A value-at-risk approach to optimisation of warranty policy

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Abstract

In the real world, a manufacturer may produce many products, which may have common components installed. Consequently, the frequencies of the warranty claims of those products are statistically dependent. Warranty policy optimisation in the existing research, however, has not considered such statistical dependence, which may increase bias in decision making. This paper is the first attempt to collectively optimises warranty policy for a set of different products, produced by one manufacturer, whose failures are statistically dependent, using tools borrowed from financial mathematics (i.e., value-at-risk theory and copula). We prove the existence of the optimal solutions for different scenarios. Numerical examples are used to validate the applicability of the proposed methods.

Keywords: (T) value-at-risk, warranty policy optimisation, mean-risk, copulas.

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1 Introduction

Warranty is essentially offered with most durable products, which may be legally mandated or market driven: the European Union (EU) passed legislation requiring a two-year warranty for all products sold in Europe (S. Wu, 2014b). Warranty expense is an important part of a manufacturer's operating expense, for example, the automotive industry is the most warranty-intensive of all: the total automotive OEM (original equipment manufacturer) warranty claims paid by the entire U.S.-based manufacturers is $10,097 million in 2015 (WarrantyWeek, 2016). To improve a manufacturer's operating expense management, an efficient warranty policy management supported by warranty data analysis and modelling is necessary. It is known that the number of warranty claims and the associated cost are uncertain, and products with longer warranty periods may attract more buyers than those with shorter warranty periods. From a

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manufacturer's perspective, however, providing a longer warranty period implies more resources that are needed to handle warranty claims. Hence, there is a need to develop approaches to optimising the warranty price and the warranty length.

In the literature, many methods aiming to optimise the warranty price and the warranty length of an individual product have been proposed. Fig. 1 illustrates the evolution of the research in warranty policy optimisation, which shows that the research evolves from simple and unrealistic assumptions to more complex and realistic ones.
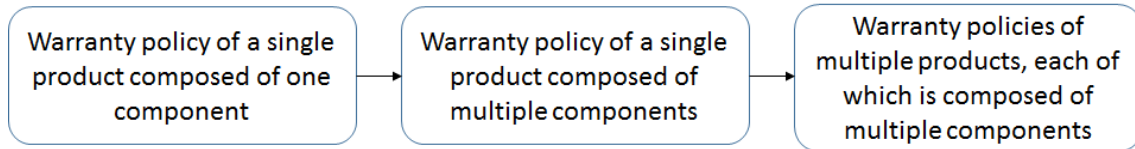

Fig. 1. Evolution of warranty policy optimisation

At the early stage, many researchers attempt to find the optimal price and warranty length, assuming that the product is composed of only one component (see the most-left rectangle in Fig. 1). At the same time, some other factors, such as production rate, market competition and demand, etc., are also considered. Ladany and Shore (2007) address a method to determine the optimal warranty period with considering the products lifetime and market demand. Lin, Wang, and Chin (2009) optimise the price, warranty length and production rate of a one component system dynamically. Wu, Chou, and Huang (2009) develop a decision model to determine the optimal price, the length of warranty and the production rate to maximise profit based on the pre-determined life cycle in a static demand market. Aggrawal, Anand, Singh, and Singh (2014) present a method to optimise warranty price and the length of warranty for a product based on a two-dimensional innovation diffusion model, and estimate the overall maximum profit for the manufacturer. Wei, Zhao, and Li (2015) investigates the optimal strategies on product price and the length of warranty of two products produced by two manufacturers and sold by one dealer. Yazdian, Shahanaghi, and Makui (2016) jointly optimises the acquisition price, re-manufacturing degree, selling price and the length of warranty of re-manufacturing products under linear and non-linear demand functions. Lei, Liu, and Shum (2017) price warranty policy of a single product dynamically with considering consumer learning, i.e. consumers' demand reacting on the warranty price change; and find warranty sales do not generate profit directly though are profitable overall.

The one-component assumption of a single product may be too unrealistic. Researchers then consider the assumption that a product is composed of multiple components (see the middle rectangle in Fig. 1). Huang, Liu, and Murthy (2007) develop a model to determine the optimal product reliability, price and warranty strategy to achieve the maximum total integrated profit for a general repairable multi-component product sold under a free replacement-repair warranty strategy. Matis, Jayaraman, and Rangan (2008) explore the optimal price and pro rate warranty length for a multi-component product with considering the different repair options on the com-

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ponents. Bai and Pham (2006) investigate optimisation of warranty policies for single products composed of multiple components. Liu, Wu, and Xie (2015) also investigate the warranty cost for a single product consisting of multiple components; meanwhile, as an improvement, the failure interactions between the components are considered. Ahmadi (2016) addresses an optimal replacement problem for complex multi-component systems by determining an optimal operating time which balances income and cost to maximizes the expected profit over a cycle. Adkins and Paxson (2017) construct a general replacement model for a multi-component product with considering the salvage value and depreciation in operating. Chen, Lo, and Weng (2017) seek to maximize the total profit per item of a multi-component product through optimally determine the production run length and the warranty period.

All of the literature mentioned above solely maximises the profit of individual products produced by a manufacturer. Little research, however, has been devoted to optimizing warranty policy of a set of products collectively. However, in real world, a manufacturer may assemble different types of product by sharing some key components; then, the manufacturer has to deal with the warranty policies for multiple products considered as multi-component systems, which is the third stage of warranty policy optimisation research (see the most-right rectangle in Fig. 1) and investigated in this paper.

The warranty claim arrival processes of products may not be statistically independent because the claims may have common causes such as similar design, same production lines and same types of components installed in the products. For example, Apple, iPhone 6, iPhone 6 Plus and iPad Mini 4, have the same type of CPU. If any design or quality problems happen on one of the products, warranty claims of the other products will crop up during a short period.

As can be seen from the above literature review, however, little attention has been paid to collectively optimise warranty policies for a portfolio of different products. This motivates us to develop novel approaches to filling in the knowledge gap.

We propose to optimise warranty policy through maximising product profit. The method collectively optimises the warranty price and the warranty length of a set of different products whose warranty claims are statistically dependent, considering the uncertainty of the product profit. The value-at-risk theory is borrowed to manage the uncertainty. The dependence is modelled by *copulas*, a tool from the probability theory. The use of copulas provides a more flexible tool to model more complicated dependence than a simple method such as covariance estimation.

The novelty of this paper lies in the fact that it is the first attempt to collectively optimise the warranty policies of a portfolio of products from a manufacturer's perspective.

The rest sections of the paper are structured as follows. Section 2 lists the assumptions of the optimisation problems. It also formulates the profits and warranty costs of individual products and portfolio of products, respectively. Section 3 investigates the existence of the optimal solutions for the different optimisation problems and uses copulas to model the dependence among warranty claims of different products. Section 4 offers numerical examples to illustrate

the proposed methods and to validate the applicability of the proposed methods. Section 5 concludes the paper and proposes our future work.

2 Formulation of the problem

Assume a manufacturer offers non-renewing free replacement warranty (NFRW) policies. Under an NFRW policy, the manufacturer provides its customers with repair or replacement at no cost within the warranty period; the original warranty is not altered upon a failed item; and the manufacturer only guarantees satisfactory service on the item within the original warranty period. Assume that repair time is negligible and the repair is minimal repair. Products are new at $t = 0$ when they are sold. The number of claims follows the non-homogeneous Poisson process (NHPP). The numbers of warranty claims and the claim cost are statistically independent.

The notations in Table 1 are used throughout this paper.

Table 1

Notation table

$X_{k,i}$	Cost of the ith warranty claim of product k
$N_k(t)$	Number of warranty claims of product k within time interval $(0, t)$
$S_k(t)$	Total cost of warranty claims of product k within time interval $(0, t)$
P_k	Price of product k
T_k	Warranty length of product k
$\boldsymbol{P}$	Vector of prices of products
$\boldsymbol{T}$	Vector of lengths of warranties
λ_k	Parameter of the claim arrival process of product k
μ_k	Expected cost per claim of product k, mean of $X_{k,i}$
σ_k	Standard deviation of $X_{k,i}$
M_k	Sales volume of product k, which is a function of P_k and T_k
$\boldsymbol{M}$	Vector of all sales volume
ω_k	Profit of product k, which is a function of P_k and T_k
Ω	Total profit of the manufacturer's products portfolio

2.1 The sales volume and profit

For a product, there are two critical marketing variables: the selling price and the warranty length (Chen et al., 2017). The sales volume of a product is negatively related to its selling

price and positively related to its warranty length. Denote P_k and T_k as the selling price P_k and the warranty length T_k of product k, respectively. Both P_k and T_k can influence the sales volume, M_k, and profit, ω_k. In what follows, the profit of product k is the revenue deducting the warranty cost, i.e. $\omega_k = M_k P_k - S_k(T_k)$, where $S_k(T_k)$ is the aggregated warranty cost of product k within T_k.

In the literature, the sales volume of product k, M_k, is expressed by a function of product price P_k and length of warranty T_k in different forms, including linear (Lin et al., 2009; Yazdian et al., 2016) and non-linear ones (Huang et al., 2007; Ladany & Shore, 2007; Xie, Liao, & Zhu, 2014). For simplicity, a linearity form, introduced by Yazdian et al. (2016), is used in this paper. The sales volume is defined by

$$M_k = A_k - \beta_k P_k + \eta_k T_k, \tag{1}$$

where $A_k(> 0)$ is a constant relating to the market size of product k, and $\beta_k(> 0)$ and $\eta_k(> 0)$ are the price and length of warranty elasticities, respectively.

2.2   The distribution of the aggregated warranty cost

Suppose a manufacturer produces n products. The aggregated warranty cost of product k follows a stochastic process $\{S_k(T_k)\}_{T_k \geq 0}$ over the warranty period $(0, T_k)$, which is expressed by the following equation,

$$S_k(T_k) = \sum_{j=1}^{N_k(T_k)} X_{k,j}, \tag{2}$$

where $X_{k,j}$ is the cost of the j-th claim of product k and $N_k(T_k)$ is the number of claims during $(0, T_k)$. $\{X_{k,1}, X_{k,2}, \ldots, X_{k,j}\}$ are independent and identically distributed random variables which have finite values on the positive half-line $\mathbb{R}_{>0}$ with the probabilities $P(X_{k,j})$. The cost of claims is assumed to follow the log-normal distribution. The counting process $N_k(T_k)$ is assumed to take a form of the NHPP with cumulative intensity $M_k \Lambda_k(T_k)$, and $P(N_k(T_k) = n) = \frac{(M_k \Lambda_k(T_k))^n}{n!} e^{-M_k \Lambda_k(T_k)}$. $N_k(t)$ and $X_{k,i}$, are assumed to be statistically independent.

The expected value of $S_k(T_k)$ is given by

$$\mathrm{E}[S_k(T_k)] = \mathrm{E}[N_k(T_k)]\mathrm{E}[X_k] = M_k \Lambda_k(T_k)\mu_k, \tag{3}$$

and the variance of $S_k(T_k)$ is given by

$$\begin{aligned}
\mathrm{Var}\left[S_k(T_k)\right] &= \mathrm{E}[N_k(T_k)]\mathrm{Var}[X_k] + \mathrm{Var}[N_k(T_k)]\mathrm{E}[X_k]^2 \\
&= M_k \Lambda_k(T_k)(\mathrm{Var}[X_k] + \mathrm{E}[X_k]^2) \\
&= M_k \Lambda_k(T_k)(\sigma_k^2 + \mu_k^2),
\end{aligned} \tag{4}$$

where μ_k and σ_k are the mean and the variance of X_k, respectively.

Denote F_{S_k} and f_{S_k} as the probability density function (pdf) and the cumulative distribution function (cdf) of $S_k(T_k)$, respectively. The characteristic function of X_k is $\varphi_{X_k}(t) = \int_{-\infty}^{\infty} f_{X_k}(x)e^{itx}dx$, where i is a unit imaginary number. Denote $\varphi_{S_k}(t)$ as the characteristic function of S_k. The probability generation function of $N_k(T_k)$ is $\psi_k(s) = \sum_{n=0}^{\infty} s^n p_{k,n}$, where $p_{k,n} = \Pr\{N(T_k) = n\}$. According to the Levy-Khintchine formula, one can obtain

$$\varphi_{S_k}(t) = \sum_{n=0}^{\infty} (\varphi_{X_k}(t))^n p_{k,n} = e^{M_k \Lambda_k(T_k)(\varphi_{X_k}(t)-1)}. \tag{5}$$

Then, the density of $S_k(T_k)$ can be calculated through the inverse Fourier transform, that is

$$f_{S_k}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{S_k}(t)e^{-itx}dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{M_k \Lambda_k(T_k)(\varphi_{X_k}(t)-1)-itx}dt.$$

Apparently, $S_k(T_k)$ is non-negative. According to Luo and Shevchenko (2009), the probability density and cumulative distribution function of $S_k(T_k)$ are given by

$$f_{S_k}(z) = \frac{2}{\pi} \int_0^{\infty} Re[\varphi_{S_k}(t)]\cos(tz)dt, \tag{6}$$

and

$$F_{S_k}(z) = \frac{2}{\pi} \int_0^{\infty} Re[\varphi_{S_k}(t)]\frac{\sin(tz)}{t}dt, \tag{7}$$

respectively, where $z \geq 0$.

In practice, computing the density and cumulative distribution through the Fourier transform requires high computing power to deal with the underflow and overflow problems. Another popular approach is to use an approximating distribution to avoid direct calculation of Eq. (7) and Eq. (6). According to Bee (2016), if the distribution of X_k is subexponential, the compound $S_k(T_k)$ inherits the subexponentiality property from X_k. In this case, X_k is assumed to follow the log-normal distribution, which is subexponential. Hence, $S_k(T_k)$ is also subexponential, and X_k and $S_k(T_k)$ are tail equivalent. For details about the definition of *tail equivalent*, the reader is referred to Bee (2016).

To make this paper concise, we use the log-normal distribution to approximate $S_k(T_k)$.

2.3 Product profit

In this subsection, we derive the profit of products for different scenarios.

2.3.1 The profit of one product

The profit of product k is given by

$$\omega_k = M_k P_k - S_k(T_k). \tag{8}$$

Denote F_{ω_k} and f_{ω_k} as the cdf and pdf of ω_k, respectively. Then

$$
\begin{aligned}
F_{\omega_k}(z) &= P[\omega_k \le z] \\
&= P[M_k P_k - S_k(T_k) \le z] \\
&= P[S_k(T_k) \ge M_k P_k - z] \\
&= 1 - F_{S_k}(M_k P_k - z),
\end{aligned}
\tag{9}
$$

which can be calculated on the basis of Eq. (7).

The expected value of ω_k is

$$
\mathrm{E}[\omega_k] = M_k[P_k - \Lambda_k(T_k)\mu_k],
\tag{10}
$$

and the variance of ω_k is

$$
\mathrm{Var}[\omega_k] = \mathrm{Var}[S_k(t)] = M_k \Lambda_k(T_k)(\sigma_k^2 + \mu_k^2).
\tag{11}
$$

2.3.2 The profit of a portfolio of products

The total profit of a portfolio consisting of N products is given by

$$
\Omega = \sum_{k=1}^{N} \omega_k = \sum_{k=1}^{N} [M_k P_k - S_k(T_k)].
\tag{12}
$$

The distribution of the profit of product portfolio can be expressed by

$$
F_\Omega(z) = P\{\Omega \le z\} = 1 - F^{(N)}(z),
\tag{13}
$$

where $F^{(N)}(z)$ is the N-fold convolution of the distribution of ω_k. The expected total profit is

$$
\mathrm{E}[\Omega] = \sum_{k=1}^{N} \mathrm{E}[\omega_k] = \sum_{k=1}^{N} M_k[P_k - \Lambda_k(T_k)\mu_k].
\tag{14}
$$

As we mentioned in Section 1.2, the warranty claim arrival processes of the products produced by the same manufacturer may be correlated. Hence the variance of the portfolio profit is related to the correlation among the products. If the claim arrival processes of different products are linearly correlated, the variance of Ω can be calculated based on the covariance matrix of the portfolio, which is

$$
\mathrm{Var}[\Omega] = \boldsymbol{I}^T \boldsymbol{V} \boldsymbol{I},
\tag{15}
$$

where $\boldsymbol{I}^T = [1, 1, \ldots, 1]$ and

$$\boldsymbol{V} = \begin{bmatrix} \text{Var}(S_1(T_1)) & \text{Cov}(S_1(T_1), S_2(T_2)) & \ldots & \text{Cov}(S_1(T_1), S_n(T_n)) \\ \text{Cov}(S_2(T_2), S_1(T_1)) & \text{Var}(S_2(T_2)) & \ldots & \text{Cov}(S_2(T_2), S_n(T_n)) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{Cov}(S_n(T_n), S_1(T_1)) & \text{Cov}(S_n(T_n), S_2(T_2)) & \ldots & \text{Var}(S_n(T_n)) \end{bmatrix}, \quad (16)$$

where $\text{Cov}(S_k(T_k), S_i(T_i)) = \rho_{k,i}\sqrt{\text{Var}(S_k(T_k))\text{Var}(S_i(T_i))}$, and $\rho_{k,i}$ is the Pearson correlation coefficient. The warranty costs of different products may have different types of dependence such as a rank correlation or a tail-dependence. In such cases, the metrics that can only measure a linear correlation may be inappropriate in the case where the relationship of the variables is not linear. As such, we employ a powerful tool, *copula*, to model the dependence among the products, which will be discussed in Section 3.3. According to Boubaker and Sghaier (2013), Kendall's tau and copula parameters, especially the parameters of the Archimedean copulas, can be used as substitutes (and more comprehensive metrics) for measuring both the linear and the nonlinear relationships in the covariance matrix in portfolio optimisation.

3 Mean-risk optimisation

In this paper, the mean-risk optimisation is used to maximise the expected profit under an acceptable risk level. The objective functions are illustrated in Eq. (10) and Eq. (14) for one product and multiple products scenarios, respectively. The constraints of optimisation problems are given for different risk measures.

In the one product scenario, the manufacturer aims to maximise the following function:

$$\text{E}[\omega_k] = M_k[P_k - \Lambda_k(T_k)\mu_k].$$

Suppose in this case, the NHPP following the cumulative intensity $\Lambda_k(T_k) = a_k T_k^{b_k}$ (where $b_k > 1$) are used. Then, the objective function is defined by

$$\text{E}[\omega_k] = A_k P_k - A_k \mu_k a_k T_k^{b_k} - \beta_k P_k^2 + \beta_k \mu_k P_k a_k T_k^{b_k} + \eta_k P_k T_k - \eta_k \mu_k a_k T_k^{b_k+1}. \quad (17)$$

In case both of P_k and T_k are decision variables, based on the properties of this function, we have Proposition 1.

Proposition 1 *Depending on whether P_k and T_k are known, one can prove the following results.*

- *If both of P_k and T_k are decision variables, the global maxima of $\text{E}[\omega_k]$ does not exist.*
- *If T_k (or P_k) is the decision variable and P_k (or T_k) is known, the global maxima of the*

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function $\mathrm{E}[\omega_k](T_k)$ *(or* $\mathrm{E}[\omega_k](P_k)$*) exists.*

The proof of Proposition 1 is presented in the Appendix.

In a multiple product scenario, the objective function to be maximised is

$$\mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k(P_k - \Lambda_k(T_k)\mu_k).$$

Since the prices and sales volumes are assumed to be mutually independent in this paper, Proposition 1 is also valid in the multiple products scenario. The dependence among the warranty claims of the products is reflected in the constraints of optimisations.

3.1 The risk measure

In finance, Artzner, Delbaen, Eber, and Heath (1999) define risk as the variability of the future value of a position due to uncertain events. Babaei, Sepehri, and Babaei (2015) point out that risk is used to characterise the situation in which a portfolio is exposed to vulnerabilities and enforces losses to the institutions. Risk measures are introduced for the requirement of quantifying the losses that may be incurred. The variance of a random variable is considered as a risk measure by the overwhelming influential models of portfolio selection. However, since the variance is a symmetric risk measure, researchers turn to using downside risk measures, such as Value-at-Risk (VaR) and Conditional VaR (Expected Shortfall), which can reflect a better notion of risk (Babaei et al., 2015). Furthermore, the variance as a risk measure is normally applied under the assumption that the correlation is linear. Such an assumption is not imposed in the VaR and CVaR theories. In this paper, we focus on maximising the profit under the mean-risk framework, in which the risk may be measured by the variance, VaR or CVaR.

3.2 One product scenario

In the one product scenario, the manufacturer aims to maximise the expected profit of one product at an acceptable risk level. The optimisation problems with different risk measures are discussed in the following subsections.

3.2.1 Mean-variance framework

A mean-variance framework aims to maximise the expected profit of product k under a given/acceptable value of variance. The optimisation problem is defined by

$$\max \quad \mathrm{E}[\omega_k] = M_k[P_k - \Lambda_k(T_k)\mu_k],$$

$$s.t. \quad \mathrm{Var}[\omega_k] = M_k\Lambda_k(T_k)(\sigma_k^2 + \mu_k^2) \leq \psi,$$

$$M_k = A_k - \beta_k P_k + \eta_k T_k \geq 0,$$

$$P_k > 0,$$

$$T_k \geq L_T,$$

where ψ is the acceptable risk level of the manufacturer, and L_T is the legal minimum limit of the length of warranty, e.g. the manufacturers should provide at least 2-year warranty in Europe. According to Proposition 1, only one variable, either P_k or T_k, is treated as the decision variable in optimisation. Regarding this mean-variance optimisation, for example, if P_k is the decision variable, the constraint is $P_k \geq \frac{1}{\beta_k}\left(A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2 + \mu_k^2)}\right)$, and the objective function can achieve the global maxima at $P_k = \frac{A_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k}{2\beta_k}$. If $\frac{1}{\beta_k}\left(A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2 + \mu_k^2)}\right) < \frac{A_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k}{2\beta_k}$, the objective function is maximized at $P_k = \frac{A_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k}{2\beta_k}$, otherwise, the objective function is maximized at $P_k = \frac{1}{\beta_k}\left(A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2 + \mu_k^2)}\right)$. Then, we have Proposition 2.

Proposition 2 *If T_k (or P_k) is the decision variable, P_k (or T_k) is known and the power law parameter $b_k > 1$, then the optimal solution, which maximises the expected profit of product k under a given variance level, exists.*

The proof of Proposition 2 can be find in the Appendix.

3.2.2 Mean-VaR framework

In general, value-at-risk is the α-quantile of a distribution, where α is a given confidence level. In this case, ω_k is the profit of product k, we defined the $\text{VaR}_\alpha(\omega_k)$ as the minimum profit at the $(1-\alpha)$ level, i.e. there is a $(1-\alpha)$ probability that the profit of product k will be greater than $\text{VaR}_\alpha(\omega_k)$. The $\text{VaR}_\alpha(\omega_k)$ can be expressed as

$$\text{VaR}_\alpha(\omega_k) = F_{\omega_k}^{-1}(\alpha). \tag{18}$$

According to Eq. (9),

$$\text{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1 - \alpha). \tag{19}$$

Then, the optimisation problem can be defined by

$$\max \quad \text{E}[\omega_k] = M_k(P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \text{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1 - \alpha) \geq \phi,$$

$$M_k \geq 0, \quad P_k > 0, \quad T_k \geq L_T,$$

where α is a given confidence level, and ϕ is the minimum profit at the $(1-\alpha)$ confidence level set by the manufacturer. If $\phi < 0$, the maximum loss should be less than $-\phi$ at the $(1-\alpha)$ confidence level. If P_k is the decision variable and T_k is known, then $M_k P_k$ in $\text{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1 - \alpha) \geq \phi$ is a parabola with a negative coefficient on the quadratic term, and $F_{S_k}^{-1}(1 - \alpha)$ can be derived from the distribution of warranty cost S_k. If T_k is known, the mean

and the variance of S_k are $(A_k - \beta_k P_k + \eta_k T_k)\Lambda_k(T_k)\mu_k$ and $(A_k - \beta_k P_k + \eta_k T_k)\Lambda_k(T_k)(\sigma_k^2 + \mu_k^2)$, respectively. Both of the quantities monotonously decrease with respect of P_k. $F_{S_k}^{-1}(1 - \alpha)$ decreases monotonously with P_k as well. As such, the feasible range of P_k is $(\max(0, P_l), P_u)$, where P_l and P_u are defined by $\text{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1 - \alpha) \geq \phi$, respectively.

3.2.3 Mean-CVaR framework

Denote $\text{CVaR}_\alpha(\omega_k)$ as the conditional value-at-risk, where α is a given probability. if the profit exceeds the $\text{VaR}_\alpha(\omega_k)$ on the left tail, $\text{CVaR}_\alpha(\omega_k)$ is the expected profit of the product k. In other word, $\text{CVaR}_\alpha(\omega_k)$ is the expected profit of the product k in the worst $\alpha\%$ of cases. In this paper, $\text{CVaR}_\alpha(\omega_k)$ is expressed as

$$\text{CVaR}_\alpha(\omega_k) = \frac{1}{\alpha} \int_{-\infty}^{\text{VaR}_\alpha(\omega_k)} z f_{\omega_k}(z)dz, \tag{20}$$

where $f_{\omega_k}(z)$ is the pdf of ω_k. The distribution of ω_k is determined by the distribution of S_k. Hence $\text{CVaR}_\alpha(\omega_k)$ can be calculated by

$$\text{CVaR}_\alpha(\omega_k) = M_k P_k - \frac{1}{\alpha} \int_{F_{S_k}^{-1}(1-\alpha)}^{+\infty} z f_{S_k}(z)dz, \tag{21}$$

where $f_{S_k}(z)$ and $F_{S_k}^{-1}(1 - \alpha)$ are defined by Eq. (6) and Eq. (7), respectively. They can be approximated by the log-normal distribution, as shown in Section 4. The optimisation problem can therefore be expressed by

$$\max \quad \text{E}[\omega_k] = M_k(P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \text{CVaR}_\alpha(\omega_k) = M_k P_k - \frac{1}{\alpha} \int_{F_{S_k}^{-1}(1-\alpha)}^{+\infty} z f_{S_k}(z)dz \geq \delta,$$

$$M_k \geq 0, P_k > 0, T_k \geq LT,$$

where α is a given confidence level and δ is the acceptable minimum level of $\text{CVaR}_\alpha(\omega_k)$ set by the manufacturer.

Referring to Propositions 1 and 2, and the above discussions in Section 3.2.2 and 3.2.3, one can obtain the following proposition.

Proposition 3 *For T_k and P_k, if one of them is the decision variable and the other is known, then the optimal solution, which maximises the expected profit of product k under a given value-at-risk level or conditional value-at-risk level, exists.*

3.3 Multiple product scenario

The preceding section investigates the scenarios of the optimisation problems of individual products under a risk-informed consideration.

Assuming that a manufacturer produces N products, one can easily estimate the expected

total profit through estimating and then summing the expected profit of each individual product. Considering the uncertainty in estimation of the total profit of the N products, one may consider the statistical dependence among the warranty claim arrival processes. In this section, the portfolio optimisation is investigated under the mean-variance, mean-VaR and mean-CVaR frameworks.

3.3.1 Mean-Variance framework

Under the mean-variance framework for the N products, the optimisation problem is defined by

$$\max \quad \mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k (P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \mathrm{Var}[\Omega] = \sum_{k=1}^{N} M_k T_k \lambda_k (\sigma_k^2 + \mu_k^2) + 2 \sum_{i \neq j}^{N} \rho_{i,j} \sqrt{M_i M_j T_i T_j \lambda_i \lambda_j (\sigma_i^2 + \mu_i^2)(\sigma_j^2 + \mu_j^2)} \leq \psi.$$

In this optimisation problem, the dependence among the warranty claims of the different products is measured with the Pearson correlation coefficient $\rho_{i,j}$, and the objective function is the sum of the expected profits of the N products. According to Proposition 2, one can obtain the following proposition.

Proposition 4 *For $\boldsymbol{T}$ and $\boldsymbol{P}$, if one of them is the decision variable vector and the other is known, then the optimal solution, which maximises the expected total profit $\mathrm{E}[\Omega]$ of the product portfolio under a given variance level, exists.*

3.3.2 Mean-VaR framework

Under the mean-VaR framework for the N products, the optimisation problem is defined by

$$\max \quad \mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k (P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \mathrm{VaR}_{\alpha}[\Omega] \geq \phi.$$

In practice, one may need to obtain the optimal solution. According to Babaei et al. (2015), even though the definition of the VaR is intuitive and easy to interpret, calculating the VaR of a portfolio is not easy. In financial mathematics, to calculate the portfolio VaR, there are three commonly used methods, which are the variance-covariance, stochastic simulation, and historical simulation methods. The variance-covariance method assumes that the risk factors are jointly normally distributed. The normality assumption may not hold in our case and therefore the variance-covariance method is not applicable. The VaR of the total profit of the product portfolio will be calculated using the stochastic simulation method in this research. The historical simulation method will be investigated in our future work. To conduct an exact and efficient simulation, copula, an important tool in the probability theory is borrowed.

Copulas are widely used in constructing multivariate distributions and formalising the dependence structures between random variables, whatever discrete or continuous. Abe Sklar

first introduced the notion of copula in 1959 (Sklar, 1959). In recent years, copulas have attracted considerable attention in both theoretical and application aspects. Sklar's theorem states that any cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula that describes the dependence structure between the variables (Sklar, 1959). Assume $(X_1, ..., X_d)$ is a given vector of random variables, its cumulative distribution function is $H(x_1, ..., x_d) = P(X_1 \leq x_1, ..., X_d \leq x_d))$, and its marginals are $F_k(x_k) = P(X_k \leq x_k)$, where $k = 1, ...d$. Sklar proved that $H(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$, where $C(.)$ is a copula. Copulas are useful in statistical applications because they allow one to estimate the marginals and the copula separately when modelling and estimating the distribution of a random vector. It has recently been used in modelling warranty claims (S. Wu, 2014a).

Denote the joint distribution of the products' number of claims by

$$H(z_1, z_2, \ldots, z_k) = C(F_{N_1}(z_1), F_{N_2}(z_2), \ldots, F_{N_k}(z_k)), \tag{22}$$

where $C(.)$ is a copula, and F_{N_k} is the CDF of the number of claims of product k. The density of the joint distribution is given by

$$h(z_1, z_2, \ldots, z_N) = c(F_{N_1}(z_1), F_{N_2}(z_2), \ldots, F_{N_k}(z_k)) \prod_{k=1}^{n} f_{N_k}(z_k), \tag{23}$$

where $c(.)$ is the density of copula $C(.)$.

Then, the joint distribution of the products' profits can be simulated based on Eq. (6), Eq. (7), Eq. (22) and Eq. (23).

Let $F_\Omega(z)$ be the distribution of the total profit of the products, where $\Omega = \sum_{k=1}^{N} \omega_k$. In a copula-based model, the VaR of the total profit can be calculated through simulation. It is clear that calculating $F_\Omega(z)$ is mainly a numerical issue (Bernard & Vanduffel, 2015).

3.3.3 Mean-CVaR framework

Under the mean-CVaR framework, the optimisation problem is defined by

$$\max \quad \mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k(P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \mathrm{CVaR}_\alpha[\Omega] \geq \delta.$$

This optimisation problem implies that the manufacturer aims to maximise the total profit under the constraint that the expected extreme profit at confidence level α is not less than δ. This constraint can also be expressed by the copula and marginal distributions.

Based on the property of the above objective function and Proposition 3, one can derive the following proposition

Proposition 5 *For $\boldsymbol{T}$ and $\boldsymbol{P}$, if one of them is the decision variable vector and the other is*

known, then the optimal solution, which maximises the expected total profit $\mathrm{E}[\Omega]$ *of the product portfolio under a given value-at-risk level or conditional value-at-risk level, exists.*

4 Numerical examples

4.1 One product scenario

Assume that the cost of each warranty claim of one product, X_1, follows a log-normal distribution with mean $\mu_1 = 200$ and standard deviation $\sigma = 40$; assume the warranty claim arrival process is a Non-homogeneous Poisson process with cumulative intensity $\Lambda_1(T_1) = 0.004T_1^{1.04}$; and the sales volume of this product is defined by $M_1 = 1,000 - 0.2P_1 + 0.13T_1$. In Table 2, L_{μ_1}

Table 2
Parameters for one product

Log-normal			
$\mu_1 = 200$	$\sigma_1 = 40$	$L_{\mu_1} = 5.2787$	$L_{\sigma_1} = 0.1980$
NHPP			
$a = 0.004$		$b = 1.04$	
Sales volume			
$A_1 = 1,000$	$\beta_1 = 0.2$	$\eta_1 = 0.13$	

and L_{σ_1} are logarithmised mean and standard deviation respectively, and they are calculated by

$$L_{\mu_1} = \ln\left(\frac{\mu_1}{\sqrt{1 + \frac{\sigma_1^2}{\mu_1^2}}}\right),$$

and

$$L_{\sigma_1} = \sqrt{\ln\left(1 + \frac{\sigma_1^2}{\mu_1^2}\right)}.$$

4.1.1 Mean-Variance

Under the mean-variance framework, the optimisation problem is

$$\max \quad \mathrm{E}[\omega_1] = (1,000 - 0.2P_1 + 0.13T_1)(P_1 - 0.004T_1^{1.04} \times 200)$$

$$s.t. \quad \mathrm{Var}[\omega_1] = (1,000 - 0.2P_1 + 0.13T_1) \times 0.004T_1^{1.04} \times (200^2 + 40^2) \leq \psi.$$

Suppose $T_1 = 720$. The above objective function is parabola and has the global maximum $\mathrm{E}[\omega_1] = 1,113,257.59$ at $P_1 = 3,108.70$ with the corresponding sales volume $M_1 = 472$. Meanwhile, the feasible range of P_1 is $[5,468 - 0.000032\psi, 5,468]$, which means: $P_1 = 3,108.70$ is the optimal solution if the variance limitation $\psi \geq 147,103,140.52$.

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If P_1 is known, for example, let $P_1 = 3,000$. The global maxima is $\mathrm{E}[\omega_1] = 1,200,600$ when $T_1 \approx 32$. The sales volume and the variance of total profit at $T_1 \approx 32$ are $M_1 = 404$ and $\mathrm{Var}[\omega_1] = 2,472,080.61$, respectively. Furthermore, the function of $\mathrm{Var}[\omega_1]$ monotonously increases with T_1. Hence, if the variance of the total profit is less than ψ with $\psi \geq 2,472,080.61$, $T_1 = 32$ is the optimal solution. However, in practice, the length of warranty is regulated by the authorities, such as at least 2 years warranty is required in Europe, this regulation may also be considered in optimisation.

4.1.2 Mean-VaR

Under the mean-VaR framework, given that the confidence level $\alpha = 0.05$, the optimisation problem is

$$\max \quad \mathrm{E}[\omega_1] = (1,000 - 0.2P_1 + 0.13T_1)(P_1 - 200 \times 0.004T_1^{1.04})$$

$$s.t. \quad \mathrm{VaR}_{0.05}(\omega_1) = (1,000 - 0.2P_1 + 0.13T_1)P_1 - F_{S_1}^{-1}(0.95) \geq \phi.$$

In this scenario, the effects of P_1 and T_1 on $\mathrm{VaR}_{0.05}$ should be investigated. However, the closed form of the cdf of S_1 is difficult to obtain, and we cannot obtain the close form of $F_{S_1}^{-1}(0.95)$ either. Then a log-normal distribution with mean, $\mathrm{E}[S_1] = 0.8T_1^{1.04} \times (1,000 - 0.2P_1 + 0.13T_1)$, and variance, $\mathrm{Var}[S_1] = 166.4T_1^{1.04} \times (1,000 - 0.2P_1 + 0.13T_1)$, is used to approximate the distribution $F_{S_1}(x)$.

Initially, we focus on the situation that T_1 is known. According to Section 4.1.1, if $T_1 = 720$, the objective function has a global maximum $\mathrm{E}[\omega_1] = 1,113,257.59$ at $P_1 = 3,108.70$. then we have

$$\mathrm{VaR}_{0.05}(\omega_1) = (1,093.6 - 0.2P_1)P_1 - F_{S_1}^{-1}(0.95). \tag{24}$$

If the constraint on $\mathrm{VaR}_{0.05}(\omega_1)$ is not considered, the feasible range of P_1 is $[0, 5,468]$. Within this range, the relationship between P_1 and the first term in Eq. (24), i.e. the revenue $(1,093.6 - 0.2P_1)P_1$, is illustrated in Fig. 2. The relationship between P_1 and the second term $F_{S_1}^{-1}(0.95)$ in Eq. (24), i.e. the right tail 5% VaR of warranty cost, is shown in Fig. 3. The relationship between P_1 and $\mathrm{VaR}_{0.05}(\omega_1)$ is illustrated in Fig. 4.

Fig. 3 reveals that the right tail 5% VaR of warranty cost monotonically decreases with P_1 within the feasible range of P_1. Then, the relationship between $\mathrm{VaR}_{0.05}(\omega_1)$ and P_1 is determined by the total revenue which is a quadratic function of P_1. Hence, with the optimal constraint, one can derive a feasible range of P_1, (p_{l1}, p_{u1}), where p_{l1} and p_{u1} are determined by ϕ.

In accordance with the market behaviour, one may set market price P_1, said $P_1 = 3,000$. The optimal length of warranty T_1 can then be determined and one can obtain

$$\mathrm{VaR}_{0.05}(\omega_1) = 3,000 \times (400 + 0.13T_1) - F_{S_1}^{-1}(0.95). \tag{25}$$

Fig. 5 and Fig. 6 reveal the effects of T_1 on the right tail 5% VaR of the warranty cost and $\mathrm{VaR}_{0.05}(\omega_1)$. The values of T_1 in these figures are discrete values from 2 years (720 days) to 10 years (3,600 days) with half of a year step (180 days). Then the optimal constraint gives a
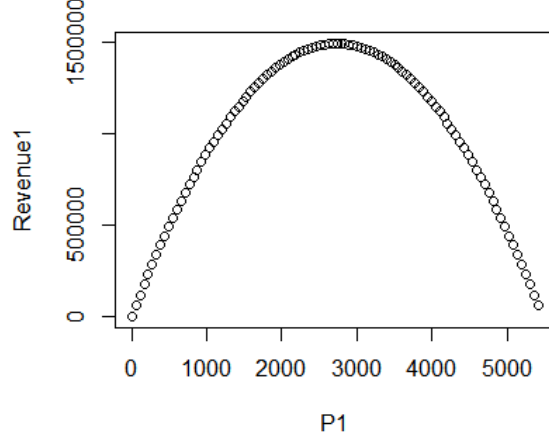
15

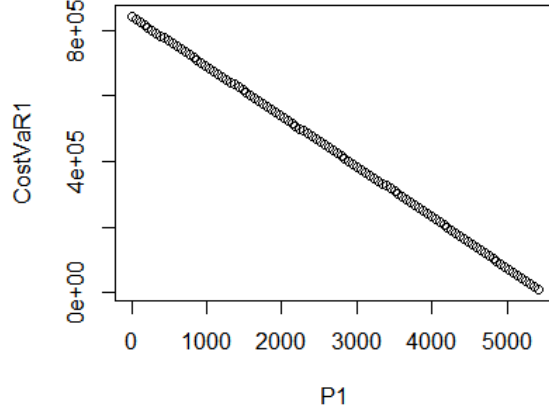Fig. 2. Total revenue (on the Y-axis) against price P_1 (on the X-axis).

Fig. 3. Right tail 5% VaR of warranty cost (on the Y-axis) against P_1 (on the X-axis).

feasible range of T_1, $[720, t_u]$, where t_u is determined by ϕ.

4.1.3 Mean-CVaR

Under the mean-CVaR framework, the objective function is the same as that under the mean-VaR framework, but the constraint is given by

$$\text{CVaR}_{0.05}(\omega_1) = (1{,}000 - 0.2P_1 + 0.13T_1)P_1 - \frac{1}{0.05} \int_{F_{S_1}^{-1}(0.95)}^{+\infty} z f_{S_1}(z)dz \geq \delta. \qquad (26)$$

The first term of Eq. (26) also is the total revenue of Product 1, and the second term is the right tail 5% CVaR of the total warranty cost of Product 1. f_{S_1} can also be approximated by the log-normal distribution.

If $T_1 = 720$ is known, the relationship between P_1 and $\text{CVaR}_{0.05}(\omega_1)$ is illustrated in Fig. 7.
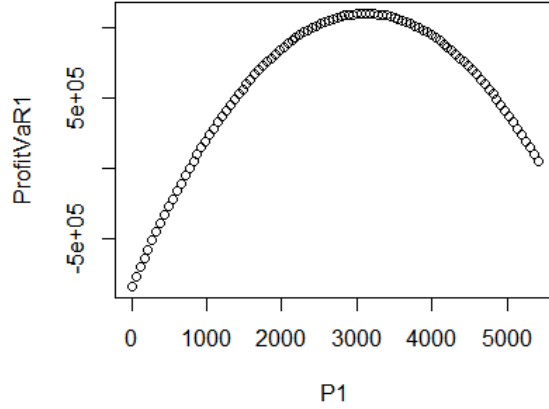
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Fig. 4. $\mathrm{VaR}_{0.05}(\omega_1)$ (on the Y-axis) against P_1 (on the X-axis).
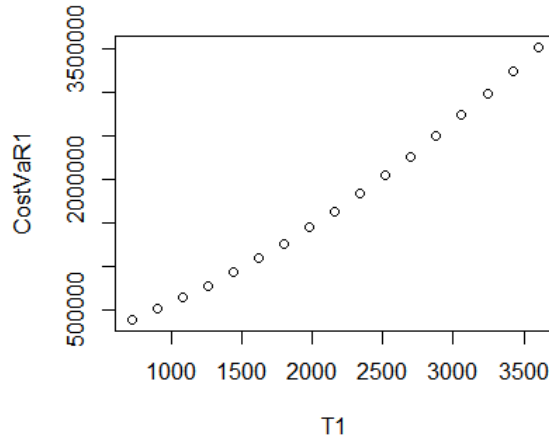

Fig. 5. Right tail 5% VaR of warranty cost (on the Y-axis) against T_1 (on the X-axis).

The feasible range of P_1 under the constraint is $[cp_l, cp_u]$, where both cp_l and cp_u are determined by δ.

If $P_1 = 3,000$ is known and $T_1 = 720 + 180v$ (for $v = 1, 2, ..., 16$), the relationship between T_1 and $\mathrm{CVaR}_{0.05}(\omega_1)$ is illustrated in Fig. 8. Then the feasible range of T_1 under the constraint is $[720, ct_u]$ where ct_u is determined by δ.

4.2 Three product scenario

Assume 7 types of components are installed in 3 products, each of which is composed of 4 or 5 components, as illustrated in Table 3. The components in each product are structured in series, which implies: if a component in an product item fails, the item fails.

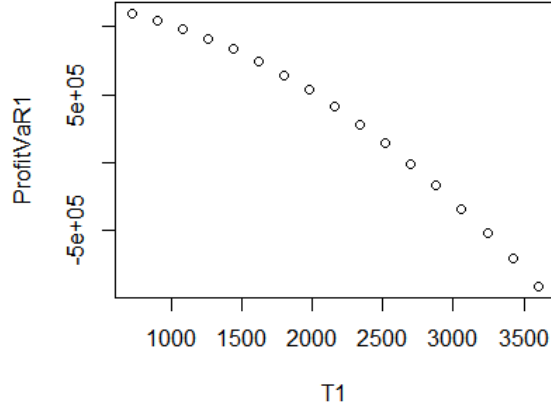Assume that the claim arrival processes of the components are NHPPs with a cumulative

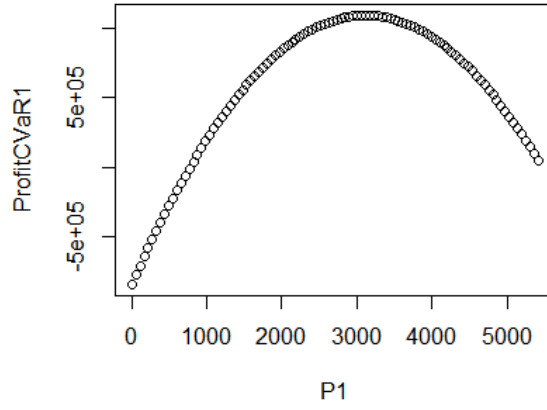Fig. 6. VaR$_{0.05}(\omega_1)$ (on the Y-axis) against T_1 (on the X-axis).

Fig. 7. CVaR1 (on the Y-axis) against P_1 (on the X-axis).

Table 3
Components of products

Product	Components						
	C1	C2	C3	C4	C5	C6	C7
Product1	✓	✓		✓	✓	✓	
Product2	✓		✓		✓	✓	
Product3	✓			✓		✓	✓

failure intensity, $\Lambda(t) = at^b$. The parameters in the claim arrival process models and the warranty claim costs of the components are presented in Table 4. Then, the claim arrival processes of the products are NHPPs. As such their intensities are given by $\Lambda_1(t) = a_1 t^{b_1} + a_2 t^{b_2} + a_4 t^{b_4} + a_5 t^{b_5} + a_6 t^{b_6}$, $\Lambda_2(t) = a_1 t^{b_1} + a_3 t^{b_3} + a_5 t^{b_5} + a_6 t^{b_6}$ and $\Lambda_3(t) = a_1 t^{b_1} + a_4 t^{b_4} + a_6 t^{b_6} + a_7 t^{b_7}$,
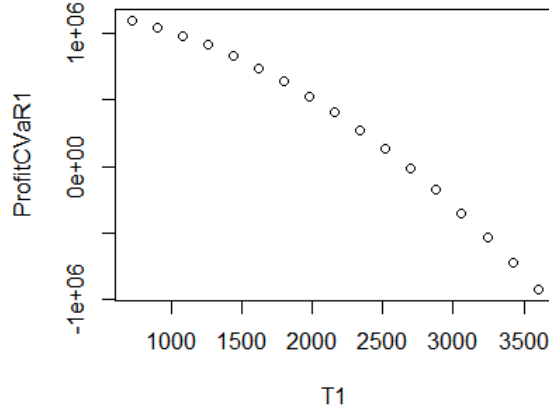
18

Fig. 8. $\mathrm{CVaR}_{0.05}(\omega_1)$ (on the Y-axis) against P_1 (on the X-axis).

Table 4
Parameters of Components

Components	a	b	Cost of each claim
C1	0.0037	1.02	100
C2	0.0037	1.01	120
C3	0.0028	1.02	80
C4	0.0028	1.02	90
C5	0.0024	1.03	65
C6	0.0019	1.03	60
C7	0.0014	1.04	50

respectively.

According to the above setting, one can generate three data sets, each of which contains claim times and cost of claims of M_k items of product k. These three data sets are correlated, based on which we can estimate the correlation parameters of our model.

Assume all products have 2-year (720 days) warranty, i.e. $T_1 = T_2 = T_3 = 720$. The products' sales volume and warranty cost parameter are then given in Table 5.

The expected warranty cost of the three products are, $\mathrm{E}[S_1] = 879.73 \times (1,093.6 - 0.2P_1)$, $\mathrm{E}[S_2] = 601.65 \times (2,079.2 - 0.5P_2)$ and $\mathrm{E}[\omega_3] = 572.91 \times (2,672 - 0.6P_3)$, respectively. The variances of the warranty costs of the three products are $\mathrm{Var}[\omega_1] = 83,376.02 \times (1,093.6 - 0.2P_1)$, $\mathrm{Var}[\omega_2] = 49,020.67 \times (2,079.2 - 0.5P_2)$ and $\mathrm{Var}[\omega_3] = 48,238.49 \times (2,672 - 0.6P_3)$, respectively. Based on these expected values and variances, the warranty cost distributions, $F_{S_1}(z_1)$, $F_{S_2}(z_2)$ and $F_{S_3}(z_3)$, can be approximated.

To model the dependence among the products, one may construct a trivariate copula. In

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Table 5
Parameters of 3 products

	Parameters of sales volume			Parameters of warranty cost	
	A_k	β_k	η_k	μ_k	σ_k
Product 1	1,000	0.2	0.13	89.36	21.99
Product 2	2,000	0.5	0.11	78.22	15.95
Product 3	2,600	0.6	0.10	79.04	20.20

practice, there are many different copula families existing, a suitable copula can be constructed or selected in two steps. In the first step, referring to the physical situation of the products or the features of empirical operating, a proper copula family can be selected. For example, if the dependence is linear, a copula from the elliptical family can be selected; and if a rank correlation is found in the data, a copula from the Archimedean family can be selected. Additionally, the form of marginal distribution, the tail-dependence, etc. all can influence copula selection. In the second step, the goodness-of-fit of the initially selected copulas can be compared by mean squared errors, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The details of copula selection in warranty data analysis will be investigated in future research.

In this case, considering the non-elliptical marginal distributions and the potential upper tail-dependence, a trivariate Gumbel copula is simply constructed as an example. Then the joint distribution of the number of claims is

$$
\begin{aligned}
H(z_1, z_2, z_3) &= C(u_1, u_2, u_3; \theta_1, \theta_2) \\
&= \exp\{-[(-\ln u_1)^{\theta_1} + [(-\ln u_2)^{\theta_2} + (-\ln u_3)^{\theta_2}]^{\frac{\theta_1}{\theta_2}}]^{\frac{1}{\theta_1}}\},
\end{aligned} \tag{27}
$$

where $u_1 = F_{N_1}(z_1)$, $u_2 = F_{N_2}(z_2)$ and $u_3 = F_{N_3}(z_3)$. With simulation, $\theta_1 = \frac{1}{1-0.59} \approx 2.44$ and $\theta_2 = \frac{1}{1-0.71} = 3.45$.

The density of the joint distribution is given by

$$
h(z_1, z_2, z_3) = c(u_1, u_2, u_3) f_{N_1}(z_1) f_{N_2}(z_2) f_{N_3}(z_3). \tag{28}
$$

The cumulative distribution function of the total number of claims of the three products, $\boldsymbol{N} = N_1 + N_2 + N_3$, of the manufacturer is given by,

$$
F_{\boldsymbol{N}}(z) = \sum_{z_1=0, z_2 \leq z-z_1}^{z} h(z_1, z_2, z - z_1 - z_2) \tag{29}
$$

The VaR and the CVaR of the total profit Ω can be determined based on Eqs. (27), (28), and (29). In Table 6, the $\text{VaR}_{0.05}$ and the $\text{CVaR}_{0.05}$ of the total profit Ω with different values of θ_2 and θ_1 are presented.

In Table 6, θ_1 represents the correlation between the profit of Products 1 and the profit of

Table 6
The VaR and CVaR of Ω with different dependences

$P_1 = 3,173.86,\ P_2 = 2,380.02,\ P_3 = 2,513.12$				
Copula parameter	$\theta_1 = 1,\ \theta_2 = 1$	$\theta_1 = 2.44,\ \theta_2 = 3.25$	$\theta_1 = 2.44,\ \theta_2 = 4$	$\theta_1 = 3,\ \theta_2 = 4$
$\text{VaR}_{0.05}(\Omega)$	4,884,922	4,859,833	4,859,018	4,858,978
$\text{CVaR}_{0.05}(\Omega)$	4,873,141	4,855,315	4,854,950	4,853,788

Products 2 and 3; and θ_2 represents the correlation between the profits of Products 2 and 3. In the second column, $\theta_1 = \theta_2 = 1$ implies that the profits of the three products are not correlated; and in the following 3 columns, larger values of θ_1 and θ_2 indicate a stronger correlations. The $\text{VaR}_{0.05}$ and the $\text{CVaR}_{0.05}$ of the total profit Ω in the dependent scenarios (Column 3, 4 and 5) are larger than that in the independent scenario (Column 2); and in the dependent scenarios these two values increase with the correlations among the profits of products being stronger. This result implies: if a manufacturer ignores the dependence among warranty claims of products, it will underestimate the upcoming total warranty cost. Such ignorance may cause bias in decision making.

5 Conclusions

In the real world, a manufacturer normally produces many different products that have common components installed. Consequently, the frequencies of warranty claims of different products are statistically dependent, which conflicts the fact that the existing methods in the literature solely focus on individual products and ignore the claim dependence.

This paper proposes a method to collectively optimise warranty policy for a portfolio of different products. Using the value-at-rick theory, it attempts to maximise the total profit of a set of products through optimising the warranty price and the warranty length. The numerical example shows that the dependence problem can be properly addressed with the proposed method.

The paper only investigates the optimisation of warranty policy for a portfolio of products covered by a one-dimensional warranty policy. That is, the warranty only covers one dimension, which can be either the usage or the age dimension, but not both. For some products (see Ye and Murthy (2016), for example), however, a warranty policy may cover both age and usage (e.g., the warranty of a car may cover both age and mileage), which is called a two-dimensional warranty policy. Our future work aims to investigate the optimisation problem for products with a two-dimensional warranty coverage.

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Appendix

Proof of Proposition 1.

Proof. The Hessian matrix of $\mathrm{E}[\omega_k]$ is

$$\boldsymbol{H}_{f(P_k,T_k)} = \begin{bmatrix} -2\beta_k & a_k b_k \beta_k \mu_k T_k^{b_k-1} + \eta_k \\ a_k b_k \beta_k \mu_k T_k^{b_k-1} + \eta_k & -a_k b_k \mu_k T_k^{b_k-2}[(b_k-1)(A_k - \beta_k P_k + \eta_k T_k) + 2\eta_k T_k] \end{bmatrix},$$

and the eigenvalues of $\boldsymbol{H}_{\mathrm{E}[\omega_k]}$ are

$$x_1 = -\{2\beta_k + a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}$$
$$+ \sqrt{\{2\beta_k - a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}^2 + 4(a_k b_k \beta_k \mu_k T^{b-1} + \eta_k)^2},$$

and

$$x_2 = -\{2\beta_k + a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}$$
$$- \sqrt{\{2\beta_k - a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}^2 + 4(a_k b_k \beta_k \mu_k T^{b-1} + \eta_k)^2},$$

respectively. It can be seen that $x_1 > 0$ and $x_2 < 0$. As such, the Hessian matrix is indefinite. As a result, the global minima of $\mathrm{E}[\omega_k]$ does not exist.

Similarly, the Hessian matrix of $\mathrm{Var}[\omega_k]$ is

$$\boldsymbol{H}_{\mathrm{Var}[\omega_k]} = \begin{bmatrix} 0 & -\beta_k(\sigma_k^2 + \mu_k^2)a_k b_k T_k^{b_k-1} \\ -\beta_k(\sigma_k^2 + \mu_k^2)a_k b_k T_k^{b_k-1} & a_k b_k(\sigma_k^2 + \mu_k^2)[2\eta_k T_k^{b_k-1} + (A_k - \beta_k P_k + \eta_k T_k)(b_k-1)T_k^{b_k-2}] \end{bmatrix},$$

and the eigenvalues of $\boldsymbol{H}_{\mathrm{Var}[\omega_k]}$ are $x_1 = a_k b_k T_k^{b_k-1}(\sigma_k^2 + \mu_k^2)\eta_k + a_k b_k T_k^{b_k-1}(\sigma_k^2 + \mu_k^2)\sqrt{\eta_k^2 + \beta_k^2} > 0$ and $x_2 = a_k b_k T_k^{b_k-1}(\sigma_k^2 + \mu_k^2)\eta_k - a_k b_k T_k^{b_k-1}(\sigma_k^2 + \mu_k^2)\sqrt{\eta_k^2 + \beta_k^2} < 0$, which implies that the Hessian matrix is indefinite and the feasible region of (P_k, T_k) defined by the constraint is therefore infinite.

Consequently, the global maxima of the objective function does not exist. $\square$

Proof of Proposition 2.

Proof. If P_k is known, the first order derivative of the objective function $\mathrm{E}[\omega_k](T_k)$ is given by

$$\frac{d\mathrm{E}[\omega_k](T_k)}{dT_k} = -\eta_k \mu_k a_k(b_k+1)T_k^{b_k} - (A_k - \beta_k P_k)\mu_k a_k b_k T_k^{b_k-1} + \eta_k P_k,$$

and the second order derivative of $E[\omega_k](T_k)$ is given by

$$\frac{d^2 E[\omega_k](T_k)}{dT_k^2} = -a_k b_k \mu_k T_k^{b_k-2}[(b_k - 1)(A_k - \beta_k P_k + \eta_k T_k) + 2\eta_k T_k] < 0,$$

as a result, the function $E[\omega_k](T_k)$ is concave for $T_k \geq 0$.

If P_k is known, the first order derivative of $Var[\omega_k](T_k)$ is

$$\frac{dVar[\omega_k](T_k)}{dT_k} = a_k(\sigma_k^2 + \mu_k^2)[\eta_k T_k^{b_k} + (A_k - \beta_k P_k + \eta_k T_k)b_k T_k^{b_k-1}] > 0,$$

and the second order derivative of $Var[\omega_k](T_k)$ is

$$\frac{d^2 Var[\omega_k](T_k)}{dT_k^2} = a_k b_k(\sigma_k^2 + \mu_k^2)[2\eta_k T_k^{b_k-1} + (A_k - \beta_k P_k + \eta_k T_k)(b_k - 1)T_k^{b_k-2}] > 0,$$

which implies that the function $Var[\omega_k](T_k)$ is convex and monotonously increases for $T_k \geq 0$ and that the feasible range of T_k defined by $Var[\omega_k](T_k) \leq \psi$ is finite. Hence, when P_k is known. As a result, the solution of the optimisation problem exists.

If T_k is known, the first order derivative of the objective function $E[\omega_k](P_k)$ is given by

$$\frac{dE[\omega_k](P_k)}{dP_k} = A_k - 2\beta_k P_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k;$$

and the second order derivative of the objective function $E[\omega_k](P_k)$ is given by

$$\frac{d^2 E[\omega_k](P_k)}{dP_k^2} = -2\beta_k < 0.$$

The fact that the second order derivative is negative implies: the objective function $E[\omega_k](P_k)$ achieves the global maxima at $P_k = \frac{A_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k}{2\beta_k}$. If T_k is known, the constraint is $P_k \geq \frac{A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2 + \mu_k^2)}}{\beta_k}$. Hence, the optimal solution exists if T_k is known. $\quad\square$

Proposition 3, 4, 5, 6 and 7 can be established based on Propositions 1 and 2.