On the existence of homoclinic type solutions of a class of inhomogenous second order Hamiltonian systems

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Abstract

We show the existence of homoclinic type solutions of second order Hamiltonian systems of the type $\ddot{q}(t) + \nabla_q V(t, q(t)) = f(t)$, where $t \in \mathbb{R}$, the C^1-smooth potential $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies a relaxed superquadratic growth condition, its gradient is bounded in the time variable, and the forcing term $f\colon \mathbb{R} \to \mathbb{R}^n$ is sufficiently small in the space of square integrable functions. The idea of our proof is to approximate the original system by time-periodic ones, with larger and larger time-periods. We prove that the latter systems admit periodic solutions of mountain-pass type, and obtain homoclinic type solutions of the original system from them by passing to the limit (in the topology of almost uniform convergence) when the periods go to infinity.

1 Introduction

During the past two decades there have been numerous applications of methods from the calculus of variations to find periodic, homoclinic and heteroclinic solutions for Hamiltonian systems. Many of the striking results that have been obtained by variational methods can be found in the well-known monographs of Ambrosetti and Coti Zelati [3], Ekeland [8], Hofer and Zehnder [9], Mawhin and Willem [16], as well as in the review articles of Rabinowitz [19, 20].
The aim of this paper is to prove the existence of solutions of the second order Hamiltonian system

$$\begin{cases} \ddot{q}(t) + \nabla_q V(t, q(t)) = f(t), \ t \in \mathbb{R}, \\ \lim\limits_{t \to \pm\infty} q(t) = \lim\limits_{t \to \pm\infty} \dot{q}(t) = 0, \end{cases} \tag{1}$$

where the C^1-smooth potential $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies a relaxed superquadratic growth condition, its gradient $V_q\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is uniformly bounded in the time variable on every compact subset of $\mathbb{R}^n$, and the norm of the forcing term $f\colon \mathbb{R} \to \mathbb{R}^n$ in the space of square integrable functions is smaller than a bound that we state below in our main theorem.
As homoclinic type solutions are global in time, it is reasonable to use global methods to find them rather than approaches based on their initial value problems. The homogenous systems of (1), i.e. when $f \equiv 0$, have been studied extensively under the assumption of superquadratic or subquadratic growth of the potential $V(t, q)$ as $|q| \to \infty$. Indeed, there are many results on homoclinic solutions for subquadratic Hamiltonian systems (cf. e.g. [20]). The first variational results for homoclinic solutions of first order Hamiltonian systems with superquadratic growth were found by V. Coti Zelati, I. Ekeland and E. Séré in [6] for time-periodic Hamiltonians. Corresponding results for second order Hamiltonian systems were obtained in [18] and [7]. S. Alama and Y.Y. Li [1] showed that asymptotic periodicity in time actually suffices to get a homoclinic solution, and E. Serra, M. Tarallo and S. Terracini [21] weakened their periodicity condition to almost periodicity in

the sense of Bohr. Finally, Hamiltonian systems with superquadratic non-periodic potentials were investigated for example by P. Montecchiari and M. Nolasco [17], A. Ambrosetti and M. Badiale [4], and by the second author of this paper in [11, 13, 14].
Our purpose is to generalize Theorem 1.1 of [5], which deals with the existence of solutions of the inhomogeneous systems (1) under the rather restrictive assumption that the potential V is of the special form

$$V(t,q) = -\frac{1}{2}|q|^2 + a(t)G(q),$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a continuous positive bounded function and $G\colon \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and satisfies the Ambrosetti-Rabinowitz superquadratic growth condition. Here, instead, the potential is of the more general form

$$V(t,q) = -K(t,q) + W(t,q)$$

with C^1-smooth potentials K and W such that

(C_1) the maps $\nabla_q K$ and $\nabla_q W$ are uniformly bounded in the time variable $t \in \mathbb{R}$ on every compact subset of $\mathbb{R}^n$,

(C_2) there exist two positive constants b_1, b_2 such that for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$

$$b_1|q|^2 \le K(t,q) \le b_2|q|^2,$$

(C_3) $K(t,q) \le (q, \nabla_q K(t,q)) \le 2K(t,q)$ for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$,

(C_4) $\nabla_q W(t,q) = o(|q|)$ as $|q| \to 0$ uniformly in $t \in \mathbb{R}$,

(C_5) there is a constant $\mu > 2$ such that for all $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \setminus \{0\}$

$$0 < \mu W(t,q) \le (q, \nabla_q W(t,q)),$$

(C_6) $m := \inf\{W(t,q)\colon t \in \mathbb{R} \ \wedge \ |q| = 1\} > 0$.

Here and subsequently, we denote by $(\cdot,\cdot)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the standard inner product in $\mathbb{R}^n$ and by $|\cdot|$ its induced norm.
Let us point out that under the above assumptions the Hamiltonian system (1) has the trivial solution when the forcing term f vanishes. Therefore it is reasonable to suppose that homoclinic type solutions exist when f is sufficiently small. Our main result affirms this hypothesis and it also gives an answer to the question how large the forcing term can be.

Theorem 1.1. *Set $M := \sup\{W(t,q)\colon t \in \mathbb{R} \ \wedge \ |q| = 1\}$ and $\bar{b}_1 := \min\{1, 2b_1\}$. Let us assume that $M < \frac{1}{2}\bar{b}_1$ and $(C_1) - (C_6)$ are satisfied. If the forcing term f is continuous, bounded, and moreover*

$$\left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{\frac{1}{2}} < \frac{\sqrt{2}}{4} \left(\bar{b}_1 - 2M \right), \tag{2}$$

then the inhomogenous system (1) possesses at least one solution.

The idea of our proof, which we give in the following second section, is to approximate the original system (1) by time-periodic ones, with larger and larger time-periods. We show that the approximating systems admit periodic solutions of mountain-pass type, and obtain a homoclinic type solution of the original system from them by passing to the limit (in the topology of almost uniform convergence) when the periods go to infinity. Finally, we discuss some examples of Theorem 1.1 in Section 3.

2 Proof of Theorem 1.1

For each $k \in \mathbb{N}$, let $E_k = W^{1,2}_{2k}(\mathbb{R}, \mathbb{R}^n)$ be the Sobolev space of $2k$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ and the standard norm

$$\|q\|_{E_k} = \left(\int_{-k}^{k} \left(|\dot{q}(t)|^2 + |q(t)|^2 \right) dt \right)^{\frac{1}{2}}.$$

We begin with the following estimate that is crucial in the main part of our proof below.

Lemma 2.1. *For every $\zeta \in \mathbb{R}$ and $q \in E_k$ we have*

$$\int_{-k}^{k} W(t, \zeta q(t)) dt \geq m|\zeta|^{\mu} \int_{-k}^{k} |q(t)|^{\mu} dt - 2km.$$

Proof. Note at first that the assertion is obviously true if $q = 0$ or $\zeta = 0$. Hence we can assume in the rest of the proof that $\zeta \neq 0$ and $q \neq 0$. Then it follows from (C_5) that, for every $q \neq 0$ and $t \in \mathbb{R}$, the function $z\colon (0, +\infty) \to \mathbb{R}$ defined by

$$z(\zeta) = W\left(t, \frac{q}{\zeta}\right) \zeta^{\mu}$$

is non-increasing. Hence, for every $t \in \mathbb{R}$,

$$W(t, q) \leq W\left(t, \frac{q}{|q|}\right) |q|^{\mu}, \text{ if } 0 < |q| \leq 1 \tag{3}$$

and

$$W(t, q) \geq W\left(t, \frac{q}{|q|}\right) |q|^{\mu}, \text{ if } |q| \geq 1. \tag{4}$$

We now fix $\zeta \in \mathbb{R} \setminus \{0\}$, $q \in E_k \setminus \{0\}$ and set

$$A_k = \{t \in [-k, k]\colon |\zeta q(t)| \leq 1\},$$
$$B_k = \{t \in [-k, k]\colon |\zeta q(t)| \geq 1\}.$$

By (4), we get

$$\begin{aligned} \int_{-k}^{k} W(t, \zeta q(t)) dt &\geq \int_{B_k} W(t, \zeta q(t)) dt \geq \int_{B_k} W\left(t, \frac{\zeta q(t)}{|\zeta q(t)|}\right) |\zeta q(t)|^{\mu} dt \\ &\geq m \int_{B_k} |\zeta q(t)|^{\mu} dt \geq m \int_{-k}^{k} |\zeta q(t)|^{\mu} dt - m \int_{A_k} |\zeta q(t)|^{\mu} dt \\ &\geq m|\zeta|^{\mu} \int_{-k}^{k} |q(t)|^{\mu} dt - 2km, \end{aligned}$$

which completes the proof. □

Further, to prove Theorem 1.1, we need the following approximative method.

Theorem 2.2 (Approximative Method, [15])**.** *Let $f\colon \mathbb{R} \to \mathbb{R}^n$ be a non-trivial, bounded, continuous and square integrable map. Assume that $V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a C^1-smooth potential such that $\nabla_q V\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is uniformly bounded in t on every compact subset of $\mathbb{R}^n$, i.e.*

$$\forall\ L > 0\ \exists\ C > 0\ \forall\ q \in \mathbb{R}^n\ \forall\ t \in \mathbb{R}\ \ |q| \leq L \Rightarrow |\nabla_q V(t, q)| \leq C.$$

Suppose that for each $k \in \mathbb{N}$ the boundary value problem

$$\begin{cases} \ddot{q}(t) + \nabla_q V_k(t, q(t)) = f_k(t), \\ q(-k) - q(k) = \dot{q}(-k) - \dot{q}(k) = 0, \end{cases}$$

where $f_k\colon \mathbb{R} \to \mathbb{R}^n$ *stands for the* $2k$*-periodic extension of* $f|_{[-k,k)}$ *to* $\mathbb{R}$ *and* $V_k\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ *denotes the* $2k$*-periodic extension of* $V|_{[-k,k)\times\mathbb{R}^n}$ *to* $\mathbb{R}\times\mathbb{R}^n$*, has a periodic solution* $q_k \in E_k$ *and* $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ *is a bounded sequence in* $\mathbb{R}$*. Then there exists a subsequence* $\{q_{k_j}\}_{j\in\mathbb{N}}$ *converging in the topology of* $C^2_{loc}(\mathbb{R},\mathbb{R}^n)$ *to a function* $q \in W^{1,2}(\mathbb{R},\mathbb{R}^n)$ *which is a solution of*

$$\ddot{q}(t) + \nabla_q V(t,q(t)) = f(t),\ t\in\mathbb{R}.$$

The approximative method was introduced by Paul H. Rabinowitz in [18] for homogenous second order Hamiltonian systems with a time-periodic potential. Later, the second author of this paper extended it to inhomogenous time-periodic Hamiltonian systems (see [10] and [12]), and more recently, Robert Krawczyk generalized it to the case of aperiodic potentials.
Let us now consider for $k\in\mathbb{N}$ the boundary value problems

$$\begin{cases} \ddot{q}(t) - \nabla_q K_k(t,q(t)) + \nabla_q W_k(t,q(t)) = f_k(t), \\ q(-k) - q(k) = \dot{q}(-k) - \dot{q}(k) = 0, \end{cases} \tag{5}$$

where $f_k\colon \mathbb{R}\to\mathbb{R}^n$ stands for the $2k$-periodic extension of $f|_{[-k,k)}$ to $\mathbb{R}$, and $K_k\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $W_k\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ are the $2k$-periodic extensions of $K|_{[-k,k)\times\mathbb{R}^n}$ and $W|_{[-k,k)\times\mathbb{R}^n}$ to $\mathbb{R}\times\mathbb{R}^n$.
As we have already mentioned in the introduction, our proof consists of two steps. First, we show the existence of solutions of (5), and second, we use Theorem 2.2 to find a solution of (1).
For our first step, let us consider the functionals $I_k\colon E_k\to\mathbb{R}$ given by

$$I_k(q) = \int_{-k}^{k} \left(\frac{1}{2}|\dot{q}(t)|^2 + K_k(t,q(t)) - W_k(t,q(t))\right) dt + \int_{-k}^{k} (f_k(t),q(t))dt. \tag{6}$$

Standard arguments show that $I_k \in C^1(E_k,\mathbb{R})$, and

$$I_k'(q)v = \int_{-k}^{k} \left((\dot{q}(t),\dot{v}(t)) + (\nabla_q K_k(t,q(t)) - \nabla_q W_k(t,q(t)),v(t))\right) dt + \int_{-k}^{k} (f_k(t),v(t))dt. \tag{7}$$

Moreover, the critical points of the functional I_k are classical $2k$-periodic solutions of (5), and we now show their existence by using the Mountain Pass Theorem. Let us recall the latter result before proceeding with our proof.

Theorem 2.3 (Mountain Pass Theorem, [2])**.** *Let* E *be a real Banach space and* $I\colon E\to\mathbb{R}$ *a* C^1*-smooth functional. If* I *satisfies the following conditions:*

- (i) $I(0) = 0$*,*
- (ii) *every sequence* $\{u_j\}_{j\in\mathbb{N}}$ *in* E *such that* $\{I(u_j)\}_{j\in\mathbb{N}}$ *is bounded in* $\mathbb{R}$ *and* $I'(u_j)\to 0$ *in* E^* *as* $j\to+\infty$ *contains a convergent subsequence (the Palais-Smale condition),*
- (iii) *there exist constans* $\rho,\alpha>0$ *such that* $I|_{\partial B_\rho(0)} \geq \alpha$*,*
- (iv) *there exists* $e\in E\setminus\bar{B}_\rho(0)$ *such that* $I(e)\leq 0$*,*

where $B_\rho(0)$ *is the open ball of radius* ρ *about* 0 *in* E*, then* I *possesses a critical value* $c\geq\alpha$ *given by*

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)), \tag{8}$$

where

$$\Gamma = \{g\in C([0,1],E)\colon\ g(0)=0,\ g(1)=e\}.$$

We now denote by $L^\infty_{2k}(\mathbb{R},\mathbb{R}^n)$ the space of $2k$-periodic essentially bounded functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|q\|_{L^\infty_{2k}} = \operatorname{ess\,sup}\{|q(t)|\colon t\in[-k,k]\}.$$

It is well known that for each $k \in \mathbb{N}$ and $q \in E_k$

$$\|q\|_{L^\infty_{2k}} \leq \sqrt{2}\|q\|_{E_k}. \tag{9}$$

Furthermore, we will write $L^2_{2k}(\mathbb{R}, \mathbb{R}^n)$ for the Hilbert space of $2k$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ and with the norm

$$\|q\|_{L^2_{2k}} = \left(\int_{-k}^{k} |q(t)|^2 dt \right)^{\frac{1}{2}}.$$

Note that by (2),

$$\|f_k\|_{L^2_{2k}} < \frac{\sqrt{2}}{4} \left(\bar{b}_1 - 2M \right). \tag{10}$$

The following lemma shows the existence of a solution of (5) and is the main part of the first step of our proof.

Lemma 2.4. *For each $k \in \mathbb{N}$, the functional I_k has a critical value of mountain pass type.*

Proof. We let $k \in \mathbb{N}$ be fixed and note at first that it is evident by (C_2) and (C_5) that $I_k(0) = 0$, which shows (i) in Theorem 2.3.
For checking the Palais-Smale condition (ii), we consider a sequence $\{u_j\}_{j\in\mathbb{N}} \subset E_k$ such that $\{I_k(u_j)\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and $I'_k(u_j) \to 0$ in E_k^* as $j \to \infty$. Then there exists a constant $C_k > 0$ such that for all $j \in \mathbb{N}$

$$|I_k(u_j)| \leq C_k \tag{11}$$

and

$$\|I'_k(u_j)\|_{E_k^*} \leq C_k. \tag{12}$$

Now, we will first show that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in the Hilbert space E_k. Using (6) and (C_5) we get

$$\begin{aligned} 2I_k(u_j) \geq \int_{-k}^{k} \left(|\dot{u}_j(t)|^2 + 2K_k(t, u_j(t)) \right) dt - \frac{2}{\mu} \int_{-k}^{k} (\nabla_q W_k(t, u_j(t)), u_j(t)) dt \\ + 2 \int_{-k}^{k} (f_k(t), u_j(t)) dt. \end{aligned}$$

From (7) and (C_3) it follows that

$$\begin{aligned} I'_k(u_j)u_j \leq \int_{-k}^{k} \left(|\dot{u}_j(t)|^2 + 2K_k(t, u_j(t)) \right) dt - \int_{-k}^{k} (\nabla_q W_k(t, u_j(t)), u_j(t)) dt \\ + \int_{-k}^{k} (f_k(t), u_j(t)) dt. \end{aligned}$$

Thus

$$\begin{aligned} 2I_k(u_j) - \frac{2}{\mu} I'_k(u_j)u_j \geq \left(1 - \frac{2}{\mu}\right) \int_{-k}^{k} \left(|\dot{u}_j(t)|^2 + 2K_k(t, u_j(t)) \right) dt \\ + \left(2 - \frac{2}{\mu}\right) \int_{-k}^{k} (f_k(t), u_j(t)) dt, \end{aligned}$$

and by (C_2) we have

$$2I_k(u_j) - \frac{2}{\mu} I'_k(u_j)u_j \geq \left(1 - \frac{2}{\mu}\right) \bar{b}_1 \|u_j\|^2_{E_k} + \left(2 - \frac{2}{\mu}\right) \int_{-k}^{k} (f_k(t), u_j(t)) dt.$$

Finally, aplying the Hölder inequality, as well as (10), (11) and (12), we obtain

$$\left(1-\frac{2}{\mu}\right)\bar{b}_1\|u_j\|_{E_k}^2-\frac{2C_k}{\mu}\|u_j\|_{E_k}-\frac{\sqrt{2}}{4}(\bar{b}_1-2M)\left(2-\frac{2}{\mu}\right)\|u_j\|_{E_k}-2C_k\le 0.$$

Since $\mu>2$ we conclude that $\{u_j\}$ is bounded.
Going to a subsequence if necessary, we can assume that there exists a function $u\in E_k$ such that $u_j\rightharpoonup u$ weakly in E_k as $j\to+\infty$. Hence $u_j\to u$ uniformly on $[-k,k]$, which implies that

$$\begin{aligned}(I_k'(u_j)-I_k'(u))\,(u_j-u)&\to 0,\\ \|u_j-u\|_{L^2_{2k}}&\to 0\end{aligned}\tag{13}$$

and

$$\begin{aligned}\int_{-k}^{k}(\nabla_q K_k(t,u_j(t))-\nabla_q W_k(t,u_j(t)),u_j(t)-u(t))dt\\ -\int_{-k}^{k}(\nabla_q K_k(t,u(t))-\nabla_q W_k(t,u(t)),u_j(t)-u(t))dt\to 0\end{aligned}$$

as $j\to+\infty$. On the other hand, it is readily seen that

$$\begin{aligned}\|\dot{u}_j-\dot{u}\|_{L^2_{2k}}^2&=(I_k'(u_j)-I_k'(u))(u_j-u)\\ &+\int_{-k}^{k}(\nabla_q K_k(t,u_j(t))-\nabla_q W_k(t,u_j(t)),u_j(t)-u(t))dt\\ &-\int_{-k}^{k}(\nabla_q K_k(t,u(t))-\nabla_q W_k(t,u(t)),u_j(t)-u(t))dt,\end{aligned}$$

and consequently

$$\|\dot{u}_j-\dot{u}\|_{L^2_{2k}}\to 0.\tag{14}$$

By (13) and (14), we see that $\|u_j-u\|_{E_k}\to 0$, and thus I_k satisfies the Palais-Smale condition.
To show (iii), we set

$$\varrho=\frac{\sqrt{2}}{2}$$

and assume that $q\in E_k$ such that $\|q\|_{E_k}=\varrho$. Note that $\|q\|_{L^\infty_{2k}}\le 1$ by (9). Thus, we can apply (3) to obtain

$$\int_{-k}^{k}W(t,q(t))dt\le\int_{-k}^{k}W\left(t,\frac{q(t)}{|q(t)|}\right)|q(t)|^\mu dt\le M\int_{-k}^{k}|q(t)|^2dt\le M\|q\|_{E_k}^2=\frac{1}{2}M.$$

From this, (C_2) and (2), we get

$$\begin{aligned}I_k(q)&\ge\frac{1}{2}\bar{b}_1\|q\|_{E_k}^2-\frac{1}{2}M-\|f_k\|_{L^2_{2k}}\|q\|_{E_k}\\ &\ge\frac{1}{4}(\bar{b}_1-2M)-\frac{\sqrt{2}}{2}\|f\|_{L^2}\\ &=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{4}(\bar{b}_1-2M)-\|f\|_{L^2}\right)\equiv\alpha>0\end{aligned}\tag{15}$$

To complete the proof, we have to show (iv), i.e. we need to find $e_k\in E_k$ such that $\|e_k\|_{E_k}>\rho$ and $I_k(e_k)\le 0$.

Let

$$\bar{b}_2 = \max\{1, 2b_2\}.$$

Combining (6) and Lemma 2.1 gives

$$I_k(\zeta q) \leq \frac{\bar{b}_2\zeta^2}{2}\|q\|_{E_k}^2 - m|\zeta|^\mu \int_{-k}^{k} |q(t)|^\mu dt + |\zeta| \cdot \|f_k\|_{L^2_{2k}} \|q\|_{E_k} + 2km \tag{16}$$

for all $\zeta \in \mathbb{R} \setminus \{0\}$ and $q \in E_k \setminus \{0\}$.
We now let $Q \in E_1$ be such that $Q \neq 0$ and $Q(-1) = Q(1) = 0$. It follows from (16) that $\|\zeta Q\|_{E_1} > \rho$ and $I_1(\zeta Q) < 0$ for $\zeta \in \mathbb{R} \setminus \{0\}$ large enough. Hence, if we define $e_1(t) = \zeta Q(t)$ and for each $k \geq 2$,

$$e_k(t) = \begin{cases} e_1(t) & \text{for } t \in [-1, 1], \\ 0 & \text{for } t \in [-k, -1) \cup (1, k], \end{cases} \tag{17}$$

then $e_k \in E_k$, and $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \rho$ as well as $I_k(e_k) = I_1(e_1) < 0$.
In summary, it follows from Theorem 2.3 that the action functional I_k has a critical value $c_k \geq \alpha$ given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k(g(s)), \tag{18}$$

where

$$\Gamma_k = \{g \in C([0, 1], E_k) \colon g(0) = 0,\ g(1) = e_k\}.$$

□

In what follows, we let q_k be a critical point for the corresponding critical value c_k that we have found in Lemma 2.4. The functions q_k, $k \in \mathbb{N}$, are solutions of (5) and as second step of our proof of Theorem 1.1, we now want to apply Theorem 2.2 to this sequence of functions.

Lemma 2.5. *The sequence $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}} \subset \mathbb{R}$ is bounded.*

Proof. We set

$$M_0 = \max_{s \in [0,1]} I_1(se_1).$$

and conclude from (17) and (18) that

$$c_k \leq M_0 \tag{19}$$

for each $k \in \mathbb{N}$. By assumption,

$$c_k = I_k(q_k) = I_k(q_k) - \frac{1}{2}I_k'(q_k)q_k = \int_{-k}^{k} K_k(t, q_k(t)) - \frac{1}{2}(\nabla_q K_k(t, q_k(t)), q_k(t))\, dt$$
$$+ \int_{-k}^{k} \frac{1}{2}(\nabla_q W_k(t, q_k(t)), q_k(t)) - W_k(t, q_k(t))\, dt + \frac{1}{2}\int_{-k}^{k} (f_k(t), q_k(t))dt.$$

Applying (C_3) and (C_5) we obtain

$$c_k \geq \left(\frac{\mu}{2} - 1\right) \int_{-k}^{k} W_k(t, q_k(t))dt + \frac{1}{2}\int_{-k}^{k} (f_k(t), q_k(t))dt.$$

Furthermore, it follows from (6) and (C_2) that

$$\int_{-k}^{k} W_k(t, q_k(t))dt \geq \frac{1}{2}\bar{b}_1\|q_k\|_{E_k}^2 + \int_{-k}^{k} (f_k(t), q_k(t))dt - I_k(q_k).$$

Using that $I_k(q_k) = c_k$, the previous two inequalities give

$$\frac{1}{2}\bar{b}_1\|q_k\|_{E_k}^2 - \frac{\mu - 1}{\mu - 2}\|f_k\|_{L^2_{2k}}\|q_k\|_{E_k} - \frac{\mu}{\mu - 2}c_k \leq 0,$$

which implies by (2) and (19) that

$$\frac{1}{2}\bar{b}_1\|q_k\|_{E_k}^2 - \frac{\sqrt{2}}{4}\left(\bar{b}_1 - 2M\right)\frac{\mu-1}{\mu-2}\|q_k\|_{E_k} - \frac{\mu}{\mu-2}M_0 \leq 0.$$

Hence there is $M_1 > 0$ such that for each $k \in \mathbb{N}$,

$$\|q_k\|_{E_k} \leq M_1.$$

□

Now, using Theorem 2.2 we see that there exists a solution $q\colon \mathbb{R} \to \mathbb{R}^n$ of (1) such that $q(t) \to 0$ as $|t| \to \infty$.

All what is left to show for the proof of Theorem 1.1 is that actually $\dot{q}(t) \to 0$ as $|t| \to \infty$. This, however, follows from the inequality

$$|\dot{q}(t)| \leq \sqrt{2}\left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(|\dot{q}(s)|^2 + |\ddot{q}(s)|^2\right) ds\right)^{\frac{1}{2}}, \quad t \in \mathbb{R}, \tag{20}$$

which can be found in [10] (Inequality (28), p. 385). Indeed, we just need to note that by (1), (C_2), (C_4) and (2)

$$\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} |\ddot{q}(s)|^2 ds \to 0, \quad |t| \to \infty.$$

If now $|t|$ goes to ∞ in (20) we see that $|\dot{q}(t)| \to 0$ as $|t| \to \infty$. Consequently, q is a solution of (1) and the proof of Theorem 1.1 is complete.

3 One-dimensional Examples

In this section we present examples for $n = 1$ satisfying the assumptions of Theorem 1.1, and the graphs of their approximating solutions q_k of (5) for increasing values of k.

Example 3.1. *Consider $K\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $W\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ given by*

$$K(t,q) = \frac{t^2+1}{t^2+2}q^2,$$

$$W(t,q) = \frac{t^2+12}{3t^2+27}q^4$$

and

$$f(t) = \frac{1}{36}e^{-t^2},$$

where $t, q \in \mathbb{R}$. One can easily check that K, W and f satisfy the assumptions of Theorem 1.1.

Example 3.2. *Let $K\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $W\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ be given by*

$$K(t,q) = \left(\frac{1}{8}sin(t) + \frac{1}{8}sin(\sqrt{2}t) + \frac{3}{4}\right)q^2,$$

$$W(t,q) = \frac{1}{4}q^4$$

and

$$f(t) = \frac{1}{32}e^{-t^2},$$

where $t, q \in \mathbb{R}$. It is immediate that K, W and f satisfy the assumptions of Theorem 1.1.

Example 3.3. *Consider* $K\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $W\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *and* $f\colon \mathbb{R}\to\mathbb{R}$ *given by*

$$K(t,q)=q^2,$$

$$W(t,q)=\frac{10}{33}q^4\left(arctg^2\left(\frac{q^2}{t^2+1}\right)+1\right)$$

and

$$f(t)=\frac{1+t^2}{10}e^{-t^2},$$

where $t,q\in\mathbb{R}$. *Again, it is readily seen that* K, W *and* f *satisfy the assumptions of Theorem 1.1.*

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