

ROTA-BAXTER OPERATORS ON THE POLYNOMIAL ALGEBRAS, INTEGRATION AND AVERAGING OPERATORS

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ABSTRACT. Rota-Baxter operators are an algebraic abstraction of integration. Following this classical connection, we study the relationship between Rota-Baxter operators and integrals in the case of the polynomial algebra $\mathbf{k}[x]$. We consider two classes of Rota-Baxter operators, monomial ones and injective ones. For the first class, we apply averaging operators to determine monomial Rota-Baxter operators. For the second class, we make use of the double product on Rota-Baxter algebras.

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1. INTRODUCTION

Rota-Baxter operators are deeply rooted in analysis. Their study originated from the work of G. Baxter [6] in 1960 on Spitzer's identity [28] in fluctuation theory. More fundamentally, the notion of Rota-Baxter operator is an algebraic abstraction of the *integration by parts formula* of calculus. Throughout the 1960s, Rota-Baxter operators were studied by well-known analysts such as Atkinson [2]. In the 1960s and 1970s, the works of Rota and Cartier [8, 24] led the study of Rota-Baxter operators into algebra and combinatorics. In the 1980s, the Rota-Baxter operator for Lie algebras was independently discovered by mathematical physicists as the operator form of the classical Yang-Baxter equation [27]. In the late 1990s, the operator appeared again as a fundamental algebraic structure in the work of Connes and Kreimer on renormalization of quantum field theory [10]. The present century witnesses a remarkable renaissance of Rota-Baxter operators through systematic algebraic studies with wide applications to combinatorics, number theory, operads and mathematical physics [1, 3, 5, 4, 10, 11, 12, 16, 20]. See [14] for a brief introduction and [15] for a more detailed treatment.

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Recently, Rota-Baxter operator related structures, including differential Rota-Baxter algebras [17] and *integro-differential algebras* [23], were introduced in the algebraic study of calculus, especially in boundary problems for linear differential equations [13, 19]. The upshot is that the Green's operator of such a boundary problem can be represented by suitable operator rings based on an integro-differential algebra.

In this paper, we revisit the analysis origin of Rota-Baxter operators to study how their algebraic properties are linked with their analytic appearance. We focus on the polynomial algebra $\mathbb{R}[x]$, which plays a central role both in analysis where it is taken as approximation of analytic functions, and in algebra where it is the free object in the category of commutative algebras. This algebra, together with the standard integral operator, is also the free commutative Rota-Baxter algebra on the empty set or, in other words, the initial object in the category of commutative Rota-Baxter algebras. Thus it provides an ideal testing ground for the interaction between analytically defined Rota-Baxter operators and the algebraically defined Rota-Baxter operators.

One natural question in this regard is when an algebraically defined Rota-Baxter operator on $\mathbb{R}[x]$ can be realized in analysis. It is a classical fact that the Riemann integral with variable upper limit is a Rota-Baxter operator of weight zero on $\mathbb{R}[x]$. This remains true when the integral operator is pre-multiplied by any polynomial. We might call these Rota-Baxter operators on $\mathbb{R}[x]$ *analytically modelled*. It is easy to see that such operators are injective. We conjecture that all injective Rota-Baxter operators on $\mathbb{R}[x]$ are indeed analytically modelled. We provide evidence for this conjecture by exploring two classes of such operators.

The first comprises what we call *monomial Rota-Baxter operators* over an arbitrary integral domain \mathbf{k} of characteristic zero, meaning Rota-Baxter operators P with $P(x^n) = ax^k$, where both $a \in \mathbf{k}$ and $k \in \mathbb{N}$ may depend on n . We classify monomial Rota-Baxter operators on $\mathbf{k}[x]$ and show that all injective monomial Rota-Baxter operators are analytically modelled. The second class is restricted to $\mathbf{k} = \mathbb{R}$ and contains those operators that satisfy a *differential law* $\partial \circ P = r$, where the right-hand side denotes the multiplication operator induced by an arbitrarily monomial $r \in \mathbb{R}[x]$. We show that any injective Rota-Baxter operator is of this form and, provided r is monomial, analytically modelled.

In Section 2 we discuss general algebraic properties of Rota-Baxter operators that will be used in subsequent sections. In Section 3 we focus on monomial Rota-Baxter operators. While determining these operators, we prove that all injective monomial Rota-Baxter operators are analytically modelled. In Section 4, we study injective Rota-Baxter operators in general (on the real polynomial ring). We first show that injective Rota-Baxter operators are precisely those that satisfy a differential law. Then we prove that, in the monomial case, they are analytically modelled.

2. GENERAL CONCEPTS AND PROPERTIES

Notation. If M is a monoid we write $M^\times = \{x \in M \mid x \neq 0_M\}$ for the semigroup of nonzero elements. In particular, the monoid of natural numbers (nonnegative integers) is denoted by \mathbb{N} , so \mathbb{N}^\times is the semigroup of positive integers. The notation $l \mid k$ signifies that l is a divisor of k .

We use \mathbf{k} to denote a commutative ring with identity 1 unless otherwise specified. All \mathbf{k} -algebras in this paper are assumed to be commutative and with a unit 1_A that will be identified with $1_{\mathbf{k}}$ through the structure map $\mathbf{k} \rightarrow A$.

We start by collecting some general properties of Rota-Baxter operators for later use. First we give the definition of a Rota-Baxter \mathbf{k} -algebra of arbitrary weight [6, 15, 25].

Definition 2.1. Let λ be a given element of \mathbf{k} . A **Rota-Baxter \mathbf{k} -algebra of weight λ** , or simply an **RBA of weight λ** , is a pair (R, P) consisting of a \mathbf{k} -algebra R and a linear operator $P: R \rightarrow R$ that satisfies the **Rota-Baxter equation**

$$(1) \quad P(u)P(v) = P(uP(v)) + P(P(u)v) + \lambda P(uv), \quad \text{for all } u, v \in R.$$

Then P is called a **Rota-Baxter operator of weight λ** . If R is only assumed to be a nonunitary \mathbf{k} -algebra, we call R a nonunitary Rota-Baxter \mathbf{k} -algebra of weight λ .

Observe first that the standard integration operator $J_0: \mathbf{k}[x] \rightarrow \mathbf{k}[x]$, given by $x^n \mapsto x^{n+1}/(n+1)$, is a (prototypical) Rota-Baxter operator of weight 0. Of course the choice of initialization point is irrelevant, so for any $a \in \mathbf{k}$ there is another weight 0 Rota-Baxter operator $J_a: \mathbf{k}[x] \rightarrow \mathbf{k}[x]$, given by $x^n \mapsto (x^{n+1} - a^{n+1})/(n+1)$. In this paper we shall only be concerned with the weight 0 case, so from now on the term ‘‘Rota-Baxter operator’’ is to be understood as ‘‘Rota-Baxter operator of weight 0’’.

Recall that from a derivation δ on a commutative \mathbf{k} -algebra R one can produce a new derivation $r\delta$ by postmultiplying with any $r \in R$. Analogously, from a Rota-Baxter operator P on R one obtains a new Rota-Baxter operator Pr by *premultiplying* with any $r \in R$. Indeed, we have

$$(Pr)(u)(Pr)(v) = P(ru)P(rv) = P(ruP(rv)) + P(P(ru)rv) = (Pr)(u)(Pr)(v) + (Pr)((Pr)(u)v)$$

for any $u, v \in R$. Applying this to $R = \mathbf{k}[x]$, we obtain the family J_{ar} of *analytically modelled* Rota-Baxter operators on $\mathbf{k}[x]$, where $a \in \mathbf{k}$ and $r \in \mathbf{k}[x]$ are arbitrary. As we will show in Theorem 4.9, in the case of monomials r , this family exhausts the injective Rota-Baxter operators.

Let $\text{End}(R) := \text{End}_{\mathbf{k}}(R)$ denote the \mathbf{k} -module of linear operators on R . Then the subset $\text{RBO}(R)$ of $\text{End}(R)$ consisting of Rota-Baxter operators $P: R \rightarrow R$ is closed under *multiplications by scalars* $c \in \mathbf{k}$ since in that case $Pc = cP$. In the case of derivations on R more is true since they form a \mathbf{k} -module (in fact a Lie algebra) while in general the sum of two Rota-Baxter operators is not a Rota-Baxter operator. This motivates the following terminology.

Definition 2.2.

- (a) We call two Rota-Baxter operators $P_1, P_2 \in \text{RBO}(R)$ **compatible** if $c_1P_1 + c_2P_2$ are in $\text{RBO}(R)$ for all $c_1, c_2 \in \mathbf{k}$.
- (b) Let $P \in \text{RBO}(R)$. Then $Q \in \text{End}(R)$ is called **consistent** with P if $P - Q$ is in $\text{RBO}(R)$.
- (c) For $P, Q \in \text{End}(R)$ we define the bilinear form $RB(P, Q): R \otimes R \rightarrow R$ by

$$RB(P, Q)(u, v) := P(u)Q(v) - P(uQ(v)) - Q(P(u)v), \quad u, v \in R.$$

Thus $P \in \text{RBO}(R)$ means that $RB(P, P) = 0$ on $R \otimes R$.

Recall that for a Rota-Baxter algebra (R, P) , the multiplication

$$\star_P : R \otimes R \rightarrow R, \quad u \star_P v := P(u)v + uP(v) \text{ for all } u, v \in R,$$

is an associative product on R , called the **double multiplication** [15, Thm. 1.1.17]. Moreover, $P: (R, \star_P) \rightarrow R$ is then a homomorphism of nonunitary Rota-Baxter algebras.

If A is a \mathbf{k} -module, its (linear) *dual* is denoted by A^* . If A is moreover a \mathbf{k} -algebra, we use the notation

$$A^\bullet := \{\phi \in A^* \mid \phi(uv) = \phi(u)\phi(v)\}$$

for the set of *multiplicative functionals*. Through the structure map $\mathbf{k} \rightarrow A$ we may also view the elements of A^* as \mathbf{k} -linear operators from A to \mathbf{k} , and those of A^\bullet as \mathbf{k} -algebra homomorphisms from A to \mathbf{k} .

Proposition 2.3.

(a) Two Rota-Baxter operators $P_1, P_2 \in \text{RBO}(R)$ are compatible if and only if $RB(P_1, P_2) + RB(P_2, P_1) = 0$. This will be the case in particular when

$$P_1(u)P_2(v) = P_1(uP_2(v)) + P_2(P_1(u)v) \quad \text{and} \quad P_2(u)P_1(v) = P_2(uP_1(v)) + P_1(P_2(u)v)$$

holds for all $u, v \in R$.

(b) Let $P \in \text{RBO}(R)$ and $Q \in \text{End}(R)$ be given. Then Q is consistent with P if and only if

$$RB(Q, Q) = RB(P, Q) + RB(Q, P).$$

is satisfied.

(c) Let P be in $\text{RBO}(R)$. The set of $f \in R^*$ that are consistent with P equals $(R, \star_P)^\bullet$.

Proof. (a) For arbitrary $c_1, c_2 \in \mathbf{k}$, the bilinear form $RB(c_1P_1 + c_2P_2, c_1P_1 + c_2P_2)$ is given by

$$c_1^2 RB(P_1, P_1) + c_1c_2(RB(P_1, P_2) + RB(P_2, P_1)) + c_2^2 RB(P_2, P_2),$$

which simplifies to $c_1c_2(RB(P_1, P_2) + RB(P_2, P_1))$ since $P_1, P_2 \in \text{RBO}(R)$.

(b) Since $P \in \text{RBO}(R)$ we have

$$RB(P - Q, P - Q) = -RB(P, Q) - RB(Q, P) + RB(Q, Q),$$

and hence the conclusion.

(c) Using that P is a linear operator and f a linear functional, we have

$$RB(f, f) = -f(u)f(v), \quad RB(f, P)(u, v) = -f(uP(v)), \quad RB(P, f)(u, v) = -f(P(u)v).$$

Thus by Item (b) we conclude that f is consistent with P if and only if

$$f(u)f(v) = f(P(u)v + uP(v)) = f(u \star_P v),$$

which is what we need. □

3. MONOMIAL ROTA-BAXTER OPERATORS ON $\mathbf{k}[x]$

In this section, we determine the Rota-Baxter operators on $\mathbf{k}[x]$ that send monomials to monomials and determine the analytically modelled ones. Throughout this section, we assume that \mathbf{k} is an integral domain containing \mathbb{Q} .

3.1. General properties. We first give general criteria for a monomial linear operator to be a Rota-Baxter operator before specializing in the following sections to the two cases of nondegenerate and degenerate operators.

Definition 3.1.

(a) A linear operator P on $\mathbf{k}[x]$ is called **monomial** if for each $n \in \mathbb{N}$, we have

$$(2) \quad P(x^n) = \beta(n)x^{\theta(n)} \quad \text{with} \quad \beta: \mathbb{N} \rightarrow \mathbf{k} \quad \text{and} \quad \theta: \mathbb{N} \rightarrow \mathbb{N}.$$

If $\beta(n) = 0$, the value of $\theta(n)$ does not matter; by convention we set $\theta(n) = 0$ in this case.

(b) A monomial operator is called **degenerate** if $\beta(n) = 0$ for some $n \in \mathbb{N}$.

Let A be a nonempty set and let B be a set containing a distinguished element 0 . For a map $\phi: A \rightarrow B$ we define its **zero set** as $\mathcal{Z}_\phi := \{a \in A \mid \phi(a) = 0\}$ to be the **zero set** of ϕ . Then and its **support** as $\mathcal{S}_\phi := A \setminus \mathcal{Z}_\phi$. Thus a monomial linear operator P on $\mathbf{k}[x]$ is nondegenerate if and only if $\mathcal{Z}_\beta = \emptyset$. As the following lemma shows, for a Rota-Baxter operator P , degeneracy at $n \in \mathbb{N}$ occurs whenever P is constant on the corresponding monomial.

Lemma 3.2. *Let P be a monomial Rota-Baxter operator on $\mathbf{k}[x]$ and let $n \in \mathbb{N}$. If $P(x^n)$ is in \mathbf{k} , then $P(x^n) = 0$. In other words, $\mathcal{S}_\beta = \mathcal{S}_\theta$, and hence $\mathcal{Z}_\beta = \mathcal{Z}_\theta$.*

Proof. If $P(x^n) = c$ is a nonzero constant, we have

$$P(x^n)P(x^n) = c^2 \neq 2c^2 = 2P(x^n P(x^n)).$$

Hence P is not a Rota-Baxter operator, and we must have $c = 0$. \square

Theorem 3.3. *Let P be a monomial linear operator on $\mathbf{k}[x]$ defined by $P(x^n) = \beta(n)x^{\theta(n)}$, $n \in \mathbb{N}$. Then P is a Rota-Baxter operator if θ and β satisfy the following conditions*

(a) $\mathcal{Z}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{Z}_\beta$;

(b) *We have*

$$(3) \quad \theta(m) + \theta(n) = \theta(m + \theta(n)) = \theta(\theta(m) + n),$$

$$(4) \quad \beta(m)\beta(n) = \beta(m + \theta(n))\beta(n) + \beta(n + \theta(m))\beta(m),$$

for all $m, n \in \mathcal{S}_\beta$.

Under the assumption that $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$, if P is a Rota-Baxter operator then the above conditions hold.

Proof. Since P is a monomial linear operator on $\mathbf{k}[x]$, the Rota-Baxter relation in Eq. (1) is equivalent to

$$(5) \quad \beta(m)\beta(n)x^{\theta(m)+\theta(n)} = \beta(m + \theta(n))\beta(n)x^{\theta(m+\theta(n))} + \beta(\theta(m) + n)\beta(m)x^{\theta(\theta(m)+n)}, \quad \text{for all } m, n \in \mathbb{N}.$$

Suppose (a) and (b) hold. Since \mathbb{N} is the disjoint union of \mathcal{Z}_β and \mathcal{S}_β , we can verify Eq. (5) by considering the following four cases:

$$m, n \in \mathcal{Z}_\beta; \quad m \in \mathcal{Z}_\beta, n \in \mathcal{S}_\beta; \quad m \in \mathcal{S}_\beta, n \in \mathcal{Z}_\beta; \quad m, n \in \mathcal{S}_\beta.$$

In the first case we have $\beta(m) = \beta(n) = 0$. Thus Eq. (5) holds. In the second case, we have $\beta(m) = 0$ and so Eq. (5) becomes $\beta(m + \theta(n))\beta(n) = 0$. Then Eq. (5) follows from Item (a). The third case can be treated similarly. In the last case, Eq. (5) follows from Eqs. (3) and (4). Thus P is a Rota-Baxter operator on $\mathbf{k}[x]$.

Now assume that $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$ and suppose that P is a Rota-Baxter operator. Then Eq. (5) holds. Taking $m \in \mathcal{Z}_\beta$ and $n \in \mathcal{S}_\beta$, we obtain $0 = \beta(m + \theta(n))\beta(n)x^{\theta(m+\theta(n))}$. Since $\beta(n) \neq 0$, we must have $\beta(m + \theta(n)) = 0$, proving (a). Taking $m, n \in \mathcal{S}_\beta$, we have $\beta(m + \theta(n)) \neq 0$ and $\beta(\theta(m) + n) \neq 0$ by the assumption. Then all the coefficients in Eq. (5) are nonzero. Thus the degrees of the monomials must be the same; this yields Eq. (3), and Eq. (4) follows. \square

By symmetry, only one of the two identities (3) is needed. Note also that by definition $A + \emptyset = \emptyset$ for any set A so that $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$ and $\mathcal{Z}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{Z}_\beta$ are automatic in the nondegenerate case. Otherwise, we have the following constraint on \mathcal{S}_β .

Lemma 3.4. *If P is a degenerate monomial Rota-Baxter operator on $\mathbf{k}[x]$, then \mathcal{S}_β is either empty or infinite. The same applies to \mathcal{Z}_β .*

Proof. Suppose $\mathcal{S}_\beta \neq \emptyset$ and $|\mathcal{S}_\beta| = t < \infty$. Then we may assume that

$$\mathcal{S}_\beta = \{m_i \in \mathbb{N} \mid 1 \leq i \leq t, m_1 < \dots < m_t\}.$$

By Eq. (5), we have $\beta(m_t)^2 = 2\beta(m_t)\beta(m_t + \theta(m_t))$. Since $\beta(m_t) \neq 0$, we have $\beta(m_t) = 2\beta(m_t + \theta(m_t))$, and so $\beta(m_t + \theta(m_t)) \neq 0$. Thus $m_t + \theta(m_t)$ is in \mathcal{S}_β . By Lemma 3.2, we have $\theta(m_t) \geq 1$. Then $m_t + \theta(m_t) > m_t$, a contradiction. Thus either $\mathcal{S}_\beta = \emptyset$ or $|\mathcal{S}_\beta| = \infty$.

On the other hand, let $\mathcal{Z}_\beta \neq \emptyset$. If $\mathcal{Z}_\beta = \mathbb{N}$, then it is certainly infinite. If $\mathcal{Z}_\beta \neq \mathbb{N}$, then take $k \in \mathcal{S}_\beta$. Since $\mathcal{S}_\theta = \mathcal{S}_\beta$ by Lemma 3.2, we have $\theta(k) > 0$. By Theorem 3.3(a), we have $\mathcal{Z}_\beta + \theta(k) \subseteq \mathcal{Z}_\beta$. This implies that \mathcal{Z}_β is infinite. \square

We now give a general setup for constructing monomial Rota-Baxter operators on $\mathbf{k}[x]$. This setup will be applied in Section 3.2 to construct nondegenerate monomial Rota-Baxter operators and in Section 3.3 to construct degenerate monomial Rota-Baxter operators.

Theorem 3.5. *Let \mathcal{S} be a subset of \mathbb{N} .*

(a) *Let the maps $\theta : \mathcal{S} \rightarrow \mathbb{N}^\times$ and $\beta : \mathcal{S} \rightarrow \mathbf{k}^\times$ satisfy the following conditions.*

(i) *We have $\mathcal{S} + \theta(\mathcal{S}) \subseteq \mathcal{S}$ and $\mathbb{N} \setminus \mathcal{S} + \theta(\mathcal{S}) \subseteq \mathbb{N} \setminus \mathcal{S}$.*

(ii) *The equations (3) and (4) are fulfilled for all $m, n \in \mathcal{S}$.*

Extend θ and β to \mathbb{N} by defining $\theta(n) = 0$ and $\beta(n) = 0$ for $n \in \mathbb{N} \setminus \mathcal{S}$. Then $P : \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ defined by $P(x^n) = \beta(n)x^{\theta(n)}$, $n \in \mathbb{N}$, is a Rota-Baxter operator on $\mathbf{k}[x]$.

(b) *Let $\theta : \mathcal{S} \rightarrow \mathbb{N}^\times$ satisfy Eq. (3) and $\mathbb{N} \setminus \mathcal{S} + \theta(\mathcal{S}) \subseteq \mathbb{N} \setminus \mathcal{S}$. Extend θ to \mathbb{N} by defining $\theta(n) = 0$ for $n \in \mathbb{N} \setminus \mathcal{S}$. For any $c \in \mathbf{k}^\times$, define $\beta : \mathbb{N} \rightarrow \mathbf{k}$ by*

$$(6) \quad \beta(n) = \begin{cases} c/\theta(n), & n \in \mathcal{S}, \\ 0, & n \notin \mathcal{S}. \end{cases}$$

Then $P : \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ defined by $P(x^n) = \beta(n)x^{\theta(n)}$ is a Rota-Baxter operator on $\mathbf{k}[x]$.

Proof. (a) This follows from Theorem 3.3.

(b) Under the assumption, we have for $m, n \in \mathcal{S}$ that

$$\begin{aligned} \beta(m + \theta(n))\beta(n) + \beta(\theta(m) + n)\beta(m) &= \frac{c^2}{\theta(m + \theta(n))\theta(n)} + \frac{c^2}{\theta(\theta(m) + n)\theta(m)} \\ &= \frac{c^2}{(\theta(m) + \theta(n))\theta(n)} + \frac{c^2}{(\theta(m) + \theta(n))\theta(m)} \\ &= \frac{c}{\theta(m)} \frac{c}{\theta(n)} \\ &= \beta(m)\beta(n). \end{aligned}$$

Thus θ and β satisfy the conditions in Theorem 3.3 for P to be a Rota-Baxter operator on $\mathbf{k}[x]$. \square

3.2. Nondegenerate case. As mentioned earlier, for a nondegenerate monomial linear operator P on $\mathbf{k}[x]$, the conditions $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$ and $\mathcal{Z}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{Z}_\beta$ are automatic. Thus we obtain the following characterization of nondegenerate monomial Rota-Baxter operators from Theorems 3.3 and 3.5.

Corollary 3.6.

(a) *Let P be a nondegenerate monomial linear operator on $\mathbf{k}[x]$ as in Eq. (2). Then P is a Rota-Baxter operator if and only if the sequences θ and β satisfy the equations (3) and (4) for all $m, n \in \mathbb{N}$. In this case, $\theta(n) \neq 0$ for all $n \in \mathbb{N}$.*

(b) *If a sequence $\theta : \mathbb{N} \rightarrow \mathbb{N}$ is nonzero and satisfies Eq. (3), then for any $c \in \mathbf{k}^\times$, the map $\beta : \mathbb{N} \rightarrow \mathbf{k}$ given by $\beta(n) := c/\theta(n)$ satisfies Eq. (4) and hence gives a Rota-Baxter operator on $\mathbf{k}[x]$.*

Equation (3) characterizes θ as an averaging operator defined as follows.

Definition 3.7.

(a) A map $\theta : S \rightarrow S$ on a semigroup S is called an **averaging operator** if

$$\theta(m\theta(n)) = \theta(m)\theta(n) \quad \text{for all } m, n \in S.$$

(b) A linear map $\Theta : R \rightarrow R$ on a \mathbf{k} -algebra R is called an **averaging operator** if Θ is an averaging operator on the multiplicative semigroup of R .

The study of averaging operators can be tracked back to Reynolds and Birkhoff [7, 22]. We refer the reader to [18] and the references therein for further details.

By Corollary 3.6, a nondegenerate monomial operator P on $\mathbf{k}[x]$ is a Rota-Baxter operator if and only if the map θ is an averaging operator on the semigroup $(\mathbb{N}, +)$, and the corresponding \mathbf{k} -linear operator $\Theta : x^n \mapsto x^{\theta(n)}$ makes $(\mathbf{k}[x], \Theta)$ into an averaging algebra. We write \mathcal{A} for the set of all nondegenerate averaging operators, i.e. sequences $\theta : \mathbb{N} \rightarrow \mathbb{N}^\times$ satisfying Eq. (3). We describe \mathcal{A} as the first step to determine nondegenerate monomial Rota-Baxter operators on $\mathbf{k}[x]$. We denote the free semigroup over \mathbb{N}^\times by $S(\mathbb{N}^\times)$, so the elements $\sigma \in S(\mathbb{N}^\times)$ are *finite sequences* $(\sigma_0, \dots, \sigma_{d-1})$ of positive numbers having any length $d > 0$.

Theorem 3.8. *There is a bijective correspondence $\Phi : \mathcal{A} \rightarrow S(\mathbb{N}^\times)$ given by*

$$\Phi(\theta) = \left(\theta(0), \dots, \theta(d-1) \right) / d \quad \text{with} \quad d := \min \{ j \in \mathbb{N}^\times \mid \theta(r+j) = \theta(r) + j \text{ for all } r \in \mathbb{N} \}$$

whose inverse maps $\sigma := (\sigma_0, \dots, \sigma_{d-1}) \in S(\mathbb{N}^\times)$ to the map $\theta : \mathbb{N} \rightarrow \mathbb{N}^\times$ defined by $\theta(n) = (\ell + \sigma_j)d$ for $n = \ell d + j$ with $\ell \in \mathbb{N}$ and $0 \leq j < d$. Moreover, we have $\text{im}(\theta) = d\mathbb{N}_{\geq s}$ for $s := \min(\sigma)$.

Proof. First consider $\theta \in \mathcal{A}$. Defining the map $\tilde{\theta} := \theta - \text{id}_{\mathbb{N}} : \mathbb{N} \rightarrow \mathbb{Z}$, one obtains from Eq. (3) that $\tilde{\theta}(m+\theta(n)) = \tilde{\theta}(m)$ for all $m, n \in \mathbb{N}$. Hence $\tilde{\theta}$ is periodic, and d is well-defined as the primitive period of $\tilde{\theta}$. Since every $\theta(n)$ is also a period of $\tilde{\theta}$, this implies $\text{im}(\theta) \subseteq d\mathbb{N}^\times$ so that the given map $\Phi : \mathcal{A} \rightarrow S(\mathbb{N}^\times)$ is well-defined.

Next let us write Ψ for the assignment $\sigma \mapsto \theta$ defined above. By checking Eq. (3) one sees that this yields a well-defined map $\Psi : S(\mathbb{N}^\times) \rightarrow \mathcal{A}$.

Now we prove $\Phi \circ \Psi = \text{id}_{S(\mathbb{N}^\times)}$, so let $\theta : \mathbb{N} \rightarrow \mathbb{N}^\times$ be the map defined as above by a given sequence $(\sigma_0, \dots, \sigma_{d-1}) \in S(\mathbb{N}^\times)$. Since $\tilde{\theta}(n) = \sigma_j d - j$ for $n = \ell d + j$, we see that d is a period of the map $\tilde{\theta}$. Assume d is greater than its primitive period d' . Then we have $d = kd'$ for $k > 1$, and

$$\sigma_0 kd' = \theta(0) = \tilde{\theta}(0) = \tilde{\theta}(d') = \sigma_{d'} d - d' = (\sigma_{d'} k - 1) d'$$

implies $\sigma_0 k = \sigma_{d'} k - 1$, which contradicts $k > 1$. We conclude that d is the primitive period of $\tilde{\theta}$, so the definition of Φ recovers the correct value of d . Moreover, for $j = 0, \dots, d-1$ we have $\theta(j) = \sigma_j d$, which implies $\Phi(\theta) = \sigma$ as required.

It remains to prove the converse relation $\Psi \circ \Phi = \text{id}_{\mathcal{A}}$. Taking an arbitrary $\theta \in \mathcal{A}$, we must prove that it coincides with the sequence θ' defined by $\theta'(\ell d + j) = (\ell + \theta(j)/d)d = \ell d + \theta(j)$ for any $\ell \in \mathbb{N}$ and $0 \leq j < d$. For these values we must then show that $\theta(\ell d + j) = \ell d + \theta(j)$, which is equivalent to $\tilde{\theta}(\ell d + j) = \tilde{\theta}(j)$. The latter is ensured since we know that $\tilde{\theta}$ has primitive period d .

As noted above, $\text{im}(\theta) \subseteq d\mathbb{N}^\times$ so $\theta/d : \mathbb{N} \rightarrow \mathbb{N}^\times$ is well-defined. We must show $\text{im}(\theta/d) = \mathbb{N}_{\geq s}$. The inclusion from left to right follows since $(\theta/d)(\ell d + j) = \ell + \sigma_j \geq \sigma_j \geq s$. Now let $n \geq s$ be given and write $s = \sigma_j$ for some $j = 0, \dots, d-1$. Then $\ell := n - \sigma_j \in \mathbb{N}$ is such that $(\theta/d)(\ell d + j) = n$, which established the inclusion from right to left. \square

As sequences, the relation between $\theta : \mathbb{N} \rightarrow \mathbb{N}^\times$ and $\sigma : \{0, \dots, d-1\} \rightarrow \mathbb{N}^\times$ can be written as $\theta/d = (\sigma, \sigma + 1, \sigma + 2, \dots)$, where $1, 2, \dots$ designate constant sequences of length d . More

precisely, we have

$$\theta/d = (\sigma_0, \dots, \sigma_{d-1}, \sigma_0 + 1, \dots, \sigma_{d-1} + 1, \dots).$$

Theorem 3.8 yields the following construction algorithm for the map θ from a nondegenerate monomial Rota-Baxter operator.

Algorithm 3.9. Every sequence $\theta: \mathbb{N} \rightarrow \mathbb{N}$ corresponding to a nondegenerate monomial Rota-Baxter operator on $\mathbf{k}[x]$ can be generated as follows:

- (a) Let $d \in \mathbb{N}^\times$ be given. For each $j = 0 \dots d-1$ fix $\sigma_j \in \mathbb{N}^\times$;
- (b) For $n \in \mathbb{N}$ with $n = \ell d + \bar{n}$ where $\bar{n} \in \{0 \dots d-1\}$ is the remainder of n modulo d , define

$$\theta(n) := n + \sigma_{\bar{n}}d - \bar{n} = \ell d + \sigma_{\bar{n}}d.$$

We consider two *extreme cases* of Algorithm 3.9 of particular interest:

Case 1: If $d = 1$ one can only choose $\theta(0) \neq 0$ so that $\theta(n) = n + \theta(0)$ for all $n \in \mathbb{N}$.

Case 2: For $d > 1$ and $\sigma_j = 1, 0 \leq j \leq d-1$ we have $\theta(n) = n + d - \bar{n} = (\ell + 1)d$ with $n = \ell d + \bar{n}$.

Example 3.10. Setting $d = 2$ and $\sigma_0 = \sigma_1 = 1$, we choose the sequence β according to Corollary 3.6(b) with $c = 2$. Then the \mathbf{k} -linear map $P: \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ by

$$P(x^{2k}) = \frac{x^{2k+2}}{k+1} \quad \text{and} \quad P(x^{2k+1}) = \frac{x^{2k+2}}{k+1}$$

is a nondegenerate Rota-Baxter operator on $\mathbf{k}[x]$.

We determine next all β for the sequences θ coming from the above two extreme cases.

Theorem 3.11.

- (a) Let $d = 1$ with $\theta(n) = n + k$ for some $k \in \mathbb{N}^\times$. Then $\beta: \mathbb{N} \rightarrow \mathbf{k}$ satisfies Eq. (4) if and only if $\beta(n) = \frac{c}{\theta(n)}$ for some $c \in \mathbf{k}^\times$.
- (b) Let $d > 1$ be given with $\theta(n) = n + d - \bar{n}$. Then $\beta: \mathbb{N} \rightarrow \mathbf{k}$ satisfies Eq. (4) if and only if it is defined as follows: Fix $c_j \in \mathbf{k}^\times$ and assign $\beta(j) := 1/c_j$ for $0 \leq j \leq d-1$. Then for any $n \in \mathbb{N}$ with $n = \ell d + \bar{n}$ define $\beta(n) = \frac{\beta(\bar{n})}{\ell+1}$.

Proof. (a) For a θ of the given form, by Eq. (4), we have

$$(7) \quad \beta(n)\beta(0) = \beta(n+k)(\beta(0) + \beta(n)).$$

Set $\beta(0) := a$ for some $a \in \mathbf{k}^\times$ and write $c := ka$. Then $\beta(0) = \frac{c}{k}$ and c is in \mathbf{k}^\times . We next prove $\beta(n) = \frac{c}{n+k}$ by induction on $n \geq 0$. The base case $n = 0$ is true. Assume $\beta(n) = \frac{c}{n+k}$ has been proved for $n \geq 0$. By Eq. (7), we obtain

$$(8) \quad \beta(n+1-k)\beta(0) = \beta(n+1)(\beta(0) + \beta(n+1-k)).$$

Since $k \geq 1$, we have $n+1-k \leq n$. By the induction hypothesis, we get $\beta(n+1-k) = \frac{c}{n+1}$. Then by Eq. (8) we have

$$\beta(n+1) = \frac{\frac{c^2}{k(n+1)}}{\frac{c}{k} + \frac{c}{n+1}} = \frac{c}{n+1+k}.$$

This completes the induction. Thus $\beta(n) = \frac{c}{\theta(n)}$ for some $c \in \mathbf{k}^\times$ and all $n \in \mathbb{N}$.

The converse follows from Theorem 3.6(b).

(b) Taking $\gamma(n) = 1/\beta(n)$, Eq. (4) is equivalent to

$$(9) \quad \frac{\gamma(m)}{\gamma(m+\theta(n))} + \frac{\gamma(n)}{\gamma(\theta(m)+n)} = 1.$$

Thus we just need to show that, for a fixed sequence θ in the theorem, a sequence $\gamma : \mathbb{N} \rightarrow \mathbf{k}$ satisfies Eq. (9) if and only if γ is defined by $\gamma(n) = (\ell + 1)\gamma(\bar{n})$ if $n = \ell d + \bar{n}$, where the $\gamma(\bar{n}) \in \mathbf{k}^\times$ for $\bar{n} \in \{0 \dots d - 1\}$ are arbitrarily preassigned.

(\implies) Take $m = 0$ and $n = \ell d$ with $\ell \geq 0$ in Eq. (9). After simplifying we obtain

$$\gamma((\ell + 1)d) = \gamma(\ell d) + \gamma(0).$$

Then by an induction on ℓ , we obtain

$$(10) \quad \gamma(\ell d) = (\ell + 1)\gamma(0).$$

Next note that for $n = \ell d + \bar{n}$ we have

$$(11) \quad \theta(n) = \ell d + d.$$

Then for $j \in \{0, \dots, d - 1\}$, taking $m = 0$ and $n = \ell d + j$ in Eq. (9) we obtain

$$1 = \frac{\gamma(0)}{\gamma(\theta(\ell d + j))} + \frac{\gamma(\ell d + j)}{\gamma(\theta(0) + \ell d + j)} = \frac{\gamma(0)}{\gamma(\ell d + d)} + \frac{\gamma(\ell d + j)}{\gamma(d + \ell d + j)}.$$

This gives

$$\gamma((\ell + 1)d + j) = \frac{\ell + 2}{\ell + 1}\gamma(\ell d + j)$$

and recursively yields

$$\gamma(\ell d + j) = (\ell + 1)\gamma(j).$$

(\impliedby) Conversely, suppose a sequence β is given by $\gamma(n) = (\ell + 1)\gamma(\bar{n})$ if $n = \ell d + \bar{n}$, for preassigned $\gamma(\bar{n})$ as specified above. Then for any $m, n \in \mathbb{N}$ with $m = kd + \bar{m}$ and $n = \ell d + \bar{n}$, by Eq. (11) we obtain

$$\begin{aligned} \frac{\gamma(m)}{\gamma(m + \theta(n))} + \frac{\gamma(n)}{\gamma(\theta(m) + n)} &= \frac{\gamma(kd + \bar{m})}{\gamma(kd + \bar{m} + \theta(\ell d + \bar{n}))} + \frac{\gamma(\ell d + \bar{n})}{\gamma(\theta(kd + \bar{m}) + \ell d + \bar{n})} \\ &= \frac{\gamma(kd + \bar{m})}{\gamma(kd + \bar{m} + \ell d + d)} + \frac{\gamma(\ell d + \bar{n})}{\gamma(kd + d + \ell d + \bar{n})} \\ &= \frac{(k + 1)\gamma(\bar{m})}{(k + \ell + 2)\gamma(\bar{m})} + \frac{(\ell + 1)\gamma(\bar{n})}{(k + \ell + 2)\gamma(\bar{n})} = 1. \end{aligned}$$

This is Eq. (9). \square

In the special case of *polynomial sequences* $\theta : \mathbb{N} \rightarrow \mathbb{N}$ and $\alpha = 1/\beta : \mathbb{N} \rightarrow \mathbf{k}$, the range of possibilities can be drastically narrowed down.

Theorem 3.12. *Suppose \mathbf{k} is a field containing \mathbb{Q} . Let $P : \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ be a nondegenerate monomial linear operator with $P(x^n) = \frac{1}{\alpha(n)}x^{\theta(n)}$ for $n \in \mathbb{N}$, and assume $\theta(n)$ as well as $\alpha(n)$ are polynomials. Then P is a Rota-Baxter operator if and only if*

$$(12) \quad \theta(n) = n + k \quad \text{and} \quad \alpha(n) = c(n + k)$$

for some $k \in \mathbb{N}^\times$ and some $c \in \mathbf{k}^\times$.

Proof. By Corollary 3.6, the operator P defined by Eq. (12) is a Rota-Baxter operator. So we just need to show that any Rota-Baxter operator given by Eq. (2) with polynomial sequences $\theta(n)$ and $\alpha(n)$ must satisfy the conditions in Eq. (12). Since P is a Rota-Baxter operator, Eq. (3) gives the characteristic relation $2\theta(n) = \theta(\theta(n) + n)$. But θ and α are polynomials with $\deg \theta$ and $\deg \alpha$ respectively. Checking degrees, let us first assume $\deg \theta \geq 2$. In this case we have

$$\deg 2\theta = \deg \theta < (\deg \theta)^2 = \deg \theta(\theta(n) + n),$$

which contradicts the characteristic relation. Thus we have $\deg \theta \leq 1$, and we can write $\theta(n) = sn + k$ for some $s, k \in \mathbb{N}$. Now the characteristic relation becomes $2(sn + k) = s(sn + n + k) + k$ or equivalently $(sn + k)(s - 1) = 0$. If $s \neq 1$ we obtain $sn + k = 0$ for all $n \in \mathbb{N}$. But then $s = k = 0$, and P is the zero operator, which contradicts the hypothesis that P is nondegenerate. Therefore $s = 1$ and hence $\theta(n) = n + k$ as claimed in Eq. (12).

For deriving the second condition of Eq. (12), we specialize Eq. (4) to obtain $2\alpha(n) = \alpha(\theta(n)+n)$ and hence the recursion $2\alpha(n) = \alpha(2n + k)$. Set $\ell = \deg \alpha$ and suppose the leading coefficient of α is $c \in \mathbf{k}^\times$. Now taking leading coefficients of the recursion, we get $2c = 2^\ell c$ and thus $\ell = 1$. This means we can write $\alpha(n) = cn + c_0$ for some $c \in \mathbf{k}^\times$ and $c_0 \in \mathbf{k}$. Substituting this into the recursion leads to $2(cn + c_0) = c(2n + k) + c_0$ and hence $\alpha(n) = c(n + k)$ as claimed in Eq. (12). It remains to show that $k \neq 0$. But this follows because $P(1) = x^k/c_k$ so that necessarily $ck \neq 0$. \square

Next we investigate injective monomial Rota-Baxter operators and show them to be analytically modelled. We note first that if P is degenerate, then there exists $n_0 \in \mathbb{N}$ such that $\beta(n_0) = 0$, and then $P(x^{n_0}) = 0$. Thus $\ker(P) \neq \{0\}$ and P is not injective. Thus any injective monomial Rota-Baxter operator is nondegenerate.

Theorem 3.13. *Let P be a monomial Rota-Baxter operator on $\mathbf{k}[x]$. The following statements are equivalent.*

- (a) *The operator P is injective.*
- (b) *The θ as in Eq. (2) from P satisfies $\theta(n) = n + k$ for some $k \in \mathbb{N}^\times$.*
- (c) *There are $k \in \mathbb{N}^\times$ and $c \in \mathbf{k}^\times$ such that $P(x^n) = c \int_0^x t^{n+k-1} dt$ and hence $P = cJ_0 x^{k-1}$.*

Proof. (a) \Rightarrow (b): Assume that P is an injective monomial Rota-Baxter operator. Then P is nondegenerate. By Algorithm 3.9, there are $d \geq 1$ and $\sigma_j \in \mathbb{N}^\times$ for $j \in \{0, \dots, d-1\}$ such that $\theta(n) = \ell d + \sigma_{\bar{n}} d$ where $n = \ell d + \bar{n}$ and \bar{n} is the remainder of n modulo d . Suppose $d > 1$. Without loss of generality, we may assume $\sigma_0 \geq \sigma_1$ so that $n := (\sigma_0 - \sigma_1)d + 1 > 0$. Since $\theta(0) = \sigma_0 d$ we have

$$\theta(n) = \theta((\sigma_0 - \sigma_1)d + 1) = (\sigma_0 - \sigma_1)d + \sigma_1 d = \theta(0),$$

hence θ is not injective. This forces $d = 1$. Then by the first case considered after Algorithm 3.9, we have $\theta(n) = n + k$ for fixed $k \geq 1$.

(b) \Rightarrow (c): For a θ of the given form, by Theorem 3.11(a), we have $\beta(n) = c/\theta(n)$ for some $c \in \mathbf{k}^\times$. Thus

$$P(x^n) = \beta(n)x^{\theta(n)} = \frac{c}{n+k}x^{n+k} = c \int_0^x t^{n+k-1} dt,$$

as needed.

(c) \Rightarrow (a): Since $P(x^n) = c \int_0^x t^{n+k-1} dt = \frac{c}{n+k}x^{n+k}$ for all $n \in \mathbb{N}$, the operator P is injective. \square

3.3. Degenerate case. We next apply Theorem 3.5 to construct degenerate monomial Rota-Baxter operators on $\mathbf{k}[x]$ when \mathcal{S}_β is either $k\mathbb{N}$ where $k \geq 1$ or is $\mathbb{N} \setminus (k\mathbb{N})$ where $k \geq 2$.

Proposition 3.14. *Let $P(x^n) = \beta(n)x^{\theta(n)}$, $n \in \mathbb{N}$, define a monomial linear operator on $\mathbf{k}[x]$ such that $\mathcal{S}_\beta = k\mathbb{N}$ for some $k > 0$. Then P is a Rota-Baxter operator on $\mathbf{k}[x]$ if and only if $\theta(km) = \tilde{\theta}(m) \in \mathcal{S}_\beta^\times$ and $\beta(km) = \tilde{\beta}(m)$, $m \geq 0$ for maps $\tilde{\theta} : \mathbb{N} \rightarrow \mathbb{N}$ and $\tilde{\beta} : \mathbb{N} \rightarrow \mathbf{k}$ that satisfy the following*

equations

$$(13) \quad \tilde{\theta}(m_1) + \tilde{\theta}(m_2) = \tilde{\theta}(m_1 + \frac{1}{k}\tilde{\theta}(m_2)) = \tilde{\theta}(\frac{1}{k}\tilde{\theta}(m_1) + m_2),$$

$$(14) \quad \tilde{\beta}(m_1)\tilde{\beta}(m_2) = \tilde{\beta}(m_1 + \frac{1}{k}\tilde{\theta}(m_2))\tilde{\beta}(m_2) + \tilde{\beta}(m_2 + \frac{1}{k}\tilde{\theta}(m_1))\tilde{\beta}(m_1) \quad \text{for all } m_1, m_2 \in \mathbb{N}.$$

Proof. Since $\mathcal{S}_\beta = \{km \mid m \in \mathbb{N}\}$, we have $\mathcal{Z}_\beta = \{km + i \mid 1 \leq i \leq k-1, m \in \mathbb{N}\}$. Suppose P is a Rota-Baxter operator on $\mathbf{k}[x]$. Then by Eq. (1), we have $P(x^{km_1+i}P(x^{km_2})) = 0$ for all $m_1, m_2 \in \mathbb{N}$ and $1 \leq i \leq k-1$. Thus $\beta(km_2)\beta(km_1 + i + \theta(km_2)) = 0$. Since $\beta(km_2) \neq 0$, we have $\beta(km_1 + i + \theta(km_2)) = 0$. Then $km_1 + i + \theta(km_2)$ is in \mathcal{Z}_β , and then $i + \theta(km_2)$ is in \mathcal{Z}_β for $1 \leq i \leq k-1$. Suppose that there exists $m_0 \in \mathbb{N}$ such that $\theta(km_0) \not\equiv 0 \pmod{k}$. Then there exists $1 \leq i_0 \leq k-1$ such that $i_0 + \theta(km_0) \equiv 0 \pmod{k}$. So $i_0 + \theta(km_0)$ is in \mathcal{S}_β by the definition of \mathcal{S}_β . This is a contradiction to the fact proved above that $i + \theta(km_2)$ is in \mathcal{Z}_β for $1 \leq i \leq k-1$.

Thus $\theta(km)$ is in \mathcal{S}_β^\times for all $m \in \mathbb{N}$. So $kn + \theta(km)$ is in \mathcal{S}_β for all $n, m \in \mathbb{N}$. By Theorem 3.3, Eqs. (3) and (4) hold. Let $\tilde{\theta}(m) := \theta(km)$ and let $\tilde{\beta}(m) := \beta(km)$, $m \in \mathbb{N}$. Thus $\tilde{\beta}(m) \neq 0$ for all $m \in \mathbb{N}$. Then by Eqs. (3) and (4), Eqs. (13) and (14) hold. This is what we want. The converse follows from Theorem 3.5(a). \square

Proposition 3.14 gives a large class of monomial Rota-Baxter operators on $\mathbf{k}[x]$ with $\mathcal{S}_\beta = k\mathbb{N}$, reducing to Corollary 3.6 for $k = 1$. On the other hand, Theorem 3.5 also gives the following result on monomial Rota-Baxter operators on $\mathbf{k}[x]$ where \mathcal{S}_β is now complementary to Proposition 3.14.

Proposition 3.15. *Let $P(x^n) = \beta(n)x^{\theta(n)}$ be a monomial linear operator on $\mathbf{k}[x]$ with $\mathcal{S}_\beta = \mathbb{N} \setminus k\mathbb{N}$ for some $k \geq 2$.*

- (a) *For any $t \in \mathbb{N}^\times$ one obtains a degenerate monomial RBO by setting $\theta(km + i) = k(m + t)$ and $\theta(km) = 0$ for $m \in \mathbb{N}$ and $1 \leq i \leq k-1$, choosing β as in Theorem 3.5(b).*
- (b) *Assume that $\theta(i) = k$ for $1 \leq i \leq k-1$. Then θ corresponds to a degenerate monomial RBO on $\mathbf{k}[x]$ if and only if $\theta(km + i) = k(m + 1)$ for all $m \in \mathbb{N}$ and $1 \leq i \leq k-1$.*

Proof. (a) By our assumption on \mathcal{S}_β , we have $\mathcal{Z}_\beta = \{km \mid m \in \mathbb{N}\}$. By assumption $\theta(km + i) = k(m + t)$ for all $m \in \mathbb{N}$ and $1 \leq i \leq k-1$, hence we obtain $\mathcal{Z}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{Z}_\beta$. Since

$$\theta(km_1 + i_1) + \theta(km_2 + i_2) = k(m_1 + m_2 + 2t)$$

and

$$\theta(km_1 + i_1 + \theta(km_2 + i_2)) = \theta(k(m_1 + m_2 + t) + i_1) = k(m_1 + m_2 + 2t)$$

for all $m_1, m_2 \in \mathbb{N}$ and $1 \leq i_1, i_2 \leq k-1$, we have Eq. (3). Thus we may apply Theorem 3.5(b) to obtain a degenerate RBO P on $\mathbf{k}[x]$.

(b) Assume first that P is a monomial RBO on $\mathbf{k}[x]$. Then by Eq. (5), $\beta(km+i)\beta(km+\theta(km+i)) = 0$ for all $m \in \mathbb{N}$ and $1 \leq i \leq k-1$. Since $\beta(km+i) \neq 0$, we have $\beta(km+\theta(km+i)) = 0$, so $km+\theta(km+i)$ is in \mathcal{Z}_β . From $\mathcal{Z}_\beta = k\mathbb{N}$ we infer $\theta(km+i) \in \mathcal{Z}_\beta$. Thus $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$. By Theorem 3.3, Eq. (3) holds. We now prove that $\theta(km+i) = k(m+1)$ by induction on $m \geq 0$. The base case $m = 0$ is immediate from our assumption. Assume $\theta(km+i) = k(m+1)$ has been proved for $m \geq 0$. By Eq. (3), we have

$$\theta(k(m+1)+i) = \theta(km+i+\theta(i)) = \theta(km+i) + \theta(i).$$

By the induction hypothesis, we get $\theta(k(m+1)+i) = k(m+2)$. This completes the proof.

Conversely, by $\theta(km+i) = k(m+1)$ and Item (a), we obtain a degenerate RBO P on $\mathbf{k}[x]$. \square

Example 3.16. Taking $k = 2$ in Proposition 3.14 and Proposition 3.15, we obtain the following degenerate monomial Rota-Baxter operators on $\mathbf{k}[x]$.

- (a) $P(x^{2k}) = \frac{x^{2(k+1)}}{k+1}$ and $P(x^{2k+1}) = 0$ for all $k \in \mathbb{N}$.
- (b) $P(x^{2k}) = 0$ and $P(x^{2k+1}) = \frac{x^{2(k+1)}}{k+1}$ for all $k \in \mathbb{N}$.

The above examples may also be regarded as special cases of the following result.

Proposition 3.17. *Let $P_0 \in \text{RBO}(R)$ for a \mathbf{k} -algebra R . Assume ϕ is a \mathbf{k} -linear operator on R such that $E := P_0(\text{im}(\phi))$ is a nonunitary \mathbf{k} -subalgebra. If ϕ is a homomorphism of the E -module R then $P_0 \circ \phi$ is also a Rota-Baxter operator on R .*

Proof. This follows immediately since

$$\begin{aligned} (P_0 \circ \phi)(a)(P_0 \circ \phi)(b) &= P_0(\phi(a))P_0(\phi(b)) = P_0(\phi(a)P_0(\phi(b))) + P_0(P_0(\phi(a))\phi(b)) \\ &= (P_0 \circ \phi)(a(P_0 \circ \phi)(b)) + (P_0 \circ \phi)((P_0 \circ \phi)(a)b), \end{aligned}$$

for all $a, b \in R$. □

For $R = \mathbf{k}[x]$ let $\phi: f(x) \mapsto (f(x) + f(-x))/2$ be the projector onto the \mathbf{k} -subspace spanned by the even monomials and set $P_0 = 2J_0x$. Then

$$(P_0 \circ \phi)(x^n) := \begin{cases} \frac{x^{2(k+1)}}{k+1} & \text{if } n = 2k, \\ 0 & \text{if } n = 2k + 1, \end{cases}$$

for all $k \in \mathbb{N}$ so that $P_0 \circ \phi$ is the same as P in Example 3.16(a). On the other hand, choosing ϕ as the projector $f(x) \mapsto (f(x) - f(-x))/2$ onto the space of odd monomials and setting $P_0 = 2J_0$ yields

$$(P_0 \circ \phi)(x^n) := \begin{cases} 0 & \text{if } n = 2k, \\ \frac{x^{2(k+1)}}{k+1} & \text{if } n = 2k + 1, \end{cases}$$

for all $k \in \mathbb{N}$ so that $P_0 \circ \phi$ is the same as P in Example 3.16(b). In both cases, E is the nonunitary algebra of nonconstant even monomials.

Proposition 3.18. *Let $P(x^n) = \beta(n)x^{\theta(n)}$ be a nonzero degenerate monomial RBO on $\mathbf{k}[x]$ satisfying the condition $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$.*

(a) *There exists a map $\sigma: \mathbb{N} \rightarrow \mathcal{S}_\beta$ such that $P_0(x^n) := P(x^{\sigma(n)})$ defines a nondegenerate monomial RBO on $\mathbf{k}[x]$.*

(b) *We have*

$$(15) \quad \mathcal{S}_\beta = C \uplus (s_1 + e\mathbb{N}) \uplus \cdots \uplus (s_k + e\mathbb{N}),$$

where $C \subset \mathbb{N}$ is finite, $k < e \in \mathbb{N}^\times$, and $s_1, \dots, s_k \in \mathcal{S}_\beta$ are incongruent modulo e (in the sense that $x - y \notin e\mathbb{Z}$) such that $s_1 - e, \dots, s_k - e \notin C$. Moreover, there exists a finite set $E \subset \mathcal{S}_\beta$ such that θ is determined uniquely by its values on E .

Proof. Since P is nonzero, both $\mathcal{S}_\beta \neq \emptyset$ and $\mathcal{Z}_\beta \neq \emptyset$ are infinite by Lemma 3.4. From Eq. (3) and the condition $\mathcal{S}_\beta + \theta(\mathcal{S}_\beta) \subseteq \mathcal{S}_\beta$ we see that $T := \theta(\mathcal{S}_\beta)$ is additively closed. As in the proof of Theorem 3.8 one checks that $\theta - \text{id}_{\mathbb{N}}$ is periodic on \mathcal{S}_β with primitive period d and $T \subseteq d\mathbb{N}^\times$ so that $d \mid e := \text{gcd}(T)$. Hence T/e is a numerical semigroup [26, Prop. 10.1], meaning a subsemigroup of \mathbb{N}^\times with a finite complement $G \subseteq \mathbb{N}^\times$ of so-called gaps. Thus we obtain $T = e\mathbb{N}^\times \setminus eG$. We write $f \in \mathbb{N}$ for the Frobenius number of T/e , meaning the greatest element of G for $G \neq \emptyset$ and $f = 0$ otherwise.

(a) Fix an element s of \mathcal{S}_β . We define $\sigma: \mathbb{N} \rightarrow \mathcal{S}_\beta$ as follows. For $n \in \mathbb{N}$, write $n = \ell e + r$ with $\ell \geq 0$ and $0 \leq r < e$. Define $\sigma(\ell e + r) := (f + \ell)e + s$. Then $\sigma: \mathbb{N} \rightarrow \mathcal{S}_\beta$ follows from the condition $\mathcal{S}_\beta + T \subseteq \mathcal{S}_\beta$ since $(f + \ell)e \in T$ for all $\ell > 0$. We show now that

$$(16) \quad \sigma(n + \theta(\sigma(m))) = \sigma(n) + \theta(\sigma(m))$$

for all $m, n \in \mathbb{N}$. We have $\theta(\sigma(m)) = te \in T$ for some $t \notin G$, and we may write $n = \ell e + r$ with $0 \leq r < e$ and $\ell \geq 0$. Then one computes $\sigma(r) + (f + \ell + t)e$ for both sides of Eq. (16).

Let us now prove that P_0 satisfies Eq. (1) or equivalently $RB(P_0, P_0) = 0$. Since the latter is a symmetric bilinear form and $\mathbf{k}[x]$ has characteristic zero, the polarization identity implies that it suffices to prove $RB(P_0, P_0)(u, u) = 0$ for all $u \in \mathbf{k}[x]$. Of course we may restrict ourselves to the canonical basis $u = x^n$, so it remains to show $P(x^{\sigma(n)})^2 = 2P_0(x^n P(x^{\sigma(n)}))$. Applying the definition of P , this is equivalent to

$$\beta(\sigma(n))^2 x^{2\theta(\sigma(n))} = 2\beta(\sigma(n)) P_0(x^{n+\theta(\sigma(n))}),$$

and we may use Eq. (16) to expand the right-hand side further to

$$2\beta(\sigma(n))\beta(\sigma(n) + \theta(\sigma(n))) x^{\theta(\sigma(n)+\theta(\sigma(n)))}.$$

But now we may apply Eqs. (3) and (4) of Theorem 3.3 to conclude that this is equal to the left-hand side. Hence P_0 is indeed a monomial RBO on $\mathbf{k}[x]$. Clearly $P_0(x^n) = P(x^{\sigma(n)}) \neq 0$ since $\sigma(n) \in \mathcal{S}_\beta$, so P_0 is nondegenerate.

(b) For defining s_1, \dots, s_k , consider first the sets $\Sigma_i := \mathcal{S}_\beta \cap (i + e\mathbb{N})$ for $i \in \{0, \dots, e-1\}$. Suppressing the empty ones, we reindex the rest as $\Sigma_1, \dots, \Sigma_k$ for $1 \leq k \leq e$. Then for any $i \in \{1, \dots, k\}$ there exists $\sigma_i \in \Sigma_i$ such that $\sigma_i + e\mathbb{N} \subseteq \Sigma_i$. Indeed, one may choose $\sigma_i = \sigma'_i + (f+1)e$ for any $\sigma'_i \in \Sigma_i$ since then $(f+1)e \in T$, and the hypothesis $\mathcal{S}_\beta + T \subseteq \mathcal{S}_\beta$ implies the required condition $\sigma_i + e\mathbb{N} \subseteq \Sigma_i$. Let $s_i \in \Sigma_i$ be minimal such that the condition is satisfied; this implies in particular $s_i - e \notin \mathcal{S}_\beta$. Then clearly $\Sigma_i = C_i \uplus (s_i + e\mathbb{N})$ for finite sets $C_i \subset \mathbb{N}$. Now define $C := C_1 \cup \dots \cup C_k$ to obtain the decomposition (15). We must have $k < e$ since otherwise $\mathcal{Z}_\beta \subseteq \{0, \dots, \max(s_1, \dots, s_e)\}$ is finite, contradicting Lemma 3.4. Finally, note that $E := \mathcal{S}_\beta \setminus (\mathcal{S}_\beta + T)$ is bounded by $\max(s_1, \dots, s_k) + (f+1)e$ and hence finite. Clearly, θ is determined on $\mathcal{S}_\beta \setminus E$ by Eq. (3). \square

4. INJECTIVE ROTA-BAXTER OPERATORS ON $\mathbf{k}[x]$

For now let \mathbf{k} be an arbitrary field of characteristic zero. An important subclass of Rota-Baxter operators P on $\mathbf{k}[x]$ are those associated with the standard derivation ∂ in the sense that $\partial \circ P = 1_{\mathbf{k}[x]}$. We generalize this for arbitrary $r \in \mathbf{k}[x]^\times$ to the *differential law* $\partial \circ P = r$, where r denotes the corresponding multiplication operator. Thus we define

$$(17) \quad \text{RBO}_r(\mathbf{k}[x]) := \{P \in \text{RBO}(\mathbf{k}[x]) \mid \partial \circ P = r\}.$$

Let us now show that the class of all operators satisfying a differential law actually coincides with the class of all injective operators, which we denote by $\text{RBO}_*(\mathbf{k}[x])$.

Theorem 4.1. *We have $\text{RBO}_*(\mathbf{k}[x]) = \bigcup_{r \in \mathbf{k}[x]^\times} \text{RBO}_r(\mathbf{k}[x])$.*

Proof. The inclusion from right to left is simple as $P(f) = 0$ implies $\partial(P(f)) = rf = 0$ and hence $f = 0$ since $\mathbf{k}[x]$ is an integral domain.

Now let $P: \mathbf{k}[x] \rightarrow \mathbf{k}[x]$ be an injective Rota-Baxter operator. Then there exists a linear map $D: \text{im}(P) \rightarrow \mathbf{k}[x]$ with $D \circ P = 1_{\mathbf{k}[x]}$. Adjoining \mathbf{k} as constants, one can immediately check that D is a derivation on the unitary subalgebra $J := \mathbf{k} + \text{im}(P)$. Note that $P(1) \notin \mathbf{k}$

since $P(1) = c$ implies $c^2 = P(1)^2 = 2P(P(1)) = 2c^2$ and hence $c = 0$, contradicting injectivity. This means $\mathbf{k} \subsetneq J$. Since $J \subseteq \mathbf{k}[x]$ is an integral domain, D extends uniquely to a derivation on the fraction field $K \subseteq \mathbf{k}(x)$ of the ring J . By Lüroth's theorem [9, Thm. 11.3.4], the intermediate field $\mathbf{k} \subset K \subseteq \mathbf{k}(x)$ is a simple transcendental extension of \mathbf{k} , so there exists $\phi \in \mathbf{k}(x) \setminus \mathbf{k}$ with $K = \mathbf{k}(\phi)$. But then $K \subseteq \mathbf{k}(x)$ is an algebraic field extension [29, §73], so the derivation D extends uniquely to $\mathbf{k}(x)$ according to [9, Thm. 11.5.3]. But it is well known [21, Prop. 1.3.2] that every \mathbf{k} -derivation on $\mathbf{k}[x]$ is a multiple of the canonical derivation, so we must have $D = \psi \partial$ for $\psi := D(x)$. Then $D \circ P = 1$ on $\mathbf{k}[x]$ implies that $\psi \cdot P(1)' = 1$, so we obtain $D = r^{-1} \partial$ with $r := P(1)' \in \mathbf{k}[x]$ and then also $\partial \circ P = r$. \square

Thus the study of injective Rota-Baxter operators on $\mathbf{k}[x]$ reduces to the study of $\text{RBO}_r(\mathbf{k}[x])$. As noted in Section 2, all standard integral operators J_a are in $\text{RBO}_1(\mathbf{k}[x])$; more generally, the analytically modelled operators $J_a r$ are in $\text{RBO}_r(\mathbf{k}[x])$. It is thus tempting to speculate that $\text{RBO}_r(\mathbf{k}[x])$ is exhausted by the $J_a r$. For the special case $\mathbf{k} = \mathbb{R}$ and $r = x^k$ this will be proved at the end of this section in Theorem 4.9. For the moment, let \mathbf{k} be an arbitrary field containing \mathbb{Q} .

From integration over the reals, it is well known that the difference between two indefinite integrals is always a definite integral, which may be interpreted as a *measure*. This generalizes to the algebraic setting in the following way.

Lemma 4.2. *Let $r \in \mathbf{k}[x]^\times$ and $a \in \mathbf{k}$ be arbitrary. Then $P \in \text{End}(\mathbf{k}[x])$ satisfies the differential law $\partial \circ P = r$ if and only if $J_a r - P \in \mathbf{k}[x]^*$.*

Proof. Since $\partial \circ J_a r = r$, a linear operator P on $\mathbf{k}[x]$ satisfies $\partial \circ P = r$ if and only if $\partial \circ \mu = 0$ for $\mu := J_a r - P$. The latter identity holds if and only if $\text{im}(\mu)$ is contained in $\ker(\partial) = \mathbf{k}$. \square

In analogy to the reals, we call the above linear functional μ the *associated measure* of P . Then the lemma says that the linear operators satisfying the differential law are classified by their associated measures in the sense that

$$\{P \in \text{End}(\mathbf{k}[x]) \mid \partial \circ P = r\} = J_a r + \mathbf{k}[x]^*,$$

where the initialized point a may be chosen arbitrarily (typically $a = 0$). But in the real case, a measure is more than an arbitrary linear functional; for the algebraic situation this is captured in the following result. Here and henceforth we employ the abbreviation $\star_{r,a}$ for $\star_{J_a r}$, and \star_r for $\star_{r,0}$.

Theorem 4.3. *Let $r \in \mathbf{k}[x]^\times$ and $a \in \mathbf{k}$ be arbitrary. Then the map defined by $P \mapsto J_a r - P$ is a bijection between $\text{RBO}_r(\mathbf{k}[x])$ and $(\mathbf{k}[x], \star_{r,a})^\bullet$.*

Proof. By Lemma 4.2 and Proposition 2.3(c), we obtain an surjective map

$$\text{RBO}_r(\mathbf{k}[x]) \rightarrow (\mathbf{k}[x], \star_{r,a})^\bullet, \quad P \mapsto J_a r - P.$$

The map is injective since $J_a r - P = J_a r - \tilde{P}$ implies $P = \tilde{P}$. \square

Thus the above classification of operators satisfying differential law may be refined to

$$\text{RBO}_r(\mathbf{k}[x]) = J_a r - (\mathbf{k}[x], \star_{r,a})^\bullet.$$

For working out a more explicit description, we specialize to the monomial case $r = x^k$, where we use the abbreviation \star_k for \star_{x^k} . To this end, we will determine $(\mathbf{k}[x], \star_k)^\bullet$, starting with $k = 0$.

Theorem 4.4.

- (a) For any $k \in \mathbb{N}$, we have the isomorphism $(\mathbf{k}[x], \star_k) \cong x^{k+1}\mathbf{k}[x]$ of nonunitary algebras.
 (b) There is a bijection $(\mathbf{k}[x], \star_0)^\bullet \rightarrow \mathbf{k}$ that sends μ to $\mu(1)$. In particular, the value $a := \mu(1) \in \mathbf{k}$ determines μ uniquely by

$$(18) \quad \mu(x^n) = \frac{1}{n+1} a^{n+1}$$

for all $n \in \mathbb{N}$. Moreover, the codimension of $\ker(\mu)$ equals 0 for $a = 0$, and 1 for $a \neq 0$.

Proof. (a) Note that $\{u_n := nx^{n-k-1} \mid n \geq k+1\}$ is a \mathbf{k} -linear basis of $\mathbf{k}[x]$ with

$$\begin{aligned} u_m \star_k u_n &= mx^{m-k-1} J_0(x^k \cdot nx^{n-k-1}) + nx^{n-k-1} J_0(x^k \cdot mx^{m-k-1}) \\ &= mx^{m-k-1} x^n + nx^{n-k-1} x^m = (m+n)x^{m+n-k-1} = u_{m+n}. \end{aligned}$$

Thus the \mathbf{k} -linear map induced by $u_n \mapsto x^n$ ($n \geq k+1$) is an isomorphism $(\mathbf{k}[x], \star_k) \rightarrow x^{k+1}\mathbf{k}[x]$ of nonunitary \mathbf{k} -algebras as claimed.

(b) Since $x\mathbf{k}[x]$ is the free nonunitary commutative \mathbf{k} -algebra on x , so is $(\mathbf{k}[x], \star_0)$ by the isomorphism from (a). Then the bijection follows from the universal property of free nonunitary commutative \mathbf{k} -algebra on x . Note that under the isomorphism from (a), the generator x of $x\mathbf{k}[x]$ corresponds to the generator $1 = u_1$ of $(\mathbf{k}[x], \star_0)$.

To prove Eq. (18), we use induction on n . For the base case $n = 0$, we have $\mu(1) = a$ by the definition of a . Now suppose Eq. (18) has been proved for a fixed n . Since

$$1 \star_0 x^n = J_0(x^n) + x^n J_0(1) = \frac{n+2}{n+1} x^{n+1}$$

and μ is an \mathbf{k} -algebra homomorphism, we have

$$\mu\left(\frac{n+2}{n+1} x^{n+1}\right) = \mu(1 \star_0 x^n) = \mu(1)\mu(x^n) = \frac{1}{n+1} a^{n+2},$$

applying the induction hypothesis in the last step. Thus we obtain $\mu(x^{n+1}) = \frac{1}{n+2} a^{n+2}$, and the induction is complete. The last statement follows since the codimension of $\ker(\mu)$ equals the dimension of $\text{im}(\mu)$ and μ is surjective if and only if $\mu(1) \neq 0$. \square

At this juncture, the results accumulated are sufficient for classifying all Rota-Baxter operators P satisfying the differential relation $\partial \circ P = 1_{\mathbf{k}[x]}$. This is an important special case since it states that all *indefinite integrals* are analytically modelled.

Theorem 4.5. *We have $\text{RBO}_1(\mathbf{k}[x]) = \{J_a \mid a \in \mathbf{k}\}$.*

Proof. The inclusion from right to left is clear, so assume $P \in \text{RBO}_1(\mathbf{k}[x])$. By Theorem 4.3, there exists $\mu \in (\mathbf{k}[x], \star_0)^\bullet$ such that $P = J_0 - \mu$. Setting now $a := \mu(1)$, Theorem 4.4 asserts that $\mu(x^n) = \frac{1}{n+1} a^{n+1}$ for all $n \in \mathbb{N}$. Then

$$P(x^n) = J_0(x^n) - \mu(x^n) = \frac{x^{n+1} - a^{n+1}}{n+1} = J_a(x^n)$$

so that $P = J_a$, and the inclusion from left to right is established. \square

For classifying the Rota-Baxter operators P with $\partial \circ P = x^k$ ($k > 0$) we must determine all algebra homomorphisms μ with respect to the multiplication \star_k . At this point, we have to restrict ourselves to the field $\mathbf{k} = \mathbb{R}$ since we shall make use of the order on the *reals* in the next two lemmas.

Lemma 4.6. *Let $\mu: (\mathbb{R}[x], \star_{2\ell+1}) \rightarrow \mathbb{R}$ be an \mathbb{R} -algebra homomorphism with $\ell \geq 0$. Then we have $\mu(1) \geq 0$.*

Proof. Since $1 \star_{2\ell+1} 1 = 2 J_0(x^{2\ell+1}) = x^{2\ell+2}/(\ell+1)$ and μ is an \mathbb{R} -algebra homomorphism, we obtain $c^2 = \mu(1 \star_{2\ell+1} 1) = \mu(x^{2\ell+2}/(\ell+1))$, where we have set $c := \mu(1)$. Hence we get the relation $\mu(x^{2\ell+2}) = (\ell+1)c^2$. We have also

$$1 \star_{2\ell+1} x^{2\ell+2} = J_0(x^{4\ell+3}) + x^{2\ell+2} J_0(x^{2\ell+1}) = \frac{3}{4\ell+4} x^{4\ell+4},$$

which implies by the \mathbb{R} -algebra homomorphism property and the previous relation that

$$(19) \quad \mu(x^{4\ell+4}) = \frac{4}{3}(\ell+1)^2 c^3.$$

Next we observe that $x^{\ell+1} \star_{2\ell+1} x^{\ell+1} = 2x^{\ell+1} J_0(x^{3\ell+2}) = (2/3)x^{4\ell+4}/(\ell+1)$. Setting $\tilde{c} := \mu(x^{\ell+1})$, this yields yet another relation

$$(20) \quad \mu(x^{4\ell+4}) = \frac{3}{2}(\ell+1)\tilde{c}^2.$$

Combining Eqs. (19) and (20), we obtain $\frac{4}{3}(\ell+1)^2 c^3 = \frac{3}{2}(\ell+1)\tilde{c}^2$ and thus $c = \sqrt[3]{\frac{9}{8(\ell+1)}\tilde{c}^2} \geq 0$. \square

Lemma 4.7. *Let $\mu: (\mathbb{R}[x], \star_k) \rightarrow \mathbb{R}$ be an \mathbb{R} -algebra homomorphism for $k \in \mathbb{N}$. Then there exists a number $a \in \mathbb{R}$ such that $\mu(1) = a^{k+1}/(k+1)$.*

Proof. We set $c := \mu(1)$ and $a := \sqrt[k+1]{(k+1)c}$. If $k = 2\ell + 1$ with $\ell \in \mathbb{N}$, Lemma 4.6 implies that $c \geq 0$ and we may extract an even root to obtain $a \in \mathbb{R}$. If on the other hand $k = 2\ell$ for $\ell \in \mathbb{N}$, the root in a is odd and hence clearly $a \in \mathbb{R}$ also in this case. \square

The number a ensured by the previous lemma serves to characterize the associated measure μ of the Rota-Baxter operator underlying the double product \star_k . Analytically speaking, Analytically speaking, $\mu(1)$ is the Riemann integral over $[0, a]$.

Proposition 4.8. *Let $\mu: (\mathbb{R}[x], \star_k) \rightarrow \mathbb{R}$ be an \mathbb{R} -algebra homomorphism. Then there exists a number $a \in \mathbb{R}$ such that $\mu(x^n) = a^{n+k+1}/(n+k+1)$ for all $n \in \mathbb{N}$. In particular, μ is uniquely determined by a .*

Proof. We prove the claim by induction on $n \in \mathbb{N}$. In the base case $n = 0$, Lemma 4.7 yields $\mu(1) = a^{k+1}/(k+1)$. Suppose now the claim has been proved up to a fixed n . Since

$$1 \star_k x^{n-k} = J_0(x^n) + x^{n-k} J_0(x^k) = \frac{n+k+2}{(n+1)(k+1)} x^{n+1}$$

and μ is an \mathbb{R} -algebra homomorphism, we have

$$\mu\left(\frac{n+k+2}{(n+1)(k+1)} x^{n+1}\right) = \mu(1 \star_k x^{n-k}) = \mu(1)\mu(x^{n-k}) = \frac{1}{(n+1)(k+1)} a^{n+k+2},$$

where we have applied the induction hypothesis in the last step since $n-k \leq n$. But this gives immediately $\mu(x^{n+1}) = a^{n+k+2}/(n+k+2)$, which completes the induction. \square

Since the number a of the proposition above characterizes the associated measures, we obtain now the *desired classification* of the Rota-Baxter operators P on $\mathbb{R}[x]$ that satisfy the differential relation $\partial \circ P = x^k$. The number a plays the role of the initialization point of the integral (we regain the standard integral J_0 for $a = 0$ since then the associated measure is zero).

Theorem 4.9. *We have $\text{RBO}_{x^k}(\mathbb{R}[x]) = \{J_a x^k \mid a \in \mathbb{R}\}$ for any $k \in \mathbb{N}$.*

Proof. The inclusion from right to left is clear, so assume $P \in \text{RBO}_{x^k}(\mathbb{R}[x])$. Then Theorem 4.3 yields an \mathbb{R} -algebra homomorphism $\mu: (\mathbb{R}[x], \star_k) \rightarrow \mathbb{R}$ such that $P = J_0 x^k - \mu$. By Proposition 4.8, there exists a number $a \in \mathbb{R}$ such that $\mu(x^n) = a^{n+k+1}/(n+k+1)$. Thus we have

$$P(x^n) = J_0(x^{n+k}) - \mu(x^n) = (x^{n+k+1} - a^{n+k+1})/(n+k+1) = J_a(x^{n+k}),$$

so that $P = J_a x^k$, and the inclusion from left to right is established. \square

As mentioned earlier, it is tempting to generalize the above result from monomials to *arbitrary polynomials*. Together with Theorem 4.1, this would imply that

$$\text{RBO}_*(\mathbf{k}[x]) = \bigcup_{r \in \mathbf{k}[x]^\times} \text{RBO}_r(\mathbf{k}[x]) = \{J_a r \mid a \in \mathbf{k}, r \in \mathbf{k}[x]^\times\},$$

for the case $\mathbf{k} = \mathbb{R}$. The missing inclusion is as follows.

Conjecture 4.10. *We have $\text{RBO}_r(\mathbb{R}[x]) \subseteq \{J_a r \mid a \in \mathbb{R}\}$ for any $r \in \mathbb{R}[x]^\times$.*

In the rest of this paper, we add some preliminary results in support of this conjecture. Let us call a Rota-Baxter operator P on $\mathbb{R}[x]$ *initialized* at a point $a \in \mathbb{R}$ if $\text{ev}_a \circ P$ is the zero operator, where $\text{ev}_a: \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ denotes evaluation at a . The typical case is when $P = J_a r$. It is easy to see that Conjecture 4.10 is equivalent to the claim that all Rota-Baxter operators in $\text{RBO}_r(\mathbb{R}[x])$ are initialized. Indeed, if P is initialized at a , then we may multiply the differential law $\partial \circ P = r$ by J_a from the left to obtain $P = J_a r$ since we have $J_a \partial = 1_{\mathbb{R}[x]} - \text{ev}_a$. So for proving Conjecture 4.10 one has to determine the initialization point a from a given Rota-Baxter operator P and $r \in \mathbb{R}[x]^\times$. If P is already known to be of the form $J_a r$, this can be done as follows.

Lemma 4.11. *For the Rota-Baxter operator $P = J_a r$ with $a \in \mathbb{R}$ and $r \in \mathbb{R}[x]^\times$ we have*

$$(21) \quad a = \frac{P(2xr' + r) - xr^2}{P(2r') - r^2},$$

provided $r(a) \neq 0$. On the other hand, if $r(a) = 0$ then $P = (r - J_a r') \circ J_0$.

Proof. Let us first consider the generic case $r(a) \neq 0$. Using the differential law $\partial \circ P = r$, one sees immediately that numerator and denominator are both constants since they vanish under ∂ . Moreover, the denominator cannot be zero since we have

$$P(2r') = \int_a^x (r^2)' = r^2 - r(a)^2 \neq r^2$$

by the assumption of genericity. Integrating $(r^2 r^{(i)})' = 2r r' r^{(i)} + r^2 r^{(i+1)}$ from a to x , we obtain

$$(22) \quad r^2 r^{(i)} - r(a)^2 r^{(i)}(a) = P(2r' r^{(i)} + r r^{(i+1)}).$$

Assuming r has degree n , we can write

$$r = 1 + r_1 x + r_2 \frac{x^2}{2!} + \cdots + r_n \frac{x^n}{n!}$$

so that $r^{(n-1)} = r_{n-1} + r_n x$ and $r^{(n)} = r_n$. Substituting $i = n-1$ and $r(a)^2 = r^2 - P(2r')$ in Eq. (22), we obtain the relation

$$(23) \quad (r_{n-1} + r_n x) r^2 - (r^2 - P(2r'))(r_{n-1} + r_n a) = P(2r_{n-1} r' + 2r_n x r' + r_n r),$$

which simplifies to $(x-a)r^2 = P(2xr' - 2ar' + r)$. Solving this for a gives Eq. (21).

Now assume $r(a) = 0$. Then for $f \in \mathbb{R}[x]$ we obtain

$$P_f' = \int_a^x r f' = [r f]_a^x - \int_a^x r' f = r f - J_a r' f$$

and hence by $(J_0 f)' = f$ the required identity $P f = (r J_0) f - (J_a r' J_0) f = (r - J_a r') J_0(f)$. \square

Lemma 4.11 suggests the following strategy for proving Conjecture 4.10. Given an arbitrary $P \in \text{RBO}_r(\mathbb{R}[x])$, we determine first the denominator of Eq. (21). If it vanishes, we try to find $\tilde{P} \in \text{RBO}_{r'}(\mathbb{R}[x])$ with $P = (r - \tilde{P}) \circ J_0$, and we use induction on the degree of r to handle \tilde{P} . In the generic case of non-vanishing denominator, we compute the value of a from Eq. (21), and it suffices to prove that P is initialized at a . For doing this, the first step would be to ascertain that $r(a)^2 = r^2 - P(2r')$. This would imply that $P(r')$ vanishes at $x = a$ and hence also $P(2xr' + r)$ by Eq. (23). Using the Rota-Baxter axiom and the above relations, one can produce polynomials p such that $P(p)$ vanishes at $x = a$. If this is done for sufficiently many polynomials p to generate $\mathbb{R}[x]$ as a real vector space, we are done. Here is an example of a class of polynomials where one can infer vanishing at $x = a$ provided $r(a)^2 = r^2 - P(2r')$ has been established. For $P = J_a r$, it recovers the fact that $J_a(r' r^{2k+1}) = (2k+2)^{-1} J_a((r^{2k+2})') = (2k+2)^{-1}(r^{2k+2} - r(a)^{2k+2})$.

Lemma 4.12. *Let $P \in \text{RBO}_r(\mathbb{R}[x])$ be arbitrary. Then we have $P(r' r^{2k}) = (2k+2)^{-1}(r^{2k+2} - c^{k+1})$ for $c := r^2 - P(2r') \in \mathbb{R}$ and all $k \geq 0$.*

Proof. We use induction on k . The base case $k = 0$ is immediate from the definition of c . Now assume the claim for all degrees below a fixed $k > 0$; we prove it for k . By the Rota-Baxter axiom and the definition of c we have

$$\begin{aligned} P(r')^{k+1} &= (k+1)! P_{r'}^{k+1}(1) = (k+1)! P(r' P_{r'}^k(1)) = (k+1) P(r' P(r')^k) \\ &= 2^{-k}(k+1) P(r'(r^2 - c)^k), \end{aligned}$$

where $P_{r'} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ is defined by $P_{r'}(p) := P(r' p)$. Substituting the defining relation of c on the left-hand side, we obtain $(r^2 - c)^{k+1} = 2(k+1) P(r'(r^2 - c)^k)$, so the binomial theorem yields

$$(2k+2) P(r' r^{2k}) = (r^2 - c)^{k+1} - 2(k+1) \sum_{l=0}^{k-1} \binom{k}{l} (-c)^{k-l} P(r' r^{2l}).$$

Applying the induction hypothesis leads to

$$\begin{aligned} (2k+2) P(r' r^{2k}) &= (r^2 - c)^{k+1} - (k+1) \sum_{l=0}^{k-1} \binom{k}{l} \frac{(-c)^{k-l}}{l+1} ((r^2)^{l+1} - c^{l+1}) \\ &= (r^2 - c)^{k+1} + (r^{2k+2} - c^{k+1}) - (k+1) \sum_{l=0}^k \binom{k}{l} \frac{(-c)^{k-l}}{l+1} ((r^2)^{l+1} - c^{l+1}). \end{aligned}$$

For evaluating the above sum, just note that integrating $(x-c)^k$ from α to β and using the binomial theorem gives

$$\frac{(\beta - c)^{k+1} - (\alpha - c)^{k+1}}{k+1} = \sum_{l=0}^k \binom{k}{l} \frac{(-c)^{k-l}}{l+1} (\beta^{l+1} - \alpha^{l+1}),$$

which may be evaluated at $(\alpha, \beta) = (c, r^2)$ in the previous sum to obtain

$$(2k+2) P(r' r^{2k}) = (r^2 - c)^{k+1} + r^{2k+2} - c^{k+1} - (r^2 - c)^{k+1} = r^{2k+2} - c^{k+1},$$

which completes the induction. \square

We conclude with a simple result about the double product \star in the general case of $J_a r$. This lemma is a kind of analogy (though not a generalization) of Theorem 4.4(a). In fact, the two results coincide for $r = x$.

Lemma 4.13. *Let \star be the double product corresponding to the Rota-Baxter operator $J_a r$ and set $\rho = r(a)$. Then the non-unitary subalgebra of $(\mathbf{k}[x], \star)$ generated by $u_n = nr^{n-2}r'$ ($n \geq 2$) is isomorphic to the non-unitary subalgebra of $(\mathbf{k}[x], \cdot)$ generated by $x^n - \rho^n$ ($n \geq 2$).*

Proof. The double product of the basis elements u_m ($m \geq 2$) and u_n ($n \geq 2$) is given by

$$\begin{aligned} u_m \star u_n &= mn r^{m-2} r' J_a r^{n-1} r' + mn r^{n-2} r' J_a r^{m-1} r' = m r^{m-2} r' (r^n - \rho^n) + n r^{n-2} r' (r^m - \rho^m) \\ &= u_{m+n} - \rho^n u_m - \rho^m u_n, \end{aligned}$$

so the \mathbf{k} -linear map ϕ defined by $\phi(u_m) = x^m - \rho^m$ is a homomorphism of nonunitary \mathbf{k} -algebras since we have

$$(x^m - \rho^m)(x^n - \rho^n) = (x^{m+n} - \rho^{m+n}) - \rho^n(x^m - \rho^m) - \rho^m(x^n - \rho^n).$$

The map ϕ is clearly bijective as it maps a \mathbf{k} -basis to a \mathbf{k} -basis. \square

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REFERENCES

- [1] M. Aguiar, On the associative analog of Lie bialgebras, *J. Algebra*. **244** (2001), 492-532. [1](#)
- [2] F. V. Atkinson, Some aspects of Baxter's functional equation, *J. Math. Anal. and Applications* **7** (1963), 1-30. [1](#)
- [3] C. Bai, A unified algebraic approach to the classical Yang-Baxter equations, *J. Phys. A: Math. Theor.* (2007) **40** 11073-11082. [1](#)
- [4] C. Bai, L. Guo and X. Ni, Nonablian generalized Lax pairs, the classical Yang-Baxter equation and postLie algebra, *Comm. Math. Phys.* **297** (2010) 553-596. [1](#)
- [5] C. Bai, O. Bellier, L. Guo and X. Ni, Splitting of operations, Manin products and Rota-Baxter operators, *Int. Math. Res. Not. IMRN.* (2012); doi: 10.1093/imrn/rnr266, [arXiv:1106.6080](#). [1](#)
- [6] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.* **10** (1960), 731-742. [1](#), [2](#)
- [7] G. Birkhoff, Moyennes de fonctions bornées, *Coil. Internat. Centre Nat. Recherche Sci. (Paris), Algèbre Théorie Nombres* **24** (1949), 149-153. [7](#)
- [8] P. Cartier, On the structure of free Baxter algebras, *Adv. Math.* **9** (1972), 253-265. [1](#)
- [9] P. M. Cohn, *Basic Algebra: Groups, Rings and Fields*, Springer, New York 2005. [14](#)
- [10] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, *Comm. Math. Phys.*, **210** (2000), 249-273. [1](#)
- [11] K. Ebrahimi-Fard, L. Guo and D. Kreimer, Spitzer's Identity and the Algebraic Birkhoff Decomposition in pQFT, *J. Phys. A: Math. Gen.* **37** (2004), 11037-11052. [1](#)
- [12] K. Ebrahimi-Fard, L. Guo and D. Manchon, Birkhoff type decompositions and the Baker-Campbell-Hausdorff recursion, *Comm. in Math. Phys.* **267** (2006), 821-845. [1](#)
- [13] X. Gao, L. Guo and S. Zheng, Gröbner-Shirshov bases and free commutative integro-differential algebras, *Jour. Algebra and Its Appl.* **13** (2014), 1350160. [2](#)
- [14] L. Guo, WHAT IS a Rota-Baxter algebra? *Notice Amer. Math. Soc.* **56** (2009), 1436-1437. [1](#)

- [15] L. Guo, *An Introduction to Rota-Baxter Algebra*, International Press (US) and Higher Education Press (China), 2012. [1](#), [2](#), [3](#)
- [16] L. Guo, W. Keigher, Baxter algebras and shuffle products, *Adv. Math.*, **150** (2000), 117-149. [1](#)
- [17] L. Guo and W. Keigher, On differential Rota-Baxter algebras, *J. Pure Appl. Algebra*, **212** (2008), 522-540. [2](#)
- [18] L. Guo and J. Pei, Averaging algebra, Schröder numbers and rooted trees, [arXiv:1401.7386](#). [7](#)
- [19] L. Guo, G. Regensburger and M. Rosenkranz, On integro-differential algebras. *J. Pure Appl. Algebra* **218** (2014), 456-473. [2](#)
- [20] L. Guo and B. Zhang, Renormalization of multiple zeta values, *J. Algebra*. **319** (2008), 3770-3809. [1](#)
- [21] A. Nowicki, *Polynomial Derivations and their Rings of Constants*, Nicolaus Copernicus University, Torun, 1994. [14](#)
- [22] O. Reynolds, On the dynamic theory of incompressible viscous fluids, *Phil. Trans. Roy. Soc. A* **136** (1895), 123-164. [7](#)
- [23] M. Rosenkranz and G. Regensburger, Solving and factoring boundary problems for linear ordinary differential equations in differential algebra, *J. Symbolic Comput.* **43** (2008), 515-544. [2](#)
- [24] G.-C. Rota, Baxter algebras and combinatorial identities I, II, *Bull. Amer. Math. Soc.* **75** (1969), 325-329, 330-334. [1](#)
- [25] G.-C. Rota, Baxter operators, an introduction, In: “Gian-Carlo Rota on Combinatorics, Introductory papers and commentaries”, Joseph P.S. Kung, Editor, Birkhäuser, Boston, 1995. [2](#)
- [26] J. C. Rosales, P. A. García-Sánchez, *Finitely Generated Commutative Monoids*, Nova Science Publisher, New York, 1991. [12](#)
- [27] M. A. Semenov-Tian-Shansky, What is a classical r -matrix?, *Funct. Ana. Appl.*, **17** (1983) 259-272. [1](#)
- [28] F. Spitzer, A combinatorial lemma and its application to probability theory, *Trans. Amer. Math. Soc.* **82** (1956), 323-339. [1](#)
- [29] L.B. van der Waerden, *Algebra I* (in German), Springer, 9th edition, 1993. [14](#)

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