x_1 x_n - \underline{x} A^{-1} \underline{x}^T$ for $\underline{x} = [x_{n-1}, \ldots, x_2]$ for invertible $A \in \mathrm{Sym}_{n-2}(\mathbb{F})$. If $S \leq O(V, \delta)$ is a Sylow p-subgroup then*

$$S \cong E(n, A, q) \rtimes S'$$

where S' is a Sylow p-subgroup of $O(V', A^{-1})$. Thence

$$S \cong \mathbb{Z}_p^{r(n-2)} \rtimes \mathbb{Z}_p^{r(n-4)} \rtimes \cdots \rtimes \mathbb{Z}_p^{r\varepsilon}$$

where $\varepsilon := 2$ if n is even and $\varepsilon := 1$ otherwise. In particular S is a bireflection group.

Proof. Given $\sigma^A(\underline{c}) \in E(n, A, q)$ and $M' := \mathrm{diag}(1, M, 1)$ for $M \in S'$ we see that $\sigma^A(\underline{c}) M' = M' \sigma^A(\underline{c} M)$ since $M^T A^{-1} M = A^{-1}$ by definition. Thus $E(n, A, q) \rtimes S' \leq S$ and so

$$|E(n, A, q) \rtimes S'| = q^{n-2} |S'|.$$

It is then a matter of noting that for a general n-dimensional vector space W and quadratic form Q, a Sylow p-subgroup $\Sigma \leq O(W, Q)$ has order

$$|\Sigma| = \begin{cases} q^{\frac{n(n-2)}{4}} & n \text{ even}, \\ q^{\frac{(n-1)^2}{4}} & n \text{ odd} \end{cases}$$

(see, for instance [28]). Thus $|E(n, A, q) \rtimes S'| = |S|$.

Via continued application of this we acquire (2.6.9) noting that if $n = 3$ then $S \cong \mathbb{Z}_p^r$ and if $n = 4$ then $S \cong \mathbb{Z}_p^{2r}$.

We see S is a bireflection group since each elementary abelian p-group copy in S is (up to factoring out appropriate copies of the irreducible representation) of socle-type $(1, m-2, 1)$. Such representations consist entirely of bireflections (with the identity nominally so) and so the result follows. $\square$

Conclusion

This section marks the threshold of our classifications in full generality. Having thus far undertaken our studies in arbitrary dimension we begin to focus our efforts down and recap what we have learned in a more orderly fashion.

2.7 The Four-Dimensional Representations

The ulterior motive for all of our representation theory delving thus far has been to plunge the depths of the four-dimensional modular representations of $\mathbb{Z}_p^r$ in a setting of convenient generality. In this section we fully classify all such representations distinguishing them by socle tabloid and thence parameterising via covering homomorphisms.

Note that we omit any socle tabloid with a 1-box in row 1 due to its redundancy (see Section 2.2).

2.7.1 Socle-Length 2 Representations

We have remarked before that any natural isomorphism from $(\mathbb{F}^{m_1 m_2}, +)$ to $U_{(m_1,m_2)}(\mathbb{F})$ automatically yields a covering homomorphism for the $\mathbb{Z}_p^r$-representations with socle-type (m_1, m_2). Below we provide an example of these covering homomorphisms for each socle tabloid.

$$\sigma_{2|111} : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}), \quad \sigma_{2|111}(\underline{c}) = \begin{bmatrix} 1 & c_1 & c_2 & c_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|c|c|} \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

$$\sigma_{22|11} : (\mathbb{F}^4, +) \to GL_4(\mathbb{F}), \quad \sigma_{22|11}(\underline{c}) = \begin{bmatrix} 1 & 0 & c_1 & c_2 \\ 0 & 1 & c_3 & c_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$$

$$\sigma_{222|1} : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}), \quad \sigma_{222|1}(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & c_3 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

2.7.2 Socle-Length 3 Representations

We move on to classifying all modular $\mathbb{Z}_p^r$-representations in $GL_4(\mathbb{F})$ whose socle series has length 3. We immediately acquire the following from Corollary 2.3.8.

Proposition 2.7.1. *There exist no modular $\mathbb{Z}_p^r$-representations with socle tabloid*

$$\delta = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \textit{or} \quad \delta^* = \begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 \\ \hline\end{array}.$$

Note that this result implies that no modular representation of $\mathbb{Z}_p^r$ has socle-type $(1,1,2)$. This also excludes one possible case of socle-type $(2,1,1)$, the remainder of which are comfortably covered by Theorem 2.5.1 and recapped here.

Proposition 2.7.2. *The homomorphism*

$$\sigma_{211} : (\mathbb{F}^3,+) \to GL_4(\mathbb{F}), \quad \sigma_{211}(\underline{c}) := \begin{bmatrix} 1 & 0 & 0 & c_3 \\ 0 & 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a covering homomorphism for all modular $\mathbb{Z}_p^r$-representations with socle-type $(2,1,1)$. Furthermore if $W, W' \leq (\mathbb{F}^3,+)$ are such that $\sigma_{211}(W)$ and $\sigma_{211}(W')$ are conjugate with socle-type $(2,1,1)$, then there exist constants $\underline{\alpha} \in \mathbb{F}^* \times \mathbb{F}^2, \beta \in \mathbb{F}$ and $\gamma \in \mathbb{F}^*$ such that

$$W' = M(\alpha_1,\alpha_2,\alpha_3,\beta,\gamma)W := \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1^2 & \beta \\ \alpha_3 & 0 & \gamma \end{bmatrix} W.$$

Furthermore these linear equivalence matrices $M(\alpha_1,\alpha_2,\alpha_3,\beta,\gamma)$ form a group with structure

$$M(\mathbb{F}^* \times \mathbb{F}^2, \mathbb{F}, \mathbb{F}^*) \cong (\mathbb{F}^*)^2 \ltimes (\mathbb{F} \ltimes \mathbb{F}^2).$$

Proof. The homomorphism arises from Theorem 2.5.1. The equivalence is given in Proposition 2.5.2. The structure of the equivalence group is then easily verifiable, observing that

$$\begin{aligned} M(\mathbb{F}^* \times \mathbb{F}^2, \mathbb{F}, \mathbb{F}^*) &= \{M(\alpha_1,0,0,0,\gamma) \mid \alpha_1,\gamma \in \mathbb{F}^*\} \\ &\quad \ltimes \Big(\{M(1,0,\alpha_3,0,1) \mid \alpha_3 \in \mathbb{F}\} \ltimes \{M(1,\alpha_2,0,\beta,1) \mid \alpha_2,\beta \in \mathbb{F}\}\Big). \end{aligned}$$

$\square$

All which remains is the following classification of all socle-type $(1,2,1)$ representations given in Section 2.5.2.

Proposition 2.7.3. *For $\mathbb{F}$ a field of characteristic $p > 2$ the homomorphisms*

$$\sigma_{3|21|1} : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}) \quad \sigma_{3|21|1}(\underline{c}) = \begin{bmatrix} 1 & 2c_2 & 2c_1 & c_3 + c_2^2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|c|} \cline{1-1} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} \\ \hline 2 & 1 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad or$$

$$\sigma_{3|22|1} : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}) \quad \sigma_{3|22|1}(\underline{c}) = \begin{bmatrix} 1 & 2c_2 & 2c_1 & c_3 + c_2^2 + c_1^2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|c|} \cline{1-1} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

are covering homomorphisms for all modular $\mathbb{Z}_p^r$-representations over $\mathbb{F}$ with socle-type $(1,2,1)$. If instead $\mathbb{F}$ has characteristic $p = 2$ then the homomorphism

$$\sigma'_{3|22|1} : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}) \quad \sigma'_{3|22|1}(\underline{c}) = \begin{bmatrix} 1 & c_1 & c_2 & c_3 + c_1c_2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{array}{|c|c|} \cline{1-1} \multicolumn{1}{|c|}{3} & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline \multicolumn{1}{|c|}{1} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

is a covering homomorphism for these representations over $\mathbb{F}$.

2.7.3 Socle-Length 4 Representations

The socle-length 4 representations in $GL_4(\mathbb{F})$ consist only of those of socle-type $(1,1,1,1)$. Their classification and equivalences follow immediately from Theorem 2.4.3 and Proposition 2.4.8, which we recap here.

Proposition 2.7.4. *The homomorphism*

$$\sigma_{1^4} : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}), \quad \sigma_{1^4}(\underline{c}) = \begin{bmatrix} 1 & 3c_1 & 3(c_1^2 + c_2) & c_1^3 + 3c_1c_2 + c_3 \\ 0 & 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a covering homomorphism for all modular $\mathbb{Z}_p^r$-representations with socle-type $(1,1,1,1)$. Furthermore if $W, W' \leq (\mathbb{F}^3, +)$ such that $\sigma(W)$ and $\sigma(W')$ are con-

jugate with socle-type 1^n then there exists an $\underline{\alpha} \in \mathbb{F}^* \ltimes \mathbb{F}^2$ such that

$$W' = \begin{bmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & \alpha_1^2 & 0 \\ \alpha_3 & 3\alpha_1\alpha_2 & \alpha_1^3 \end{bmatrix} W.$$

2.7.4 The Atlas of 4-Dimensional Representations

For completion we provide a table of all socle tabloids in dimension 4 and the covering homomorphism associated to each. We refer to this as the *atlas* of dimension 4, if the reader forgives our blatant theft from the hands of group theorists, topologists and, prior to them, cartographers.

The reader may wish to consider socle tabloids δ and δ^* as a single case, since the homomorphism(s) governing δ^* are easily recoverable from those of δ. However for completeness we include all required homomorphisms here explicitly.

Furthermore any diagram whose representations contain a trivial free summand (i.e. a 1-box in the first row) shall be singled out as unimportant, since by Corollary 2.3.4 they exist in the image of some other homomorphism automatically.

- $\begin{ytableau} 1 & 1 & 1 & 1 \end{ytableau}$: I_4

- $\begin{ytableau} 2 & 2 & 2 \\ 1 \end{ytableau}$: $\sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & c_3 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $\begin{ytableau} 2 \\ 1 & 1 & 1 \end{ytableau}$: $\sigma(\underline{c}) = \begin{bmatrix} 1 & c_1 & c_2 & c_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $\begin{ytableau} 2 & 2 & 1 \\ 1 \end{ytableau}$: Decomposable - $\begin{ytableau} 2 & 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} 1 \end{ytableau}$ $\xleftrightarrow{*}$ $\begin{ytableau} 2 & 1 \\ 1 & 1 \end{ytableau}$: Decomposable - $\begin{ytableau} 2 \\ 1 & 1 \end{ytableau} \oplus \begin{ytableau} 1 \end{ytableau}$

- $\begin{ytableau} 2 & 1 & 1 \\ 1 \end{ytableau}$: Decomposable - $\begin{ytableau} 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} 1 & 1 \end{ytableau}$

- $\begin{ytableau} 2 & 2 \\ 1 & 1 \end{ytableau}$: $\sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & c_1 & c_2 \\ 0 & 1 & c_3 & c_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $\begin{ytableau} 3 & 3 \\ 2 \\ 1 \end{ytableau} \xleftrightarrow{*} \begin{ytableau} 3 \\ 2 \\ 1 & 1 \end{ytableau}$: Do not exist

- $\begin{ytableau} 3 & 2 \\ 2 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_1 & c_3 + c_1^2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{*}{\longleftrightarrow} \begin{ytableau} 3 \\ 2 & 1 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 2c_2 & 2c_1 & c_3 + c_2^2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $\begin{ytableau} 3 & 1 \\ 2 \\ 1 \end{ytableau}$: Decomposable - $\begin{ytableau} 3 \\ 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} 1 \end{ytableau}$

- $\begin{ytableau} 3 \\ 2 & 2 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 2c_2 & 2c_1 & c_3 + c_2^2 + c_1^2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & c_1 & c_2 & c_3 + c_1 c_2 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $\begin{ytableau} 4 \\ 3 \\ 2 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 3c_1 & 3(c_1^2 + c_2) & c_1^3 + 3c_1 c_2 + c_3 \\ 0 & 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Atlas Overlap

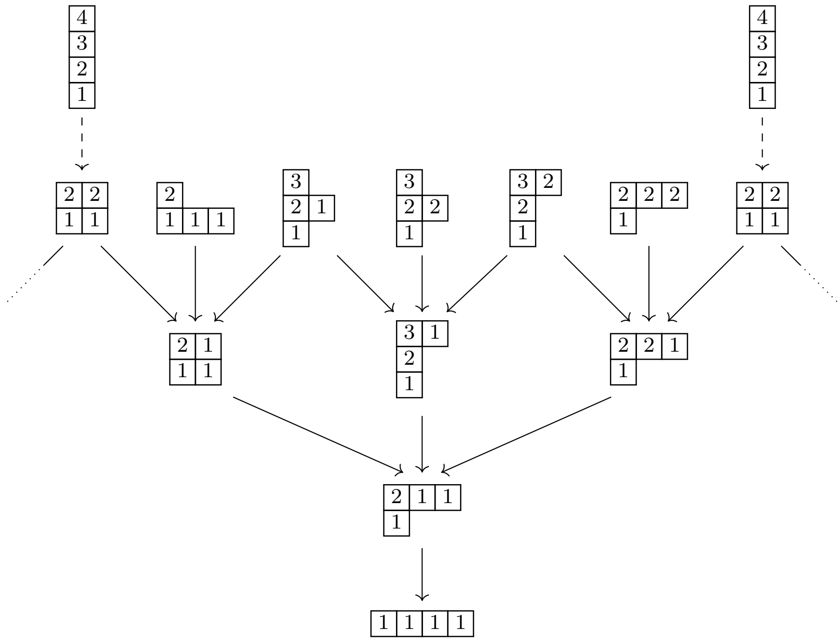


Figure 2.1: Diagram demonstrating how the covering homomorphisms of the four-dimensional atlas overlap in their images.

We have remarked before that covering homomorphisms often spread themselves wide enough to encompass more representations than their intended purview.

Indeed Corollary 2.3.4 tells us that any representation with a $\boxed{1}$ in the first row of its socle tabloid is automatically caught in the net of a covering homomorphism for the tabloid given by replacing that box with a $\boxed{2}$. Dually it also sits inside a covering homomorphism attached to the tabloid acquired by moving $\boxed{1}$ to the second row. Thus our covering homomorphisms often have a nontrivial overlap.

Considering all of the homomorphisms given in the atlas we proffer figure 2.1. This illustrates the overlap between the maps of the four dimensional atlas, where an arrow $\delta \to \delta'$ indicates that the covering homomorphism for δ also induces all representations with tabloid δ' up to equivalence. The dashed line corresponds to only a partial covering.

Note that the diagram loops at the sides and thus may be better demonstrated as a three-dimensional graph, something the author capitulated on within the LaTeX architecture.

Conclusion

Largely the intent of the representation theoretic part of this document was to construct the four-dimensional modular $\mathbb{Z}_p^r$-representations for use in the latter invariant theoretic part. In the process our methods have been generalised to larger families. It is by this virtue that we may in fact continue the process into dimension 5.

2.8 The Five-Dimensional Representations

Having classified all 4-dimensional representation we proceed to dimension 5. Between sections 2.4, 2.5 and 2.6 we have already classified all representations in this dimension via covering homomorphisms. We give a summary of the 5-dimensional atlas covering all socle tabloids in dimension 5, their deconstructions and associated covering homomorphisms.

2.8.1 The Atlas of 5-Dimensional Representations

Here we list the socle tabloids in dimension 5 along with their deconstructions and provide covering homomorphisms for each.

Before doing so we recap each of the tableoids

$$\begin{array}{|l}3\,3\,3\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}3\\2\\1\,1\,1\end{array} \quad \begin{array}{|l}3\,3\,2\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}3\\2\,1\\1\,1\end{array} \quad \begin{array}{|l}3\,3\,1\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}3\,1\\2\\1\,1\end{array} \quad \begin{array}{|l}3\,3\\2\,1\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}3\,2\\2\\1\,1\end{array}$$

$$\begin{array}{|l}3\,3\\2\\1\,1\end{array} \quad \begin{array}{|l}4\,4\\3\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}4\\3\\2\\1\,1\end{array} \quad \begin{array}{|l}4\,3\\3\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}4\\3\\2\,1\\1\end{array} \quad \begin{array}{|l}4\\3\,3\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}4\\3\\2\,2\\1\end{array}$$

which have no associated representations since either they or their duals violate the results of Section 2.3.2 or 2.6.3. We also separate the tableoids

$$\begin{array}{|l}1\,1\,1\,1\,1\end{array}, \quad \begin{array}{|l}2\,2\,2\,1\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}2\,1\\1\,1\,1\end{array}, \quad \begin{array}{|l}2\,2\,1\,1\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}2\,1\,1\\1\,1\end{array}, \quad \begin{array}{|l}2\,1\,1\,1\\1\end{array}$$

$$\begin{array}{|l}2\,2\,1\\1\,1\end{array}, \quad \begin{array}{|l}3\,2\,1\\2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}3\,1\\2\,1\\1\end{array}, \quad \begin{array}{|l}3\,1\,1\\2\\1\end{array}, \quad \begin{array}{|l}3\,1\\2\,2\\1\end{array}, \quad \begin{array}{|l}4\,1\\3\\2\\1\end{array}$$

since any representation associated to one of these shall contain a trivial free summand (i.e. a 1-box in the first row). We ignore them since by Corollary 2.3.4 such representations exist in the image of some other covering homomorphism.

We now proceed to provide the atlas of 5-dimensional representations. For brevity we consider socle tableoids δ and δ^* as a single case, since the covering homomorphisms governing δ^* are easily recoverable from those of δ.

Furthermore we omit the domain on which the covering homomorphism is defined since it is of the form $(\mathbb{F}^d, +)$ where the value of d can be easily observed from the homomorphism itself. The lower limit for the characteristic p of the fields over which these representations exist should also be determinable from the homomorphism, as any prime for which the image induces the desired socle tableoid.

Note also that B_i denotes the ith complete exponential Bell polynomial.

1. $\begin{array}{|l}2\,2\,2\,2\\1\end{array} \xleftrightarrow{*} \begin{array}{|l}2\\1\,1\,1\,1\end{array} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & 0 & c_4 \\ 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. $\begin{array}{|l}2\,2\,2\\1\,1\end{array} \xleftrightarrow{*} \begin{array}{|l}2\,2\\1\,1\,1\end{array} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & c_5 & c_6 \\ 0 & 1 & 0 & c_3 & c_4 \\ 0 & 0 & 1 & c_1 & c_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

3. $\begin{ytableau} 3 & 2 & 2 \\ 2 \\ 1 \end{ytableau} \overset{*}{\longleftrightarrow} \begin{ytableau} 3 \\ 2 & 1 & 1 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & 2c_1 & c_4 + c_1^2 \\ 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

4. $\begin{ytableau} 3 & 3 \\ 2 & 2 \\ 1 \end{ytableau} \overset{*}{\longleftrightarrow} \begin{ytableau} 3 \\ 2 & 2 \\ 1 & 1 \end{ytableau}$ (from Example 2.6.5)

(a) $\delta_{(1)} = \begin{ytableau} 2 & 2 \\ 1 & 1 \end{ytableau}$ indecomposable: $\sigma_{S_1}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_2 & 0 & c_4 + c_2^2 \\ 0 & 1 & c_1 & c_2 & c_3 + c_1c_2 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\delta_{(2)} = \begin{ytableau} 2 \\ 1 \end{ytableau} \oplus \begin{ytableau} 2 \\ 1 \end{ytableau}$ decomposable: $\sigma_{S_2}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_2 & 0 & c_4 + c_2^2 \\ 0 & 1 & 0 & 2c_1 & c_3 + c_1^2 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

5. $\begin{ytableau} 3 & 2 \\ 2 & 2 \\ 1 \end{ytableau} \overset{*}{\longleftrightarrow} \begin{ytableau} 3 \\ 2 & 2 & 1 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & c_1 & c_2 & c_4 + c_1c_2 \\ 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

6. $\begin{ytableau} 3 & 2 \\ 2 & 1 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_1 & c_5 & c_3 + c_1^2 \\ 0 & 1 & 0 & c_4 & c_2 \\ 0 & 0 & 1 & 0 & c_1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

7. $\begin{ytableau} 3 \\ 2 & 2 & 2 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 2c_1 & 2c_2 & 2c_3 & c_4 + 2c_1c_3 + c_2^2 \\ 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

8. $\begin{ytableau} 4 & 2 \\ 3 \\ 2 \\ 1 \end{ytableau} \overset{*}{\longleftrightarrow} \begin{ytableau} 4 \\ 3 & 1 \\ 2 \\ 1 \end{ytableau} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 3c_1 & 3(c_1^2 + c_2) & c_1^3 + 3c_1c_2 + c_4 \\ 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$9. \ \ytableaushort{4,32,2,1} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 3c_3 & 3c_1 & 3(c_1^2+c_2) & c_3^2+c_1^3+3c_1c_2+c_4 \\ 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 1 & 2c_1 & c_1^2+c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$10. \ \ytableaushort{5,4,3,2,1} : \sigma(\underline{c}) = \begin{bmatrix} 1 & 4B_1(\underline{c}) & 6B_2(\underline{c}) & 4B_3(\underline{c}) & B_4(\underline{c}) \\ 0 & 1 & 3B_1(\underline{c}) & 3B_2(\underline{c}) & B_3(\underline{c}) \\ 0 & 0 & 1 & 2B_1(\underline{c}) & B_2(\underline{c}) \\ 0 & 0 & 0 & 1 & B_1(\underline{c}) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2.8.2 Overlap in the Dimension 5 Atlas

Since our given covering homomorphisms overlap in image we present diagrams 2.2 and 2.3 which illustrate these overlaps. Each arrow $\delta \to \delta'$ indicates that the covering homomorphisms for δ also acts as covering homomorphisms for δ'. We also display partial inclusions using dashed arrows.

We denote by $\delta := \ytableaushort{33,22,1}_{\,d.}$ the tabloid $\ytableaushort{33,22,1}$ with the additional structural data that $\delta_{(1)} = \ytableaushort{22,11} = \ytableaushort{2,1} \oplus \ytableaushort{2,1}$ is decomposable, and similarly for its dual. For the tabloid $\ytableaushort{33,22,1}$ with this submodule indecomposable we denote $\ytableaushort{33,22,1}_{\,ind.}$, and similarly for the dual.

Conclusion

The studious reader might be able to predict the title of the following section. Whilst in $GL_4(\mathbb{F})$ and $GL_5(\mathbb{F})$ our prior calculations and formulations sufficed to fully describe our representations, explorations in $GL_6(\mathbb{F})$ require rather more original calculation.

2.9 The Six-Dimensional Representations

Having exhausted the modular $\mathbb{Z}_p^r$-representations in $GL_4(\mathbb{F})$ and $GL_5(\mathbb{F})$ we naturally progress to $GL_6(\mathbb{F})$. Unfortunately our earlier skills and tool-set are not quite as effective in this new environment. Additional work is required to classify all six-dimensional representations. We posit here those representations

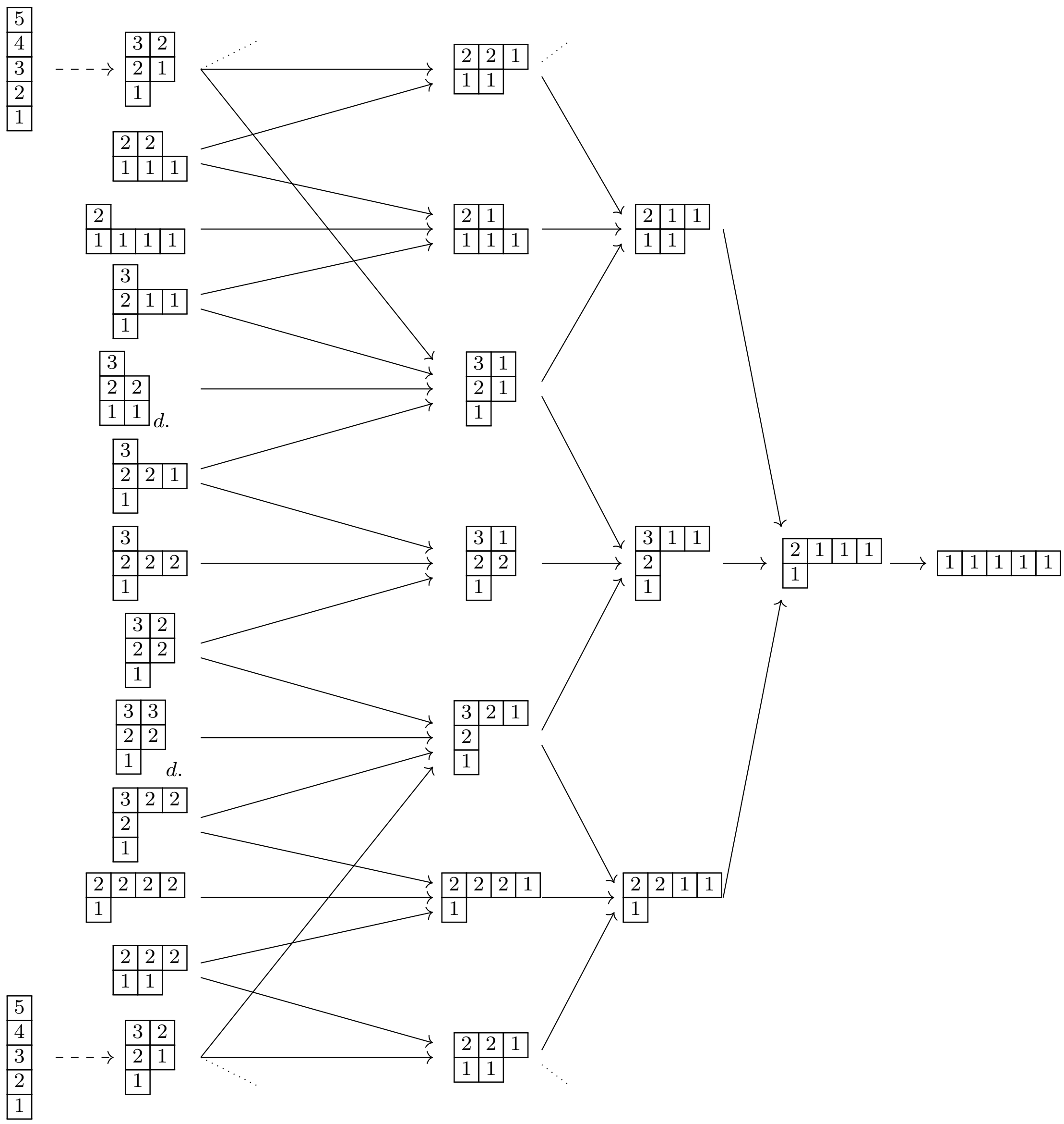


Figure 2.2: A diagram illustrating overlaps in the 5-dimensional atlas.

whose covering homomorphisms do not follow from prior results, omitting a full atlas for brevity.

2.9.1 Representations with Trivial Free Summands

We begin by giving a glancing appraisal of the representations with trivial free summands, those whose socle tabloids contain 1-boxes in their first rows. By

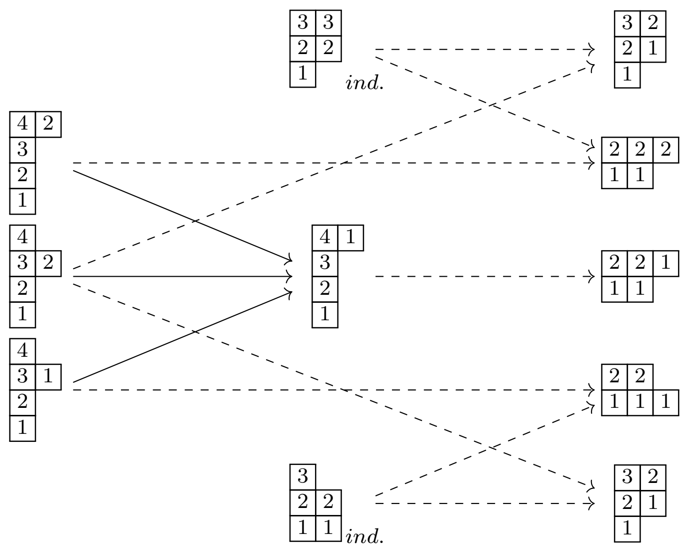
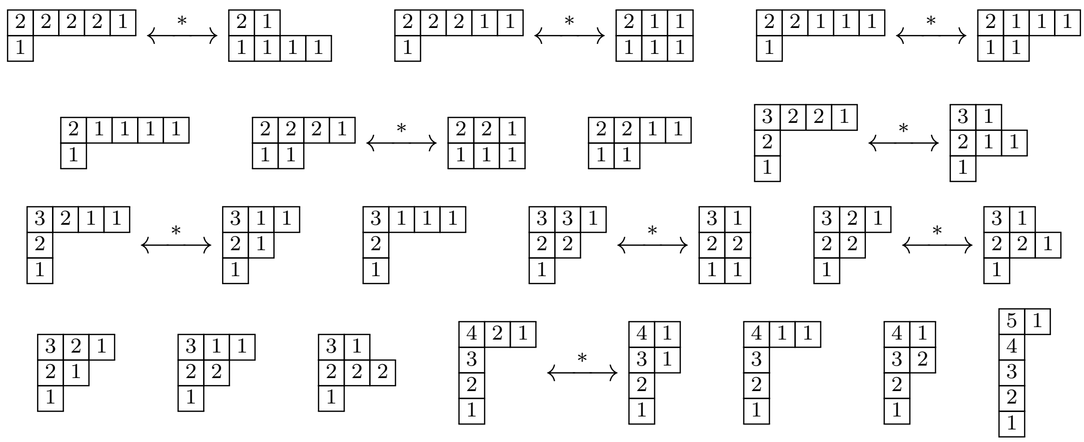


Figure 2.3: A diagram illustrating partial overlaps in the 5-dimensional atlas.

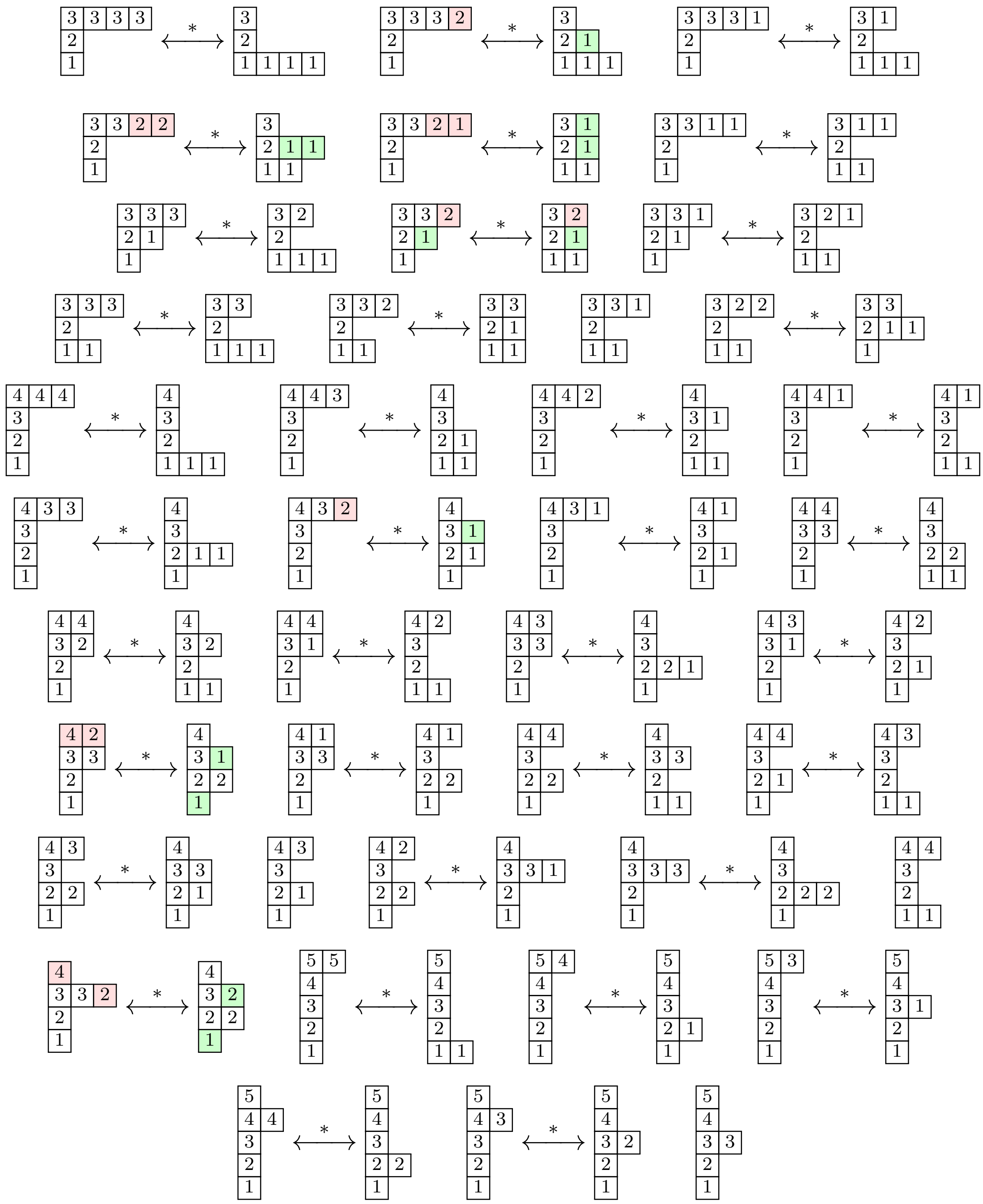
Corollary 2.3.4 the equivalence classes of these representations exist in the image of covering homomorphisms for other tabloids. These tabloids are included for the sake of completion, listed below with their corresponding duals.



2.9.2 Tabloids without Representations

Below we posit all six-box tabloids which satisfy the necessary requirements to be socle tabloids (as in Lemma 2.2.7) but fail to have any associated representations.

Given a proposed doubly-conforming basis $\mathcal{B}$ for a module with tabloid δ we colour in pink boxes of δ corresponding to the subset $\mathcal{B}_p$ and green the boxes corresponding to the subset $\mathcal{B}_g$ of $\mathcal{B}$. The resulting representation $\langle \mathcal{B} \setminus \mathcal{B}_g \rangle / \langle \mathcal{B}_p \rangle$ or its dual then violates either the results of Section 2.3.2 or 2.6.3.



2.9.3 Tabloids with Representations

We examine those socle tabloids with associated representations in dimension 6, but whose classification does not follow from prior explicit or inductive arguments.

Socle-Type (3,2,1), Dual-Type (1,2,3)

Suppose $\mathbb{Z}_p^r \cong G \leq GL(V)$ is a modular representation with socle tabloid

$$\delta = \ytableaushort{333,22,1}\,.$$

Distinct cases to consider are whether $V_{(1)} \sim \delta_{(1)} = \ytableaushort{222,11}$ is indecomposable or $\delta_{(1)} = \ytableaushort{22,1} \oplus \ytableaushort{2,1}$. At first this appears as two cases covered by distinct homomorphisms. The following, however, breaks this illusion.

Proposition 2.9.1. *Suppose $\mathbb{Z}_p^r \cong G \leq GL(V)$ has socle tabloid $\ytableaushort{333,22,1}$. Then the submodule $V_{(1)} := (V^*/\operatorname{Soc} V^*)^*$ is indecomposable.*

Proof. Suppose instead that $V_{(1)} \sim \delta_{(1)} = \ytableaushort{22,1} \oplus \ytableaushort{2,1}$ decomposes. By choosing a doubly-conforming basis $\{v_1, \ldots, v_6\}$ upon which G shall act its elements take the form

$$\begin{bmatrix} 1 & 0 & 0 & c_{14} & 0 & c_{16} \\ 0 & 1 & 0 & 0 & c_{25} & c_{26} \\ 0 & 0 & 1 & 0 & c_{35} & c_{36} \\ 0 & 0 & 0 & 1 & 0 & c_{46} \\ 0 & 0 & 0 & 0 & 1 & c_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the submodule with socle tabloid $\ytableaushort{2,1}$ is $\langle v_1, v_4 \rangle$. One can then observe that the quotient $V/\langle v_1, v_4 \rangle$ has socle tabloid $\ytableaushort{33,2,1}$. Such a representation cannot exist by Corollary 2.3.8 and so neither can the representation we began with. $\square$

This leaves us with a single case to examine.

Proposition 2.9.2. *The homomorphism*

$$\sigma : (\mathbb{F}^5, +) \to GL_6(\mathbb{F}), \quad \sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 0 & c_1 & c_2 & c_5 + c_1c_2 \\ 0 & 1 & 0 & 2c_2 & 0 & c_4 + c_2^2 \\ 0 & 0 & 1 & 0 & 2c_1 & c_3 + c_1^2 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

covers all modular $\mathbb{Z}_p^r$-representations with socle tabloid $\begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$.

Proof. By Theorem 2.6.3 a complete set of covering homomorphisms for the representations with dual-type $(1, 2, 3)$ for $p > 2$ correspond to the incongruent orbits of 3-dimensional subspaces of $\mathrm{Sym}_2(\mathbb{F})$ acted on by $GL_2(\mathbb{F})$. However $\mathrm{Sym}_2(\mathbb{F}) \cong \mathbb{F}^3$ and as such there is only one orbit consisting of $\mathrm{Sym}_2(\mathbb{F})$ itself. By taking our basis to be

$$\mathrm{Sym}_2(\mathbb{F}) = \mathrm{span}_{\mathbb{F}} \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \right\}$$

we acquire the result. Since the $p = 2$ case requires 3-dimensional subspaces of $\mathrm{Sym}_2^0(\mathbb{F}) \cong \mathbb{F}$ the non-existence is clear. □

Socle-Type (2,3,1), Dual-Type (1,3,2) Representations

We examine the first instance for which multiple covering homomorphisms are required per tabloid deconstruction.

Proposition 2.9.3. *The homomorphisms $\sigma_{S_1}, \ldots, \sigma_{S_6} : (\mathbb{F}^5, +) \to GL_6(\mathbb{F})$,*

$$\sigma_{S_1}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_3 & 2c_2 & 0 & c_5 + c_3^2 + c_2^2 \\ 0 & 1 & 0 & 0 & 2c_1 & c_4 + c_1^2 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_{S_2}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_3 & 0 & 0 & c_5 + c_3^2 \\ 0 & 1 & c_1 & 2c_2 & c_3 & c_4 + c_2^2 + c_1c_3 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_{S_3}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_3 & 2c_2 & 0 & c_5 + c_2^2 + c_3^2 \\ 0 & 1 & 2c_3 & 0 & 2c_1 & c_4 + c_3^2 + c_1^2 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_{S_4}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_3 & 2c_2 & 0 & c_5 + c_3^2 + c_2^2 \\ 0 & 1 & 2c_3 + c_1 & 0 & c_3 & c_4 + c_3^2 + c_1c_3 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\sigma_{S_5}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_3+c_1 & 2c_2 & c_3 & c_5+c_1c_3+c_2^2+c_3^2 \\ 0 & 1 & c_2 & c_3 & 0 & c_4+c_2c_3 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_{S_6}(\underline{c}) = \begin{bmatrix} 1 & 0 & c_2 & c_3 & 0 & c_5+c_2c_3 \\ 0 & 1 & 0 & c_1 & c_2 & c_4+c_1c_2 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

form a complete set of covering homomorphisms for all modular $\mathbb{Z}_p^r$-representations with socle tabloid $\begin{matrix}\boxed{3}\boxed{3}\phantom{\boxed{2}}\\ \boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\phantom{\boxed{2}\boxed{2}}\end{matrix}$.

This is an immediate consequence of Theorem 2.6.3 and the following, the proof of which is relegated to Appendix A.

Lemma (Lemma A.1)**.** *The incongruent orbits of non-degenerate dimension* 2 *subspaces in* $\mathrm{Sym}_3(\mathbb{F})$ *are represented by*

$$S_1 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad S_2 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

$$S_3 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad S_4 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

$$S_5 := \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \quad S_6 := \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

We remark that the only covering homomorphism required in the $p = 2$ case is σ_6 since it corresponds to the only subspace of alternate matrices. Indeed σ_g is the only homomorphism given which induces representations with the given socle tabloid over such a field.

Socle-Type (2,2,2), Dual-Type (2,2,2) Representations

Suppose $\mathbb{Z}_p^r \cong G \leq GL(V)$ has socle tabloid $\delta = \begin{matrix}\boxed{3}\boxed{3}\\ \boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\end{matrix}$. Recall from Corollary 2.3.11 that in this instance $V_{(1)}$ is decomposable if and only if $V^{(1)}$ is also decomposable. These deconstructions carry a single covering homomorphism each.

Proposition 2.9.4. *The homomorphisms $\sigma d., \sigma_{ind.} : (\mathbb{F}^6, +) \to GL_6(\mathbb{F})$ given by*

$$\sigma_{d.}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_1 & 0 & c_5 + c_1^2 & c_6 \\ 0 & 1 & 0 & 2c_2 & c_3 & c_4 + c_2^2 \\ 0 & 0 & 1 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\sigma_{ind.}(\underline{c}) = \begin{bmatrix} 1 & 0 & 2c_1 & 2c_2 & c_5 + c_1^2 & c_6 + 2c_1c_2 \\ 0 & 1 & 0 & 2c_1 & c_3 & c_4 + c_1^2 \\ 0 & 0 & 1 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & 1 & 0 & c_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

form a complete set of covering homomorphisms for the modular $\mathbb{Z}_p^r$-representations with socle tabloid $\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$.

Proof. Consider a representation $\mathbb{Z}_p^r \cong G \leq GL(V)$ with socle tabloid $\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$ acting on a doubly conforming basis. Given any two $C = [c_{i,j}], D = [d_{i,j}] \in G$ we see that

$$[C, D] = \begin{bmatrix} 0 & \begin{bmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{bmatrix} \begin{bmatrix} d_{35} & d_{36} \\ d_{45} & d_{46} \end{bmatrix} - \begin{bmatrix} d_{13} & d_{14} \\ d_{23} & d_{24} \end{bmatrix} \begin{bmatrix} c_{35} & c_{36} \\ c_{45} & c_{46} \end{bmatrix} \\ 0 & 0 \end{bmatrix} = 0 \tag{2.17}$$

In order to preserve the socle tabloid there must be elements C, D such that

$$\begin{vmatrix} c_{13} & c_{14} \\ c_{23} & c_{24} \end{vmatrix} \neq 0 \quad \text{and} \quad \begin{vmatrix} d_{35} & d_{36} \\ d_{45} & d_{46} \end{vmatrix} \neq 0.$$

Hence by (2.17) C satisfies both conditions. Thus $(C - I_6)^2 \neq 0$ and thus $p > 2$. We may then choose a basis $\{v_1, \ldots, v_6\}$ for V such that G contains

$$J = \sigma_{d.}(1, 1, 0, 0, 0, 0) = \sigma_{ind.}(1, 0, 0, 0, 0, 0) = \begin{bmatrix} I_2 & 2I_2 & I_2 \\ 0 & I_2 & I_2 \\ 0 & 0 & I_2 \end{bmatrix}.$$

For some other element D to commute with this we require from (2.17) that

$$\begin{bmatrix} d_{13} & d_{14} \\ d_{23} & d_{24} \end{bmatrix} = 2 \begin{bmatrix} d_{35} & d_{36} \\ d_{45} & d_{46} \end{bmatrix}$$

If both $V_{(1)}$ and $V^{(1)}$ decompose choose a corresponding basis which reflects this form

$$\begin{bmatrix} 1 & 0 & 2d_{35} & 0 & d_{15} & d_{16} \\ 0 & 1 & 0 & 2d_{46} & d_{25} & d_{26} \\ 0 & 0 & 1 & 0 & d_{35} & 0 \\ 0 & 0 & 0 & 1 & 0 & d_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and exponentiate to acquire the image of the homomorphism $\sigma_{d.}$.

If neither $V_{(1)}$ nor $V^{(1)}$ decompose then up to permuting v_5 and v_6 we must have $\langle v_1, v_2, v_3, v_4, v_6 \rangle \sim \begin{array}{|c|c|} \hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} =: \delta'$ with $\delta'_{(1)} = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$ indecomposable. This subrepresentation sits in the image of the homomorphism 4(a) in Section 2.8.1 up to equivalence. An alternative but equivalent homomorphism can be given by $\sigma : (\mathbb{F}^4, +) \to GL_5(\mathbb{F})$ where

$$\sigma(\underline{c}) := \begin{bmatrix} 1 & 0 & 2c_1 & 2c_2 & c_4 + 2c_1c_2 \\ 0 & 1 & 0 & 2c_1 & c_3 + c_1^2 \\ 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Applying the same basis change to the respective elements in V we may preserve the form of J and as such the representation elements take the form

$$\begin{bmatrix} 1 & 0 & 2d_{46} & 2d_{36} & d_{15} & d_{16} \\ 0 & 1 & 0 & 2d_{46} & d_{25} & d_{26} \\ 0 & 0 & 1 & 0 & d_{46} & d_{36} \\ 0 & 0 & 0 & 1 & 0 & d_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

the exponential of which yields the image of $\sigma_{ind.}$ thus completing the proof. $\square$

Socle-Type (2,2,1,1), Dual-Type (1,2,2,1)

In this section we consider the representations with socle tabloid $\begin{array}{|c|c|}\hline 4&3\\ \hline 3&2\\ \hline 2\\ \cline{1-1} 1\\ \cline{1-1}\end{array}$.

Proposition 2.9.5. *The homomorphisms $\sigma_i : (\mathbb{F}^5,+) \to GL_6(\mathbb{F})$ given by*

$$\sigma_1(\underline{c}) := \begin{bmatrix} 1 & 0 & 3c_1 & 0 & 3(c_1^2+c_3) & c_1^3+3c_1c_3+c_5 \\ 0 & 1 & 0 & 2c_2 & 0 & c_4+c_2^2 \\ 0 & 0 & 1 & 0 & 2c_1 & c_3+c_1^2 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad V_{(1)} \sim \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline 1\\ \hline\end{array} \oplus \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}$$

$$\sigma_2(\underline{c}) := \begin{bmatrix} 1 & 0 & 3c_1 & 0 & 3(c_1^2+c_3) & c_1^3+3c_1c_3+c_5 \\ 0 & 1 & 0 & c_1 & c_2 & c_4+c_1c_2 \\ 0 & 0 & 1 & 0 & 2c_1 & c_3+c_1^2 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} V_{(1)} \textit{ indec.} \\ \mathrm{Soc}_2(V) \sim \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} \oplus \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} \end{array}$$

$$\sigma_3(\underline{c}) := \begin{bmatrix} 1 & 0 & 3c_1 & 2c_2 & 3(c_1^2+c_3) & c_1^3+3c_1c_3+c_2^2+c_5 \\ 0 & 1 & 0 & c_1 & c_2 & c_4+c_1c_2 \\ 0 & 0 & 1 & 0 & 2c_1 & c_3+c_1^2 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{array}{l} V_{(1)} \textit{ indec.} \\ \mathrm{Soc}_2(V) \textit{ indec.} \end{array}$$

are a complete set of covering homomorphisms for the modular $\mathbb{Z}_p^r$-representations with socle tabloid $\delta :=$ $\begin{array}{|c|c|}\hline 4&3\\ \hline 3&2\\ \hline 2\\ \cline{1-1} 1\\ \cline{1-1}\end{array}$.

Proof. Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle tabloid δ and act on a doubly conforming basis. Observing that $\delta^{(1)} = \begin{array}{|c|c|}\hline 3&2\\ \hline 2\\ \cline{1-1} 1\\ \cline{1-1}\end{array}$ we may choose a basis such that $c_{35} = 2c_{56}$ for all $C = [c_{i,j}] \in G$, in particular showing $p > 2$. Further since $\delta_{(1)} = \begin{array}{|c|c|}\hline 3&2\\ \hline 2&1\\ \hline 1\\ \cline{1-1}\end{array}$ we may further specify our basis to take $c_{13} = 2c_{35} = 4c_{56}$. Since $(C - I_6)^3[1,6] = 8c_{56}^3$ is nonzero for some element it follows that $p > 3$.

Let C, D be such that $\begin{vmatrix} c_{46} & d_{46} \\ c_{56} & d_{56} \end{vmatrix} \neq 0$ as must exist since $\dim(\mathrm{Soc}\, V^*) = 1$. Up to labelling, choice of basis for V and by enforcing commutativity we may take these to be

$$C = \begin{bmatrix} 1 & 0 & 3 & c_1 & 0 & 0 \\ 0 & 1 & 0 & 2c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 & d_1 & 3d_3 + c_1 & d_5 \\ 0 & 1 & 0 & 2d_2 & 2c_2 & d_4 \\ 0 & 0 & 1 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Denote the basis upon which these act by $\mathcal{B} = \{v_1, \ldots, v_6\}$. In order to commute with both C and D any other element must then adopt the form

$$E = \begin{bmatrix} 1 & 0 & 3e_1 & d_1e_2 + (3d_3 + c_1)e_1 - 3e_1d_3 & c_1e_2 + 3e_3 & e_5 \\ 0 & 1 & 0 & 2c_2e_1 + 2d_2e_2 & 2c_2e_2 & e_4 \\ 0 & 0 & 1 & 0 & 2e_1 & e_3 \\ 0 & 0 & 0 & 1 & 0 & e_2 \\ 0 & 0 & 0 & 0 & 1 & e_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

for $(e_1, e_2, e_3, e_4, e_5) \in \mathbb{F}^5$.

Suppose $V_{(1)}$ decomposes. Then $\mathcal{B}$ can be chosen such that $c_1 = c_2 = d_1 = 0$ and thus $d_2 \neq 0$. By altering this basis to $\{v_1, d_2v_2, v_3, v_4, v_5, v_6\}$ we acquire the unrefined form of the image of σ_1, from which this part follows.

Suppose instead that $V_{(1)}$ is indecomposable. Then $c_2 \neq 0$ and $d_2 = 0$. If $\mathrm{Soc}_2(V)$ decomposes then we may choose $d_1 = c_1 = 0$. Thereafter by acting on the basis $\{v_1, 2c_2v_2, v_3, v_4, v_5, v_6\}$ we acquire the unrefined group corresponding to the refined image of σ_2, from which the result follows.

If instead $\mathrm{Soc}_2(V)$ is indecomposable then we demand that $d_1 \neq 0$. Then by acting upon the basis $\{v_1, \sqrt{\frac{2}{d_1}}(c_1v_1 + 2c_2v_2), v_3, \sqrt{\frac{2}{d_1}}v_4, v_5, v_6\}$ we acquire the unrefined form of the image of σ_3, thus concluding the proof. $\square$

Socle-Type (1,2,2,1), Dual-Type (1,2,2,1)

In this section we examine the representations with socle tabloid $\begin{array}{|c|c|}\cline{1-1} 4 \\ \hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$.

Proposition 2.9.6. *The homomorphisms $\sigma_1, \sigma_2 : (\mathbb{F}^5, +) \to GL_6(\mathbb{F})$ given by*

$$\sigma_1(\underline{c}) := \begin{bmatrix} 1 & 3c_2 & 3c_1 & 3(c_2^2+c_4) & 3(c_1^2+c_3) & c_1^3+c_2^3+3(c_1c_3+c_2c_4)+c_5 \\ 0 & 1 & 0 & 2c_2 & 0 & c_2^2+c_4 \\ 0 & 0 & 1 & 0 & 2c_1 & c_1^2+c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sigma_2(\underline{c}) := \begin{bmatrix} 1 & 2c_1 & c_2 & c_1^2+c_3 & 2(c_1c_2+c_4) & c_1^2c_2+2c_1c_4+c_2c_3+c_5 \\ 0 & 1 & 0 & c_1 & c_2 & c_1c_2+c_4 \\ 0 & 0 & 1 & 0 & 2c_1 & c_1^2+c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

are covering homomorphisms for all modular $\mathbb{Z}_p^r$-representations with socle tabloid $\delta := \begin{array}{|c|c|}\hline 4 \\ \hline 3 & 3 \\ \hline 2 & 2 \\ \hline 1 \\ \hline \end{array}$.

Proof. Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle tabloid δ and act on a doubly conforming basis. Suppose $V_{(1)}^{(1)}$ is decomposable. Then $V^{(1)}$ is covered by homomorphism 4(b) of the 5-dimensional atlas in Section 2.8.1. Thus $p > 2$ and we may choose our basis such that the elements of G are in the form

$$C := \begin{bmatrix} 1 & c_9 & c_8 & c_7 & c_6 & c_5 \\ 0 & 1 & 0 & 2c_2 & 0 & c_4 \\ 0 & 0 & 1 & 0 & 2c_1 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let C, D be two such elements satisfying $\begin{vmatrix} c_2 & d_2 \\ c_1 & d_1 \end{vmatrix} \neq 0$, as must exist since $\dim(\operatorname{Soc} V^*) = 1$. By choosing our basis $\mathcal{B} = \{v_1, \ldots, v_6\}$ for V appropriately, and noting that they must commute, we may take these elements to be

$$C := \begin{bmatrix} 1 & 0 & c_8 & c_7 & c_6 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, D := \begin{bmatrix} 1 & d_9 & 0 & d_7 & c_8d_3+c_7 & d_5 \\ 0 & 1 & 0 & 2 & 0 & d_4 \\ 0 & 0 & 1 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For a general element E to commute with C we require that $e_8 = c_8e_1$. In particular this requires $c_8 \neq 0$ which, since $(C - I_6)^3[1,6] = 2c_8$, implies that $p > 3$. Using this we may further alter $\mathcal{B}$ to fix $[c_8, c_7, c_6] = [3, 0, 0]$ and $d_4 = 0$.

Thence E commutes with C and D only if $[e_9, e_8, e_7, e_6] = [d_9e_2, 3e_1, d_9e_4, 3e_3]$. By altering the basis one final time to $\{v_1, \sqrt{\frac{2}{d_9}}v_2, v_3, \sqrt{\frac{2}{d_9}}v_4, v_5, v_6\}$ we see the resulting action of E lies in the unrefined form of σ_1 in the statement.

Suppose instead that $V_{(1)}^{(1)}$ is indecomposable. Then $V^{(1)}$ is covered by homomorphism 4(a) of Section 2.8.1 and so $p > 2$. We thus choose our basis to write our representation elements in the form

$$C := \begin{bmatrix} 1 & c_9 & c_8 & c_7 & c_6 & c_5 \\ 0 & 1 & 0 & c_1 & c_2 & c_4 \\ 0 & 0 & 1 & 0 & 2c_1 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Once again let $C, D \in G$ satisfy $\begin{vmatrix} c_2 & d_2 \\ c_1 & d_1 \end{vmatrix} \neq 0$ as must exist since $\dim(\operatorname{Soc} V^*) = 1$. By choosing our dual basis $\mathcal{B}^* = \{x_6, \ldots, x_1\}$ for V^* appropriately, noting that our elements must commute, we take these elements to act by

$$C := \begin{bmatrix} 1 & 2d_8 & c_8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, D := \begin{bmatrix} 1 & 0 & d_8 & d_7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For an E to commute with both C and D we require that $[e_9, e_8, e_7, e_6] = [2d_8e_1, c_8e_1 + d_8e_2, d_7e_2 + d_8e_3, c_8e_3 + 2d_8e_4]$. In particular we observe that d_8 is nonzero.

If $p > 3$ we elect to use the dual basis

$$\left\{ \frac{1}{d_8}x_6 + \frac{c_8d_7}{6d_8^2}x_5, x_5 + \frac{c_8}{3d_8}x_4, x_4 + \frac{d_7}{2d_8}x_3, x_3 + \frac{c_8}{3d_8}x_2, x_2, x_1 \right\}$$

on which our elements act in the form of the unrefined group equal to the image of σ_2.

If instead $p = 3$ then since $(C - I_6)^3[1,6] = 2c_8$ we have $c_8 = 0$. Thus by acting upon the dual basis $\{\frac{x_6}{d_8}, x_5, x_4 + \frac{d_7}{2d_8}x_3, x_3, x_2, x_1\}$ then we once again arrive in the unrefined group equal to the image of σ_2. Thus the result follows. $\square$

2.9.4 Seeding the Six-Dimensional Atlas

All of the six-dimensional tabloids without trivial free summands and with valid representations are given here:

$$\ytableaushort{22222,1} \longleftrightarrow \ytableaushort{2,11111} \qquad \ytableaushort{2222,11} \longleftrightarrow \ytableaushort{22,1111} \qquad \ytableaushort{222,111} \qquad \ytableaushort{3222,2,1} \longleftrightarrow \ytableaushort{3,2111,1}$$

$$\ytableaushort{333,22,1} \longleftrightarrow \ytableaushort{3,22,111} \qquad \ytableaushort{332,22,1} \longleftrightarrow \ytableaushort{3,221,11} \qquad \ytableaushort{322,22,1} \longleftrightarrow \ytableaushort{3,2211,1} \qquad \ytableaushort{322,21,1} \longleftrightarrow \ytableaushort{32,211,1}$$

$$\ytableaushort{33,222,1} \longleftrightarrow \ytableaushort{3,222,11} \qquad \ytableaushort{32,222,1} \longleftrightarrow \ytableaushort{3,2221,1} \qquad \ytableaushort{33,221,1} \longleftrightarrow \ytableaushort{32,22,11} \qquad \ytableaushort{32,221,1} \qquad \ytableaushort{33,22,11} \qquad \ytableaushort{3,2222,1}$$

$$\ytableaushort{422,3,2,1} \longleftrightarrow \ytableaushort{4,311,2,1} \qquad \ytableaushort{43,32,2,1} \longleftrightarrow \ytableaushort{4,32,21,1} \qquad \ytableaushort{42,32,2,1} \longleftrightarrow \ytableaushort{4,321,2,1} \qquad \ytableaushort{42,31,2,1} \qquad \ytableaushort{4,33,22,1} \qquad \ytableaushort{4,322,2,1}$$

$$\ytableaushort{52,4,3,2,1} \longleftrightarrow \ytableaushort{5,41,3,2,1} \qquad \ytableaushort{5,42,3,2,1} \qquad \ytableaushort{6,5,4,3,2,1}$$

Whilst it has become tradition to provide an atlas in each given dimension we take the liberty of omitting it here for brevity. We trust the reader understands that the more taxing work is complete. Compiling an atlas of all 57 covering homomorphisms (when $\mathbb{F} = \overline{\mathbb{F}}$) is hereafter a matter of going through each tabloid and applying the appropriate results.

2.10 Conclusion

In this chapter we focused our efforts on explicitly describing the modular representations in as much generality as possible. We discerned the notion of socle tabloids and their deconstructions to rend inequivalent families of representations apart. Thence we generalised methods of identifying equivalence classes with certain additive vector groups in order to parameterise them comfortably. This, however, does not answer all questions which might arise. Hence we collect here some of the open questions following from our work.

2.10.1 Further Problems for Covering Homomorphisms

By specifying a socle tabloid δ and the decomposition of each of the quotient- and sub-module's tabloids $\delta_{(i)}^{(j)}$ we have been able to divide representations into manageable families in all dimensions up to and including 6.

If the reader wishes to continue into dimension 7 the characteristic of the field begins to play a role beyond limiting Jordan block sizes we have seen thus far. This stems from Theorem 2.6.3 and the fact that the number of incongruent 3-dimensional subspaces of $\mathrm{Sym}_3(\mathbb{F})$ is larger when $p = 3$ than otherwise.

To what extent does the characteristic then interfere with the number of covering homomorphisms required for a given family of representations? How much information is required to ensure a family has a predictable number of covering homomorphisms? Does the wildness of the representation type make this unpredictable or grow predictably?

Once the families have been divided the question still remains to determine the unrefined groups. There exists a body of work regarding commuting upper-triangular matrices which may aid in this endeavour. Furthermore any reader familiar with Lie group and algebras will have noticed the similarity between these and the constructions in Section 2.1. We would like to make use of these to solidify our processes so that we need only focus on combinatorial data.

2.10.2 Further Problems for Equivalences

Suppose we have our family of representations and covering homomorphisms $\sigma_i : (\mathbb{F}^d, +) \to GL(V)$. We wish to know which subgroup $\sigma_i(W)$ are conjugate and how to determine this from the $W \leq \mathbb{F}^d$ chosen.

Conjugacy of subgroups in the image of σ_i by the stabiliser corresponds to a linear action on the W. Since we express each covering homomorphism as $\sigma = \exp \circ \tau$ where $\tau(W)$ is an additive group of commuting matrices, conjugacy classes of $\sigma(W)$ correspond to conjugacy classes of $\tau(W)$. Whilst it is easier to calculate in the latter form, since the entries of $\tau(\underline{c})$ are linear, deciphering the equivalence actions and their resulting group structure is still not a trivial task. Were we to understand this linear action the problem of acquiring inequivalent orbits of representations then amounts to determining the orbits of the linear action.

Furthermore representations may be equal as sets but inequivalent as representations. Our prime impetus in this classification problem is to study the invariant rings, which does not care for the representation's arrangement. Hence we may precompose any representation with an automorphism to acquire the same matrix group but a possibly inequivalent representation.

In the case of covering homomorphisms all such representations shall be induced by the same homomorphism. As such determining conjugacy of subgroups of the image gains more importance from this invariant theoretic perspective. This is made more significant given how large the automorphism group of $\mathbb{Z}_p^r$ is compared to other groups.

The questions of immediate value are the following: Can we determine the group structure of the linear equivalence map for any given covering homomorphism without lengthy calculation? How easily can we determine the orbits of these maps to distinguish between equivalent representations? Can we then distinguish between the further orbits of the right action of the automorphism group $GL_r(\mathbb{F}_p)$ thereby identifying potentially inequivalent representations with equal invariant rings?

2.10.3 Further Problems for Socle Tabloids

Since the socle tabloid is a new construction many of its uses and properties are likely yet to be determined.

All work constructing and deconstructing socle tabloids holds for general modular p-group representations. Our work has been centered fully upon elementary abelian p-groups. How much use can these tabloids have in the more general representation theoretic world beyond our own purview?

The representation type of $\mathbb{Z}_p^r$ (except when $r = 1$ or $p = r = 2$) is wild and as a consequence determining a complete classification of all indecomposables is generally considered impossible. If the socle tabloid of a representation cannot be decomposed then neither can the representation itself. However the converse does not hold.

Given a decomposable socle tabloid with specific deconstruction what additional information must be specified about a representation with this data in order to ensure it is (in)decomposable? If this information is relatively simple to procure/specify it would make the degree-by-degree process of identifying indecomposables a more manageable, if still ultimately unfinishable, task.

Another notion not touched upon in this chapter is the effect of tensor products of representations and the resulting socle tabloids. Experiments in this area show that the resulting tabloid is not necessarily predictable from the initial tabloids alone. However a 'generic' tensor of tabloids may be defined as follows:

Given two tabloids δ, γ the *generic tensor* $\delta \otimes \gamma$ can be defined to contain an $(i + j - 1)$-box in row $r + s - 1$ for every i-box in row r of δ and j-box in row s of γ. It seems we may then formulate socle tabloids of tensors of representations as being either the generic tabloid tensor or some 'degenerate' form thereafter. Full examination of this idea is required to either deny it or fully formulate into a useful result.

Moving On

The ultimate aim of this representation theory is to best equip ourselves to delve into the depths of their invariant theory. In the following chapter we begin this descent using the methods developed.

Chapter 3

Invariant Theory

3.1 Preliminaries

In this chapter we direct our focus upon the invariant theory of modular $\mathbb{Z}_p^r$-representations. Whilst the prior chapter went into details on the representation theory side the reader may consider the two chapters distinct. Any results used from the prior chapter will be explicitly referenced. Instead we introduce our chapter specific notation as follows.

3.1.1 Notation and Standard Results

We consider modular $\mathbb{Z}_p^r$-representations V as left $\mathbb{F}\mathbb{Z}_p^r$-modules with dual module $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ over fields of characteristic $\mathrm{char}(\mathbb{F}) = p$. This induces a matrix group $G \leq GL(V)$ given by decoding the action as a left multiplication on column vectors in V. More specifically for $g \in G, v \in V$ and $x \in V^*$ we have $(x \cdot g)(v) := x(g \cdot v)$, thereby making V^* a right $\mathbb{F}\mathbb{Z}_p^r$-module.

Given an $\mathbb{F}\mathbb{Z}_p^r$-module $V = \langle v_1, \dots, v_n \rangle$ we denote by $\mathbb{F}[V] := S(V^*)$ the symmetric algebra of the dual space onto which we extend the action of $\mathbb{Z}_p^r$, and thus the action of G, multiplicatively. We write $\mathbb{F}[V] = \mathbb{F}[x_1, \dots, x_n]$ where x_i is dual to v_{n-i+1}. We focus on the invariant ring

$$\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] \mid f \cdot g = f, \ \forall g \in G\}.$$

The aim is often to acquire explicit generating sets for the invariant ring, or learn more about its structural properties.

We remark that the invariant ring of a representation depends only on its image, the order in which the elements are presented posing no significance. Hence we often simply refer to the invariants of a matrix group rather than a representation, hence the notation $\mathbb{F}[V]^G$.

Given an $\underline{\alpha} \in \mathbb{N}_0^n$ we denote $\underline{x}^{\underline{\alpha}} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Unless otherwise stated we use the graded reverse lexicographic ordering on $\mathbb{F}[V]$ with $x_1 < \cdots < x_n$, that is $\underline{x}^{\underline{\alpha}} < \underline{x}^{\underline{\beta}}$ if $\deg(\underline{x}^{\underline{\alpha}}) < \deg(\underline{x}^{\underline{\beta}})$ or if they have the same degree but the first nonzero element in $\underline{\alpha} - \underline{\beta}$ is positive.

Let $f \in \mathbb{F}[V]$. We define the *lead monomial* of f, denoted $LM(f)$, to be the largest monomial under our given monomial ordering appearing in f with non-zero coefficient $LC(f) \neq 0$, called the *lead coefficient* of f. Subsequently the *lead term* of f is defined $LT(f) := LC(f)LM(f)$. We augment these with an element subscript x_i, for example $LT_{x_i}(f)$, when viewing f as a single-variable polynomial in x_i.

The *Krull dimension* of a ring is equal to the supremum of the lengths of ascending chains of prime ideals of the ring. It is known, for instance in Example 2.3.1 of [12], that polynomial rings $\mathbb{F}[x_1, \ldots, x_n]$ have Krull dimension n.

Many of the following common definitions and results may be found in a variety of sources. We shall cite [12] specifically, the following to be found in Section 2.6 of [12].

Definition 3.1.1. A sequence of homogeneous elements $f_1, \ldots, f_n$ in a finitely generated algebra $A \subseteq \mathbb{F}[V]$ of Krull dimension n is called a *homogeneous system of parameters* if A is a finitely generated $\mathbb{F}[f_1, \ldots, f_n]$-module.

It is prudent to note that a subset of $\mathbb{F}[V]^G$ is a homogeneous system of parameters for the invariant ring if and only if it is a homogeneous system of parameters for $\mathbb{F}[V]$ whenever G is finite (see [12] Corollary 3.0.6). Thus the Krull dimension of $\mathbb{F}[V]^G$ equals the Krull dimension of $\mathbb{F}[V]$, that is $\dim(V)$.

By [25] any finitely generated graded connected $\mathbb{F}$-algebra has a homogeneous system of parameters and thus we may always find one for our invariant rings. We often use the following to verify whether a proposed set constitutes a homogeneous system of parameters, as given in Lemma 2.6.3 of [12].

Lemma 3.1.2. *Let $f_1, \ldots, f_n \in \mathbb{F}[V]^G$ be homogeneous and denote $\overline{V} := V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$. Then $\{f_1, \ldots, f_n\}$ form a homogeneous system of parameters for $\mathbb{F}[V]^G$ if and* *only if*

$$\mathcal{V}_{\overline{V}}(f_1, \ldots, f_n) := \{v \in \overline{V} \mid f_i(v) = 0,\ \forall\, i \in [\![1, n]\!]\} = \{0\}.$$

It follows from Lemma 3.1.2 that a sequence of homogeneous invariants $f_1, \ldots, f_n$ with lead terms $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ respectively (under any monomial ordering) form a homogeneous system of parameters for $\mathbb{F}[V]^G$. It is thus useful to consider the following constructions.

Definition 3.1.3. For $G \le GL_n(\mathbb{F})$ and $f \in \mathbb{F}[V]$ define the stabiliser of f under G by $G_f := \{\sigma \in G \mid f \cdot \sigma = f\}$. If $H \le G$ we define the *relative norm* and *relative transfer* of f by

$$N_H^G(f) := \prod_{\sigma \in G/H} f \cdot \sigma, \qquad Tr_H^G(f) := \sum_{\sigma \in G/H} f \cdot \sigma$$

where $\sigma \in G/H$ represents running over coset representatives of G/H in G. Thus we define the *norm* and *transfer* of f by

$$N^G(f) := N_{G_f}^G(f) = \prod_{\sigma \in G/G_f} f \cdot \sigma, \qquad Tr^G(f) := Tr_{G_f}^G(f) = \sum_{\sigma \in G/G_f} f \cdot \sigma.$$

Often we denote $N_i^G := N^G(x_i)$.

The norm as we define it is the product of the elements in the orbit of f by G, the transfer being the sum thereof. We remark that some authors choose to define the norm as $N_{\{Id\}}^G(f) = N^G(f)^{|G_f|}$ however we remove the stabiliser to avoid this redundancy, and similarly for the transfer.

By choosing a socle-conforming basis for a p-group P we act upper-triangularly and thus $LT(N_i^P) = x_i^{p^{\alpha_i}}$ for some α_i. Hence by Lemma 3.1.2 $N_1^P, \ldots, N_n^P$ always form a homogeneous system of parameters.

3.1.2 Invariant Ring Structure

Many of the open problems within modular invariant theory circulate around classifying which groups have invariant rings with a specific structure. We recount those relevant to our studies.

Arguably the simplest and most desirable invariant rings are those for which a homogeneous systems of parameters act as a generating set.

Lemma 3.1.4. *A homogeneous system of parameters* $\{f_1, \dots, f_n\} \subset \mathbb{F}[V]^G$ *for* $n = \dim(V)$ *satisfies* $\mathbb{F}[V]^G = \mathbb{F}[f_1, \dots, f_n]$ *if and only if* $\prod_{i=1}^n \deg(f_i) = |G|$.

In this case we say that $\mathbb{F}[V]^G$ is a *polynomial* invariant ring, since it is a free polynomial ring over $\mathbb{F}$. In modular invariant theory many groups will have more complicated invariant rings than polynomial. There is one particular family of groups with polynomial invariant rings which are well studied.

Definition 3.1.5. Let $P \leq GL(V)$ be a modular P-group representation. We say P is *Nakajima* with Nakajima basis $\mathcal{B} \subset V^*$ if $\mathcal{B} = \{x_1, \dots, x_n\}$ satisfies

$$P = \{\sigma_n \sigma_{n-1} \cdots \sigma_1 \mid \sigma_i \in P_i\} \quad \text{where} \quad P_i := \{\sigma \in P \mid x_j \cdot \sigma = x_j \ \forall j \neq i\}.$$

This, as hinted before, satisfies the following.

Lemma 3.1.6. *Let* $P \leq GL(V)$ *be a p-group with ordered dual basis* $\mathcal{B} = \{x_1, \dots, x_n\} \subset V^*$ *whose dual is socle-conforming (i.e.* P *acts upper-triangularly). Then* P *is a Nakajima group with Nakajima basis* $\mathcal{B}$ *if and only if*

$$\mathbb{F}[V]^P = \mathbb{F}[N_1^P, \dots, N_n^P].$$

Arguably the polynomial structure is the simplest and most convenient, to be generated as an algebra by $\dim(V)$ many algebraically independent invariants. It is an ongoing problem to determine precisely those modular representations with polynomial invariant rings. See for instance [4]

We call an element $g \in G \leq GL(V)$ a *reflection* if $\dim(V^g) = \dim(V) - 1$ and say G is a *reflection group* if it is generated by reflections. We note the following well known result of Serre [26].

Lemma 3.1.7. *Let* $G \leq GL(V)$ *be finite. If* $\mathbb{F}[V]^G$ *is polynomial then* G *is a reflection group.*

As an extreme example we shall see in Section 3.2 that representations for which $\dim(V^G) = \dim(V) - 1$ have polynomial invariant rings. However not all reflection groups have polynomial invariant rings and thus we require more tools to analyse the structure.

For a graded Noetherian ring R a sequence $r_1, \dots, r_s \in R$ is called a *regular sequence of length* s if

- $(r_1, \ldots, r_s)R \neq R$,
- r_1 is not a zero divisor in R and
- for each $1 < i \leq s$ the element r_i is not a zero divisor of $R/(r_1, \ldots, r_{i-1})R$.

A regular sequence is called *maximal* if it not the start of a longer regular sequence. Any two maximal regular sequences in R have the same length, which we define to be the *depth* of R.

We say that $\mathbb{F}[V]^G$ is *Cohen-Macaulay* if the depth of $\mathbb{F}[V]^G$ is equal to $\dim(V) = n$ (i.e. its Krull dimension). This is – in a subjective, computational sense – a second-best scenario to polynomial. In particular we have the following, exemplified in Theorem 4.3.5 of [4].

Lemma 3.1.8. *The invariant ring $\mathbb{F}[V]^G$ is Cohen-Macaulay if and only if it is free as a module over $\mathbb{F}[f_1, \ldots, f_n]$ for any homogeneous system of parameters $f_1, \ldots, f_n$.*

It is known ([12] Proposition 9.2.4) that if $H < G$ has index $|G : H|$ invertible in $\mathbb{F}$ and $\mathbb{F}[V]^H$ is Cohen-Macaulay then so is $\mathbb{F}[V]^G$. Hence the study of p-group invariants is of great interest in this area.

Practically speaking we'd like to know how many generators we'd need to generate this module.

Proposition 3.1.9 ([14] Theorem 3.7.1)**.** *Let $f_1, \ldots, f_n$ be a homogeneous system of parameters for $\mathbb{F}[V]^G$. Then $\mathbb{F}[V]^G$ is a free $\mathbb{F}[f_1, \ldots, f_n]$-module if and only if it has $\prod_{i=1}^n \deg(f_i)/|G|$ many module generators. If it is not a free $\mathbb{F}[f_1, \ldots, f_n]$-module then more generators are required.*

By the well known result of [18] non-modular invariant rings are always Cohen-Macaulay. Unfortunately in the case of modular p-group representations very few have Cohen-Macaulay invariants. In practice for $P \leq GL(V)$ a p-group, $\mathbb{F}[V]^P$ being Cohen-Macaulay is preferable since the number of generators required as a module over a homogeneous system of parameters does not depend on the characteristic of the field over which the representation was defined, a property not shared in the non-Cohen-Macaulay case.

An element $g \in GL(V)$ is called a *bireflection* if $\dim(V^g) \geq \dim(V) - 2$. We then acquire the analogue to Lemma 3.1.7 from Kemper [21].

Lemma 3.1.10. *If the ring $\mathbb{F}[V]^G$ is Cohen-Macaulay then G is generated by bireflections.*

In cases where we choose to prove the non-Cohen-Macaulay-ness of a ring we shall employ the contrapositive of the following. We call a subset of a homogeneous system of parameters a partial homogeneous system of parameters.

Lemma 3.1.11. *Let $A \subseteq \mathbb{F}[V]$ be a finitely generated algebra. Then A is Cohen-Macaulay if and only if every partial homogeneous system of parameters for A is a regular sequence in A.*

We say a ring R is *regular* if the size of every minimal generating set for the maximal ideal of the localisation of R at any prime ideal is equal to its Krull dimension. Thence a ring R a *complete intersection* if we may write $R \cong R'/(r_1, \ldots, r_m)$ where R' is a regular ring and $r_1, \ldots, r_m \in R'$ is a regular sequence. An invariant ring $\mathbb{F}[V]^G$ with $\mathbb{F}[f_1, \ldots, f_n]$-module generators $h_1, \ldots, h_s$ is a complete intersection if the ideal of relations between $f_1, \ldots, f_n, h_1, \ldots, h_s$ is minimally generated by s relations.

It can be shown (see [16] chapter 18) that complete intersections are Cohen-Macaulay. In [11] it was conjectured that the invariant rings of all three-dimensional modular $\mathbb{Z}_p^r$-representations are Cohen-Macaulay if and only if they are complete intersections, a result which remains open. We continue this examination into higher dimensions in this chapter.

3.1.3 Invariant Fields

We denote the fraction field of $\mathbb{F}[V]$ by $\mathbb{F}(V) := \mathrm{Quot}(\mathbb{F}[V])$ and thus the *field of invariants* $\mathbb{F}(V)^G := \mathrm{Quot}(\mathbb{F}[V]^G) = \mathrm{Quot}(\mathbb{F}[V])^G$. For $P \leq GL(V)$ a modular p-group representation $\mathbb{F}(V)^P$ is purely transcendental (see, for instance, [24]) and so there exist algebraically independent $f_1, \ldots, f_n \in \mathbb{F}(V)^G$ such that $\mathbb{F}(V)^G = \mathbb{F}(f_1, \ldots, f_n)$.

We often find it simpler to construct a generating set for $\mathbb{F}(V)^G$ than for $\mathbb{F}[V]^G$ and would thus prefer some translation tool between the two.

Proposition 3.1.12 ([7], Theorem 2.4)**.** *Let $P \leq GL(V)$ be an upper-triangular p-group representation. Choose homogeneous invariants $\phi_1, \ldots, \phi_n$ such that $\phi_m \in$*

$\mathbb{F}[x_1, \ldots, x_m]^P$ *has minimal positive* x_m*-degree for* $m = 1, \ldots, n$*. Then* $\mathbb{F}(V)^P = \mathbb{F}(\phi_1, \ldots, \phi_n)$*. Furthermore*

$$\mathbb{F}[x_1, \ldots, x_m]^P[LC_{x_m}(\phi_m)^{-1}] = \mathbb{F}[x_1, \ldots, x_{m-1}]^P[\phi_m, LC_{x_m}(\phi_m)^{-1}].$$

We shall often make use of this result, particularly in instances where we know $\mathbb{F}[x_1, \ldots, x_{n-1}]^G$ and a choice of ϕ_n such that $LC_{x_n}(\phi_n) = x_1^\alpha$ and so $\mathbb{F}[V]^G[x_1^{-1}] = \mathbb{F}[x_1, \ldots, x_{n-1}]^G[\phi_n, x_1^{-1}]$. Then we may use the following algorithm outlined in [11].

The SAGBI/Divide-by-x_1 Algorithm

Given a subalgebra $A \subseteq \mathbb{F}[V]$ the *lead term algebra* of A, denoted $LT(A)$ is the algebra generated by the lead terms of all elements in A. A set $\mathcal{B} \subseteq A$ is a *SAGBI basis* (Subalgebra Analogue for a Gröebner Basis of Ideals) for A if the lead terms (or monomials) of $\mathcal{B}$ generate $LT(A)$ as an algebra.

Let $\mathcal{B} = \{f_1, \ldots, f_m\} \subset \mathbb{F}[V]^G$ be a set of homogeneous elements. A *tête-à-tête* is a pair of monomials in the f_i denoted $f^I := f_1^{i_1} \cdots f_m^{i_m}, f^J := f_1^{j_1} \cdots f_m^{j_m}$ such that $LT(f^I) = LT(f^J)$. A tête-à-tête is *non-trivial* if $i_k j_k = 0$ for all $k = 1, \ldots, m$, that is f^I and f^J share no factors in $\mathcal{B}$.

Given a tête-à-tête f^I, f^J we *subduct* their difference against $\mathcal{B}$ as follows: We take $F_0 = f^I - f^J$ and inductively construct F_{k+1} by subtracting from F_k a monomial in the elements of $\mathcal{B}$ with the same lead term as F_k. We continue this until F_k has a lead monomial not constructible from the lead monomials in $\mathcal{B}$.

By definition all non-trivial tête-à-têtes in a SAGBI basis subduct to zero.

The slightly renamed *SAGBI/Divide-by-*x_1 *algorithm* from [11] is as follows: Given a set $\mathcal{B} \subset \mathbb{F}[V]$ we subduct each tête-à-tête arising from its elements and for each nonzero subduction f with lead monomial $\underline{x}^{\underline{a}}$ we append $x_1^{-a_1} f \in \mathbb{F}[V]$ to $\mathcal{B}$. We continue this process until $\mathcal{B}$ has been augmented to a SAGBI basis for the algebra it generates.

For our purposes the power of this algorithm arises when we prepare the following setup.

Proposition 3.1.13 ([11], Theorem 2.2)**.** *Let* $\mathbb{Z}_p^r \cong G \leq GL(V)$ *be upper-triangular. Let* $\mathcal{B} \subset \mathbb{F}[V]^G$ *contain* x_1 *and generate an algebra* A *such that* $A[x_1^{-1}] = \mathbb{F}[V]^G[x_1^{-1}]$*. Suppose there exist* $f_2, \ldots, f_n \in \mathcal{B}$ *such that* $LM(f_i) = x_i^{a_i}$

for some $a_i \in \mathbb{N}$. Then applying the SAGBI/Divide-by-x_1 algorithm to $\mathcal{B}$ yields a SAGBI basis for $\mathbb{F}[V]^G$.

3.1.4 Trivial Free Summands

Let $G \leq GL(V)$ be a representation with $V = V' \oplus V_1$ with V_1 a trivial free module. Then a basis $\{v_1, \ldots, v_n\}$ for V may be chosen such that $v_1 \in V^G$ and $x_n \in (V^*)^G$. Then

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, \ldots, x_{n-1}]^G[x_n]$$

and so this reduces to an $(n-1)$-dimensional problem. Henceforth we elect to ignore representations with trivial free summands. For more information on such representations see Section 2.2.

Chapter Outline

Having dealt with preliminary results and notation we move on to consider the invariants of modular $\mathbb{Z}_p^r$-representations. Sections 3.2 and 3.4 each introduce a general method of constructing invariants of which we shall make heavy use.

By way of example for each, sections 3.3 and 3.5 examine specific families of representations and how their invariants may be constructed using these methods.

Finally Section 3.6 deals with all modular $\mathbb{Z}_p^2$-representations in dimension 4, providing SAGBI bases for those Cohen-Macaulay invariant rings and generating sets for localised invariant rings (and thus invariant fields) which satisfy the criteria of Proposition 3.1.13 otherwise.

3.2 Recovery Functions, Matrix Minors and Hyperplane Groups

The method of constructing invariants as the determinants and minors of matrices stretches back to at least the work of Dickson [15]. The trend continued and developed, a branch of which ended with constructions in [11] for elements in $\mathbb{F}(V)^{\mathbb{Z}_p^r}$. This section aims to examine this construction in generality before applying to a notable example.

In [8] a method for inductively generating generators for hyperplane group invariant rings was outlined. In the language of this document such groups are all those modular $\mathbb{Z}_p^r$-representations with socle-type $(n-1,1)$. Here we provide an explicit construction for a set of generating invariants for these groups using the matrix minor method.

3.2.1 Recovery Functions

The modular $\mathbb{Z}_p^r$-representations we examine often arise from some homomorphism $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$. In the parlance of section 2.1 we call these *covering homomorphisms*. In this situation we identify each matrix group $\mathbb{Z}_p^r \cong G \leq GL_n(\mathbb{F})$ with a $d \times r$ matrix $C = [\underline{c}_1^T \ \cdots \ \underline{c}_r^T]$ such that G is generated by $\sigma(\underline{c}_i) = \sigma(c_{1,i}, \dots, c_{m,i})$. We say that C generates G with respect to σ. It is beneficial for us to be able to encode this information in an invariant-theoretic context.

Definition 3.2.1. Let $C \in M_{d,r}(\mathbb{F})$ generate $G \leq GL(V)$ with respect to $\sigma : (\mathbb{F}^d, +) \to GL_n(\mathbb{F})$. We call $\delta_i \in \mathbb{F}(V)$ a **recovery function** for G if

$$\delta_i \cdot (\sigma(\underline{c}_j) - 1) = c_{i,j}, \quad \forall j \in [\![1, r]\!].$$

We call a full set of recovery functions $\delta_1, \dots, \delta_d$ a **complete recovery set**.

Suppose we have a complete recovery set $\delta_1, \dots, \delta_d$. By defining $\underline{r}_i := [c_{i,1}, \dots, c_{i,r}, \delta_i]$ we construct an $(r+1) \times (r+1)$ matrix with rows given by the $\underline{r}_i$ and their pth powers. The resulting determinant shall be an element of $\mathbb{F}(V)^G$, since the final column shall be mapped to the jth column under the action of $\sigma(\underline{c}_j)$ by design. The variety with which we can construct these matrices is key to constructing useful invariants.

3.2.2 Hyperplane Groups

We demonstrate the effectiveness of recovery functions on a prominent example. Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(m, 1)$ for $m = n - 1$ and act on a socle-conforming basis. Thus G is in the image of the homomorphism

$$\sigma : (\mathbb{F}^m, +) \to U_{(m,1)}(\mathbb{F}) \leq GL_{m+1}(\mathbb{F}), \qquad \sigma(\underline{c}) = \begin{bmatrix} \begin{matrix} & c_m \\ I_m & \vdots \\ & c_1 \end{matrix} \\ \begin{matrix} 0 & 1 \end{matrix} \end{bmatrix}$$

where $\underline{c} = (c_1, \ldots, c_m)$. Let G be generated by $C = [\underline{c}_1^T, \ldots, \underline{c}_r^T]$ with respect to σ and denote $\sigma_i := \sigma(\underline{c}_i)$. We assume no relation between the rows of C so that x_1 is the only monic invariant of degree 1.

Since $x_{i+1} \cdot \sigma_j = x_{i+1} + c_{i,j}x_1$ it follows that the $\delta_i := x_{i+1}/x_1$ for $i = 1, \ldots, m$ form a complete recovery set for G. Consider an $A = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ such that $\sum_{i=1}^m \alpha_i = r$ and construct the matrix

$$\Gamma := \Gamma_A := \left[\begin{array}{cccc:c} c_{1,1} & c_{1,2} & \cdots & c_{1,r} & \delta_1 \\ c_{1,1}^p & c_{1,2}^p & \cdots & c_{1,r}^p & \delta_1^p \\ \vdots & \vdots & & \vdots & \vdots \\ c_{1,1}^{p^{\alpha_1-1}} & c_{1,2}^{p^{\alpha_1-1}} & \cdots & c_{1,r}^{p^{\alpha_1-1}} & \delta_1^{p^{\alpha_1-1}} \\ & & \vdots & & \\ c_{m,1} & c_{m,2} & \cdots & c_{mr} & \delta_m \\ c_{m,1}^p & c_{m,2}^p & \cdots & c_{mr}^p & \delta_m^p \\ \vdots & \vdots & & \vdots & \vdots \\ c_{m,1}^{p^{\alpha_m-1}} & c_{m,2}^{p^{\alpha_m-1}} & \cdots & c_{m,r}^{p^{\alpha_m-1}} & \delta_m^{p^{\alpha_m-1}} \\ \hdashline c_{1,1}^{p^{\alpha_1}} & c_{1,2}^{p^{\alpha_1}} & \cdots & c_{1,r}^{p^{\alpha_1}} & \delta_1^{p^{\alpha_1}} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{m,1}^{p^{\alpha_m}} & c_{m,2}^{p^{\alpha_m}} & \cdots & c_{m,r}^{p^{\alpha_m}} & \delta_m^{p^{\alpha_m}} \end{array}\right] = \left[\begin{array}{c} \underline{r}_1 \\ \underline{r}_1^p \\ \vdots \\ \underline{r}_1^{p^{\alpha_1-1}} \\ \vdots \\ \underline{r}_m \\ \underline{r}_m^p \\ \vdots \\ \underline{r}_m^{p^{\alpha_m-1}} \\ \hdashline \underline{r}_1^{p^{\alpha_1}} \\ \vdots \\ \underline{r}_m^{p^{\alpha_m}} \end{array}\right] \in M_{r+m,r+1}(\mathbb{F}).$$

The $(r+1)\times(r+1)$ minors of this matrix shall exist in $\mathbb{F}[V]^G[x_1^{-1}] \subset \mathbb{F}(V)^G$: The action of any $(\sigma(\underline{c}_i) - 1)$ upon any such minor shall affect only terms contributed by the final column, each of which are mapped to their counterparts in the ith column, thus causing the minor to vanish.

Denote by $\Gamma[S_1|S_2]$ the minor of Γ constructed from rows indexed by S_1 and columns indexed by S_2. Similarly we define $\Gamma\{S_1|S_2\}$ to be the minor with rows indexed by $[\![1, r+m]\!] \setminus S_1$ and columns indexed by $[\![1, r+1]\!] \setminus S_2$.

We shall find the following of use:

$$\Delta = \Delta_A := \Gamma[1, \ldots, r \mid 1, \ldots, r] \in \mathbb{F},$$

$$\widetilde{f}_i = \widetilde{f}_{i,A} := \Gamma[1, \ldots, r, r+i \mid 1, \ldots, r+1] \in \mathbb{F}[V]^G[x_1^{-1}].$$

We then define $f_{i,A} \in \mathbb{F}[V]^G$ to be the polynomial obtained from $\widetilde{f}_{i,A}$ by minimally clearing the x_1 denominator. Precisely how useful these invariants are relies on how A is chosen.

We inductively construct a useful choice of A as follows: Beginning with $A := (0, \ldots, 0)$, for j from 1 to r increase the entry $A_i \mapsto A_i + 1$ if

$$\dim(\mathrm{span}_{\mathbb{F}_p}\{c_{i,1}, \ldots, c_{i,j}\}) > A_i$$

and A_i is minimal amongst these choices. If A is constructed in this fashion then we say it is *C-compatible.*

The remainder of this section is reserved for the proof of the following, which acts as an explicit alternative to the inductive work of [8].

Theorem 3.2.2. *Let C generate G under σ as above and A be C-compatible. Then $\Delta_A \neq 0$ and $\mathbb{F}[V]^G = \mathbb{F}[x_1, f_{1,A}, \ldots, f_{m,A}]$.*

We prove by induction on $r = \mathrm{rank}(G)$ that $\Delta_A \neq 0$, $\deg(f_{j,A}) = p^{A_j}$, and $x_1, f_{1,A}, \ldots, f_{m,A}$ is a h.s.o.p for $\mathbb{F}[V]^G$. Theorem 3.2.2 would then follow from these by Lemma 3.1.4.

Proof of Theorem 3.2.2

When $r = m$ the only C-compatible choice is $A = (1, 1, \ldots, 1)$ and thus by assumption $\Delta_A = \det(C) \neq 0$. The $f_{i,A} = N_{i+1}^G = x_{i+1}^p - x_i$ form a homogeneous system of parameters and thus the result holds.

Now suppose the result holds up to and including representations of rank $r - 1$. Let $C \in M_{d,r}(\mathbb{F})$ generate $G \leq GL_{m+1}(\mathbb{F})$. Create $\widehat{C}$ by removing the final column of C and $\widehat{G} \leq G$ the group generated by $\widehat{C}$ under σ. By induction there exists a $\widehat{C}$-compatible $\widehat{A}$ such that $\Delta_{\widehat{A}} \neq 0$ and $\mathbb{F}[V]^{\widehat{G}} = \mathbb{F}[x_1, f_{1,\widehat{A}}, \ldots, f_{m,\widehat{A}}]$ with $\deg(f_{i,\widehat{A}}) = p^{\widehat{\alpha}_i}$.

Since $\widehat{G} < G$ we have $\mathbb{F}[V]^G \subset \mathbb{F}[V]^{\widehat{G}}$. Hence for some $1 \leq i \leq m$ we have $f_{i,\widehat{A}} \cdot (\sigma_r - 1) \neq 0$. Choose such an i with minimal degree, and thus a minimal $\widehat{\alpha}_i$, and define the consequently C-compatible

$$A = (\alpha_1, \ldots, \alpha_m) := (\widehat{\alpha}_1, \ldots, \widehat{\alpha}_i + 1, \ldots, \widehat{\alpha}_m).$$

Since $f_{i,\widehat{A}}$ is a p-polynomial in each δ_i we see that $f_{i,\widehat{A}} \cdot (\sigma_r - 1)$ becomes

$$\pm \begin{vmatrix} c_{1,1} & \cdots & c_{1,r-1} & \delta_1 \\ \vdots & & \vdots & \vdots \\ c_{1,1}^{p^{\alpha_1-1}} & \cdots & c_{1,r-1}^{p^{\alpha_1-1}} & \delta_1^{p^{\alpha_1-1}} \\ & \vdots & & \\ c_{i,1} & \cdots & c_{i,r-1} & \delta_i \\ \vdots & & \vdots & \vdots \\ c_{i,1}^{p^{\alpha_i}} & \cdots & c_{i,r-1}^{p^{\alpha_i}} & \delta_i^{p^{\alpha_i}} \\ & \vdots & & \\ c_{m,1} & \cdots & c_{m,r-1} & \delta_m \\ \vdots & & \vdots & \vdots \\ c_{m,1}^{p^{\alpha_m-1}} & \cdots & c_{m,r-1}^{p^{\alpha_m-1}} & \delta_m^{p^{\alpha_m-1}} \end{vmatrix} \cdot (\sigma_r - 1) = \pm \begin{vmatrix} c_{1,1} & \cdots & c_{1,r-1} & c_{1,r} \\ \vdots & & \vdots & \vdots \\ c_{1,1}^{p^{\alpha_1-1}} & \cdots & c_{1,r-1}^{p^{\alpha_1-1}} & c_{1,r}^{p^{\alpha_1-1}} \\ & \vdots & & \\ c_{i,1} & \cdots & c_{i,r-1} & c_{i,r} \\ \vdots & & \vdots & \vdots \\ c_{i,1}^{p^{\alpha_i}} & \cdots & c_{i,r-1}^{p^{\alpha_i}} & c_{i,r}^{p^{\alpha_i}} \\ & \vdots & & \\ c_{m,1} & \cdots & c_{m,r-1} & c_{m,r} \\ \vdots & & \vdots & \vdots \\ c_{m,1}^{p^{\alpha_m-1}} & \cdots & c_{m,r-1}^{p^{\alpha_m-1}} & c_{m,r}^{p^{\alpha_m-1}} \end{vmatrix}$$

and so is equal to $\pm\Delta_A$. Thus $\Delta_A = \pm f_{i,\widehat{A}} \cdot (\sigma_r - 1) \neq 0$.

To continue we require understanding of the relationship between the $f_{j,A}$ and the $f_{j,\widehat{A}}$. Technical lemmas contained in appendix B aid in this.

Lemma (B.2)**.** *For $M \in \mathbb{F}^{n\times r}$ with $n > r$ let $S_1, S_2 \subseteq [\![1, n]\!]$ be subsets of size $r-1$, $R := [\![1, r]\!]$ and $1 \leq c_1 < c_2 \leq r$ be integers. Then*

$$\begin{aligned} M\big[S_1 \mid R \setminus c_1\big]M\big[S_2 \mid R \setminus c_2\big] &- M\big[S_1 \mid R \setminus c_2\big]M\big[S_2 \mid R \setminus c_1\big] \\ &= (-1)^r \sum_{i=1}^{r-1} (-1)^i M\big[S_1, S_{2,i} \mid R\big]M\big[S_2 \setminus S_{2,i} \mid R \setminus \{c_1, c_2\}\big]. \end{aligned}$$

Corollary (B.3)**.** *Let $M \in \mathbb{F}^{n\times n}$ for $n \geq 3$ and $r_1, r_2, c_1, c_2 \in [\![1, n]\!]$ with $r_1 < r_2$ and $c_1 < c_2$. Then*

$$\det(M)M\{r_1, r_2|c_1, c_2\} = M\{r_1|c_1\}M\{r_2|c_2\} - M\{r_1|c_2\}M\{r_2|c_1\}$$

Using Corollary B.3 we write $f_{j,A}$ in terms of the previously constructed $f_{j,\widehat{A}}$. Recall that i was chosen such that $\widetilde{f}_{i,\widehat{A}} \cdot (\sigma_r - I_n) \neq 0$.

Corollary 3.2.3. *For $j \neq i$ we have*

$$f_{j,A} = \Delta_{\widehat{A}}^{-1}\Big(\Delta_A f_{j,\widehat{A}} - x_1^{p^{\alpha_j-(\alpha_i-1)}} f_{i,\widehat{A}}\big[\widetilde{f}_{j,\widehat{A}} \cdot (\sigma_r - 1)\big]\Big)$$

Proof. Apply Corollary B.3 to the matrix constructing $\widetilde{f}_{j,A}$ with the final two columns, the final row and the row corresponding to the highest power of δ_i. Clear denominators to acquire the result, noting that if $\alpha_j < \alpha_i - 1$ then by assumption $\widetilde{f_{j,\widehat{A}}} \cdot (\sigma - 1) = 0$. $\square$

This result implies that $\deg(f_{A,j}) = \deg(f_{\widehat{A},j}) = p^{\alpha_j}$ for $j \neq i$. We need now an expression for $f_{i,A}$ in terms of the $\widetilde{f}_{j,\widehat{A}}$.

Corollary 3.2.4. *Define* $\beta_i := \sum_{j=1}^{i} \alpha_j$ *and denote by* Γ_i *the matrix taken from* Γ *to construct* $\widetilde{f}_{i,A}$*, i.e.* $\det(\Gamma_i) = \Gamma[1, \ldots, r, r+i \mid 1, \ldots, r+1]$. *Then*

$$\Delta_A \widetilde{f}^p_{i,\widehat{A}} - \Delta^p_A \widetilde{f}_{i,\widehat{A}} = \sum_{j=1}^{m} (-1)^{\beta_i - \beta_j} \Gamma_i\{\beta_j, r+1 \mid r, r+1\}^p \widetilde{f}_{j,A}.$$

Proof. Apply Lemma B.2 to Γ using

$$S_1 = [\![1, r]\!], \quad S_2 = [\![1, r+m]\!] \setminus \{1, \beta_1 + 1, \ldots, \beta_{m-1} + 1\}, \quad c_1 = r, \quad c_2 = r+1.$$

This yields

$$\begin{aligned}\Gamma\big[[\![1,r]\!] \mid 1, \ldots, r-1, r+1\big]\Gamma\big[S_2 \mid [\![1,r]\!]\big] - \Gamma\big[[\![1,r]\!] \mid [\![1,r]\!]\big]\Gamma\big[S_2 \mid 1, \ldots, r-1, r+1\big] \\ = (-1)^{r+1} \sum_{j=1}^{r} (-1)^j \Gamma\big[[\![1,r]\!], S_{2,j} \mid [\![1, r+1]\!]\big]\Gamma\big[S_2 \setminus \{S_{2,j}\} \mid [\![1, r-1]\!]\big].\end{aligned}$$

Hence the choice of S_1 and S_2 ensures that

$$\Gamma\big[S_2 \mid S_1\big] = (-1)^{\sum_{j=1}^{m-1} \beta_m - \beta_j - j}\Gamma\big[S_1 \mid S_1\big]^p.$$

Since $\Gamma\big[S_1 \mid S_1\big] = \Delta_A$ and $\Gamma\big[S_1 \mid 1, \ldots, r-1, r+1\big] = (-1)^{\beta_m - \beta_i} \widetilde{f}_{i,\widehat{A}}$ we have

$$\begin{aligned}(-1)^{\beta_m - \beta_i + \sum_{j=1}^{m-1} \beta_m - \beta_j - (n-j)} \left(\Delta^p_A \widetilde{f}_{i,\widehat{A}} - \Delta_A \widetilde{f}^p_{i,\widehat{A}}\right) \\ = (-1)^{r+1} \sum_{j=1}^{r} (-1)^j \Gamma\big[S_1, S_{2,j} \mid [\![1, r+1]\!]\big]\Gamma\big[S_2 \setminus \{S_{2,j}\} \mid [\![1, r-1]\!]\big].\end{aligned}$$

Note that $\Gamma\big[S_1, S_{2,j} \mid [\![1, r+1]\!]\big]$ is zero unless $r < S_{2,j} \leq r+m$ that is for $j = r-m+1, \ldots, r$. In this case

$$\Gamma\big[S_1, S_2 \mid [\![1,r]\!]\big] = \Gamma\big[1, \ldots, r, r+k \mid [\![1, r+1]\!]\big] = \widetilde{f}_{k,A}.$$

On the flip side we have

$$\Gamma\big[S_2\backslash\{r{+}k\} \bigm| [\![1,r{-}1]\!]\big] = (-1)^{\beta_n+\beta_k+(n-k)+(\sum_{j=1}^{n-1}\beta_n-\beta_j-(n-j))}\Gamma_i\{\beta_k,r{+}1 \bigm| r,r{+}1\}^p$$

the change in sign accounting for the reordering of rows. Thus we obtain

$$\left(\Delta_A^p\widetilde{f}_{i,\widehat{A}} - \Delta_A\widetilde{f}^p_{i,\widehat{A}}\right) = (-1)^{r+1}\sum_{j=1}^{n}(-1)^{\beta_j-\beta_i+r}\Gamma_i\{\beta_j,r+1 \bigm| r,r+1\}^p\widetilde{f}_{j,A}.$$

The result follows. $\square$

It follows from Corollary 3.2.4 that we may write

$$\widetilde{f}_{i,A} = \Delta_{\widehat{A}}^{-p}\Big(\Delta_A\widetilde{f}^p_{i,\widehat{A}} - \Delta^p_A\widetilde{f}_{i,\widehat{A}} - \sum_{\substack{j=1,\dots,n\\ j\neq i}}(-1)^{\beta_i-\beta_j}\Gamma_i\{\beta_j,r{+}1 \bigm| r,r{+}1\}^p\widetilde{f}_{j,A}\Big). \quad (3.1)$$

Note in particular that since $\Gamma_i\{\beta_j,r+1 \bigm| r,r+1\}$ is (up to sign) the coefficient of $\delta_j^{p^{\alpha_j-1}}$ in $\widetilde{f}_{i,\hat{A}}$ these vanish whenever $\alpha_i < \alpha_j$. Hence it follows by induction that $\deg(f_{i,A}) = \deg(f_{i,\hat{A}})^p = p^{\alpha_i}$ and so the degrees have been confirmed.

All that remains is to show that $x_1, f_{1,A}, \dots, f_{m,A}$ form a h.s.o.p. We do this using Lemma 3.1.2 by showing that the set of $v \in \overline{V}$ which vanish on these elements is zero.

Suppose $x_1(v) = f_{1,A}(v) = \dots = f_{m,A}(v) = 0$. Then since v vanishes on $f_{j,A}$ for $j \neq i$, we clear denominators in (3.1) (noting that the coefficients of $\widetilde{f}_{j,A}$ vanish if $\alpha_i < \alpha_j$) to acquire

$$\Delta_A(f_{i,\widehat{A}}(v))^p - (x_1(v))^p\Delta^p_A f_{i,\widehat{A}}(v) = \Delta_A(f_{i,\widehat{A}}(v))^p = 0.$$

Hence it follows that $f_{i,\widehat{A}}$ vanishes at v. Then from Corollary 3.2.3 we have

$$0 = f_{j,A}(v) = \Delta_{\widehat{A}}^{-1}\Big(f_{j,\widehat{A}}(v)\Delta_A - x_1^{p^{\alpha_j-\alpha_i}}(v)f_{i,\hat{A}}(v)\big[\widetilde{f}_{j,\widehat{A}}\cdot(\sigma-1)\big]\Big) = \Delta_{\widehat{A}}^{-1}\Delta_A f_{j,\widehat{A}}(v)$$

and so v vanishes on all of the $f_{j,\widehat{A}}$. Since these, x_1 included, form a h.s.o.p for $\mathbb{F}[V]^{\widehat{G}}$ it follows that $v = 0$ by the vanishing criterion for h.s.o.p's. Thus it follows that the $f_{j,A}$ and x_1 form a h.s.o.p for $\mathbb{F}[V]^G$. The product of the degrees finalise induction step and so the proof of Theorem 3.2.2.

3.2.3 SAGBI Bases

In practice it is often useful to acquire a SAGBI basis for our invariant rings. All invariants constructed from Γ are p-polynomials in each variable except possibly x_1, but their lead monomials may be dependent.

Proposition 3.2.5. *Let C generate G and $\mathbb{F}[V]^G = \mathbb{F}[x_1, f_{1,A}, \dots, f_{m,A}]$ be as constructed above. Apply the SAGBI/Divide-by-x_1 algorithm to the set $\{x_1, f_{1,A}, \dots, f_{m,A}\}$ to acquire $\{x_1, h_1, \dots, h_k\}$. Then $k = m$ and $\{x_1, h_1, \dots, h_k\}$ is a SAGBI basis for $\mathbb{F}[V]^G$ with $LT(h_i) = x_{i+1}^{p^{\alpha_i}}$ up to relabeling.*

Proof. Assume there is a tête-à-tête between the generators, lest we terminate immediately. By construction each monomial in each $f_{i,A}$ is of the form $x_1^{\alpha} x_j^{p^{\beta}}$ for some $j \geq 1$ and $\alpha, \beta \in \mathbb{N}_0$. Hence this tête-à-tête difference is of the form $f_{i,A} - f_{j,A}^{p^{\gamma}}$. Subduct this difference, using only pth powers of the $f_{j,A}$, until we acquire an invariant $x_1^{\phi_i} f'_{i,A}$ with lead term not in the lead term algebra. We then replace f_i with f'_i and so $\mathbb{F}[V]^G = \mathbb{F}[x_1, f_{1,A}, \dots, f'_{i,A}, \dots, f_{m,A}]$.

By construction the only terms that appear in the subduction are also of the form $x_1^{\alpha} x_j^{p^{\beta}}$. The lead term of $x_1^{\phi_i} f'_i$ must therefore be of the form $x_1^{\phi_i} x_{\iota}^{p^{\theta}}$. If $\phi_i > 0$ then $\deg(f'_i) < \deg(f_i)$. However

$$|G| = \prod_{j=1}^{n} \deg(f_j) > \deg(f_{1,A}) \cdots \deg(f_{i-1,A}) \deg(f'_{i,A}) \deg(f_{i+1,A}) \cdots \deg(f_{m,A})$$

which contradicts Lemma 3.1.4. Hence $LM(f'_i) = x_{\iota}^{p^{\theta}}$ doesn't exist in the lead term algebra of the original set. We then repeat the process having replaced f_i with f'_i until no tête-à-tête remains. The result then follows. $\square$

Conclusion

The definition of recovery functions and their resulting matrix minor invariants shall be of use for future examples, although we stem the enthusiasms of anybody expecting such a clean classification as that of the hyperplane example given. Whilst these invariant rings were polynomial we shan't always be so fortunate, as the invariant rings for many representations aren't even Cohen-Macaulay.

3.3 Invariants of Socle-Type $(m, 1, \ldots, 1)$ Representations

Whilst the generalisation of recovery functions has been fruitful we have thus far focused on representations with low-length socle series. Here we examine the opposition in extremis, considering those representations classified in Section 2.4 and 2.5. The reader need not familiarise themselves with these sections beyond that which we claim here, although is welcome to do so.

In general we shall provide a construction for generators of the invariant field for all representations with socle-type $(m, \ldots, 1)$ for $m \geq 1$. In many such cases the invariant ring cannot be Cohen-Macaulay.

3.3.1 Recovery Functions

Let $W \leq (\mathbb{F}^{n-1}, +)$ be finite. We generate our matrix group $\mathbb{Z}_p^r \cong G \leq GL_n(\mathbb{F})$ with socle-type $M := (m, 1, \ldots, 1)$ as in Section 2.5.1 via the homomorphism

$$\sigma_M : (\mathbb{F}^{n-1}, +) \to GL_n(\mathbb{F}), \qquad \sigma_M(\underline{c}) = \begin{bmatrix} & 0 & \cdots & 0 & c_{n-1} \\ I_{m-1} & \vdots & & \vdots & \vdots \\ & 0 & \cdots & 0 & c_{n-m+1} \\ 0 & & \sigma_{1^{n-m+1}}(\underline{c}) & & \end{bmatrix}$$

where $\sigma_{1^k}(\underline{c})[i,j] = \binom{k-i}{k-j} B_{j-i}(\underline{c})$ for $B_m(\underline{c})$ the complete exponential bell polynomials.

We aim to construct recovery functions for these representations by abusing the properties of the Bell polynomials. In particular we recall the well-known additive property given in Lemma 2.4.4,

$$B_m(\underline{\gamma} + \underline{\delta}) = \sum_{i=0}^{m} \binom{m}{i} B_i(\underline{\gamma}) B_{m-i}(\underline{\delta}).$$

Definition 3.3.1. Let $\Delta(\underline{c}) := \sigma_M(\underline{c}) - 1 \in \mathbb{F}G$. For $i = 2, \ldots, n-m$ inductively define

$$\delta_1 := x_2, \qquad \delta_i := x_{i+1} x_1^{i-1} - B_i(\delta_1, \ldots, \delta_{i-1}, 0).$$

We use these to exploit the following.

Lemma 3.3.2. *For $\ell = 1, \dots, n-m$*

$$\frac{\delta_\ell}{x_1^\ell} \cdot \Delta(\underline{c}) = c_\ell.$$

Proof. We prove by induction on ℓ, $\delta_1 = x_2$ serving as a clear base. Assume the result holds for all integers $i < \ell$. Then using the additivity relation of Lemma 2.4.4 we acquire

$$\begin{aligned}
\delta_\ell \cdot \sigma_M(\underline{c}) &= x_1^{\ell-1} x_{\ell+1} \cdot \sigma_{1^n}(\underline{c}) - B_\ell(\delta_1, \dots, \delta_{\ell-1}, 0) \cdot \sigma_{1^n}(\underline{c}) \\
&= x_1^{\ell-1} \left[\sum_{i=0}^{\ell} \binom{\ell}{i} B_i(\underline{c}) x_{\ell+1-i} \right] - B_\ell(\delta_1 + c_1 x_1, \dots, \delta_{\ell-1} + c_{\ell-1} x_1^{\ell-1}, 0) \\
&= \sum_{i=0}^{\ell} \binom{\ell}{i} \left[x_1^{\ell-1} x_{\ell+1-i} B_i(\underline{c}) - B_{\ell-i}(\delta_1, \dots, \delta_{\ell-1}, 0) B_i(c_1 x_1, \dots, c_{\ell-1} x_1^{\ell-1}, 0) \right].
\end{aligned}$$

Recall from the definition of the Bell polynomials that any monomial $\underline{c}^{\underline{\alpha}}$ appearing in $B_i(\underline{c})$ satisfies $\sum_{j=1}^{i} j\alpha_j = i$. Hence it follows that

$$B_i(c_1 x_1, \dots, c_{\ell-1} x_1^{\ell-1}, 0) = x_1^i B_i(c_1, \dots, c_{\ell-1}, 0).$$

Furthermore the definition also truncates monomials in $B_i(\underline{c})$ at i. Hence

$$\begin{aligned}
B_i(c_1, \dots, c_{\ell-1}, c_\ell) &= B_i(c_1, \dots, c_{\ell-1}, 0), \qquad i < \ell, \\
B_\ell(c_1, \dots, c_{\ell-1}, c_\ell) &= c_\ell + B_\ell(c_1, \dots, c_{\ell-1}, 0).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\delta_\ell \cdot \sigma_{1^n}(\underline{c}) &= \sum_{i=0}^{\ell} \binom{\ell}{i} \left[x_1^{\ell-1} x_{\ell+1-i} B_i(\underline{c}) - x_1^i B_{\ell-i}(\delta_1, \dots, \delta_{\ell-1}, 0) B_i(c_1, \dots, c_{\ell-1}, 0) \right] \\
&= \underbrace{x_1^{\ell-1} x_{\ell+1} - B_\ell(\delta_1, \dots, \delta_{\ell-1}, 0)}_{\delta_\ell} \\
&\qquad + \sum_{i=1}^{\ell-1} \binom{\ell}{i} B_i(\underline{c}) x_1^i \underbrace{\left[x_1^{\ell-i-1} x_{\ell+1-i} - B_{\ell-i}(\delta_1, \dots, \delta_{\ell-i}) \right]}_{0} \\
&\qquad + x_1^\ell \Big(\underbrace{B_\ell(c_1, \dots, c_\ell) - B_\ell(c_1, \dots, c_{\ell-1}, 0)}_{c_\ell} \Big) \\
&= \delta_\ell + c_\ell x_1^\ell.
\end{aligned}$$

The result follows by induction. □

We note several properties of the δ_ℓ which we shall find useful. For the purpose of constructing invariant fields we shall find it pertinent to observe that

$$\delta_{\ell-1} \in \mathbb{F}[x_1, \ldots, x_\ell], \qquad LT_{x_\ell}(\delta_{\ell-1}) = x_1^{\ell-2} x_\ell.$$

It follows immediately from Lemma 3.3.2 that any representation element $\sigma_{1^n}(\underline{c})$ for which $c_\ell = 0$ leaves δ_ℓ invariant. In particular we acquire the following.

Corollary 3.3.3. *Given a symmetric power of a two-dimensional modular representation, that is with image of the form*

$$\mathbb{Z}_p^r \cong G = \langle \sigma_{1^n}(c, 0, \ldots, 0) \bigm| c \in W \rangle$$

for some $W \leq (\mathbb{F}, +)$, then

$$\mathbb{F}[V]^G[x_1^{-1}] = \mathbb{F}[x_1, N_2^G, \delta_2, \ldots, \delta_{n-1}][x_1^{-1}].$$

Furthermore applying the SAGBI/Divide-by-x_1 algorithm to the set

$$\{x_1, N_2^G, \ldots, N_n^G, \delta_2, \ldots, \delta_{n-1}\}$$

shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

Proof. By Lemma 3.3.2 each of $\delta_2, \ldots, \delta_{n-1}$ are invariants. Since the action of G on $\mathbb{F}[x_1, x_2]$ is Nakajima we have $\mathbb{F}[x_1, x_2]^G = \mathbb{F}[x_1, N^G(x_2)]$. Then since each $\delta_{i-1} \in \mathbb{F}[x_1, \ldots, x_i]^G$ is of minimal degree in x_i with $LM_{x_i}(\delta_{i-1}) = x_1^{i-2} x_i$ we apply Proposition 3.1.12 to acquire

$$\mathbb{F}[x_1, \ldots, x_i]^G[x_1^{-1}] = \mathbb{F}[x_1, \ldots, x_{i-1}]^G[\delta_{i-1}, x_1^{-1}]$$

from which we inductively acquire the construction. Adding the norms of $x_3, \ldots, x_n$ to these yields a set which satisfies the conditions of the SAGBI/Divide-by-x_1 algorithm. □

If $m > 1$ we have yet to construct a full set of recovery functions. Fortunately since $x_i \cdot (\sigma - I_n) \in \mathbb{F}[x_1]$ for all $\sigma \in G$ and $i = n - m + 2, \ldots, n$ we may use the same techniques for the remainder as were developed for hyperplane groups. By taking $\widetilde{\delta_i} := x_{i+1}$ for $i = n-m+1, \ldots, n-1$ we acquire

$$\left\{ \frac{\delta_1}{x_1}, \ldots, \frac{\delta_{n-m}}{x_1^{n-m}}, \frac{\widetilde{\delta_{n-m+1}}}{x_1}, \ldots, \frac{\widetilde{\delta_{n-1}}}{x_1} \right\}$$

as a complete set of recovery functions.

3.3.2 Invariant Fields of Socle-Type $(m, 1, \ldots, 1)$

Considering an arbitrary $\mathbb{Z}_p^r \cong G = \langle \sigma_M(\underline{c}_1), \ldots, \sigma_M(\underline{c}_r) \rangle$ where we denote

$$\underline{c}_i = (c_{1,i}, c_{2,i}, \ldots, c_{n-1,i}) \in \mathbb{F}^{n-i}$$

we borrow the structures of Section 3.2 to construct $\mathbb{F}(V)^G$ as follows.

If the stabiliser $G_\ell \leq G$ of $\mathbb{F}[\ell] := \mathbb{F}[x_1, \ldots, x_\ell]$ has rank ρ_ℓ then the restriction of the action of G to $\mathbb{F}[\ell]$ is equal to the restriction of the action of G/G_ℓ which has rank $r_\ell := r - \rho_\ell$. From this define $A = (\alpha_1, \ldots, \alpha_{n-1})$ so that $\alpha_m := r_{m+1} - r_m$.

We define the single-row matrices

$$R_\ell := \begin{cases} \begin{bmatrix} c_{\ell,1}, & c_{\ell,2}, & \cdots, c_{\ell,r}, & \frac{\delta_\ell}{x_1^\ell} \end{bmatrix}, & \ell = 1, \ldots, n-m, \\ \begin{bmatrix} c_{\ell,1}, & c_{\ell,2}, & \cdots, c_{\ell,r}, & \frac{x_{\ell+1}}{x_1} \end{bmatrix}, & \ell = n-m+1, \ldots, n-1. \end{cases}$$

and thus R^{p^i} by raising each entry of R to the power p^i. Thence we construct the matrix

$$\Gamma := \left[\begin{array}{c} R_1 \\ R_1^p \\ \vdots \\ R_1^{p^{\alpha_1 - 1}} \\ \vdots \\ R_{n-1} \\ R_{n-1}^p \\ \vdots \\ R_{n-1}^{p^{\alpha_{n-1}-1}} \\ \hdashline R_1^{p^{\alpha_1}} \\ \vdots \\ R_{n-1}^{p^{\alpha_{n-1}}} \end{array} \right]$$

From this we construct the minors

$$\widetilde{f_i} := \Gamma[1, \dots, r, r+i \mid 1, \dots, r+1] \in \mathbb{F}[V]^G[x_1^{-1}]$$

and thence define f_i by minimally clearing the denominator of $\widetilde{f_i}$.

Theorem 3.3.4. *Let $G = \sigma_M(W) \leq GL_n(\mathbb{F})$ have socle-type $M = (m, 1, \dots, 1)$ for some $W \leq \mathbb{F}^{n-1}$. Then*

$$\mathbb{F}[V]^G[x_1^{-1}] = \mathbb{F}[x_1, f_1, \dots, f_{n-1}][x_1^{-1}].$$

Consequently applying the SAGBI/Divide-by-x_1 algorithm to the set

$$\{x_1, f_2, \dots, f_{n-1}, N_3^G, \dots, N_n^G\}$$

shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

The remainder of this section is devoted to the proof of this result. Naturally we use Proposition 3.1.12 in this proof. Consequently it is sufficient to show that $f_\ell \in \mathbb{F}[x_1, \dots, x_{\ell+1}]^G$ and is of minimal positive degree in $x_{\ell+1}$.

Lemma 3.3.5. *For each $\ell = 1, \dots, n-1$ we have $f_\ell \in \mathbb{F}[\ell+1]^G$.*

Proof. Note that, up to a sign, the coefficient of $\left(\frac{\delta_i}{x_1^i}\right)^{p^\alpha}$ or $\left(\frac{x_{i+1}}{x_1}\right)^{p^\alpha}$ in f_ℓ is an $r \times r$ minor of Γ.

If $i > \ell$ then the submatrix yielding this minor contains, by construction, $\alpha_1 + \cdots + \alpha_\ell + 1 = r_{\ell+1} + 1$ rows containing powers of $c_{j,k}$ for $j = 1, \dots, \ell$. Since $r_{\ell+1}$ is the rank of the group acquired by restricting the action of G to $\mathbb{F}[\ell+1]$ it thus follows that this minor must vanish, as a relation must exist between these elements. Since the δ_i (and trivially the x_{i+1}) contain only the variables $x_1, \dots, x_{i+1}$ the result then follows. □

All that remains is to show that each f_ℓ has minimal positive $x_{\ell+1}$ degree in $\mathbb{F}[\ell+1]^G$. Observe that $\Gamma[\,1, \dots, r \mid 1, \dots, r\,] \neq 0$ by the choice of A. Since this is the coefficient of $({}^{\delta_\ell}/{}_{x_1^\ell})^{p^{\alpha_\ell}}$ or $({}^{x_\ell}/{}_{x_1})^{p^{\alpha_\ell}}$ it follows that

$$\deg_{x_{\ell+1}}(f_\ell) \geq \begin{cases} \deg_{x_{\ell+1}}(\delta_\ell^{p^{\alpha_\ell}}) = p^{\alpha_\ell} \\ \deg_{x_{\ell+1}}(x_{\ell+1}^{p^{\alpha_\ell}}) = p^{\alpha_\ell} \end{cases}$$

Moreover the degree cannot exceed this since no power of δ_i or x_{i+1} for $i > \ell$ occurs in f_ℓ and all other terms appearing do not contain $x_{\ell+1}$, and thus must be equal.

If $r_{\ell+1} = r_\ell$ then $\alpha_\ell = 0$ and so $\deg_{x_{\ell+1}}(f_\ell) = 1$ is thus minimal.

If instead $r_{\ell+1} > r_\ell$ then by definition there exists a subgroup of G of rank $r_{\ell+1} - r_\ell$ which acts trivially on $\mathbb{F}[\ell]$ but not on $\mathbb{F}[\ell+1]$. By construction this subgroup acts on the variables $x_1, \ldots, x_{\ell+1}$ by

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & c_\ell \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

In particular this subgroup $G_{\ell+1}$ is Nakajima and thus

$$\mathbb{F}[\ell+1]^{G_{\ell+1}} = \mathbb{F}[x_1, \ldots, x_\ell, N_{\ell+1}^{G_{\ell+1}}].$$

Since $\mathbb{F}[\ell+1]^G$ is a subring of this polynomial ring it follows that

$$\deg(N_{\ell+1}^{G_{\ell+1}}) = p^{r_{\ell+1}-r_\ell} = p^{\alpha_\ell} = \deg_{x_{\ell+1}}(f_\ell)$$

is the minimal positive $x_{\ell+1}$ degree of an element in $\mathbb{F}[\ell+1]^G$. Thus the proof of Theorem 3.3.4 follows through continued application of Proposition 3.1.12.

Conclusion

Despite the enthusiasms of the last two sections not all problems in invariant theory can be solved using recovery functions. We have restricted our attentions thus far to groups for which we may always obtain a complete set of recovery functions. In the next section we delve into worlds where this is not always possible.

3.4 Invariant Induction in Socle-Length 2

Denote by $\sigma_{m,n-m} : (M_{m,n-m}(\mathbb{F}), +) \to U_{(m,n-m)}(\mathbb{F})$ the homomorphism

$$\sigma_{m,n-m}(A) = \begin{bmatrix} I_m & A \\ 0 & I_{n-m} \end{bmatrix}$$

Then $\sigma_{m,n-m}$ contains in its image the image of all $\mathbb{Z}_p^r$-representations with socle-type $(m, n-m)$ up to equivalence.

Those representations with socle-type $(1, n-1)$ are Nakajima groups and so by Lemma 3.1.6 have known polynomial invariant rings. Similarly socle-type $(n-1, 1)$ representations are hyperplane groups and thus as in Section 3.2 their invariant rings are also polynomial.

In section 8.2 of [12] an example of a reflection group with socle-type $(2, 2)$ is shown to have non-polynomial invariant ring. Hence we cannot always be so fortunate in this sphere.

This section is aimed at generalising a method of inductively constructing invariants by rank to better understand the invariant ring structure of these representations. Our efforts are at least partly aimed at acquiring a semi-reliable measure of the complexity these rings obtain.

3.4.1 Socle Length 2 Representations, Invariants and Reflections

Much of the impetus behind the work in this section arises from the work of Broer [6] which we paraphrase here.

Corollary 3.4.1. *Let $H \leq G$ be elementary abelian p-groups with $[G : H] = p$. Then*

$$\mathbb{F}[V]^H = \mathbb{F}[V]^G[u]$$

is a hypersurface for some $u \in \mathbb{F}[V]^H$ if and only if $\mathbb{F}[V]^G$ is a direct summand of $\mathbb{F}[V]^H$ as an $\mathbb{F}[V]^G$-module. This holds, for instance, if $\mathbb{F}[V]^G$ is polynomial.

Recalling that $g \in G$ is a *bireflection* if $\dim(V^g) \geq \dim(V) - 2$, this leads to the following.

Proposition 3.4.2. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(m, n-m)$ and be generated by reflections $\sigma_1, \ldots, \sigma_{r-1}$ and a bireflection τ. Let $G' := \langle \sigma_1, \ldots, \sigma_{r-1} \rangle \leq G$ be Nakajima. By defining*

$$\phi(\sigma) := (\sigma - I_n)V$$

for $\sigma \in G$, suppose

1. *$\phi(\tau) = \phi(\sigma_i) + \phi(\sigma_j)$ for some $i \neq j$,*
2. *$\phi(\tau) = \phi(\sigma_i) + \mathbb{F}v$ for some $v \notin \sum_{i=1}^{r-1} \phi(\sigma_i)$, or*
3. *$\phi(\tau) \cap \sum_{i=1}^{r-1} \phi(\sigma_i) = \{0\}$,*

i.e. $\phi(\tau)$ intersects with not more than two independent $\phi(\sigma)$ non-trivially. Then $\mathbb{F}[V]^G$ is a hypersurface.

Similarly if τ is a reflection with $\phi(\tau) \neq \phi(\sigma_i)$ for all i and $\phi(\tau) \in \phi(\sigma_i) + \phi(\sigma_j)$ for some $i \neq j$ then $\mathbb{F}[V]^G$ is a hypersurface.

Proof. By Lemma 3.1.6 $\mathbb{F}[V]^{G'}$ is polynomial. The above cases tell us that we may choose a Nakajima basis for G' under which τ acts nontrivially on only two of the dual basis elements, which we may assume to be x_{n-1} and x_n.

Let τ_i for $i = 1, 2$ be reflections defined by

$$x_{n+1-i} \cdot \tau_i := x_{n+1-i} \cdot \tau$$

and trivial elsewhere. Then $\tau = \tau_1 \tau_2$. Defining $G^+ := \langle \sigma_1, \ldots, \sigma_{r-1}, \tau_1, \tau_2 \rangle$ we have $[G^+ : G] = p$. By construction G^+ is Nakajima and so $\mathbb{F}[V]^{G^+}$ is polynomial. Hence by Corollary 3.4.1 we have $\mathbb{F}[V]^G$ is a hypersurface over $\mathbb{F}[V]^{G^+}$ as required. □

The criteria in the previous proposition were to ensure that $\phi(\tau)$ did not intersect with any more than two $\phi(\sigma_i)$ nontrivially. Thus under the basis given for G', τ does not differ from I_n in more than two rows.

Whilst this may seem peculiarly specific we include it for the hope of an extension to the inclusion of more bireflections – perhaps with an aim to show bireflection $(m, n-m)$ groups are complete intersections – and for the following.

Corollary 3.4.3. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(2,2)$ for $r \geq 2$. Suppose the index p subgroup $G' \leq G$ is Nakajima. Then either G is again Nakajima or $\mathbb{F}[V]^G$ is a hypersurface.*

Proof. Let $G' = \langle \sigma_1, \ldots, \sigma_{r-1} \rangle$ where σ_i are reflections, and $G = \langle \sigma_1, \ldots, \sigma_{r-1}, \tau \rangle$. If G is Nakajima the result follows from Lemma 3.1.6.

Suppose G is not Nakajima. By Proposition 3.4.2 either criterion 1 (if there exists $i \neq j$ with $\phi(\sigma_i) \neq \phi(\sigma_j)$) or criterion 2 (if all $\phi(\sigma_i)$ are equal) holds, from which the result follows. $\square$

Theorem A of [20] puts pay to the prospect of general type $(m, n-m)$ representations following this pattern.

Having discovered a collection of hypersurface invariant rings it would be advantageous to explicitly define their generators. Indeed since the polynomial ring from which they grow is a Nakajima invariant ring, the only mystery is the additional generator which yields the hypersurface.

3.4.2 Inductive Difference Invariants

Let $H \leq G \leq GL(V)$ be elementary abelian p-groups with $|G : H| = p$. Then $\mathbb{F}[V]^G \subset \mathbb{F}[V]^H$. Let $\tau \in G \setminus H$ and write $\Delta_\tau := \tau - 1 \in \mathbb{F}G$. For $f_1, f_2 \in \mathbb{F}[V]^H$ we define

$$\mathcal{R}(f_1, f_2, \tau) := \gcd\left(f_1 \cdot \Delta_\tau, f_2 \cdot \Delta_\tau\right)^{-1} \begin{vmatrix} f_1 & f_1 \cdot \Delta_\tau \\ f_2 & f_2 \cdot \Delta_\tau \end{vmatrix}.$$

Since $\mathbb{F}[V]^H$ is a unique factorisation domain we define gcd to be the common monic divisor of greatest degree with respect to our usual ordering. If $f_i \cdot \Delta_\tau \in \mathbb{F}[V]^G$ then $\mathcal{R}(f_1, f_2, \tau) \in \mathbb{F}[V]^G$. We thus refer to these as *inductive difference invariants* of G since this process allows us to inductively construct invariants from those of subgroups $H \leq G$ of decreasing index.

Such a construction has been used implicitly in many situations, an example of which we reiterate here.

Example 3.4.4. For a matrix $A \in M_2(\mathbb{F})$ we denote by $\underline{\left| A \right.} := \begin{bmatrix} I_2 & A \\ 0 & I_2 \end{bmatrix}$. Consider the group

$$G = \left\langle \sigma_1 := \underline{\left|\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}\right.}\,,\ \sigma_2 := \underline{\left|\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}\right.}\,,\ \tau := \underline{\left|\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right.} \right\rangle_{\mathbb{F}_p}.$$

It has been shown (for instance in [12]) that the invariant ring $\mathbb{F}_p[V]^G$ is a hypersurface given by

$$\mathbb{F}_p[V]^G = \mathbb{F}_p[x_1, x_2, N_3^G, N_4^G][h]$$

$$h = x_1(x_3^p - x_1^{p-1}x_3) + x_2(x_4^p - x_2^{p-1}x_4).$$

Note that by setting $H = \langle \sigma_1, \sigma_2 \rangle_{\mathbb{F}_p}$ we have

$$N_3^H \cdot (\tau - 1) = (x_1 + x_2)^p - x_1^{p-1}(x_1 + x_2) = x_2(x_2^{p-1} - x_1^{p-1}) \in \mathbb{F}[V]^G$$
$$N_4^H \cdot (\tau - 1) = (x_1 + x_2)^p - x_2^{p-1}(x_1 + x_2) = x_1(x_1^{p-1} - x_2^{p-1}) \in \mathbb{F}[V]^G$$

and so

$$\mathcal{R}\left(N_3^H, N_4^H, \tau\right) = -x_1 N_3^H - x_2 N_4^H = -h \in \mathbb{F}[V]^G.$$

Thus we have $\mathbb{F}_p[V]^G = \mathbb{F}_p[x_1, x_2, N_3^G, N_4^G][\mathcal{R}\left(N_3^H, N_4^H, \tau\right)]$.

To obtain the main result of this section we effectively generalise this example of [12] and in doing so precisely describe the situation as posited in Corollary 3.4.3. For this though, we need the following.

Proposition 3.4.5 ([12] Prop. 11.0.1)**.** *Let $G \leq G^+$ be groups acting on an integral domain R of characteristic $[G^+ : G] = p$. Let $\sigma \in G^+ \setminus G$. Suppose there exists a $y \in R^G$ such that*

- *the polynomial $x := y \cdot (\sigma - 1)$ lies in R^{G^+} and*
- *the set $(\sigma - 1)R^G$ lies in the principal ideal $(x)R$.*

Then $R^G = R^{G^+}[y]$.

Using this we prove what our narrative might suggest predictable.

Proposition 3.4.6. *Let $H \leq G \leq GL(V)$ be elementary abelian p-groups with $[G : H] = p$ under a socle-conforming basis with socle-type $(2, 2)$. Suppose $H = \langle \sigma_1, \ldots, \sigma_r \rangle$ and $\tau \in G \setminus H$. If H is Nakajima and G is not then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G][\mathcal{R}(N_3^H, N_4^H, \tau)].$$

Proof. Ensuring that $\{x_1, x_2, x_3, x_4\}$ is a Nakajima basis for H, define τ_1, τ_2 as acting trivially on V^*, except

$$x_4 \cdot \tau_1 := x_4 \cdot \tau \qquad x_3 \cdot \tau_2 := x_3 \cdot \tau.$$

Then by taking $G^+ := \langle H, \tau_1, \tau_2 \rangle$ we acquire $H < G < G^+$ by assumption. Furthermore G^+ is Nakajima with invariant ring

$$\mathbb{F}[V]^{G^+} = \mathbb{F}[x_1, x_2, N_3^G, N_4^G].$$

We use Proposition 3.4.5 with G, G^+ as their namesakes, $y := \mathcal{R}(N_3^H, N_4^H, \tau)$ and $\sigma = \tau_1$. Since $N_3^G \cdot (\tau_1 - 1) = 0$ we have

$$\begin{aligned} x_G &:= \mathcal{R}(N_3^H, N_4^H, \tau) \cdot (\tau_1 - 1) \\ &= \frac{N_3^H \cdot (\tau_2 - 1) N_4^H \cdot (\tau_1 - 1)}{\gcd(N_3^H \cdot (\tau_2 - 1), N_4^H \cdot (\tau_1 - 1))} \in \mathbb{F}[x_1, x_2] \subset \mathbb{F}[V]^{G^+} \end{aligned}$$

and so we need only show that each element in $\mathbb{F}[V]^G \cdot (\tau_1 - 1)$ is divisible by x_G.

Since H is Nakajima we have

$$\mathbb{F}[V]^G \subset \mathbb{F}[V]^H = \mathbb{F}[x_1, x_2, N_3^H, N_4^H].$$

Let $m := x_1^{a_1} x_2^{a_2} (N_3^H)^{a_3} (N_4^H)^{a_4}$ be an arbitrary 'monomial' in $\mathbb{F}[V]^H$. Then

$$\begin{aligned} m \cdot (\tau_1 - 1) &= x_1^{a_1} x_2^{a_2} (N_3^H)^{a_3} \left[(N_4^H \cdot \tau_1)^{a_4} - (N_4^H)^{a_4} \right] \\ &= x_1^{a_1} x_2^{a_2} (N_3^H)^{a_3} \left[(N_4^H + (N_4^H \cdot (\tau_1 - 1)))^{a_4} - (N_4^H)^{a_4} \right] \\ &= x_1^{a_1} x_2^{a_2} (N_3^H)^{a_3} \left[\sum_{i=1}^{a_4} \tbinom{a_4}{i} (N_4^H)^{a_4 - i} (N_4^H \cdot (\tau_1 - 1))^i \right]. \end{aligned}$$

Hence either $N_4^H \cdot (\tau_1 - 1)$ divides $m \cdot (\tau_1 - 1)$ or the latter is zero. Similarly we may see that

$$m \cdot (\tau_2^{-1} - 1) = x_1^{a_1} x_2^{a_2} (N_4^H)^{a_4} \left[\sum_{i=1}^{a_3} \tbinom{a_3}{i} (-1)^i (N_3^H)^{a_3 - i} (N_3^H \cdot (\tau_2 - 1))^i \right].$$

Hence either $N_3^H \cdot (\tau_2 - 1)$ divides $m \cdot (\tau_2^{-1} - 1)$ or the latter is zero.

Let $f \in \mathbb{F}[V]^G \subseteq \mathbb{F}[V]^H$. Then we note that

$$f = f \cdot \tau = f \cdot (\tau_1 \tau_2), \quad \implies \quad f \cdot (\tau_1 - 1) = f \cdot (\tau_2^{-1} - 1).$$

Hence by the above argument $N_4^H \cdot (\tau - 1)$ and $N_3^H \cdot (\tau - 1)$ divide $f \cdot (\tau_1 - 1)$ and thus the x_G divides all $f \cdot (\tau_1 - 1)$ as required. □

Whilst this particular case is conveniently covered by the $\mathcal{R}$ polynomials, their influence stretches further as we begin to examine the invariant fields of an arbitrary representation with socle-type $(2,2)$.

3.4.3 The Field Of Fractions of Socle-Type $(2,2)$ Invariants

We base the constructions of this section upon the following specification of Proposition 3.1.12.

Corollary 3.4.7. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ have socle-type $(2,2)$ acting upon a socle-conforming basis. Let $f_4 \in \mathbb{F}[V]^G$ be such that $\deg_{x_4}(f_4) \neq 0$ is minimal. Then*

$$\mathbb{F}[V]^G[LC_{x_4}(f_4)^{-1}] = \mathbb{F}[x_1, x_2, N_3^G, f_4][LC_{x_4}(f_4)^{-1}].$$

Proof. Let $f \in \mathbb{F}[V]^G$. By Proposition 3.1.12 there exists an integer $k \geq 0$ such that $LC_{x_4}(f_4)^k f \in \mathbb{F}[x_1, x_2, x_3]^G[f_4]$.

The action of G on $\mathbb{F}[x_1, x_2, x_3]$ is either trivial – in which case G is Nakajima and the result follows trivially – or has socle-type $(1,2)$. Hence $\mathbb{F}[x_1, x_2, x_3]^G = \mathbb{F}[x_1, x_2, N_3^G]$ and so $LC_{x_4}(f_4)^k f \in \mathbb{F}[x_1, x_2, N_3^G, f_4]$ which yields the result. $\square$

This gives an implicit description of the desired field of fractions. Whilst appealing the task remains to calculate f_4. We do so by use of the $\mathcal{R}$ invariants.

Proposition 3.4.8. *Let $\mathbb{Z}_p^r \cong G = \langle \sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s \rangle \leq GL(V)$ have socle-type $(2,2)$ such that $H_0 := \langle \sigma_1, \ldots, \sigma_r \rangle$ is a maximal Nakajima subgroup of G. Choose a socle-conforming basis for V arising from a Nakajima basis of H_0. Construct the subgroups*

$$H_i := \langle \sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_i \rangle \qquad i = 1, \ldots, s$$

and inductively define the polynomials

$$h_i := \mathcal{R}(h_{i-1}, N_3^{H_{i-1}}, \tau_i)$$

for $i = 1, \ldots, s$, where $h_0 := N_4^{H_0}$. Then

$$\mathbb{F}[V]^G[LC_{x_4}(h_s)^{-1}] = \mathbb{F}[x_1, x_2, N_3^G, h_s][LC_{x_4}(h_s)^{-1}]$$

where, in particular, $LC_{x_4}(h_s) \in \mathbb{F}[x_1, x_2]$. Furthermore h_s may be written as the sum of a p-polynomial in x_3 and a p-polynomial in x_4, both with coefficients in $\mathbb{F}[x_1, x_2]$.

Proof. We wish to show that $\deg_{x_4}(h_s) > 0$ is minimal in $\mathbb{F}[V]^G$ and then apply Corollary 3.4.7.

By upper-triangularity the action of any element $\sigma \in G$ on x_3 yields no x_4 terms. Since $\mathbb{F}[V]^G \subset \mathbb{F}[V]^{H_0} = \mathbb{F}[x_1, x_2, N_3^{H_0}, N_4^{H_0}]$ a lower bound for the x_4 degree of a polynomial in $\mathbb{F}[V]^G$ is $\deg_{x_4}(N_4^{H_0})$. Then we observe that

$$h_1 = \underbrace{\gcd(N_3^{H_0} \cdot \Delta_{\tau_1}, N_4^{H_0} \cdot \Delta_{\tau_1})}_{\in \mathbb{F}[x_1,x_2]}{}^{-1}(\underbrace{N_3^{H_0} \cdot \Delta_{\tau_1}}_{\in \mathbb{F}[x_1,x_2]} \underbrace{N_4^{H_0}}_{\in \mathbb{F}[x_1,x_2,x_4]} - \underbrace{N_4^{H_0} \cdot \Delta_{\tau_1}}_{\in \mathbb{F}[x_1,x_2]} \underbrace{N_3^{H_0}}_{\in \mathbb{F}[x_1,x_2,x_3]}).$$

Since $N_3^{H_0}$ and $N_4^{H_0}$ are p-polynomials in x_3 and x_4 respectively it follows that $h_1 \cdot (\tau_2 - 1) \in \mathbb{F}[x_1, x_2]$. Furthermore $\deg_{x_4}(h_1) = \deg_{x_4}(N_4^{H_0})$ and so $LC_{x_4}(h_1)$ is a polynomial in $\mathbb{F}[x_1, x_2]$. Hence if $s = 1$ the result follows.

Applying the same arguments inductively through $h_1, \ldots, h_s$, since

$$h_i = \underbrace{\gcd(N_3^{H_{i-1}} \cdot \Delta_{\tau_i}, h_{i-1} \cdot \Delta_{\tau_i})}_{\in \mathbb{F}[x_1,x_2]}{}^{-1}(\underbrace{N_3^{H_{i-1}} \cdot \Delta_{\tau_i}}_{\in \mathbb{F}[x_1,x_2]} \underbrace{h_{i-1}}_{\in \mathbb{F}[V]} - \underbrace{h_{i-1} \cdot \Delta_{\tau_i}}_{\in \mathbb{F}[x_1,x_2]} \underbrace{N_3^{H_{i-1}}}_{\in \mathbb{F}[x_1,x_2,x_3]})$$

it follows h_i is a p-polynomial in x_3 and x_4, $LC_{x_4}(h_i) \in \mathbb{F}[x_1, x_2]$ and $\deg_{x_4}(h_i) = \deg_{x_4}(h_{i-1})$. Hence by induction we obtain that $\deg_{x_4}(h_s) = \deg_{x_4}(N_4^{H_0})$ is minimal and that $LC_{x_4}(h_s) \in \mathbb{F}[x_1, x_2]$ from which the result follows. □

The issues with calculating the invariant rings hereafter lie largely in the complexity of $LC_{x_4}(h_s)$. For instance when $N_3^G \in \mathbb{F}[x_1, x_3]$ we can see that $LC_{x_4}(h_s) \in \mathbb{F}[x_1]$. Short of being constant this is one of the more preferable examples since we may then apply the SAGBI/Divide-by-x_1 algorithm (as in Section 3.1.3) to acquire a SAGBI basis. In general, however, we may not be so fortunate.

For now we consider a case whose invariant rings follow almost instantaneously from this result.

3.4.4 Invariants of Socle-Type $(2, 2)$ Vector Representations

Consider the group

$$\left\langle \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \mid c \in W \leq \mathbb{F} \right\rangle \leq GL(V_2).$$

Then $2V_2 := V_2 \oplus V_2$ has socle-type $(2, 2)$. The invariants $\mathbb{F}[mV]^G$ are often referred to as vector invariants. Generating invariants have been calculated in the case $G \cong \mathbb{Z}_p$ by Campbell and Hughes in [9] and subsequently by Campbell, Shank and Wehlau in [10]. The elementary abelian case generally remains open, but for $2V_2$ we can provide a classification here.

Proposition 3.4.9. *Let $\mathbb{Z}_p^r \cong G \leq GL(2V_2)$ be as above. Define $\delta := x_2x_3 - x_1x_4 \in \mathbb{F}[V]^G$. Then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G, \delta].$$

Furthermore these elements form a SAGBI basis subject to the sole relation

$$x_2^{p^r} N_3^G - x_1^{p^r} N_4^G = \sum_{i=0}^{r} \gamma_i (x_1x_2)^{p^r - p^i} \delta^{p^i}$$

where $N_3^G = \sum_{i=0}^{r} \gamma_i x_1^{p^r - p^i} x_3^{p^i}$.

Proof. Since δ is invariant and $\deg_{x_4}(\delta) = 1$ it follows from Proposition 3.1.12 that $\mathbb{F}[V]^G[x_1^{-1}] = \mathbb{F}[x_1, x_2, N_3^G, \delta][x_1^{-1}]$. Thus we may, and shall, apply the SAGBI/Divide-By-x_1 algorithm to the set $\{x_1, x_2, N_3^G, N_4^G, \delta\}$.

The only tête-à-tête in this set corresponds to the difference $x_2^{p^r} N_3^G - \delta^{p^r}$. Subducting this difference yields the relation above and thus the algorithm terminates. We hence acquire a SAGBI basis for $\mathbb{F}[V]^G$ by Proposition 3.1.13. $\square$

Conclusion

This section primarily focused on the hypersurface invariants of socle-length 2 representations and methods of constructing generators using the $\mathcal{R}$ polynomials. Such constructions shall prove useful for representations beyond this purview, as shall be seen almost immediately.

3.5 Invariants of Socle Length 3 Representations

Hitherto our invariant theoretic constructions revolved around the recovery functions of sections 3.2 and 3.3 and the inductive methods of Section 3.4. Here we consider a wider case in which both methods have their advantages.

Unlike socle-length 2 there is yet to exist an explicit construction for all $\mathbb{Z}_p^r$-representations with socle-length 3. Nevertheless many families existing therein have been classified. We focus on the invariants of a prominent family.

3.5.1 Hook Groups

We call a p-group $P \leq GL(V)$ a *hook group* if there exists some P-stable subspace $U < V$ with $\dim(U) = \dim(V) - 1$ such that

$$\{(\sigma - 1) \cdot v \mid \sigma \in P, v \in U\} \subseteq \mathbb{F}u$$

for some $u \in U$. This allows us to write the elements of P in the form

$$\begin{bmatrix} 1 & \gamma_{1,2} \cdots \gamma_{1,n-1} & \gamma_{1,n} \\ & & \gamma_{2,n} \\ 0 & I_{n-2} & \vdots \\ & & \gamma_{n-1,n} \\ 0 & 0 & 1 \end{bmatrix}.$$

We have seen hook groups as Sylow p-subgroups of the finite orthogonal groups in Section 2.6.4. From the perspective of invariant theory the fact that hook groups consist solely of bireflections is of intrigue to the Cohen-Macaulay problem.

One can see from the structure of hook groups that their socle series has length 2 or 3. The latter demands a socle-type of the form $(m_1, m_2, 1)$ and dual-type $(n_1, n_2, 1)$. Such p-groups may not necessarily be elementary abelian.

Applying the techniques of sections 2.3.1 and 2.5 we can wholly classify the elementary abelian hook groups as follows: The images of $\mathbb{Z}_p^r$-representations with socle-type $(\mu_1, \mu_2, 1)$ and dual-type $(\nu_1, \nu_2, 1)$ exist up to conjugacy in the image of $\sigma : (\mathbb{F}^{n+m_1m_3-1}, +) \to GL_n(\mathbb{F})$ where

$$\sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 2v_1(\underline{c}) & v_3(\underline{c}) & c_{m_1+m_2+1} + v_1(\underline{c})v_1(\underline{c})^T \\ 0 & I_{m_1} & 0 & C & v_2(\underline{c})^T \\ 0 & 0 & I_{m_2} & 0 & v_1(\underline{c})^T \\ 0 & 0 & 0 & I_{m_3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$v_1(\underline{c}) = [c_{m_2}, \ldots, c_1], \quad v_2(\underline{c}) = [c_{m_1+m_2}, \ldots, c_{m_2+1}], \quad v_3(\underline{c}) = [c_{m_1+m_2+2}, \ldots, c_{n-1}],$$

with $C \in M_{m_1,m_3}(\mathbb{F})$ independent. Socle-length 3 elementary abelian hook groups are these representations for which $C \equiv 0$. Those with socle length 2 have $m_2 = 0$ and are automatically elementary abelian.

3.5.2 Hyperplane Invariants of Hook Groups

Explicitly given an ordered generating set $\sigma_1, \ldots, \sigma_r$ for an elementary abelian hook group G we denote

$$G_i^j := \langle \sigma_i, \sigma_{i+1}, \ldots, \sigma_j \rangle$$

with $G^j := G_1^j$ for simplicity. Choose this generating set and $\rho_2 \in \mathbb{N}$ such that $G_{\rho_2+1}^r$ is the stabiliser of $\mathbb{F}[x_1, \ldots, x_{n-1}]$. Thus $\mathbb{F}[x_1, \ldots, x_{n-1}]^G = \mathbb{F}[x_1, \ldots, x_{n-1}]^{G^{\rho_2}}$ and this restricted action of G^{ρ_2} is that of a hyperplane group. Thus we use the work of Section 3.2 as follows.

If $\dim(\mathrm{Soc}_1 V^*) = m$ then define the $c_{i,j} \in \mathbb{F}$ by

$$x_{m+i} \cdot (\sigma_j - 1) =: x_1 c_{i,j}.$$

Thence for $k = 1, \ldots, \rho_2$ construct the matrices $C^{(k)} := [c_{i,j}] \in M_{n-m-1,k}(\mathbb{F})$ which generate G^k acting on $\mathbb{F}[x_1, \ldots, x_{n-1}]$.

As per Section 3.2 we choose $C^{(k)}$-compatible sequences $\underline{\alpha}^{(k)} = (\alpha_1^{(k)}, \ldots, \alpha_{n-m-1}^{(k)}) \in \mathbb{N}_0^{n-m-1}$. Using these we construct the matrices $\Gamma^{(k)}$ as in Section 3.2.2 with minors

$$\widetilde{f_i^{(k)}} := \Gamma^{(k)}\big[1, \ldots, k, k+i \bigm| 1, \ldots, k+1\big]$$

and so the invariants $f_i^{(k)}$ by clearing the denominator of $\widetilde{f_i^{(k)}}$. Then by Theorem

3.2.2 we have

$$\mathbb{F}[x_1, \ldots, x_{n-1}]^{G^k} = \mathbb{F}[x_1, \ldots, x_m, f_1^{(k)}, \ldots, f_{n-m-1}^{(k)}].$$

Remark. We shall require use of all $\Gamma^{(k)}$ in later sections. Astute readers shall recall that the process of constructing a $C^{(\rho_2)}$-compatible $\underline{\alpha}^{\rho_2}$ already requires constructing $C^{(k)}$-compatible $\underline{\alpha}^{(k)}$ for all $k = 1, \ldots, \rho_2 - 1$. Hence little additional calculation is required to construct these.

It remains to reintroduce x_n into consideration. It follows from Proposition 3.1.12 that, should we find an $F \in \mathbb{F}[V]^G$ which has minimal positive degree in x_n, we we have

$$\mathbb{F}[V]^G[LC_{x_n}(F)^{-1}] = \mathbb{F}[x_1, \ldots, x_m, f_1^{(\rho_2)}, \ldots, f_{n-m-1}^{(\rho_2)}, F][LC_{x_n}(F)^{-1}]$$

3.5.3 Integral Hook Invariants

Momentarily suppose that our elementary abelian hook group $G \leq GL(V)$ has socle length 3. Under the appropriate basis for V our elements are of the form

$$\sigma(\underline{c}) = \begin{bmatrix} 1 & 0 & 2v_1(\underline{c}) & v_3(\underline{c}) & c_{m_1+m_2+1} + v_1(\underline{c})v_1(\underline{c})^T \\ 0 & I_{m_1} & 0 & 0 & v_2(\underline{c})^T \\ 0 & 0 & I_{m_2} & 0 & v_1(\underline{c})^T \\ 0 & 0 & 0 & I_{m_3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

for $v_1(\underline{c}) := [c_{m_2}, \ldots, c_1], v_2(\underline{c}) = [c_{m_1+m_2}, \ldots, c_{m_2+1}]$ and $v_3(\underline{c}) = [c_{m_1+m_2+2}, \ldots, c_{n-1}]$.

Such representations require elements $\sigma(\underline{c}_1), \ldots, \sigma(\underline{c}_{m_2})$ such that the matrix $[v_1(\underline{c}_1)^T \ \cdots v_1(\underline{c}_{m_2})^T]$ is invertible. By choosing our basis appropriately we ensure that these each satisfy $v_3(\underline{c}_i) = 0$ and $(\underline{c}_i)_{m_1+m_2+1} = 0$. Then the polynomial

$$\delta := x_1 x_n - \left(\sum_{i=m_3+2}^{m_3+m_2+1} x_i^2 \right)$$

is invariant under the action of these elements, since

$$\delta \cdot (\sigma(\underline{c}) - 1) = v_3(\underline{c}) \begin{bmatrix} x_{m_3+1} \\ \vdots \\ x_2 \end{bmatrix} x_1 + c_{m_1+m_2+1} x_1^2 \in \mathbb{F}[x_1, \ldots, x_{m_3+1}] \subset \mathbb{F}[V]^G.$$

In summary by choosing our basis appropriately we ensure that the subgroup of elements fixing δ is at least of order p^{m_2}. With foresight we suggest that the more computationally convenient choices are those for which this set is maximised.

If instead our representation has socle-length 2 our elements take the form

$$\gamma := \left[\begin{array}{c|cccc} \multirow{4}{*}{I_{m_1+1}} & \gamma_{1,m_1+2} & \cdots & \gamma_{1,n-1} & \gamma_{1,n} \\ & 0 & \cdots & 0 & \gamma_{2,n} \\ & \vdots & \vdots & \vdots & \vdots \\ & 0 & \cdots & 0 & \gamma_{m_1+1,n} \\ \hline 0 & & & I_{m_3+1} & \end{array}\right].$$

Such a representation must contain an element whose Jordan normal form contains two 2×2 Jordan blocks. We then choose our basis to fix

$$\begin{bmatrix} 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in G.$$

This element then leaves $\delta' := x_1x_n - x_2x_{n-1}$ invariant since

$$\delta' \cdot (\gamma - 1) = x_1 \sum_{i=1}^{m_3+1} \gamma_{1,n+1-i}x_i - \gamma_{2,n}x_1x_2 \in \mathbb{F}[x_1, \ldots, x_{m_3+1}] \subset \mathbb{F}[V]^G.$$

Hence for any given elementary abelian hook group $G \leq GL(V)$ we may choose a socle-conforming basis for V and a homogeneous $\delta \in \mathbb{F}[V]$ such that

- $LT_{x_n}(\delta) = x_1x_n$,
- $\delta \cdot (\sigma - 1) \in \mathbb{F}[x_1, \ldots, x_m] \subset \mathbb{F}[V]^G$ for all $\sigma \in G$,
- $\{\sigma \in G \mid \delta \cdot \sigma = \delta\}$ is nontrivial.

Combining this with the thus far obtained hyperplane work and the inductive $\mathcal{R}$ constructions of Section 3.4.2 we may construct invariant field generators as follows.

3.5.4 Invariant Fields of Hook Groups

Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ be an elementary abelian hook group. Let $m := m_3 + 1 = \dim(\mathrm{Soc}(V^*))$ and choose a basis for V and a generating set $\sigma_1, \ldots, \sigma_r$ for G such that

$$G^{\rho_1} = \langle \sigma_1, \ldots, \sigma_{\rho_1} \rangle = \{\sigma \in G \mid \delta \cdot \sigma = \delta\}$$

for δ as given in Section 3.5.3 and

$$G^r_{\rho_2+1} = \langle \sigma_{\rho_2+1}, \ldots, \sigma_r \rangle = \{\sigma \in G \mid x_i \cdot \sigma = x_i,\ \forall i = 1, \ldots, n-1\}.$$

Note that $G^r_{\rho_2+1}$ may be trivial but we choose our basis for V such that G^{ρ_1} is not. Denote by $\Delta_i := \sigma_i - 1 \in \mathbb{F}G$.

Define also the G^k-invariants $f_1^{(k)}, \ldots, f_{n-m-1}^{(k)}$ for $k = 1, \ldots, \rho_2$ as given in Section 3.5.2.

We iteratively transform δ from being an invariant under G^{ρ_1} to being invariant under larger G^j as follows: Defining $\delta_{\rho_1} := \delta$ we iterate

$$\begin{aligned} \delta_{j+1} &:= \mathcal{R}(\delta_j, f_{i_j}^{(j)}, \sigma_{j+1}) \\ &= \gcd(\delta_j \cdot \Delta_{j+1}, f_{i_j}^{(j)} \cdot \Delta_{j+1})^{-1} \begin{vmatrix} \delta_j & \delta_j \cdot \Delta_{j+1} \\ f_{i_j}^{(j)} & f_{i_j}^{(j)} \cdot \Delta_{j+1} \end{vmatrix} \end{aligned}$$

for $j = \rho_1, \ldots, \rho_2 - 1$, where the i_j are chosen such that $f_{i_j}^{(j)} \cdot \Delta_{j+1} \neq 0$. Then we have the following.

Theorem 3.5.1. *Let $G \leq GL(V)$ be as above. Then*

$$\mathbb{F}[V]^G[x_1^{-1}] = \mathbb{F}[x_1, \ldots, x_m, f_1^{(\rho_2)}, \ldots, f_{n-m-1}^{(\rho_2)}, N^G_{G^{\rho_2}}(\delta_{\rho_2})][x_1^{-1}].$$

Hence applying the SAGBI/Divide-by-x_1 algorithm to the set

$$\{x_1, \ldots, x_m, f_1^{(\rho_2)}, \ldots, f_{n-m-1}^{(\rho_2)}, N^G_{G^{\rho_2}}(\delta_{\rho_2}), N^G_{m+1}, \ldots, N^G_n\}$$

shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

Proof. Since $G^r_{\rho_2+1}$ acts non-trivially on all x_i besides x_n it is a Nakajima subgroup. Hence

$$\mathbb{F}[V]^G \subset \mathbb{F}[V]^{G^r_{\rho_2+1}} = \mathbb{F}[x_1, \ldots, x_{n-1}, N^G_{G^{\rho_2}}(x_n)].$$

In particular the minimal possible x_n degree for an element in $\mathbb{F}[V]^G$ is $\deg(N^G_{G^{\rho_2}}(x_n)) =$

$|G^r_{\rho_2+1}|$. Thus were we to show that

$$\mathrm{LM}_{x_n}(\delta_{\rho_2}) = x_1^{\alpha} x_n$$

for some $\alpha \in \mathbb{N}$ then $N^G_{G^{\rho_2}}(\delta_{\rho_2}) \in \mathbb{F}[V]^G$ will have minimal positive degree in x_n and lead coefficient a power of x_1. The result would then follow from Proposition 3.1.12 and Proposition 3.1.13.

By construction $\delta_{\rho_1} \in \mathbb{F}[V]^{G^{\rho_1}}$ is such that

$$\mathrm{LM}_{x_n}(\delta_{\rho_1}) = x_1 x_n \quad \text{and} \quad \delta_{\rho_1} \cdot (\sigma - 1) \in \mathbb{F}[x_1, \ldots, x_m] \subset \mathbb{F}[V]^G$$

for all $\sigma \in G$. We use this as a base of induction to show that each $\delta_j \in \mathbb{F}[V]^{G^j}$ satisfies $\mathrm{LM}_{x_n}(\delta_j) = x_1^{\alpha_j} x_n$ and that $\delta_j \cdot (\sigma - 1) \in \mathbb{F}[x_1, \ldots, x_m]$ for all $\sigma \in G$.

Recall that

$$\delta_{j+1} = \underbrace{\gcd(\delta_j \cdot \Delta_{j+1}, f^{(j)}_{i_j} \cdot \Delta_{j+1})}_{\in \mathbb{F}[x_1]}{}^{-1} \Big[\delta_j \underbrace{(f^{(j)}_{i_j} \cdot \Delta_{j+1})}_{\in \mathbb{F}[x_1]} - f^{(j)}_{i_j} \underbrace{(\delta_j \cdot \Delta_{j+1})}_{\in \mathbb{F}[x_1, \ldots, x_m]}\Big].$$

since the $f^{(j)}_{i_j}$ are p-polynomials in $x_{m+1}, \ldots, x_{n-1}$. In particular both $f^{(j)}_{i_j} \cdot \Delta_{j+1}$ and $\delta_j \cdot \Delta_{j+1}$ are invariant under G^{j+1} and so $\delta_{j+1} \in \mathbb{F}[V]^{G^{j+1}}$. By induction we see $\mathrm{LM}_{x_n}(\delta_{j+1}) = x_1^{\alpha_{j+1}} x_n$. Furthermore for an arbitrary $\sigma \in G$ and $\Delta := \sigma - 1$

$$\delta_{j+1} \cdot \Delta = \underbrace{\gcd(\delta_j \cdot \Delta_{j+1}, f^{(j)}_{i_j} \cdot \Delta_{j+1})^{-1}}_{\in \mathbb{F}[x_1]} \Big[\underbrace{(\delta_j \cdot \Delta)}_{\in \mathbb{F}[x_1, \ldots, x_m]} \underbrace{(f^{(j)}_{i_j} \cdot \Delta_{j+1})}_{\in \mathbb{F}[x_1]} - \underbrace{(f^{(j)}_{i_j} \cdot \Delta)}_{\in \mathbb{F}[x_1]} \underbrace{(\delta_j \cdot \Delta_{j+1})}_{\in \mathbb{F}[x_1, \ldots, x_m]}\Big]$$

and so $\delta_{j+1} \cdot \Delta \in \mathbb{F}[x_1, \ldots, x_m]$. Thus by induction for each j we have $\delta_j \in \mathbb{F}[V]^{G^j}$ and at each stage their lead monomials in x_n are $\mathrm{LM}_{x_n}(\delta_j) = x_1^{\alpha_j} x_n$. In particular this holds for δ_{ρ_2} and thus the result follows. $\square$

We have now procured generators for the invariant field of an arbitrary elementary abelian hook group, and thence a method of procuring a SAGBI basis. We note that, for instance, all representations with socle-type $(2, 1, 1)$ and $(1, 2, 1)$ are hook groups and thus this process directly aids our efforts with the four-dimensional representations.

3.5.5 Cohen-Macaulayness in Hook Groups

Since a group must be generated by bireflections to have a Cohen-Macaulay invariant ring it follows that hook groups, consisting entirely of bireflections, are of interest in this area. Whilst all three-dimensional $\mathbb{Z}_p^r$-representations necessarily have Cohen-Macaulay invariants – as either hyperplane groups, Nakajima groups or with codimension 2 socle – those in higher dimensions have no such assurance.

From Section 2.2 we may associate a socle tabloid to any given representation, a combinatorial generalisation object encoding the socle- and dual-type. Unfortunately this section is devoted to illustrating the following.

Theorem 3.5.2. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$. Then whether or not $\mathbb{F}[V]^G$ is Cohen-Macaulay is not discernible from the socle tabloid of G alone.*

By Section 2.5.2 the image of any $\mathbb{Z}_p^r$-representation with socle-type $(1, 2, 1)$ and dual-type $(2, 1, 1)$ exists up to conjugacy in the image of the homomorphism

$$\sigma : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}), \quad \sigma(c_1, c_2, c_3) = \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 + c_3 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We consider groups of the form

$$G := \left\langle \sigma(1,0,0) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \sigma(c_1, c_2, 0) = \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$$

for $c_2 \neq 0$. It is important here to distinguish between the cases where $c_1^p = c_1$ or otherwise.

Case 1: $c_1^p = c_1$

We may easily generalise this case as follows. Consider a representation of the form

$$G := \langle \sigma_1 := \sigma(1,0,0), \sigma(0, c_1, 0), \dots, \sigma(0, c_{r-1}, 0) \rangle \cong \mathbb{Z}_p^r.$$

In particular the subgroup $H := \langle \sigma(0, c_1, 0), \ldots, \sigma(0, c_{r-1}, 0)\rangle$ is Nakajima and thus $\mathbb{F}[V]^H = \mathbb{F}[x_1, x_2, x_3, N_4^H]$. Denote $N_4^H = \sum_{i=0}^{r-1} \gamma_i x_4^{p^i} x_2^{p^{r-1}-p^i}$. Since

$$x_1 x_4^{p^i} \cdot (\sigma_1 - 1) = x_1(2x_3^{p^i} - x_1^{p^i}) = (2x_3^{p^i+1} - x_1^{p^i-1}x_3^2) \cdot (\sigma_1 - 1)$$

we construct the invariant

$$\begin{aligned} F &:= \mathcal{R}\left(N_4^H,\ \sum_{i=0}^{r-1} \gamma_i (2x_3^{p^i+1} - x_1^{p^i-1}x_3^2) x_2^{p^{r-1}-p^i},\ \sigma_1\right) \\ &= x_1 N_4^H - \sum_{i=0}^{r-1} \gamma_i (2x_3^{p^i+1} - x_1^{p^i-1}x_3^2) x_2^{p^{r-1}-p^i} \in \mathbb{F}[V]^G \end{aligned}$$

which has minimal x_4-degree in $\mathbb{F}[V]^G$. Thus it follows that applying the SAGBI/Divide-by-x_1 algorithm to the set $\{x_1, x_2, N_3^G, N_4^G, F\}$ shall yield a SAGBI basis.

Proposition 3.5.3. *Let $\mathbb{Z}_p^r \cong G \leq GL(V)$ and $F \in \mathbb{F}[V]^G$ be given as above. Then $\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G, F]$.*

Whilst one might wish to apply the SAGBI/Divide-By-x_1 algorithm here the resulting relation is far from clean, soon growing expensive in both time and pencil lead. We instead adapt a proof that the invariant ring of the indecomposable 3-dimensional cyclic module is a hyperplane. Despite the difference in rank and dimension the proof is almost identical to that given in [12, Section 4.10.2] as the action of σ_1 on $\mathbb{F}[V]^H$ is the action of a 3×3 Jordan block on a polynomial algebra, albeit ill-graded.

Since $\mathbb{F}[V]^G \subset \mathbb{F}[V]^H = \mathbb{F}[x_1, x_2, x_3, N_4^H]$ we informally refer to a G-invariant's N_4^H-degree. Hence any invariant in $\mathbb{F}[V]^G$ shall have x_4-degree of the form dp^{r-1} for $d \in \mathbb{N}_0$.

We begin the proof by showing that any invariant with small enough N_4^H degree may be written as a polynomial in $\{x_1, x_2, N_3^G, F\}$.

Lemma 3.5.4. *Let $f \in \mathbb{F}[V]^G$ have x_4-degree $jp^{r-1} < p^r$. Write*

$$f = \sum_{i=0}^{j} \alpha_i (N_4^H)^i \quad \textit{for} \quad \alpha_i \in \mathbb{F}[x_1, x_2, x_3, F]$$

such that $\deg_{x_4}(\alpha_i (N_4^H)^i) < jp^{r-1}$ for each $i < j$. Then $f \in \mathbb{F}[x_1, x_2, N_3^G, F]$.

Proof. We prove by induction on j where

$$f = \alpha_0 \in \mathbb{F}[x_1, x_2, x_3, F]^G = \mathbb{F}[x_1, x_2, N_3^G, F]$$

acts as a base of induction. For $j > 0$ denote $\Delta := \sigma_1 - 1 \in \mathbb{F}G$ so that

$$f \cdot \Delta = \sum_{i=0}^{j} (\alpha_i \cdot \Delta)(N_4^H)^i + (\alpha_i \cdot \sigma_1)\big((N_4^H)^i \cdot \Delta\big) = 0.$$

Since our action is upper-triangular and N_4^H is a p-polynomial in x_4 we have $\deg_{x_4}((N_4^H)^j \cdot \Delta) < \deg_{x_4}((N_4^H)^j)$. We assumed that this was the only term of our expression of f which achieves this degree in x_4 and so $\alpha_j \cdot \Delta = 0$. Consequently $\alpha_j \in \mathbb{F}[V]^G$.

Alternatively we rewrite $f \cdot \Delta$ to acquire

$$\begin{aligned} f \cdot \Delta &= (\alpha_j \cdot \Delta)\left((N_4^H)^j \cdot \sigma_1\right) + \alpha_j\left((N_4^H)^j \cdot \Delta\right) + (\alpha_{j-1} \cdot \Delta)(N_4^H)^{j-1} + \dots \\ &= j\alpha_j (N_4^H)^{j-1}(N_4^H \cdot \Delta) + (\alpha_{j-1} \cdot \Delta)(N_4^H)^{j-1} + (\text{lower } x_4 \text{ order terms}) = 0 \end{aligned}$$

and so $\alpha_j(N_4^H \cdot \Delta) = (-j^{-1}\alpha_{j-1}) \cdot \Delta$. Thus

$$\alpha_j \in (\mathbb{F}[x_1, x_2, x_3, F]) \cdot \Delta \subset x_1\mathbb{F}[x_1, x_2, x_3, F]$$

since $F \in \mathbb{F}[V]^G$ and $x_3 \cdot \Delta = x_1$. So we write $\alpha_j = x_1\widetilde{\alpha_j}$ for some $\widetilde{\alpha}_j \in \mathbb{F}[x_1, x_2, x_3, F]$. We also, for convenience, write $\widetilde{F} = F - x_1 N_4^H \in \mathbb{F}[x_1, x_2, x_3]$. Then

$$\alpha_j (N_4^H)^j = \widetilde{\alpha}_j (x_1 N_4^H)(N_4^H)^{j-1} = \widetilde{\alpha}_j (F - \widetilde{F})(N_4^H)^{j-1}$$

and so

$$f = (\widetilde{\alpha}_j (F - \widetilde{F}) + \alpha_{j-1})(N_4^H)^{j-1} + \sum_{i=0}^{j-1} \alpha_i (N_4^H)^i$$

and thus the result follows by applying the induction hypothesis. □

What remains is to examine those whose x_4-degree meets or exceeds N_4^G. Since N_4^G is monic we may reduce this case by straight-forward 'division'.

Lemma 3.5.5. *If $f \in \mathbb{F}[V]^G$ is of the form $f = f_Q N_4^G + f_R$ for $\deg_{x_4}(f_R) < \deg_{x_4}(N_4^G)$ then $f_Q, f_R \in \mathbb{F}[V]^G$.*

Proof. Let $\tau \in G$. Then $f = f \cdot \tau = (f_Q \cdot \tau)N_4^G + (f_R \cdot \tau)$. Since our action is upper-triangular $\deg_{x_4}(f_R) = \deg_{x_4}(f_R \cdot \tau)$. Then by the uniqueness of remainders $f_R = f_R \cdot \tau$ and thus $f_Q = f_Q \cdot \tau$. $\square$

Hence we hereafter write our invariants in the form $f = f_Q N_4^G + f_R$. Using Lemma 3.5.4 the remainder may be written

$$f_R = \sum_{i=0}^{p-1} f_{R,i}(N_4^H)^i, \qquad f_{R,i} \in \mathbb{F}[x_1, x_2, x_3].$$

These coefficients are of interest for the following reason.

Lemma 3.5.6. *Let $f = f_Q N_4^G + \sum_{i=0}^{p-1} f_{R,i}(N_4^H)^i$ be as above. Then $x_1^i \mid f_{R,i}$.*

Proof. By Lemma 3.5.5 we know f_R is invariant and thus

$$f_R = f_R \cdot \sigma_1 = \sum_{i=0}^{p-1}(f_{R,i} \cdot \sigma_1)\left((N_4^H)^i \cdot \sigma_1\right) = \sum_{i=0}^{p-1}(f_{R,i} \cdot \sigma_1)\sum_{j=0}^{i}\binom{i}{j}(N_4^H)^j\left(N_4^H \cdot \Delta\right)^{i-j}.$$

We compare coefficients of powers of N_4^H in the above equation to that in f_R itself. For a fixed j we have

$$f_{R,j} = \sum_{i=j}^{p-1}\binom{i}{j}\left(N_4^H \cdot \Delta\right)^{i-j}(f_{R,i} \cdot \sigma_1).$$

Rearranging this and acting by σ_1^{-1} we acquire

$$(j+1)(N_4^H \cdot (\sigma_1^{-1} - 1))f_{R,j+1} = f_{R,j} \cdot (1 - \sigma_1^{-1}) + \sum_{i=j+2}^{p-1}\binom{i}{j}\left((N_4^H \cdot \Delta)^{i-j} \cdot \sigma_1^{-1}\right) f_{R,i}. \tag{3.2}$$

Since x_1, x_2 are invariant under σ_1 and $x_3 \cdot \sigma^{-1} = -x_1$ it follows that

$$\mathbb{F}[x_1, x_2, x_3] \cdot \sigma_1^{-1} \subset x_1\mathbb{F}[x_1, x_2, x_3].$$

Hence if $f_{R,j}$ is divisible by x_1^k then $x_1^{k+1} \mid f_{R,j} \cdot (\sigma_1^{-1} - 1)$. Coupling this with an induction-reminiscent assumption and (3.2) we see that

$$x_1^k \mid f_{R,j} \quad \text{and} \quad x_1^{k+1} \mid f_{R,i} \;\; \forall i = j+2, \ldots, p-1 \quad \Longrightarrow \quad x_1^{k+1} \mid f_{R,j+1}. \tag{3.3}$$

Using this and proceeding by induction on t shows that $x_1^t \mid f_{R,t}$ and $x_1^{t+1} \mid f_{R,i}$ for all $t+1 < i < p$, thus providing the result.

For $t = 0$ we take (3.2) with $j = p-2$

$$(p-1)\left(N_4^H \cdot (\sigma_1^{-1} - 1)\right) f_{R,p-1} = f_{R,p-2}(1 - \sigma_1^{-1})$$

whose right hand side is divisible by x_1. However $N_4^H \cdot (\sigma_1^{-1} - 1)$ is not, having lead monomial $x_3^{p^{r-1}}$. Thus it follows that $x_1 \mid f_{R,p-1}$.

Applying implication (3.3) to this with $k = 0$ and $j = p-3$ yields $x_1 \mid f_{R,p-2}$. Continuing to apply (3.3) for decreasing values of k eventually yields $x_1 \mid f_{R,i}$ for all $i = 1, \ldots, p-1$. This acts as our base of induction.

Now assume $x_1^{t-1} \mid f_{R,t-1}$ and $x_1^t \mid f_{R,i}$ for all $i = t+1, \ldots, p-1$ for some value of t.

- Applying (3.3) with $k = t-1, j = t-1$ yields $x_1^t \mid f_{R,t}$.
- Applying (3.3) with $k = t$ for j from $p-3$ down to $t+1$ (in that order) yields $x_1^{t+1} \mid f_{R,j}$.

Thence the result follows by induction. □

With these lemmas in place we are now equipped to prove Proposition 3.5.3.

Proof of Proposition 3.5.3. Given $f \in \mathbb{F}[V]^G$ write it in the form $f = f_Q N_4^G + f_R$ for $\deg_{x_4}(f_R) < p^r$, where we view elements of $\mathbb{F}[V]^G \subset \mathbb{F}[V]^H$ as polynomials in N_4^H. Our proposed generators are not exempt from this and so we write

$$F = x_1 N_4^H + \widetilde{F}, \qquad \widetilde{F} \in \mathbb{F}[x_1, x_2, x_3].$$

By Lemma 3.5.6 we may write $f_R = \sum_i \widetilde{f_{R,i}} (x_1 N_4^H)^i$ and so we may nontrivially divide f_R by F. Thus we write $f_R = \widetilde{f_Q} F + \widetilde{f_R}$ where $\deg_{x_4}(\widetilde{f_R}) = 0$. Utilising the same arguments as Lemma 3.5.5 indicate that $\widetilde{f_Q}, \widetilde{f_R} \in \mathbb{F}[V]^G$.

So we have

$$f = f_Q N_4^G + \widetilde{f_Q} F + \widetilde{f_R}, \qquad \widetilde{f_R} \in \mathbb{F}[x_1, x_2, x_3]^G = \mathbb{F}[x_1, x_2, N_3^G].$$

All that remains to show is that f_Q and $\widetilde{f_Q}$ belong to $\mathbb{F}[x_1, x_2, N_3^G, N_4^G, F]$.

If $\deg_{x_4}(\widetilde{f_Q}) = 0$ then $\widetilde{f_Q} \in \mathbb{F}[x_1, x_2, x_3]^G = \mathbb{F}[x_1, x_2, N_3^G]$ as required. Otherwise apply the same deconstruction of f to $\widetilde{f_Q}$ to acquire $\widetilde{f_Q} = \widetilde{\widetilde{f_Q}}F + \widetilde{\widetilde{f_R}}$. Then $\deg_{x_4}(\widetilde{\widetilde{f_Q}}) < \deg_{x_4}(\widetilde{f_Q})$. A simple induction argument then yields what we desire.

The same argument may be applied to f_Q: If $\deg_{x_4}(f_Q) = 0$ then the result holds. Otherwise deconstruct $f_Q = f_Q' N_4^G + \widetilde{f_Q}' F + \widetilde{f_R}'$. For this we have shown that $\widetilde{f_Q}', \widetilde{f_R}' \in \mathbb{F}[x_1, x_2, N_3^G, F]$ and $\deg_{x_4}(f_Q') < \deg_{x_4}(f_Q)$ from which the result follows by induction.

We have shown that $f_Q, \widetilde{f_Q}, \widetilde{f_R} \in \mathbb{F}[x_1, x_2, N_3^G, N_4^G, F]$ and thus so must f. $\square$

Our aim was to determine the invariants of the group $G = \langle \sigma(1,0,0), \sigma(c_1, c_2, 0)\rangle$ with $c_1^p = c_1$. This being the case we may instead write $G = \langle \sigma(1,0,0), \sigma(0, c_2, 0)\rangle$. Thus the invariant theory of G is covered by Proposition 3.5.3 and it consequently has Cohen-Macaulay invariants.

It would be remiss of us to include this example and Theorem 3.5.2 if the $c_1^p \neq c_1$ case were to have the same fate.

Case 2: $c_1^p \neq c_1$

Let $\mathbb{Z}_p^2 \cong G \leq GL(V)$ be of the form

$$\left\langle \sigma_1 = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$$

for which $c_2 \neq 0$ and $c_1^p \neq c_1$. We show that $\mathbb{F}[V]^G$ is not Cohen-Macaulay. For convenience define $\delta := x_3^2 - x_1x_4 \in \mathbb{F}[V]^{\sigma_1}$. Then

$$F_1 := \mathcal{R}(N_3^{\sigma_1}, \delta, \sigma_2) \in \mathbb{F}[V]^G$$

is an invariant with degree 1 in x_4. Thus $\mathbb{F}[V]^G[x_1^{-1}] = \mathbb{F}[x_1, x_2, N_3^G, F_1][x_1^{-1}]$ and so applying the SAGBI/Divide-by-x_1 algorithm to $\{x_1, x_2, N_3^G, N_4^G, F_1\}$ shall yield a SAGBI basis.

For convenience denote $H = \langle \sigma_1 \rangle$. We enlist the aid of the invariant

$$F_2 := N_H^G(\delta) = x_3^{2p} - x_1^p x_4^p - (c_2x_1x_2)^{p-1}(x_3^2 - x_1x_4).$$

This has lead term x_3^{2p} and thus we consider the h.s.o.p $\{x_1, x_2, F_2, N_4^G\}$ for $\mathbb{F}[V]^G$. One may easily verify that

$$F_1 F_2^{\frac{p-1}{2}} - c_2 x_2 N_3^G \in x_1^{p-1}\mathbb{F}[V]^G.$$

Then in the image of the quotient $\mathbb{F}[V]^G/(x_1, x_2)\mathbb{F}[V]^G$ yields $\overline{F_1 F_2^{\frac{p-1}{2}}} = 0$. We then use this to show that F_2 is a zero divisor in $\mathbb{F}[V]^G/(x_1, x_2)\mathbb{F}[V]^G$ and thus $\{x_1, x_2, F_2\}$ is not a regular sequence. Since it is a partial h.s.o.p for $\mathbb{F}[V]^G$ it would then follow by Lemma 3.1.11 that $\mathbb{F}[V]^G$ is not Cohen-Macaulay.

We need only show that $\overline{F_1 F_2^{\frac{p-3}{2}}}$ is not in the ideal. This requires the following slightly technical lemmas.

Lemma 3.5.7. *Let $m = \delta^{a_1}(N_3^H)^{a_2}(N_4^H)^{a_3}$ for $a_1 < p$ and $\deg(m) < p^2$.*

1. *If $a_3 > 0$ then $LM(m \cdot (\sigma_2 - 1)) = x_3^{2a_1+p(a_2+1)} x_4^{p(a_3-1)}$,*
2. *If $a_3 = 0$ and $a_1 > 0$ then $LM(m \cdot (\sigma_2 - 1)) = x_1 x_2 x_3^{2(a_1-1)+pa_2}$,*
3. *If $a_3 = a_1 = 0$ then $LM(m \cdot (\sigma_2 - 1)) = x_1^p x_3^{p(a_2-1)}$.*

Proof. Note that

$$\begin{aligned} m \cdot (\sigma_2 - 1) &= (\delta \cdot \sigma_2)^{a_1}(N_3^H \cdot \sigma_2)^{a_2}(N_4^H \cdot \sigma_2)^{a_3} - \delta^{a_1}(N_3^H)^{a_2}(N_4^H)^{a_3} \\ &= \sum_{\substack{i=0,\ldots,a_1 \\ j=0,\ldots,a_2 \\ k=0,\ldots,a_3 \\ (i,j,k)\neq(a_1,a_2,a_3)}} \binom{a_1}{i}\binom{a_2}{j}\binom{a_3}{k}\delta^i(\delta\cdot\Delta)^{a_1-i}(N_3^H)^j(N_3^H\cdot\Delta)^{a_2-j}(N_4^H)^k(N_4^H\cdot\Delta)^{a_3-k}. \end{aligned}$$

The largest term possible in the format of the summand above would be for $(i, j, k) = (a_1, a_2, a_3)$ which is, alas, absent. Hence the process of determining the largest term possible is a game of damage limitation, in choosing which of i, j or k to reduce by 1 in order to minimise the drop in term order.

Note that $LM(\delta \cdot \Delta) = x_1 x_2$, $LM(N_3^H \cdot \Delta) = x_1^p$ and $LM(N_4^H \cdot \Delta) = x_3^p$. Hence if $a_3 \neq 0$ the best case is to examine $(i, j, k) = (a_1, a_2, a_3 - 1)$ and thus

$$\begin{aligned} LM(m \cdot (\sigma_2 - 1)) &= LM(\delta^{a_1} N_3^{a_2} N_4^{a_3-1}(N_4^H \cdot \Delta)) \\ &= x_3^{2a_1} x_3^{pa_2} x_4^{p(a_3-1)} x_3^p = x_3^{2a_1+p(a_2+1)} x_4^{p(a_3-1)}. \end{aligned}$$

Otherwise if $a_3 = 0$ but $a_1 \neq 0$ then the next best case is to take $(i, j, k) = (a_1 - 1, a_2, 0)$ to acquire

$$LM(m \cdot (\sigma_2 - 1)) = LM(\delta^{a_1 - 1}(\delta \cdot \Delta)N_3^{a_2}) = x_3^{2(a_1-1)} x_1 x_2 x_3^{pa_2} = x_1 x_2 x_3^{2(a_1-1)+pa_2}.$$

Finally if $a_1 = a_3 = 0$ then our only choice is to take $(i, j, k) = (0, a_2 - 1, 0)$ with lead monomial

$$LM(m \cdot (\sigma_2 - 1)) = LM(N_3^{a_2 - 1}(N_3^H \cdot \Delta)) = x_1^p x_3^{p(a_2-1)}.$$

□

Lemma 3.5.8. *Let G and F_1, F_2 be as above. Then $F_1 F_2^{\frac{p-3}{2}} \notin (x_1, x_2)\mathbb{F}[V]^G$.*

Proof. Note that $LM(F_1 F_2^{\frac{p-3}{2}}) = x_2 x_3^{p(p-2)}$. Hence if $F_1 F_2^{\frac{p-3}{2}} \in (x_1, x_2)\mathbb{F}[V]^G$ then there must exist an invariant $f \in \mathbb{F}[V]^G$ with lead term $x_3^{p(p-2)}$. We prove that no such element exists.

Since $\mathbb{F}[V]^G \subset \mathbb{F}[V]^H = \mathbb{F}[x_1, x_2, N_3^H, N_4^H, \delta]$ any invariant of G may be written in terms of these given generators. The only relation governing these allows us to ensure that any power of δ of at least p may be rewritten in other terms.

Suppose $f \in \mathbb{F}[V]^G$ has lead term $x_3^{p(p-2)}$. Then we may write an expression for f in the generators of $\mathbb{F}[V]^H$ with $(N_3^H)^{p-2}$ as a 'lead term'. Observe that $LM((N_3^H)^{p-2} \cdot (\sigma_2 - 1)) = x_1^p x_3^{p(p-3)}$. Lemma 3.5.7 tells us, however, that there exists no other monomial in $\mathbb{F}[x_1, x_2, N_3^H, N_4^H, \delta]$ which yields the same lead monomial when acted upon by $\sigma_2 - 1$.

Moreover since each monomial yields a different lead term under this action no sum of monomials in these elements could yield this lead term either. As such no polynomial in $\mathbb{F}[V]^H$ can cancel the terms we demand. Hence $(N_2^H)^{p-2}$ cannot be the 'lead term' of an invariant under G. The result then follows. □

It thus follows from this that $\mathbb{F}[V]^G$ is non-Cohen-Macaulay: since $F_1 F_2^{\frac{p-3}{2}} \notin (x_1, x_2)\mathbb{F}[V]^G$ and $F_1 F_2^{\frac{p-1}{2}} \in (x_1, x_2)\mathbb{F}[V]^G$ we see that x_1, x_2, F_2, whilst being part of an h.s.o.p, do not form a regular sequence, thus violating the Cohen-Macaulay property.

Conclusion

This section dealt with the invariant fields of hook groups, remarkable for their property of being entirely populated by bireflections. It is known that if a representation's invariant ring is to be Cohen-Macaulay it must be generated by bireflections. Whilst we have demonstrated that any potential converse relation is fickle, these examples may be of use in helping determine a sufficient requirement.

3.6 Invariants of Modular Four-Dimensional $\mathbb{Z}_p^r$-Representations

In this section we collate our efforts toward the invariants of 4-dimensional modular $\mathbb{Z}_p^r$-representations. We restrict our focus to rank 2 although we make explicit some of the techniques which extend easily to further ranks. We borrow heavily from Section 2.7 in which we parameterised all four-dimensional modular $\mathbb{Z}_p^r$-representations.

Here we determine which 4-dimensional modular representations of $\mathbb{Z}_p^2$ have Cohen-Macaulay invariant rings and in such cases provide generating sets in the form of SAGBI bases. In each non-Cohen-Macaulay case we provide generators for the localised invariant ring and thence sets upon which one can apply the SAGBI/Divide-By-x_1 algorithm to procure a SAGBI basis.

3.6.1 Representation Theory Recap

For ease of translation from Section 2.7 we recap the notion of a socle tabloid.

If $P \leq GL(V)$ is a modular p-group representation, by Proposition 2.2.2 we may construct a basis for V which conforms to the socle series of V and whose dual conforms to the socle series of V^*, up to some permutation. Given such a basis we construct the *socle tabloid* of P (or V) as the tabloid containing $\boxed{j}$ in row i for every v in the basis with strict inclusions $v \in \mathrm{Soc}_i(V)$ and $v^* \in \mathrm{Soc}_j(V^*)$. Proposition 2.2.5 tells us that this tabloid is irrespective of basis chosen.

In Section 2.7.4 we assigned a single homomorphism $\sigma_\delta : (\mathbb{F}^d, +) \to GL_4(\mathbb{F})$ for each valid socle tabloid δ in dimension 4. Collectively the images of these homomorphisms contain the images of all four-dimensional $\mathbb{Z}_p^r$ representations up to conjugacy.

As in Section 2.7.4 we omit any representations with trivial free summands since their invariants reduce to a lower-dimensional problem.

3.6.2 Invariants in Socle-Length 2

Representations with socle-type $(1,3)$ are Nakajima and thus their invariant rings are polynomial and known. Those representations with socle-type $(3,1)$ are hyperplane representations and thus their invariants are dealt with both in [8] and Section 3.2.

Those which remain have socle-type $(2,2)$ and so socle tabloid $\begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}$. Since these have a codimension 2 socle their invariants must be Cohen-Macaulay. An incomplete treatise on these exists within Section 3.4 although not particularly systematically. We focus on those representation with rank 2.

Recall from earlier sections the notation

$$\underline{\big|\; C\;} := \begin{bmatrix} I_2 & C \\ 0 & I_2 \end{bmatrix}, \quad \mathcal{R}(f_1, f_2, \sigma) := \gcd(f_1 \cdot \Delta, f_2 \cdot \Delta)^{-1} \begin{vmatrix} f_1 & f_1 \cdot \Delta \\ f_2 & f_2 \cdot \Delta \end{vmatrix}$$

for $f, g \in \mathbb{F}[V], \sigma \in G$ and $\Delta := \sigma - 1$. We shall find use in the element

$$\delta := \mathcal{R}\left(x_3, x_4, \underline{\big|\; I_2\;}\right) = x_2x_3 - x_1x_4.$$

Proposition 3.6.1. *Let $\mathbb{Z}_p^2 \cong H \leq GL(V)$ have socle tabloid $\begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array}$. Then H is conjugate to G where G is one of the following:*

1. $\left\langle \underline{\left|\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}\right.}, \underline{\left|\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}\right.} \right\rangle$, *in which case* $\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G]$.

2. $\left\langle \sigma_1 := \underline{\left|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right.}, \sigma_2 := \underline{\left|\begin{matrix} \alpha & 0 \\ 0 & 0 \end{matrix}\right.} \right\rangle$ *for* $\alpha^p \neq \alpha$. *Then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G][\mathcal{R}(N_3^{\langle\sigma_1\rangle}, N_4^{\langle\sigma_1\rangle}, \sigma_2)]$$

3. $\left\langle \underline{\left|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right.}, \underline{\left|\begin{matrix} \alpha & 0 \\ 0 & \alpha \end{matrix}\right.} \right\rangle$ *for* $\alpha^p \neq \alpha$. *Then* $V = 2V_2$ *and thus*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G][\delta].$$

4. $\left\langle \sigma_1 := \left\lfloor\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right. , \sigma_2 := \left\lfloor\begin{matrix} \alpha & 0 \\ 0 & \beta \end{matrix}\right. \right\rangle$ *for* $\alpha^p \neq \alpha, \beta^p \neq \beta$ *and* $\alpha \neq \beta$. *Then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G][\mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2), \mathcal{R}(N_4^{\langle\sigma_1\rangle}, \delta, \sigma_2)].$$

5. $\left\langle \sigma_1 := \left\lfloor\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right. , \sigma_2 := \left\lfloor\begin{matrix} \alpha & 1 \\ 0 & \alpha \end{matrix}\right. \right\rangle$ *for* $\alpha^p \neq \alpha$. *Then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_4^G, R_1 := \mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2), R_2].$$

where R_2 *is acquired by subducting the tête-à-tête difference* $N_3^G - R_1^p$.

Furthermore $\mathbb{F}[V]^H$ *is a complete intersection.*

Proof. Two matrices $\lfloor C$ and $\lfloor D$ are equivalent only if $D = A_1 C A_2^{-1}$ for some $A_1, A_2 \in GL_2(\mathbb{F})$.

Any representation with socle tabloid $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 1 & 1 \\ \hline\end{array}$ must contain an element $\lfloor C$ for which $\det(C) \neq 0$, lest the dimension of $\mathrm{Soc}(V)$ or $\mathrm{Soc}(V^*)$ rise. We thus choose our basis such that our first generator is $\lfloor I_2$ and our second generator is chosen up to similarity of its corner matrix.

We then take our second generator $\lfloor D$ such that D is in Jordan normal form. The possibilities here yield cases 1 – 5.

Case 1, in which D is singular with eigenvalue in $\mathbb{F}_p$, is immediate from the Nakajima property and Lemma 3.1.6.

Case 2, in which D is singular with eigenvalue outside of $\mathbb{F}_p$, has a nontrivial Nakajima subgroup and its invariant ring follows immediately from Proposition 3.4.6.

Case 3, in which D has repeated eigenvalues, is the vector invariant case and thus follows from Proposition 3.4.9.

The remaining cases – non-repeating eigenvalues, and a 2×2 Jordan Block – require more work.

Since δ is invariant under $\sigma_1 := \lfloor I_2$ we have use of the invariant $\mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2)$. By construction this has degree 1 in x_4 and so by Proposition 3.4.8 applying the SAGBI/Divide-by-x_1 algorithm to the set $\{x_1, x_2, N_3^G, N_4^G, \mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2)\}$ shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

For case 4 we apply the algorithm to the extended set

$$\mathcal{B}_4 := \{x_1, x_2, N_3^G, N_4^G, \mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2), \mathcal{R}(N_4^{\langle\sigma_1\rangle}, \delta, \sigma_2)\}.$$

Note that (up to choice of coefficient)

$$R_1 := \mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2) = (\beta-\alpha)x_2(x_3^p - x_1^{p-1}x_3) + (\beta-\beta^p)x_1^{p-1}(x_2x_3 - x_1x_4)$$

$$R_2 := \mathcal{R}(N_4^{\langle\sigma_1\rangle}, \delta, \sigma_2) = (\beta-\alpha)x_1(x_4^p - x_2^{p-1}x_4) + (\alpha-\alpha^p)x_2^{p-1}(x_2x_3 - x_1x_4).$$

Hence one may observe the relations

$$R_1^p = \frac{(\beta-\beta^p)^p}{\alpha-\beta}x_1^{p^2-1}R_2 + \frac{(\beta-\beta^p)^p-(\beta-\alpha^p)}{\beta-\alpha}x_1^{p(p-1)}x_2^{p-1}R_1 + (\beta-\alpha)^p x_2^p N_3^G, \tag{3.4}$$

$$R_2^p = \frac{(\alpha-\alpha^p)^p}{\beta-\alpha}x_2^{p^2-1}R_1 + \frac{(\alpha-\alpha^p)^p-(\alpha-\beta^p)}{\alpha-\beta}x_1^{p-1}x_2^{p(p-1)}R_2 + (\beta-\alpha)^p x_1^p N_4^G. \tag{3.5}$$

Subducting the tête-à-tête difference $(\beta-\alpha)^{-p}R_1^p - x_2^pN_3^G$ yields relation (3.4). Subducting the tête-à-tête difference $(\alpha-\alpha^p)^{-p}R_2^p - (\beta-\alpha)^{-1}R_1x_2^{p^2-1}$ yields relation (3.5). Any other tête-à-tête is equivalent to a combination of these and so subduct to zero. Thus by Proposition 3.1.13 $\mathcal{B}_4$ is a SAGBI basis for $\mathbb{F}[V]^G$.

Finally in case 5 we begin with the set consisting of x_1, x_2,

$$\begin{aligned}
N_3^G &= x_3^{p^2} - x_1^{p(p-1)}x_3^p - (\alpha^p-\alpha)^{p-1}x_1^{p(p-1)}(x_3^p - x_1^{p-1}x_3)\\
N_4^G &= x_4^{p^2} - x_2^{p(p-1)}x_4^p - ((\alpha^p-\alpha)x_2^p - x_1x_2^{p-1} + x_1^p)^{p-1}(x_4^p - x_2^{p-1}x_4)\\
R_1 &:= \mathcal{R}(N_3^{\langle\sigma_1\rangle}, \delta, \sigma_2) = (\alpha^p-\alpha)x_1^{p-2}(x_2x_3 - x_1x_4) + x_3^p - x_1^{p-1}x_3.
\end{aligned}$$

Preempting the initial subduction of this set we define

$$R_2 := x_2^{p+1}x_3 - x_1\left(\frac{x_2^px_3}{\alpha^p-\alpha} + x_2^px_4\right) + \frac{x_1^2x_4^p + x_1^px_2x_3 - x_1^{p+1}x_4}{\alpha^p-\alpha} \in \mathbb{F}[V]^G.$$

Then the first subduction yields the relation

$$N_3^G - R_1^p = (\alpha^p-\alpha)^p x_1^{p(p-2)}x_2^pR_1 + (\alpha^p-\alpha)^{p+1}x_1^{(p+1)(p-2)}R_2 + (\alpha^p-\alpha)^{p-1}x_1^{p(p-1)}R_1.$$

Thus by introducing R_2 to our set N_3^G is redundant and can be removed. Hence we continue with the set $\{x_1, x_2, N_4^G, R_1, R_2\}$. The only tête-à-tête arising from this set yields the rather unpleasant expression

$$\begin{aligned}R_2^p - x_2^{p(p+1)} R_1 = (\alpha - \alpha^p) x_1^{p-2} x_2^{p^2} R_2 + \frac{x_1^p x_2^{p^2} R_1}{(\alpha - \alpha^p)^p} + \frac{x_1^{2(p-1)} x_2^{p(p-1)} R_2}{(\alpha^p - \alpha)^{p-1}} + \frac{x_1^{2p} N_4^G}{(\alpha^p - \alpha)^p} \\ + x_1^{2(p-1)} \left(x_2^p + \frac{x_1^p - x_1 x_2^{p-1}}{\alpha^p - \alpha} \right)^{p-1} R_2 + \frac{x_1^{p^2} x_2^p R_1}{(\alpha^p - \alpha)^p} - \frac{x_1^{p^2+p-2} R_2}{(\alpha^p - \alpha)^{p-1}}.\end{aligned}$$

Verifying this is a matter of computation, but it yields the end of the algorithm and thus the confirmation of the result. $\square$

3.6.3 Socle-Type (2,1,1) Invariants

In this section we calculate SAGBI bases for the invariant rings for each modular representation of $\mathbb{Z}_p^2$ with socle-type $(2, 1, 1)$. The only case to examine are those representations with socle tabloid $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$.

Recall the polynomial $\delta_2 := x_1 x_3 - x_2^2$ from the study of socle-type $(1, 1, 1)$ representations. Since this action sits within our own here, δ_2 ought to be equally useful.

Proposition 3.6.2. *Let $\mathbb{Z}_p^2 \cong H \leq GL(V)$ have socle tabloid $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 \\ \cline{1-1} 1 \\ \cline{1-1}\end{array}$. Then H is conjugate to one of the following.*

- $G = \left\langle \sigma_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \sigma_2 := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$ *in which case*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, x_3]^{\langle \sigma_1 \rangle}[N_4^{\langle \sigma_2 \rangle}] = \mathbb{F}[x_1, \delta_2, N_2^{\langle \sigma_1 \rangle}, N_3^{\langle \sigma_1 \rangle}, N_4^{\langle \sigma_2 \rangle}].$$

- $G = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2c & c^2 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$ *in which case*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, \delta_2, N_3^G, N_4^G, F_1, F_2]$$

where $F_1 = x_2^p - x_1^{p-1}((c^p - c)x_4 + x_2)$ is generated as in Theorem 3.3.4 and

$$F_2 := \frac{\delta_2^p + F_1^2}{2(c^p - c)x_1^{p-1}} = x_1^p x_3^p - x_2^{2p} + (x_2^p - x_1^{p-1}((c^p - c)x_4 + x_2))^2$$

Proof. As per Section 2.7 our representation may be written in the form

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2c & c^2 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$$

for some $c \in \mathbb{F}$. If $c^p = c$ then we are in case 1. Any socle-type $(m, 1, \ldots, 1)$ representation G generated by $W \leq (\mathbb{F}^{n-1}, +)$ which can be written $W = \{\underline{c} + \underline{d} \mid \underline{c} \in W_1, \underline{d} \in W_2\}$ for

$$\begin{aligned} W_1 &= \{\underline{c} \in W \mid c_{n-m+1} = \cdots = c_{n-1} = 0\}, \\ W_2 &= \{\underline{d} \in W \mid d_1 = \cdots = d_{n-m} = 0\} \end{aligned}$$

then satisfies

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, \ldots, x_{n+m-1}]^{\sigma_{1^{n-m+1}}(W_1)}[N_{n-m+2}^{\sigma(W_2)}, \ldots, N_n^{\sigma(W_2)}].$$

Thus the problem reduces to examining a socle-type 1^{n-m+1} representation. In this case the action of $\langle \sigma_1 \rangle$ is of a 3-dimensional cyclic representation, the invariants of which are well known (see for instance [12] Theorem 4.10.1).

Case 2, in which $c^p \neq c$, requires more work. We recall Theorem 3.3.4: The maximal minors $\Gamma[1, 2, 3 \mid 1, 2, 3]$, $\Gamma[1, 2, 4 \mid 1, 2, 3]$ and $\Gamma[1, 2, 5 \mid 1, 2, 3]$ of the matrix

$$\Gamma := \begin{bmatrix} 1 & c & x_2/x_1 \\ 1 & c^p & x_2^p/x_1^p \\ 1 & c^{p^2} & x_2^{p^2}/x_1^{p^2} \\ 0 & 0 & \delta/x_1^2 \\ 0 & 1 & x_4/x_1 \end{bmatrix}$$

yield, when their denominators are cleared, the invariants

$$\begin{aligned} f_1 &= (c^p - c)x_2^{p^2} - (c^{p^2} - c)x_1^{p^2-p}x_2^p + (c^p - c)^p x_1^{p^2-1}x_2 = (c^p - c)N_2^G \\ f_2 &= (c^p - c)\delta \\ f_3 &= -x_2^p + x_1^{p-1}((c^p - c)x_4 + x_2) = -F_1. \end{aligned}$$

Thus applying the SAGBI/Divide-By-x_1 algorithm to the set $\{x_1, N_2^G, N_3^G, N_4^G, \delta, F_1\}$ shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

Observe that

$$F_1^p - N_2^G = (c^p - c)^{p-1} x_1^{p(p-1)} (F_1 - (c^p - c) N_4^G)$$

and thus we may remove N_2^G from consideration. The only remaining tête-à-tête in the set yields the difference $f_1^2 + \delta^p$. This difference has lead term $x_1^{p-1} x_2^p x_4$ which we cannot cancel with our current elements. Hence we define F_2 from this process as in the statement.

With the subsequent set $\{x_1, N_3^G, N_4^G, \delta, F_1, F_2\}$ the only remaining nontrivial tête-à-tête arising from this subducts to

$$X := F_2^p + F_1^p N_4^G + \frac{F_1^{p+1}}{(c^p - c)^p} - \frac{c^{p^2} - c}{(c^p - c)^p} x_1^{p-1} F_1^{p-1} F_2 + \frac{x_1^{p-1} \delta^{(p^2+1)/2}}{c^p - c}$$

with $LT(X) = \frac{x_1^p x_3^{p^2}}{2(c^p - c)^p}$. By replacing N_3^G with $x_1^{-p} X$ we acquire our SAGBI basis and thus the result follows. $\square$

3.6.4 Socle-Type (1,2,1) Invariants

Recall from Section 2.7.4 that representations with socle-type $(1, 2, 1)$ may have either dual-type $(1, 2, 1)$ or $(2, 1, 1)$. We consider these separately.

Case 1: Dual-type $(1, 2, 1)$

Recall from Section 2.7.4 that $\mathbb{Z}_p^2$-representations with both socle-type and dual-type $(1, 2, 1)$ can exist for $p = 2$ and thus have their own specialised homomorphism. However we opt to distinguish between the $p = 2$ and $p > 2$ case by electing different computationally convenient forms.

Proposition 3.6.3. *Let $\mathbb{Z}_p^2 \cong H \leq GL(V)$ have socle tabloid $\begin{array}{|c|c} \hline 3 & \multicolumn{1}{c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline 1 & \multicolumn{1}{c}{} \\ \hline \end{array}$. Then H is equivalent to one of the following:*

1. $G = \left\langle \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$ *for $p = 2$. Then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, N_2^G, N_3^G, N_4^G, \delta, F]$$

where $\delta := x_2x_3 + x_1x_4$ *and* $F := (\delta^2 + N_2^G N_3^G)/x_1$.

2. $G = \left\langle \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 + c_2^2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$ *for* $p > 2$, $c_2 \neq 0$. *Then*

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, N_2^G, N_4^G, \delta, f_1, f_2]$$

for the invariants

$$\delta := x_3^2 + x_2^2 - x_1x_4, \qquad f_1 := x_3^p - x_1^{p-1}\left(x_3 + \frac{c_1^p - c_1}{c_2}x_2\right),$$

$$f_2 := \frac{1}{2x_1^{p-1}}\left[\delta^p - f_1^2 - (N_2^G)^2 - 2x_1^{p-1}\delta^{(p+1)/2}\right]$$

In both cases the given generating set is a SAGBI basis and $\mathbb{F}[V]^G$ *is a complete intersection.*

Proof. The equivalences follow from the classification in Section 2.7.4.

If $p = 2$ then we arrive in case 1. By augmenting Theorem 3.5.1 by including the requisite norms and applying the SAGBI/Divide-by-x_1 algorithm to the set $\{x_1, N_2^G, N_3^G, N_4^G, \delta\} \subset \mathbb{F}[V]^G$ we acquire a SAGBI basis. Since

$$N_2^G = x_2^2 + x_1x_2, \qquad N_3^G = x_3^2 + x_1x_3, \qquad \delta = x_2x_3 + x_1x_4$$

the only tête-à-tête in this set yields difference $\delta^2 - N_2^G N_3^G$. We then take

$$F := x_1^{-1}(\delta^2 + N_2^G N_3^G) = x_2x_3^2 + x_2^2x_3 + x_1(x_4^2 + x_2x_3).$$

to add to our set. The remaining nontrivial tête-à-tête difference in this set eventually subducts to

$$x_1^{-2}\Big(F^2 + N_2^G N_3^G(N_3^G + N_2^G) + x_1F(N_3^G + N_2^G + \delta)\Big) = x_4^4 + L.O.T.s.$$

Replacing N_4^G with this yields a SAGBI basis from which the result follows.

Now suppose $p > 2$ and define δ as in the statement. Since $LT(\delta) = x_3^2$ and $f_1 \in \mathbb{F}[x_1, x_2, x_3]^G$ has minimal degree $\deg_{x_3}(f_1) = p$, by Theorem 3.5.1 if we apply the SAGBI/Divide-by-x_1 algorithm to the set $\{x_1, N_2^G, N_4^G, \delta, f_1\}$ we shall acquire a SAGBI basis for $\mathbb{F}[V]^G$.

The sole nontrivial tête-à-tête in our set is $LT(\delta^p) = LT(f_1^2)$, subducting the difference of which yields our construction for f_2. Subducting the remaining tête-à-tête difference – as detailed in appendix C – reduces to an invariant with lead term of the form $x_1^p x_4^{p^2}$. Factoring out the x_1^p from this yields an invariant with lead term $x_4^{p^2}$ with which we can replace N_4^G thus effectively terminating the process. $\square$

Case 2: Dual-Type $(2,1,1)$

It remains to examine the $\mathbb{Z}_p^2$-representations with socle-type $(1,2,1)$ and dual-type $(2,1,1)$. Fortunately the graft has already been undertaken in Section 3.5.5.

Proposition 3.6.4. *Let $\mathbb{Z}_p^2 \cong G \le GL(V)$ have socle tabloid $\begin{array}{|c|c} \cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline 1 & \\ \cline{1-1} \end{array}$. Then up to conjugacy*

$$G = \left\langle \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle.$$

- *If $c_1^p = c_1$ then we may take $c_1 = 0$. If we then define*

$$F := x_1 N_4^{\langle \sigma(0,c_2,0) \rangle} - \left[(2x_3^{p+1} - x_1^{p-1}x_3^2) - c_2^{p-1}x_2^{p-1}x_3^2\right]$$

  then

$$\mathbb{F}[V]^G = \mathbb{F}[x_1, x_2, N_3^G, N_4^G, F]$$

  is a complete intersection and the given generating set is a SAGBI basis.

- *If $c_1^p \neq c_1$ then $\mathbb{F}[V]^G$ is not Cohen-Macaulay, as shown in Section 3.5.5. Defining $\delta = x_3^2 - x_1x_4$, applying the SAGBI/Divide-By-x_1 algorithm to the set*

$$\{x_1, x_2, N_3^G, N_4^G, \mathcal{R}(N_2^{\sigma(1,0,0)}, \delta, \sigma(c_1, c_2, 0))\}$$

  yields a SAGBI basis for $\mathbb{F}[V]^G$.

Proof. By Section 2.5.2 any $\mathbb{Z}_p^r$-representation with socle-type $(1,2,1)$ and dual-type $(2,1,1)$ exists up to equivalence in the image of the homomorphism

$$\sigma : (\mathbb{F}^3, +) \to GL_4(\mathbb{F}), \quad \sigma(c_1, c_2, c_3) = \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 + c_3 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then any $\mathbb{Z}_p^2$-representation is of the form $G = \langle \sigma(1,0,0), \sigma(c_1, c_2, 0) \rangle$ up to equivalence. If $c_1^p = c_1$ then the invariants of G come under the purview of Proposition 3.5.3. Otherwise if $c_1^p \neq c_1$ the result is a reiteration of Section 3.5.5. □

3.6.5 Socle-Type (1,1,1,1) Invariants

Representations with socle-type $(1,1,1,1)$ have images, up to conjugacy, within the image of $\sigma_{1^4} : (\mathbb{F}^3, +) \to GL(V)$ where

$$\sigma_{1^4}(c_1, c_2, c_3) = \begin{bmatrix} 1 & 3c_1 & 3(c_1^2 + c_2) & c_1^3 + 3c_1c_2 + c_3 \\ 0 & 1 & 2c_1 & c_1^2 + c_2 \\ 0 & 0 & 1 & c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since there must exist an element $\sigma_{1^4}(\underline{c})$ in such representations with $c_1 \neq 0$ the group cannot be generated by bireflections, violating the conditions of Lemma 3.1.10. Thus we do not have Cohen-Macaulay invariant rings. However we may always, up to conjugacy, take this element to be $\sigma_{1^4}(1,0,0)$.

Recall from Section 3.3.2 the following constructions. Define $\delta_1 := x_2$ and

$$\delta_\ell := x_{\ell+1} x_1^{\ell-1} - B_\ell(\delta_1, \ldots, \delta_{\ell-1}, 0)$$

where $B_m(\underline{c})$ are the exponential Bell polynomials. Explicitly we shall use

$$\delta_1 = x_2, \quad \delta_2 = x_1x_3 - x_2^2, \quad \delta_3 = x_1^2 x_4 + 2x_2^3 - 3x_1x_2x_3.$$

Given a representation $G = \langle \sigma_{1^4}(1,0,0), \sigma_{1^4}(c_1,c_2,c_3) \rangle$ and $\underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$

for which $\sum \alpha_i = 2$, construct the matrix

$$\Gamma_{\underline{\alpha}} := \left[\begin{array}{ccc} 1 & c_1 & \delta_1/x_1 \\ & \vdots & \\ 1 & c_1^{p^{\alpha_1-1}} & (\delta_1/x_1)^{p^{\alpha_1-1}} \\ 0 & c_2 & \delta_2/x_1^2 \\ & \vdots & \\ 0 & c_2^{p^{\alpha_2-1}} & (\delta_2/x_1^2)^{p^{\alpha_2-1}} \\ 0 & c_3 & \delta_3/x_1^3 \\ & \vdots & \\ 0 & c_3^{p^{\alpha_3-1}} & (\delta_3/x_1^3)^{p^{\alpha_3-1}} \\ \hdashline 1 & c_1^{p^{\alpha_1}} & (\delta_1/x_1)^{p^{\alpha_1}} \\ 0 & c_2^{p^{\alpha_2}} & (\delta_2/x_1^2)^{p^{\alpha_2}} \\ 0 & c_3^{p^{\alpha_3}} & (\delta_3/x_1^3)^{p^{\alpha_3}} \end{array}\right].$$

From this we construct $\widetilde{f_{\underline{\alpha},i}} := \Gamma[1,2,i+2|1,2,3] \in \mathbb{F}[V]^G[x_1^{-1}]$ and thence clear denominators to acquire $f_{\underline{\alpha},i}$ for $i = 1,2,3$.

Proposition 3.6.5. *Let $\mathbb{Z}_p^2 \cong H \leq GL(V)$ have socle-type $(1,1,1,1)$. Then H is conjugate to*

$$\langle \sigma_{1^4}(1,0,0), \sigma_{1^4}(c_1,c_2,c_3)\rangle$$

for $\underline{c} \neq (\gamma,0,0)$ for any $\gamma \in \mathbb{F}_p$.

- *If $c_1^p \neq c_1$ we let $\underline{\alpha} = (2,0,0)$. Then applying the SAGBI/Divide-By-x_1 algorithm to the set*

  $$\{x_1, N_2^G, N_3^G, N_4^G, f_{\underline{\alpha},2}, f_{\underline{\alpha},3}\}$$

  shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

- *If $c_1^p = c_1$ and $c_2 \neq 0$ then take $c_1 = 0$. We take $\underline{\alpha} = (1,1,0)$. Then applying the SAGBI/Divide-By-x_1 algorithm to the set*

  $$\{x_1, N_2^G, N_3^G, N_4^G, f_{\underline{\alpha},2} = c_2N^G(\delta_2), f_{\underline{\alpha},3} = c_2\delta_3 - c_3x_1\delta_2\}$$

  shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

- *If $c_1^p = c_1$, $c_2 = 0$ and $c_3 \neq 0$ than we take $c_1 = 0$. We take $\underline{\alpha} = (1,0,1)$.*

Then applying the SAGBI/Divide-By-x_1 algorithm to the set

$$\{x_1, N_2^G, N_3^G, N_4^G, f_{\underline{\alpha},2} = c_3\delta_2, f_{\underline{\alpha},3} = N^G(\delta_3)\}$$

shall yield a SAGBI basis for $\mathbb{F}[V]^G$.

These results are corollaries of Theorem 3.3.4. All one need verify is the proposed structures for $f_{\underline{\alpha},2}$ and $f_{\underline{\alpha},3}$.

Conclusion

This section culminates the work of the document into examining the invariants of all modular $\mathbb{Z}_p^2$-representations. A variety of directions lay beyond this work: One may wish to determine the non-C-M invariant rings, extend to rank 3 representations, or jump to 5-dimensional representations in rank 2. Some of these cases have already been partially examined in this work and in the literature. Alas we conclude here.

3.7 Conclusion of Invariant Theory

Here we conclude the document with some thoughts as to the work on invariants we have procured. We examine where this work may assist in open problems and the further questions this work raises.

3.7.1 Cohen-Macaulayness in Dimension 4

The progenitor of this document [11] classified all 3-dimensional modular representations of $\mathbb{Z}_p^r$. Every such representation has a Cohen-Macaulay invariant ring, something that fails to hold for dimension 4.

In order for a modular representation to have a Cohen-Macaulay invariant ring it must be generated by bireflections. The extreme case of this for which $\dim(V^G) = \dim(V) - 2$ is sufficient to ensure Cohen-Macaulay. The representations with socle-type $(2,2)$ and $(2,1,1)$ all possess this feature, whilst those with socle-type $(3,1)$ and $(1,3)$ have polynomial invariant ring. Socle-type $(1,1,1,1)$ representations never have Cohen-Macaulay invariants.

It is socle-type $(1, 2, 1)$ which remains outstanding. Section 3.5.5 illustrates that examples therein exist even in rank 2 with and without the Cohen-Macaulay property. We would like to know under what conditions such representations have Cohen-Macaulay invariant rings. Since the only non-Cohen-Macaulay case given exists under the socle tabloid $\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, that is dual-type $(2, 1, 1)$, perhaps are those with tabloid $\begin{array}{|c|c|}\cline{1-1} 3 & \multicolumn{1}{c}{} \\ \hline 2 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ more well-behaved? Since the latter have a full set of recovery functions perhaps this may be explored further.

Section 3.5 focused on hook groups for this reason; Since hook groups consist entirely of bireflections we'd like to know when exactly these have Cohen-Macaulay invariant rings, $(1, 2, 1)$ acting as the smallest uncertain prototype.

3.7.2 Ideal $(m, 1, \ldots, 1)$ Field Generators

In Section 3.3.2 we provide a construction for invariant field generators for representations with socle-type $(m, 1, \ldots, 1)$. Furthermore we provide extensions to this upon which one may apply the SAGBI/Divide-by-x_1 algorithm to acquire a SAGBI basis for the invariant ring. To ensure that this proceeds smoothly we would desire an initial set with elements of minimal degree and with lead terms which shall ultimately appear in the SAGBI basis.

Using recovery functions we constructed a matrix and a nonnegative integer sequence used to acquire minors existing in the invariant field. In general the largest integer chosen dictates how large the degree of our generators become. Our method affords us no room for movement in these integers.

In [11], by use of the Plücker relations, we may lessen the degrees in the three-dimensional case. We do not know to what extent this may be generalised. Indeed their work focused on the 'generic case' whereas in Section 3.3.2 we are more specific. How or whether this generalises to arbitrary socle-type $(m, 1, \ldots, 1)$ is not understood and yet is most desirable for computation and further understanding.

3.7.3 Symmetric Power Invariants

Given a two-dimensional $\mathbb{Z}_p^r$-representation on V_2 we consider the symmetric powers $\mathrm{Symm}_m(V_2)$. These exists in our framework as the socle-type 1^n representations with image of the form

$$\left\langle \sigma_{1^n}(c, 0, \ldots, 0) \;\middle|\; c \in W \right\rangle$$

for some $W \leq (\mathbb{F}, +)$ and σ_{1^n} as defined in Section 2.4. The invariants of the symmetric square case were dealt with in [11] and shown to be hypersurfaces.

The cyclic case – i.e where $W = \mathbb{F}_p$ – is particularly notable, since the indecomposable representations are all of this form. Their invariant rings, after being conjectured in [27], were shown to be have SAGBI bases generated by norms, transfers and integral (i.e. characteristic-independent) elements in [29].

One might hope that extension into arbitrarily ranked elementary abelian p-group symmetric powers we might acquire a similar result, say generation by relative norms, relative transfers and integral invariants. To this end [17] provides a degree bound of $|G|-\dim(V)$ above which such constructions suffice, and below which we might expect only integral invariants to exist. This is not, however, the case.

In dimension $n = 4$ for $W = \mathbb{F}_{p^2}$ we already see examples with SAGBI bases with lead terms not acquirable from these constructions alone. For instance when $p = 5$ there exists an element with lead term $x_2^3 x_3^{18}$. If $H := \sigma_{1^4}(\mathbb{F}_5, 0, \ldots, 0)$ then we may construct such an invariant in the form $\mathrm{Tr}_H^G\big((N_4^H)^3 \cdot \mathrm{Tr}^H(x_3^3 x_4^4)\big)/x_1$ but one can show through exhaustive calculation that such a lead term is unobtainable through relative norms and relative transfers alone.

Since this lead term does not appear for $p = 7$ however we cannot consider this an integral invariant either. This case has its own elusive lead terms, for instance, $x_2^5 x_3^{40}$.

The author has yet to find a situation in which including elements of the form $\mathrm{Tr}_H^G((N_4^H)^i, \mathrm{Tr}^H(x_3^j x_4^k))/x_1$ does not complete the SAGBI basis in rank 2. The cyclic calculations utilise the structural properties of the symmetric powers explored in [2] and [19]. Alas these decompositions no longer necessarily hold in the elementary abelian p-group case. We believe this goes some way toward explaining this irregularity. Naturally more careful examination is required before conclusions can be drawn.

Appendices

Appendix A

Incongruent Two-Dimensional Subspaces of $\mathrm{Sym}_3(\mathbb{F})$

We prove the following result which is of use classifying the modular $\mathbb{Z}_p^r$-representations with dual-type $(1, 3, 2)$. We work over an algebraically closed field $\mathbb{F} = \overline{\mathbb{F}}$ of characteristic $p > 2$.

Define $\mathrm{Sym}_m(\mathbb{F})$ denotes the set of all $m \times m$ symmetric matrices over $\mathbb{F}$. A subspace $S \leq \mathrm{Sym}_m(\mathbb{F})$ is called degenerate if $\ker(S) := \cap_{M \in S} \ker(M) \neq \{0\}$. Let $GL_3(\mathbb{F})$ act on $Sym_m(\mathbb{F})$ by congruence.

Lemma A.1. *Any nondegenerate two-dimensional subspace of* $\mathrm{Sym}_3(\mathbb{F})$ *is* $GL_3(\mathbb{F})$*-congruent to a space generated by one of the following:*

$$S_1 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad S_2 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

$$S_3 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad S_4 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}$$

$$S_5 := \left\{ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \quad S_6 := \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$$

Proof. Suppose our subspace $S \leq \mathrm{Sym}_3(\mathbb{F})$ contains an element of rank 1. Up to congruence we may choose our basis to be

$$D_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A := \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Furthermore we can ensure that the submatrix $A' := \begin{bmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{bmatrix}$ is diagonal with 0 and 1 entries. This cannot have rank 0 lest we acquire a degenerate case. If $\mathrm{rank}(A') = 2$ then up to congruence we have $A' = I_2$. Thence

$$\begin{bmatrix} -1 & 0 & 0 \\ a_{12} & 0 & 1 \\ a_{13} & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 1 & 0 \\ a_{13} & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ a_{12} & 0 & 1 \\ a_{13} & 1 & 0 \end{bmatrix} = \begin{bmatrix} -a_{12}^2 - a_{13}^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and, by observing that this doesn't alter D_1, we acquire the orbit of S_1.

If instead $\mathrm{rank}(A') = 1$ we may take $A' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $a_{13} \neq 0$ to ensure non-degeneracy. By observing that the congruence action

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a_{12}}{a_{13}} & \frac{1}{a_{13}} \end{bmatrix}^T \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{12} & 1 & 0 \\ a_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{a_{12}}{a_{13}} & \frac{1}{a_{13}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

does not alter D_1 we thus acquire the orbit of S_2.

Now consider the case where our subspace contains no element of rank 1. Given two elements of rank 3 an $\mathbb{F}$-linear combination exists of rank 2 (since $p > 2$), and thus we can ensure a basis of the form

$$D_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A := \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Once again we can diagonalise the block diagonal area of A not covered by D_2. This leaves only a_{33} and so we can ensure that this entry is either 0 or 1.

Case 1: If $a_{33} \neq 0$ then by using the congruence matrix $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{a_{13}}{a_{33}} & -\frac{a_{23}}{a_{33}} & -\frac{1}{\sqrt{a_{33}}} \end{bmatrix}$

we can choose basis

$$D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thence to preserve D_2 we can choose $A' = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ up to orthogonal congruence. Since our subspace contains no rank 1 elements A' is nonzero. Furthermore by Theorem 27 of [1] two symmetric matrices are orthogonally congruent if and only if they are similar (when $p > 2$ and $\mathbb{F} = \overline{\mathbb{F}}$). Yet further, since

$$\det(A' + \alpha I_2) - \det(A') = \alpha(\mathrm{tr}(A') + \alpha)$$

we can always add multiples of D_2 to A to ensure A' has rank 1.

If A' is diagonalisable then we acquire the case

$$S_3 := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Otherwise A' is similar to a matrix of the form $\begin{bmatrix} 1 & \sqrt{-1} \\ \sqrt{-1} & -1 \end{bmatrix}$. This is, the reader may agree, aesthetically unpleasant. By noting that

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 0 & 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & \sqrt{-1} & 0 \\ \sqrt{-1} & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & \sqrt{-1} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

we instead acquire the S_4 case.

Case 2: If $a_{33} = 0$ then since we care only for non-degenerate orbits one of a_{13} and a_{23} is nonzero. By permuting basis and scaling A we can take $a_{23} = 1$. Assuming that $a_{13} \neq \pm\sqrt{-1}$ we may take the matrix E defined as

$$\begin{bmatrix} a_{13} & a_{13}\sqrt{-1} & 1 \\ 1 & \sqrt{-1} & -a_{13} \\ \frac{(a_{13}^2-1)(a_{22}-a_{11})-4a_{12}a_{13}}{2(a_{13}^2+1)} - 1 & -\sqrt{-1}\frac{(a_{13}^2-1)(a_{11}-a_{22}-2)+4(a_{12}a_{13}-1)}{2(a_{13}^2+1)} & \frac{a_{13}(a_{22}-a_{11})+a_{12}(a_{13}^2-1)}{a_{13}^2+1} \end{bmatrix}$$

under which

$$\frac{a_{13}^2a_{22} - 2a_{12}a_{13} + a_{11}}{2(a_{13}^2+1)^2}E^TD_2E - \frac{1}{2(a_{13}^2+1)}E^TAE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{a_{13}^2+1}E^TD_2E = \begin{bmatrix} 1 & \sqrt{-1} & 0 \\ \sqrt{-1} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and thus this case is equivalent to S_4.

If instead $a_{13} = \pm\sqrt{-1}$ but $a_{11} \neq a_{22} \pm 2a_{12}\sqrt{-1}$ then taking E to be

$$\frac{1}{8}\begin{bmatrix} 4a_{13} & -8a_{13} & 0 \\ -4 & -8 & 0 \\ -10a_{12}a_{13} + 3(a_{11}-a_{22}) & 16a_{12}a_{13} - 4(a_{11}-a_{22}) & -16a_{12}a_{13} + 8(a_{11}-a_{22}) \end{bmatrix}$$

we acquire

$$E^TD_2E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{E^T(A - (a_{11} - 2a_{12}a_{13})D_2)E}{a_{22} - a_{11} + 2a_{12}a_{13}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

thus yielding S_5.

Finally if $a_{13} = \pm\sqrt{-1}$ and $a_{11} = a_{22} \pm 2a_{12}\sqrt{-1}$ by taking

$$E := \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \pm 1 & \pm 1 \\ 0 & \sqrt{-1} & -\sqrt{-1} \\ -\sqrt{-1} & \mp a_{12} & \mp a_{12} \end{bmatrix}$$

we then acquire

$$E^TD_2E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad E^T(A - a_{22}D_2)E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

thus yielding S_6. The result then follows from straight-forward verification that none of these six cases are congruent. $\square$

Appendix B

Plücker Relations and Determinant Combinatorics

In this section we examine some consequences of the well-known Plücker relations and their applications to hyperplane invariants examined in Section 3.2.

For an $m \in \mathbb{N}$ denote by $\underline{m}$ the ordered sequence $1, \ldots, m$. If a sequence S has entries in $\underline{m}$ we write $S \subset \underline{m}$. We denote by S_i the ith element of the sequence and by $S \setminus S_i$ the sequence acquired by removing the ith term from S.

For an $n \times m$ matrix M and sequences $R \subset \underline{n}$ and $C \subset \underline{m}$ of the same length, we denote by $M[R \mid C]$ the determinant of the matrix formed from the rows and columns of M indexed by R and C respectively. Similarly we denote by $M\{R \mid C\}$ the determinant of the matrix formed by *removing* the rows and columns from M indexed by R and C respectively.

We utilise the Plücker relations in the following form.

Lemma B.1 (Plücker Relations)**.** *Let M be a $n \times m$ matrix with $n > m$. Let $R, S \subset \underline{n}$ be sequences of length $m-1$ and $m+1$ respectively. Then*

$$\sum_{i=1}^{m+1} M[R, S_i \mid \underline{m}]\, M[S \setminus S_i \mid \underline{m}] = 0.$$

Using this we posit the following, which the author suspects exists elsewhere buried unknown, deep within some combinatorist's archives.

Lemma B.2. *Let $M = [m_{i,j}]$ be an $n \times r$ matrix with $n > r$. Let $R, S \subset \underline{n}$ be* *sequences of length* $r-1$ *and* $1 \le c_1 < c_2 \le r$ *be integers. Then*

$$\begin{aligned} &M[R \mid \underline{r} \setminus c_1]M[S \mid \underline{r} \setminus c_2] - M[R \mid \underline{r} \setminus c_2]M[S \mid \underline{r} \setminus c_1] \\ &\qquad = (-1)^r \sum_{i=1}^{r-1} (-1)^i M[R, S_i \mid \underline{r}] M[S \setminus S_i \mid \underline{r} \setminus \{c_1, c_2\}]. \end{aligned}$$

Proof. For the sake of the page width we adopt the notation

$$S^{\{i_1,\dots,i_s\}} := S \setminus \{S_{i_1}, \dots, S_{i_s}\}.$$

The proof is a matter of expanding the right hand side of the proposed formula with the Laplace expansion, applying the Plücker relations to the fallout and simplifying to obtain the left hand side.

Expanding the minors $M\big[R, S_i \mid \underline{r}\big]$ along columns c_1 and c_2 on the right hand side of the proposed equation we acquire

$$\begin{aligned} M\big[R, S_k \mid \underline{r}\big] &= (-1)^{r+c_2} M\big[R \mid \underline{r}^{\{c_2\}}\big] m_{S_k,c_2} + \sum_{j=1}^{r-1} (-1)^{j+c_2} M\big[R^{\{j\}}, S_k \mid \underline{r}^{\{c_2\}}\big] m_{R_j,c_2} \\ &= (-1)^{r+c_2} m_{S_k,c_2} \sum_{j=1}^{r-1} (-1)^{j+c_1} M\big[R^{\{j\}} \mid \underline{r}^{\{c_1,c_2\}}\big] m_{R_j,c_1} \\ &\quad + \sum_{j=1}^{r-1} (-1)^{j+c_2} m_{R_j,c_2} \Bigg((-1)^{r+c_1-1} M\big[R^{\{j\}} \mid \underline{r}^{\{c_1,c_2\}}\big] m_{S_k,c_1} \\ &\qquad + \left(\sum_{i=1}^{j-1} - \sum_{i=j+1}^{r-1} \right) (-1)^{i+c_1} M\big[R^{\{i,j\}}, S_k \mid \underline{r}^{\{c_1,c_2\}}\big] m_{R_i,c_1} \Bigg). \end{aligned}$$

Thus the right hand side of our proposed equation becomes

$$\begin{aligned} &\sum_{k=1}^{r-1} (-1)^{r+k} M\big[S^{\{k\}} \mid \underline{r}^{\{c_1,c_2\}}\big] M\big[R, S_k \mid \underline{r}\big] \\ &= \sum_{k=1}^{r-1} \sum_{j=1}^{r-1} \sum_{i=1}^{j-1} (-1)^{r+c_1+c_2+i+j+k} m_{R_j,c_2} m_{R_i,c_1} M[S^{\{k\}} \mid \underline{r}^{\{c_1,c_2\}}] M[R^{\{i,j\}}, S_k \mid \underline{r}^{\{c_1,c_2\}}] \\ &- \sum_{k=1}^{r-1} \sum_{j=1}^{r-1} \sum_{i=j+1}^{r-1} (-1)^{r+c_1+c_2+i+j+k} m_{R_j,c_2} m_{R_i,c_1} M\big[S^{\{k\}} \mid \underline{r}^{\{c_1,c_2\}}\big] M\big[R^{\{i,j\}}, S_k \mid \underline{r}^{\{c_1,c_2\}}\big] \end{aligned}$$

$$+\sum_{k=1}^{r-1}\sum_{j=1}^{r-1}(-1)^{c_1+c_2+j+k}m_{R_j,c_1}m_{S_k,c_2}M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]M[R^{\{j\}}\mid \underline{r}^{\{c_1,c_2\}}]$$
$$-\sum_{k=1}^{r-1}\sum_{j=1}^{r-1}(-1)^{c_1+c_2+j+k}m_{R_j,c_2}m_{S_k,c_1}M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]M[R^{\{j\}}\mid \underline{r}^{\{c_1,c_2\}}].$$

Applying the Plücker relations as in Lemma B.1 to the matrix constructed by removing columns c_1 and c_2 from M with the sequences $R^{\{i,j\}}$ and S we acquire

$$\sum_{k=1}^{r-1}(-1)^k M\big[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}\big]M\big[R^{\{i,j\}},S_k\mid \underline{r}^{\{c_1,c_2\}}\big]=0.$$

Thus the right-hand side of our proposed equation continues to reduce:

$$\sum_{j=1}^{r-1}\sum_{i=1}^{j-1}(-1)^{r+c_1+c_2+i+j}m_{R_j,c_2}m_{R_i,c_1}\underbrace{\sum_{k=1}^{r-1}(-1)^k M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]M[R^{\{i,j\}},S_k\mid \underline{r}^{\{c_1,c_2\}}]}_{=0}$$
$$-\sum_{j=1}^{r-1}\sum_{i=j+1}^{r-1}(-1)^{r+c_1+c_2+i+j}m_{R_j,c_2}m_{R_i,c_1}\underbrace{\sum_{k=1}^{r-1}(-1)^k M\big[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}\big]M\big[R^{\{i,j\}},S_k\mid \underline{r}^{\{c_1,c_2\}}\big]}_{=0}$$
$$+\sum_{k=1}^{r-1}\sum_{j=1}^{r-1}(-1)^{c_1+c_2+j+k}\left(m_{R_j,c_1}m_{S_k,c_2}-m_{R_j,c_2}m_{S_k,c_1}\right)M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]M[R^{\{j\}}\mid \underline{r}^{\{c_1,c_2\}}]$$
$$=\sum_{k=1}^{r-1}\sum_{j=1}^{r-1}(-1)^{c_1+c_2+j+k}\left(m_{R_j,c_1}m_{S_k,c_2}-m_{R_j,c_2}m_{S_k,c_1}\right)M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]M[R^{\{j\}}\mid \underline{r}^{\{c_1,c_2\}}]$$
$$=\left(\sum_{j=1}^{r-1}(-1)^{j+c_1}m_{R_j,c_1}M[R^{\{j\}}\mid \underline{r}^{\{c_1,c_2\}}]\right)\left(\sum_{k=1}^{r-1}(-1)^{k+c_2}m_{S_k,c_2}M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]\right)$$
$$\quad-\left(\sum_{j=1}^{r-1}(-1)^{j+c_2}m_{R_j,c_2}M[R^{\{j\}}\mid \underline{r}^{\{c_1,c_2\}}]\right)\left(\sum_{k=1}^{r-1}(-1)^{k+c_1}m_{S_k,c_1}M[S^{\{k\}}\mid \underline{r}^{\{c_1,c_2\}}]\right)$$
$$=\left(M\big[R\mid \underline{r}^{\{c_2\}}\big]\right)\left(-M\big[S\mid \underline{r}^{\{c_1\}}\big]\right)-\left(M\big[R\mid \underline{r}^{\{c_1\}}\big]\right)\left(-M\big[S\mid \underline{r}^{\{c_2\}}\big]\right)$$
$$=M\big[R\mid \underline{r}^{\{c_1\}}\big]M\big[S\mid \underline{r}^{\{c_2\}}\big]-M\big[R\mid \underline{r}^{\{c_2\}}\big]M\big[S\mid \underline{r}^{\{c_1\}}\big]$$

as required. □

Whilst this result in its full generality is useful in the text we shall, on occasion, only require the following corollary which would surprise the author greatly were it to turn out to be original.

Corollary B.3. *Let A be an $n \times n$ matrix $(n \geq 3)$ and $1 \leq r_1 < r_2 \leq n$ and $1 \leq c_1 < c_2 \leq n$. Then*

$$\det(A)A\{r_1, r_2|c_1, c_2\} = A\{r_1|c_1\}A\{r_2|c_2\} - A\{r_1|c_2\}A\{r_2|c_1\} \tag{B.1}$$

Proof. Apply Lemma B.2 in the case $r = n$, and $R = \underline{r} \setminus r_1$ and $S = \underline{r} \setminus r_2$. $\square$

Appendix C

Subduction Calculations for Socle-Type $(1, 2, 1)$, Rank 2

Let $\mathbb{F} = \overline{\mathbb{F}}$ have characteristic $p > 2$ and define a the rank 2 matrix group

$$\mathbb{Z}_p^2 \cong G = \left\langle \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2c_1 & 2c_2 & c_1^2 + c_2^2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right\rangle$$

for $c_2 \neq 0$ as in Proposition 3.6.3. Consider the set $\{x_1, N_2^G, N_4^G, \delta, f_1, f_2\}$ where

$$\delta := x_3^2 + x_2^2 - x_1 x_4, \qquad f_1 := x_3^p - x_1^{p-1}\left(x_3 + \frac{c_1^p - c_1}{c_2} x_2\right),$$

$$f_2 := \frac{1}{2x_1^{p-1}}\left[\delta^p - f_1^2 - (N_2^G)^2 - 2x_1^{p-1}\delta^{(p+1)/2}\right].$$

Since $LT(\delta) = x_3^2$ and $f_1 \in \mathbb{F}[x_1, x_2, x_3]^G$ has minimal degree $\deg_{x_3}(f_1) = p$, by Theorem 3.5.1 if we apply the SAGBI/Divide-by-x_1 algorithm to the set $\{x_1, N_2^G, N_4^G, \delta, f_1\}$ we shall acquire a SAGBI basis for $\mathbb{F}[V]^G$. Here we show that all tête-à-tête differences arising from this set subduct to zero and thus it serves as a SAGBI basis for $\mathbb{F}[V]^G$.

One may observe that the tête-à-tête difference $\delta^p - f_1^2$ naturally gives rise to the construction of f_2. The only remaining tête-à-tête difference is given by either

$$\begin{array}{ll} c_2^p f_2^p + (c_1^p - c_1)^p N_2^G f_1^p & \text{if } \ c_1^p \neq c_1, \quad \text{or} \\ f_2^p + (N_2^G)^2 \delta^{\frac{p(p-1)}{2}} & \text{if } \ c_1^p = c_1. \end{array}$$

We undertake each of these cases together denoting for the sake of brevity

$$\widehat{f_1} := \frac{c_1^p - c_1}{c_2} f_1 + c_2^{p-1} N_2^G, \qquad \widehat{f_2} := f_2 + \delta^{(p+1)/2}.$$

Lemma C.1. *The invariant*

$$\begin{aligned}
F := {} & c_2^p f_2^p + (c_1^p - c_1)^p N_2^G f_1^p - c_2^p \Big[\delta^p - (N_2^G)^2\Big]^{\frac{p+1}{2}} + c_2^p \delta^{\frac{p(p+1)}{2}} - c_2^{p^2} (N_2^G)^{p+1} \\
& + x_1^{p-1} c_2^{2p-1} \left[\sum_{i=1}^{(p+1)/2} (-1)^{i+\frac{p+1}{2}} \binom{\frac{p+1}{2}}{i} \delta^i \widehat{f_2}^{\,p+1-2i} (\widehat{f_1}^{\,2} + f_1^2)^{i-1} \right. \\
& \left. + \widehat{f_2} \sum_{i=0}^{(p-1)/2} (-1)^{i+\frac{p+1}{2}} \widehat{f_1}^{\,2i} f_1^{p-1-2i} + N_2^G \widehat{f_2} \sum_{i=1}^{(p-1)/2} (-1)^{i+1} \widehat{f_1}^{\,2i-1} f_1^{p-1-2i} \right] \\
= {} & c_2^p \widehat{f_2}^{\,p} - c_2^p N_2^G \widehat{f_1}^{\,p} - c_2^p \Big[\delta^p - (N_2^G)^2\Big]^{\frac{p+1}{2}} + \frac{x_1^{p-1} c_2^{2p-1}}{\widehat{f_1}^{\,2} + f_1^2} \left[\Big(\delta(\widehat{f_1}^{\,2} + f_1^2) - \widehat{f_2}^{\,2}\Big)^{\frac{p+1}{2}} \right. \\
& \left. + \widehat{f_2} \Big((-1)^{\frac{p-1}{2}} \widehat{f_2}^{\,p} + (-1)^{\frac{p+1}{2}} f_1^{p+1} - \widehat{f_1}^{\,p+1} + (-1)^{\frac{p+1}{2}} \widehat{f_1}^{\,p} N_2^G + f_1^{p-1} \widehat{f_1} N_2^G \Big) \right],
\end{aligned} \tag{C.1}$$

satisfies $LM(F) = x_1^p x_4^{p^2}$.

Remark. The latter version of the relation is acquired from the former by liberal application of the binomial theorem and the well-known relation

$$\sum_{i=0}^{\alpha} X^i Y^{\alpha-i} = \frac{X^{\alpha+1} - Y^{\alpha+1}}{X - Y}.$$

We posit both forms since the first is more indicative of the subduction we would undertake, yet the latter is far more straightforward to manipulate. Should this relation hold replacing N_4^G with $x_1^{-p} F$ in our proposed set effectively terminates the algorithm, from which the result shall then follow. What remains is to prove the relation.

Proof of Lemma C.1. Denoting by $L.O.T.s$ any term below $x_1^p x_4^{p^2}$ in the graded reverse lexicographic order,

$$[\delta^p - (N_2^G)^2]^{(p+1)/2} - \delta^{p(p+1)/2} = x_3^{p(p+1)} + (c_2 x_1)^{p-1} x_2^{p+1} x_3^{p(p-1)} - (x_3^{2p} + x_2^{2p})^{\frac{p+1}{2}} + L.O.T.s$$

and thus the x_1-free terms in (C.1) amount to

$$\begin{aligned}
&c_2^p\widehat{f_2}^{\,p} - c_2^p N_2^G \widehat{f_1}^{\,p} - c_2^p\Big[\delta^p - (N_2^G)^2\Big]^{\frac{p+1}{2}} \\
&= \Big[c_2^p x_3^{p^2+p} + c_2^{p^2} x_2^{p^2+p} + (c_1^{p^2} - c_1^p)x_2^p x_3^{p^2} - \tfrac{1}{2}c_2^p x_1^p x_4^{p^2}\Big] \\
&\quad - (c_2^p x_2^p - c_2^{2p-1}x_1^{p-1}x_2)\Big[\tfrac{(c_1^p-c_1)^p}{c_2^p}x_3^{p^2} + c_2^{p^2-p}x_2^{p^2}\Big] \\
&\quad - c_2^p\Big[x_3^{2p} + 2c_2^{p-1}x_1^{p-1}x_2^{p+1}\Big]^{\frac{p+1}{2}} + L.O.T.s \\
&= +c_2^{2p-1}x_1^{p-1}\left[\tfrac{c_1^{p^2}-c_1^p}{c_2^p}x_2x_3^{p^2} + c_2^{p^2-p}x_2^{p^2+1} - x_2^{p+1}x_3^{p^2-p}\right] - \tfrac{1}{2}c_2^p x_1^p x_4^{p^2} + L.O.T.s.
\end{aligned}$$

We then show that the bracketed terms in (C.1) cancel the bracketed terms given above by showing that

$$\begin{aligned}
&\frac{1}{\widehat{f_1}^{\,2} + f_1^2}\Big[\widehat{f_2}\Big((-1)^{\frac{p-1}{2}}\widehat{f_2}^{\,p} + (-1)^{\frac{p+1}{2}}f_1^{p+1} - \widehat{f_1}^{\,p+1} + (-1)^{\frac{p+1}{2}}\widehat{f_1}^{\,p}N_2^G + f_1^{p-1}\widehat{f_1}N_2^G\Big) \\
&+ \Big(\delta(\widehat{f_1}^{\,2} + f_1^2) - \widehat{f_2}^{\,2}\Big)^{\frac{p+1}{2}}\Big] = -\frac{c_1^{p^2}-c_1^p}{c_2^p}x_2x_3^{p^2} - c_2^{p^2-p}x_2^{p^2+1} + x_2^{p+1}x_3^{p^2-p} + L.O.T.s \\
&= x_2\left(x_3^{p-1}x_2 - \widehat{f_1}\right)^p + L.O.T.s
\end{aligned}$$

where hereafter $L.O.T.S$ refers to any monomial below $x_1x_4^{p^2}$.

As a warning we compel the reader to verify that no $x_1x_4^{p^2}$ term can possibly appear in above expression, as such terms may interfere we the desired lead term of F.

Firstly we note that $\widehat{f_2} = x_3^{p+1} + x_2\widehat{f_1}$, allowing us to write

$$\delta(\widehat{f_1}^{\,2} + f_1^2) - \widehat{f_2}^{\,2} = x_3^2(\widehat{f_1} - x_2x_3^{p-1})^2 + L.O.T.s$$

and

$$\widehat{f_2}^{\,p} = f_1^{p+1} + N_2^G\widehat{f_1}^{\,p} + L.O.T.s.$$

Using these we simplify thus:

$$\begin{aligned}
&\frac{1}{\widehat{f_1}^{\,2} + f_1^2}\Big[\widehat{f_2}\Big((-1)^{\frac{p-1}{2}}\widehat{f_2}^{\,p} + (-1)^{\frac{p+1}{2}}f_1^{p+1} - \widehat{f_1}^{\,p+1} + (-1)^{\frac{p+1}{2}}\widehat{f_1}^{\,p}N_2^G + f_1^{p-1}\widehat{f_1}N_2^G\Big) \\
&+ \Big(\delta(\widehat{f_1}^{\,2} + f_1^2) - \widehat{f_2}^{\,2}\Big)^{\frac{p+1}{2}}\Big]
\end{aligned}$$

$$= \frac{1}{\widehat{f_1}^2 + f_1^2} \left[\widehat{f_1}(x_3^{p+1} + x_2 \widehat{f_1}) \left(x_3^{p^2-p} x_2^p - \widehat{f_1}^p \right) + x_3^{p+1} (\widehat{f_1} - x_2 x_3^{p-1})^{p+1} \right] + L.O.T.s$$
$$= \frac{1}{\widehat{f_1}^2 + f_1^2} \left[x_2 \left(\widehat{f_1}^2 + x_3^{2p} \right) \left(x_3^{p-1} x_2 - \widehat{f_1} \right)^p \right] + L.O.T.s$$
$$= x_2 \left(x_3^{p-1} x_2 - \widehat{f_1} \right)^p + L.O.T.s$$

Thus the bracketed terms of (C.1) have the desired form, F has the desired lead term, our set forms a SAGBI basis and thus the result follows. □

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