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# Dimensional reduction on a sphere

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(Dated: September 21<sup>st</sup> 2005)

The question of the dimensional reduction of two-dimensional (2d) quantum models on a sphere to one-dimensional (1d) models on a circle is addressed. A possible application is to look at a relation between the 2d anyon model and the 1d Calogero-Sutherland model, which would allow for a better understanding of the connection between 2d anyon exchange statistics and Haldane exclusion statistics. The latter is realized microscopically in the 2d LLL anyon model and in the 1d Calogero model. In a harmonic well of strength  $\omega$  or on a circle of radius  $R$  – both parameters  $\omega$  and  $R$  have to be viewed as long distance regulators – the Calogero spectrum is discrete. It is well known that by confining the anyon model in a 2d harmonic well and projecting it on a particular basis of the harmonic well eigenstates, one obtains the Calogero-Moser model. It is then natural to consider the anyon model on a sphere of radius  $R$  and look for a possible dimensional reduction to the Calogero-Sutherland model on a circle of radius  $R$ . First, the free one-body case is considered, where a mapping from the 2d sphere to the 1d chiral circle is established by projection on a special class of spherical harmonics. Second, the  $N$ -body interacting anyon model is considered : it happens that the standard anyon model on the sphere is not adequate for dimensional reduction. One is thus lead to define a new spherical anyon-like model deduced from the Aharonov-Bohm problem on the sphere where each flux line pierces the sphere at one point and exits it at its antipode.

## I. INTRODUCTION

There has been some interest in relating the two dimensional anyon model [1–4] to the one dimensional Calogero model [5–8]. These studies [9, 10] were motivated by the fact that both models describe particles with non standard statistics, which interpolate from Bose-Einstein statistics to Fermi-Dirac statistics. In the anyon case, exchange statistics is at work, based on the non trivial braiding properties of the  $N$ -body configuration space; in the Calogero case, Haldane-exclusion statistics is manifest [11], based on Hilbert space counting arguments. Although a priori different, a non ambiguous relation has been established [12–14] between the anyon model and the Calogero model using as long distance regularisation a confining harmonic well: by projecting the anyon model on a particular subspace of the 2d harmonic well Hilbert space, one obtains the 1d Calogero model in a harmonic well (Calogero-Moser model). This mapping relates Haldane-exclusion statistics to anyon-braiding statistics, as already foreseen in the LLL-anyon model [15, 16], where Haldane thermodynamics are realized microscopically. A more intuitive way to look at the harmonic projection is to notice that the 2d harmonic quantum numbers on which the projection is made would be those of the lowest Landau level if a magnetic field were present. In other words, a zero magnetic field limit has been taken, which indeed becomes meaningful via the harmonic well regularisation : it follows that the vanishing magnetic field limit of the LLL-anyon model is the Calogero model [12].

The present work discusses whether a similar correspondence can be established with a different regularisation scheme, for example by modifying the topology of the 2d plane to a sphere (see also a previous attempt to dimensionally reduce Laughlin wave functions on a cylinder [17]). One should obtain a compact anyon model with a discrete spectrum, which might be dimensionally reducible to the Calogero-Sutherland (C-S) model on the circle. Note that a direct relation between the Calogero-Moser and the C-S models has already been discussed in Ref. 18. The C-S model is a prime example of a solvable model with exclusion statistics, and therefore its relation to the anyon model seems natural. This motivates our respective analysis of the anyon model on a sphere of radius  $R$ , which, by projection on a special class of spherical harmonics, might yield the C-S model on a circle of the same radius.

As a first step, the free cases of a quantum mechanical particle on the circle and on the sphere are analyzed in section II. A dimensional reduction scheme from the sphere to a chiral circle is achieved, which mimicks the harmonic dimensional reduction scheme. As a second step, the question of defining the anyon model on a sphere is addressed. The anyon model on the sphere has been discussed by various authors [19–23], yet basic differences with regard to the C-S model immediately show up. In the latter model, the statistical parameter is a continuous coupling parameter, while in the spherical anyon model, only discrete  $N$ -dependent statistical parameters are allowed due to Dirac quantization of the total flux on the sphere. This restriction may also be regarded as a property of the braid group on the sphere, since a loop made by a particle encircling all others is contractible on the sphere and has trivial braiding properties. Another important difference lies in the scaling of the spectrum with the statistical parameter. In the anyon model, the integrable part of the spectrum scales linearly [21], whereas the spectrum of the C-S model is quadratic. One is thus lead to propose a new anyon-like model on the sphere in section III, whose properties will be discussed. Finally,

section IV gives a summary of the results obtained so far.

## II. NON INTERACTING CASES

### A. A reminder: harmonic dimensional reduction

In the free case, it is known how to map a 2d particle in a harmonic well on a 1d particle in a harmonic well. One starts from the 2d Hamiltonian

$$H = -2 \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} + \frac{1}{2} \omega^2 z \bar{z} \quad (1)$$

with spectrum

$$E_{nm} = \omega(2n + |m| + 1), \quad (2)$$

where  $n \in \mathbb{N}$  is the radial quantum number and  $m \in \mathbb{Z}$  is the orbital quantum number. Specializing to the particular subset of quantum numbers, i.e., on each degenerate energy level one picks up the state of maximal angular momentum  $l \geq 0$  and radial quantum number  $n = 0$  (notice that these are the LLL quantum numbers if a magnetic field were present)

$$\langle z, \bar{z} | 0, l \rangle = z^l e^{-\frac{1}{2} \omega z \bar{z}}, \quad l \geq 0 \quad (3)$$

with spectrum

$$E_l = \omega(l + 1) \quad (4)$$

and projecting the Hamiltonian on this subspace – i.e., assuming  $\Psi = f(z) e^{-\frac{1}{2} \omega z \bar{z}}$  – gives

$$\omega \left( 1 + z \frac{\partial}{\partial z} \right) f(z) = E f(z). \quad (5)$$

Note that one might have as well used the subspace  $n = 0$  and  $l \leq 0$ , that is the eigenfunction  $\bar{\Psi} = f(\bar{z}) e^{-\frac{1}{2} \omega z \bar{z}}$  to obtain the eigenvalue equation

$$\omega (1 + \bar{z} \bar{\partial}) f(\bar{z}) = E f(\bar{z}). \quad (6)$$

In the thermodynamic limit  $\omega \rightarrow 0, l \rightarrow \infty$  with  $\omega l$  fixed, the physical picture is that of states with a probability density significant only increasingly close to the edge of the 2d plane, thus mimicking particles on the 1d boundary of the 2d sample.

(5) (and (6)) is nothing but an eigenvalue equation for a 1d harmonic well Hamiltonian in the coherent state representation. Consider the 1d harmonic oscillator Hamiltonian

$$H = -\frac{1}{2} \left( \frac{d}{dx} \right)^2 + \frac{1}{2} \omega^2 x^2 = \omega \left( a^\dagger a + \frac{1}{2} \right). \quad (7)$$

Coherent states  $|\alpha\rangle$  are eigenstates of the annihilation operator

$$a = \sqrt{\frac{\omega}{2}} x + \frac{i}{\sqrt{2\omega}} p_x. \quad (8)$$

Applying  $a^\dagger$  to the coherent states  $|\alpha\rangle$  in the canonical basis  $|n\rangle$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \quad (9)$$

one obtains

$$a^\dagger |\alpha\rangle = \left( \frac{\partial}{\partial \alpha} + \frac{\bar{\alpha}}{2} \right) |\alpha\rangle. \quad (10)$$

This leads to the Hamiltonian in the coherent states basis

$$H = \omega \left( \alpha \frac{\partial}{\partial \alpha} + \frac{\alpha \bar{\alpha}}{2} + \frac{1}{2} \right), \quad (11)$$

which may be rewritten as (5) via the non-unitary transformation (up to a zero-point energy shift)

$$\tilde{H} = e^{\frac{|\alpha|^2}{2}} H e^{-\frac{|\alpha|^2}{2}} = \omega \left( \frac{1}{2} + \alpha \frac{\partial}{\partial \alpha} \right). \quad (12)$$

The mapping between the canonical states basis and the coherent states basis (Bargman transform) maps  $\alpha^n$  (eigenstates in the coherent basis) on the Hermite polynomials  $H_n(x)$  (eigenstates in the configuration space)

$$\alpha^n = \frac{1}{\sqrt{2^n}} \int_{-\infty}^{\infty} dx H_n(x) e^{-x^2 + \sqrt{2}\alpha x - \frac{\alpha^2}{2}}. \quad (13)$$

In the sequel, a similar dimensional reduction is outlined which establishes a mapping from a 2d particle on a sphere to a 1d chiral particle on a circle.

### B. Free particle on the circle

The Hamiltonian of a free particle on a circle is

$$H = \frac{1}{2mR^2} \hat{L}^2, \quad \vec{r} = \cos \phi \vec{e}_x + \sin \phi \vec{e}_y \quad (14)$$

where  $\hat{L} = -i \frac{\partial}{\partial \phi}$  is the angular momentum operator in the coordinate representation. For simplicity of notations, one sets  $m = R = 1$ . The eigenstates  $|l\rangle$  with momentum  $l \in \mathbb{Z}$  are

$$\langle \phi | l \rangle = \frac{1}{\sqrt{2\pi}} e^{il\phi} \quad (15)$$

with eigenvalues  $E_l = \frac{1}{2} l^2$  ( $l \in \mathbb{Z}$  enforces canonical single valued wave functions). One now introduces coherent states on the circle [24], starting from the unitary operator

$$U = e^{i\hat{\phi}}, \quad (16)$$

which defines a ladder operator on the basis  $|l\rangle$

$$U|l\rangle = |l+1\rangle, \quad (17)$$

since  $[\hat{L}, U] = U$ . By analogy with the coherent states of the harmonic oscillator, one introduces the operator

$$\exp\{i(\hat{\phi} + i\hat{L})\} = U \exp\left\{-\hat{L} - \frac{1}{2}\right\} \quad (18)$$

and its eigenstates  $|\xi\rangle$ , which are the coherent states. In the canonical basis  $\{|l\rangle\}$

$$|\xi\rangle = \sum_l \xi^{-l} e^{-\frac{1}{2}l^2} |l\rangle. \quad (19)$$

One then calculates the action of the angular momentum operator on  $|\xi\rangle$

$$\hat{L}|\xi\rangle = \sum_l l \xi^{-l} e^{-\frac{1}{2}l^2} |l\rangle = -\xi \frac{\partial}{\partial \xi} \sum_l \xi^{-l} e^{-\frac{1}{2}l^2} |l\rangle = -\xi \frac{\partial}{\partial \xi} |\xi\rangle. \quad (20)$$

Thus, the Hamiltonian in the coherent states representation is

$$H = \frac{1}{2} \left( \xi \frac{\partial}{\partial \xi} \right)^2, \quad (21)$$

with eigenstates  $\psi_l(\xi) = \xi^l$  and eigenvalues  $E_l = \frac{1}{2} l^2$  with  $l \in \mathbb{Z}$ .

This mapping from the basis of angular momentum to coherent eigenstates establishes an equivalent on the circle of the Bargman transform (13) for the 1d harmonic well. Directly evaluating  $\langle l|\xi\rangle$  or inserting the completeness relation  $I = \frac{1}{\pi} \int d\phi |\phi\rangle \langle \phi|$  gives its analogue

$$\xi^l = \frac{1}{2\pi^2} e^{\frac{1}{2}l^2} \int_0^{2\pi} d\phi e^{-il\phi} \theta_3\left(\frac{1}{2}(\phi + i \ln \xi) \middle| \frac{i}{2\pi}\right). \quad (22)$$

### C. Free particle on the sphere

The Hamiltonian of a free particle on the sphere is again (14) but for the 2d angular momentum operator of a particle confined on a sphere of radius  $R$ . As above, one sets  $m = R = 1$ . The eigenstates are the spherical harmonics  $Y_l^m(\theta, \phi) = \langle \theta, \phi | l, m \rangle$ , with  $l = 0, 1, 2, \dots$  and  $-l \leq m \leq l$ . Their energies are degenerate in  $m$  with the eigenvalue equation

$$H|l, m\rangle = \frac{1}{2}l(l+1)|l, m\rangle. \quad (23)$$

On the circle, the spectrum has a twofold degeneracy on each energy level (except at  $l = 0$ ). One wishes to select a subspace of the 2d Hilbert space  $\{|l, m\rangle, l \in \mathbb{N}, |m| \leq l\}$  to establish a mapping on the circle. This subspace has to be characterized by a single quantum number. It is natural to consider either  $\{|l, l\rangle\}$  or  $\{|l, -l\rangle\}$  in analogy with the two subspaces discussed for the harmonic well in II A. In configuration space, these eigenstates rewrite

$$\langle \theta, \phi | l, l \rangle = Y_l^l(\theta, \phi) = \sin^l \theta e^{il\phi} \quad \text{or} \quad \langle \theta, \phi | l, -l \rangle = Y_l^{-l}(\theta, \phi) = \sin^l \theta e^{-il\phi}. \quad (24)$$

The physical picture is that of states with a probability density significant only increasingly close to the equator with increasing angular momentum  $l$ , thus mimicking a particle on the equatorial circle.

To achieve the relation with the coherent state representation on the circle, one rewrites the Hamiltonian on the sphere in stereographic coordinates  $(z, \bar{z})$ , with  $z = \cot(\theta/2)e^{i\phi}$

$$H = -\frac{1}{2}(1 + z\bar{z})^2 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}. \quad (25)$$

The wave functions (24) rewrite as

$$\langle z, \bar{z} | l, +l \rangle = \left( \frac{z}{1 + z\bar{z}} \right)^l \equiv u^l, \quad \text{and} \quad \langle z, \bar{z} | l, -l \rangle = \left( \frac{\bar{z}}{1 + z\bar{z}} \right)^l \equiv \bar{u}^l. \quad (26)$$

Note that in the thermodynamic limit,  $R \rightarrow \infty$ ,  $u^l$  or  $\bar{u}^l$  reproduces the lowest Landau states in the presence of a magnetic monopole, since if one reintroduces the  $R$  dependence,

$$u = \frac{z}{1 + \frac{z\bar{z}}{4R^2}} \xrightarrow{R \rightarrow \infty} z, \quad \text{and} \quad \bar{u} \rightarrow \bar{z}.$$

Projecting on these particular Hilbert spaces  $\{|l, +l\rangle\}$  or  $\{|l, -l\rangle\}$  means that the Hamiltonian acts on functions of  $u$  only or  $\bar{u}$  only. One obtains either

$$H = \frac{1}{2} \left( u^2 \left( \frac{\partial}{\partial u} \right)^2 + 2u \frac{\partial}{\partial u} \right) \quad (27)$$

or

$$H = \frac{1}{2} \left( \bar{u}^2 \left( \frac{\partial}{\partial \bar{u}} \right)^2 + 2\bar{u} \frac{\partial}{\partial \bar{u}} \right). \quad (28)$$

These Hamiltonians are basically (21), up to a zero point shift for the energy and the angular momentum

$$H_w = \frac{1}{2} \left[ \left( w \frac{\partial}{\partial w} + \frac{1}{2} \right)^2 - \frac{1}{4} \right] = \frac{1}{2} \left[ \left( \hat{L}_w + \frac{1}{2} \right)^2 - \frac{1}{4} \right], \quad (29)$$

where  $w$  is either  $u$  or  $\bar{u}$ , and spectrum  $E_l = \frac{1}{2} \left( \left( l + \frac{1}{2} \right)^2 - \frac{1}{4} \right)$  with  $l \in \mathbb{N}$ . Therefore, the chiral Hilbert subspaces allow for a dimensional reduction from the two dimensional problem on the sphere to the one dimensional chiral problem on the circle, as can be seen from (14) and (29), with spectra which are both quadratic in the single quantum number of either problem.

### III. A GENERALIZED AHARONOV-BOHM MODEL ON THE SPHERE

As exposed in the introduction, we are interested in a model on the sphere, which might be dimensionally reducible to the Calogero-Sutherland model. Since the latter features exclusion statistics, a connection with the anyon model appears possible. Though, the spectrum of the anyon model as defined in [1–4] scales linearly with the interaction parameter, whereas the C-S spectrum scales quadratically.

On the plane, anyons are defined either by the nontrivial monodromy of the  $N$ -body wave function with a free Hamiltonian, or equivalently, via a singular gauge transformation, by a multivalued  $N$ -body wave function (bosonic by convention), with an interacting Aharonov-Bohm  $N$ -body Hamiltonian describing a situation where each particle carries a (statistical) flux line of strength  $\Phi = \alpha\Phi_0$  (where  $\Phi_0$  is the flux quantum). In this section, we construct a model on the sphere starting from the Aharonov-Bohm problem and the principle of flux attachment, which can be traced back to a geometrical definition of the statistical phase between two particles.

The standard Aharonov-Bohm problem consists of a particle coupled to a single flux-tube piercing the plane at the origin. The Aharonov-Bohm problem on the sphere, considered in Ref. 25, consists of a particle coupled to a flux piercing the sphere at the southpole and exiting at the northpole. A spectrum quadratic in  $\alpha$  has been obtained, with a continuous Aharonov-Bohm coupling parameter  $\alpha$ , since there is no Dirac quantization condition, given that the total flux through the sphere is null.

#### A. Single particle case: interaction with a flux line $\Phi$ entering the sphere at the South pole and exiting at the North pole

Consider a charged particle on a sphere (spherical coordinates : polar angle  $\theta$ , azimuthal angle  $\phi$ ). The circulation of the potential vector along each azimuthal circle (constant  $\theta$ ) has to be equal to the flux  $\Phi$ . Therefore

$$\vec{A} = \frac{\Phi}{2\pi} \vec{\partial}\phi. \quad (30)$$

The Aharonov-Bohm Hamiltonian  $H^{AB}$  is obtained by substitution of the multivalued phase  $\exp(i\alpha\phi)$  in the free Hamiltonian  $H^o = -\frac{1}{2R^2}\Delta$  (in units  $\hbar = m = 1$ ), i.e.,  $\Psi_o = \exp(i\alpha\phi)\Psi$ , with  $\Phi = \alpha\Phi_o$  (up to  $\delta$  contact terms at the South and North poles)

$$H^{AB} = -\frac{1}{2R^2} \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \left( \frac{\partial}{\partial\phi} + i\alpha \right)^2 \right]. \quad (31)$$

The eigenfunctions of (31) are found to be [25] ( $x = \cos\theta$ )

$$\Psi = e^{im\phi} P_\lambda^{-\mu}(x), \quad (32)$$

where the Legendre polynomials  $P_\lambda^{-\mu}(x)$  are related to the hypergeometric function

$$P_\lambda^{-\mu}(x) = \frac{1}{\Gamma(1+\mu)} \left( \frac{1-x}{1+x} \right)^{\frac{\mu}{2}} F_H \left( -\lambda, \lambda+1; \mu+1; \frac{1-x}{2} \right) \quad (33)$$

and  $\mu = |m + \alpha|$ , with  $m$  the angular quantum number of  $L_z$ . Eq. (33) yields vanishing boundary conditions for the wave function, imposed at the South and North poles where the flux line pierces the sphere, for  $\lambda = \mu + k$  with  $k$  a positive integer ; the spectrum is  $E_\lambda = \lambda(\lambda + 1)$ , thus it contains terms quadratic in  $\alpha$ , which are also present in the Calogero-Sutherland spectrum.

Stereographically projected coordinates  $(z, \bar{z})$  on the projective plane containing the South pole turn out to be useful ( $r$  is distance from the South pole in the projective plane,  $\rho = r/(2R) = \cotg(\theta/2)$  is the rescaled radial coordinate, we set  $R = 1$ ):

$$z = \rho e^{i\phi}. \quad (34)$$

One has

$$\cos\theta = \frac{\rho^2 - 1}{\rho^2 + 1}, \quad \sin\theta = \frac{2\rho}{\rho^2 + 1}, \quad \rho^2 = \frac{1+x}{1-x}, \quad (35)$$

as well as

$$\frac{1-x}{2} = \frac{1}{1+z\bar{z}} \quad \text{and} \quad \frac{1+x}{2} = \frac{z\bar{z}}{1+z\bar{z}}. \quad (36)$$

Looking at (32), and starting rather from the free Hamiltonian (in projective coordinates)

$$H^o = -\frac{1}{2}(1+z\bar{z})^2 \partial \bar{\partial}, \quad (37)$$

the eigenstates are

$$\Psi^o = e^{i(m+\alpha)\phi} \tan(\theta/2)^{|m+\alpha|} \frac{1}{\Gamma(1+\mu)} F_H \left( -\lambda, \lambda+1; \mu+1; \frac{1-x}{2} \right), \quad (38)$$

which is either, if  $m+\alpha \geq 0$ ,

$$\Psi^o = F^{m+\alpha} \frac{1}{\Gamma(1+\mu)} F_H \left( -\lambda, \lambda+1; \mu+1; \frac{1-x}{2} \right), \quad (39)$$

or if  $m+\alpha \leq 0$

$$\Psi^o = \bar{F}^{-m-\alpha} \frac{1}{\Gamma(1+\mu)} F_H \left( -\lambda, \lambda+1; \mu+1; \frac{1-x}{2} \right). \quad (40)$$

In (39,40),  $F = 1/\bar{z}$  simultaneously encodes on the sphere both the A-B phase and the analogous of a ‘‘short distance’’ behavior : it defines the South-North pole Aharonov-Bohm problem (on the plane, one has  $F = z$  for the A-B problem with a vortex at the origin).

The appearance of  $F$  in the eigenstates is not accidental.  $\phi$  being an harmonic function on the sphere, the Cauchy-Riemann equations yield the function  $F = |F| \exp(i\phi)$ : in the local coordinate system spanned by the vectors  $\partial_\theta \leftrightarrow \partial_x$  and  $\partial_\phi / \sin \theta \leftrightarrow \partial_y$ :

$$\partial_\theta \ln |F| = \frac{1}{\sin \theta} \quad \frac{1}{\sin \theta} \partial_\phi \ln |F| = 0 \quad (41)$$

Therefore,  $|F| = \tan \frac{\theta}{2}$  and then  $F = \tan \frac{\theta}{2} \exp(i\phi) = 1/\bar{z}$ .

As an illustration of (38), consider the simple case  $k = 0$ , i.e.,  $\lambda = |m+\alpha|$ , where the hypergeometric function rewrites

$$F_H \left( -\lambda, \lambda+1; \lambda+1, \frac{1-x}{2} \right) = \left( 1 - \frac{1-x}{2} \right)^\lambda = \left( \frac{1+x}{2} \right)^\lambda = \left( \frac{z\bar{z}}{1+z\bar{z}} \right)^\lambda. \quad (42)$$

If  $m+\alpha \geq 0$

$$\Psi^o = \left( \frac{z}{1+z\bar{z}} \right)^{m+\alpha}, \quad (43)$$

whereas if  $m+\alpha \leq 0$

$$\Psi^o = \left( \frac{\bar{z}}{1+z\bar{z}} \right)^{-m-\alpha}, \quad (44)$$

with eigenvalue  $E = |m+\alpha|(|m+\alpha|+1)$ . For  $-1/2 < \alpha \leq 1/2$ , the ground state is either (44) with  $m = 0$ , if  $-1/2 < \alpha < 0$  or (43) with  $m = 0$ , if  $0 < \alpha < 1/2$ .

One explicitly sees that although  $F$  is singular at the South pole, the wave function is still regular both at the South and North poles – as it should – because of appropriate terms in the hypergeometric function.

Proceeding as on the plane, one can bypass the A-B Hamiltonian and define a new Hamiltonian  $\tilde{H}$ , directly obtained from  $H^o$  by extracting  $F^\alpha$  (or  $\bar{F}^{-\alpha}$ ) in  $\Psi^o$ , i.e.,  $\Psi^o = F^\alpha \tilde{\Psi}$ , and consequently

$$\tilde{H} = -\frac{1}{2}(1+z\bar{z})^2 \left( \partial \bar{\partial} - \alpha \frac{1}{z} \partial \right). \quad (45)$$

Or, rather taking advantage of (43) (or (44)), one can define a Hamiltonian  $\tilde{H}'$  obtained from  $H^o$  by extracting  $(\frac{z}{1+z\bar{z}})^\alpha$  (or  $(\frac{\bar{z}}{1+z\bar{z}})^{-\alpha}$ ), i.e.,  $\Psi^o = (\frac{z}{1+z\bar{z}})^\alpha \tilde{\Psi}'$ , which yields

$$\tilde{H}' = -\frac{1}{2}(1+z\bar{z})^2 \left( \partial \bar{\partial} + \alpha \frac{1}{z(1+z\bar{z})} \bar{\partial} - \alpha \frac{z}{1+z\bar{z}} \partial \right) + \frac{1}{2} \alpha (\alpha + 1). \quad (46)$$

Both  $\tilde{H}$  and  $\tilde{H}'$  have simple form.

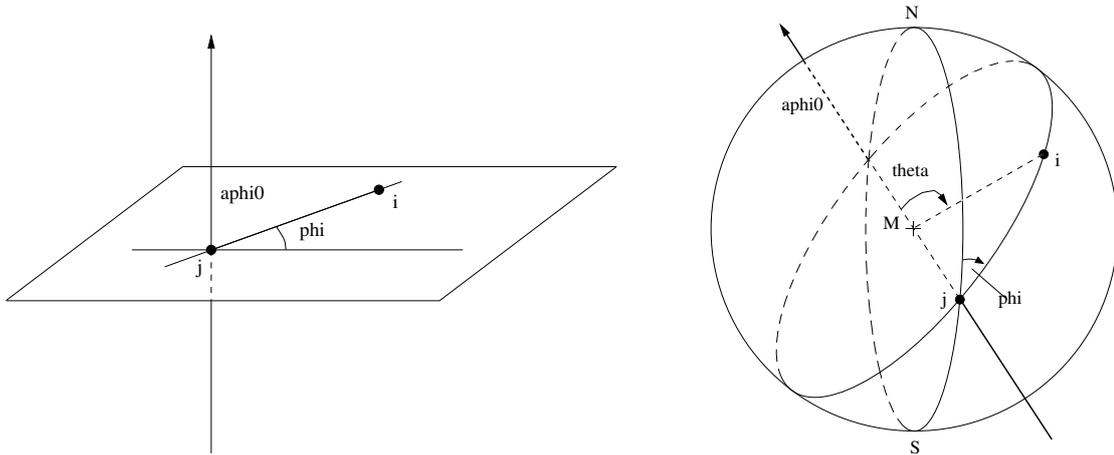


FIG. 1: On the plane, the relative angle  $\phi_{ij}$  of a vortex  $z_j$  and a particle  $z_i$  is the angle between a reference geodesic – i.e., a straight line, here the horizontal line – passing through the vortex  $z_j$  and the geodesic passing through both the vortex  $z_j$  and the particle  $z_i$ . On the sphere,  $\phi_{ij}$  is the angle between a reference geodesic – i.e., a great circle of the sphere, here passing through both poles and the vortex  $z_j$  – and the geodesic passing through both the vortex  $z_j$  and the particle  $z_i$  (it also passes through the antipodes  $-1/\bar{z}_j, -1/\bar{z}_i$ ). Equivalently, one can consider the azimuthal angle of the particle  $i$  in the coordinate system where the South-North axis coincides with the flux line  $j$ : this is again  $\phi_{ij}$ , up to a constant.

### B. Single particle case : interaction with a flux line $\Phi$ entering the sphere at a point and exiting at the antipode

In order to generalize to the  $N$ -body problem, consider now a particle at position  $z_i$  coupled to a vortex  $\Phi$  entering the sphere at  $z_j$  and exiting at its antipode  $-1/\bar{z}_j$ . This problem is identical to the precedent, the spectrum is the same; it remains to rewrite the wave functions and the corresponding function  $F_{ij}$  which generalizes  $F$  in the appropriate coordinates  $z_i$  and  $z_j$ .

It turns out that the A-B phase on the sphere has an analogous geometric interpretation to the one on the plane as depicted in FIG. 1. One has

$$\cot \phi_{ij} = \frac{\cos \theta_i \sin \theta_j - \sin \theta_i \cos \theta_j \cos(\phi_i - \phi_j)}{\sin \theta_i \sin(\phi_i - \phi_j)}, \quad (47)$$

or

$$\phi_{ij}(z_i, z_j) = \frac{1}{2i} \ln \frac{(\bar{z}_i z_j + 1)(z_i - z_j)\bar{z}_j}{(\bar{z}_i \bar{z}_j + 1)(\bar{z}_i - \bar{z}_j)z_j}. \quad (48)$$

This function is harmonic with respect to  $z_i$  and  $z_j$ . The Cauchy Riemann equations yield the “short distance” behavior

$$|F_{ij}| = \sqrt{\frac{(1 + z_i \bar{z}_j)(1 + \bar{z}_i z_j)}{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}}, \quad (49)$$

leading to the function

$$F_{ij} = \sqrt{z_j \bar{z}_j} \frac{\bar{z}_i + \frac{1}{z_j}}{\bar{z}_i - \bar{z}_j}, \quad (50)$$

which, in the limiting case ( $z_i = z, z_j \rightarrow 0$ ), coincides with  $F = 1/\bar{z}$  (up to a constant shift in the phase). Clearly, this function describes a vortex-antivortex pair on the projected plane, at position  $z_j$  and  $-1/\bar{z}_j$ , which are, not surprisingly, the positions where the vortex pierces the sphere. The Aharonov-Bohm Hamiltonian follows from the free one  $H_i^o$  for particle  $i$ , which is given by

$$H_i^o = -\frac{1}{2}(1 + z_i \bar{z}_i)^2 \partial_i \bar{\partial}_i, \quad (51)$$

and the substitution of the phase  $\exp(\phi_{ij}(z_i, z_j))$  in  $\Psi_i^o$ . Rather, one can directly construct the Hamiltonian  $\tilde{H}_i$  by substitution of the function (50), i.e.,  $\Psi_i^o = (F_{ij})^\alpha \tilde{\Psi}_i$

$$\begin{aligned}\tilde{H}_i &= F_{ij}^{-\alpha} H_i^o F_{ij}^\alpha \\ &= -\frac{1}{2}(1 + z_i \bar{z}_i)^2 \left\{ \partial_i \bar{\partial}_i + \alpha(\partial_i \ln F_i) \bar{\partial}_i + \alpha(\bar{\partial}_i \ln F_i) \partial_i + \alpha^2 (\partial_i F_i) (\bar{\partial}_i F_i) + \alpha \partial_i \bar{\partial}_i F_i \right\} \\ &= -\frac{1}{2}(1 + z_i \bar{z}_i)^2 \left\{ \partial_i \bar{\partial}_i - \alpha \frac{1 + z_j \bar{z}_j}{(1 + \bar{z}_i z_j)(\bar{z}_i - \bar{z}_j)} \partial_i \right\}.\end{aligned}\quad (52)$$

The eigenfunctions are deduced from (39,40) replacing  $F$  by  $F_{ij}$  as well as the argument of the hypergeometric function  $x$  by  $x_{ij}$  given by

$$x_{ij} = \cos \theta_{ij} = \frac{2|z_i - z_j|^2}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)} - 1. \quad (53)$$

We note that

$$\frac{1 + x_{ij}}{2} = \frac{|z_i - z_j|^2}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)}, \quad \text{and} \quad \frac{1 - x_{ij}}{2} = \frac{|1 + z_i \bar{z}_j|^2}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)}. \quad (54)$$

A particular subset of the solutions possesses again a simple analytic form when  $\lambda = \mu$ , i.e.,  $k = 0$

$$\Psi_i^o = \begin{cases} (F_{ij})^{m+\alpha} \left(\frac{1+x_{ij}}{2}\right)^{m+\alpha} &= \left(\frac{(1+\bar{z}_i z_j)(z_i - z_j)}{(1+z_i \bar{z}_i)(1+z_j \bar{z}_j)}\right)^{m+\alpha}, & m + \alpha > 0 \\ (\bar{F}_{ij})^{-m-\alpha} \left(\frac{1+x_{ij}}{2}\right)^{-m-\alpha} &= \left(\frac{(1+z_i \bar{z}_j)(\bar{z}_i - \bar{z}_j)}{(1+z_i \bar{z}_i)(1+z_j \bar{z}_j)}\right)^{-m-\alpha}, & m + \alpha < 0, \end{cases} \quad (55)$$

which are deformations of (43) and (44). As previously, and taking advantage of (55), one can define a Hamiltonian  $\tilde{H}'_i$  with the prescription  $\Psi_i^o = \left(\frac{(1+\bar{z}_i z_j)(z_i - z_j)}{(1+z_i \bar{z}_i)(1+z_j \bar{z}_j)}\right)^\alpha \tilde{\Psi}'_i$  giving

$$\tilde{H}'_i = -\frac{1}{2}(1 + z_i \bar{z}_i)^2 \left\{ \partial_i \bar{\partial}_i + \alpha \frac{1 + \bar{z}_i z_j}{(1 + z_i \bar{z}_i)(z_i - z_j)} \bar{\partial}_i - \alpha \frac{z_i - z_j}{(1 + z_i \bar{z}_i)(1 + \bar{z}_i z_j)} \partial_i \right\} + \frac{1}{2} \alpha(\alpha + 1). \quad (56)$$

### C. Generalization to the N-body case

Consider now a system of  $N$  identical particles of charge  $e$  and attached flux tubes  $\alpha\phi_0$  (now with  $\alpha \in [-1, 1]$ ) piercing the sphere at the positions of the particles and exiting at their antipodes. In this spherical model, the total flux through the sphere is null, thus there is no Dirac quantization condition on  $\alpha$ . Locally, the relative phase of two particles (47) is anyon-like, however globally – and in contrast to the planar anyon model, where  $\phi_{ij} = \phi_{ji} + \pi$  – here the phase between two particles is not symmetric. Consequently, it is not possible to write a global phase for the many-particle wave function. Nonetheless, each particle must see all the fluxes carried by the other particles, thus the phase for particle  $i$  is

$$\phi_i = \sum_{j \neq i} \phi_{ij}. \quad (57)$$

Consequently, the A-B Hamiltonian  $H_i^{AB}$  for particle  $i$  coupled to all other particles follows as usual from  $H_i^o$  by extracting the multivalued phase  $\exp(i\alpha\phi_i)$ . The global A-B Hamiltonian of the system is obtained as the sum of the  $H_i^{AB}$ 's

$$H^{AB} = \sum_i H_i^{AB}. \quad (58)$$

Contrary to the phase, the “short-distance” behavior, i.e., the absolute value of (50), is symmetric under the exchange of particles,  $|F_{ij}| = |F_{ji}|$ , as it should for a “distance”. One can thus substitute in the A-B Hamiltonian  $H^{AB}$  a global

“short-distance” behavior  $\prod_{i<j} |F_{ij}|^\alpha$  (here one restricts to  $\alpha \in [0, 1]$ ) to obtain a global  $\tilde{H}$  Hamiltonian. Equivalently, one can directly start from the free Hamiltonian  $H_i^o$  for particle  $i$  and substitute  $\prod_{j,j\neq i} F_{ij}^\alpha$  to obtain

$$\tilde{H}_i = -\frac{1}{2}(1 + z_i \bar{z}_i)^2 \left\{ \partial_i \bar{\partial}_i - \alpha \sum_{j\neq i} \frac{1 + z_j \bar{z}_j}{(1 + \bar{z}_i z_j)(\bar{z}_i - \bar{z}_j)} \partial_i \right\}. \quad (59)$$

Then  $\tilde{H} = \sum_i \tilde{H}_i$ . Or, following the lines which lead to (56), substitute  $\prod_{j,j\neq i} \left( \frac{(1 + \bar{z}_i z_j)(z_i - z_j)}{(1 + z_i \bar{z}_i)(1 + z_j \bar{z}_j)} \right)^\alpha$  to obtain  $\tilde{H}'_i$ , then

$$\begin{aligned} \tilde{H}'_N &= \sum_i \tilde{H}'_i = \\ &= -\frac{1}{2} \sum_i^N (1 + z_i \bar{z}_i)^2 \left\{ \partial_i \bar{\partial}_i + \alpha \sum_{j\neq i} \frac{1 + \bar{z}_i z_j}{(z_i - z_j)(1 + z_i \bar{z}_i)} \bar{\partial}_i - \alpha \sum_{j\neq i} \frac{z_i - z_j}{(1 + \bar{z}_i z_j)(1 + z_i \bar{z}_i)} \partial_i \right\} \\ &+ \frac{\alpha(\alpha + 1)}{2} N(N - 1) + \frac{1}{2} \alpha^2 \sum_i \sum_{\substack{j,k \\ j\neq i, k\neq\{i,j\}}} \frac{(1 + \bar{z}_i z_k)(z_i - z_j)}{(z_i - z_k)(1 + \bar{z}_i z_j)}. \end{aligned} \quad (60)$$

This Hamiltonian has a complicated structure, as can be seen in particular for the 3-body  $\alpha^2$  term.

#### D. The two-body case and its ground state

In the 2-anyon case things simplify, since the  $\alpha^2$  term is now merely a c-number:

$$\begin{aligned} \tilde{H}'_2 &= -\frac{1}{2}(1 + z_1 \bar{z}_1)^2 \left\{ \partial_1 \bar{\partial}_1 + \alpha \frac{1 + \bar{z}_1 z_2}{(1 + z_1 \bar{z}_1)(z_1 - z_2)} \bar{\partial}_1 - \alpha \frac{z_1 - z_2}{(1 + z_1 \bar{z}_1)(1 + \bar{z}_1 z_2)} \partial_1 \right\} \\ &- \frac{1}{2}(1 + z_2 \bar{z}_2)^2 \left\{ \partial_2 \bar{\partial}_2 + \alpha \frac{1 + \bar{z}_2 z_1}{(1 + z_2 \bar{z}_2)(z_2 - z_1)} \bar{\partial}_2 - \alpha \frac{z_2 - z_1}{(1 + z_2 \bar{z}_2)(1 + \bar{z}_2 z_1)} \partial_2 \right\} + \alpha(\alpha + 1) \end{aligned} \quad (61)$$

It is immediate that the ground state is  $\tilde{\Psi}_{GS} = 1$ , with energy  $E_{GS} = \alpha(\alpha + 1)$ , i.e., for the A-B Hamiltonian

$$\Psi_{GS}^{AB} = \left( \frac{|1 + \bar{z}_1 z_2| |z_1 - z_2|}{(1 + z_1 \bar{z}_1)(1 + z_2 \bar{z}_2)} \right)^\alpha. \quad (62)$$

Note that when  $\alpha \rightarrow 1$ , the 2-body wave function has a fermionic behavior, since, when  $z_1 \simeq z_2$ , it vanishes as  $|z_1 - z_2|$  (also when  $z_1 \simeq -1/\bar{z}_2$  it vanishes as  $|1 + \bar{z}_1 z_2|$ ). Likewise, note that  $\phi_{12} \simeq \phi_{21}$  up to a constant, which confirms that the model is locally anyon-like.

Apart from the ground state, no other particular solutions are known so far. Among other approaches,<sup>1</sup> we explored whether there are eigenstates based on the set of solutions of the one-body problem.

## IV. CONCLUSION

A dimensional reduction scheme was proposed, which relates a quantum mechanical free particle on the sphere of radius  $R$  to a chiral particle on a circle of same radius. This projection parallels a somewhat analogous projection between a 2d particle in a harmonic well and a 1d particle in a harmonic well. Further a generalized Aharonov-Bohm

<sup>1</sup> e.g., another possible approach to the search of eigenstates of the problem is a different rewriting of  $\tilde{H}'_2$  in analogy to the anyon problem on the plane, where one may separate an angular momentum part from an interacting 2-body term :

$$\tilde{H}'_2 = \underbrace{-\frac{1}{2}(1 + z_1 \bar{z}_1)^2 \left\{ \partial_1 \bar{\partial}_1 - \alpha \frac{z_1 \partial_1 + \bar{z}_1 \bar{\partial}_1}{1 + z_1 \bar{z}_1} \right\}}_{\text{one-body, pseudo-angular-momentum part}} + \alpha \frac{1}{z_1 - z_2} \bar{\partial}_1 + \alpha \frac{1}{\bar{z}_1 + \frac{1}{z_2}} \partial_1 \Big\} + (1 \leftrightarrow 2) + \alpha(\alpha + 1)$$

model on the sphere was defined, with in mind its dimensional reduction to the Calogero-Sutherland model. This model was found to have an anyon-like character and has interesting properties in that it allows for continuous values of its coupling parameter  $\alpha$ , since it is not subject to a Dirac quantization condition. Furthermore, the corresponding  $N$ -body Hamiltonian may be expressed in a simple form in the stereographic projection coordinates. Its construction follows a geometrical analogy with the one on the plane. However, and contrary to the plane where the anyon model is defined from a free Hamiltonian by a global multi-valued statistical phase, here the phase of each particle accounts for its interactions with the flux lines attached to the other particles. Adding up the vector potentials in the interacting description yields the Aharonov-Bohm Hamiltonian. This model does not share all the usual properties of anyon exchange statistics due to a global phase asymmetry  $\phi_{ij} \neq \phi_{ji}$ ; in particular the limit  $\alpha = 1$  does not correspond to free fermions. Yet, at short distance, the phase is symmetric  $\phi_{ij} \simeq \phi_{ji}$  and this locally anyon-like character is also reflected in the fermionic short distance behavior for the 2-body ground state.

Despite its relative simplicity, eigenstates of the model seem rather complicated to construct in the general case. Consequently, the question whether the proposed model allows for a dimensional reduction towards the Calogero-Sutherland model is still open.

### Acknowledgements

We thank Alain Comtet for suggesting the question of the relation between the anyon and Calogero-Sutherland models, as well as for his participation in the early stages of this work. Likewise, we thank Jean Desbois for helpful discussions. S. M. acknowledges the hospitality of the Laboratoire de Physique Théorique et Modèles Statistiques Orsay and support by RFBR grant N 04-02-17087.

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