A Permutation-free Calculus for Lax Logic

Jacob M. Howe

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Computer Science Division
University of St Andrews, Scotland, KY16 9SS
jacob@dcs.st-and.ac.uk
http://www-theory.cs.st-and.ac.uk/~jacob

1 Introduction

There has been recent interest in a modal logic of Curry (see [Cur52]), now called Lax Logic. Here the modality ($\circ$, *somehow*) has some of the properties of both necessity and possibility. For more about Lax Logic see, for example, [FM97], [BBdP95].

The work of Herbelin ([Her95]), developed by Dyckhoff & Pinto ([DP96], [DP98]), introduces a sequent calculus, whose proofs can be translated in a 1-1 manner to normal natural deductions. This simple Gentzen system (which we call MJ following Dyckhoff and Pinto; Herbelin called it LJT) gives an efficient syntax-directed calculus for enumerating proofs, a task which is considerably harder in the natural deduction calculus itself.

In this paper the same 'permutation-free' techniques used to develop MJ are applied to Lax Logic, giving a 'permutation-free' calculus for Lax Logic. As our starting point we take the above cited papers of Fairtlough & Mendler and of Benton, Bierman & de Paiva.

2 Natural Deduction

First we give the natural deduction calculus for propositional Lax Logic. This is taken directly from [BBdP95], and can be seen in Figure 1.

We now look at the normalisation steps. Again these are taken directly from [BBdP95]. As the reduction rules for the intuitionistic connectives are completely standard, we do not include them here, concentrating instead on those involving the modality. We give these reductions in a tree rather than sequent style.

First the β-reduction:

–

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{\circ A}\,(\circ_{\mathcal{I}}) \quad \begin{matrix}[A]\\ \vdots\\ \circ B\end{matrix}}{\circ B}\,(\circ_{\varepsilon}) \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ A\\ \vdots\\ \circ B\end{matrix}$$

Now we give the commuting conversions (or c-reductions) involving the modality:

–

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \circ A\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ \circ B\end{matrix}}{\circ B}\,(\circ_{\varepsilon}) \quad \begin{matrix}[B]\\ \vdots\\ \circ C\end{matrix}}{\circ C}\,(\circ_{\varepsilon}) \quad\rightsquigarrow\quad \dfrac{\begin{matrix}\vdots\\ \circ A\end{matrix}\quad \dfrac{\begin{matrix}[A]\\ \vdots\\ \circ B\end{matrix}\quad \begin{matrix}[B]\\ \vdots\\ \circ C\end{matrix}}{\circ C}\,(\circ_{\varepsilon})}{\circ C}\,(\circ_{\varepsilon})$$

$$\dfrac{}{\Gamma, P \vdash P}\,(ax) \qquad \dfrac{}{\Gamma \vdash \top}\,(\top)$$

$$\dfrac{\Gamma, P \vdash Q}{\Gamma \vdash P \supset Q}\,(\supset_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash P \supset Q \quad \Gamma \vdash P}{\Gamma \vdash Q}\,(\supset_{\varepsilon})$$

$$\dfrac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\,(\wedge_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P}\,(\wedge_{\varepsilon 1}) \qquad \dfrac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}\,(\wedge_{\varepsilon 2})$$

$$\dfrac{\Gamma \vdash P}{\Gamma \vdash P \vee Q}\,(\vee_{\mathcal{I}1}) \qquad \dfrac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q}\,(\vee_{\mathcal{I}2})$$

$$\dfrac{\Gamma \vdash P \vee Q \quad \Gamma, P \vdash R \quad \Gamma, Q \vdash R}{\Gamma \vdash R}\,(\vee_{\varepsilon})$$

$$\dfrac{\Gamma \vdash P}{\Gamma \vdash \circ P}\,(\circ_{\mathcal{I}}) \qquad \dfrac{\Gamma \vdash \circ P \quad \Gamma, P \vdash \circ Q}{\Gamma \vdash \circ Q}\,(\circ_{\varepsilon})$$

Figure 1: Sequent style presentation of natural deduction for Lax Logic

–

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad\begin{matrix}[A]\\ \vdots\\ \circ C\end{matrix}\quad\begin{matrix}[B]\\ \vdots\\ \circ C\end{matrix}}{\circ C}\,(\vee_\varepsilon)\quad\begin{matrix}[C]\\ \vdots\\ \circ D\end{matrix}}{\circ D}\,(\circ_\varepsilon)$$

$$\rightsquigarrow\quad\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad\dfrac{\begin{matrix}[A]\\ \vdots\\ \circ C\end{matrix}\quad\begin{matrix}[C]\\ \vdots\\ \circ D\end{matrix}}{\circ D}\,(\circ_\varepsilon)\quad\dfrac{\begin{matrix}[B]\\ \vdots\\ \circ C\end{matrix}\quad\begin{matrix}[C]\\ \vdots\\ \circ D\end{matrix}}{\circ D}\,(\circ_\varepsilon)}{\circ B}\,(\vee_\varepsilon)$$

Definition 1 *A natural deduction is said to be in* β, c-**normal form** *when no* β-*reductions and no* c-*reductions are applicable.*

We now give a presentation of a restricted version of natural deduction for Lax Logic. In this calculus, the only proofs are those that are in β, c-normal form. This calculus has two kinds of 'sequents', differentiated by their consequence relations,$\rhd$ and $\rhd\!\rhd$. Rules are applicable only when the premisses are of a certain kind, and the conclusions are then of one kind or the other. Thus the valid deductions are restricted. This calculus, which we shall call NLAX, is given in Figure 2.

Proposition 1 *The calculus NLAX only allows deductions to which no* β-*reductions and no* c-*reductions are applicable. Moreover, it allows all* β, c-*normal deductions.*

PROOF: By inspection one can see that deductions to which one could apply a reduction to are not allowed in NLAX because they would involve a rule application with a premiss of an incorrect category.

It is easy to see that by use of the (M) rule, all other deductions are possible. ■

3 Term Assignment

In this section we give a term assignment system for NLAX. Moggi gave a λ-calculus, which he called the *c*omputational λ-calculus. This calculus naturally matches Lax Logic, as can be seen in Figure 3. More about the computational λ-calculus can be found in [BBdP95].

We give this again using an abstract syntax with explicit constructors that we prefer. First we give a translation of Moggi's terms to ours, and then we give yet another presentation of natural deduction for Lax Logic, this time annotated with terms in our syntax, in Figure 4.

Translation: Moggi's terms $\rightsquigarrow$ our terms

$$x \rightsquigarrow var(x)$$

$$* \rightsquigarrow *$$

$$\lambda x.e \rightsquigarrow \lambda x.e$$

$$e\ f \rightsquigarrow ap(e, f)$$

$$(e, f) \rightsquigarrow pr(e, f)$$

$$fst(e) \rightsquigarrow fst(e)$$

$$snd(e) \rightsquigarrow snd(e)$$

$$inl(e) \rightsquigarrow i(e)$$

$$inr(e) \rightsquigarrow j(e)$$

$$case\ e\ of\ inl(x)\ \rightarrow\ f \mid inr(y)\ \rightarrow\ g \rightsquigarrow wn(e, x.f, y.g)$$

$$val(e) \rightsquigarrow smhi(e)$$

$$\frac{}{\Gamma, P \rhd P}\ (ax) \qquad \frac{}{\Gamma \rhd\!\!\rhd \top}\ (\top) \qquad \frac{\Gamma \rhd P}{\Gamma \rhd\!\!\rhd P}\ (M)$$

$$\frac{\Gamma, P \rhd\!\!\rhd Q}{\Gamma \rhd\!\!\rhd P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \rhd P \supset Q \quad \Gamma \rhd\!\!\rhd P}{\Gamma \rhd Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd P \quad \Gamma \rhd\!\!\rhd Q}{\Gamma \rhd\!\!\rhd P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \rhd P \wedge Q}{\Gamma \rhd P}\ (\wedge_{\varepsilon 1}) \qquad \frac{\Gamma \rhd P \wedge Q}{\Gamma \rhd Q}\ (\wedge_{\varepsilon 2})$$

$$\frac{\Gamma \rhd\!\!\rhd P}{\Gamma \rhd\!\!\rhd P \vee Q}\ (\vee_{\mathcal{I}1}) \qquad \frac{\Gamma \rhd\!\!\rhd Q}{\Gamma \rhd\!\!\rhd P \vee Q}\ (\vee_{\mathcal{I}2})$$

$$\frac{\Gamma \rhd P \vee Q \quad \Gamma, P \rhd\!\!\rhd R \quad \Gamma, Q \rhd\!\!\rhd R}{\Gamma \rhd\!\!\rhd R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd P}{\Gamma \rhd\!\!\rhd \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \rhd \circ P \quad \Gamma, P \rhd\!\!\rhd \circ Q}{\Gamma \rhd\!\!\rhd \circ Q}\ (\circ_{\varepsilon})$$

Figure 2: NLAX: Sequent style presentation for normal natural deduction for Lax Logic

$$\frac{}{\Gamma, x : P \vdash x : P}\ (ax) \qquad \frac{}{\Gamma \vdash * : \top}\ (\top)$$

$$\frac{\Gamma, x : P \vdash e : Q}{\Gamma \vdash \lambda x.e : P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \supset Q \quad \Gamma \vdash f : P}{\Gamma \vdash e\ f : Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P \quad \Gamma \vdash f : Q}{\Gamma \vdash (e, f) : P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \wedge Q}{\Gamma \vdash fst(e) : P}\ (\wedge_{\varepsilon 1}) \qquad \frac{\Gamma \vdash e : P \wedge Q}{\Gamma \vdash snd(e) : Q}\ (\wedge_{\varepsilon 2})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash inl(e) : P \vee Q}\ (\vee_{\mathcal{I}1}) \qquad \frac{\Gamma \vdash e : Q}{\Gamma \vdash inr(e) : P \vee Q}\ (\vee_{\mathcal{I}2})$$

$$\frac{\Gamma \vdash e : P \vee Q \quad \Gamma, x : P \vdash f : R \quad \Gamma, y : Q \vdash g : R}{\Gamma \vdash case\ e\ of\ inl(x)\ \rightarrow\ f \mid inr(y)\ \rightarrow\ g : R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash val(e) : \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : \circ P \quad \Gamma, x : P \vdash f : \circ Q}{\Gamma \vdash let\ x\ \Leftarrow\ e\ in\ f : \circ B}\ (\circ_{\varepsilon})$$

Figure 3: Sequent style presentation of natural deduction for Lax Logic, with Moggi's computational λ terms

$$let\ x \ \Leftarrow\ e\ in\ f \rightsquigarrow smhe(e, x.f)$$

We are interested in the normal natural deductions for Lax Logic as canonical proofs. We now restrict the terms that can be built, in order that they match our restricted natural deduction calculus NLAX, giving us canonical proof objects. (That is, no reductions will be applicable at the term level; the term reductions match the β- and c-reductions for types given earlier). The terms come in two syntactic categories, A and N. V is the category of variables. The extra constructor $an(A)$ matches the (M) rule of NLAX.

A terms:

$$var(x) \mid ap(A, N) \mid fstA \mid snd(A)$$

N terms:

$$* \mid an(A) \mid \lambda V.N \mid pr(N, N) \mid i(N) \mid j(N)$$

$$wn(A, V.N, V.N) \mid smhi(N) \mid smhe(A, V.N)$$

In Figure 5 we give one final presentation of a natural deduction calculus for Lax Logic, this time NLAX together with proof annotations.

$$\frac{}{\Gamma, x : P \vdash var(x) : P}\ (ax) \qquad \frac{}{\Gamma \vdash * : \top}\ (\top)$$

$$\frac{\Gamma, x : P \vdash e : Q}{\Gamma \vdash \lambda x.e : P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \supset Q \quad \Gamma \vdash f : P}{\Gamma \vdash ap(e, f) : Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P \quad \Gamma \vdash f : Q}{\Gamma \vdash pr(e, f) : P \wedge Q}\ (\wedge_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : P \wedge Q}{\Gamma \vdash fst(e) : P}\ (\wedge_{\varepsilon 1}) \qquad \frac{\Gamma \vdash f : P \wedge Q}{\Gamma \vdash snd(e) : Q}\ (\wedge_{\varepsilon 2})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash i(e) : P \vee Q}\ (\vee_{\mathcal{I}1}) \qquad \frac{\Gamma \vdash e : Q}{\Gamma \vdash j(e) : P \vee Q}\ (\vee_{\mathcal{I}2})$$

$$\frac{\Gamma \vdash e : P \vee Q \quad \Gamma, x : P \vdash f : R \quad \Gamma, y : Q \vdash g : R}{\Gamma \vdash wn(e, x.f, y.g) : R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \vdash e : P}{\Gamma \vdash smhi(e) : \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : \circ P \quad \Gamma, x : P \vdash f : \circ Q}{\Gamma \vdash smhe(e, x.f) : \circ Q}\ (\circ_{\varepsilon})$$

Figure 4: Sequent style presentation of natural deduction for Lax Logic

$$\frac{}{\Gamma, x; P \rhd var(x) : P}\ (ax) \qquad \frac{}{\Gamma \rhd\!\!\rhd * : \top}\ (\top)$$

$$\frac{\Gamma, x : P \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd \lambda x.N : P \supset Q}\ (\supset_{\mathcal{I}}) \qquad \frac{\Gamma \rhd A : P \supset Q \quad \Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd ap(A, N) : Q}\ (\supset_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd N_1 : P \quad \Gamma \rhd\!\!\rhd N_2 : Q}{\Gamma \rhd\!\!\rhd pr(N_1, N_2) : P \wedge Q}\ (\wedge_{\mathcal{I}})$$

$$\frac{\Gamma \rhd A : P \wedge Q}{\Gamma \rhd fst(A) : P}\ (\wedge_{\varepsilon 1}) \qquad \frac{\Gamma \rhd A : P \wedge Q}{\Gamma \rhd snd(A) : Q}\ (\wedge_{\varepsilon 2})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd i(N) : P \vee Q}\ (\vee_{\mathcal{I}1}) \qquad \frac{\Gamma \rhd\!\!\rhd N : Q}{\Gamma \rhd\!\!\rhd j(N) : P \vee Q}\ (\vee_{\mathcal{I}2})$$

$$\frac{\Gamma \rhd A : P \vee Q \quad \Gamma, x_1 : P \rhd\!\!\rhd N_1 : R \quad \Gamma, x_2 : Q \rhd\!\!\rhd N_2 : R}{\Gamma \rhd\!\!\rhd wn(A, x_1.N_1, x_2.N_2) : R}\ (\vee_{\varepsilon})$$

$$\frac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd smhi(N) : \circ P}\ (\circ_{\mathcal{I}}) \qquad \frac{\Gamma \rhd A : \circ P \quad \Gamma, x : P \rhd\!\!\rhd N : \circ Q}{\Gamma \rhd\!\!\rhd smhe(A, x.N) : \circ Q}\ (\circ_{\varepsilon})$$

Figure 5: NLAX with proof annotations

$$\frac{}{\Gamma, P \Rightarrow P}\ (ax) \qquad \frac{}{\Gamma \Rightarrow \top}\ (\top) \qquad \frac{\Gamma, P, P \Rightarrow R}{\Gamma, P \Rightarrow R}\ (C)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma, Q \Rightarrow R}{\Gamma, P \supset Q \Rightarrow R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\ (\wedge_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}1}) \qquad \frac{\Gamma, Q \Rightarrow R}{\Gamma, P \wedge Q \Rightarrow R}\ (\wedge_{\mathcal{L}2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\ (\vee_{\mathcal{R}2}) \qquad \frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma, P \vee Q \Rightarrow R}\ (\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow \circ P}\ (\circ_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \circ R}{\Gamma, \circ P \Rightarrow \circ R}\ (\circ_{\mathcal{L}})$$

Figure 6: Sequent Calculus for Lax Logic

4 Sequent Calculus

The stated aim of this paper is to present a sequent calculus for Lax Logic whose proofs naturally correspond in a 1-1 way to normal natural deductions for Lax Logic - i.e. the proofs of NLAX. In this section we give such a sequent calculus, but first we remind the reader of the sequent calculus as presented in [FM97] and [BBdP95]. This can be seen in Figure 6.

In fact, our presentation is slightly different from both those cited. The calculus in [BBdP95] doesn't mention structural rules, and so presumably the contexts in that paper are sets. [FM97] have both weakening and contraction on both the left and the right, plus exchange. Here the only structural rule we consider (or need) is contraction on the left. The contexts in our presentation are multisets. We leave all discussion of cut until later in the paper.

We now present a new sequent calculus which we call PFLAX ('permutation-free' Lax Logic). This calculus has two forms of judgment, $\Gamma \Rightarrow R$ and $\Gamma \xrightarrow{Q} R$, where the place above the single arrow with the privileged formula in it is known as the 'stoup'. The calculus is displayed in Figure 7.

The stoup is a form of focusing: the formula in the stoup is always principal in the premiss unless it is a disjunction or a 'somehow' formula. One might ask

$$\frac{}{\Gamma \xrightarrow{P} P}\,(ax) \qquad \frac{}{\Gamma \Rightarrow \top}\,(\top) \qquad \frac{\Gamma, P \xrightarrow{P} R}{\Gamma, P \Rightarrow R}\,(C)$$

$$\frac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\,(\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \supset Q} R}\,(\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P \quad \Gamma \Rightarrow Q}{\Gamma \Rightarrow P \wedge Q}\,(\wedge_{\mathcal{R}}) \qquad \frac{\Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{P \wedge Q} R}\,(\wedge_{\mathcal{L}1}) \qquad \frac{\Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \wedge Q} R}\,(\wedge_{\mathcal{L}2})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}1}) \qquad \frac{\Gamma \Rightarrow Q}{\Gamma \Rightarrow P \vee Q}\,(\vee_{\mathcal{R}2}) \qquad \frac{\Gamma, P \Rightarrow R \quad \Gamma, Q \Rightarrow R}{\Gamma \xrightarrow{P \vee Q} R}\,(\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow P}{\Gamma \Rightarrow \circ P}\,(\circ_{\mathcal{R}}) \qquad \frac{\Gamma, P \Rightarrow \circ R}{\Gamma \xrightarrow{\circ P} \circ R}\,(\circ_{\mathcal{L}})$$

Figure 7: The Sequent Calculus PFLAX

why we do not formulate the $(\circ_{\mathcal{L}})$ rule as follows

$$\frac{\Gamma \xrightarrow{P} \circ R}{\Gamma \xrightarrow{\circ P} \circ R} (\circ_{\mathcal{L}})$$

To answer this, we point out that the resulting calculus would not then match normal natural deductions in the manner we would like. We also invite them to consider proofs of the sequent $\circ\circ(P \wedge Q) \Rightarrow \circ(Q \wedge P)$.

5 Term Assignment for Sequent Calculus

We give a term assignment system for PFLAX. This we get by extending that given in [Her95], [DP96], [DP98]. The term calculus has two syntactic categories, M and Ms. V is the category of variables.

M::=

$$* \mid (V; Ms) \mid \lambda V.M \mid pair(M, M) \mid inl(M) \mid inr(M) \mid smhr(M)$$

Ms::=

$$[] \mid M :: Ms \mid p(Ms) \mid q(Ms) \mid when(V.M, V.M) \mid smhl(V.M)$$

These terms can easily be attached to PFLAX, as seen in Figure 8.

6 Results

Having presented the calculi for Lax Logic, we now prove that they have the properties we claim for them. We prove soundness and adequacy for PFLAX, and the equivalence of the term calculi. These results prove the desired correspondence.

The full details of these proofs are rather repetitive: therefore we only give the proofs for the $\supset, \circ$ fragment of Lax Logic. The rest of the calculus is the same as for intuitionistic logic as presented in [DP96], and the reader is referred to that paper for the remaining cases.

We start by giving pairs of functions that define translations between the term assignment systems for natural deduction and sequent calculus.

Sequent Calculus $\rightarrow$ Natural Deduction:

$\theta : M \rightarrow N$

$$\frac{}{\Gamma \xrightarrow{P} [] : P}\ (ax) \qquad \frac{}{\Gamma \Rightarrow * : \top}\ (\top) \qquad \frac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\ (C)$$

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \supset Q}\ (\supset_{\mathcal{R}}) \qquad \frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} (M :: Ms) : R}\ (\supset_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma \Rightarrow M_2 : Q}{\Gamma \Rightarrow pair(M_1, M_2) : P \wedge Q}\ (\wedge_{\mathcal{R}})$$

$$\frac{\Gamma \xrightarrow{P} Ms : R}{\Gamma \xrightarrow{P \wedge Q} p(Ms) : R}\ (\wedge_{\mathcal{L}1}) \qquad \frac{\Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \wedge Q} q(Ms) : R}\ (\wedge_{\mathcal{L}2})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow inl(M) : P \vee Q}\ (\vee_{\mathcal{R}1}) \qquad \frac{\Gamma \Rightarrow M : Q}{\Gamma \Rightarrow inr(M) : P \vee Q}\ (\vee_{\mathcal{R}2})$$

$$\frac{\Gamma, x_1 : P \Rightarrow M_1 : R \quad \Gamma, x_2 : Q \Rightarrow M_2 : R}{\Gamma \xrightarrow{P \vee Q} when(x_1.M_1, x_2.M_2) : R}\ (\vee_{\mathcal{L}})$$

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow smhr(M) : \circ P}\ (\circ_{\mathcal{R}}) \qquad \frac{\Gamma, x : P \Rightarrow M : \circ R}{\Gamma \xrightarrow{\circ P} smhl(x.M) : \circ R}\ (\circ_{\mathcal{L}})$$

Figure 8: The Sequent Calculus PFLAX, with Term Assignment

$\theta(x; Ms) = \theta'(var(x), Ms)$
$\theta(\lambda x.M) = \lambda x.\theta(M)$
$\theta(smhr(M)) = smhi(\theta(M))$

$\theta' : A \times Ms \to N$

$\theta'(A, []) = an(A)$
$\theta'(A, M :: Ms) = \theta'(ap(A, \theta(M)), Ms)$
$\theta'(A, smhl(x.Ms)) = smhe(A, x.\theta(M))$

Natural Deduction to Sequent Calculus:

$\psi : N \to M$

$\psi(an(A)) = \psi'(A, [])$
$\psi(\lambda x.N) = \lambda x.\psi(N)$
$\psi(smhe(A, x.N)) = \psi'(A, smhl(x.\psi(N)))$
$\psi(smhi(N)) = smhr(\psi(N))$

$\psi' : A \times Ms \to M$

$\psi'(var(x), Ms) = (x; Ms)$
$\psi'(ap(A, N), Ms) = \psi'(A, \psi(N) :: Ms)$

We now prove two lemmas showing the equivalence of the term calculi.

Lemma 1

i) $\psi(\theta(M)) \;=\; M$

ii) $\psi(\theta'(A, Ms)) \;=\; \psi'(A, Ms)$

PROOF: The proof is by simultaneous structural induction on M and Ms.
Case 1. $M = (x; Ms)$

$\psi(\theta(x; Ms))$	=	$\psi(\theta'(var(x), Ms))$	def θ
	=	$\psi'(var(x), Ms)$	ind ii)
	=	$(x; Ms)$	def ψ'

Case 2. $M = \lambda x.M$

$$
\begin{array}{rcll}
\psi(\theta(\lambda x.M)) & = & \psi(\lambda x.\theta(M)) & \text{def } \theta \\
 & = & \lambda x.\psi(\theta(M)) & \text{def } \psi \\
 & = & \lambda x.M & \text{ind i)}
\end{array}
$$

Case 3. $M = smhr(M)$

$$
\begin{array}{rcll}
\psi(\theta(smhr(M))) & = & \psi(smhi(\theta(M))) & \text{def } \theta \\
 & = & smhr(\psi(\theta(M))) & \text{def } \psi \\
 & = & smhr(M) & \text{ind i)}
\end{array}
$$

Case 4. $Ms = []$

$$
\begin{array}{rcll}
\psi(\theta'(A, [])) & = & \psi(an(A)) & \text{def } \theta \\
 & = & \psi'(A, []) & \text{def } \psi'
\end{array}
$$

Case 5. $Ms = M :: Ms$

$$
\begin{array}{rcll}
\psi(\theta'(A, M :: Ms)) & = & \psi(\theta'(ap(A, \theta(M)), Ms)) & \text{def } \theta' \\
 & = & \psi'(ap(A, \theta(M)), Ms) & \text{ind ii)} \\
 & = & \psi'(A, \psi(\theta(M)) :: Ms) & \text{def } \psi' \\
 & = & \psi'(A, M :: Ms) & \text{ind i)}
\end{array}
$$

Case 6. $Ms = smhl(x.M)$

$$
\begin{array}{rcll}
\psi(\theta'(A, smhl(x.M))) & = & \psi(smhe(A, x.\theta(M))) & \text{def } \theta' \\
 & = & \psi'(A, smhl(x.\psi(\theta(M)))) & \text{def } \psi \\
 & = & \psi'(A, smhl(x.M)) & \text{ind i)}
\end{array}
$$

■

Lemma 2

i) $\theta(\psi(N)) \ = \ N$

ii) $\theta(\psi'(A, Ms)) \ = \ \theta'(A, Ms)$

PROOF: By simultaneous structural induction on N and A.

Case 1. $N = an(A)$

$$
\begin{array}{rcll}
\theta(\psi(an(A)) & \theta(\psi'(A, [])) & \text{def } \psi & \\
 & = & \theta'(A, []) & \text{ind ii)} \\
 & = & an(A) & \text{def } \theta'
\end{array}
$$

Case 2. $N = \lambda x.N$

$$
\begin{array}{lcll}
\theta(\psi(\lambda x.N)) & = & \theta(\lambda x.\psi(N)) & \text{def } \psi \\
 & = & \lambda x.theta(\psi(N)) & \text{def } \theta \\
 & = & \lambda x.N & \text{ind i)}
\end{array}
$$

Case 3. $N = smhi(N)$

$$
\begin{array}{lcll}
\theta(\psi(smhi(N))) & = & \theta(smhr(\psi(N))) & \text{def } \psi \\
 & = & smhi(\theta(\psi(N))) & \text{def } \theta \\
 & = & smhi(N) & \text{ind i)}
\end{array}
$$

Case 4. $N = smhe(A, x.N)$

$$
\begin{array}{lcll}
\theta(\psi(smhe(A, x.N))) & = & \theta(\psi'(A, smhl(x.\psi(N)))) & \text{def } \psi \\
 & = & \theta'(A, smhl(x.\psi(N))) & \text{ind ii)} \\
 & = & smhe(A, x.\theta(\psi(N))) & \text{def } \theta' \\
 & = & smhe(A, x.N) & \text{ind i)}
\end{array}
$$

Case 5. $A = var(x)$

$$
\begin{array}{lcll}
\theta(\psi'(var(x), Ms)) & = & \theta(x; Ms) & \text{def } \psi' \\
 & = & \theta'(var(x), Ms) & \text{def } \theta
\end{array}
$$

Case 6. $A = ap(A, N)$

$$
\begin{array}{lcll}
\theta(\psi'(ap(A, N), Ms)) & = & \theta(\psi'(A, \psi(N) :: Ms)) & \text{def } \psi' \\
 & = & \theta'(A, \psi(N) :: Ms) & \text{ind ii)} \\
 & = & \theta'(ap(A, \theta(\psi(N))), Ms) & \text{def } \theta' \\
 & = & \theta'(ap(A, N), Ms) & \text{ind i)}
\end{array}
$$

■

Now we prove soundness and adequacy theorems.

Theorem 1 *(*SOUNDNESS*) The following rules are admissible:*

$$
\frac{\Gamma \Rightarrow M : R}{\Gamma \rhd\!\!\rhd \theta(M) : R}\ i) \qquad \frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(A, Ms) : R}\ ii)
$$

PROOF: By simultaneous structural induction on M and Ms.

Case 1. $M = (x; Ms)$

We have a derivation ending in:

$$
\frac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\ (C)
$$

and we know that

$$x : P \Rightarrow var(x) : P$$

is deducible.

So we have:

$$\frac{x : P \rhd var(x) : P \quad \Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(var(x), Ms) : R}\ (ii)$$

and we know that

$$\theta'(var(x), Ms) = \theta(x; Ms)$$

Case 2. $M = \lambda x.M$

We have a derivation ending in

$$\frac{\Gamma, x : P \Rightarrow M : Q}{\Gamma \Rightarrow \lambda x.M : P \supset Q}\ (\supset_{\mathcal{R}})$$

whence

$$\frac{\dfrac{\Gamma, x : P \Rightarrow M : Q}{\Gamma, x : P \rhd\!\!\rhd \theta(M) : Q}\ i)}{\Gamma \rhd\!\!\rhd \lambda x.\theta(M) : P \supset Q}\ (\supset_{\mathcal{I}})$$

and we know that

$$\lambda x.\theta(M) = \theta(\lambda x.M)$$

Case 3. $M = smhr(M)$

We have a derivation ending as follows

$$\frac{\Gamma \Rightarrow M : P}{\Gamma \Rightarrow smhr(M) : \circ P}\ (\circ_{\mathcal{R}})$$

whence

$$\frac{\dfrac{\Gamma \Rightarrow M : P}{\Gamma \rhd\!\!\rhd \theta(M) : P}\ i)}{\Gamma \rhd\!\!\rhd smhi(\theta(M)) : \circ P}\ (\circ_{\mathcal{I}})$$

and we know that

$$smhi(\theta(M)) = \theta(smhr(M))$$

Case 4. $Ms = []$

We have a deduction and a derivation:

$$\Gamma \rhd A : P \qquad \frac{}{\Gamma \xrightarrow{P} [] : P}\ (ax)$$

From the deduction, we obtain:

$$\frac{\Gamma \rhd A : P}{\Gamma \rhd\!\!\rhd an(A) : P} \ (M)$$

and since

$$an(A) = \theta'(A, [])$$

we have what we require.

Case 5. $Ms = M :: Ms$

We have a derivation ending in

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} M :: Ms : R} \ (\supset_{\mathcal{L}})$$

whence

$$\frac{\dfrac{\Gamma \rhd A : P \supset Q \quad \dfrac{\Gamma \Rightarrow M : P}{\Gamma \rhd\!\!\rhd \theta(M) : P} \ i)}{\Gamma \rhd ap(A, \theta(M)) : Q} \ (\supset_{\varepsilon}) \quad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \rhd\!\!\rhd \theta'(ap(A, \theta(M)), Ms) : R} \ ii)$$

and we know that

$$\theta'(ap(A, \theta(M)), Ms) = \theta'(A, M :: Ms)$$

Case 6. $Ms = smhl(x.Ms)$

We have a derivation ending

$$\frac{\Gamma, x : P \Rightarrow M : \circ Q}{\Gamma \xrightarrow{\circ P} smhl(x.M) : \circ Q} \ (\circ_{\mathcal{L}})$$

whence

$$\frac{\Gamma \rhd A : \circ P \quad \dfrac{\Gamma, x : P \Rightarrow M : \circ Q}{\Gamma, x : P \rhd\!\!\rhd \theta(M) : \circ Q} \ i)}{\Gamma \rhd\!\!\rhd smhe(A, x.\theta(M)) : \circ Q} \ (\circ_{\varepsilon})$$

and we know that

$$smhe(A, x.\theta(M)) = \theta'(A, smhl(x.M))$$

■

Theorem 2 *(*ADEQUACY*) The following rules are admissible:*

$$\frac{\Gamma \rhd\!\rhd N : R}{\Gamma \Rightarrow \psi(N) : R}\ i) \qquad \frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \Rightarrow \psi'(A, Ms) : R}\ ii)$$

PROOF: By simultaneous structural induction on A and N.

Case 1. $N = an(A)$

We have a deduction ending

$$\frac{\Gamma \rhd A : P}{\Gamma \rhd\!\rhd an(A) : P}\ (M)$$

We know that we can derive

$$\frac{}{\Gamma \xrightarrow{P} [] : P}\ (ax)$$

hence we have

$$\frac{\Gamma \rhd A : P \quad \Gamma \xrightarrow{P} [] : P}{\Gamma \Rightarrow \psi'(A, []) : P}\ ii)$$

We know that

$$\psi'(A, []) = \psi(an(A))$$

Case 2. $N = \lambda x.N$

We have a deduction ending

$$\frac{\Gamma, x : P \rhd\!\rhd N : Q}{\Gamma \rhd\!\rhd \lambda x.N : P \supset Q}\ (\supset_{\mathcal{I}})$$

whence

$$\frac{\dfrac{\Gamma, x : P \rhd\!\rhd N : Q}{\Gamma, x : P \Rightarrow \psi(N) : Q}\ i)}{\Gamma \Rightarrow \lambda x.\psi(N) : P \supset Q}\ (\supset_{\mathcal{R}})$$

and we know that

$$\lambda x.\psi(N) = \psi(\lambda x.N)$$

Case 3. $N = smhe(A, x.N)$

We have a deduction ending in

$$\frac{\Gamma \rhd A : \circ P \quad \Gamma, x : P \rhd\!\rhd N : \circ Q}{\Gamma \rhd\!\rhd smhe(A, x.N) : \circ Q}\ (\circ_{\mathcal{E}})$$

whence

$$\dfrac{\Gamma \rhd A : \circ P \quad \dfrac{\dfrac{\Gamma, x : P \rhd\!\!\rhd N : \circ Q}{\Gamma, x : P \Rightarrow \psi(N) : \circ Q}\, i)}{\Gamma \xrightarrow{\circ P} smhl(x.\psi(N)) : \circ Q}\,(\circ_{\mathcal{L}})}{\Gamma \Rightarrow \psi'(A, smhl(x.\psi(N))) : \circ Q}\, ii)$$

and we know that

$$\psi'(A, smhl(x.\psi(N))) = \psi(smhe(A, x.N))$$

Case 4. $N = smhi(N)$

We have a deduction ending in

$$\dfrac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \rhd\!\!\rhd smhi(N) : \circ P}\,(\circ_{\mathcal{I}})$$

whence

$$\dfrac{\dfrac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \Rightarrow \psi(N) : P}\, i)}{\Gamma \Rightarrow smhr(\psi(N)) : \circ P}\,(\circ_{\mathcal{R}})$$

and we know that

$$smhr(\psi(N)) = \psi(smhi(N))$$

Case 5. $A = var(x)$

We can extend to

$$\dfrac{\Gamma, x : P \xrightarrow{P} Ms : R}{\Gamma, x : P \Rightarrow (x; Ms) : R}\,(C)$$

and since

$$(x; Ms) = \psi'(var(x), Ms)$$

we have the result without further ado.

Case 6. $A = ap(A, N)$

We have a deduction ending in

$$\dfrac{\Gamma \rhd A : P \supset Q \quad \Gamma \rhd N : P}{\Gamma \rhd ap(A, N) : Q}\,(\supset_{\mathcal{E}})$$

whence

$$\dfrac{\Gamma \rhd A : P \supset Q \qquad \dfrac{\dfrac{\Gamma \rhd\!\!\rhd N : P}{\Gamma \Rightarrow \psi(N) : P}\, i) \qquad \Gamma \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{P \supset Q} \psi(N) :: Ms : R}\,(\supset_{\mathcal{L}})}{\Gamma \Rightarrow \psi'(A, \psi(N) :: Ms) : R}\, ii)$$

and we know that

$$\psi'(A, \psi(N) :: Ms) = \psi'(ap(A, N), Ms)$$

■

Theorem 3 *The normal natural deductions of Lax Logic (the proofs of NLAX) are in 1-1 correspondence to the proofs of PFLAX.*

PROOF: Immediate from theorems 1 and 2 and lemmas 1 and 2. ■

7 Cut Elimination

Now we move onto a study of cut in PFLAX. In the usual sequent calculus, cut may be formulated as follows:

$$\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow Q}{\Gamma \Rightarrow Q}\,(cut)$$

In PFLAX, we need four cut rules because of the two judgement forms.

$$\dfrac{\Gamma \xrightarrow{Q} P \quad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{Q} R}\,(cut_1) \qquad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{Q} R}{\Gamma \xrightarrow{Q} R}\,(cut_2)$$

$$\dfrac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{P} R}{\Gamma \Rightarrow R}\,(cut_3) \qquad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow R}{\Gamma \Rightarrow R}\,(cut_4)$$

These have associated terms:

$$M ::= \ cut_1^P(Ms, Ms) \mid cut_2^P(M, V.Ms)$$

$$Ms ::= \ cut_3^P(M, Ms) \mid cut_4^P(M, V.M)$$

And we can give the cut rules again with the proof terms:

$$\dfrac{\Gamma \xrightarrow{Q} Ms_1 : P \quad \Gamma \xrightarrow{P} Ms_2 : R}{\Gamma \xrightarrow{Q} cut_1^P(Ms_1, Ms_2) : R}\,(cut_1)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma, x : P \xrightarrow{Q} Ms : R}{\Gamma \xrightarrow{Q} cut_2^P(M, x.Ms) : R}\,(cut_2)$$

$$\frac{\Gamma \Rightarrow M : P \quad \Gamma \xrightarrow{P} Ms : R}{\Gamma \Rightarrow cut_3^P(M, Ms) : R}\,(cut_3)$$

$$\frac{\Gamma \Rightarrow M_1 : P \quad \Gamma, x : P \Rightarrow M_2 : R}{\Gamma \Rightarrow cut_4^P(M_1, x.M_2) : R}\,(cut_4)$$

We now give reduction rules for PFLAX^{cut}. As in the previous section, we restrict ourselves to the $\supset, \circ$ fragment of the logic, in order to prevent repetition of results that can be found elsewhere ([DP96]). Here we give reductions without terms, together with the associated term reductions.

Case 1. $cut_1^P([], Ms) \rightsquigarrow Ms$

$$\frac{\dfrac{}{\Gamma \xrightarrow{P} P}\,(ax) \qquad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{P} R}\,(cut_1) \qquad \rightsquigarrow \qquad \Gamma \xrightarrow{P} R$$

Case 2. $cut_1^P(M :: Ms_1, Ms_2) \rightsquigarrow M :: cut_1^P(Ms_1, Ms_2)$

$$\frac{\dfrac{\Gamma \Rightarrow A \quad \Gamma \xrightarrow{B} P}{\Gamma \xrightarrow{A \supset B} P}\,(\supset_{\mathcal{L}}) \qquad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{A \supset B} R}\,(cut_1)$$

$$\rightsquigarrow \qquad \frac{\Gamma \Rightarrow A \qquad \dfrac{\Gamma \xrightarrow{B} P \quad \Gamma \xrightarrow{P} R}{\Gamma \xrightarrow{B} R}\,(cut_1)}{\Gamma \xrightarrow{A \supset B} R}\,(\supset_{\mathcal{L}})$$

Case 3. $cut_1^{\circ P}(smhl(x.M), Ms) \rightsquigarrow smhl(x.cut_3^{\circ P}(M, Ms))$

$$\frac{\dfrac{\Gamma, A \Rightarrow \circ P}{\Gamma \xrightarrow{\circ A} \circ P}\,(\circ_{\mathcal{L}}) \qquad \Gamma \xrightarrow{\circ P} \circ R}{\Gamma \xrightarrow{\circ A} \circ R}\,(cut_1) \qquad \rightsquigarrow \qquad \frac{\dfrac{\Gamma, A \Rightarrow \circ P \quad \Gamma \xrightarrow{\circ P} \circ R}{\Gamma, A \Rightarrow \circ R}\,(cut_3)}{\Gamma \xrightarrow{\circ A} \circ R}\,(\circ_{\mathcal{L}})$$

Case 4. $cut_2^P(M, x.[]) \rightsquigarrow []$

$$\frac{\Gamma \Rightarrow P \qquad \dfrac{}{\Gamma, P \xrightarrow{R} R}\,(ax)}{\Gamma \xrightarrow{R} R}\,(cut_2) \qquad \rightsquigarrow \qquad \frac{}{\Gamma \xrightarrow{R} R}\,(ax)$$

Case 5. $cut_2^P(M_1, x.(M_2 :: Ms)) \rightsquigarrow cut_4^P(M_1, x.M_2) :: (cut_2^P(M_1, x.Ms))$

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Gamma, P \Rightarrow A \quad \Gamma, P \xrightarrow{B} R}{\Gamma, P \xrightarrow{A \supset B} R}\,(\supset_{\mathcal{L}})}{\Gamma \xrightarrow{A \supset B} R}\,(cut_2)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow A}{\Gamma \Rightarrow A}\,(cut_4) \qquad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{B} R}{\Gamma \xrightarrow{B} R}\,(cut_2)}{\Gamma \xrightarrow{A \supset B} R}\,(\supset_{\mathcal{L}})$$

Case 6. $cut_2^P(M_1, x_1.smhl(x_2.M_2)) \rightsquigarrow smhl(x_2.cut_4^P(M_1, x_1.M_2))$

$$\dfrac{\Gamma \Rightarrow P \qquad \dfrac{\Gamma, P, A \Rightarrow \circ R}{\Gamma, P \xrightarrow{\circ A} \circ R}\,(\circ_{\mathcal{L}})}{\Gamma \xrightarrow{\circ A} \circ R}\,(cut_2) \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P, A \Rightarrow \circ R}{\Gamma, A \Rightarrow \circ R}\,(cut_4)}{\Gamma \xrightarrow{\circ A} \circ R}\,(\circ_{\mathcal{L}})$$

Case 7. $cut_3^P((x; Ms_1), Ms_2) \rightsquigarrow (x; cut_1^P(Ms_1, Ms_2))$

$$\dfrac{\dfrac{\Gamma, A \xrightarrow{A} P}{\Gamma, A \Rightarrow P}\,(C) \qquad \Gamma, A \xrightarrow{P} R}{\Gamma, A \Rightarrow R}\,(cut_3)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \xrightarrow{A} P \quad \Gamma, A \xrightarrow{P} R}{\Gamma, A \xrightarrow{A} R}\,(cut_3)}{\Gamma, A \Rightarrow R}\,(C)$$

Case 8. $cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \rightsquigarrow cut_3^Q(cut_4^P(M_2, x.M_1), Ms)$

$$\dfrac{\dfrac{\Gamma, P \Rightarrow Q}{\Gamma \Rightarrow P \supset Q}\,(\supset_{\mathcal{R}}) \qquad \dfrac{\Gamma \Rightarrow P \quad \Gamma \xrightarrow{Q} R}{\Gamma \xrightarrow{P \supset Q} R}\,(\supset_{\mathcal{L}})}{\Gamma \Rightarrow R}\,(cut_3)$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow Q}{\Gamma \Rightarrow Q}\,(cut_4) \qquad \Gamma \xrightarrow{Q} R}{\Gamma \Rightarrow R}\,(cut_3)$$

Case 9. $cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \rightsquigarrow cut_4^P(M_1, x.M_2)$

$$\dfrac{\dfrac{\Gamma \Rightarrow P}{\Gamma \Rightarrow \circ P}\,(\circ_{\mathcal{R}}) \qquad \dfrac{\Gamma, P \Rightarrow \circ R}{\Gamma \xrightarrow{\circ P} \circ R}\,(\circ_{\mathcal{L}})}{\Gamma \Rightarrow \circ R}\,(cut_3) \qquad \rightsquigarrow \qquad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow \circ R}{\Gamma \Rightarrow \circ R}\,(cut_4)$$

Case 10. $cut_3^P(M, []) \rightsquigarrow M$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{}{\Gamma \xrightarrow{P} P}\,(ax)}{\Gamma \Rightarrow P}\,(cut_3) \quad \rightsquigarrow \quad \Gamma \Rightarrow P$$

Case 11. $cut_4^P(M, x.(y; Ms)) \rightsquigarrow (y; cut_2^P(M, x.Ms))$

$$\dfrac{\Gamma, A \Rightarrow P \quad \dfrac{\Gamma, P, A \xrightarrow{A} R}{\Gamma, P, A \Rightarrow R}\,(C)}{\Gamma, A \Rightarrow R}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma, A \Rightarrow P \quad \Gamma, A, P \xrightarrow{A} R}{\Gamma, A \xrightarrow{A} R}\,(cut_2)}{\Gamma, A \Rightarrow R}\,(C)$$

Case 12. $cut_4^P(M, x.(x; Ms)) \rightsquigarrow cut_3^P(M, cut_2^P(M, x.Ms))$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P \xrightarrow{P} R}{\Gamma, P \Rightarrow R}\,(C)}{\Gamma \Rightarrow R}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma \Rightarrow P \quad \Gamma, P \xrightarrow{P} R}{\Gamma \xrightarrow{P} R}\,(cut_2)}{\Gamma \Rightarrow R}\,(cut_3)$$

Case 13. $cut_4^P(M_1, x.\lambda y.M_2) \rightsquigarrow \lambda y.cut_4^P(M_1, x.M_2)$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P, A \Rightarrow B}{\Gamma, P \Rightarrow A \supset B}\,(\supset_{\mathcal{R}})}{\Gamma \Rightarrow A \supset B}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow P}{\Gamma, A \Rightarrow P}\,(W) \quad \Gamma, A, P \Rightarrow B}{\Gamma, A \Rightarrow B}\,(cut_4)}{\Gamma \Rightarrow A \supset B}\,(\supset_{\mathcal{R}})$$

Case 14. $cut_4^P(M_1, x.smhr(M_2)) \rightsquigarrow smhr(cut_4^P(M_1, x.M_2))$

$$\dfrac{\Gamma \Rightarrow P \quad \dfrac{\Gamma, P \Rightarrow A}{\Gamma, P \Rightarrow \circ A}\,(\circ_{\mathcal{R}})}{\Gamma \Rightarrow \circ A}\,(cut_4) \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow P \quad \Gamma, P \Rightarrow A}{\Gamma \Rightarrow A}\,(cut_4)}{\Gamma \Rightarrow \circ A}\,(\circ_{\mathcal{R}})$$

Notice that we used the following lemma:

Lemma 3 *The following rules are admissible in PFLAX:*

$$\dfrac{\Gamma \Rightarrow R}{\Gamma, P \Rightarrow R}\,(W) \qquad \dfrac{\Gamma \xrightarrow{Q} R}{\Gamma, P \xrightarrow{Q} R}\,(W)$$

We summarise the term reductions:

1. $cut_1^P([], Ms) \rightsquigarrow Ms$

2. $cut_1^P(M :: Ms_1, Ms_2) \rightsquigarrow M :: cut_1^P(Ms_1, Ms_2)$

3. $cut_1^{\circ P}(smhl(x.M), Ms) \rightsquigarrow smhl(x.cut_3^{\circ P}(M, Ms))$

4. $cut_2^P(M, x.[]) \rightsquigarrow []$

5. $cut_2^P(M_1, x.(M_2 :: Ms)) \rightsquigarrow cut_4^P(M_1, x.M_2) :: (cut_2^P(M_1, x.Ms))$

6. $cut_2^P(M_1, x_1.smhl(x_2.M_2)) \rightsquigarrow smhl(x_2.cut_4^P(M_1, x_1.M_2))$

7. $cut_3^P((x; Ms_1), Ms_2) \rightsquigarrow (x; cut_1^P(Ms_1, Ms_2))$

8. $cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \rightsquigarrow cut_3^Q(cut_4^P(M_2, x.M_1), Ms)$

9. $cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \rightsquigarrow cut_4^P(M_1, x.M_2)$

10. $cut_3^P(M, []) \rightsquigarrow M$

11. $cut_4^P(M, x.(y; Ms)) \rightsquigarrow (y; cut_2^P(M, x.Ms))$

12. $cut_4^P(M, x.(x; Ms)) \rightsquigarrow cut_3^P(M, cut_2^P(M, x.Ms))$

13. $cut_4^P(M_1, x.\lambda y.M_2) \rightsquigarrow \lambda y.cut_4^P(M_1, x.M_2)$

14. $cut_4^P(M_1, x.smhr(M_2)) \rightsquigarrow smhr(cut_4^P(M_1, x.M_2))$

Definition 2 *A* **simple cut** *instance is an instance of cut with cut free premisses.*

Definition 3 *The* **weight** *of a simple cut instance is the ordered quadruple:*

$$(|A|, cutno., h_1, h_2)$$

where:

- $|A|$ *is the size of the cut formula, defined as usual.*
- $cutno.$ *is the type of the cut (i.e. 1, 2, 3, 4)*
- h_1 *is the height of the derivation of the right premiss*
- h_2 *is the height of the derivation of the left premiss*

we make the convention that $cut_1 = cut_3 < cut_2 = cut_4$

Now we prove the theorem.

Theorem 4 *(*WEAK CUT ELIMINATION*) The rules* $cut_1, cut_2, cut_3, cut_4$ *are admissible in PFLAX.*

PROOF: We give a weak cut reduction strategy:

– pick any simple cut instance and reduce

– recursively reduce any simple cut instances in the result

By induction on the weight of the cut instance, and induction on the number of simple cut instances, this strategy terminates.

This can easily be seen by inspection. ∎

8 Strong Normalisation

In this section we prove that the cut reduction system strongly normalises, giving us another proof of cut elimination for PFLAX.

We prove strong normalisation using the recursive path-ordering techniques from term rewriting ([Der82]). This technique for proving strong normalisation is attractive because it is purely syntactic; reasoning is about the structure of the terms themselves rather than about a mapping of terms into tuples of natural numbers.

Again we restrict ourselves to the $\circ, \supset$ fragment of Lax Logic.

8.1 Termination Using the Recursive Path-Ordering

We define two partial orders, one on term constructors (or operators), $>$, and one on terms, $\succ$. This second partial order, the recursive path-ordering, is defined in terms of the first. Given that $>$ has some simple properties (transitivity, irreflexivity, well-foundedness), the recursive path-ordering theorem tells us that $\succ$ is well-founded; that is, there is no infinite decreasing sequence $\alpha_1 \succ \alpha_2 \succ \ldots$. Finally we show for any reduction $\xi \rightsquigarrow \xi'$, that $\xi \succ \xi'$. By the well foundedness of $\succ$, every reduction sequence terminates; the cut reduction rules are strongly normalising.

Definition 4 *We define the* **recursive path-ordering**.

Let F be a set of operators, $f, g \in F$. Let T(F) be the set of terms over F, $s, t \in T(F)$. We also write terms as $f(s_1, \ldots s_n)$, where $f(s_1, \ldots, s_n)$ is built from operator f applied to terms $s_1, \ldots, s_n$.

Let $>$ be a transitive, irreflexive partial ordering on F. Then $\succ$ is defined recursively on T(F) as follows:

$$s = f(s_1, \ldots, s_m) = g(t_1, \ldots, t_n) = t$$

iff

$i)$ $s_i \succeq t$ *for some* $i \in \{1, ..., m\}$

or $ii)$ $f > g$ *and* $s \succ t_j$ *for every* $j \in \{1, ..., n\}$

or $iii)$ $f = g$ *and* $[s_1, ..., s_m] \succ\!\!\succ [t_1, ..., t_n]$

We have used the following abbreviations: $\succeq$ for $\succ$ or equivalent up to permutation of subterms; $\succ\!\!\succ$ for the extension of $\succ$ to finite multisets.

Definition 5 *A relation* $\supset$ *on set* K *is* **well-founded** *iff there are no infinite decreasing sequences of* K*-terms* $\kappa_1 \supset \kappa_2 \supset ...$.

Theorem 5 (RECURSIVE PATH-ORDERING THEOREM) *If* $>$ *is well founded, then* $\succ$ *is well-founded.*

8.2 Strong Normalisation for PFLAX

We apply the recursive path ordering technique to the term assignment system of PFLAX.

The operators are the term constructors of PFLAX; that is, the constructors ;, λ, ::, [ ], $smhl$, $smhr$, together with those for cut. The cut constructors are in fact an infinite family of constructors parametrised by the formulae of Lax Logic, i.e. the constructors are cut_i^P where P ranges over the formulae of Lax Logic.

$$Op = \{cut_i^P \mid i \in \{1, 2, 3, 4\},\ P \text{ a formula}\} \cup \{;, \lambda, ::, [\,], smhl, smhr\}$$

The terms over Op are the proof terms of PFLAX^{cut}.

If we write $f(s_1, ..., s_n)$, f is the top term constructor and $s_1, ..., s_n$ are the immediate subterms.

We define a partial ordering on term constructors:

- if P and Q are formulae then $P > Q$ if Q is a subterm of P (i.e. $>$ is the subterm ordering)
- $cut_i^P > cut_j^Q$ if $P > Q$, $i, j \in \{1, 2, 3, 4\}$
- $cut_4^P, cut_2^P > cut_3^P, cut_1^P$
- we put $cut_1^P = cut_3^P$ and $cut_2^P = cut_4^P$
- $cut_i^P >; , \lambda, ::, [\,], smhl, smhr$
- $;, \lambda, ::, [\,], smhl, smhr$ are ordered equally.

Proposition 2 *The ordering* $>$ *on Op is transitive, irreflexive and well-founded.*

PROOF: Transitivity and irreflexivity and obvious.

We have an infinite number of term constructors, so it is possible that we could have an infinite decreasing sequence of them:

$$cut^P_{i_1} > cut^Q_{i_2} > \ldots.$$

As either the cut suffix or the size of the cut formula must decrease, the length of the sequence is bounded (by twice the size of P). ■

Corollary 1 *$\succ$ is well founded for the terms of PFLAX.*

PROOF: By the recursive path-ordering theorem. ■

Lastly we show for each cut reduction $\alpha \rightsquigarrow \alpha'$, that $\alpha \succ \alpha'$.

Proposition 3 *For each cut reduction $\alpha \rightsquigarrow \alpha'$, $\alpha \succ \alpha'$ holds.*

PROOF: We analyse case by case.

Case 1.
$cut^P_1([], Ms) \succ Ms$
since $Ms \succeq Ms$

Case 2.
$cut^P_1(M :: Ms_1, Ms_2) \succ M :: cut^P_1(Ms_1, Ms_2)$
since $cut^P_1 >::$ and
$cut^P_1(M :: Ms_1, Ms_2) \succ M$
since $M :: Ms_1 \succeq M$
$cut_1(M :: Ms_1, Ms_2) \succ cut^P_1(Ms_1, Ms_2)$
since $cut^P_1 = cut^P_1$ and
$[M :: Ms_1, Ms_2] \succ\!\!\succ [Ms_1, Ms_2]$

Case 3.
$cut^{\circ P}_1(smhl(x.M), Ms) \succ smhl(x.cut^{\circ P}_3(M, Ms))$
since $cut^{\circ P}_1 > smhl$ and
$cut^{\circ P}_1(smhl(x.M), Ms) \succ cut^{\circ P}_3(M, Ms)$
since $cut^{\circ P}_1 = cut^{\circ P}_3$ and
$[smhl(x.M), Ms] \succ\!\!\succ [M, Ms]$

Case 4.
$cut^P_2(M, x.[]) \succ []$
since $[] \succeq []$

Case 5.

$cut_2^P(M_1, x.(M_2 :: Ms)) \succ cut_4^P(M_1, x.M_2) :: cut_2^P(M_1, x.Ms)$
 since $cut_2^P >::$ and
  $cut_2^P(M_1, x.(M_2 :: Ms)) \succ cut_4^P(M_1, x.M_2)$
   since $cut_2^P = cut_4^P$ and
    $[M_1, M_2 :: Ms] \succ\!\!\succ [M_1, M_2]$
  $cut_2^P(M_1, x.(M_2 :: Ms)) \succ cut_2^P(M_1, x.Ms)$
   since $cut_2^P = cut_2^P$ and
    $[M_1, M_2 :: Ms] \succ\!\!\succ [M_1, M_2]$

Case 6.

$cutr_2^P(M_1, x_1.smhl(x_2.M_2)) \succ smhl(x_2.cut_4^P(M_1, x_1.M_2))$
 since $cut_2^P > smhl$ and
  $cut_2^P(M_1, x_1.smhl(x_2.M_2)) \succ cut_4^P(M_1, x_1.M_2)$
   since $cut_2^P = cut_4^P$ and
    $[M_1, smhl(x_2.M_2)] \succ\!\!\succ [M_1, M_2]$

Case 7.

$cut_3^P((x; Ms_1), Ms_2) \succ (x; cut_1^P(Ms_1, Ms_2))$
 since $cut_3^P >;$ and
  $cut_3^P((x; Ms_1), Ms_2) \succ cut_1^P(Ms_1, Ms_2)$
   since $cut_3^P = cut_1^P$ and
    $[(x; Ms_1), Ms_2] \succ\!\!\succ [Ms_1, Ms_2]$

Case 8.

$cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \succ cut_3^Q(cut_4^P(M_2, x.M_1), Ms)$
 since $cut_3^{P \supset Q} > cut_3^Q$ and
  $cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \succ cut_4^P(M_2, x.M_1)$
   since $cut_3^{P \supset Q} > cut_4^P$ and
    $cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \succ M_2$
     since $M_2 :: Ms \succeq M_2$
    $cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \succ M_1$
     since $\lambda x.M_1 \succeq M_1$
  $cut_3^{P \supset Q}(\lambda x.M_1, M_2 :: Ms) \succ Ms$
   since $M_2 :: Ms \succeq Ms$

Case 9.

$cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \succ cut_4^P(M_1, x.M_2)$
 since $cut_3^{\circ P} > cut_4^P$ and
  $cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \succ M_1$
   since $smhr(M_1) \succeq M_1$

$cut_3^{\circ P}(smhr(M_1), smhl(x.M_2)) \succ M_2$
since $smhl(x.M_2) \succeq M_2$

Case 10.
$cut_3^P(M, []) \succ M$
since $M \succeq M$

Case 11.
$cut_4^P(M, x.(y; Ms)) \succ (y; cut_2^P(M, x.Ms))$
since $cut_4^P > ;$ and
$cut_4^P(M, x.(y; Ms)) \succ cut_2^P(M, x.Ms)$
since $cut_4^P = cut_2^P$ and
$[M, (y; Ms)] \succ\!\succ [M, Ms]$

Case 12.
$cut_2^P(M, x.(x; Ms)) \succ cut_3^P(M, cut_2^P(M, x.Ms))$
since $cut_2^P > cut_3^P$ and
$cut_2^P(M, x.(x; Ms)) \succ M$
since $M \succeq M$
$cut_2^P(M, x.(x; Ms)) \succ cut_2^P(M, x.Ms)$
since $cut_2^P = cut_2^P$ and
$[M, (x; Ms)] \succ\!\succ [M, Ms]$

Case 13.
$cut_4^P(M_1, x_1.\lambda x_2.M_2) \succ \lambda x_2.cut_4^P(M_1, x_2.M_2)$
since $cut_4^P > \lambda$ and
$cut_4^P(M_1, x_1.\lambda x_2.M_2) \succ cut_4^P(M_1, x_1.M_2)$
since $cut_4^P = cut_4^P$ and
$[M_1, \lambda x_2.M_2] \succ\!\succ [M_1, M_2]$

Case 14.
$cut_4^P(M_1, x.smhr(M_2)) \succ smhr(cut_4^P(M_1, x.M_2))$
since $cut_4^P > smhr$ and
$cut_4^P(M_1, x.smhr(M_2)) \succ cut_4^P(M_1, x.M_2)$
since $cut_4^P = cut_4^P$ and
$[M_1, smhr(M_2)] \succ\!\succ [M_1, M_2]$

■

Theorem 6 *The cut reduction system for PFLAX strongly normalises.*

PROOF: Immediate from Corollary 1 and Proposition 3. ■

9 Conclusion and Related Work

We have presented a Gentzen system for propositional Lax Logic whose proofs correspond in a 1-1 way to the normal natural deductions. This calculus is syntax-directed and hence suitable for use in proof enumeration. Although we have only presented the calculus for the propositional fragment of the logic, the results easily extend to cover first-order quantifiers. More about quantified Lax Logic can be found in [FW97].

We have also applied these 'permutation-free' techniques to Intuitionistic Linear Logic, [How97a]. Owing to the nature of the introduction rule for the exponential, the resulting sequent calculus, SILL, is complicated. We are currently refining this work. The intuitionistic calculus MJ has been used as the basis for work on propositional theorem proving, [How97b].

Lax Logic has recently been used in hardware verification, see for example, [FM94]. It has also been applied in constraint logic programming, [FMW97]. 'Permutation-free' calculi are natural extensions to logic programming when logic programming is thought of as backward proof search on hereditary Harrop formulae.

References

[BBdP95] N. Benton, G. M. Bierman, and V. de Paiva. Computational Types from a Logical Perspective I. Technical Report TR-365, University of Cambridge, 1995.

[Cur52] H. B. Curry. The Elimination Theorem When Modality is Present. *Journal of Symbolic Logic*, 17(4):249–65, 1952.

[Der82] N. Dershowitz. Orderings for Term-Rewriting Systems. *Theoretical Computer Science*, 17:279–301, 1982.

[DP96] R. Dyckhoff and L. Pinto. A Permutation-free Sequent Calculus for Intuitionistic Logic. Research Report CS/96/9, University of St Andrews, 1996.

[DP98] R. Dyckhoff and L. Pinto. Cut elimination and a permutation-free sequent calculus for intuitionistic logic. *Studia Logic*, 60:107–118, 1998.

[FM94] M. Fairtlough and M. Mendler. An Intuitionistic Modal Logic with Applications to the Formal Verification of Hardware. In *Computer Science Logic*, pages 354–68. Springer, 1994.

[FM97] M. Fairtlough and M. Mendler. Propositional Lax Logic. *Information and Computation*, 137(1), 1997.

[FMW97] M. Fairtlough, M. Mendler, and M. Walton. First-order Lax Logic as a Framework for Constraint Logic Programming. Technical Report MIPS-9714, University of Passau, 1997.

[FW97] M. Fairtlough and M. Walton. Quantified Lax Logic. Technical Report CS-97-11, University of Sheffield, 1997.

[Her95] H. Herbelin. A λ-calculus Structure Isomorphic to Gentzen-style Sequent Calculus Structure. In L Pacholski and J Tiuryn, editors, *Proceedings of the 1994 workshop Computer Science Logic*, volume 933 of *Springer Lecture Notes in Computer Science*, pages 61–75, 1995.

[How97a] J.M. Howe. A Sequent Calculus for Intuitionistic Linear Logic. Unpublished, 1997.

[How97b] J.M. Howe. Two Loop Detection Mechanisms: a Comparison. *Springer LNAI*, 1227:188–200, 1997. Proceedings of TABLEAUX'97.