The ς-Semantics: A Comprehensive Semantics for Functional Programs

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May 9, 1996

Abstract

A comprehensive semantics for functional programs is presented, which generalizes the well-known call-by-value and call-by-name semantics. By permitting a separate choice between call-by value and call-by-name for every argument position of every function and parameterizing the semantics by this choice we abstract from the parameter-passing mechanism. Thus common and distinguishing features of all instances of the ς-semantics, especially call-by-value and call-by-name semantics, are highlighted. Furthermore, a property can be validated for all instances of the ς-semantics by a single proof. This is employed for proving the equivalence of the given denotational (fixed-point based) and two operational (reduction based) definitions of the ς-semantics. We present and apply means for very simple proofs of equivalence with the denotational ς-semantics for a large class of reduction-based ς-semantics. Our basis are simple first-order constructor-based functional programs with patterns.

Keywords: functional programming language, denotational and operational semantics, call-by-value, call-by-name, strictness, pattern-matching.

Contents

1 Introduction

Two essentially different parameter-passing-mechanisms exist for the evaluation of functional programs: call-by-value (cbv) and call-by-name (cbn)[1]. Unfortunately, formal definitions of the two arising semantics are generally rather independent of each other: Firstly, the leftmost-innermost and the leftmost-outermost reduction strategy, which are often associated with cbv- and cbn-evaluation, respectively, look rather incomparable. Secondly, denotational definitions of cbv-semantics are usually inelegant modifications of their cbn counterpart ([FH88]). This independence is in contrast with the fact that many modern functional programming languages even use a mixture of both parameter-passing-mechanisms, e.g. ML is based on cbv-evaluation but includes cbn-evaluated streams (lazy lists) for input/output and the cbn-language HASKELL permits cbv-annotations for efficiency reasons ([MTH90, HJW$^+$92, MH$^+$96]).

Therefore we define one unifying, generalized semantics by permitting a separate choice between cbv and cbn for every parameter position of every function. This semantics, named ς-semantics[2], is parameterized by this choice (ς) and thus includes the well-known cbv- and cbn-semantics as special instances.

Hereby we see that the mentioned association with leftmost-innermost and leftmost-outermost reduction is rather misleading. Instead we will see that the two semantics use different instances of the program for their operational semantics and leftmost-innermost reduction is even a kind of outermost-reduction. The ς-semantics highlights the true differences between cbv- and cbn-evaluation. Even more important is that common features are stressed. This ameliorates our understanding of the mentioned prevailing mixtures of cbv- and cbn-semantics in existing languages. Furthermore, abstracting from the parameter-passing-mechanism provides the means for stating, analysing, and proving validity of semantic properties of functional programs in general. Here we use this to prove the well-definedness and the equivalence of the three definitions, one denotational and two operational, which we give for the ς-semantics (hence in particular cbv- and cbn-semantics are dealt with by the same proofs). In the proof of equivalence we employ rather general techniques which even provide the means for simple proofs of equivalence with the denotational semantics for operational, reduction based ς-semantics which may be defined in the future, e.g. for increased efficiency.

[1]Call-by-need evaluation is just a more efficient semantic equivalent of cbn-evaluation. This efficiency aspect will just be touched in Chapter 10.

[2]ς is a variation of the Greek letter sigma.

Functional Programs and Data Types

Our basis are simple first-order constructor-based functional programs with patterns like the following:

```
add(x,Zero)        →  x
add(x,Succ(y))     →  Succ(add(x,y))

mult(x,Zero)       →  Zero
mult(x,Succ(y))    →  add(x,mult(x,y))
```

We regard a program as a specification of a data type, which in the first-order case is simply an algebra consisting of a carrier set and a set of operations. One part of the data type is already defined by the signature and the semantics of the programming language and not by the particular program. In the example we assume the carrier set to contain the constructor terms $\texttt{Zero}, \texttt{Succ(Zero)}, \ldots$, and the constructor symbols to denote themselves (free interpretation). Only the meaning of the symbols `add` and `mult` is given by the program. Consequently, the semantics of a programming language defines a base data type $\mathrm{BDT} = \langle \text{carrier}, \text{base operations} \rangle$, and the semantics of a particular program P is the extension of this data type by additional operations: $\mathrm{DT}(P) = \langle \text{carrier}, \text{base operations} \cup \text{defined operations} \rangle$ The ς-semantics is modular, that is extending a program does not modify but only extend its data type.

Constructors and Pattern Matching

We use constructor-based (also called algebraic) data types, because on the one hand only potentially non-flat data types fully expose the relationship between cbv- and cbn-parameter-passing, and on the other hand all data types used in modern functional programming languages are covered[3]. These languages permit the user only to define new constructor-based data types and also the predefined data types like lists, characters and numbers can be regarded as constructor-based (with some additional base operations which are not considered here), being implemented in a special, more efficient way[4].

For common functional programming languages pattern-matching is only syntactic sugar. Priority rules for choosing a reduction rule (first-fitting in HASKELL, best-fitting in HOPE; [FH88]) give complicated semantic definitions and make equational reasoning rather difficult ([Tho89, Tho95]). In contrast, precisely regarding pattern-matching as foundation leads us to a simple definition of the ς-semantics.

Starting-Points

The denotational semantics we define is a fixed-point semantics like those for recursive applicative program schemes ([Vui74a, Vui74b, Niv75, DS76, BL77, GTWW77, Cou90]). While the

[3]However, some non-free data types like e.g. sets cannot be completely modelled by constructor-based data types.

[4]With this point of view arbitrary precision integers require an infinite signature.

concept of a base data type is central to this field, viewing the semantics of a whole program as a data type is inspired by algebraic specifications ([Wir90]). We define reduction semantics as operational semantics, but due to pattern-matching we cannot use the simple reduction semantics of recursive applicative program schemes. Instead, we apply notions and results from term rewriting systems ([DJ90, Klo92, Hue80, HO80, HL91]), especially for proving the equivalence of the denotational and operational ς-semantics. For proving the equivalence of the two operational ς-semantics we generalize the proofs in [O'D77] and [BK86] that parallel-outermost and leftmost-outermost reductions are terminating whenever possible.

Structure of the Paper

In Section 2 we introduce preliminary concepts and notations. Afterwards, the syntax of our programs is defined in Section 3. Subsequently, we consider in Section 4 properties we expect any semantics to possess, and, to prepare the generalization, we define the standard cbv- and cbn-semantics for our programs in Section 5. In Section 6 the ς-semantics is defined. Then we prove in Section 7 the equivalence of the three given definitions of the ς-semantics. Subsequently we prove in Section 8 some properties of our ς-semantics, especially those we consider desirable in Section 4. In Section 9 we discuss the use of ς-semantics for modelling the mixed strictnesses of modern functional programming languages and in Section 10 we consider more efficient reduction strategies than those given, as basis for realistic implementations. We conclude with a summary and some remarks in Section 11.

To avoid tiresome details many proofs are only sketched. Full proofs and additional examples are given in [Chi95].

2 Basic Definitions and Properties

We denote the natural numbers by $\mathbb{N}$, the positive natural numbers by $\mathbb{N}_+$ and the set $\{1, 2, \ldots, n\}$ by $[n]$ for any $n \in \mathbb{N}$. $\mathbb{B} = \{\mathsf{tt}, \mathsf{ff}\}$ is the set of boolean values. If M is a set, then M^* denotes the set of words over M with ε as empty word.

The notions partial order, mapping over partial orders, algebra, ordered algebra, and finite and infinite term are standard and may be found in e.g. [Wec92].

Both [DJ90] and [Klo92] give comprehensive surveys of term rewriting systems, the latter also of almost orthogonal term rewriting systems. However, our presentation is quite different in order to introduce the concept of an instance of a term rewriting system, which will be the basis of our operational ς-semantics, in a simple way.

2.1 Partial Orders

A *partial order* $\mathfrak{A} = \langle A, \leq_{\mathfrak{A}} \rangle$ consists of a non-empty set A and a reflexive, antisymmetric, and transitive relation $\leq_{\mathfrak{A}} \subseteq A \times A$. We use various symbols like $\leq, \subseteq, \preceq, \sqsubseteq$, and $\trianglelefteq$ for different partial order relations.

Let T be a subset of A. In case of their existence, the *least element of* T is denoted by $\mathrm{Least}_{\mathfrak{A}}(T)$ and the *least upper bound* of T by $\bigsqcup_{\mathfrak{A}} T$. T is an *(ω-)chain* iff it is non-empty,finite or countably infinite, and totally ordered. We often write chains as sequences $T = (a_i)_{i\in\mathbb{N}}$ such that $a_0 \leq a_1 \leq a_2 \leq \ldots$. The partial order $\mathfrak{A}$ is an *(ω-)complete partial order (a cpo)* iff every chain $T \subseteq A$ and the empty set have a least upper bound. Then $\mathfrak{A}$ has the least element $\perp_{\mathfrak{A}} := \bigsqcup \emptyset$.

A subset $T \subseteq A$ is *cofinal in a subset* $T' \subseteq A$ iff for all $t \in T$ there exists a $t' \in T$ such that $t \leq t'$. If $\mathfrak{A}$ is a cpo, $T, T' \subseteq A$ are chains, and T is cofinal in T', then $\bigsqcup T \leq \bigsqcup T'$.

Let $\mathfrak{A}$ be a cpo. An element a of A is *ω-compact* iff for every chain $T \subseteq A$ the property $a \leq \bigsqcup T$ implies the existence of an $a' \in T$ such that $a \leq a'$. A cpo $\mathfrak{A}$ is *ω-inductive* iff for every element $a \in A$ there exists a chain $T \subseteq A$ of ω-compact elements such that $a = \bigsqcup T$.

2.2 Mappings over Partial Orders

If A is a set and $\mathfrak{B} = \langle B, \leq_{\mathfrak{B}} \rangle$ a partial order, then the *canonical partial order of the mapping space*, $\langle (A \rightarrow B), \preceq \rangle$, is defined by $\varphi \preceq \psi :\Longleftrightarrow \forall a \in A.\ \varphi(a) \leq_{\mathfrak{B}} \psi(a)$.

Assuming $\mathfrak{A}$ and $\mathfrak{B}$ are partial orders, a mapping $\varphi : A \rightarrow B$ is *monotonic* iff $a \leq_{\mathfrak{A}} a'$ implies $\varphi(a) \leq_{\mathfrak{B}} \varphi(a')$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be cpos. A mapping $\varphi : A \rightarrow B$ is *strict* iff $\varphi(\perp_{\mathfrak{A}}) = \perp_{\mathfrak{B}}$. A mapping φ is *(ω-)continuous* iff for every chain $T \subseteq A$ the least upper bound of $\varphi(T) := \{\varphi(t) \mid t \in T\}$ exists and $\bigsqcup \varphi(T) = \varphi(\bigsqcup T)$. The set of (ω-)continuous mappings is denoted by $[A \rightarrow B]$. The *canonical partial order of (ω-)continuous mappings*, $\langle [A \rightarrow B], \preceq \rangle$, is complete.

Let φ be a mapping from a set A to itself. A *fixed-point* of φ is an element $a \in A$ such that $\varphi(a) = a$. The well-known fixed-point theorem of Knaster and Tarski states that if $\mathfrak{A}$ is a cpo and $\varphi : A \rightarrow A$ is continuous, then φ has a least fixed point given by $\mathrm{Fix}(\varphi) := \bigsqcup_{i\in\mathbb{N}} \varphi^i(\perp_{\mathfrak{A}})$.

Until here we considered only mappings with one argument. However, for $\varphi, \psi : A_1 \times \ldots \times A_n \rightarrow A$ we define $\varphi \preceq \psi :\Longleftrightarrow \forall a_1 \in A_1, \ldots, a_n \in A_n.\ \varphi(a_1, \ldots, a_n) \leq_{\mathfrak{A}} \psi(a_1, \ldots, a_n)$ and then the generalization of other previously defined notions is straightforward.

2.3 Algebras

A *signature* Σ is a set of operation symbols. Associated with every $f \in \Sigma$ is a natural number denoting its *arity*. We write $f^{(n)}$ for an operation symbol f of arity n and $\Sigma_n \subseteq \Sigma$ is the subset of all such operation symbols. A *constant* is an operation symbol of arity 0.

A *(Σ-)algebra* $\mathfrak{A} = \langle A, \alpha \rangle$ consists of a non-empty set A and a mapping $\alpha : \bigcup_{n\in\mathbb{N}} \Sigma_n \rightarrow (A^n \rightarrow A)$ which assigns a mapping (an operation) of arity n to every operation symbol of arity n. We generally write $f^{\mathfrak{A}}$ for $\alpha(f)$. The class of all (Σ-)algebras is denoted by Alg_{Σ}.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-algebras. A mapping $h : A \rightarrow B$ is a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ iff it preserves operations, i.e. $h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n))$ for all $a_1, \ldots, a_n \in A$ and $f^{(n)} \in \Sigma$. $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* iff such a homomorphism h and its inverse h^{-1} exist.

Let K be a class of Σ-algebras and $X \subseteq A$ for an algebra $\mathfrak{A} \in K$. $\mathfrak{A}$ is *free in K relative to X* iff every mapping φ from X to an algebra $\mathfrak{B}$ of K can uniquely be extended to a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. In the case $X = \emptyset$ we say that $\mathfrak{A}$ is *initial in K*. Since all algebras relatively free in K to X are isomorphic, we do not distinguish them in the following and just speak of the relatively free to X and the initial algebra in K (if any exists at all). For $K = \mathrm{Alg}_\Sigma$ we speak of *the absolutely free and the absolutely initial* Σ*-algebra.*

2.4 Ordered Algebras

The triple $\mathfrak{A} = \langle A, \leq, \alpha\rangle$ is an *ordered* (Σ*-)algebra* iff $\langle A, \leq\rangle$ is a partial order and $\langle A, \alpha\rangle$ a Σ-algebra such that all operations $\alpha(f)$ are monotonic w.r.t. the partial order. An ordered algebra may be viewed both as algebra and as partial order by ignoring $\leq$ and α, respectively.

The set of all ordered Σ-algebras with partial order $\langle A, \leq\rangle$ is denoted by $\mathrm{Alg}_\Sigma\langle A, \leq\rangle$. Its *canonical partial order* $\langle \mathrm{Alg}_\Sigma\langle A, \leq\rangle, \sqsubseteq\rangle$ is given by $\mathfrak{A} \sqsubseteq \mathfrak{B} :\Longleftrightarrow \forall f \in \Sigma.\ f^{\mathfrak{A}} \preceq f^{\mathfrak{B}}$.

An ordered Σ-algebra $\langle A, \leq, \alpha\rangle$ is *(ω-)complete* iff $\langle A, \leq\rangle$ is complete and the operations $f^{\mathfrak{A}}$ are continuous for all $f \in \Sigma$. The class of all complete Σ-algebras is denoted by $\mathrm{Alg}^\infty_{\Sigma,\perp}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be complete ordered Σ-algebras. An *(ω-)morphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a strict continuous homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. For complete algebras the notions of *isomorphism*, *freedom* and *initiality* are defined w.r.t. morphisms instead of homomorphisms.

2.5 Finite and Infinite Terms

In the following Σ is a signature with at least one constant and X a finite or countably infinite *set of variables* with $X \vee \Sigma = \emptyset$.

We define the Σ*-term algebra* $\mathcal{T}_\Sigma(X) = \langle \mathrm{T}_\Sigma(X), \tau\rangle$ to be the absolutely free Σ-algebra (i.e. in Alg_Σ) relative to X. Hence the Σ*-ground term algebra* $\mathcal{T}_\Sigma = \langle \mathrm{T}_\Sigma, \tau\rangle := \mathcal{T}_\Sigma(\emptyset)$ is the absolutely initial Σ-algebra. Similarly we define the *algebra of infinite partial* Σ*-terms* $\mathcal{T}^\infty_{\Sigma,\perp}(X) = \langle \mathrm{T}^\infty_{\Sigma,\perp}(X), \trianglelefteq, \tau^\infty\rangle$ to be the free (ω-complete) Σ-algebra in $\mathrm{Alg}^\infty_{\Sigma,\perp}$ to X, and $\mathcal{T}^\infty_{\Sigma,\perp} = \langle \mathrm{T}^\infty_{\Sigma,\perp}, \trianglelefteq, \tau^\infty\rangle := \mathcal{T}^\infty_{\Sigma,\perp}(\emptyset)$ is the *initial* Σ*-algebra in* $\mathrm{Alg}^\infty_{\Sigma,\perp}$.

These abstract definitions make us independent of any concrete term representations. However, we generally write terms in the common prefix notation with parenthesis, e.g. $f(g(x), a)$. Infinite partial terms may be visualized as particular partial mappings $\mathbb{N}^*_+ \rightharpoonup \Sigma \mathbin{\dot\cup} X \mathbin{\dot\cup} \{\perp\}$ (cf. positions below).

In the following we use the meta symbols t, f, x and their variations for respectively terms ($\mathrm{T}^\infty_{\Sigma,\perp}(X)$), signature symbols and variables. We often write tuples of terms $(t_1, \ldots, t_n) \in (\mathrm{T}^\infty_{\Sigma,\perp}(X))^n$ as vectors $\vec{t}$, using the same meta symbol (here t) for the vector $\vec{t}$ and its indexed elements t_i. Moreover we abbreviate terms $f(t_1, \ldots, t_n)$ by $f(\vec{t})$.

The freeness of $\mathcal{T}^\infty_{\Sigma,\perp}(X)$ in $\mathrm{Alg}^\infty_{\Sigma,\perp}$ guarantees unique decomposability of terms, i.e. $t = \perp$ or $t = x$ or there exist $f^{(n)} \in \Sigma$ and terms $t_1, \ldots, t_n$ so that $t = f(\vec{t})$.

The order $\langle \mathrm{T}^\infty_{\Sigma,\perp}(X), \trianglelefteq\rangle$ of infinite partial terms is characterized by $\perp \trianglelefteq t$ and $t'_1 \trianglelefteq t''_1, \ldots, t'_n \trianglelefteq t''_n \Longrightarrow f(\vec{t'}) \trianglelefteq f(\vec{t''})$.

Due to the existing unique homomorphism from $\mathcal{T}_\Sigma(X)$ to $\mathcal{T}^\infty_{\Sigma,\perp}(X)$ the terms $\mathrm{T}_\Sigma(X)$ can be considered as subset of $\mathrm{T}^\infty_{\Sigma,\perp}(X)$. Furthermore $\perp$ can be considered as a new constant ($\notin \Sigma \mathbin{\dot\cup} X$) so that $\mathcal{T}^\infty_{\Sigma,\perp}(X)$ is a $\Sigma \mathbin{\dot\cup} \{\perp\}$-algebra (it is not a free $\Sigma \mathbin{\dot\cup} \{\perp\}$-algebra in Alg_Σ or $\mathrm{Alg}^\infty_{\Sigma \mathbin{\dot\cup} \{\perp\},\perp}$!). Defining the set of *finite partial* Σ*-terms* $\mathrm{T}_{\Sigma,\perp}(X) := \mathrm{T}_{\Sigma \mathbin{\dot\cup} \{\perp\}}(X)$ we have $\mathrm{T}_\Sigma(X) \subset \mathrm{T}_{\Sigma,\perp}(X) \subseteq \mathrm{T}^\infty_{\Sigma,\perp}(X)$. The terms of $\mathrm{T}_{\Sigma,\perp}(X)$ are exactly the ω-compact elements of $\mathrm{T}^\infty_{\Sigma,\perp}(X)$ and $\mathcal{T}^\infty_{\Sigma,\perp}(X)$ is ω-inductive.

Let $t, t' \in \mathrm{T}^{\infty}_{\Sigma,\perp}(X)$ be terms. The *set of variables appearing in a term* t, that is the least subset Y of X with $t \in \mathrm{T}^{\infty}_{\Sigma,\perp}(Y)$, is denoted by $\mathrm{Var}(t)$. A list $u \in \mathbb{N}^*_+$ of positive natural numbers is called *position*. The *set of positions of a term* t is the least subset $\mathrm{Pos}(t)$ of $\mathbb{N}^*_+$ with $\varepsilon \in \mathrm{Pos}(t)$ and $t = f^{(n)}(\vec{t}), u \in \mathrm{Pos}(t_i), i \in [n] \Longrightarrow i.u \in \mathrm{Pos}(t)$. The *symbol at a position* $u \in \mathrm{Pos}(t)$ *in a term* t, $t(u)$, is defined by[5] $\perp(\varepsilon) := \perp$, $x(\varepsilon) := x$, $f(\vec{t})(\varepsilon) := f$ and $f(\vec{t})(i.u) := t_i.u$. The *subterm of a term* t *at a position* $u \in \mathrm{Pos}(t)$, t/u, is given by $t/\varepsilon := t$ and $f(\vec{t})/i.u := t_i/u$. The term $t[u \leftarrow t']$, created by *inserting the term* t' *at the position* $u \in \mathbb{N}^*_+$ *in* t, is defined by $t[\varepsilon \leftarrow t'] := t'$, $f^{(n)}(\vec{t})[i.u \leftarrow t'] := f(t_1, \ldots, t_i[u \leftarrow t'], \ldots, t_n)$ if $i \in [n]$, and $t[i.u \leftarrow t'] := t$ otherwise. Finally the *set of positions of a term* t *with a symbol* $g \in \Sigma \mathbin{\dot\cup} X \mathbin{\dot\cup} \{\perp\}$ is defined by $\mathrm{Pos}(g, t) := \{u \in \mathrm{Pos}(t) \mid t(u) = g\}$. A term t is *linear* iff it does not contain multiple occurrences of the same variable, i.e. $|\mathrm{Pos}(x, t)| \leq 1$ for all $x \in X$. We employ the operations just defined for tuples as well. Since $\mathrm{Pos}(\vec{t}) := \{i.u_i \in \mathbb{N}^*_+ \mid u_i \in t_i\}$, we have $\varepsilon \notin \mathrm{Pos}(\vec{t})$ and hence $\mathrm{T}^{\infty}_{\Sigma,\perp}(t)$ and $(\mathrm{T}^{\infty}_{\Sigma,\perp}(t))^1$ have to be distinguished.

We define two partial orders over positions: The *prefix order* $\langle \mathbb{N}^*_+, \leq \rangle$ is given by $u \leq v :\Longleftrightarrow \exists w \in \mathbb{N}^*_+.\ u.w = v.$ and the *lexicographic order* $\langle \mathbb{N}^*_+, \leq_{\mathrm{lex}} \rangle$ by $u \leq_{\mathrm{lex}} v :\Longleftrightarrow u \leq v \vee \exists w, u', v' \in \mathbb{N}^*_+.\ \exists i, j \in \mathbb{N}_+.\ i < j \wedge u = w.i.u' \wedge v = w.j.v'$. Two positions u and v are *independent*, written $u \parallel v$, iff neither $u \leq v$ nor $v \leq u$.

A mapping $\sigma : X \to \mathrm{T}_\Sigma(X)$ with $\sigma(x) \neq x$ for only finitely many $x \in X$ is a *substitution*. We write $[t_1/x_1, \ldots, t_n/x_n]$ for σ if $\sigma(x_i) = t_i$ for all $i \in [n]$, and similarly $[t/Y]$ if $\sigma(x) = t$ for all $x \in Y \subseteq X$, under the condition that $\sigma(x) = x$ for all other $x \in X$. Its unique extension to a homomorphism from $\mathcal{T}_\Sigma(X)$ to $\mathcal{T}_\Sigma(X)$ is denoted by σ as well and we usually write $t\sigma$ for the instance $\sigma(t)$ of t. If $t\sigma$ is a ground term, i.e. contains no variables, it is called *ground instance* of t.

With regard to operational semantics, the reader should note that we define substitutions only for finite terms and that applying a substitution is therefore an effective operation.

2.6 Term Rewriting Systems

A *rewrite rule* $l \to r$ is a pair of terms with $l \in \mathrm{T}_\Sigma(X)$ and $r \in \mathrm{T}_\Sigma(\mathrm{Var}(l))$. A *term rewriting system (TRS)* is a finite set of rewrite rules R. A left-hand side of a rule is called *redex scheme* and the set of all redex schemes of a TRS R is denoted by RedS_R. If $l \in \mathrm{RedS}_\mathrm{R}$ and $\sigma : \mathrm{Var}(t) \to \mathrm{T}_\Sigma$, then $l\sigma$ is a *redex of the TRS* R. Red_R is the set of all redexes of R and $\mathrm{RedPos}(t) := \{u \in \mathrm{Pos}(t) \mid t/u \in \mathrm{Red}_R\}$ is the *set of all redex positions of a term* $t \in \mathrm{T}_\Sigma$.

A term $t \in \mathrm{T}_\Sigma$ is *rewritable at a position* $u \in \mathrm{Pos}(t)$ *by a rewrite rule* $l \to r \in R$ *in a single step to a term* $t' \in \mathrm{T}_\Sigma$ iff there exists a substitution $\sigma : \mathrm{Var}(l) \to \mathrm{T}_\Sigma$ with $t/u = l\sigma$ and $t' = t[u \leftarrow r\sigma]$. Both t' and σ are uniquely determined by t, u, and $l \to r$. Hence a (simple) reduction is a tuple $A = \langle t, u, l \to r \rangle$, written as $A = (t \xrightarrow[l \to r]{u} t')$ or just $A = (t \xrightarrow[R]{u} t')$.

Let ϱ be a set of reductions, the ϱ-reductions. If ϱ is a proper subset of all reductions, then we mark ϱ-reductions by ϱ, that is we write $A = t \xrightarrow[l \to r, \varrho]{u} t'$ or $A = t \xrightarrow[R, \varrho]{u} t'$. The *$\varrho$-reduction relation* $\xrightarrow[R,\varrho]{} \subseteq \mathrm{T}_\Sigma \times \mathrm{T}_\Sigma$ is defined by $t \xrightarrow[R,\varrho]{} t' :\Longleftrightarrow \exists u \in \mathrm{RedPos}_R(t).\exists (l \to r) \in R :$ $(t \xrightarrow[l \to r]{u} t'$ is a ϱ-reduction). A *sequential ϱ-reduction strategy* is a deterministic ϱ-reduction

[5]This notation is inspired by the interpretation of terms as partial mappings $\mathbb{N}^*_+ \nrightarrow \Sigma \mathbin{\dot\cup} X \mathbin{\dot\cup} \{\perp\}$.

relation, i.e. where for every $t \in \mathrm{T}_\Sigma$ there is at most one $t' \in \mathrm{T}_\Sigma$ such that $t \xrightarrow[R,\varrho]{} t'$. We write $A = t_1 \xrightarrow[R,\varrho]{} t_2 \xrightarrow[R,\varrho]{} t_3 \xrightarrow[R,\varrho]{} \dots$ for a (finite or infinite) *ϱ-reduction sequence* $A = t_1 \xrightarrow[R,\varrho]{} t_2, t_2 \xrightarrow[R,\varrho]{} t_3, t_3 \xrightarrow[R,\varrho]{} t_4, \dots$.

$A' := \langle t, \{\langle u_i, l_i{\to}r_i\rangle \mid i \in [n]\}\rangle$ is a *(parallel) reduction*, written as $A' = t \xrightarrow[R]{U}\!\!\!\!\!\rightarrow t'$, iff $A = (t = t_1 \xrightarrow[l_1\to r_1]{u_1} t_2 \xrightarrow[l_2\to r_2]{u_2} \dots t_n = t')$ is a reduction sequence and $U := \{u_1, \dots, u_n\} \subseteq \mathrm{RedPos}_R(t)$ a set of mutually independent redex positions of t. This implies $t' = t[u_1 \leftarrow r_1\sigma_1]\dots[u_n \leftarrow r_n\sigma_n]$ with the order of the replacement being irrelevant due to the independence of the redex positions. The definitions of *parallel ϱ-reduction*, $A' = t \xrightarrow[R,\varrho]{U}\!\!\!\!\!\rightarrow t'$, *parallel ϱ-reduction relation* $\xrightarrow[R,\varrho]{}\!\!\!\!\!\rightarrow$, and *parallel ϱ-reduction sequence* and *strategy* are straightforward.

Let $\xrightarrow[\varrho]{} \subseteq T \times T$ be an arbitrary relation over a set T. The *reflexive and transitive closure* of $\xrightarrow[\varrho]{}$ is denoted by $\xrightarrow[\varrho]{*}$. The relation $\xrightarrow[\varrho]{}$ is *confluent* iff for all $t, t', t'' \in T$ with $t \xrightarrow[\varrho]{*} t'$ and $t \xrightarrow[\varrho]{*} t''$ there exists a $\hat{t} \in T$ such that $t' \xrightarrow[\varrho]{*} \hat{t}$ and $t'' \xrightarrow[\varrho]{*} \hat{t}$. An element $t \in T$ is a *ϱ-normal form* iff there is no $t' \in T$ with $t' \neq t$ and $t \xrightarrow[\varrho]{} t'$. An element $t' \in T$ is a *ϱ-normal form of* $t \in T$ iff $t \xrightarrow[\varrho]{*} t'$ and t' is a ϱ-normal form. If $\xrightarrow[\varrho]{}$ is confluent, then the ϱ-normal form of an element $t \in T$ is unique (if it exists).

We use the properties just defined for the relations $\xrightarrow[R,\varrho]{}$ and $\xrightarrow[R,\varrho]{}\!\!\!\!\!\rightarrow$. A unique ϱ-normal form of a term $t \in \mathrm{T}_\Sigma$ is denoted by $t{\downarrow}_{R,\varrho}$. A term $t \in \mathrm{T}_\Sigma$ is a normal form iff $\mathrm{RedPos}_R(t) = \emptyset$.

The reader should note that we overload notation in that respect that e.g. $t \xrightarrow[R,\varrho]{} t'$ may denote both a proposition concerning the relation $\xrightarrow[R,\varrho]{}$ and a reduction (a tuple). The respective meaning should be clear from the context.

2.7 Almost Orthogonal TRS

An *almost orthogonal TRS R* is a TRS which is left-linear, that is all its redex schemes are linear, and which fulfills the following condition of uniqueness concerning overlays of redexes:

> If $l{\to}r, l'{\to}r' \in R, u \in \mathrm{Pos}(l)$ with $l/u \notin X$ and $\sigma, \sigma' : X{\to}\mathrm{T}_\Sigma$, then $(l/u)\sigma = l'\sigma'$ implies $u = \varepsilon$ and $r\sigma = r'\sigma'$.

Almost orthogonal TRSs have the important property that in a reduction $t \xrightarrow[l\to r]{u} t'$ the term t and the position u determine uniquely — not $l{\to}r$ and σ, but — the rewrite rule instance $l\sigma{\to}r\sigma$ and thereby t'. Let R be an almost orthogonal TRS. The following frequently used property takes advantage of the lack of non-trivial overlays of redexes.

Lemma 2.1
Let $t \in \mathrm{T}_\Sigma$, $u, v \in \mathrm{RedPos}_R(t)$ with $u < v$, and $l \in \mathrm{RedS_R}$ with $t/u = l\sigma$ for a substitution σ. Then there exists a unique $w \in \mathbb{N}_+^*$ such that $u \leq u.w \leq v$ and $l/w \in X$.

Proof idea:
Show that $l/w \notin X$ for all w with $u \leq u.w \leq v$ contradicts the condition of uniqueness of almost orthogonal TRSs. □

In principal, almost orthogonal TRSs appear for the first time in [Ros73] and are treated in detail in [O'D77]. However, both references actually do not consider TRSs but so called subtree replacement systems which are special, generally infinite sets of rewrite rules without variables. Nonetheless, all ground instances of all reduction rules of an almost orthogonal TRS R together, that is all $l\sigma \rightarrow r\sigma$ with $l \rightarrow r \in R$ and $l\sigma, r\sigma \in \mathrm{T}_\Sigma$, form such a subterm replacement system. This gives us the idea to not only consider the set of all instances of the rewrite rules of a TRS, but also other instances of a TRS, which contain only particular instances of the rewrite rules of the TRS.

An *instance I of an almost orthogonal TRS* is uniquely determined by the set of redexes of the instance $\mathrm{Red}_{R,I} \subset \mathrm{Red}_R$, because variables which occur in a right-hand side occur also in the left-hand side and because of the condition of uniqueness (cf. instance predicate Q of rule schemata in [O'D77]). For $\mathrm{Red}_{R,I} = \mathrm{Red}_R$ we obtain the commonly considered canonical instance of a TRS. For a term $t \in \mathrm{T}_\Sigma$ we define its *I-redex positions* $\mathrm{RedPos}_{R,I}(t) := \{u \in \mathrm{RedPos}_R(t) \mid t/u \in \mathrm{Red}_{R,I}\}$. A reduction $A = t \xrightarrow[R]{u} t'$ is an *I-reduction*, i.e. a reduction in the instance I, iff $u \in \mathrm{RedPos}_{R,I}(t)$. I-reductions are a special kind of ϱ-reductions defined in the previous subsection. Consequently we write $A = t \xrightarrow[R,I]{u} t'$ and have the notions of sequential and parallel I-reduction relation, I-reduction strategy and I-reduction sequence.

For the study of many properties of reduction sequences it is important to follow how redexes are rearranged in a reduction. The *residual map* maps a redex position v of a term t to its residuals, i.e. redex positions of a term t' which are copies of t/v under the rearrangement caused by the reduction $t \xrightarrow[l \rightarrow r]{u} t'$ (cf. [HL91]):

$$v \setminus (t \xrightarrow[l \rightarrow r]{u} t') := \begin{cases} \emptyset & \text{, if } v = u; \\ \{v\} & \text{, if } v \parallel u \text{ or } v < u; \\ \{u.w'.v' \mid v = u.w.v', r/w' = l/w \in X\} & \text{, if } v > u. \end{cases}$$

Example 2.1

Let $R := \{f(x,y) \rightarrow x, g(x) \rightarrow f(x,x), b \rightarrow a\}$ and consider the reduction $A := f(g(b), g(a)) \xrightarrow{1} f(f(b,b), g(a))$.

The redex positions of the first term are $\varepsilon, 1, 1.1, 2$ and those of the second term are $\varepsilon, 1, 1.1, 1.2, 2$.

The rearrangement of redex positions is described by $\varepsilon \setminus A = \{\varepsilon\}, 1 \setminus A = \emptyset, 1.1 \setminus A = \{1.1, 1.2\}, 2 \setminus A = \{2\}$. □

Although the definition uses the rewrite rule $l \rightarrow r$, it actually depends only on v, t and u:

Lemma 2.2

If $A = t \xrightarrow[l \rightarrow r]{u} \hat{t}$ and $A' = t \xrightarrow[l' \rightarrow r']{u} \hat{t}$ are reductions and $v \in \mathrm{RedPos}_R(t)$, then $v \setminus A = v \setminus A'$.

Proof idea:

Simple but tiresome, using Lemma 2.1. □

Reductions preserve the independence of redex positions:

Lemma 2.3
If $t \xrightarrow{w} t'$ is a reduction and $u, v \in \mathrm{RedPos}_R(t)$, then $u \parallel v$ implies $(u \setminus t \xrightarrow{w} t') \parallel (v \setminus t \xrightarrow{w} t')$.
Proof idea:
Case analysis on the relative order of u, v and w, using lemma 2.1. □

Because of this lemma the following *extension of the residual map to parallel reductions* is well-defined:

$$u \setminus t \xrightarrow{V}\!\!\!\!\rightarrow t' := \begin{cases} \{u\} & \text{, if for all } v \in V \text{ either } v \parallel u \text{ or } u < v; \\ u \setminus t \xrightarrow{v} t'' & \text{, if } v \in V \text{ exists such that } v \leq u. \end{cases}$$

Furthermore we define the *residual map for sets of redex positions*:

$$\begin{aligned} U \setminus t \xrightarrow[R,I]{} t' &:= \bigcup \{u \setminus t \xrightarrow[R,I]{} t' \mid u \in U\}, \\ U \setminus t \xrightarrow[R,I]{}\!\!\!\!\rightarrow t' &:= \bigcup \{u \setminus t \xrightarrow[R,I]{}\!\!\!\!\rightarrow t' \mid u \in U\}. \end{aligned}$$

A set of I-redexes $\mathrm{Red}_{R,I}$ is *residually closed* iff for all I-reductions $t \xrightarrow[R,I]{} t'$ and all positions $u \in \mathrm{RedPos}_{R,I}(t)$ the inclusion $u \setminus t \xrightarrow[R,I]{} t' \subseteq \mathrm{RedPos}_{R,I}(t')$ holds. Obviously this property is not fulfilled for arbitrary sets of I-redexes. However, without it the notion of residual map is rather useless.

Lemma 2.4 Residual closure of Red_R
The set of all redexes of an almost orthogonal TRS R, Red_R, is residually closed. In particular we have $t/v = t'/v'$ for all reductions $t \xrightarrow{u} t'$ and $v \in \mathrm{RedPos}_R(t)$ with $v \not< u$ and all $v' \in v \setminus t \xrightarrow{u} t'$.
Proof idea:
Case analysis on the relative order of u and v, using lemma 2.1. □

Lemma 2.5 Parallel moves lemma
Let $\mathrm{Red}_{R,I}$ be residually closed. If $t \xrightarrow[R,I]{U}\!\!\!\!\rightarrow t'$ and $t \xrightarrow[R,I]{V}\!\!\!\!\rightarrow t''$ are reductions, then exists a unique $\hat{t} \in \mathrm{T}_\Sigma$ such that $t' \xrightarrow[R,I]{V'}\!\!\!\!\rightarrow \hat{t}$ and $t'' \xrightarrow[R,I]{U'}\!\!\!\!\rightarrow \hat{t}$ are reductions with $U' := U \setminus t \xrightarrow[R,I]{V}\!\!\!\!\rightarrow t''$ and $V' := V \setminus t \xrightarrow[R,I]{U}\!\!\!\!\rightarrow t'$, that is:

$$\begin{tikzcd} & t \arrow[dl, two heads, "U"', "R{,}I"] \arrow[dr, two heads, "V", "R{,}I"'] & \\ t' \arrow[dr, two heads, "R{,}I"', "V'"] & & t'' \arrow[dl, two heads, "U'"', "R{,}I"] \\ & \hat{t} & \end{tikzcd}$$

Proof:
[O'D77, Chi95]. □

Consequently $\xrightarrow[R,I]{}$ and $RP_{R,I}$ are confluent for residually closed redex sets $\mathrm{Red}_{R,I}$.

Finally an I-redex position of a term is *innermost/outermost* iff it is maximal/minimal w.r.t. the prefix order $\langle \mathbb{N}_+^*, \leq \rangle$ in the set of all I-redex positions of the term. The *leftmost-innermost/leftmost-outermost I-redex position of a term* is its least innermost/outermost I-redex position w.r.t. the lexicographic order $\langle \mathbb{N}_+^*, \leq_{\mathrm{lex}} \rangle$. We write $t \xrightarrow[R,I,\mathrm{li}]{} t'$, $t \xrightarrow[R,I,\mathrm{lo}]{} t'$, $t \xrightarrow[R,I,\mathrm{no}]{}$ for a reduction $t \xrightarrow[R,I]{} t'$ where u is the leftmost-innermost, the leftmost-outermost, not an outermost I-redex position of t, respectively, and we use corresponding notation for the associated I-reduction relations.

3 Abstract Syntax

Here we define the syntax of simple constructor-based first-order functional programs. Since we later define numerous different semantics for this syntax, we obtain numerous different functional programming languages.

We use a signature which distinguishes between constructor and (definable) function symbols.

Definition 3.1
Let $\mathcal{C}$ be a signature of **constructor symbols** with $\mathcal{C}_0 \neq \emptyset$ and $\mathcal{F}$ be a signature of **function symbols** so that $\mathcal{C} \cap \mathcal{F} = \emptyset$. Then $\Sigma = (\mathcal{C}, \mathcal{F})$ is a **program signature**. □

We often regard the program signature plainly as signature $\Sigma = \mathcal{F} \dot\cup \mathcal{C}$ with every operation symbol $g \in \Sigma$ nevertheless uniquely identifiable as constructor or function symbol. In all our examples constructor symbols start with a capital letter and function symbols with a small one, as in MIRANDA[6] and HASKELL. We use the same convention for meta variables denoting signature symbols, using small letters for meta variables ranging over the whole program signature.

Definition 3.2
Let Σ be a program signature. An ordered pair of terms

$$f(p_1, \ldots, p_n) \quad \to \quad r$$

with $f^n \in \mathcal{F}$, $p_1, \ldots, p_n \in \mathrm{T}_{\mathcal{C}}(X)$, $f(p_1, \ldots, p_n)$ linear, and $r \in \mathrm{T}_\Sigma(\mathrm{Var}(\vec{p}))$ is a **program rule** and $\vec{p}$ is called a **pattern**. A finite set P of program rules which fulfills the **condition of uniqueness**

$$l_1 {\to} r_1, l_2 {\to} r_2 \in P,\ l_1\sigma_1 = l_2\sigma_2 \implies r_1\sigma_1 = r_2\sigma_2$$

$(\sigma_1, \sigma_2 : X {\to} \mathrm{T}_\Sigma)$ is a **program**. The set of all programs over Σ is denoted by Prog_Σ. □

[6] MIRANDA is a trademark of Research Software Ltd.

An example of a program was already given in the introduction. As explained there, we do not want to use any priority rule for pattern matching. Our abstract definition of a program as a set of rules makes the definition of a first-fitting priority rule even impossible, whereas every program text trivially implies an order.

Under these circumstances the condition of uniqueness is needed to guarantee that a program specifies (deterministic) functions. Its significance and that of the left-linearity condition become fully clear in the proof of well-definedness of the semantics in Subsections 6.2 and 6.4.

The reader should notice that patterns do not need to be **complete** in the sense that for all $f^n \in \mathcal{F}$ and $t_1, \ldots, t_n \in \mathrm{T}_{\mathcal{C}}$ there would be $f(\vec{p}) \to r \in P$ and a substitution σ with $f(\vec{t}) = f(\vec{p})\sigma$. There may even be function symbols without any program rule.

Obviously the definition of programs implies:

Corollary 3.1
A program is an almost orthogonal TRS. □

This gives us a confluent reduction relation $\xrightarrow[P]{}$ for the operational semantics and enables us to exploit many further known properties and proof methods.

In the following P is an arbitrary program over an arbitrary program signature[7] $\Sigma = (\mathcal{C}, \mathcal{F})$.

4 Kinds of Semantics

For defining the data type $\mathrm{DT}(P)$ of a program $P \in \mathrm{Prog}_\Sigma$ we use a base data type $\mathrm{BDT}_{\mathcal{C}}$ and the program rules.

The principal idea of a denotational fixed-point semantics is to associate a transformation $\Phi_P : \mathrm{Alg}_\Sigma \to \mathrm{Alg}_\Sigma$ with the program, which fixes the function operators by doing the pattern matching and interpreting the right-hand sides of the program rules. Due to the recursive nature of the program rules the transformation Φ_P needs an algebra as input and the data type is defined as a fixed-point of Φ_P.

As usual all operators are continuous mappings over cpos, which is justified by interpreting the partial order as an order of information content (cf. appendix B.2 of [FH88]). Hence all used algebras are complete. Since the data type shall be an extension of the base data type, the transformation Φ_P is actually only defined over a set of admissible data types, the **interpretations** $\mathrm{Int}_\Sigma := \{\mathfrak{A} \in \mathrm{Alg}^\infty_{\Sigma,\perp} \mid \mathfrak{A}|_{(\mathcal{C},\emptyset)} = \mathrm{BDT}_{\mathcal{C}}\}$. Together with the canonical partial order $\sqsubseteq$ of complete algebras we get the cpo $\langle \mathrm{Int}_\Sigma, \sqsubseteq \rangle$. The data type $\mathrm{DT}(P)$ is defined as the least fixed-point of the continuous transformation Φ_P.

An operational reduction semantics is based on the computation on terms using a reduction relation $\xrightarrow[P]{}$. Therefore it does not (directly) assign a data type to a program but a **term semantics** $[\![\cdot]\!]_P : \mathrm{T}_\Sigma \to A$, a mapping from program terms T_Σ to a **computation domain** A. This computation domain is identical with the carrier set of the data type $\mathrm{DT}(P) = \langle A, \alpha \rangle$.

[7]We sometimes write P_Σ to state that P is a program over the signature Σ, since this information cannot be deduced from the pure set of program rules.

A data type fixes a term semantics as well: If $\mathfrak{A} = \langle A, \alpha \rangle \in \mathrm{Alg}_\Sigma$, $Y \subseteq X$, and $\beta : Y \rightarrow A$ is a valuation, then the **algebraic term semantics** $[\![\cdot]\!]^{\mathrm{alg}}_{\mathfrak{A},\beta} : \mathrm{T}_\Sigma(Y) \rightarrow A$ is the unique homomorphism from $\mathrm{T}_\Sigma(Y)$ to $\mathfrak{A}$ which extends β. Conversely, an algebra $\mathfrak{A}_{[\![\cdot]\!]} = \langle A, \alpha \rangle$ is a **data type of the term semantics** $[\![\cdot]\!] : \mathrm{T}_\Sigma \rightarrow A$, if $f^{\mathfrak{A}_{[\![\cdot]\!]}}(a_1, \dots a_n) = [\![f(t_1, \dots, t_n)]\!]$ for all $f^{(n)} \in \Sigma$, $t_1, \dots, t_n \in \mathrm{T}_\Sigma$, and $a_1, \dots, a_n \in A$ with $[\![t_1]\!] = a_1, \dots, [\![t_n]\!] = a_n$. Evidently we have $[\![\cdot]\!] = [\![\cdot]\!]^{\mathrm{alg}}_{\mathfrak{A}_{[\![\cdot]\!]}}$.

In general there are several data types for one term semantics, because there may not be for every $a \in A$ a $t \in \mathrm{T}_\Sigma$ with $[\![t]\!] = a$. We see in Section 7 that for the ς-semantics there is a one-to-one relationship though, i.e. the ς-data type could be defined using only the ς-term semantics. Nonetheless, there exist many reduction semantics for one fixed-point semantics, since many different reduction strategies fix the same term semantics.

A term semantics $[\![\cdot]\!] : \mathrm{T}_\Sigma \rightarrow A$ is **compositional** (or invariant) iff for all $t', t'' \in \mathrm{T}_\Sigma$ and $t \in \mathrm{T}_\Sigma(\{\Box\})$ we have $[\![t']\!] = [\![t'']\!] \implies [\![t[t'/\Box]]\!] = [\![t[t''/\Box]]\!]$. Compositionality reflects the nature of terms as compound objects and only a compositional term semantics engenders a semantic equality ($t \simeq t' \iff [\![t]\!] = [\![t']\!]$) which is a congruence. Algebraic term-semantics are compositional, because they are homomorphisms. Likewise, data types of term semantics $[\![\cdot]\!]$ exist only for compositional $[\![\cdot]\!]$, since otherwise the $f^{\mathfrak{A}_{[\![\cdot]\!]}}$ are not well-defined. Hence, an operational semantics needs to be compositional so that an equivalent denotational semantics can exist.

The following term semantics $[\![\cdot]\!]^{\mathrm{nf}}_P : \mathrm{T}_\Sigma \rightarrow \mathrm{T}_\mathcal{C} \,\dot\cup\, \{\bot\}$, naïvely defined on the basis of normal forms, is not compositional:

$$[\![t]\!]^{\mathrm{nf}}_P := \begin{cases} t\!\downarrow_P & \text{, if } t\!\downarrow_P \text{ exists and}^8 t\!\downarrow_P \in \mathrm{T}_\mathcal{C}; \\ \bot & \text{, otherwise.} \end{cases}$$

Example 4.1
Regarding the program[9]

```
list1        →  []:list1
list2        →  [[]]:list2
head(x:xs)   →  x
```

we see that neither `list1` nor `list2` have a normal form and hence $[\![\texttt{list1}]\!]^{\mathrm{nf}}_P = [\![\texttt{list2}]\!]^{\mathrm{nf}}_P = \bot$. Nonetheless we have

$$\begin{array}{lclcl} \texttt{head(}\underline{\texttt{list1}}\texttt{)} & \xrightarrow[P]{} & \underline{\texttt{head}}\texttt{([]:list1)} & \xrightarrow[P]{} & \texttt{[]} \\ \texttt{head(}\underline{\texttt{list2}}\texttt{)} & \xrightarrow[P]{} & \underline{\texttt{head}}\texttt{([[]]:list2)} & \xrightarrow[P]{} & \texttt{[[]]} \end{array}$$

and hence $[\![\texttt{head(list1)}]\!]^{\mathrm{nf}}_P = \texttt{[]} \neq \texttt{[[]]} = [\![\texttt{head(list2)}]\!]^{\mathrm{nf}}_P$ in contradiction to compositionality. □

[8] Necessary, because patterns may not be complete.
[9] We use the well-known 'sugared' syntax for lists instead of $\texttt{Nil}^{(0)}$ and $\texttt{Cons}^{(2)}$.

Finally a property deserves attention although it is guaranteed by every (reasonable) reduction or fixed-point semantics: modularity. Any program can be extended by new function symbols together with respective program rules. We expect the data type assigned to the extended program to be an extension of the data type of the original program, just as that one is an extension of the base data type (cf. hierarchical specifications in Subsection 5.4 of [Wir90]). We call a semantics $\mathrm{DT} : \mathrm{Prog}_\Sigma \to \mathrm{Alg}_\Sigma$, mapping a program to a data type, **modular** iff $\mathrm{DT}(P)|_{(\mathcal{C},\mathcal{F}')} = \mathrm{DT}(P_{(\mathcal{C},\mathcal{F}')})$ for all $P \in \mathrm{Prog}_\Sigma$ and $\mathcal{F}' \subseteq \mathcal{F}$ with $P_{(\mathcal{C},\mathcal{F}')} := \{f(\vec{p}) \to r \in P \mid f \in \mathcal{F}'\}$. This also reveals the natural characterization of the base data type as the data type of the empty program: $\mathrm{BDT}_\mathcal{C} = \mathrm{DT}(\emptyset_\mathcal{C})$.

A compositional reduction semantics defines modular data types, because the term semantics of a term is not changed by additional program rules for new function symbols.

Since a (reasonable) transformation Φ_P uses only the program rules $f(\vec{p}) \to r \in P$ for fixing the operation of the function symbol f, any semantics based on the least fixed-point of Φ_P is modular.

A well-known semantics which is not modular is the **quotient algebra semantics** ([DJ90]), which defines the data type of a program to be the free term algebra $\mathcal{T}_\Sigma$ modulo the equations (rules) of the program. Extending a program may even not only change the (old) operations of the data type but also the carrier set.

5 Cbv- and Cbn-Semantics

Preparing the generalization to ς-semantics in the next section, we define here the well-known cbv- and cbn-semantics. Some amendments are necessary due to the constructors and the pattern-matching.

We do not give any statement about well-definedness or equivalence of the reduction and fixed-point semantics, because that is done for the more general ς-semantics in Sections 6 and 7.

5.1 Cbv-Reduction Semantics

We start with the reduction semantics, since cbv and cbn are rather operational notions.

<u>**Definition 5.1**</u>
Let $t \in \mathrm{T}_\Sigma$ be a term.

- An position $u \in \mathrm{RedPos}_P(t)$ is an **innermost* redex position of the term *t*** iff $t/u = f(c_1, \ldots, c_n)$ with $c_1, \ldots, c_n \in \mathrm{T}_\mathcal{C}$.
- The **leftmost-innermost* (li*-) redex position of the term *t*** is the least innermost* redex position of t w.r.t. the lexicographical order.
- A reduction $A = t \xrightarrow[l \to r]{u} t'$ is a **leftmost-innermost* reduction**, written $A = t \xrightarrow[P,\mathrm{li}^*]{} t'$, iff u is the li*-redex position of t. Hereby the **leftmost-innermost* reduction strategy** $\xrightarrow[P,\mathrm{li}^*]{}$ is defined as well.

□

Definition 5.2
The **li*-reduction** or **cbv-reduction semantics** $[\![\cdot]\!]^{\text{red}}_{P,\text{cbv}} : \mathrm{T}_\Sigma \to \mathrm{T}_\mathcal{C} \,\dot\cup\, \{\bot\}$ is defined by

$$[\![t]\!]^{\text{red}}_{P,\text{cbv}} := \begin{cases} t\!\downarrow_{P,\text{cbv}} & \text{, if } t\!\downarrow_{P,\text{cbv}} \text{ exists and } t\!\downarrow_{P,\text{cbv}} \in \mathrm{T}_\mathcal{C}; \\ \bot & \text{, otherwise.} \end{cases}$$

□

We avoid the name leftmost-innermost reduction, since the li*-reduction strategy is not identical with the leftmost-innermost reduction strategy. The leftmost-innermost redex position of a term is not always a li*-redex position (the converse is true), due to the possibility of incomplete patterns.

Example 5.1
Let $P = \{\texttt{f(x)}\to\texttt{A}\}$ be a program with the function symbols $\texttt{f}^{(1)}$ and $\texttt{a}^{(0)}$. Then the term `f(a)` is still reducible by leftmost-innermost reduction[10]: $\texttt{f(}\underline{\texttt{a}}\texttt{)} \xrightarrow[\text{li}]{} \texttt{A}$, but `f(a)` does not contain any li*-redex. Hence $[\![\texttt{f(a)}]\!]^{\text{red}}_{P,\text{cbv}} = \bot$. □

In contrast to the normal form semantics the cbv-semantics is compositional:

Example 5.2
Using the program of Example 4.1 we still have $[\![\texttt{list1}]\!]^{\text{red}}_{P,\text{cbv}} = [\![\texttt{list2}]\!]^{\text{red}}_{P,\text{cbv}} = \bot$, but also

$$\begin{array}{lclcl} \texttt{head(}\underline{\texttt{list1}}\texttt{)} & \xrightarrow[P,\text{li}^*]{} & \texttt{head([]:}\underline{\texttt{list1}}\texttt{)} & \xrightarrow[P,\text{li}^*]{} & \ldots \\ \texttt{head(}\underline{\texttt{list2}}\texttt{)} & \xrightarrow[P,\text{li}^*]{} & \texttt{head([[]]:}\underline{\texttt{list2}}\texttt{)} & \xrightarrow[P,\text{li}^*]{} & \ldots \end{array}$$

and hence $[\![\texttt{head(list1)}]\!]^{\text{red}}_{P,\text{cbv}} = \bot = [\![\texttt{head(list2)}]\!]^{\text{red}}_{P,\text{cbv}}$ □

5.2 Cbv-Fixed-Point Semantics

Definition 5.3
The ordered $\mathcal{C}$-algebra $\text{BDT}_{\mathcal{C},\text{cbv}} := \mathcal{T}_\mathcal{C}^\bot := \langle \mathrm{T}_\mathcal{C}^\bot, \trianglelefteq, \tau\rangle$ with τ given by $\langle \mathrm{T}_\mathcal{C}, \trianglelefteq, \tau\rangle = \mathcal{T}_\mathcal{C}$ and $\mathrm{T}_\mathcal{C}^\bot := \mathrm{T}_\mathcal{C} \,\dot\cup\, \{\bot\}$ is the **cbv-base data type** over the constructor symbols $\mathcal{C}$. □

To distinguish syntax and semantics we underline meta-variables ranging over the carrier set of base types[11], e.g. $\underline{t} \in \mathrm{T}_\mathcal{C}^\bot$.

[10] We underline the redexes reduced in a reduction.
[11] In a few cases like the reduction cbv-semantics of Subsection 5.1 this distinction is not possible or too awkward.

Definition 5.4
The set $\mathrm{Int}_{\Sigma,\mathrm{cbv}} := \{\mathfrak{A} \in \mathrm{Alg}^{\infty}_{\Sigma,\perp} \mid \mathfrak{A}|_{\mathcal{C},\emptyset} = \mathrm{BDT}_{\mathcal{C},\mathrm{cbv}}\}$ is the set of **cbv-interpretations** and $\langle \mathrm{Int}_{\Sigma,\mathrm{cbv}}, \sqsubseteq \rangle$ is the canonical cpo of cbv-interpretations with least element $\perp_{\mathrm{cbv}} := \perp_{\mathrm{Int}_{\Sigma,\mathrm{cbv}}}$. □

Note that $\perp_{\mathrm{cbv}}$ and $\mathrm{BDT}_{\mathcal{C},\mathrm{cbv}}$ are not identical but $\perp_{\mathrm{cbv}}|_{\mathcal{C},\emptyset} = \mathrm{BDT}_{\mathcal{C},\mathrm{cbv}}$.

Definition 5.5
The **cbv-transformation** of $P \in \mathrm{Prog}_{\Sigma}$, $\Phi_{P,\mathrm{cbv}} : [\mathrm{Int}_{\Sigma,\mathrm{cbv}} \to \mathrm{Int}_{\Sigma,\mathrm{cbv}}]$, is defined by

$$f^{\Phi_{P,\mathrm{cbv}}(\mathfrak{A})}(\vec{\underline{t}}) := \begin{cases} [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A},\beta} & \text{, if } f(\vec{p}) \to r \in P \text{ and } \beta : \mathrm{Var}(\vec{p}) \to \mathrm{T}_{\mathcal{C}} \text{ exist} \\ & \quad \text{with } [\![p_1]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\mathrm{cbv}},\beta} = \underline{t}_1, \ldots, [\![p_n]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\mathrm{cbv}},\beta} = \underline{t}_n; \\ \perp & \text{, otherwise;} \end{cases}$$

for all $f^{(n)} \in \mathcal{F}$, $\vec{\underline{t}} \in (\mathrm{T}^{\perp}_{\mathcal{C}})^n$ and $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\mathrm{cbv}}$. □

In the cbv-transformation pattern-matching is performed by the equations $[\![p_i]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\mathrm{cbv}},\beta} = \underline{t}_i$. Since patterns consist only of constructor symbols, the base data type $\mathrm{BDT}_{\mathcal{C},\mathrm{cbv}}$ suffices here; using $\mathfrak{A}$ instead is possible but would suggest a non-existing dependence. Restricting the range of the valuation β to $\mathrm{T}_{\mathcal{C}}$ instead of $\mathrm{T}^{\perp}_{\mathcal{C}}$ assures that pattern-matching never succeeds if $\underline{t}_i = \perp$ for some $i \in [n]$. This reflects the fact that all operations are strict in the cbv-semantics.

Definition 5.6
The **cbv-fixed-point data type** $\mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbv}}(P) \in \mathrm{Int}_{\Sigma,\mathrm{cbv}}$ is defined by

$$\mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbv}}(P) := \mathrm{Fix}(\Phi_{P,\mathrm{cbv}}) = \bigsqcup_{i \in \mathbb{N}} (\Phi_{P,\mathrm{cbv}})^i(\perp_{\mathrm{cbv}})$$

and the **cbv-fixed-point (term) semantics** $[\![\cdot]\!]^{\mathrm{fix}}_{P,\mathrm{cbv}} : \mathrm{T}_{\Sigma} \to \mathrm{T}^{\perp}_{\mathcal{C}}$ by

$$[\![t]\!]^{\mathrm{fix}}_{P,\mathrm{cbv}} := [\![t]\!]^{\mathrm{alg}}_{\mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbv}}(P)}$$

□

Example 5.3 Determining a cbv-fixed-point data type
Taking the program of Example 4.1, its cbv-fixed-point data type is determined by means of the following table. Let $\mathfrak{A}_i := (\Phi_{P,\mathrm{cbv}})^i(\perp_{\mathrm{cbv}})$ for all $i \in \mathbb{N}$ and $\mathfrak{A}_\infty := \mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbv}}(P)$.

	$i = 0$	$i = 1$	…	$i = \infty$
$\texttt{list1}^{\mathfrak{A}_i}()$	$\perp$	$[\![\texttt{[]:list1}]\!]^{\mathrm{alg}}_{\perp_{\mathrm{cbv}},(\texttt{list1} \mapsto \perp)} = \perp$	…	as for $i = 1$
$\texttt{list2}^{\mathfrak{A}_i}()$	$\perp$	$[\![\texttt{[[]]:list2}]\!]^{\mathrm{alg}}_{\perp_{\mathrm{cbv}},(\texttt{list2} \mapsto \perp)} = \perp$	…	as for $i = 1$
$\texttt{head}^{\mathfrak{A}_i}$	$\underline{t} \mapsto \perp$	$\begin{pmatrix} \perp \mapsto \perp \\ \texttt{[]} \mapsto \perp \\ \underline{t}_1\texttt{:}\underline{t}_2 \mapsto \underline{t}_1 \end{pmatrix}$	…	as for $i = 1$

with $\underline{t} \in \mathrm{T}^{\perp}_{\mathcal{C}}$ and $\underline{t}_1, \underline{t}_2 \in \mathrm{T}_{\mathcal{C}}$. □

5.3 Cbn-Fixed-Point Semantics

While the cbv-semantics enforces strictness of all operations, the cbn-semantics even defines non-strict constructor operations, thus permitting partial and infinite data structures.

Definition 5.7
The ordered $\mathcal{C}$-algebra $\mathrm{BDT}_{\mathcal{C},\mathrm{cbn}} := \mathcal{T}^{\infty}_{\mathcal{C},\perp}$ is the **cbn-base data type** over the constructor symbols $\mathcal{C}$. $\square$

Definition 5.8
The set $\mathrm{Int}_{\Sigma,\mathrm{cbn}} := \{\mathfrak{A} \in \mathrm{Alg}^{\infty}_{\Sigma,\perp} \mid \mathfrak{A}|_{\mathcal{C},\emptyset} = \mathrm{BDT}_{\mathcal{C},\mathrm{cbn}}\}$ is the set of **cbn-interpretations** and $\langle \mathrm{Int}_{\Sigma,\mathrm{cbn}}, \sqsubseteq\rangle$ is the canonical cpo of cbv-interpretations with least element $\perp_{\mathrm{cbn}} := \perp_{\mathrm{Int}_{\Sigma,\mathrm{cbn}}}$. $\square$

Definition 5.9
The **cbn-transformation** of $P \in \mathrm{Prog}_{\Sigma}$, $\Phi_{P,\mathrm{cbn}} : [\mathrm{Int}_{\Sigma,\mathrm{cbn}} \to \mathrm{Int}_{\Sigma,\mathrm{cbn}}]$, is defined by

$$f^{\Phi_{P,\mathrm{cbn}}(\mathfrak{A})}(\vec{\underline{t}}) := \begin{cases} [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A},\beta} & , \text{ if } f(\vec{p}) \to r \in P \text{ and } \beta : \mathrm{Var}(\vec{p}) \to \mathrm{T}^{\infty}_{\mathcal{C},\perp} \text{ exist} \\ & \quad \text{with } [\![p_1]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\mathrm{cbn}},\beta} = \underline{t}_1, \ldots, [\![p_n]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\mathrm{cbn}},\beta} = \underline{t}_n; \\ \perp & , \text{ otherwise;} \end{cases}$$

for all $f^{(n)} \in \mathcal{F}$, $\vec{\underline{t}} \in (\mathrm{T}^{\infty}_{\mathcal{C},\perp})^n$ and $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\mathrm{cbn}}$. $\square$

The valuation β of the cbn-Transformation ranges over the full computation domain $\mathrm{T}^{\infty}_{\mathcal{C},\perp}$, so that function operations can be non-strict.

Definition 5.10
The **cbn-fixed-point data type** $\mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbn}}(P) \in \mathrm{Int}_{\Sigma,\mathrm{cbn}}$ is defined by

$$\mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbn}}(P) := \mathrm{Fix}(\Phi_{P,\mathrm{cbn}}) = \bigsqcup_{i \in \mathbb{N}} (\Phi_{P,\mathrm{cbn}})^i(\perp_{\mathrm{cbn}})$$

and the **cbn-fixed-point (term) semantics** $[\![\cdot]\!]^{\mathrm{fix}}_{P,\mathrm{cbn}} : \mathrm{T}_{\Sigma} \to \mathrm{T}^{\infty}_{\mathcal{C},\perp}$ by

$$[\![t]\!]^{\mathrm{fix}}_{P,\mathrm{cbn}} := [\![t]\!]^{\mathrm{alg}}_{\mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbn}}(P)}$$

$\square$

Example 5.4 Determining a cbn-fixed-point data type
Taking anew the program of Example 4.1, its cbn-fixed-point data type is determined by means of the following table. Let $\mathfrak{A}_i := (\Phi_{P,\mathrm{cbn}})^i(\perp_{\mathrm{cbn}})$ for all $i \in \mathbb{N}$ and $\mathfrak{A}_\infty = \mathrm{DT}^{\mathrm{fix}}_{\mathrm{cbn}}(P)$.

	$i=0$	$i=1$	$i=2$	$\ldots$	$i=\infty$
$\texttt{list1}^{\mathfrak{A}_i}()$	$\perp$	`[]`$:\perp$	`[]`$:$`[]`$:\perp$	$\ldots$	`[[],[],`$\ldots$`]`
$\texttt{list2}^{\mathfrak{A}_i}()$	$\perp$	`[[]]`$:\perp$	`[[]]`$:$`[[]]`$:\perp$	$\ldots$	`[[[]],[[]],`$\ldots$`]`
$\texttt{head}^{\mathfrak{A}_i}$	$\underline{t} \mapsto \perp$	$\begin{pmatrix} \perp \mapsto \perp \\ \texttt{[]} \mapsto \perp \\ \underline{t}_1 : \underline{t}_2 \mapsto \underline{t}_1 \end{pmatrix}$	as for $i=1$	$\ldots$	as for $i=1$

with $\underline{t}, \underline{t}_1, \underline{t}_2 \in \mathrm{T}^{\infty}_{\mathcal{C},\bot}$.

The equations $[\![\texttt{head(list1)}]\!]^{\text{fix}}_{P,\text{cbn}} = \texttt{[]}$ and $[\![\texttt{head(list2)}]\!]^{\text{fix}}_{P,\text{cbn}} = \texttt{[[]]}$ do not violate compositionality, because $[\![\texttt{list1}]\!]^{\text{fix}}_{\text{P},\text{cbn}}$ and $[\![\texttt{list2}]\!]^{\text{fix}}_{\text{P},\text{cbn}}$ differ. □

5.4 Cbn-Reduction Semantics

In general the leftmost-outermost (lo-) reduction strategy is associated with cbn-semantics, but due to the patterns and the non-flat base data type this strategy is not complete for our cbn-semantics:

Example 5.5 Incompleteness of lo-reduction
Considering the program

$$\begin{array}{lcl} \texttt{and(x, False)} & \to & \texttt{False} \\ \texttt{a} & \to & \texttt{False} \\ \texttt{undef} & \to & \texttt{undef} \end{array}$$

we have $[\![\texttt{and(undef,a)}]\!]^{\text{fix}}_{P,\text{cbn}} = \texttt{and}^{\mathrm{DT}^{\text{fix}}_{\text{cbn}}}(\bot, \texttt{False}) = \texttt{False}$, but

$$\texttt{and(}\underline{\texttt{undef}}\texttt{,a)} \xrightarrow[P,\text{lo}]{} \texttt{and(}\underline{\texttt{undef}}\texttt{,a)} \xrightarrow[P,\text{lo}]{} \ldots.$$

□

Therefore we use the parallel-outermost (po-) reduction strategy.

Definition 5.11
A reduction $A = t \overset{U}{\underset{P}{\twoheadrightarrow}} t'$ is a **po-reduction**, written $A = t \xrightarrow[P,\text{po}]{} t'$, iff U is the set of all outermost redex positions of t. This also defines the po-reduction strategy $\underset{P,\text{po}}{\twoheadrightarrow}$. □

Example 5.6
Using the program of Example 5.5 above, we obtain

$$\texttt{and(}\underline{\texttt{undef}}\texttt{,}\underline{\texttt{a}}\texttt{)} \underset{P,\text{po}}{\twoheadrightarrow} \underline{\texttt{and}}\texttt{(undef,False)} \underset{P,\text{po}}{\twoheadrightarrow} \texttt{False}$$

as desired. □

Simply using the po-normal form to define the cbn-reduction semantics — analogously to the cbv-semantics — is not sufficient, due to the computation domain $\mathrm{T}^{\infty}_{\mathcal{C},\bot}$. Obviously an infinite constructor term can never be the result of a computation, but it can be approximated to arbitrary precision.

Definition 5.12
The algebraic term semantics with respect to the algebra $\bot_{\text{cbn}}$,

$$[\![\cdot]\!]^{\text{alg}}_{\bot_{\text{cbn}}} : \mathrm{T}_{\Sigma} \to \mathrm{T}^{\infty}_{\mathcal{C},\bot},$$

is called **semantic cbn-approximation**. □

Definition 5.13
The **po-reduction** or **cbn-reduction semantics** $[\![\cdot]\!]^{\text{po}}_{P,\text{cbn}} : \mathrm{T}_\Sigma \to \mathrm{T}^\infty_{\mathcal{C},\perp}$ is defined by

$$[\![t]\!]^{\text{po}}_{P,\text{cbn}} := \bigsqcup\{[\![t']\!]^{\text{alg}}_{\perp_\text{cbn}} \mid t \xrightarrow[P,\text{po}]{*} t'\}.$$

□

Example 5.7 Approximating a cbn-reduction semantics
Using the program of Example 5.5, we have the following approximation

$$\begin{tikzcd}
\texttt{list1} \arrow[r, two heads, "P{,}\text{po}"'] \arrow[d, "{[\![\cdot]\!]^{\text{alg}}_{\perp_\text{cbn}}}"] & \texttt{[]:list1} \arrow[r, two heads, "P{,}\text{po}"'] \arrow[d, "{[\![\cdot]\!]^{\text{alg}}_{\perp_\text{cbn}}}"] & \texttt{[]:[]:list1} \arrow[r, two heads, "P{,}\text{po}"'] \arrow[d, "{[\![\cdot]\!]^{\text{alg}}_{\perp_\text{cbn}}}"] & \texttt{[]:[]:[]:list1} \arrow[r, two heads, "P{,}\text{po}"'] \arrow[d, "{[\![\cdot]\!]^{\text{alg}}_{\perp_\text{cbn}}}"] & \cdots \\
\perp & \texttt{[]:}\perp & \texttt{[]:[]:}\perp & \texttt{[]:[]:[]:}\perp & \cdots
\end{tikzcd}$$

□

The reader should notice that the semantic cbn-approximation is computable, because it is characterized by

$$\begin{aligned}
[\![G(t_1,\ldots,t_n)]\!]^{\text{alg}}_{\perp_\text{cbn}} &= G([\![t_1]\!]^{\text{alg}}_{\perp_\text{cbn}},\ldots,[\![t_n]\!]^{\text{alg}}_{\perp_\text{cbn}}) \\
[\![f(t_1,\ldots,t_n)]\!]^{\text{alg}}_{\perp_\text{cbn}} &= \perp
\end{aligned}$$

for all $G^{(n)} \in \mathcal{C}$ and $f^{(n)} \in \mathcal{F}$.

Also, if for a $t \in \mathrm{T}_\Sigma$ we have $[\![t]\!]^{\text{po}}_{P,\text{cbn}} = \underline{t} \in \mathrm{T}_\mathcal{C}$, then the po-normal form $t{\downarrow_{P,\text{po}}}$ exists and $\underline{t} = t{\downarrow_{P,\text{po}}}$, as we prove in Section 8.

6 Definition of the ς-Semantics

Looking at the definitions of the cbv- and the cbn-fixed-point semantics the fact stands out that both definitions consist mainly of two rather independent parts: the base data type and the transformation. The idea to exchange these between cbv- and cbn-semantics immediately suggests itself. The cbn-transformation can be applied to cbv-interpretations with only little adaptions, likewise the cbv-transformation can be applied to cbn-interpretations.

The two resulting mixed semantics are not even that unusual. The first one is more closely related to the cbn-semantics of recursive applicative program schemes than our cbn-semantics. In the theory of recursive applicative program schemes 'cbn' refers just to the evaluation of function operations. The major part of the literature ([Vui74a], [Man74], [Niv75], [DS76] and [LS87]) considers only flatly ordered base data types (interpretations), because the flat order permits simpler proofs.

The second mixture has even been implemented in versions of the functional programming language Hope ([FH88]), thereby combining the expressiveness of infinite data structures with the efficiency of cbv-evaluation.

However, defining these additional semantics would have the unpleasant consequence that we would have to prove all properties of semantics four times, e.g. the equivalence of fixed-point and reduction semantics. Besides, finding reduction semantics for the two new semantics would not be easy.

The solution is a further generalization, blurring the dividing-lines. We introduce a new parameter ς which states for every argument position of every operation symbol, if the operation shall be strict at that argument position.

Definition 6.1
A mapping $\varsigma : \bigcup_{n\in\mathbb{N}} \Sigma_n \rightarrow \mathbb{B}^n$ is a **forced strictness** for the signature Σ. For $g \in \Sigma$ the boolean vector $\varsigma(g)$ is called **forced strictness** of g. The symbol $g^{(n)} \in \Sigma$ is **forcedly strict** in $\underline{\vec{t}} \in (\mathrm{T}^{\infty}_{\mathcal{C},\perp})^n$ iff there exists an $i \in [n]$ with $\varsigma(g)_i = \mathbf{tt}$ and $\underline{t}_i = \perp$. $\Box$

Admittedly the forced strictness ς gives us an arbitrary number of semantics, depending on the signature, but they can be handled simultaneously in a simple, uniform way.

In all the following sections ς is an arbitrary forced strictness.

6.1 ς-Fixed-Point Semantics

The definition is completely analogous to that of the cbv- and cbn-fixed-point semantics. The exact relationship is discussed in Subsection 6.5.

Definition 6.2
The ordered algebra $\mathrm{BDT}_{\mathcal{C},\varsigma} := \mathrm{T}_{\mathcal{C},\varsigma} := \langle \mathrm{T}_{\mathcal{C},\varsigma}, \trianglelefteq, \tau\rangle$ with

- $\mathrm{T}_{\mathcal{C},\varsigma}$ being the least subset of $\mathrm{T}^{\infty}_{\mathcal{C},\perp}$ satisfying
  - $G^{(n)} \in \mathcal{C}$, $\underline{\vec{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$, G not forcedly strict for $\underline{\vec{t}} \implies G(\underline{\vec{t}}) \in \mathrm{T}_{\mathcal{C},\varsigma}$ and
  - $T \subseteq \mathrm{T}_{\mathcal{C},\varsigma}$ is a chain $\implies \bigsqcup T \in \mathrm{T}_{\mathcal{C},\varsigma}$, and
- τ defined by
$$\tau(G)(\underline{\vec{t}}) = \begin{cases} G(\underline{\vec{t}}) & \text{, if } G \text{ is not forcedly strict for } \underline{\vec{t}}; \\ \perp & \text{, otherwise;} \end{cases}$$
  for all $G^{(n)} \in \mathcal{C}$, $\underline{\vec{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$

is the **ς-base data type** over the constructor symbols $\mathcal{C}$. $\Box$

Using simply $\mathrm{T}^{\infty}_{\mathcal{C},\perp}$ instead of $\mathrm{T}_{\mathcal{C},\varsigma}$ is an alternative but would disagree with our aim that the cbv-semantics shall be an instance of the ς-semantics. Besides, the elements of $\mathrm{T}^{\infty}_{\mathcal{C},\perp} \setminus \mathrm{T}_{\mathcal{C},\varsigma}$ could never be denoted by syntactic terms.

Definition 6.3
The set $\mathrm{Int}_{\Sigma,\varsigma} := \{\mathfrak{A} \in \mathrm{Alg}^{\infty}_{\Sigma,\perp} \mid \mathfrak{A}|_{\mathcal{C},\emptyset} = \mathrm{BDT}_{\mathcal{C},\varsigma}\}$ is the set of **ς-interpretations** and $\langle \mathrm{Int}_{\Sigma,\varsigma}, \sqsubseteq \rangle$ is the canonical cpo of ς-interpretations with least element $\perp_\varsigma := \perp_{\mathrm{Int}_{\Sigma,\varsigma}}$. $\Box$

Since pattern matching is more complicated in the context of forced strictness, we define it separately.

Definition 6.4
A term tuple $\underline{\vec{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$ is **semantically ς-matchable** with a redex scheme $f^{(n)}(\vec{p}) \in \mathrm{RedS}_P$ by a variable mapping $\beta : \mathrm{Var}(\vec{p}) \to \mathrm{T}_{\mathcal{C},\varsigma}$ iff

(i) f is not forcedly strict for $\underline{\vec{t}}$,

(ii) $p_1[\bot/\mathrm{Var}(p_1)] \trianglelefteq \underline{t}_1, \ldots, p_n[\bot/\mathrm{Var}(p_n)] \trianglelefteq \underline{t}_n$, and

(iii) $[\![p_1]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta} = \underline{t}_1, \ldots, [\![p_n]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta} = \underline{t}_n$.

□

The necessity of the new **order condition** (ii) is explained in the next subsection.

Definition 6.5
The **ς-transformation** of $P \in \mathrm{Prog}_\Sigma$, $\Phi_{P,\varsigma} : [\mathrm{Int}_{\Sigma,\varsigma} \to \mathrm{Int}_{\Sigma,\varsigma}]$, is defined by

$$f^{\Phi_{P,\varsigma}(\mathfrak{A})}(\underline{\vec{t}}) := \begin{cases} [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A},\beta} & , \text{ if } \underline{\vec{t}} \text{ is semantically } \varsigma\text{-matchable with} \\ & \quad \text{the left-hand side of a program rule } f(\vec{p}) \to r \in P \\ & \quad \text{by a valuation } \beta : \mathrm{Var}(\vec{p}) \to \mathrm{T}_{\mathcal{C},\varsigma}; \\ \bot & , \text{ otherwise;} \end{cases}$$

for all $f^{(n)} \in \mathcal{F}$, $\underline{\vec{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$ and $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\varsigma}$. □

Definition 6.6
The **ς-fixed-point data type** $\mathrm{DT}^{\mathrm{fix}}_\varsigma(P) \in \mathrm{Int}_{\Sigma,\varsigma}$ is defined by

$$\mathrm{DT}^{\mathrm{fix}}_\varsigma(P) := \mathrm{Fix}(\Phi_{P,\varsigma}) = \bigsqcup_{i \in \mathbb{N}} (\Phi_{P,\varsigma})^i(\bot_\varsigma)$$

and the **ς-fixed-point (term) semantics** $[\![\cdot]\!]^{\mathrm{fix}}_{P,\varsigma} : \mathrm{T}_\Sigma \to \mathrm{T}_{\mathcal{C},\varsigma}$ by

$$[\![t]\!]^{\mathrm{fix}}_{P,\varsigma} := [\![t]\!]^{\mathrm{alg}}_{\mathrm{DT}^{\mathrm{fix}}_\varsigma(P)}$$

□

6.2 Well-Definedness of the ς-Fixed-Point Semantics

When proving the well-definedness of the ς-fixed-point semantics we also justify some details of the given definition which are not straightforward or where alternatives are feasible.

Lemma 6.1 Continuity of forced strictness
Let $g^{(n)} \in \Sigma$, $T = (\vec{\underline{t}}_j)_{j \in \mathbb{N}}$ a chain with $\underline{t}_{i,j} \in \mathrm{T}_{\mathcal{C},\varsigma}$ and $\vec{\underline{t}} = \bigsqcup T$. The symbol g is forcedly strict for $\vec{\underline{t}}$ iff g is forcedly strict for all $\vec{\underline{t}}_j \in T$.
Proof idea:
Case analysis on g being forcedly strict and g not being forcedly strict for $\vec{\underline{t}}$. □

Corollary 6.2
The constructor operations of the ς-base data type are continuous, i.e. $\mathrm{BDT}_{\mathcal{C},\varsigma} \in \mathrm{Alg}^{\infty}_{\Sigma,\perp}$. □

Lemma 6.3
The canonically ordered set of ς-interpretations $\langle \mathrm{Int}_{\Sigma,\varsigma}, \sqsubseteq \rangle$ is a cpo.
Proof idea:
Every ς-interpretation is a cpo and all have the same carrier set and order. □

Now we prove that — similar to operational confluence — the function operations resulting from an application of the ς-transformation are true mappings, i.e. exactly one result term is respectively associated with every argument term tuple. Here the order condition of semantic ς-matching proves to be necessary. The following example illustrates this.

Example 6.1
We regard the program

$$\begin{array}{lcl} \mathtt{f(G(x))} & \rightarrow & \mathtt{A} \\ \mathtt{f(H(x))} & \rightarrow & \mathtt{B} \end{array}$$

with the forced strictness $\varsigma(\mathtt{f}) := (\mathsf{ff})$ and $\varsigma(\mathtt{G}) := \varsigma(\mathtt{H}) := (\mathsf{tt})$. With $\beta(\mathtt{x}) := \perp$ we have

$$[\![\mathtt{G(x)}]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta} = \mathtt{G}^{\mathrm{BDT}_{\mathcal{C},\varsigma}}(\perp) = \perp \quad \text{and} \quad [\![\mathtt{H(x)}]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta} = \mathtt{H}^{\mathrm{BDT}_{\mathcal{C},\varsigma}}(\perp) = \perp.$$

Since $\mathtt{f}$ is not forcedly strict for the argument term $\perp$, we would get

$$\mathtt{A} = [\![\mathtt{A}]\!]^{\mathrm{alg}}_{\perp_\varsigma,\beta} = \mathtt{f}^{\Phi_{P,\varsigma}(\perp_\varsigma)}(\perp) = [\![\mathtt{B}]\!]^{\mathrm{alg}}_{\perp_\varsigma,\beta} = \mathtt{B}$$

without the order condition. □

Lemma 6.4 Semantic unification implies syntactic unification
Let $p, p' \in \mathrm{T}_{\mathcal{C}}(X)$ be linear patterns with $\mathrm{Var}(p) \cap \mathrm{Var}(p') = \emptyset$, $\beta : \mathrm{Var}(p) \rightarrow \mathrm{T}_{\mathcal{C},\varsigma}$ and $\beta' : \mathrm{Var}(p') \rightarrow \mathrm{T}_{\mathcal{C},\varsigma}$ valuations, and $\underline{t} \in \mathrm{T}_{\mathcal{C},\varsigma}$. If

$$p[\perp/\mathrm{Var}(p)] \trianglelefteq \underline{t}, \quad p'[\perp/\mathrm{Var}(p')] \trianglelefteq \underline{t} \quad \text{and} \quad [\![p]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta} = \underline{t} = [\![p']\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta}$$

then exist two substitutions

$$\sigma : \mathrm{Var}(p) \rightarrow \mathrm{T}_{\mathcal{C}}(\mathrm{Var}(p) \,\dot{\cup}\, \mathrm{Var}(p')) \quad \text{and} \quad \sigma' : \mathrm{Var}(p') \rightarrow \mathrm{T}_{\mathcal{C}}(\mathrm{Var}(p) \,\dot{\cup}\, \mathrm{Var}(p')),$$

and a valuation

$$\hat{\beta} : (\mathrm{Var}(p) \mathbin{\dot{\cup}} \mathrm{Var}(p'))\to \mathrm{T}_{\mathcal{C},\varsigma}$$

so that

$$\begin{array}{rcl} \forall x \in \mathrm{Var}(p).\ \beta(x) & = & [\![x\sigma]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\hat{\beta}} \\ \forall x \in \mathrm{Var}(p').\ \beta'(x) & = & [\![x\sigma']\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\hat{\beta}} \\ p\sigma & = & p'\sigma'. \end{array}$$

Proof idea:
Parallel structural induction on p and p'. □

Corollary 6.5
The function operations resulting from a ς-transformation are well-defined, i.e. if $\vec{\underline{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$ with $n \in \mathbb{N}$ and $\vec{\underline{t}}$ is semantically ς-matchable with the left-hand side of a reduction rule $f^{(n)}(\vec{p})\to r \in P$ by $\beta : \mathrm{Var}(\vec{p})\to \mathrm{T}_{\mathcal{C},\varsigma}$ and with the left-hand side of a reduction rule $f^{(n)}(\vec{p}')\to r' \in P$ by $\beta' : \mathrm{Var}(\vec{p}')\to \mathrm{T}_{\mathcal{C},\varsigma}$, then $[\![r]\!]^{\mathrm{alg}}_{\mathfrak{A},\beta} = [\![r']\!]^{\mathrm{alg}}_{\mathfrak{A},\beta'}$ for all $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\varsigma}$. □

Finally, we have to prove that the algebras resulting from a ς-transformation are continuous, that is they are ς-interpretations, and that the ς-transformation is continuous. First we show the continuity of semantic ς-matching.

Lemma 6.6 Characterization of semantic ς-matching
The term tuple $\vec{\underline{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$ is semantically ς-matchable with $f(\vec{p}) \in \mathrm{RedS}_P$ by $\beta : \mathrm{Var}(\vec{p})\to \mathrm{T}_{\mathcal{C},\varsigma}$ iff conditions (i) and (ii) of semantic ς-matchability (def. 6.4) are fulfilled and $\beta(x) = \vec{\underline{t}}/u$ with u given by $\{u\} = \mathrm{Pos}(x, \vec{p})$ for all $x \in \mathrm{Var}(\vec{p})$.
Proof idea:
Structural induction on the pattern. □

The reader should notice that u is only well-defined, because patterns are linear.

Lemma 6.7 Continuity of semantic ς-matching
Let $f^{(n)}(\vec{p}) \in \mathrm{RedS}_P$, $T = (\vec{\underline{t}}_j)_{j\in\mathbb{N}}$ a chain with $\underline{t}_{i,j} \in \mathrm{T}_{\mathcal{C},\varsigma}$ and $\vec{\underline{s}} := \bigsqcup T$.

- The term tuple $\vec{\underline{s}}$ is semantically ς-matchable with $f(\vec{p})$ iff a $\underline{t}_k \in T$ exists which is semantically ς-matchable with $f(\vec{p})$.
- If all $\vec{\underline{t}}_j \in T$ are semantically ς-matchable with $f(\vec{p})$ by respective $\beta_j : \mathrm{Var}(\vec{p})\to \mathrm{T}_{\mathcal{C},\varsigma}$, then $\vec{\underline{s}}$ is semantically ς-matchable with $f(\vec{p})$ by $\beta = \bigsqcup_{j\in\mathbb{N}} \beta_j$.

Proof idea:
Straightforward, using the previous lemma. □

The following two rather general lemmas are the basis for the last two lemmas of this subsection.

Lemma 6.8 **Continuity of the algebraic term semantics w.r.t. the valuation**

Let Σ be a signature, $\langle A,\leq\rangle$ a cpo, $\mathfrak{A}\in \mathrm{Alg}^{\infty}_{\Sigma,\perp}(\langle A,\leq\rangle)$ and $t\in \mathrm{T}_{\Sigma}(X)$. Then $[\![t]\!]^{\mathrm{alg}}_{\mathfrak{A},\cdot} : (X\to A)\to A$ is continuous w.r.t valuations of the canonical cpo $\langle(X\to A),\leq\rangle$.

Proof idea:
Structural induction on t. □

Lemma 6.9 **Continuity of the algebraic term semantics w.r.t the algebra**

Let Σ be a signature and $\langle A,\leq\rangle$ a cpo. Let $I\subseteq \mathrm{Alg}^{\infty}_{\Sigma,\perp}(\langle A,\leq\rangle)$ such that the canonical partial order $\langle I,\sqsubseteq\rangle$ is continuous. Let $\beta : X\to A$ and $t\in \mathrm{T}_{\Sigma}(X)$. Then $[\![t]\!]^{\mathrm{alg}}_{\cdot,\beta} : I\to A$ is continuous w.r.t. algebras of I.

Proof idea:
Structural induction on t. □

Lemma 6.10 **Continuity of the results of a ς-transformation**

$\Phi_{P,\varsigma}(\mathfrak{A})\in \mathrm{Int}_{\Sigma,\varsigma}$ for all $\mathfrak{A}\in \mathrm{Int}_{\Sigma,\varsigma}$.

Proof idea:
Show continuity of the function operations of $\Phi_{P,\varsigma}(\mathfrak{A})$ by using Lemma 6.8. □

Lemma 6.11 **Continuity of the ς-transformation**

$$\Phi_{P,\varsigma} : [\mathrm{Int}_{\Sigma,\varsigma}\to \mathrm{Int}_{\Sigma,\varsigma}].$$

Proof idea:
Show that $f^{\bigsqcup \Phi_{P,\varsigma}(T)}(\underline{\vec{t}})$ exists and $f^{\bigsqcup \Phi_{P,\varsigma}(T)}(\underline{\vec{t}}) = f^{\Phi_{P,\varsigma}(\bigsqcup T)}(\underline{\vec{t}})$ for all $f^{(n)}\in\mathcal{F}$, $\underline{\vec{t}}\in(\mathrm{T}_{\mathcal{C},\varsigma})^n$ and chains $T=(\mathfrak{A}_j)_{j\in\mathbb{N}}\subseteq \mathrm{Int}_{\Sigma,\varsigma}$ using Lemma 6.9. □

Looking at the definition of the ς-transformation the question arises, why we set the result of a function operation to $\perp$, when the argument terms are not semantically ς-matchable with any left-hand side. Simply leaving this value unchanged is an obvious alternative. We define $\Phi^{*}_{P,\varsigma}$ by

$$f^{\Phi^{*}_{P,\varsigma}(\mathfrak{A})}(\underline{\vec{t}}) := \begin{cases} [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A},\beta} & , \text{ if } \underline{\vec{t}} \text{ is semantically } \varsigma\text{-matchable with} \\ & \text{ the left-hand side of a program rule } f(\vec{p})\to r\in P \\ & \text{ by a valuation } \beta : \mathrm{Var}(\vec{p})\to \mathrm{T}_{\mathcal{C},\varsigma}; \\ f^{\mathfrak{A}}(\underline{\vec{t}}) & , \text{ otherwise.} \end{cases}$$

It is easy to prove that $\bigsqcup \Phi^{*}_{P,\varsigma}(\perp_{\varsigma})$ exists and even equals the least fixed-point of $\Phi_{P,\varsigma}$. However, $\Phi^{*}_{P,\varsigma}$ does not map ς-interpretations to ς-interpretations; the resulting algebra may be more general. To remedy this, we can define $\mathrm{Int}^{*}_{\Sigma,\varsigma}$ just like $\mathrm{Int}_{\Sigma,\varsigma}$, with the exception that operations do not need to be continuous. Using $\mathrm{Int}^*_{\Sigma,\varsigma}$ as domain and range of $\Phi^*_{P,\varsigma}$ solves the problem, but then $\Phi^*_{P,\varsigma}$ is no longer continuous and we cannot apply the fixed-point theorem of Tarski. Actually $\bigsqcup \Phi^*_{P,\varsigma}(\bot_\varsigma) = \mathrm{Fix}(\Phi_{P,\varsigma})$ exists as mentioned and can even be proved to be the least fixed-point of $\Phi^*_{P,\varsigma}$, but simple proofs of this repeatedly refer to $\Phi_{P,\varsigma}$, so that we better persevere with it.

6.3 ς-Reduction Semantics

For the cbv- and the cbn-semantics we were able to fall back on the well-known innermost and outermost reduction strategies. However, ς-semantics mixes the two argument evaluation mechanisms and hence there is no obvious suitable reduction strategy. Especially the variable strictness of the constructor operations are irritating, because there are no reduction rules for constructor symbols.

We approach the problem by examining which kinds of reductions are sound w.r.t. the ς-fixed-point semantics.

Definition 6.7
A **reduction** $t \xrightarrow[l\longrightarrow r]{u} t'$ **is sound** w.r.t. a term semantics $[\![\cdot]\!] : \mathrm{T}_\Sigma \to A$ iff $[\![t]\!] = [\![t']\!]$. □

Example 6.2
Using the program

```
positive(Succ(x))  →  Succ(Zero)
inf                →  Succ(inf)
```

the outermost reduction $\underline{\texttt{positive}}\texttt{(Succ(inf))} \longrightarrow \texttt{Succ(Zero)}$ is *not* sound w.r.t. the cbv-fixed-point semantics: $[\![\texttt{positive(Succ(inf))}]\!]^{\mathrm{fix}}_{P,\mathrm{cbv}} = \bot \neq \texttt{Succ(Zero)} = [\![\texttt{Succ(Zero)}]\!]^{\mathrm{fix}}_{P,\mathrm{cbv}}$. □

The example illustrates what can easily be gained from the definition of the ς-transformation: Only forced strictness may cause a reduction to be unsound w.r.t. a ς-fixed-point semantics.

Obviously, reducing a redex $f(\vec{\underline{t}})$ is sound if $[\![\underline{t}_i]\!]^{\mathrm{fix}}_{P,\sigma} \neq \bot$ for all forcedly strict argument positions i. Naturally this condition is sufficient but not necessary; the instantiated right-hand side of a program rule may have the value $\bot$ as well and for general reductions the context of the redex is moreover relevant. However, considering only redexes proves to be sufficient for our purposes.

Since the arguments $\underline{t}_i$ generally comprise function symbols and its ς-fixed-point semantics is unknown when reduction takes place, $[\![\underline{t}_i]\!]^{\mathrm{fix}}_{P,\sigma}$ can only be approximated.

Definition 6.8
The algebraic term semantics with respect to the algebra $\bot_\varsigma$, $[\![\cdot]\!]^{\mathrm{alg}}_{\bot_\varsigma} : \mathrm{T}_\Sigma \to \mathrm{T}_{\mathcal{C},\varsigma}$, is called **semantic ς-approximation**. □

Since the semantic ς-approximation has a purely syntactic definition (cf. the cbn-approximation in Subsection 5.4), it can be employed in an operational semantics.

Due to the monotonicity of the algebraic term semantics w.r.t. the algebra in accordance with Lemma 6.9, $\underline{t} \trianglelefteq [\![t]\!]^{\mathrm{alg}}_{\perp_\varsigma}$ implies $t \trianglelefteq [\![t]\!]^{\mathrm{fix}}_{P,\varsigma}$. Hence we can define a meaningful notion of syntactic ς-matching in analogy to semantic ς-matching (def. 6.4). Using that, we define the set of ς-redexes, whose reduction is sound w.r.t. the ς-fixed-point semantics. This soundness is proved in Subsection 7.2.

Definition 6.9
A term tuple $\vec{t} \in (\mathrm{T}_\Sigma)^n$ is **syntactically ς-matchable** with a redex scheme $f^{(n)}(\vec{p}) \in \mathrm{RedS}_P$ by a substitution $\sigma : \mathrm{Var}(\vec{p}) \to \mathrm{T}_{\mathcal{C},\varsigma}$ iff

(i) f is not forcedly strict for $([\![t_1]\!]^{\mathrm{alg}}_{\perp_\varsigma} \ldots [\![t_n]\!]^{\mathrm{alg}}_{\perp_\varsigma})$,

(ii) $p_1[\perp/\mathrm{Var}(p_1)] \trianglelefteq [\![t_1]\!]^{\mathrm{alg}}_{\perp_\varsigma}, \ldots, p_n[\perp/\mathrm{Var}(p_n)] \trianglelefteq [\![t_n]\!]^{\mathrm{alg}}_{\perp_\varsigma}$, and

(iii) $f(\underline{\vec{t}}) = f(\vec{p})\sigma$.

□

Definition 6.10 (Cf. V- and N-redexes in [Cou90])
A term $f(\vec{t}) \in \mathrm{T}_\Sigma$ is a **ς-redex** of a redex scheme $f(\vec{p}) \in \mathrm{RedS}_P$ iff $\vec{t}$ is syntactically ς-matchable with $f(\vec{p})$ by some substitution σ. The set of all ς-redexes of a program P is denoted by $\mathrm{Red}_{P,\varsigma}$. □

The set of ς-redexes defines an instance of a program viewed as TRS. This instance is denoted by ς as well. Thus we obtain notions like ς-redex position, ς-reduction and ς-reduction relation.

These we employ for a very simple definition of ς-reduction semantics. We define our ς-reduction semantics like the po-reduction semantics as least upper bound of results of finite reduction sequences. Only, instead of po-reduction sequences all ς-reduction sequences are considered.

Definition 6.11
The **global ς-reduction semantics**, $[\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma} : \mathrm{T}_\Sigma \to \mathrm{T}_{\mathcal{C},\varsigma}$, is defined by

$$[\![t]\!]^{\mathrm{red}}_{P,\varsigma} := \bigsqcup \{ [\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t' \}.$$

□

The global ς-reduction semantics will be of central importance in the equivalence proofs of the next section. However, it is not suitable for practical purposes, since efficient implementations require deterministic reduction strategies. Hence we define a generalization of the po-reduction semantics.

Definition 6.12
The **po-ς-reduction semantics**, $[\![\cdot]\!]^{\text{po}}_{P,\varsigma} : \mathrm{T}_\Sigma \to \mathrm{T}_{\mathcal{C},\varsigma}$, is defined by

$$[\![t]\!]^{\text{po}}_{P,\varsigma} := \bigsqcup \{ [\![t']\!]^{\text{alg}}_{\perp_\varsigma} \mid t \xrightarrow[P,\varsigma,\text{po}]{*} t' \}.$$

□

6.4 Connection with the Previously Defined Cbv- and Cbn-Semantics

By a simple trick of notation we can make fully explicit that the cbv- and cbn-semantics defined in the previous section are special instances of the ς-semantics.

Definition 6.13
The forced strictnesses $\text{cbv}, \text{cbn} \in \bigcup_{n \in \mathbb{N}} \Sigma_n \to \mathbb{B}^n$ are defined by

$$\text{cbv}(g)_i := \mathsf{tt} \quad \text{and} \quad \text{cbn}(g)_i := \mathsf{ff}$$

for all $g^{(n)} \in \Sigma$ and $i \in [n]$. □

For the fixed-point semantics it is quite clear that the definitions of Section 5 are special instances of those given for the ς-semantics, e.g. the definition of the ς-base data type gives $\text{BDT}_{\mathcal{C},\text{cbv}} = \mathcal{T}^{\perp}_{\mathcal{C}}$ and $\text{BDT}_{\mathcal{C},\text{cbn}} = \mathcal{T}^{\infty}_{\mathcal{C},\perp}$. Just semantic cbv- and cbn-matching was not defined in Section 5. Nonetheless it is simple to prove that in the special cases of $\varsigma = \text{cbv}$ and $\varsigma = \text{cbn}$ the ς-matching conditions (i) and (iii) imply the order condition (ii), and that therefore semantic cbv-/cbn-matching is equivalent to the conditions given in the definition of the cbv-/cbn-transformation in Section 5.

The case of the reduction semantics is slightly more complicated. Obviously all redexes are cbn-redexes ($\text{Red}_{P,\text{cbn}} = \text{Red}_P$) and therefore the po-reduction semantics and the po-cbn-reduction semantics are identical.

Nonetheless, there is no similar correspondence for the cbv-semantics. This is not because of the renunciation of using semantic approximations for the li*-reduction semantics. For all ς such that the ς-base data type is flatly ordered, a definition of the ς-reduction semantics using normal forms w.r.t the employed reduction relation is equivalent to a definition using semantic approximations. Moreover, cbv-redexes, outermost cbv-redexes, and innermost* redexes are the same. However, the li*-reduction semantics reduces only the leftmost of the generally several redexes reduced in parallel by the po-cbv-reduction semantics. Consequently, we call the li*-reduction also lo-cbv-reduction. A lo-ς-reduction semantics is not defined, because in general it is not complete as was shown for $\varsigma = \text{cbn}$ in Example 5.5. However, for $\varsigma = \text{cbv}$ it is, and we prove this in Subsection 7.7.

From the perspective of the ς-semantics we see that the difference between the reduction semantics of the cbv- and of the cbn-semantics is only seemingly based on the positions of redexes. Instead, the fundamental difference is the kind of redexes which are reduced.

6.5 Well-Definedness of the ς-Reduction Semantics

We start with a property needed in the proof of the subsequent lemma.

Lemma 6.12 **Commutativity of semantic approximation and replacement of subterms**

$$[\![t[u \leftarrow t']]\!]^{\text{alg}}_{\perp_\varsigma} = [\![t]\!]^{\text{alg}}_{\perp_\varsigma}[u \leftarrow [\![t']\!]^{\text{alg}}_{\perp_\varsigma}]$$

for all $t, t' \in \mathrm{T}_\Sigma$ and $u \in \mathrm{Pos}(t)$.

Proof idea:
Structural induction on u. □

Besides proving the well-definedness of the ς-reduction semantics, the next two lemmas are of major importance for the ς-semantics.

Already in Subsection 5.4 we employed repeated reduction to approximate the semantic value of a term. The next lemma proves that reduction can only lead to a gain of information (w.r.t the cpo $\langle \mathrm{T}_{\mathcal{C},\varsigma}, \trianglelefteq\rangle$) and never to a loss of it.

Lemma 6.13 **Gain of information by reduction**

$$\begin{array}{lcl} t \xrightarrow[P]{} t' & \Longrightarrow & [\![t]\!]^{\text{alg}}_{\perp_\varsigma} \trianglelefteq [\![t']\!]^{\text{alg}}_{\perp_\varsigma} \\ t \xrightarrow[P,\text{no}]{} t' & \Longrightarrow & [\![t]\!]^{\text{alg}}_{\perp_\varsigma} = [\![t']\!]^{\text{alg}}_{\perp_\varsigma} \end{array}$$

Proof:
The existence of a reduction $t \xrightarrow[f(\vec{p})\longrightarrow r]{u} t'$ implies $t/u = f(\vec{p})\sigma$ and $t'/u = r\sigma$ for some substitution σ. We have $[\![f(\vec{p})\sigma]\!]^{\text{alg}}_{\perp_\varsigma} = \perp_\varsigma \trianglelefteq [\![r\sigma]\!]^{\text{alg}}_{\perp_\varsigma}$. Using the previous lemma we obtain

$$\begin{array}{lclcl} [\![t]\!]^{\text{alg}}_{\perp_\varsigma} & = & [\![t[u \leftarrow f(\vec{p})\sigma]]\!]^{\text{alg}}_{\perp_\varsigma} & = & [\![t]\!]^{\text{alg}}_{\perp_\varsigma}[u \leftarrow [\![f(\vec{p})\sigma]\!]^{\text{alg}}_{\perp_\varsigma}] \\ & \trianglelefteq & [\![t]\!]^{\text{alg}}_{\perp_\varsigma}[u \leftarrow [\![r\sigma]\!]^{\text{alg}}_{\perp_\varsigma}] & = & [\![t']\!]^{\text{alg}}_{\perp_\varsigma}. \end{array}$$

If $u \notin \mathrm{Outer}_P(t)$, then there is a position $v < u$ with $t(v) \in \mathcal{F}$. Hence $[\![t/v]\!]^{\text{alg}}_{\perp_\varsigma} = \perp_\varsigma$ and therefore $u \notin \mathrm{Pos}([\![t]\!]^{\text{alg}}_{\perp_\varsigma})$. This implies

$$[\![t]\!]^{\text{alg}}_{\perp_\varsigma} = [\![t]\!]^{\text{alg}}_{\perp_\varsigma}[u \leftarrow [\![f(\vec{p})\sigma]\!]^{\text{alg}}_{\perp_\varsigma}] = [\![t]\!]^{\text{alg}}_{\perp_\varsigma}[u \leftarrow [\![r\sigma]\!]^{\text{alg}}_{\perp_\varsigma}] = [\![t']\!]^{\text{alg}}_{\perp_\varsigma}.$$

□

This lemma is also valid for simple and parallel ς-reduction, since for example $t \xrightarrow[P,\varsigma,\text{no}]{} t'$ implies $t \xrightarrow[P,\text{no}]{} t'$.

Lemma 6.14
The set of ς-redexes $\mathrm{Red}_{P,\varsigma}$ is residually closed.

Proof:
Let $t \xrightarrow{u} t'$ be a reduction and $v \in \mathrm{RedPos}_{P,\varsigma}(t)$. Due to Lemma 2.4 we have $v \setminus t \xrightarrow{u} t' \subseteq \mathrm{RedPos}_P(t')$.

$\underline{v \parallel u, u \leq v}$: $t/v = t/\hat{v}$ for all $\hat{v} \in v \setminus t \xrightarrow{u} t'$ and hence $v \setminus t \xrightarrow{u} t' \subseteq \mathrm{RedPos}_{P,\varsigma}(t')$.

$\underline{v < u}$: Then $u = v.k.u'$ for some $k \in \mathbb{N}_+$ and $u' \in \mathbb{N}_+^*$. Since $v \in \mathrm{RedPos}_{P,\varsigma}(t)$, we have $t/v = f(\vec{t})$ and $t'/v = f(\vec{t'})$ with $t_k \xrightarrow{u'} t'_k$ and $t_i = t'_i$ for all other $i \in [n]$. Hence we can deduce from $\vec{t}$ being syntactically ς-matchable with some $f(\vec{p}) \in \mathrm{RedS}_P$ that $\vec{t'}$ is syntactically ς-matchable with $f(\vec{p})$ as well. Therefore $v \setminus t \xrightarrow{u} t' = \{v\} \subseteq \mathrm{RedPos}_{P,\varsigma}(t')$.

□

According to Section 2 this residual closure implies the confluence of the ς-reduction relations $\xrightarrow[P,\varsigma]{}$ and $\xrightarrow[P,\varsigma]{}\!\!\!\!\!\rightarrow$.

Lemma 6.15 Well-definedness of the ς-reduction semantics
The sets $T_{\mathrm{red}} := \{[\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\}$ and $T_{\mathrm{po}} := \{[\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma} \mid t \xrightarrow[P,\varsigma,\mathrm{po}]{*}\!\!\!\!\!\rightarrow t'\}$ have respective least upper bounds for all $t \in \mathrm{T}_\Sigma$.

Proof:
Since T_Σ is countable, $T_{\mathrm{red}}, T_{\mathrm{po}} \subseteq \{[\![t]\!]^{\mathrm{alg}}_{\perp_\varsigma} \mid t \in \mathrm{T}_\Sigma\}$ are countable as well.

$\underline{T_{\mathrm{red}}}$: Let $[\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma}, [\![t'']\!]^{\mathrm{alg}}_{\perp_\varsigma} \in T_{\mathrm{red}}$. Consequently $t \xrightarrow[P,\varsigma]{*} t'$ and $t \xrightarrow[P,\varsigma]{*} t''$. Due to the confluence of the ς-reduction relation there is a $\hat{t} \in \mathrm{T}_\Sigma$ with $t' \xrightarrow[P,\varsigma]{*} \hat{t}$ and $t'' \xrightarrow[P,\varsigma]{*} \hat{t}$. Together with Lemma 6.13 about the gain of information by reduction we get $[\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma} \trianglelefteq [\![\hat{t}]\!]^{\mathrm{alg}}_{\perp_\varsigma}$ and $[\![t'']\!]^{\mathrm{alg}}_{\perp_\varsigma} \trianglelefteq [\![\hat{t}]\!]^{\mathrm{alg}}_{\perp_\varsigma}$. Hence T_{red} is directed.

$\underline{T_{\mathrm{po}}}$: From the lemma about gain of information we deduce immediately that T_{po} is even a chain.

Since T_{red} and T_{po} are countable directed subsets of $\mathrm{T}_{\mathcal{C},\varsigma}$, their respective least upper bounds exist. □

7 Equivalence of the Definitions of the ς-Semantics

We prove in this section that the three definitions of the ς-semantics, that is those of the ς-fixed-point semantics, the global ς-reduction semantics, and the po-ς-reduction semantics, are equivalent. We do this by proving separately soundness and completeness.

Definition 7.1
A term semantics $[\![\cdot]\!] : \mathrm{T}_\Sigma \rightarrow \mathrm{T}_{\mathcal{C},\varsigma}$ is **sound** w.r.t. another term semantics $[\![\cdot]\!]' : \mathrm{T}_\Sigma \rightarrow \mathrm{T}_{\mathcal{C},\varsigma}$ iff $[\![\cdot]\!] \preceq [\![\cdot]\!]'$ and **complete** iff $[\![\cdot]\!]' \preceq [\![\cdot]\!]$. □

Subsections 7.1 and 7.2 establish basic properties which are used subsequently. Afterwards, the soundness of the two ς-reduction semantics w.r.t. the ς-fixed-point semantics and the soundness of the po- w.r.t. the global ς-reduction semantics are shown in Subsection 7.3. Subsection 7.4 contains the proof of completeness of the global ς-reduction semantics w.r.t. the ς-fixed-point semantics. In Subsection 7.5 we give a comprehensive method for proving the completeness of any reduction semantics based on a so called Π-fair reduction strategy w.r.t. a global reduction semantics. We apply this in Subsections 7.6 and 7.7 to show respectively the completeness of the po- w.r.t. the global ς-reduction semantics and that of the li*- w.r.t. the global cbv-reduction semantics. The overall result of this section is:

Proposition 7.1 **Equivalence of the definitions of the ς-semantics**

$$[\![\cdot]\!]^{\mathrm{fix}}_{P,\varsigma} = [\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma} = [\![\cdot]\!]^{\mathrm{po}}_{P,\varsigma}.$$

Proof:
$[\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\mathrm{fix}}_{P,\varsigma}$ and $[\![\cdot]\!]^{\mathrm{po}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\mathrm{fix}}_{P,\varsigma}$ proved by Lemma 7.8.
$[\![\cdot]\!]^{\mathrm{fix}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma}$ proved by Lemma 7.10.
$[\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\mathrm{po}}_{P,\varsigma}$ proved by Lemma 7.19. □

Hence we can define:

Definition 7.2
The ς-interpretation $\mathrm{DT}_\varsigma(P) := \mathrm{DT}^{\mathrm{fix}}_\varsigma(P)$ is called **ς-data type** and the mapping $[\![\cdot]\!]_{P,\varsigma} := [\![\cdot]\!]^{\mathrm{fix}}_{P,\varsigma} = [\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma} = [\![\cdot]\!]^{\mathrm{po}}_{P,\varsigma}$ is called **ς-(term-)semantics**. □

7.1 Syntactic and Semantic ς-Matching

First, we need to prove some basic properties.

Lemma 7.2 **Characterization of syntactic ς-matching** (cf. lem. 6.6)
The term tuple $\vec{t} \in (\mathrm{T}_\Sigma)^n$ is syntactically ς-matchable with $f(\vec{p}) \in \mathrm{RedS}_P$ by a substitution $\sigma : \mathrm{Var}(\vec{p}) \to \mathrm{T}_\Sigma$ iff conditions (i) and (ii) of semantic ς-matchability (def. 6.4) are fulfilled for the semantically approximated term tuple $([\![t_1]\!]^{\mathrm{alg}}_{\perp_\varsigma}, \ldots, [\![t_n]\!]^{\mathrm{alg}}_{\perp_\varsigma})$ and $x\sigma = \vec{t}/u_x$ with u_x given by $\{u_x\} := \mathrm{Pos}(x, \vec{p})$ for all $x \in \mathrm{Var}(\vec{p})$.
Proof idea:
Structural induction on $\vec{p}$. □

The reader should notice again that the linearity of patterns is crucial for the well-definedness of u_x.

Lemma 7.3 **Commutativity of term semantics and forming of subterms**
Let $c \in \mathrm{T}_\mathcal{C}(X)$, $\beta : \mathrm{Var}(c) \to \mathrm{T}_{\mathcal{C},\varsigma}$ and $u \in \mathrm{Pos}(c) \cap \mathrm{Pos}([\![c]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta})$. Then

$$[\![c]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta}/u = [\![c/u]\!]^{\mathrm{alg}}_{\mathrm{BDT}_{\mathcal{C},\varsigma},\beta}.$$

Proof idea:
Structural induction on u. □

The next proposition is of fundamental importance for the equality of the ς-fixed-point and the ς-reduction semantics.

Proposition 7.4 Syntactic and semantic ς-matching
Let $f^{(n)}(\vec{p}) \in \mathrm{RedS}_P$ and $\vec{t} \in (\mathrm{T}_\Sigma)^n$.

1. The following three statements are equivalent.

   a) $\vec{t}$ is syntactically ς-matchable with $f(\vec{p})$ (by a substitution $\sigma : X \to \mathrm{T}_\Sigma$).

   b) $(\llbracket t_1 \rrbracket^{\mathrm{alg}}_{\perp_\varsigma}, \ldots, \llbracket t_n \rrbracket^{\mathrm{alg}}_{\perp_\varsigma})$ is semantically ς-matchable with $f(\vec{p})$.

   c) For all interpretations $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\varsigma}$ the tuple $(\llbracket t_1 \rrbracket^{\mathrm{alg}}_{\mathfrak{A}}, \ldots, \llbracket t_n \rrbracket^{\mathrm{alg}}_{\mathfrak{A}})$ is semantically ς-matchable with $f(\vec{p})$ (by a respective valuation $\beta_{\mathfrak{A}} : X \to \mathrm{T}_{\mathcal{C},\varsigma}$).

2. If the statements above are fulfilled, then $\beta_{\mathfrak{A}}(x) = \llbracket x\sigma \rrbracket^{\mathrm{alg}}_{\mathfrak{A}}$ for all $x \in \mathrm{Var}(\vec{p})$ and even $\llbracket t \rrbracket^{\mathrm{alg}}_{\mathfrak{A},\beta_{\mathfrak{A}}} = \llbracket t\sigma \rrbracket^{\mathrm{alg}}_{\mathfrak{A}}$ for all $t \in \mathrm{T}_\Sigma(\mathrm{Var}(\vec{p}))$.

Proof idea:

1. a) $\Longleftrightarrow$ b) follows from the characterization of syntactic and semantic ς-matching (lemmas 7.2 and 6.6). b) $\Longrightarrow$ c) employs the monotonicity of semantics ς-matching which is valid according to Lemma 6.7 and c) $\Longrightarrow$ b) is trivial.

2. Follows from lemmas 7.2 and 6.6 as well, using the previous lemma about commutativity.

□

The reader should note that $\vec{t}$ is not necessarily syntactically ς-matchable with $f(\vec{p})$, if $(\llbracket t_1 \rrbracket^{\mathrm{alg}}_{\mathfrak{A}}, \ldots, \llbracket t_n \rrbracket^{\mathrm{alg}}_{\mathfrak{A}})$ is only semantically matchable with $f(\vec{p})$ for some $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\varsigma}$.

7.2 Soundness of ς-Reduction

Here we prove in several steps the soundness of ς-reduction w.r.t. the ς-fixed-point semantics which was already claimed in Subsection 6.3.

Lemma 7.5
In fixed-points of the ς-transformation simple ς-reduction is sound, i.e. if $f(\vec{p}) \to r \in P$, $\sigma : X \to \mathrm{T}_\Sigma$, $\mathfrak{A} = \Phi_{P,\varsigma}(\mathfrak{A}) \in \mathrm{Int}_{\Sigma,\varsigma}$, and $f(\vec{p})\sigma$ is a ς-redex, then:

$$\llbracket f(\vec{p})\sigma \rrbracket^{\mathrm{alg}}_{\mathfrak{A}} = \llbracket r\sigma \rrbracket^{\mathrm{alg}}_{\mathfrak{A}}.$$

Proof:
Since $f(\vec{p})\sigma$ is a ς-redex, the term tuple $(p_1\sigma, \dots, p_n\sigma)$ is syntactically ς-matchable with $f(\vec{p})$ by σ.

Due to Proposition 7.4 about syntactic and semantic ς-matching we know that $(\llbracket p_1\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}, \dots, \llbracket p_n\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}})$ is semantically ς-matchable with $f(\vec{p})$ by $\beta_{\mathfrak{A}} : X \to \mathrm{T}_{\mathcal{C},\varsigma}$, which is defined by $\beta_{\mathfrak{A}}(x) := \llbracket x\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}$ for all $x \in X$; hence

$$f^{\Phi_{P,\varsigma}(\mathfrak{A})}(\llbracket p_1\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}, \dots, \llbracket p_n\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}) = \llbracket r\rrbracket^{\text{alg}}_{\mathfrak{A},\beta_{\mathfrak{A}}} \tag{1}$$

and also (prop. 7.4, item 2):

$$\llbracket r\rrbracket^{\text{alg}}_{\mathfrak{A},\beta_{\mathfrak{A}}} = \llbracket r\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}. \tag{2}$$

Since $\mathfrak{A}$ is a fixed-point of $\Phi_{P,\varsigma}$, we have

$$\llbracket f(\vec{p})\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}} = f^{\mathfrak{A}}(\llbracket p_1\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}, \dots, \llbracket p_n\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}) = f^{\Phi_{P,\varsigma}(\mathfrak{A})}(\llbracket p_1\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}, \dots, \llbracket p_n\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}). \tag{3}$$

Altogether:

$$\llbracket f(\vec{p})\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}} \overset{(3)}{=} f^{\Phi_{P,\varsigma}(\mathfrak{A})}(\llbracket p_1\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}, \dots, \llbracket p_n\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}) \overset{(1)}{=} \llbracket r\rrbracket^{\text{alg}}_{\mathfrak{A},\beta_{\mathfrak{A}}} \overset{(2)}{=} \llbracket r\sigma\rrbracket^{\text{alg}}_{\mathfrak{A}}.$$

□

Corollary 7.6
In fixed-points of the ς-transformation ς-reduction is sound, i.e. if $\mathfrak{A} = \Phi_{P,\varsigma}(\mathfrak{A}) \in \mathrm{Int}_{\Sigma,\varsigma}$, then for all $t, t' \in \mathrm{T}_\Sigma$:

$$t \xrightarrow[P,\varsigma]{*} t' \implies \llbracket t\rrbracket^{\text{alg}}_{\mathfrak{A}} = \llbracket t'\rrbracket^{\text{alg}}_{\mathfrak{A}}.$$

Proof:
Follows from the preceding lemma by the invariance of algebraic term semantics. □

Corollary 7.7
ς-reduction is sound w.r.t. the ς-fixed-point semantics, i.e.

$$t \xrightarrow[P,\varsigma]{*} t' \implies \llbracket t\rrbracket^{\text{fix}}_{P,\varsigma} = \llbracket t'\rrbracket^{\text{fix}}_{P,\varsigma}.$$

□

7.3 Soundness of the ς-Reduction Semantics

Lemma 7.8
The ς-reduction semantics are sound w.r.t. the ς-fixed-point semantics:

$$\llbracket\cdot\rrbracket^{\text{red}}_{P,\varsigma} \preceq \llbracket\cdot\rrbracket^{\text{fix}}_{P,\varsigma} \text{ and } \llbracket\cdot\rrbracket^{\text{po}}_{P,\varsigma} \preceq \llbracket\cdot\rrbracket^{\text{fix}}_{P,\varsigma}.$$

Proof:
Let $t, t' \in \mathrm{T}_\Sigma$ with $t \xrightarrow[P,\varsigma]{*} t'$. According to the soundness of ς-reduction sequences (cor. 7.7) $[\![t]\!]^{\text{fix}}_{P,\varsigma} = [\![t']\!]^{\text{fix}}_{P,\varsigma}$. Since $\bot_\varsigma \sqsubseteq \mathrm{DT}^{\text{fix}}_\varsigma(P)$ and since the algebraic term semantics is monotonic (Lemma 6.9), we have $[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \trianglelefteq [\![t']\!]^{\text{alg}}_{\mathrm{DT}^{\text{fix}}_\varsigma(P)} = [\![t']\!]^{\text{fix}}_{P,\varsigma}$. Together $[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \trianglelefteq [\![t]\!]^{\text{fix}}_{P,\varsigma}$ and therefore

$$[\![t]\!]^{\text{red}}_{P,\varsigma} = \bigsqcup\{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\} \trianglelefteq [\![t]\!]^{\text{fix}}_{P,\varsigma}$$

and

$$[\![t]\!]^{\text{po}}_{P,\varsigma} = \bigsqcup\{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \mid t \overset{*}{\underset{P,\varsigma,\text{po}}{\twoheadrightarrow}} t'\} \trianglelefteq [\![t]\!]^{\text{fix}}_{P,\varsigma}.$$

□

Lemma 7.9
The po-ς-reduction semantics is sound w.r.t. the global ς-reduction semantics:

$$[\![\cdot]\!]^{\text{po}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\text{red}}_{P,\varsigma}.$$

Proof:

$$\begin{array}{ccccccc} & \{t' | t \overset{*}{\underset{P,\varsigma,\text{po}}{\twoheadrightarrow}} t'\} & \subseteq & \{t' | t \xrightarrow[P,\varsigma]{*} t'\} & & & \\ \Longrightarrow & \{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} | t \overset{*}{\underset{P,\varsigma,\text{po}}{\twoheadrightarrow}} t'\} & \subseteq & \{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} | t \xrightarrow[P,\varsigma]{*} t'\} & \Longrightarrow & [\![t]\!]^{\text{po}}_{P,\varsigma} \trianglelefteq & [\![t]\!]^{\text{red}}_{P,\varsigma}. \end{array}$$

□

7.4 Completeness of the Global ς-Reduction Semantics

Lemma 7.10
The global ς-reduction semantics is complete w.r.t. the ς-fixed-point semantics:

$$[\![\cdot]\!]^{\text{fix}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\text{red}}_{P,\varsigma}$$

Proof:
Let $t \in \mathrm{T}_\Sigma$ and $\mathfrak{A}_i := (\Phi_{P,\varsigma})^i(\bot_\varsigma) \in \mathrm{Int}_{\Sigma,\varsigma}$ for all $i \in \mathbb{N}$. According to the next lemma $\{[\![t]\!]^{\text{alg}}_{\mathfrak{A}_i} \mid i \in \mathbb{N}\}$ is cofinal in $\{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\}$ which implies $\bigsqcup_{i\in\mathbb{N}}[\![t]\!]^{\text{alg}}_{\mathfrak{A}_i} \trianglelefteq \bigsqcup\{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\}$. Hence we have

$$[\![t]\!]^{\text{fix}}_{P,\varsigma} = [\![t]\!]^{\text{alg}}_{\mathrm{DT}^{\text{fix}}_\varsigma(P)} = [\![t]\!]^{\text{alg}}_{\bigsqcup_{i\in\mathbb{N}}\mathfrak{A}_i} \overset{\text{l. 6.9}}{=} \bigsqcup_{i\in\mathbb{N}} [\![t]\!]^{\text{alg}}_{\mathfrak{A}_i} \trianglelefteq \bigsqcup\{[\![t']\!]^{\text{alg}}_{\bot_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\} = [\![t]\!]^{\text{red}}_{P,\varsigma}.$$

□

The following lemma contains the heart of the completeness proof.

Lemma 7.11
The approximations of the ς-fixed-point semantics are cofinal in those of the global ς-reduction semantics, i.e. if $t \in \mathrm{T}_\Sigma$ and $\mathfrak{A}_i := (\Phi_{P,\varsigma})^i(\bot_\varsigma) \in \mathrm{Int}_{\Sigma,\varsigma}$ for all $i \in \mathbb{N}$, then for every $i \in \mathbb{N}$ there is a $t' \in \mathrm{T}_\Sigma$ with $t \xrightarrow[P,\varsigma]{*} t'$ and $[\![t]\!]^{\mathrm{alg}}_{\mathfrak{A}_i} \trianglelefteq [\![t']\!]^{\mathrm{alg}}_{\bot_\varsigma}$.

Proof:

$\underline{i = 0}$: With $t' := t$ we have $t \xrightarrow[P,\varsigma]{*} t'$ and $[\![t]\!]^{\mathrm{alg}}_{\mathfrak{A}_i} = [\![t]\!]^{\mathrm{alg}}_{\bot_\varsigma} = [\![t']\!]^{\mathrm{alg}}_{\bot_\varsigma}$.

$\underline{i \Rightarrow i+1}$:

$\underline{t = f(t_1, \dots, t_n)}$: $(f^{(n)} \in \mathcal{F})$.

Case 1: $([\![t_1]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}}, \dots, [\![t_n]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}})$ is semantically ς-matchable with the left-hand side of $f(\vec{p})\to r \in P$ by $\beta : X \to \mathrm{T}_{\mathcal{C},\varsigma}$.
Then

$$[\![t]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}} = f^{\mathfrak{A}_{i+1}}([\![t_1]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}}, \dots, [\![t_n]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}}) = [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta}. \tag{1}$$

In accordance with the hypotheses of the structural induction there exist $t'_1, \dots, t'_n \in \mathrm{T}_\Sigma$ with

$$t_l \xrightarrow[P,\varsigma]{*} t'_l \tag{2}$$

and

$$[\![t_l]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}} \trianglelefteq [\![t'_l]\!]^{\mathrm{alg}}_{\bot_\varsigma} \tag{3}$$

for all $l \in [n]$.
Equation (2) implies

$$t = f(t_1, \dots, t_n) \xrightarrow[P,\varsigma]{*} f(t'_1, \dots, t'_n) \tag{4}$$

The monotonicity of algebraic term semantics w.r.t. the algebra (Lemma 6.9) gives us

$$[\![t'_l]\!]^{\mathrm{alg}}_{\bot_\varsigma} \trianglelefteq [\![t'_l]\!]^{\mathrm{alg}}_{\mathfrak{A}} \tag{5}$$

for all $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\varsigma}$ and $l \in [n]$.
From (3) and (5) and the monotonicity of semantic ς-matching (Lemma 6.7) we conclude that for all ς-interpretations $\mathfrak{A} \in \mathrm{Int}_{\Sigma,\varsigma}$, especially $\mathfrak{A}_i$, $([\![t'_1]\!]^{\mathrm{alg}}_{\mathfrak{A}}, \dots, [\![t'_n]\!]^{\mathrm{alg}}_{\mathfrak{A}})$ is semantically ς-matchable with $f(\vec{p})$ by respective valuations $\beta_{\mathfrak{A}}$, and

$$\beta \preceq \beta_{\mathfrak{A}}. \tag{6}$$

With Proposition 7.4 about syntactic and semantic ς-matching follows, that $(t'_1, \dots, t'_n)$ is syntactically ς-matchable with $f(\vec{p})$ by a substitution σ and

$$[\![\hat{t}\sigma]\!]^{\mathrm{alg}}_{\mathfrak{A}_i} = [\![\hat{t}]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta_{\mathfrak{A}_i}} \tag{7}$$

for all $\hat{t} \in \mathrm{T}_\Sigma(\mathrm{Var}(\vec{p}))$.

The syntactic ς-matchability implies

$$f(t'_1,\dots,t'_n) \xrightarrow[P,\varsigma]{} r\sigma. \tag{8}$$

Equation (7) is especially valid for $\hat{t} = r$:

$$[\![r\sigma]\!]^{\mathrm{alg}}_{\mathfrak{A}_i} = [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta_{\mathfrak{A}_i}}. \tag{9}$$

From (6) and the monotonicity of algebraic term semantics w.r.t. the valuation (Lemma 6.8) we deduce

$$[\![r]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta} \trianglelefteq [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta_{\mathfrak{A}_i}}. \tag{10}$$

Finally the hypotheses of the induction over i assures the existence of a $t' \in \mathrm{T}_\Sigma$ with

$$r\sigma \xrightarrow[P,\varsigma]{*} t' \tag{11}$$

and

$$[\![r\sigma]\!]^{\mathrm{alg}}_{\mathfrak{A}_i} \trianglelefteq [\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma}. \tag{12}$$

Together (4), (8) and (11) give

$$t = f(t_1,\dots,t_n) \xrightarrow[P,\varsigma]{*} f(t'_1,\dots,t'_n) \xrightarrow[P,\varsigma]{} r\sigma \xrightarrow[P,\varsigma]{*} t' \tag{13}$$

and we have

$$[\![t]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}} \overset{(1)}{=} [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta} \overset{(10)}{\trianglelefteq} [\![r]\!]^{\mathrm{alg}}_{\mathfrak{A}_i,\beta_{\mathfrak{A}_i}} \overset{(9)}{=} [\![r\sigma]\!]^{\mathrm{alg}}_{\mathfrak{A}_i} \overset{(12)}{\trianglelefteq} [\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma}. \tag{14}$$

Case 2: Otherwise, i.e. there does not exist any matching reduction rule.
Let $t' := t$. Then

$$t \xrightarrow[P,\varsigma]{*} t'$$

and

$$[\![t]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}} = f^{\mathfrak{A}_{i+1}}([\![t_1]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}},\dots,[\![t_n]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}}) = \perp \trianglelefteq [\![t']\!]^{\mathrm{alg}}_{\perp_\varsigma}.$$

$t = G(t_1,\dots,t_n)$: ($G^{(n)} \in \mathcal{C}$).

According to the hypotheses of the structural induction there exist $t'_1,\dots,t'_n \in \mathrm{T}_\Sigma$ with

$$t_l \xrightarrow[P,\varsigma]{*} t'_l \tag{1}$$

and

$$[\![t_l]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}} \trianglelefteq [\![t'_l]\!]^{\mathrm{alg}}_{\perp_\varsigma} \tag{2}$$

for all $l \in [n]$.

Together with the monotonicity of all operations of a ς-interpretation (2) implies

$$G^{\mathfrak{A}_{i+1}}([\![t_1]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}},\dots,[\![t_n]\!]^{\mathrm{alg}}_{\mathfrak{A}_{i+1}}) \trianglelefteq G^{\mathfrak{A}_{i+1}}([\![t'_1]\!]^{\mathrm{alg}}_{\perp_\varsigma},\dots,[\![t'_n]\!]^{\mathrm{alg}}_{\perp_\varsigma}). \tag{3}$$

Furthermore, the operation of a constructor symbol is the same in all ς-interpretations:

$$G^{\mathfrak{A}_{i+1}} = G^{\mathfrak{A}_0} = G^{\perp_\varsigma}. \tag{4}$$

Let $t' := G(t'_1, \ldots, t'_n)$. Then (1) gives us

$$t = G(t_1, \ldots, t_n) \xrightarrow[P,\varsigma]{*} G(t'_1, \ldots, t'_n) = t'$$

and (3) and (4) imply

$$\begin{aligned} [\![t]\!]^{\text{alg}}_{\mathfrak{A}_{i+1}} &= G^{\mathfrak{A}_{i+1}}([\![t_1]\!]^{\text{alg}}_{\mathfrak{A}_{i+1}}, \ldots, [\![t_n]\!]^{\text{alg}}_{\mathfrak{A}_{i+1}}) \\ &\overset{(3)}{\trianglelefteq} G^{\mathfrak{A}_{i+1}}([\![t'_1]\!]^{\text{alg}}_{\perp_\varsigma}, \ldots, [\![t'_n]\!]^{\text{alg}}_{\perp_\varsigma}) \\ &\overset{(4)}{=} G^{\perp_\varsigma}([\![t'_1]\!]^{\text{alg}}_{\perp_\varsigma}, \ldots, [\![t'_n]\!]^{\text{alg}}_{\perp_\varsigma}) \\ &= [\![t']\!]^{\text{alg}}_{\perp_\varsigma}. \end{aligned}$$

□

Unfortunately it is not possible to use the same proof method for the po-ς-reduction semantics, since the step from (2) to (4) is not valid for po-ς-reduction.

7.5 Completeness of Π-Fair Reduction Semantics

We do not prove the completeness of the po-ς-reduction semantics directly, as has been done in detail in [Chi95]. The proof there is based on O'Donnell's proof that eventually outermost sequences terminate whenever possible (lemma 10 and theorem 17 in [O'D77]). In the appendix of [BK86] O'Donnell's proof is extended to so called Π-fair reduction sequences. By using that extended version we obtain a more general result, showing that any semantics based on Π-fair reduction is complete. This is subsequently applied in Subsections 7.6 and 7.7 to the po-ς- and the li*-semantics, respectively.

In this and the following subsections I is an arbitrary instance of an almost orthogonal TRS R over a signature Σ, so that $\text{Red}_{R,I}$ is residually closed. $\langle A, \trianglelefteq \rangle$ is a cpo (with $A \neq \emptyset$) and $[\![\cdot]\!]_\perp : \text{T}_\Sigma \rightarrow A$ a mapping such that $t \xrightarrow[R,I]{} t' \Longrightarrow [\![t]\!]_\perp \trianglelefteq [\![t']\!]_\perp$.

For having a concrete example, the reader may keep in mind that later R will be our program P, I an instance ς, and $[\![\cdot]\!]_\perp$ a semantic ς-approximation $[\![\cdot]\!]_{\perp_\varsigma}$.

Due to its complexity, the proof of [BK86] is not reproduced here but only outlined and cited as far as necessary for our extension. Unfortunately it considers only the smaller class of orthogonal TRSs. Nonetheless, the generalization to instances of almost orthogonal TRSs with residually closed redex sets is straightforward, because the proof is based on a variation of the parallel moves lemma (2.5), as already remarked in [BK86].

Definition 7.3 (Cf. Definition 7.2 in [BK86])
A predicate $\Pi \subseteq \mathbb{N}^* \times \text{T}_\Sigma$ with $(u, t) \in \Pi \Longrightarrow u \in \text{RedPos}_{R,I}(t)$ is **gaining** iff it has the following three properties:

I) **Preservation of a Π-redex-position.**

For all I-reductions

$$\begin{tikzcd} t_0 \arrow[r, "u_1", "R,I"'] \arrow[d, "u_2", "R,I"'] & t_1 \arrow[d, two heads, "V_2", "R,I"'] \\ t_2 \arrow[r, two heads, "V_1", "R,I"'] & t_3 \end{tikzcd}$$

with $V_1 = u_1 \setminus t_0 \xrightarrow[R,I]{u_2} t_2$, $V_2 = u_2 \setminus t_0 \xrightarrow[R,I]{u_1} t_1$, and all $w_i \in \mathrm{RedPos}_{R,I}(t_i)$ $(i \in [4])$ with

$$w_1 \in w_0 \setminus t_0 \longrightarrow t_1, \quad w_2 \in w_0 \setminus t_0 \longrightarrow t_2,$$
$$w_3 \in w_1 \setminus t_1 \longrightarrow t_3, \quad w_3 \in w_2 \setminus t_2 \longrightarrow t_3,$$

we have

$$\Pi(w_0, t_0), \Pi(w_2, t_2), \Pi(w_3, t_3) \Longrightarrow \Pi(w_1, t_1).$$

II) **¬Π-reductions do not create new Π-redex-positions.**

For every reduction $t \xrightarrow[R,I]{u} t'$ such that $\neg\Pi(u, t)$, every v' such that $\Pi(v', t')$ has a respective predecessor, i.e. a v with $v' \in v \setminus t \xrightarrow[R,I]{u} t'$ such that $\Pi(v, t)$.

III) **¬Π-reductions do not change the semantic approximation.**

For every $t \xrightarrow[R,I]{u} t'$ such that $\neg\Pi(u, t)$, we have $[\![t]\!]_\perp = [\![t']\!]_\perp$.

We write $t \xrightarrow[\Pi]{u} t'$ and $t \xrightarrow[\neg\Pi]{u} t'$ for a reduction $t \xrightarrow[R,I]{u} t'$ with $\Pi(u, t)$ and $\neg\Pi(u, t)$, respectively. □

The definition of the first property is slightly different from [BK86] where the transitive-reflexive closure $\xrightarrow{*}$ is used instead of the parallel $\longrightarrow\!\!\!\!\!\rightarrow$. Since that would require introducing the supplementary concepts of development and diagram we use our equivalent definition. The third property is added for our extension towards semantics.

<u>**Definition 7.4**</u> (Def. 7.5 in [BK86])

1. Let $A = t_1 \xrightarrow[R,I]{v_1} t_2 \xrightarrow[R,I]{v_2} \ldots$ be a (finite or infinite) reduction sequence such that $u_j \in \mathrm{RedPos}_{R,I}(t_j)$ and $u_{i+1} \in u_i \setminus t_i \xrightarrow[R,I]{v_i} t_{i+1}$ for all $i \geq j$ as far as u_i is defined. Then the sequence $u_j, u_{j+1}, u_{j+2}, \ldots$ is called a **trace** in A.

2. Let A be as in 1. and let $\Pi \subseteq \mathbb{N}^* \times \mathrm{T}_\Sigma$ be a predicate. Then a trace $u_j, u_{j+1}, \ldots$ in A is a **Π-trace** iff for all $i \geq j$ we have $\Pi(u_i, t_i)$.

3. Let A be a reduction and $\Pi \subseteq \mathbb{N}^* \times \mathrm{T}_\Sigma$ be a predicate. Then A is **Π-fair** iff A contains no infinite Π-traces.

□

$$\begin{tikzcd}
t_1 \arrow[r, "v_1"] \arrow[ddd, "\Pi"', "*"] & t_2 \arrow[r, "v_2"] \arrow[dd, "\Pi"', "*"] & t_3 \arrow[r, "v_3"] \arrow[d, "\Pi"', "*"] & \cdots \\
 & & s_3 \arrow[r, "*"] \arrow[d, "\neg\Pi"', "*"] & \cdots \\
 & s_2 \arrow[r, "*"] \arrow[d, "\neg\Pi"', "*"] & \hat{s}_3 \arrow[dd, "*"] & \\
s_1 \arrow[r, "*"] \arrow[d, "\neg\Pi"', "*"] & \hat{s}_2 \arrow[d, "*"] & & \\
t'_1 \arrow[r, "*"] & t'_2 \arrow[r, "*"] & t'_3 \arrow[r, "*"] & \cdots
\end{tikzcd}$$

Figure 1: The Π-fair reduction construction.

Lemma 7.12

If Π is a gaining predicate, $A = t_1 \xrightarrow[R,I]{v_1} t_2 \xrightarrow[R,I]{v_2} \ldots$ a Π-fair reduction sequence, and $t_1 \xrightarrow[R,I]{} t'_1$ a reduction, then there exists a **Π-fair reduction construction** as shown in figure 1 such that

1. the sequence $s_1, \hat{s}_2, s_2, \hat{s}_2, s_3, \ldots$ converges to A, that is there exists a $j \in \mathbb{N}$ with $t_i = s_i$ for all $i \geq j$, and
2. the residual reduction sequence $t'_1 \xrightarrow[R,I]{*} t'_2 \xrightarrow[R,I]{*} \ldots$ of the Π-fair reduction construction is Π-fair.

Proof:

Theorem 7.8 in [BK86] (see also lemma 17 in [O'D77]). □

Lemma 7.13 Semantic cofinality of Π-fair reduction sequences I

Let $A = t_1 \xrightarrow[R,I]{} t_2 \xrightarrow[R,I]{} \ldots$ be a Π-fair reduction sequence and $B = t_1 \xrightarrow[R,I]{} t'_1$ a reduction. Then exists an $l \in \mathbb{N}_+$ with $[\![t'_1]\!]_\perp \trianglelefteq [\![t_l]\!]_\perp$.

Proof:

We consider the Π-fair reduction construction of A and B with the identifiers used in Lemma 7.12 and prove by induction that $[\![t'_1]\!]_\perp \trianglelefteq [\![s_i]\!]_\perp$ for all $i \in \mathbb{N}_+$.

$i = 1$: By property III the reduction $s_1 \xrightarrow[\neg\Pi]{*} t'_1$ implies $[\![t'_1]\!]_\perp = [\![s_1]\!]_\perp$.

$i \Rightarrow i+1$: We have $s_i \xrightarrow{*} \hat{s}_{i+1}$ and $s_{i+1} \xrightarrow[\neg\Pi]{*} \hat{s}_{i+1}$. Our prerequisite concerning $[\![\cdot]\!]_\perp$ and property III imply $[\![s_i]\!]_\perp \trianglelefteq [\![\hat{s}_{i+1}]\!]_\perp = [\![s_{i+1}]\!]_\perp$. Together with the induction hypotheses we get $[\![t'_1]\!]_\perp \trianglelefteq [\![s_{i+1}]\!]_\perp$.

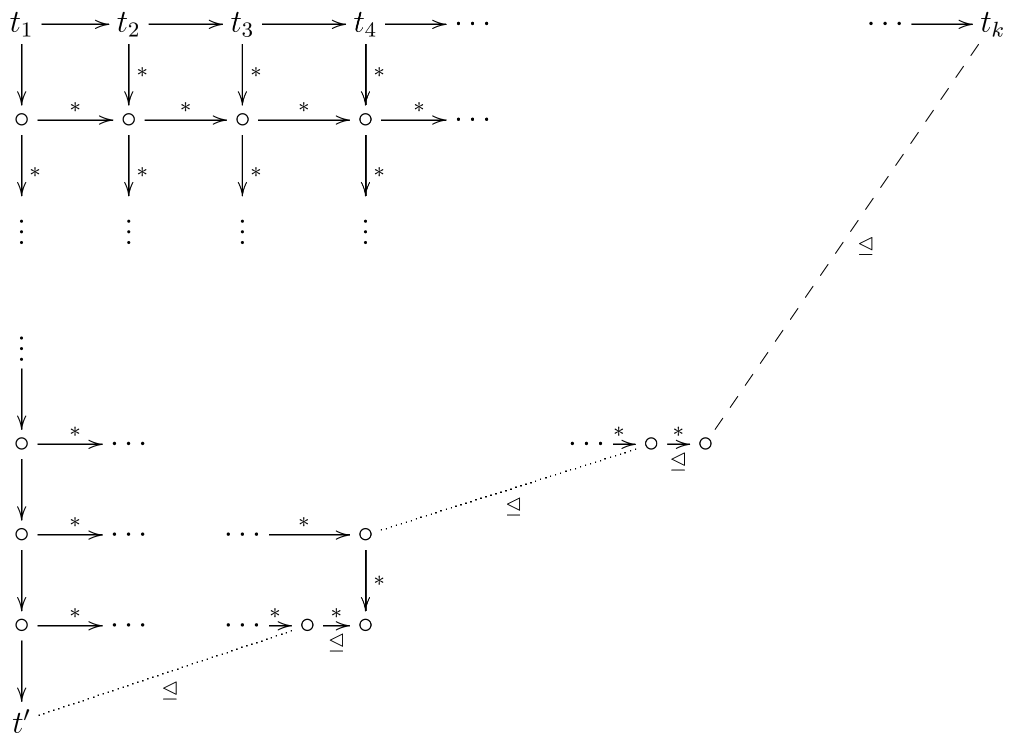

Figure 2: Proof idea of Lemma 7.14

According to Lemma 7.12 there exists an $l \in \mathbb{N}_+$ with $t_l = s_l$. Hence we have $[\![t'_1]\!]_\perp \trianglelefteq [\![s_l]\!]_\perp = [\![t_l]\!]_\perp$. □

Lemma 7.14 Semantic cofinality of Π-fair reduction sequences II
Let $A = t_1 \xrightarrow[R,I]{} t_2 \xrightarrow[R,I]{} \dots$ be a Π-fair reduction sequence and $B = t_1 \xrightarrow[R,I]{*} t'$ a reduction. Then exists a $k \in \mathbb{N}_+$ with $[\![t']\!]_\perp \trianglelefteq [\![t_k]\!]_\perp$.
Proof idea:
Induction over the length of B, employing the preceding lemma for the induction step; see figure 2. □

Thus we finally obtain:

Proposition 7.15 Completeness of Π-fair reduction semantics
Let a **global reduction semantics** $[\![\cdot]\!]^{\text{red}}_{R,I} : \mathrm{T}_\Sigma \to A$ be defined by

$$[\![t]\!]^{\text{red}}_{R,I} := \bigsqcup\{[\![t']\!]_\perp \mid t \xrightarrow[R,I]{*} t'\}$$

and let a 'successor'-function $F : \mathrm{T}_\Sigma \to \mathrm{T}_\Sigma$, fixing a unique Π-fair reduction sequence $t \xrightarrow[R,I]{*} F(t) \xrightarrow[R,I]{*} F^2(t) \xrightarrow[R,I]{*} \dots$ for every term t, define a **Π-fair reduction semantics** $[\![\cdot]\!]^{\Pi}_{R,I} : \mathrm{T}_\Sigma \to A$ by

$$[\![t]\!]^{\Pi}_{R,I} := \bigsqcup_{i \in \mathbb{N}} [\![F^n(t)]\!]_\perp$$

Then the Π-fair reduction semantics is complete w.r.t. the global reduction semantics, i.e.

$$[\![\cdot]\!]^{\mathrm{red}}_{R,I} \preceq [\![\cdot]\!]^{\Pi}_{R,I}.$$

□

7.6 Completeness of the Po-ς-Reduction Semantics

To apply the result of the last subsection we just have to prove that $\Pi^{\mathrm{o}}_{P,\varsigma}$ is a gaining predicate, where $\Pi^{\mathrm{o}}_{R,I}(u,t)$ holds iff u is an outermost I-redex position of t. Since this proof is only tedious it was moved to the appendix.

Lemma 7.16
The predicate $\Pi^{\mathrm{o}}_{P,\varsigma}$ is gaining.

Proof:
Property I: Lemma A.1. Property II: Lemma A.4 and Lemma A.2. Property III: Lemma 6.13 (gain of information). □

Corollary 7.17
The po-ς-reduction semantics is complete w.r.t. the global ς-reduction semantics, i.e. $[\![\cdot]\!]^{\mathrm{red}}_{P,\varsigma} \preceq [\![\cdot]\!]^{\mathrm{po}}_{P,\varsigma}$.

Proof:
The po-ς-reduction semantics is a $\Pi^{\mathrm{o}}_{P,\varsigma}$-fair reduction semantics. □

7.7 Completeness of the Lo-Cbv-Reduction Semantics

Analogously to the previous subsection we define a predicate $\Pi^{\mathrm{lo}}_{R,I}$ by $\Pi^{\mathrm{lo}}_{R,I}(u,t) :\Longleftrightarrow u$ is the leftmost-outermost I-redex position of t. Proving $\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ to be gaining was moved to the appendix as well. We finally obtain:

Lemma 7.18
The predicate $\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ is gaining.

Proof:
Property I: Lemma A.5. Property II: Lemma A.6. Property III: Lemma A.7. □

Corollary 7.19
The lo-cbv-reduction semantics (li*-reduction semantics) is complete w.r.t. the global cbv-reduction semantics, i.e. $[\![\cdot]\!]^{\mathrm{red}}_{P,\mathrm{cbv}} \preceq [\![\cdot]\!]^{\mathrm{lo}}_{P,\mathrm{cbv}}$. □

8 Properties of the ς-Semantics

In Section 4 we discussed some properties which we expect our semantics to have. We record that since $[\![\cdot]\!]_{P,\varsigma} = [\![\cdot]\!]^{\mathrm{alg}}_{\mathrm{DT}_\varsigma(P)}$ is an algebraic semantics, the ς-semantics is compositional. The ς-semantics is modular as well; the broad reasoning of Section 4 can easily be transformed into a formal proof for the ς-semantics.

We mentioned in Section 4 that the ς-term semantics uniquely determines the ς-data type although the equation

$$f^{\mathrm{DT}_\varsigma(P)}(\vec{\underline{t}}) = [\![f(\vec{t})]\!]_{P,\varsigma}$$

(for all $f^{(n)} \in \Sigma$, $\vec{\underline{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$, $\vec{t} \in (\mathrm{T}_\Sigma)^n$ with $[\![t_1]\!]_{P,\varsigma} = \underline{t}_1, \ldots, [\![t_n]\!]_{P,\varsigma} = \underline{t}_n$) does not give a complete definition of the ς-data type $\mathrm{DT}_\varsigma(P)$ by $[\![\cdot]\!]_{P,\varsigma}$, because there may not be for every $\underline{t} \in \mathrm{T}_{\mathcal{C},\varsigma} \setminus \mathrm{T}_\mathcal{C}$ a $t \in \mathrm{T}_\Sigma$ with $[\![t]\!]_{P,\varsigma} = \underline{t}$.

The problem can be remedied, because the ς-semantics is modular and the data type $\langle \mathrm{T}_{\mathcal{C},\varsigma}, \trianglelefteq \rangle$ is ω-inductive. If there is a term $t_\perp$ with $[\![t_\perp]\!]_{P,\varsigma} = \perp$, then for any finite partial data term $\underline{t} \in (\mathrm{T}_{\mathcal{C},\varsigma} \cap \mathrm{T}_{\mathcal{C},\perp}) \setminus \mathrm{T}_\mathcal{C}$ a program term $t \in \mathrm{T}_\Sigma$ with $[\![t]\!]_{P,\varsigma} = \underline{t}$ is constructible by using $t_\perp$ and constructor symbols. If there is no such $t_\perp$, then we extend our program P by a new function symbol $c_\perp$ to a program P', without giving a rule for $c_\perp$. Consequently $[\![c_\perp]\!]_{P',\varsigma} = \perp$. If we can solve the problem for the infinite data terms as well, then we have $\mathrm{DT}_\varsigma(P) = \mathrm{DT}_\varsigma(P')|_\Sigma$ due to modularity. Since $\langle \mathrm{T}_{\mathcal{C},\varsigma}, \trianglelefteq \rangle$ is ω-inductive, there exist for all tuples of (possibly infinite) terms $\vec{\underline{t}} \in (\mathrm{T}_{\mathcal{C},\varsigma})^n$ chains $T_1, \ldots, T_n \subseteq \mathrm{T}_{\mathcal{C},\varsigma} \cap \mathrm{T}_{\mathcal{C},\perp}$ of finite terms with $\underline{t}_1 = \bigsqcup T_1, \ldots, \underline{t}_n = \bigsqcup T_n$ and these finite terms can be denoted by program terms as just observed. Since all operations are continuous we have

$$\begin{aligned} f^{\mathrm{DT}_\varsigma(P)}(\vec{\underline{t}}) &= f^{\mathrm{DT}_\varsigma(P)}(\textstyle\bigsqcup T_1, \ldots, \bigsqcup T_n) \\ &= \textstyle\bigsqcup f^{\mathrm{DT}_\varsigma(P)}(T_1, \ldots, T_n) \quad = \quad \bigsqcup \{ [\![f(\vec{t})]\!]_{P,\varsigma} \mid \vec{t} \in (T_1, \ldots, T_n) \} \end{aligned}$$

and the ς-data type is completely defined.

It should be noted that a similar reasoning using furthermore the fact that our programs are only of first order shows, that the ς-semantics is fully abstract in the sense of Definition 2.1 of [CF92].

We noticed already w.r.t. the cbn-semantics in Subsection 5.4 that since the reduction semantics is defined as least upper bound of semantic approximations, data terms are possibly only approximated to arbitrary precision, but never actually reached as final results. Obviously this is unavoidable for infinite data terms and many partial data terms, but we do expect an operational semantics to compute all finite, total data terms.

Lemma 8.1 Computability of the ς-reduction semantics
Let $t \in \mathrm{T}_\Sigma$.

- If $[\![t]\!]^{\mathrm{red}}_{P,\varsigma} = \underline{t} \in \mathrm{T}_\mathcal{C}$, then $t{\downarrow}_{P,\varsigma}$ exists and $[\![t]\!]^{\mathrm{red}}_{P,\varsigma} = \underline{t} = t{\downarrow}_{P,\varsigma}$.
- If $[\![t]\!]^{\mathrm{po}}_{P,\varsigma} = \underline{t} \in \mathrm{T}_\mathcal{C}$, then $t{\downarrow}_{P,\mathrm{po},\varsigma}$ exists and $[\![t]\!]^{\mathrm{po}}_{P,\varsigma} = \underline{t} = t{\downarrow}_{P,\mathrm{po},\varsigma}$.

Proof:

- Let $\underline{t} = [\![t]\!]^{\text{red}}_{P,\varsigma} = \bigsqcup\{[\![t']\!]^{\text{alg}}_{\perp_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\} \in \mathrm{T}_{\mathcal{C}}$. Since $\langle \mathrm{T}_{\mathcal{C},\varsigma}, \trianglelefteq\rangle$ is ω-inductive and $\underline{t}$ is ω-compact, there exists $[\![\hat{t}]\!]^{\text{alg}}_{\perp_\varsigma} \in \{[\![t']\!]^{\text{alg}}_{\perp_\varsigma} \mid t \xrightarrow[P,\varsigma]{*} t'\}$ with $\underline{t} = [\![\hat{t}]\!]^{\text{alg}}_{\perp_\varsigma}$. Then $\underline{t} \in \mathrm{T}_{\mathcal{C}}$ implies $\hat{t} = \underline{t}$. Moreover $t \xrightarrow[P,\varsigma]{*} \hat{t} = \underline{t}$ and hence $\underline{t}$ is the ς-normal form of t.

- The proof for the po-ς-reduction semantics is analogous.

$\square$

As an aside we remark that together with the equality of the general and the po-ς-reduction semantics the equality of the ς-normal form and the po-ς-normal form of a term follows.

9 Applications of ς-Semantics

Not only do ς-semantics provide a uniform framework for cbv- and cbn-semantics, but also the mixed strictnesses permit a natural description of real functional programming languages which rarely have a pure cbv- or cbn-semantics.

The functional programming language Hope with its strict functions and non-strict constructors has already been mentioned.

All basically strict languages like (the functional subsets of) ML and Scheme have a predefined set of non-strict functions: a form of conditional (`if`) is necessary to enable universal computability, because pattern matching is not primitive, and also others like sequential logical `and` and `or` are provided. At least non-strict lists (streams) exist to permit purely functional input and output and often further non-strict constructors are provided for arbitrary infinite data structures.

For non-strict functional programming languages strictness is an issue, because cbv-evaluation proves to be more efficient than cbn-evaluation and also facilitates parallel implementations ([FH88], Chapter 13 in [PvE95]). Therefore, many modern non-strict functional programming languages provide various kinds of strictness annotations which may be generated automatically by a strictness analysis or supplied by the user ([HJW+92],[MH+96],[PvE95]). These annotations may be for arguments and/or results of functions and/or constructors in type declarations and/or at individual application points.

Strictness annotations at argument positions in type declarations coincide with our forced strictness. Hence one may prove the equivalence of the cbn-semantics and a particular ς-semantics for a particular program and then use a ς-reduction strategy for a correct implementation.

Strictness annotations at individual application points are useful when information about contexts is included. E.g. in the program

```
even(Zero)             →   Succ(Zero)
even(Succ(Zero))       →   Zero
even(Succ(Succ(x)))    →   even(x)
```

$\texttt{even}^{\mathrm{DT}_{\mathrm{cbn}}(P)}(\underline{t})$ is only unequal $\perp$ for total terms $\underline{t} \in \mathrm{T}_{\mathcal{C}}$. Notions like structure strictness (our strictness), spine strictness, and normalization strictness describe this kind of dependency on the (non-)partiality of arguments. This information can be transferred to strictness annotations at individual applications of constructors, e.g. in `even(Succ(a))` the position of `Succ` may be considered strict in its argument.

This kind of annotation is not expressible by ς-semantics. It is not the aim of this paper but would even contradict a uniform presentation of cbv- and cbn-semantics, since varying forced strictnesses of constructors would require $\mathrm{T}^{\infty}_{\mathcal{C},\perp}$ as carrier set of the data type, excluding the true cbv-semantics. However, continuing in this direction is straightforward.

10 Efficient ς-Reduction Strategies

The po-ς-reduction strategy is too costly to be used in actual implementations. Sharing common subterms yields an improvement. Even in the case of cbv-semantics it saves space and time of copying complex data structures. However, since this technique is orthogonal to and at a different level than the concept of reduction strategy, it is not discussed here.

Obviously a still complete ς-reduction strategy does not need to reduce all ς-redexes which are reduced by the po-ς-reduction strategy. Many approximations of a subset of needed ς-redexes will spring to the reader's mind. The concept of gaining predicates enables easy proofs of completeness of such ς-reduction strategies.

Restricting the set of admissible patterns of a program can lead to considerably simpler ς-reduction strategies. In Berry's example ([Ber78])

```
h(True, False, x)  →  True
h(False, x, True)  →  True
h(x, True, False)  →  True
```

all three arguments in the term $\texttt{h}(t_1\texttt{,}\ t_2\texttt{,}\ t_3\texttt{)}$ have to be reduced in parallel (alternately)[12], since the term's semantics may not be $\perp$ even if any one argument denotes $\perp$. Hence, one may only permit sequential sets of patterns (cf. strong sequentiality in [HL91]), that is those which are translatable into explicit tests; e.g.:

$$\texttt{add(x, y)} \;\rightarrow\; \texttt{case}_{\texttt{Zero},\texttt{Succ}}\texttt{(y, x, Succ(add(x, sel}_{\texttt{Succ},1}\texttt{(y))))}$$

The semantics of $\texttt{case}_{C_1,\ldots,C_n}$ and $\texttt{sel}_{C_j,i}$ is given by

$$\begin{array}{lcl} \texttt{case}_{C_1,\ldots,C_n}\texttt{(}C_j(\vec{t}), t_1, \ldots, t_n\texttt{)} & \rightarrow & t_j \\ \texttt{sel}_{C_j,i}\texttt{(}C_j\texttt{(}\vec{t}\texttt{))} & \rightarrow & t_i \end{array}$$

[12]Surprisingly, a complete sequential reduction strategy exists nonetheless ([AM94]). It is practically useless due to its inefficiency though.

with $\mathcal{C} = \{C_1, \ldots, C_n\}$, $C_j^{(m)} \in \mathcal{C}$, $i \in [m]$.

This transfers some complexity from the reduction strategy to the case expressions which replace the patterns. Locating a ς-redex in a term is still not trivial though, due to the possible forced strictness of functions and constructors. Moreover, while patterns form a natural part of the ς-semantics, `case` is a special function since it needs to be non-strict in all its arguments but the first, even in the cbv-semantics.

As an aside should be noted that for programs with tests instead of patterns the lo-cbn-reduction strategy is normalizing ([O'D77], [BK86]) but not complete: With `undef` $\rightarrow$ `undef` reducing `[A, B, undef, C]` does not yield its cbn-semantics `[A, B,` $\perp$`, C]` but `A:B:`$\perp$. However, this is exactly the result any interpreter of a modern non-strict functional programming language gives. The lo-reduction strategy becomes complete, when we define the list constructor `Cons (:)` of the output list[13] to be strict in its first argument.

11 Conclusion

In this paper we defined and studied the ς-semantics, a comprehensive semantics for functional constructor-based programs with patterns.

By introducing the forced strictness ς as parameter we obtained a single definition for both cbv- and cbn-semantics. Thus the true common and distinguishing features of these two standard semantics were highlighted. We established that the usual operational definition by innermost, respectively outermost reduction is rather deceptive. Instead, a forced strictness ς fixes an instance of the program characterized by a set of ς-redexes. Reductions in this instance are sound w.r.t. the ς-semantics and therefore serve as basis for operational ς-semantics.

We defined a denotational ς-semantics which assigns a data type to a program in a modular way and moreover ensures the compositionality of the ς-semantics. We defined two operational ς-semantics. The general ς-reduction semantics is elementary and the means by which the completeness of arbitrary Π-fair reduction semantics was proved. This result was applied to the po-ς-semantics. Furthermore, it provides a tool for proving the soundness and completeness of other, future ς-reduction semantics. In general, we are able to validate a property for all instances of the ς-semantics by a single proof.

We saw that, although implementations may translate patterns into explicit tests, the ς-semantics is naturally defined on the basis of patterns. Finally, the ς-semantics gives a simple formal foundation for the prevailing mixed strictnesses of modern functional programming languages.

We did not give a declarative ς-semantics, that is a semantics which views reduction rules as equations and uses models, in which all equations are valid, as data types. The reason is that generally ς-data types are not models, due to the forced strictness ς which does not appear in the program. Only the cbn-data type is a model. That one can even be uniquely characterized as the minimal interpretation which is a model ($\mathrm{DT}_{P,\mathrm{cbn}}(P) =$

[13]Any output of a real-world functional program is either a list of characters (string) or a list of commands for the operating system.

$\text{Least}_{\langle \text{Int}_{\Sigma,\text{cbn}}, \sqsubseteq \rangle}(\text{Int}_{\Sigma,\text{cbn}} \cap \text{Mod}(P)))$. For the cbv-data type an analogous characterization exists, if partial algebras (with true partial operations) are used instead of $\text{Alg}^{\infty}_{\Sigma,\perp}$. Nonetheless, a simple declarative ς-semantics seems not to be attainable ([Chi95]).

We considered only first-order functional programs without sorts or types. Thus the essential points were well exposed. Higher-order functional programs require more complicated semantic domains than simple algebras, and the introduction of a type system, at least to distinguish base data elements formed by constructors and functions of different orders, would be unavoidable. The concept of ς-semantics should carry over to higher-order functional programs without principal difficulties. Nonetheless, it may be worthwhile to put this into effect, since new issues may arise and further insights may be gained.

Formal semantics are not only the prerequisite for the implementation of correct interpreters and compilers and the verification of programs, but also lead us to a continuously improving understanding of the nature of programming languages.

Acknowledgement. The author is especially indebted to Thomas Noll for many fruitful discussions.

References

[AM94] S. Antoy and A. Middeldorp. A sequential reduction strategy. In *Proceedings of the 4th International Conference on Algebraic and Logic Programming*, LNCS 850, pages 168–185, Madrid, September 1994.

[Ber78] G. Berry. Stable models of typed λ-calculi. In *Proceedings of the 5th ICALP Conference*, LNCS 62, pages 73–89, Udine, Italy, 1978.

[BK86] J.A. Bergstra and J.W. Klop. Conditional rewrite rules: Confluence and termination. *Journal of Computer and System Sciences*, 32:323–362, 1986.

[BL77] G. Berry and J.-J. Lévy. Minimal and optimal computations of recursive programs. In *Fourth ACM Symposium on Principles of Programming Languages*, pages 215–226, 1977.

[CF92] R. Cartwright and M. Felleisen. Observable sequentiality and full abstraction. In *Conference Record of the 19th Annual ACM Symposium on Principles of Programming Languages, POPL*, pages 328–342, 1992.

[Chi95] O. Chitil. Denotationelle und operationelle Semantiken für konstruktorbasierte funktionale Programmiersprachen erster Ordnung. Master's thesis, Aachen University of Technology, 1995.

[Cou90] B. Courcelle. Recursive applicative program schemes. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, pages 459–492. Elsevier, Amsterdam, 1990.

[DJ90] N. Dershowitz and J.-P. Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, pages 243–320. Elsevier, Amsterdam, 1990.

[DS76] P.J. Downey and R. Sethi. Correct computation rules for recursive languages. *SIAM J. Comput.*, 5(3):378–401, September 1976.

[FH88] A.F. Field and P.G. Harrison. *Functional Programming.* Addison-Wesley, 1988.

[GTWW77] J.A. Goguen, J.W. Thatcher, E.G. Wagner, and J.B. Wright. Initial algebra semantics and continuous algebras. *JACM*, 24(1):68–95, January 1977.

[HJW$^+$92] P. Hudak, S. L. Peyton Jones, Ph. Wadler, et al. Report on the programming language Haskell, version 1.2. SIGPLAN Notices 27(5), May 1992.

[HL91] G. Huet and J.-J. Lévy. Computations in orthogonal rewriting systems I + II. In J.-L. Lassez and G. Plotkin, editors, *Computational Logic – Essays in Honor of Alan Robinson*, pages 395–443. MIT Press, 1991. Original version: Call by Need Computations in Non-Ambiguous Term Rewriting Systems, Report 359, INRIA, 1979.

[HO80] G. Huet and D.C. Oppen. Equations and rewrite rules: A survey. In R. Book, editor, *Formal Language Theory: Perspectives and Open Problems*, pages 349–405. Academic Press, 1980.

[Hue80] G. Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *JACM*, 27(4):797–821, 1980.

[Klo92] J.W. Klop. Term rewriting systems. In S. Abramsky, Dov. M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, pages 1–116. Oxford University Press, 1992.

[LS87] J. Loeckx and K. Sieber. *The Foundations of Program Verification, Part B.* Wiley–Teubner, 1987.

[Man74] Z. Manna. *Mathematical Theory of Computation*, chapter 5. MCGraw Hill, 1974.

[MH$^+$96] B.J. Moose, K. Hammond, et al. Report on the programming language Haskell, version 1.3. Technical Report YALEU/DCS/RR-1106, Yale University, 1996. URL: http://haskell.cs.yale.edu/haskell-report/haskell-report.html.

[MTH90] R. Milner, M. Tofte, and R. Harper. *The Definition of Standard ML.* MIT, 1990.

[Niv75] M. Nivat. On the interpretation of recursive polyadic program schemes. In *Symposia Mathematica 15 – Convegni del Febbraio e dell' Aprile del 1973*, pages 255–281, Rom, 1975.

[O'D77] M.J. O'Donnell. *Computing in Systems Described by Equations.* LNCS 58. Springer, 1977.

[PvE95] R. Plasmeijer and M. van Eekelen. Concurrent Clean language report, version 1.0, April 1995.

[Ros73] B.K. Rosen. Tree-manipulating systems and Church-Rosser theorems. *JACM*, 20:160–187, 1973.

[Tho89] S.J. Thompson. A logic for Miranda. *Formal Aspects of Computing*, 1:339–365, 1989.

[Tho95] S.J. Thompson. A logic for Miranda, revisited. *Formal Aspects of Computing*, 7:412–429, 1995.

[Vui74a] J. Vuillemin. Correct and optimal implementations of recursion in a simple programming language. *Journal of Computer and System Sciences*, 9:332–354, 1974.

[Vui74b] J. Vuillemin. *Syntax, sémantique et axiomatique d'un langage de programmation simple.* Thèse, Université Paris VI, 1974.

[Wec92] W. Wechler. *Universal Algebra for Computer Scientists.* Springer, 1992.

[Wir90] M. Wirsing. Algebraic specification. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science, Vol. B: Formal Models and Semantics*, pages 675–788. Elsevier, Amsterdam, 1990.

A Predicates $\Pi^{\mathrm{o}}_{P,\varsigma}$ and $\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ Are Gaining

Here we give the proofs which were omitted in Subsections 7.6 and 7.7.

As there, I is an arbitrary instance of an almost orthogonal TRS R over a signature Σ, so that $\mathrm{Red}_{R,I}$ is residually closed, $\langle A, \trianglelefteq \rangle$ is a cpo with $A \neq \emptyset$, and $[\![\cdot]\!]_\perp : \mathrm{T}_\Sigma \rightarrow A$ a mapping with $t \xrightarrow[R,I]{} t' \Longrightarrow [\![t]\!]_\perp \trianglelefteq [\![t']\!]_\perp$.

A.1 $\Pi^{\mathrm{o}}_{P,\varsigma}$ Is Gaining

We show that $\Pi^{\mathrm{o}}_{R,I}$ has property I and $\Pi^{\mathrm{o}}_{P,\varsigma}$ has property II of gaining redexes. Property III is already proved for $\Pi^{\mathrm{o}}_{P,\varsigma}$ by Lemma 6.13 about gain of information.

<u>Lemma A.1</u> **Preservation of an outermost redex-occurrence**
$\Pi^{\mathrm{o}}_{R,I}$ has property I of gaining redexes.

Proof: (Cf. Example 7.4 (ii) in [BK86])
Let $(v] := \{v' \in \mathbb{N}^*_+ \mid v' \leq v\}$ for arbitrary $v \in \mathbb{N}^*_+$. We use the same identifiers as in definition 7.3.

a) Show: $(w_1] \cap V_2 = \emptyset$.

Suppose $w'_1 \in (w_1] \cap V_2$. From $w_1 \leq w_1$, $w'_1 \in u_2 \setminus t_0 \xrightarrow{u_1} t_1$ and $w_1 \in w_0 \setminus t_0 \xrightarrow{u_1} t_1$ the definition of residuals gives us $u_2 \leq w_0$. However, $w_0 = u_2$ contradicts $w_1 \in w_0 \setminus t_0 \xrightarrow{u_2} t_2$ and $u_2 < w_0$ contradicts $\Pi^{\mathrm{o}}_{R,I}(w_0, t_0)$.

b) Show: $\Pi^{\mathrm{o}}_{R,I}(w_1, t_1)$.

Suppose $\neg\Pi^{\mathrm{o}}_{R,I}(w_1, t_1)$. Then exists w'_1 with $\Pi^{\mathrm{o}}_{R,I}(w'_1, t_1)$ and $w'_1 < w_1$. Since $(w'_1] \subseteq (w_1]$ property a) gives us $(w'_1] \cap V_2 = \emptyset$. This implies $w'_1 \setminus t_1 \xrightarrow{V_2}\!\!\!\!\rightarrow t_3 = \{w'_1\} \in \mathrm{RedPos}_{R,I}(t_3)$. Since $w'_1 < w_1$ and by a) we know that $\{w_3\} = w_1 \setminus t_1 \xrightarrow{V_2}\!\!\!\!\rightarrow t_3 = \{w_1\}$, we get $\Pi^{\mathrm{o}}_{R,I}(w_3, t_3)$ in contradiction to the assumptions.

□

Verifying property II of gaining predicates takes more effort.

Definition A.1 (Cf. def. 32 in [O'D77])
$\mathrm{Red}_{R,I}$ is **outer** iff

$$\begin{array}{l} t \xrightarrow[R,I]{w} t' \text{ is a reduction,} \\ u < v < w, v \in \mathrm{RedPos}_{R,I}(t), u \in \mathrm{RedPos}_{R,I}(t') \end{array} \quad \Longrightarrow \quad u \in \mathrm{RedPos}_{R,I}(t).$$

□

Lemma A.2 (Lemma 14 in [O'D77])
If $\mathrm{Red}_{R,I}$ is outer, then $\Pi^{\mathrm{o}}_{R,I}$ satisfies property II of gaining redexes.
Proof:
Let $t \xrightarrow[R,I]{u} t'$ be a reduction with $\neg\Pi^{\mathrm{o}}_{R,I}(u, t)$ and $\Pi^{\mathrm{o}}_{R,I}(v', t')$.

a) Show: There is no $w < v'$ with $w \in \mathrm{RedPos}_{R,I}(t)$.

Suppose such a w exists. Without loss of generality $\Pi^{\mathrm{o}}_{R,I}(w, t)$. Then $w \setminus t \xrightarrow{u} t' = \{w\} \subseteq \mathrm{RedPos}_{R,I}(t')$ in contradiction to $\Pi^{\mathrm{o}}_{R,I}(v', t')$.

b) Show: $v' \in \mathrm{RedPos}_{R,I}(t)$.

$u < v'$ contradicts a) and $u = v'$ together with a) contradicts $\neg\Pi^{\mathrm{o}}_{R,I}(u, t)$, leaving:

$v' \parallel u$: Then $t/v' = t'/v'$ and therefore $v' \in \mathrm{RedPos}_{R,I}(t)$.

$v' < u$: Since $\neg\Pi^{\mathrm{o}}_{R,I}(u, t)$ there is a $w \in \mathrm{RedPos}_{R,I}(t)$ with $w < u$, and even $v' \leq w < u$ according to a). The outer property finally gives $v' \in \mathrm{RedPos}_{R,I}(t)$.

Together a) and b) imply $\Pi^{\mathrm{o}}_{R,I}(v, t)$ and $v \setminus t \xrightarrow[R,I]{u} t' = \{v'\}$ for $v := v'$. □

It only remains to show that our redex sets are outer.

Lemma A.3 ([O'D77])
The redex set Red_R is outer.

Proof:
Let $t \xrightarrow[l\longrightarrow r]{w} t'$ be a reduction, $u < v < w$, $v \in \mathrm{RedPos}_R(t)$, and $u \in \mathrm{RedPos}_R(t')$. Hence exist $v', w' \in \mathbb{N}_+^*$ with $v = u.v'$ and $w = v.w' = u.v'.w'$. The reduction implies the existence of a substitution σ with

$$t/u = t'/u[v'.w' \leftarrow l\sigma] \qquad (1)$$
$$t'/u = t/u[v'.w' \leftarrow r\sigma]$$

Since $u \in \mathrm{RedPos}_R(t')$, there exists a redex scheme $\hat{l} \in \mathrm{RedS}_R$ and a substitution $[t_1/x_1, \ldots, t_n/x_n] : \mathrm{Var}(\hat{l}) \rightarrow \mathrm{T}_\Sigma$ with

$$t'/u = \hat{l}[t_1/x_1, \ldots, t_n/x_n] \qquad (2)$$

Since Red_R is residually closed and $v \setminus t \xrightarrow[l\longrightarrow r]{w} t' = \{v\}$, we have $v \in \mathrm{RedPos}_R(t')$. According to Lemma 2.1 there exist $v_1, v_2 \in \mathbb{N}_+^*$ with $v = u.v_1.v_2$ and $\hat{l}/v_1 \in X$, i.e. $\hat{l}/v_1 = x_k$ for a $k \in [n]$.

Altogether we obtain

$$\begin{array}{rcl} t/u & \overset{(1)}{=} & t'/u[v'.w' \leftarrow l\sigma] \\ & \overset{(2)}{=} & (\hat{l}[t_1/x_1, \ldots, t_n/x_n])[v'.w' \leftarrow l\sigma] \\ & = & \hat{l}[t_1/x_1, \ldots, t_k[v_2.w' \leftarrow l\sigma]/x_k, \ldots, t_n/x_n]. \end{array}$$

Hence $t/u \in \mathrm{Red}_R$ and therefore $u \in \mathrm{RedPos}_R(t)$. □

Lemma A.4
The redex set $\mathrm{Red}_{P,\varsigma}$ is outer.

Proof:
Let $t \xrightarrow[l\longrightarrow r,\varsigma]{w} t'$ be a ς-reduction, $u < v < w$, $v \in \mathrm{RedPos}_{P,\varsigma}(t)$, and $u \in \mathrm{RedPos}_{P,\varsigma}(t')$. Hence exist $v', w' \in \mathbb{N}_+^*$ and $k \in \mathbb{N}_+$ with $v = u.k.v'$ and $w = v.w' = u.k.v'.w'$. Since $\mathrm{Red}_{P,\varsigma} \subseteq \mathrm{Red}_P$, the previous lemma gives us $u \in \mathrm{RedPos}_P(t)$, that is

$$\begin{array}{rcl} t/u & = & f(t_1, \ldots, t_n) \\ t'/u & = & f(t'_1, \ldots, t'_n) \end{array}$$

for an $f^{(n)} \in \mathcal{F}$ and some $t_1, \ldots, t_n, t'_1, \ldots, t'_n \in \mathrm{T}_\Sigma$ with

$$t_i = t'_i \qquad (1)$$

for all $i \in [n]$ with $i \neq k$, and

$$t_k \xrightarrow[l\longrightarrow r,\varsigma]{v'.w'} t'_k.$$

Since $v' \in \mathrm{RedPos}_{P,\varsigma}(t)$, we have $t_k \xrightarrow[P,\varsigma,\mathrm{no}]{v'.w'} t'_k$. Then by Lemma 6.13 about gain of information

$$[\![t_k]\!]^{\mathrm{alg}}_{\perp_\varsigma} = [\![t'_k]\!]^{\mathrm{alg}}_{\perp_\varsigma}. \tag{2}$$

Since $u \in \mathrm{RedPos}_{P,\varsigma}(t')$, the function symbol f is not forcedly strict for $([\![t'_1]\!]^{\mathrm{alg}}_{\perp_\varsigma}, \ldots, [\![t'_n]\!]^{\mathrm{alg}}_{\perp_\varsigma})$, and $([\![t'_1]\!]^{\mathrm{alg}}_{\perp_\varsigma}, \ldots, [\![t'_n]\!]^{\mathrm{alg}}_{\perp_\varsigma})$ is semantically ς-matchable with the pattern $\vec{p}$ of a redex scheme $f(\vec{p}) \in \mathrm{RedS}_P$. Due to (1) and (2) this is valid for $([\![t_1]\!]^{\mathrm{alg}}_{\perp_\varsigma}, \ldots, [\![t_n]\!]^{\mathrm{alg}}_{\perp_\varsigma})$ as well. Hence $u \in \mathrm{RedPos}_{P,\varsigma}(t)$.

□

A.2 $\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ Is Gaining

We show that $\Pi^{\mathrm{lo}}_{R,I}$ has property I and $\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ has property II and property III of gaining redexes.

Lemma A.5 Preservation of an lo-redex occurrence
$\Pi^{\mathrm{lo}}_{R,I}$ has property I of gaining redexes.
Proof:
Using the identifiers of definition 7.3 we have $\Pi^{\mathrm{o}}_{R,I}(w_1, t_1)$ due to Lemma A.1. Let w'_1 be the lo-I-redex position of t_1 and suppose that $w_1 \neq w'_1$. Analogously to the proof of Lemma A.1 we conclude $w'_1 \in \mathrm{RedPos}_{R,I}(t_3)$ with $w'_1 <_{\mathrm{lex}} w_1 = w_3$ in contradiction to the assumption $\Pi^{\mathrm{lo}}_{R,I}(w_3, t_3)$. □

Lemma A.6
$\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ has property II of gaining redexes.
Proof:
Let $t \xrightarrow[P,\mathrm{cbv}]{u} t'$ be a reduction with $\neg\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}(u, t)$. Let v and v' be the lo-cbv-redex positions of t and t', respectively. Obviously $t \xrightarrow[P,\mathrm{cbv}]{u} t' = \{v\} \subseteq \mathrm{RedPos}_{P,\mathrm{cbv}}(t')$. To show that $v = v'$ suppose:

$v <_{\mathrm{lex}} v'$: Contradicts v' being lo in t'.

$v' <_{\mathrm{lex}} v$:

- $v' < v$: Contradicts v' being a cbv-redex, because cbv-redexes never contain any redexes as proper subterms.
- $v' \parallel v$: $v' > u$: Together with $v' <_{\mathrm{lex}} v$ this implies $u <_{\mathrm{lex}} v$ in contradiction to v being lo in t.
  - $v' \parallel u$: Because of $t/v' = t'/v'$ we have $v' \in \mathrm{RedPos}_{P,\mathrm{cbv}}(t)$. Together with $v' <_{\mathrm{lex}} v$ this contradicts v being lo in t.
  - $v' < u$: All three ordering assumptions together imply $u <_{\mathrm{lex}} v$ in contradiction to v being lo in t.

□

Lemma A.7

$\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}$ has property III of gaining redexes.

Proof:

Let o be the lo-cbv-redex position of t. $t \xrightarrow[P,\mathrm{cbv}]{u} t'$ with $\neg\Pi^{\mathrm{lo}}_{P,\mathrm{cbv}}(u,t)$ implies $o := \mathrm{LOuter}_{P,\mathrm{cbv}}(t) \neq u$ so that $o \setminus t \xrightarrow[P,\mathrm{cbv}]{u} t' = \{o\} \subseteq \mathrm{RedPos}_{P,\mathrm{cbv}}(t')$. Consequently t and t' still contain redexes and we have $[\![t]\!]^{\mathrm{alg}}_{\perp_{\mathrm{cbv}}} = \perp = [\![t']\!]^{\mathrm{alg}}_{\perp_{\mathrm{cbv}}}$. $\square$