A strong Dixmier-Moeglin equivalence for quantum Schubert cells and an open problem for quantum Plücker coordinates

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To Mam, Da, and Ro...

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Abstract

In this thesis, the algebras of primary interest are the *quantum Schubert cells* and the *quantum Grassmannians*, both of which are known to satisfy a condition on primitive ideals known as the *Dixmier-Moeglin equivalence*.

A stronger version of the Dixmier-Moeglin equivalence is introduced - a version which deals with all prime ideals of an algebra rather than just the primitive ideals. Quantum Schubert cells are shown to satisfy the *strong Dixmier-Moeglin equivalence*.

Until now, given a torus-invariant prime ideal of the quantum Grassmannian, one could not decide which quantum Plücker coordinates it contains. Presented here is a graph-theoretic method for answering this question. This may be useful for providing a full description of the inclusions between the torus-invariant prime ideals of the quantum Grassmannian and may lead to a proof that quantum Grassmannians satisfy the strong Dixmier-Moeglin equivalence.

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Chapter 1

Introduction

This thesis is organised into two main parts - one dealing with a strengthening of the notion of the Dixmier-Moeglin equivalence for quantum Schubert cells, the other providing a graph-theoretic solution to the problem of deciding whether or not a given quantum Plücker coordinate belongs to a given torus-invariant prime ideal of a quantum Grassmannian.

Dixmier and Moeglin gave an algebraic condition and a topological condition for recognising the primitive ideals among the prime ideals of the universal enveloping algebra of a finite-dimensional complex Lie algebra; they showed that the primitive, rational, and locally closed ideals coincide. In modern terminology, they showed that the universal enveloping algebra of a finite-dimensional complex Lie algebra satisfies the *Dixmier-Moeglin equivalence.* We define quantities which measure how "close" an arbitrary prime ideal of a noetherian algebra is to being primitive, rational, and locally closed; if every prime ideal is equally "close" to satisfying each of these three properties, then we say that the algebra satisfies the *strong Dixmier-Moeglin equivalence.* Using the example of the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, we show that the strong Dixmier-Moeglin equivalence is strictly stronger than the Dixmier-Moeglin equivalence. For a simple complex Lie algebra $\mathfrak{g}$ and an element w of the Weyl group of $\mathfrak{g}$, De Concini, Kac, and Procesi have constructed a subalgebra $U_q[w]$ of the quantised enveloping $\mathbb{K}$-algebra $U_q(\mathfrak{g})$. These *quantum Schubert cells* are known to satisfy the Dixmier-

Moeglin equivalence and we show that they in fact satisfy the strong Dixmier-Moeglin equivalence when q is not a root of unity. Along the way, we show that commutative affine domains, uniparameter quantum tori, and uniparameter quantum affine spaces satisfy the strong Dixmier-Moeglin equivalence.

Algebras of quantum matrices have certain subalgebras known as *partition subalgebras*, in which we define the notion of a *pseudo quantum minor* (analogously to the notion of a quantum minor of an algebra of quantum matrices). Partition subalgebras of quantum matrices admit rational torus actions and, based on results of Casteels, we develop a graph-theoretic method for deciding whether or not a given pseudo quantum minor belongs to a given torus-invariant prime ideal of a partition subalgebra of quantum matrices. The *quantum Grassmannian* $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is generated as an algebra by the maximal quantum minors of the algebra $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ of quantum $m \times n$ matrices. These generators are known as the *quantum Plücker coordinates* of the quantum Grassmannian. By a one-to-one correspondence of Launois, Lenagan, and Rigal, the torus-invariant prime ideals of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ (except the irrelevant ideal) correspond to the torus-invariant prime ideals of the partition subalgebras of the algebra $\mathcal{O}_q(M_{m,n-m}(\mathbb{K}))$ of quantum $m \times (n-m)$ matrices. Let J be a torus-invariant prime ideal of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ and let J' be the corresponding torus invariant prime ideal of the corresponding partition subalgebra of $\mathcal{O}_q(M_{m,n-m}(\mathbb{K}))$. Given a quantum Plücker coordinate α of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$, we show that the question of whether or not α belongs to J can be answered by reading off a graph. Indeed we reduce the question of whether or not α belongs to J to the question of whether or not a certain pseudo quantum minor belongs to J' and we answer this question by the graph-theoretic method (mentioned above) which we developed based on results of Casteels.

N.B. We adopt the following conventions throughout this thesis:

- $\mathbb{N}$ is the set of nonnegative integers (in particular, $0 \in \mathbb{N}$);
- $\mathbb{K}$ is an infinite field;
- $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$;
- $q \in \mathbb{K}^\times$ is not a root of unity;
- for integers $a < b$, $[\![a, b]\!]$ denotes the set of all integers x such that $a \leq x \leq b$;
- for an integer $t \geq 1$, S_t is the symmetric group on $[\![1, t]\!]$;
- all algebras are unital associative $\mathbb{K}$-algebras, unless otherwise stated;
- every ideal is two-sided, unless otherwise stated;
- for a $\mathbb{K}$-algebra R, $\mathrm{Spec}(R)$ denotes the space of prime ideals of R (which we endow with the Zariski topology) and $\mathcal{Z}(R)$ denotes the centre of R;
- all homomorphisms and (skew) derivations of $\mathbb{K}$-algebras are $\mathbb{K}$-linear;
- if we say that $R[X; \sigma, \delta]$ is an Ore extension of a $\mathbb{K}$-algebra R, the reader may assume that σ is an automorphism of R and δ is a left σ-derivation of R.
- Is I is an ideal of a ring R and x is an element of R, then $\overline{x}$ shall denote the canonical image of x in R/I, namely the coset $I + x$.
- If R is a semiprime noetherian ring, then $\mathrm{Frac}(R)$ shall denote its (semisimple artinian) total ring of fractions.

Chapter 2

Preliminaries

2.1 The algebras which appear in this thesis

2.1.1 Quantum affine spaces

Let N be a positive integer and let $\Lambda = (\lambda_{i,j}) \in \mathcal{M}_N(\mathbb{K}^\times)$ be a multiplicatively skew-symmetric matrix. The *quantum affine space* associated to Λ is denoted by $\mathcal{O}_\Lambda(\mathbb{K}^N)$ or $\mathbb{K}_\Lambda[T_1, \dots, T_N]$ and is presented as the $\mathbb{K}$-algebra with generators $T_1, \dots, T_N$ and relations

$$T_j T_i = \lambda_{j,i} T_i T_j \text{ for all } i, j \in [\![1, N]\!].$$

The algebra $\mathcal{O}_\Lambda(\mathbb{K}^N)$ can be written as the iterated skew-polynomial extension

$$\mathbb{K}[T_1][T_2; \sigma_2] \cdots [T_N; \sigma_N],$$

where, for each $j \in [\![2, N]\!]$, σ_j is the automorphism of $\mathbb{K}[T_1][T_2; \sigma_2] \cdots [T_{j-1}; \sigma_{j-1}]$ defined by $\sigma_j(T_i) = \lambda_{j,i} T_i$ for all $i \in [\![1, j-1]\!]$.

It is clear that $\mathcal{O}_\Lambda(\mathbb{K}^N)$ is a domain; it is noetherian by [18, Theorem 1.14]. There is a PBW-type $\mathbb{K}$-basis for $\mathcal{O}_\Lambda(\mathbb{K}^N)$ given by $\{T_1^{i_1} \cdots T_N^{i_N} \mid i_1, \dots, i_N \in \mathbb{N}\}$.

A quantum affine space $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[T_1, \dots, T_N]$ is called a *uniparameter* quantum affine space with parameter q if there exists an additively skew-symmetric matrix $A =$

$(a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ such that $\Lambda = (q^{a_{i,j}})$. In this case, we denote $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[T_1, \ldots, T_N]$ by $\mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$.

2.1.2 Quantum tori

Let N be a positive integer and let $\Lambda = (\lambda_{i,j}) \in \mathcal{M}_N(\mathbb{K}^\times)$ be a multiplicatively skew-symmetric matrix. The *quantum torus* associated to Λ is denoted by $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ or $\mathbb{K}_\Lambda[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$ and is presented as the $\mathbb{K}$-algebra generated by $T_1^{\pm 1}, \ldots, T_N^{\pm 1}$ with relations

$$T_i T_i^{-1} = T_i^{-1} T_i = 1 \text{ for all } i, \;\; T_j T_i = \lambda_{j,i} T_i T_j \text{ for all } i, j.$$

The algebra $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ can be written as the iterated skew-Laurent extension

$$\mathbb{K}[T_1^{\pm 1}][T_2^{\pm 1}; \sigma_2] \cdots [T_N^{\pm 1}; \sigma_N],$$

where for each $j \in [\![2, N]\!]$, σ_j is the automorphism of $\mathbb{K}[T_1^{\pm 1}][T_2^{\pm 1}; \sigma_2] \cdots [T_{j-1}^{\pm 1}; \sigma_{j-1}]$ defined by $\sigma_j(T_i) = \lambda_{j,i} T_i$ for all $i \in [\![1, j-1]\!]$.

It is clear that $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is a domain; it is noetherian by [18, Corollary 1.15]. There is a PBW-type $\mathbb{K}$-basis for $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ given by $\{T_1^{i_1} \cdots T_N^{i_N} \mid (i_1, \ldots, i_N) \in \mathbb{Z}^N\}$.

A quantum torus $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N) = \mathbb{K}_\Lambda[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$ is called a *uniparameter* quantum torus with parameter q if there exists an additively skew-symmetric matrix $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ such that $\Lambda = (q^{a_{i,j}})$. In this case, we write $\mathcal{O}_{q,A}((\mathbb{K}^\times)^N) = \mathbb{K}_{q,A}[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$ for $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N) = \mathbb{K}_\Lambda[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$.

2.1.3 Quantum matrices

Consider the variety $M_{2,2}(\mathbb{K})$ of 2×2 matrices over $\mathbb{K}$, which is simply affine 4-space $\mathbb{K}^4$ and whose coordinate ring $\mathcal{O}(M_{2,2}(\mathbb{K}))$ is simply the polynomial algebra $\mathbb{K}[a, b, c, d]$ in four indeterminates over $\mathbb{K}$. There are natural morphisms of varieties $\mathbb{K}^2 \times M_{2,2}(\mathbb{K}) \to \mathbb{K}^2$ and $M_{2,2}(\mathbb{K}) \times \mathbb{K}^2 \to \mathbb{K}^2$ given by matrix multiplication. These maps dualise to give morphisms of algebras

$$\mathcal{O}(\mathbb{K}^2) \to \mathcal{O}(\mathbb{K}^2) \otimes \mathcal{O}(M_{2,2}(\mathbb{K})) \text{ and } \mathcal{O}(\mathbb{K}^2) \to \mathcal{O}(M_{2,2}(\mathbb{K})) \otimes \mathcal{O}(\mathbb{K}^2). \tag{2.1}$$

The algebra $\mathcal{O}(\mathbb{K}^2)$ is the polynomial algebra $\mathbb{K}[x, y]$ in two variables and the morphisms (2.1) can be written in matrix form as

$$\begin{pmatrix} x & y \end{pmatrix} \mapsto \begin{pmatrix} x & y \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.2}$$

(To clarify this notation, the first map in (2.2) is given by $x \mapsto x \otimes a + y \otimes c,\ y \mapsto x \otimes b + y \otimes d$ and the second map in (2.2) is given by $x \mapsto a \otimes x + b \otimes y,\ y \mapsto c \otimes x + d \otimes y$.)

Manin [29] used this framework to arrive at a natural definition of the quantised coordinate ring of $M_{2,2}(\mathbb{K})$, which we shall denote by $\mathcal{O}_q(M_{2,2}(\mathbb{K}))$ and refer to informally as the (algebra of) 2×2 *quantum matrices*. For $\mathcal{O}_q(M_{2,2}(\mathbb{K}))$, Manin wanted an algebra with four generators a, b, c, d, and relations such that, where $\mathcal{O}_q(\mathbb{K}^2) = \mathbb{K}\langle x, y\rangle / \langle xy - qyx\rangle$ is the quantum plane (quantum affine 2-space), one has morphisms of algebras

$$\mathcal{O}_q(\mathbb{K}^2) \to \mathcal{O}_q(\mathbb{K}^2) \otimes \mathcal{O}_q(M_{2,2}(\mathbb{K})) \text{ and } \mathcal{O}_q(\mathbb{K}^2) \to \mathcal{O}_q(M_{2,2}(\mathbb{K})) \otimes \mathcal{O}_q(\mathbb{K}^2) \tag{2.3}$$

given by the formulae in (2.2). This leads to an algebra $\mathcal{O}_q(M_{2,2}(\mathbb{K}))$ with generators a, b, c, d — which we think of as lying in a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ — and the six relations

$$ab = qba,\ ac = qca,\ bd = qdb,\ cd = qdc,\ bc = cb,\ ad - da = (q - q^{-1})bc. \tag{2.4}$$

It turns out that the algebra $\mathcal{O}_q(M_{2,2}(\mathbb{K}))$ can be given the structure of a bialgebra in a natural way and that the maps (2.3) equip the quantum plane $\mathcal{O}_q(\mathbb{K}^2)$ with the structure of a left and a right comodule algebra for $\mathcal{O}_q(M_{2,2}(\mathbb{K}))$.

One can define quantum matrices $\mathcal{O}_q(M_{m,n})$ for any m, n; one thinks of the generators $X_{i,j}$ of $\mathcal{O}_q(M_{m,n})$ as lying in the matrix

$$\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{m1} & \cdots & X_{mn} \end{pmatrix}, \tag{2.5}$$

(called the matrix of canonical generators for $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$) with the relations given for all $i, k \in [\![1, m]\!]$ and all $l, j \in [\![1, n]\!]$ by

$$\begin{aligned} X_{i,j}X_{i,l} &= qX_{i,l}X_{i,j} && \text{if } j < l; \\ X_{i,j}X_{k,j} &= qX_{k,j}X_{i,j} && \text{if } i < k; \\ X_{i,j}X_{k,l} &= X_{k,l}X_{i,j} && \text{if } k < i \text{ and } j < l; \\ X_{i,j}X_{k,l} - X_{k,l}X_{i,j} &= (q - q^{-1})X_{i,l}X_{k,j} && \text{if } i < k \text{ and } j < l. \end{aligned} \tag{2.6}$$

Remark 2.1.1. *The relations* (2.6) *can be thought of as follows: taking any* 2×2 *submatrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *of the canonical matrix* (2.5), *the relations between* $a, b, c,$ *and* d *are exactly those appearing in* (2.4).

Adding the generators in lexicographical order, the algebra $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ may be expressed as an iterated Ore extension

$$\mathbb{K}[X_{1,1}] \cdots [X_{i,j}; \sigma_{i,j}, \delta_{i,j}] \cdots [X_{m,n}; \sigma_{m,n}, \delta_{m,n}], \tag{2.7}$$

where the $\sigma_{i,j}$ are automorphisms. As such, $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is clearly a domain and it is noetherian by [18, Theorem 2.6].

Definition 2.1.2. Let $I = \{i_1 < \cdots < i_t\} \subseteq [\![1, m]\!]$ and $J = \{j_1 < \ldots < j_t\} \subseteq [\![1, n]\!]$. The *quantum minor* $[I \mid J]$ of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is defined by

$$[I \mid J] = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_1, j_{\sigma(1)}} X_{i_2, j_{\sigma(2)}} \cdots X_{i_t, j_{\sigma(t)}},$$

where for $\sigma \in S_t$, the length $\ell(\sigma)$ of σ is the cardinality of the set $\{(i,j) \in [\![1,t]\!] \times [\![1,t]\!] \mid i < j \text{ and } \sigma(i) > \sigma(j)\}$.

Certain subalgebras of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ called *partition subalgebras* shall be of significant interest to us but we shall postpone their definition until we need them in Chapter 4. We postpone also the notion of a *pseudo quantum minor* of a partition subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$.

2.1.4 Quantum Grassmannians

When $m \leq n$, the $m \times n$ *quantum Grassmannian* $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ generated by all maximal (i.e. $m \times m$) quantum minors of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$; these generators are called the *quantum Plücker coordinates* of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$. Since $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ is a domain, so is $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$. The quantum Grassmannian $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is noetherian by [21, Theorem 1.1]. We shall elaborate on the construction of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ in Section 5.1.

2.1.5 Quantised enveloping algebras

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and, relative to this choice of Cartan subalgebra, choose a root system Φ of $\mathfrak{g}$. Choose an ordered base $\pi := \{\alpha_1, \ldots, \alpha_n\}$ of Φ, so that π is a basis of a real Euclidean vector space E, whose inner product we denote by $(-,-)$. Recall that the Cartan matrix of $\mathfrak{g}$ associated to the above choice of simple roots is given by $C = (c_{i,j})$, where $c_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.

Let us normalise $(-,-)$ so that short simple roots have length $\sqrt{2}$ i.e. short simple roots α satisfy $(\alpha, \alpha) = 2$. For $i \in [\![1,n]\!]$, the simplicity of $\mathfrak{g}$ guarantees that $(\alpha_i, \alpha_i) \in \{2,4,6\}$, so that $q_i := q^{(\alpha_i,\alpha_i)/2} \in \{q, q^2, q^3\}$; for nonnegative integers $k \leq p$, define

$$\begin{bmatrix} p \\ k \end{bmatrix}_{q_i} := \frac{(q_i - q_i^{-1}) \cdots (q_i^{p-1} - q_i^{1-p})(q_i^p - q_i^{-p})}{(q_i - q_i^{-1}) \cdots (q_i^k - q_i^{-k})(q_i - q_i^{-1}) \cdots (q_i^{p-k} - q_i^{k-p})}.$$

The *quantised enveloping algebra* $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{K}$ is the $\mathbb{K}$-algebra generated by $F_1, \dots, F_n$, $K_1^{\pm 1}, \dots, K_n^{\pm 1}$, $E_1, \dots, E_n$ with the following relations

$$K_i K_i^{-1} = 1 \text{ and } K_i K_j = K_j K_i;$$

$$K_i E_j K_i^{-1} = q_i^{c_{i,j}} E_j \text{ and } K_i F_j K_i^{-1} = q_i^{-c_{i,j}} F_j;$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}};$$

and the *quantum Serre relations*

$$\sum_{k=0}^{1-c_{i,j}} (-1)^k \begin{bmatrix} 1 - c_{i,j} \\ k \end{bmatrix}_{q_i} E_i^{1-c_{i,j}-k} E_j E_i^k = 0 \quad (i \neq j);$$

$$\sum_{k=0}^{1-c_{i,j}} (-1)^k \begin{bmatrix} 1 - c_{i,j} \\ k \end{bmatrix}_{q_i} F_i^{1-c_{i,j}-k} F_j F_i^k = 0 \quad (i \neq j).$$

This presentation of $U_q(\mathfrak{g})$ is analogous to Serre's presentation of $U(\mathfrak{g})$.

We denote by $U_q^+(\mathfrak{g})$ and $U_q^-(\mathfrak{g})$ the subalgebras of $U_q(\mathfrak{g})$ generated by $E_1, \dots, E_n$ and $F_1, \dots, F_n$ respectively. We denote by $U_q^0(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $K_1^{\pm 1}, \dots, K_n^{\pm 1}$.

2.1.6 Quantum Schubert cells $U_q[w]$

Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n, as in the previous subsection. The Weyl group of $\mathfrak{g}$, which we denote by $\mathcal{W}$, is the subgroup of the general linear group $\mathrm{GL}(E)$ of E generated by the reflections s_i ($i \in [\![1, n]\!]$) with reflecting hyperplanes given by $\{\beta \in E \mid (\beta, \alpha_i) = 0\}$ ($i \in [\![1, n]\!]$). For any element w of $\mathcal{W}$, De Concini, Kac, and Procesi [12] defined a quantum analogue, $U_q[w]$, of the universal enveloping algebra of the nilpotent Lie algebra $\mathfrak{n}^+ \cap \mathrm{Ad}_w(\mathfrak{n}^-)$, where Ad denotes the adjoint action. The algebra $U_q[w]$ is called a *quantum Schubert cell*.

Let us describe a construction of $U_q[w]$ which leads to an expression of $U_q[w]$ as an iterated Ore extension. The Weyl group $\mathcal{W}$ is a Coxeter group with respect to the generators $s_1, \ldots, s_n$ and we define the length, $\ell(w)$, of w to be the smallest N such that there exist $i_j \in [\![1, n]\!]$ satisfying $w = s_{i_1} \cdots s_{i_N}$. Let us fix this *reduced* expression

$$w = s_{i_1} \cdots s_{i_N}. \tag{2.8}$$

It is well known that $\beta_1 := \alpha_{i_1},\ \beta_2 := s_{i_1}(\alpha_{i_2}), \ldots, \beta_N := s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$ are distinct positive roots and are independent (up to reordering) of the chosen reduced expression for w. The construction of generators for $U_q[w]$ is analogous to the construction of $\beta_1, \ldots, \beta_N$: Let $\mathcal{B}_{\mathcal{W}}$ be the braid group of $\mathcal{W}$, which is obtained from $\mathcal{W}$ by omitting the involution relations between the generators $s_1, \ldots, s_n$. In particular, $\mathcal{B}_{\mathcal{W}}$ has generators $T_1, \ldots, T_n$ such that there is a surjective morphism $\mathcal{B}_{\mathcal{W}} \to \mathcal{W}$ which sends each T_i to s_i. Using Lusztig's action of the braid group $\mathcal{B}_{\mathcal{W}}$ on $U_q(\mathfrak{g})$ by algebra automorphisms (see [6, I.6.7]), define elements $X_1, \ldots, X_N$ of $U_q(\mathfrak{g})$ by

$$X_1 = E_{i_1},\ X_2 = T_{i_1} \cdot E_{i_2}, \ldots,\ X_N = T_{i_1} \cdots T_{i_{N-1}} \cdot E_{i_N}$$

and define $U_q[w]$ to be the subalgebra of $U_q(\mathfrak{g})$ generated by $X_1, \ldots, X_N$. It is well known that although the elements $X_1, \ldots, X_n$ of $U_q(\mathfrak{g})$ depend on the choice (2.8) of reduced expression for w, the algebra $U_q[w]$ generated by $X_1, \ldots, X_N$ is independent of this choice. By results of Lusztig and Levendorskii-Soibelman, $U_q[w]$ is a subalgebra of $U_q^+(\mathfrak{g})$ (and is in fact equal to $U_q^+(\mathfrak{g})$ if w is chosen as the unique longest element of $\mathcal{W}$) and by a result of Levendorskii-Soibelman, there are commutation relations between the generators $X_1, \ldots, X_N$ allowing $U_q[w]$ to be written as an iterated Ore extension

$$U_q[w] = k[X_1][X_2; \sigma_2; \delta_2] \cdots [X_N; \sigma_N, \delta_N], \tag{2.9}$$

where the σ_i are automorphisms, so that $U_q[w]$ is clearly a domain and is noetherian by [18, Theorem 2.6].

Example 2.1.3. *Let $\mathfrak{g} = \mathfrak{sl}_{m+n}$, $\mathcal{W} = S_{m+n}$, and $w = (1\ 2\ \cdots\ m+n)^m$. Cauchon and Mériaux [31, Proposition 2.1.1] constructed an isomorphism $U_q[w] \cong \mathcal{O}_q(M_{m,n}(\mathbb{K}))$.*

2.2 H-stratification

The notion of H-stratification, due to Goodearl and Letzter, plays a central role in this thesis. We introduce the idea here.

Let us suppose that R is a noetherian $\mathbb{K}$-algebra and that $H = (\mathbb{K}^\times)^r$ is an algebraic $\mathbb{K}$-torus[1] acting on R by automorphisms. Let us assume also that the action of H on R is rational i.e. R has a basis of H-eigenvectors whose eigenvalues $H \to k^\times$ are rational maps. We refer to the H-invariant prime ideals of R as H*-prime* ideals. We denote by H-Spec R the H*-spectrum* of R, namely the subspace of Spec R consisting of all H-prime ideals.

For an ideal I of R, $(I \colon H) := \bigcap_{h \in H} h \cdot I$ is easily checked to be the largest H-invariant ideal of R contained in I. It is well known that if P is a prime ideal of R, then $(P \colon H)$ is an H-prime ideal of R. For an H-prime ideal J of R, the H*-stratum* of Spec R associated to J is denoted by $\operatorname{Spec}_J R$ and is defined by $\operatorname{Spec}_J R = \{P \in \operatorname{Spec} R \mid (P \colon H) = J\}$. That is, $\operatorname{Spec}_J R$ is the subspace of Spec R consisting of all those prime ideals P of R with the property that J is the largest H-prime ideal (and in fact the largest H-invariant ideal) of R contained in P. The H-strata form a partition of Spec R, usually referred to as the H*-stratification*:

$$\operatorname{Spec}(R) = \bigsqcup_{J \in H-\operatorname{Spec}(R)} \operatorname{Spec}_J(R).$$

We shall later discuss the notion of H-stratification in much more detail; we shall pay particular attention to the crucial role which it plays in understanding the prime and primitive spectra of various quantum algebras.

[1]This is a slight abuse of terminology because, strictly speaking, H is not an algebraic group. Technically we should refer to H as the group of $\mathbb{K}$-rational points of the affine algebraic group $(\overline{\mathbb{K}}^\times)^r$.

Example 2.2.1. *The algebraic $\mathbb{K}$-torus $H = (\mathbb{K}^\times)^N$ acts rationally on any quantum affine space $\mathcal{O}_\Lambda(\mathbb{K}^N) = \mathbb{K}_\Lambda[T_1, \ldots, T_N]$ by automorphisms as follows:*

$$(a_1, \ldots, a_N) \cdot T_i = a_i T_i \text{ for all } i \in [\![1, N]\!] \text{ and all } (a_1, \ldots, a_N) \in H.$$

For a subset Δ of $[\![1, N]\!]$, let K_Δ be the ideal of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ generated by those T_i with $i \in \Delta$. The ideal K_Δ is clearly an H-invariant completely prime ideal of $\mathcal{O}_\Lambda(\mathbb{K}^N)$. Goodearl and Letzter have shown [17, Proposition 2.11] that all H-prime ideals of $\mathcal{O}_\Lambda(\mathbb{K}^N)$ take this form, namely that $H\text{-Spec}\,\mathcal{O}_\Lambda(\mathbb{K}^N) = \{K_\Delta \mid \Delta \subseteq [\![1, N]\!]\}$. For any $\Delta \subseteq [\![1, N]\!]$, we denote $\mathrm{Spec}_{K_\Delta} \mathcal{O}_\Lambda(\mathbb{K}^N)$ by $\mathrm{Spec}_\Delta(\mathcal{O}_\Lambda(\mathbb{K}^N)))$ and we have

$$\mathrm{Spec}_\Delta(\mathcal{O}_\Lambda(\mathbb{K}^N)) = \Big\{P \in \mathrm{Spec}\,\mathcal{O}_\Lambda(\mathbb{K}^N) \mid P \cap \{T_i \mid i \in [\![1, N]\!]\} = \{T_i \mid i \in \Delta\}\Big\}.$$

Example 2.2.2. *The algebraic torus $H = (\mathbb{K}^\times)^{m+n}$ acts rationally by automorphisms on $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ as follows:*

$$(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \cdot X_{i,j} = \alpha_i \beta_j X_{i,j} \tag{2.10}$$

for all $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \in H$ and all $(i, j) \in [\![1, m]\!] \times [\![1, n]\!]$.

2.3 Cauchon-Goodearl-Letzter extensions

Definition 2.3.1. In terminology similar to that introduced in [23, Definition 3.1], an iterated Ore extension

$$R = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N, \delta_N]$$

(where the σ_i are automorphisms and the δ_i are left σ_i-derivations) is called a *Cauchon-Goodearl-Letzter (or CGL) extension* if there exists an algebraic $\mathbb{K}$-torus $H = (\mathbb{K}^\times)^d$ acting rationally on R by automorphisms and

(i) $X_1, \dots, X_N$ are H-eigenvectors;

(ii) For all $j \in [\![2, N]\!]$, δ_j is locally nilpotent;

(iii) For all $j \in [\![2, N]\!]$, there exists $q_j \in \mathbb{K}^\times$ not a root of unity such that $\sigma_j \circ \delta_j = q_j \delta_j \circ \sigma_j$;

(iv) For all $j \in [\![2, N]\!]$ and all $i \in [\![1, j-1]\!]$, we have $\sigma_j(X_i) = \lambda_{j,i} X_i$ for some $\lambda_{j,i} \in k^\times$;

(v) The set $\{\lambda \in \mathbb{K}^\times \mid \text{there exists } h \in H \text{ such that } h \cdot X_1 = \lambda X_1\}$ is infinite;

(vi) For all $j \in [\![2, N]\!]$, there exists $h_j \in H$ such that $h_j \cdot X_j = q_j X_j$ and, for $i \in [\![1, j-1]\!]$, $h_j \cdot X_i = \lambda_{j,i} X_i$.

The CGL extension R is said to be a *uniparameter CGL extension* (with parameter q) if $\lambda_{j,i}$ is an integral power of q for all $j \in [\![2, N]\!]$ and all $i \in [\![1, j-1]\!]$.

Remark 2.3.2. *To a uniparameter CGL extension* $\mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N, \delta_N]$ *with parameter* q*, we associate a skew-symmetric integral matrix* $(a_{i,j})_{(i,j) \in [\![1,N]\!] \times [\![1,N]\!]}$ *such that* $\lambda_{j,i} = q^{a_{j,i}}$ *for all* $j \in [\![2, N]\!]$ *and all* $i \in [\![1, j-1]\!]$.

Remark 2.3.3. *Every CGL extension appearing in this thesis is uniparameter with parameter* q *or* q^{-1}.

The class of uniparameter CGL extensions contains many quantum algebras such as uniparameter quantum affine spaces, quantum matrices, and quantum Schubert cells. Uniparameter CGL extensions are clearly domains; they are noetherian by [18, Theorem 2.6]; they have finite Gelfand-Kirillov dimension[2] by [6, Lemma II.9.7]; they satisfy the noncommutative Nullstellensatz[3] over $\mathbb{K}$ by [6, Theorem II.7.17]; all prime ideals of a uniparameter CGL extension are completely prime by [6, Theorem II.6.9].

[2]For a definition of Gelfand-Kirillov dimension, see Chapter 2 of [22].

[3]For a definition of the noncommutative Nullstellensatz, see the beginning of Subsection 3.1.1.

2.4 Cauchon's deleting-derivations algorithm

Let $R = \mathbb{K}[X_1][X_2;\sigma_2,\delta_2]\cdots[X_N;\sigma_N,\delta_N]$ be a uniparameter CGL extension with parameter q, associated skew-symmetric integral matrix A, and admitting a rational action by an algebraic torus H. It is easy to check that R satisfies the conditions specified in [10, Section 3.1] and both [10, Hypothèse 4.1.1] and [10, Hypothèse 4.1.2]. In [10, Section 3], Cauchon introduced an algorithm, now known as the *deleting-derivations algorithm*, which he used to relate the prime and primitive spectra of R to those of the quantum affine space obtained by "deleting" the derivations $\delta_2,\ldots,\delta_N$.

The deleting-derivations algorithm sets $(X_1^{(N+1)},\ldots,X_N^{(N+1)}) = (X_1,\ldots,X_N)$ and, for each $j \in [\![2,N]\!]$, constructs from $(X_1^{(j+1)},\ldots,X_N^{(j+1)})$ a family $(X_1^{(j)},\ldots,X_N^{(j)})$ of elements of $\mathrm{Frac}(R)$. For each $j \in [\![2,N+1]\!]$, the subalgebra of $\mathrm{Frac}(R)$ generated by $X_1^{(j)},\ldots,X_N^{(j)}$ is denoted by $R^{(j)}$; in particular, $R^{(N+1)} = R$. By [10, Theorem 3.2.1], for each $j \in [\![1,N]\!]$, there is an isomorphism

$$\begin{aligned} &R^{(j+1)} \xrightarrow{\cong} \mathbb{K}[X_1][X_2;\sigma_2,\delta_2]\cdots[X_j;\sigma_j,\delta_j][X_{j+1},\tau_{j+1}]\cdots[X_N;\tau_N] \\ &X_i^{(j+1)} \mapsto X_i \quad \text{for all } i \in [\![1,N]\!], \end{aligned} \tag{2.11}$$

where for each $l \in [\![j+1,N]\!]$, τ_l is the automorphism which sends each X_i ($i \in [\![1,l-1]\!]$) to $\lambda_{l,i}X_i$. In particular, $R^{(2)}$ is isomorphic, by an isomorphism which sends each $X_i^{(2)}$ to T_i ($i = 1,\ldots,N$), to the quantum affine space given by

$$\mathbb{K}_{q,A}[T_1,\ldots,T_N] = \mathbb{K}[T_1][T_2;\tau_2]\cdots[T_n;\tau_N], \tag{2.12}$$

where for each $l = [\![2,N]\!]$, τ_l is the automorphism which sends each T_i ($i \in [\![1,l-1]\!]$) to $\lambda_{l,i}T_i$. Via this isomorphism, we identify $R^{(2)}$ with the quantum affine space $\mathbb{K}_{q,A}[T_1,\ldots,T_N]$.

The deleting-derivations algorithm: Suppose that $j \in [\![2,N]\!]$ and that the set $(X_1^{(j+1)},\ldots,X_N^{(j+1)})$ has been constructed. Notice that (2.11) shows in particular that the element $X_j^{(j+1)}$ of $\mathrm{Frac}(R)$ is nonzero and hence invertible. To construct $(X_1^{(j)},\dots,X_N^{(j)})$ from $(X_1^{(j+1)},\dots,X_N^{(j+1)})$, identify[4] $(X_1^{(j+1)},\dots,X_N^{(j+1)})$ with $(X_1,\dots,X_N)$ via the isomorphism (2.11) and for $i\in [\![1,N]\!]$, set

$$X_i^{(j)} := \begin{cases} X_i^{(j+1)} & \text{if } i\geq j; \\ \displaystyle\sum_{n=0}^{+\infty} \frac{(1-q_j)^{-n}}{[n]!_{q_j}} \delta_j^n \circ \sigma_j^{-n}(X_i^{(j+1)})(X_j^{(j+1)})^{-n} & \text{if } i<j, \end{cases} \tag{2.13}$$

where $[n]!_{q_j} = (1)\times(1+q_j)\times\cdots\times(1+q_j+\cdots+q_j^{n-1})$.

2.4.1 An injection $\varphi_j\colon \operatorname{Spec}(R^{(j+1)})\to\operatorname{Spec}(R^{(j)})$

In [10, Section 4.3], Cauchon constructed for each $j\in [\![2,N]\!]$ an injection

$$\varphi_j\colon \operatorname{Spec}(R^{(j+1)})\to\operatorname{Spec}(R^{(j)}).$$

We shall not describe the construction of this injection but we shall describe some of its properties which shall be useful to us.

2.4.2 Identifying several total rings of fractions

Let $j\in\{2,\dots,N\}$ and let Q be a prime ideal of $R^{(j+1)}$. Then [10, Lemme 5.3.1 and Lemma 5.3.2] give isomorphisms

$$\operatorname{Frac}(R^{(j+1)}/Q) \xrightarrow{\cong} \operatorname{Frac}(R^{(j)}/\varphi_j(Q)). \tag{2.14}$$

2.4.3 Relationships between generators

Let P be a prime ideal of R. For $j\in [\![2,N+1]\!]$, set $P^{(j)}=\varphi_j\circ\cdots\circ\varphi_N(P)\in\operatorname{Spec}(R^{(j)})$ (which gives $P^{(N+1)}=P$), and let $a_1^{(j)},\dots a_N^{(j)}$ be the canonical images of $X_1^{(j)},\dots,X_N^{(j)}$ in $R^{(j)}/P^{(j)}$. Let us denote by G the total ring of fractions of R/P (which is a division ring since all prime ideals of R are completely prime) and by varying j over $[\![2,N]\!]$

[4]We make this identification in order that the term $\delta_j^n\circ\sigma_j^{-n}(X_i^{(j+1)})$ in (2.13) makes sense.

and Q over $P^{(3)}, \dots, P^{(N+1)}$ in the isomorphism (2.14), let us identify the total ring of fractions of each noetherian domain $R^{(j)}/P^{(j)}$ ($j \in [\![2, N+1]\!]$) with the division ring G. Some immediate consequences of this setup (noted in [10, Proposition 5.4.1]) are that for each $j \in [\![2, N+1]\!]$,

- $R^{(j)}/P^{(j)}$ is the subalgebra of G generated by $a_1^{(j)}, \dots, a_N^{(j)}$;
- there is a morphism of algebras $f_j \colon R^{(j)} \to G$ which sends each $X_i^{(j)}$ ($i \in [\![1, N]\!]$) to $a_i^{(j)}$;
- the kernel of f_j is $P^{(j)}$ and its image is $R^{(j)}/P^{(j)}$.

For $j \in [\![2, N]\!]$, Cauchon [10] gives an algorithm for constructing the generators $a_1^{(j)}, \dots, a_N^{(j)}$ of the algebra $R^{(j)}/P^{(j)}$ from the generators $a_1^{(j+1)}, \dots, a_N^{(j+1)}$ of the algebra $R^{(j+1)}/P^{(j+1)}$. Indeed suppose that $j \in [\![2, N]\!]$ and that $i \in [\![1, N]\!]$. By [10, Proposition 5.4.2], when we identify[5] $(X_1^{(j+1)}, \dots, X_N^{(j+1)})$ with $(X_1, \dots, X_N)$ via the isomorphism (2.11), we have

$$a_i^{(j)} = \begin{cases} \displaystyle\sum_{n=0}^{+\infty} \frac{(1-q_j)^{-n}}{[n]!_{q_j}} \lambda_{j,i}^{-n} f_{j+1}(\delta_j^n(X_i^{(j+1)}))(a_j^{(j+1)})^{-n} & \text{if } 1 \le i < j \text{ and } a_j^{(j+1)} \neq 0; \\ a_i^{(j+1)} & \text{otherwise.} \end{cases} \tag{2.15}$$

2.4.4 The canonical injection $\varphi \colon \operatorname{Spec}(R) \to \operatorname{Spec}(R^{(2)})$

By [10, Lemma 4.2.1], the action of H on R induces an action of H on $R^{(2)} = \mathbb{K}_{q,A}[T_1, \dots, T_N]$. Although this may not be the same torus action as defined in Example 2.2.1, [10, Proposition 5.5.1] shows that both actions have the same invariant prime ideals i.e. if we let W be the power set of $[\![1, N]\!]$ and if for $\Delta \in W$, we let K_Δ be the ideal of $R^{(2)}$ generated by $\{T_i \mid i \in \Delta\}$, then $H-\operatorname{Spec}(R^{(2)}) = \{K_\Delta \mid \Delta \in W\}$. For $\Delta \in W$, denote the H-stratum of K_Δ by $\operatorname{Spec}_\Delta(R^{(2)})$, so that

$$\operatorname{Spec}_\Delta(R^{(2)}) = \left\{P \in \operatorname{Spec}(R^{(2)}) \mid P \cap \{T_i \mid i \in [\![1, N]\!]\} = \{T_i \mid i \in \Delta\}\right\}.$$

[5] We make this identification in order that the term $\delta_j^n(X_i^{(j+1)})$ in (2.15) makes sense.

Define the *canonical injection* $\varphi\colon \operatorname{Spec}(R) \to \operatorname{Spec}(R^{(2)})$ by $\varphi := \varphi_2 \circ \cdots \circ \varphi_N$ (see [10, Définition 4.4.1]) and for $\Delta \in W$, set $\operatorname{Spec}_\Delta(R) = \varphi^{-1}(\operatorname{Spec}_\Delta(R^{(2)}))$. Denote by W' the set of those $\Delta \in W$ with $\operatorname{Spec}_\Delta(R) \neq \emptyset$. The elements of W are called the *diagrams* of R and the elements of W' are called the *Cauchon diagrams* of R. For any Cauchon diagram Δ of R, the canonical injection φ restricts to a bi-increasing homeomorphism from $\operatorname{Spec}_\Delta(R)$ to $\operatorname{Spec}_\Delta(R^{(2)})$ ([10, Théorèmes 5.1.1 and 5.5.1]).

Remark 2.4.1. *Though we shall not do it, it would be more precise to call the elements of W' the* Cauchon diagrams of the CGL extension R, *rather than the* Cauchon diagrams of R; *not all ways to write R as a CGL extension yield the same Cauchon diagrams.*

By [10, Proposition 4.4.1], we have

$$\operatorname{Spec}(R) = \bigsqcup_{\Delta \in W'} \operatorname{Spec}_\Delta(R).$$

This is called the *canonical partition* of $\operatorname{Spec}(R)$ and, by [10, Théorème 5.5.2], it coincides with the partition of $\operatorname{Spec}(R)$ into H-strata. Let us make this more precise: By [10, Lemme 5.5.8 and Théorème 5.5.2], we have

(i) For any $\Delta \in W'$, there is a unique H-prime ideal J_Δ of R such that $\varphi(J_\Delta) = K_\Delta$;

(ii) $H - \operatorname{Spec}(R) = \{J_\Delta \mid \Delta \in W'\}$, so that $H - \operatorname{Spec}(R)$ is finite (having cardinality at most 2^N);

(iii) $\operatorname{Spec}_{J_\Delta}(R) = \operatorname{Spec}_\Delta(R)$ for all $\Delta \in W'$.

Remark 2.4.2. *In the current general setting of uniparameter CGL extensions, Cauchon diagrams are sets rather than diagrams. The terminology* Cauchon diagram *comes from the application of Cauchon's theory of deleting derivations to algebras of quantum matrices. Set $R = \mathcal{O}_q(M_{m,n}(\mathbb{K}))$, set $N = mn$, and recall that R can be expressed as an iterated Ore extension* (2.7) *in N indeterminates. There is a poset isomorphism $\iota\colon [\![1, N]\!] \xrightarrow{\cong} [\![1, m]\!] \times [\![1, n]\!]$, where the set $[\![1, m]\!] \times [\![1, n]\!]$ is endowed with the lexicographical order; let us identify $[\![1, N]\!]$ and $[\![1, m]\!] \times [\![1, n]\!]$ via ι. Let us identify* *any subset S of $[\![1,m]\!] \times [\![1,n]\!]$ with an $m \times n$ rectangular array of boxes such that the box in the (i,j)-position is black if $(i,j) \in S$ and white otherwise. Cauchon showed in [11, Section 3.2] that the Cauchon diagrams of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ are exactly those $m \times n$ rectangular arrays of black and white boxes satisfying the following rule:*

If a box is black, then either every box to its left is black or every box above it is black.

Chapter 3

A strong Dixmier-Moeglin equivalence for quantum Schubert cells

With the exception of Subsection 3.7.1, the material of this chapter is original and is based on joint work with Prof. Stéphane Launois and Prof. Jason Bell; the results come from [4].

It is a difficult and often intractable problem to classify the irreducible representations of an algebra. Dixmier proposed that a good first step towards tackling this problem would be to find the kernels of the irreducible representations, that is the annihilators of the simple modules, namely the primitive ideals. In any ring, every primitive ideal is prime; Dixmier [13] and Moeglin [32] gave an algebraic condition and a topological condition for deciding whether or not a given prime ideal of the universal enveloping algebra of a finite-dimensional complex Lie algebra is primitive:

- A prime ideal P of a ring R is said to be *locally closed* if the singleton set $\{P\}$ is locally closed in the Zariski topology on $\operatorname{Spec} R$, namely if $\{P\}$ is the intersection of a Zariski-open subset of $\operatorname{Spec} R$ and a Zariski-closed subset of $\operatorname{Spec} R$. (For a prime ideal P of a ring R, it is easily shown that P is locally closed if and only if

P is strictly contained in the intersection of all prime ideals of R which strictly contain P.)

- A prime ideal P of a noetherian $\mathbb{K}$-algebra R is said to be *rational* if the field extension[1] $\mathcal{Z}(\operatorname{Frac} R/P)$ of $\mathbb{K}$ is algebraic.

Dixmier and Moeglin proved that for a prime ideal of the universal enveloping algebra of a finite-dimensional complex Lie algebra, the properties of being primitive, locally closed, and rational are equivalent. In modern terminology, they proved that the universal enveloping algebra of a finite-dimensional complex Lie algebra satisfies the *Dixmier-Moeglin equivalence.*

Since the work of Dixmier and Moeglin on universal enveloping algebras of finite-dimensional complex Lie algebras, many more algebras have been shown to satisfy the Dixmier-Moeglin equivalence: [6, Corollary II.8.5] lists several quantised coordinate rings which satisfy the Dixmier-Moeglin equivalence; Bell, Rogalski, and Sierra [5] have shown that twisted homogeneous coordinate rings of projective surfaces satisfy the Dixmier-Moeglin equivalence. However, Irving [20] and Lorenz [27] have shown that there exist noetherian algebras of infinite Gelfand-Kirillov dimension for which the Dixmier-Moeglin equivalence fails. Moreover Bell, Launois, León Sánchez, and Moosa [3] have shown that there exist noetherian algebras of finite Gelfand-Kirillov dimension which do not satisfy the Dixmier-Moeglin equivalence.

Our goal is to extend the notion of the Dixmier-Moeglin equivalence to all prime ideals, in a way which captures how "close" they are to being primitive. Of course, not all non-primitive prime ideals are created equal. For example, in the polynomial ring $\mathbb{C}[x, y]$, the primitive ideals are the maximal ideals $\langle x - \alpha, y - \beta \rangle$. For this reason, we think of the prime ideal $\langle x \rangle$ as being "closer" to being primitive than the prime ideal $\langle 0 \rangle$, in the same sense that it is "closer" to being maximal — that is, the height of $\langle x \rangle$ is greater than the height of $\langle 0 \rangle$.

[1]The $\mathbb{K}$-algebra $\mathcal{Z}(\operatorname{Frac} R/P)$ is a field by the Goldie and Artin-Wedderburn theorems.

In general, given a noetherian $\mathbb{K}$-algebra R and given a prime ideal P of R, we are interested in the *primitivity degree*, $\operatorname{prim.deg} P$, of P, which we define as follows:

$$\operatorname{prim.deg} P := \inf\{\operatorname{ht} Q \mid Q \in \operatorname{Prim} R/P\},$$

where $\operatorname{Prim} R/P$ denotes the subspace of $\operatorname{Spec} R/P$ consisting of the primitive ideals of R/P. This quantity gives a measure of how close the prime ideal P is to being primitive. Clearly, P is primitive if and only if $\operatorname{prim.deg} P = 0$.

Remark 3.0.1. *We would like to have a more representation-theoretic characterisation of primitivity degree, such as a way to realise the prime ideals of a given primitivity degree as the kernels of members of a family of representations. However we have not been able to find such a characterisation.*

We use the notion of primitivity degree to extend the idea of the Dixmier-Moeglin equivalence to all prime ideals. To this end, we define generalisations of the notions of a locally closed ideal and a rational ideal.

It is easy to extend the notion of a rational ideal: for a prime ideal P of R, we define the *rationality degree*, $\operatorname{rat.deg} P$, of P to be the transcendence degree of the field extension $\mathcal{Z}(\operatorname{Frac} R/P)$ of $\mathbb{K}$. Clearly, P is rational if and only if $\operatorname{rat.deg} P = 0$.

Remark 3.0.2. *It seems reasonable to expect that, under some mild assumptions, the property that* $\operatorname{rat.deg} P = d$ *should relate to the existence of a rational ideal of height d in R/P but it seems difficult to establish such a relationship.*

In the same spirit of generalisation, we define the *local closure degree*, $\operatorname{loc.deg} P$, of a prime ideal P of R to be the smallest nonnegative integer d such that $\bigcap_{Q \in \operatorname{Spec}_{>d} R/P} Q \neq 0$, where $\operatorname{Spec}_{>d} R/P$ denotes the subspace of $\operatorname{Spec} R/P$ consisting of all prime ideals of R/P which are of height strictly greater than d. Clearly, P is locally closed if and only if $\operatorname{loc.deg} P = 0$.

Remark 3.0.3. *In the case that the noetherian $\mathbb{K}$-algebra R has finite Gelfand-Kirillov dimension, all prime ideals of R have finite height by [22, Corollary 3.16]. All of the* *algebras which will concern us in this chapter have finite Gelfand-Kirillov dimension and so we shall always use the following equivalent characterisation of local closure degree: for a prime ideal P of R,* $\operatorname{loc.deg} P$ *is the smallest nonnegative integer d such that $\bigcap_{Q \in \operatorname{Spec}_{d+1} R/P} Q \neq 0$, where* $\operatorname{Spec}_{d+1} R/P$ *denotes the subspace of* $\operatorname{Spec} R/P$ *consisting of all prime ideals of R/P which are of height $d+1$. In this context, we shall prove (in the proof of Proposition 3.1.1) that if* $P \in \operatorname{Spec} R$ *is such that* $\operatorname{loc.deg} P = d$*, then R/P has a locally closed ideal of height d.*

Definition 3.0.4. A noetherian $\mathbb{K}$-algebra R is said to satisfy the *strong Dixmier-Moeglin equivalence* (SDME) if every prime ideal P of R satisfies

$$\operatorname{loc.deg} P = \operatorname{prim.deg} P = \operatorname{rat.deg} P.$$

We remark that the strong Dixmier-Moeglin equivalence is strictly stronger than the Dixmier-Moeglin equivalence. Indeed the Dixmier-Moeglin equivalence simply says that if P is a prime ideal of a noetherian $\mathbb{K}$-algebra R, then

$$\operatorname{loc.deg} P = 0 \iff \operatorname{prim.deg} P = 0 \iff \operatorname{rat.deg} P = 0.$$

Even though the universal enveloping algebra, $U(\mathfrak{sl}_2(\mathbb{C}))$, of $\mathfrak{sl}_2(\mathbb{C})$ satisfies the Dixmier-Moeglin equivalence (as was shown in the original work of Dixmier and Moeglin), it fails to satisfy the strong Dixmier-Moeglin equivalence. Indeed, since $U(\mathfrak{sl}_2(\mathbb{C}))$ is a domain, $\langle 0 \rangle$ is a (completely) prime ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$. By [9, Remark 4.6], all prime ideals of $U(\mathfrak{sl}_2(\mathbb{C}))$ except $\langle 0 \rangle$ are primitive, so that $\operatorname{prim.deg}\langle 0 \rangle = 1$. It is well known that the centre of $U(\mathfrak{sl}_2(\mathbb{C}))$ is given by the polynomials in the Casimir element; by [14, Corollary 4.2.3], $\mathcal{Z}(\operatorname{Frac} U(\mathfrak{sl}_2(\mathbb{C})))$ is given by the rational functions in the Casimir element, so that $\operatorname{rat.deg}\langle 0 \rangle = \operatorname{tr.deg}_{\mathbb{C}} \mathcal{Z}(\operatorname{Frac} U(\mathfrak{sl}_2(\mathbb{C}))) = 1$. By [9, Theorem 4.5 and Proposition 5.13], there are infinitely many height two prime ideals in $U(\mathfrak{sl}_2(\mathbb{C}))$ and their intersection is zero, so that $\operatorname{loc.deg}\langle 0 \rangle > 1$. Since, by [9, Theorem 4.5], there are no height three prime ideals in $U(\mathfrak{sl}_2(\mathbb{C}))$, the intersection of the height three prime ideals is nonzero (in fact it is the entirety of $U(\mathfrak{sl}_2(\mathbb{C}))$), so that $\operatorname{loc.deg}\langle 0 \rangle = 2$.

The goal of this chapter is to prove that quantum Schubert cells $U_q[w]$ (see Subsection 2.1.6 for more details) satisfy the strong Dixmier-Moeglin equivalence.

It shall be useful to define a weaker version of the strong Dixmier-Moeglin equivalence which is often easy to prove and provides a useful stepping-stone to proving the strong Dixmier-Moeglin equivalence.

Definition 3.0.5. A noetherian $\mathbb{K}$-algebra R is said to satisfy the *quasi strong Dixmier-Moeglin equivalence* if every prime ideal P of R satisfies $\operatorname{loc.deg} P = \operatorname{rat.deg} P$.

With the quasi strong Dixmier-Moeglin equivalence in hand for a noetherian $\mathbb{K}$-algebra R, the problem is reduced to showing that every prime ideal P of R satisfies $\operatorname{prim.deg} P = \operatorname{rat.deg} P$. For a quantum Schubert cell $U_q[w]$, we prove this by exploiting the good behaviour of the poset of H-prime ideals of $U_q[w]$, where H is a suitable algebraic $\mathbb{K}$-torus acting rationally on $U_q[w]$ by automorphisms.

This chapter is organised as follows. First, we prove various general results about the (quasi) strong Dixmier-Moeglin equivalence (Section 3.1). Next, we consider various examples from the quantum world. Using Cauchon's theory of deleting derivations, one can relate the prime and primitive spectra of a quantum Schubert cell to those of an associated uniparameter quantum affine space, which can in turn be related via localisations to the prime and primitive spectra of a family of uniparameter quantum tori. Since there is a bi-increasing homeomorphism between the prime spectrum of a uniparameter quantum torus and the prime spectrum of its centre, which is a commutative affine domain, we are guided into a natural strategy: we shall prove the strong Dixmier-Moeglin equivalence first for commutative affine domains (Section 3.2), then for uniparameter quantum tori (Section 3.3), then for uniparameter quantum affine spaces (Section 3.5), and then for quantum Schubert cells (Section 3.7). Partial results are also obtained for a larger class of algebras — we prove in Section 3.6 that every uniparameter Cauchon-Goodearl-Letzter extension satisfies the quasi strong Dixmier-Moeglin equivalence. (Finally, we use our results to deduce that the quantum groups $\mathcal{O}_q(SL_n)$ and $\mathcal{O}_q(GL_n)$ satisfy the strong Dixmier-Moeglin equivalence.)

3.1 General results on the (quasi) SDME

In this section we prove that, under some mild assumptions, the primitivity degree of a prime ideal is bounded above by its local closure degree, and then we prove transfer results for the quasi strong Dixmier-Moeglin equivalence for an algebra and its localisations.

3.1.1 An upper bound for the primitivity degree

Some of the implications needed to prove the Dixmier-Moeglin equivalence hold in a very general setting. Recall that a noetherian $\mathbb{K}$-algebra R is said to satisfy the *noncommutative Nullstellensatz* over $\mathbb{K}$ if R is a Jacobson ring (which means that every prime ideal is an intersection of primitive ideals) and the endomorphism ring of every irreducible R-module is algebraic over $\mathbb{K}$. By [6, Lemma II.7.15], for any noetherian $\mathbb{K}$-algebra R which satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ and for any prime ideal P of R, we have

$$P \text{ is locally closed} \implies P \text{ is primitive} \implies P \text{ is rational.} \tag{3.1}$$

We have generalised the first implication above to a large class of algebras:

Proposition 3.1.1. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand-Kirillov dimension which has the property that every locally closed ideal is primitive (this is the case if, for example, R satisfies the noncommutative Nullstellensatz over $\mathbb{K}$). Then for any prime ideal P of R, we have* $\operatorname{loc.deg} P \geq \operatorname{prim.deg} P$.

Proof. Let $P \in \operatorname{Spec} R$ be such that $\operatorname{loc.deg} P = d$. We claim that the algebra $B := R/P$ has a locally closed ideal of height d. Indeed if not, then every prime ideal Q of height d in B is such that $\bigcap_{Q \subsetneqq T \in \operatorname{Spec} B} T = Q$. It follows that $\bigcap_{Q \in \operatorname{Spec}_d B} \left(\bigcap_{Q \subsetneqq T \in \operatorname{Spec} B} T \right) = \bigcap_{Q \in \operatorname{Spec}_d B} Q$, so that $\bigcap_{T \in \operatorname{Spec}_{>d} B} T = \bigcap_{Q \in \operatorname{Spec}_d B} Q$ i.e. $\bigcap_{T \in \operatorname{Spec}_{d+1} B} T = \bigcap_{Q \in \operatorname{Spec}_d B} Q$. This is a contradiction because $\operatorname{loc.deg} P = d$ implies that the intersection $\bigcap_{Q \in \operatorname{Spec}_d B} Q$ is trivial while the intersection $\bigcap_{T \in \operatorname{Spec}_{d+1} B} T$ is not.

This establishes the claim that the algebra $B = R/P$ has a locally closed ideal of height d; since this ideal is also primitive, the proof is complete. ■

We do not know whether the second implication in (3.1) can be similarly generalised but we will prove, on a case-by-case basis, that for all prime ideals P of a commutative affine domain, a uniparameter quantum torus, a uniparameter quantum affine space, or a quantum Schubert cell, we have

$$\mathrm{prim.deg}\, P = \mathrm{rat.deg}\, P.$$

We will do the same for all prime ideals P of the quantum groups $\mathcal{O}_q(SL_n)$ and $\mathcal{O}_q(GL_n)$.

3.1.2 Transferring the quasi SDME

Recall that a noetherian $\mathbb{K}$-algebra R is said to satisfy the quasi strong Dixmier-Moeglin equivalence if, for every prime ideal P of R, we have $\mathrm{loc.deg}\, P = \mathrm{rat.deg}\, P$.

Lemma 3.1.2. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand-Kirillov dimension which is a domain and in which every prime ideal is completely prime. Let $\mathcal{E}$ be a right Ore set of regular elements of R which is finitely generated as a multiplicative system. Then for any $d \in \mathbb{N} \setminus \{0\}$, we have*

$$\textstyle\bigcap_{P \in \mathrm{Spec}_d R} P \neq 0 \iff \bigcap_{Q \in \mathrm{Spec}_d R\mathcal{E}^{-1}} Q \neq 0.$$

It follows immediately that $\mathrm{loc.deg}\langle 0\rangle_R = \mathrm{loc.deg}\langle 0\rangle_{R\mathcal{E}^{-1}}$*, where* $\langle 0\rangle_R$ *and* $\langle 0\rangle_{R\mathcal{E}^{-1}}$ *denote the zero ideals of R and $R\mathcal{E}^{-1}$ respectively.*

Proof. Let $\mathcal{E}$ be generated as a multiplicative system by $x_1, \ldots, x_n$. Since all prime ideals of R are completely prime, the conditions $P \cap \mathcal{E} = \emptyset$ and $x_1, \ldots, x_n \notin P$ are equivalent for every prime ideal P of R.

By [18, Theorem 10.20], extension ($P \mapsto P\mathcal{E}^{-1}$) and contraction ($Q \mapsto Q \cap R$) are mutually inverse increasing bijections between $\{P \in \mathrm{Spec}\, R \mid P \cap \mathcal{E} = \emptyset\} = \{P \in$

$\operatorname{Spec} R \mid x_1, \ldots, x_n \notin P\}$ and $\operatorname{Spec} R\mathcal{E}^{-1}$, so that since both extension and contraction send the zero ideal to the zero ideal, we get

$$\textstyle\bigcap_{P \in \operatorname{Spec}_d R,\ x_1,\ldots,x_n \notin P} P \neq 0 \iff \bigcap_{Q \in \operatorname{Spec}_d R\mathcal{E}^{-1}} Q \neq 0. \tag{3.2}$$

We claim that

$$\textstyle\bigcap_{P \in \operatorname{Spec}_d R,\ x_1,\ldots,x_n \notin P} P \neq 0 \iff \bigcap_{P \in \operatorname{Spec}_d R} P \neq 0. \tag{3.3}$$

One implication is trivial. For the other, suppose that $\bigcap_{P \in \operatorname{Spec}_d R,\ x_1,\ldots,x_n \notin P} P \neq 0$ and choose any $0 \neq r$ which belongs to this intersection. Then $0 \neq r x_1 \cdots x_n \in \bigcap_{P \in \operatorname{Spec}_d R} P$, verifying (3.3). Now (3.2) and (3.3) immediately give the result. ∎

Lemma 3.1.3. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand-Kirillov dimension in which every prime ideal is completely prime. If R satisfies the quasi strong Dixmier-Moeglin equivalence and $\mathcal{E}$ is a right Ore set of regular elements of R which is finitely generated as a multiplicative system, then $R\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier-Moeglin equivalence.*

Proof. Every prime ideal of $R\mathcal{E}^{-1}$ takes the form $P\mathcal{E}^{-1}$ for some $P \in \operatorname{Spec} R$ with $P \cap \mathcal{E} = \emptyset$. Denoting by $\overline{\mathcal{E}}$ the image of $\mathcal{E}$ in R/P, we have

$$\begin{aligned}
\operatorname{loc.deg} P\mathcal{E}^{-1} &= \operatorname{loc.deg}\langle 0\rangle_{R\mathcal{E}^{-1}/P\mathcal{E}^{-1}} \\
&= \operatorname{loc.deg}\langle 0\rangle_{(R/P)\overline{\mathcal{E}}^{-1}} \\
&= \operatorname{loc.deg}\langle 0\rangle_{R/P} \qquad \text{(Lemma 3.1.2)} \\
&= \operatorname{loc.deg} P \\
&= \operatorname{rat.deg} P.
\end{aligned}$$

Since it is clear that $\operatorname{rat.deg} P = \operatorname{rat.deg} P\mathcal{E}^{-1}$, we are done. ∎

Proposition 3.1.4. *Let R be a noetherian $\mathbb{K}$-algebra of finite Gelfand-Kirillov dimension in which every prime ideal is completely prime. Suppose that for every $P \in \operatorname{Spec} R$, there exists a right Ore set $\mathcal{E}$ of regular elements of R/P which is finitely generated as* *a multiplicative system, such that $(R/P)\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier-Moeglin equivalence. Then R itself satisfies the quasi strong Dixmier-Moeglin equivalence.*

Proof. Choose any $P \in \operatorname{Spec} R$. We have

$$\begin{aligned} \operatorname{loc.deg} P &= \operatorname{loc.deg}\langle 0\rangle_{R/P} \\ &= \operatorname{loc.deg}\langle 0\rangle_{(R/P)\mathcal{E}^{-1}} \qquad \text{(Lemma 3.1.2)} \\ &= \operatorname{rat.deg}\langle 0\rangle_{(R/P)\mathcal{E}^{-1}}. \end{aligned}$$

Since it is clear that $\operatorname{rat.deg}\langle 0\rangle_{(R/P)\mathcal{E}^{-1}} = \operatorname{rat.deg} P$, we are done. ■

3.2 The SDME in the commutative case

If there is to be any hope that the strong Dixmier-Moeglin equivalence will hold for any quantum algebras, one should first check that it holds for commutative affine domains. Before checking this, let us introduce the useful notion of Tauvel's height formula:

Definition 3.2.1. *Tauvel's height formula* is said to hold in a $\mathbb{K}$-algebra R if for every prime ideal P of R, the following equality holds:

$$\operatorname{GK.dim} R/P = \operatorname{GK.dim} R - \operatorname{ht} P.$$

It is well known that Tauvel's height formula holds in commutative affine domains; as we shall remark later, it has also been shown to hold in several interesting quantum algebras, including all of those which interest us in this chapter.

In commutative affine domains, the notions of primitive, locally closed, and rational ideals all agree with the notion of a maximal ideal, so the following result is not surprising.

Proposition 3.2.2. *Every commutative affine domain over $\mathbb{K}$ satisfies the strong Dixmier-Moeglin equivalence.*

Proof. Let R be a commutative affine domain over $\mathbb{K}$ and let $P \in \operatorname{Spec} R$. By [34, Remark 6.9 (i), Corollary 6.49], every primitive (i.e. maximal) ideal of R/P has height $\operatorname{tr.deg}_{\mathbb{K}} \operatorname{Frac}(R/P)$, so that

$$\operatorname{prim.deg} P = \operatorname{K.dim} R/P = \operatorname{rat.deg} P. \tag{3.4}$$

If we set $d = \operatorname{prim.deg} P = \operatorname{K.dim} R/P = \operatorname{rat.deg} P$, then all maximal ideals of R/P have height d, so that $\operatorname{Spec}_{d+1} R/P$ is empty and hence $\bigcap_{Q \in \operatorname{Spec}_{d+1} R/P} Q = R/P \neq 0$. Since R is a Jacobson ring, we get $\bigcap_{Q \in \operatorname{Spec}_d R/P} Q = 0$, so that $\operatorname{loc.deg} P = d$. This completes the proof. ■

Remark 3.2.3. *Affine prime noetherian polynomial identity algebras over $\mathbb{K}$ can be shown to satisfy the strong Dixmier-Moeglin equivalence by a proof essentially the same as the proof above.*

Remark 3.2.4. *Let P be a prime ideal of a commutative affine domain R over $\mathbb{K}$. Since Gelfand-Kirillov dimension and Krull dimension agree in commutative affine domains, Tauvel's height formula gives* $\operatorname{K.dim} R/P = \operatorname{K.dim} R - \operatorname{ht} P$. *Now we conclude from Proposition 3.2.2 and equation* (3.4) *that*

$$\operatorname{loc.deg} P = \operatorname{prim.deg} P = \operatorname{rat.deg} P = \operatorname{K.dim} R - \operatorname{ht} P.$$

3.3 The SDME for uniparameter quantum tori

Consider any quantum torus $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N) = \mathbb{K}_\Lambda[T_1^{\pm 1}, \ldots, T_N^{\pm 1}]$. By [6, Corollary II.7.18], $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ and by [6, Theorem II.9.14], $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is catenary and satisfies Tauvel's height formula.

We recall from [17, Section 1] some useful facts about quantum tori. For $\underline{i} = (i_1, \ldots, i_N) \in \mathbb{Z}^N$, we set $\underline{T}^{\underline{i}} := T_1^{i_1} \cdots T_N^{i_N}$. For any $\underline{s}, \underline{t} \in \mathbb{Z}^N$, we have $\underline{T}^{\underline{s}}\underline{T}^{\underline{t}} = \sigma(\underline{s}, \underline{t})\underline{T}^{\underline{t}}\underline{T}^{\underline{s}}$, where $\sigma \colon \mathbb{Z}^N \times \mathbb{Z}^N \to \mathbb{K}^\times$ is the alternating bicharacter which sends any $((s_1, \ldots, s_N), (t_1, \ldots, t_N))$ to $\prod_{i,j=1}^N \lambda_{i,j}^{s_i t_j}$.

When S is the subgroup $\{\underline{s} \in \mathbb{Z}^N \mid \sigma(\underline{s}, -) \equiv 1\}$ of $\mathbb{Z}^N$, the centre of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is spanned over $\mathbb{K}$ by those $\underline{T}^{\underline{s}}$ with $\underline{s} \in S$. When $\underline{b_1}, \dots, \underline{b_r}$ is a basis for S, the centre of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is a commutative Laurent polynomial ring in $\underline{T}^{\underline{b_1}}, \dots, \underline{T}^{\underline{b_r}}$. Moreover, $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ is a free module over its centre with basis $\{\underline{T}^{\underline{t}}\}_{\underline{t}}$, where $\underline{t}$ runs over any transversal for S in $\mathbb{Z}^N$.

There is a bi-increasing homeomorphism, known as *extension*, from $\operatorname{Spec} \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))$ to $\operatorname{Spec} \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ given by $I \mapsto \langle I \rangle$ (where $\langle I \rangle$ denotes the ideal of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ generated by I). The inverse of this map is given by $J \mapsto J \cap \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))$ and is known as *contraction* from $\operatorname{Spec} \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ to $\operatorname{Spec} \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))$. In fact, contraction and extension define mutually inverse increasing bijections between the set of all ideals of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ and the set of all ideals of its centre.

Computing the rationality degree of a prime ideal P of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ requires study of the centre of $\operatorname{Frac}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)/P)$. The following general lemma is folklore, but we have not been able to locate it in the literature[2].

Lemma 3.3.1. *Let R be a prime noetherian ring and suppose that every nonzero ideal of R intersects $\mathcal{Z}(R)$ nontrivially. Then*

$$\mathcal{Z}(\operatorname{Frac} R) \cong \operatorname{Frac} \mathcal{Z}(R).$$

Proof. $\operatorname{Frac} \mathcal{Z}(R)$ embeds naturally into $\mathcal{Z}(\operatorname{Frac} R)$. Let $z \in \mathcal{Z}(\operatorname{Frac} R)$ and set $I = \{a \in R \mid za \in R\}$. Then I is a nonzero ideal of R and thus contains a nonzero element c of $\mathcal{Z}(R)$, which is regular in R (since R is prime) and hence is certainly regular in $\mathcal{Z}(R)$. Now $z = (zc)c^{-1} \in \operatorname{Frac} \mathcal{Z}(R)$. ■

Proposition 3.3.2. *For every prime ideal P of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$, we have*

$$\mathcal{Z}\left(\operatorname{Frac} \frac{\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)}{P}\right) \cong \operatorname{Frac} \mathcal{Z}\left(\frac{\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)}{P}\right).$$

[2]We thank Ken Goodearl for bringing this result to our attention.

Proof. Set $R = \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$ and let P be a prime ideal of R. By Lemma 3.3.1, it will suffice to show that every nonzero ideal of R/P intersects $\mathcal{Z}(R/P)$ nontrivially. This follows easily from the fact that every ideal of R is generated by its intersection with $\mathcal{Z}(R)$. ■

Proposition 3.3.3. *For any ideal I of $\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$, we have*

$$\mathcal{Z}\left(\frac{\mathcal{O}_\Lambda((\mathbb{K}^\times)^N)}{I}\right) \cong \frac{\mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))}{I \cap \mathcal{Z}(\mathcal{O}_\Lambda((\mathbb{K}^\times)^N))}.$$

Proof. Retain the notation introduced at the beginning of the current section (Section 3.3). Set $R = \mathcal{O}_\Lambda((\mathbb{K}^\times)^N)$. We may clearly assume that I is a proper ideal and that R is noncommutative. We claim that $\mathcal{Z}(R/I) = (\mathcal{Z}(R) + I)/I$. Indeed the inclusion $\mathcal{Z}(R/I) \supseteq (\mathcal{Z}(R) + I)/I$ is obvious. Suppose that $x \in R$ is central modulo I. We may choose elements $\underline{0}, \underline{i_1}, \ldots, \underline{i_n}$ of a transversal for S in $\mathbb{Z}^N$ and central elements $z_0, z_1, \ldots, z_n$ of R such that

$$x = z_0 + \sum_{a=1}^{n} z_a \underline{T}^{\underline{i_a}}.$$

Fixing any $b \in [\![1, n]\!]$, there exists j_b belonging to the chosen transversal for S in $\mathbb{Z}^N$ such that $\sigma(j_b, i_b) \neq 1$. Since $\underline{T}^{\underline{j_b}} x (\underline{T}^{\underline{j_b}})^{-1} = x$ modulo I, we have

$$\sum_{a=1}^{n} (1 - \sigma(\underline{j_b}, \underline{i_a})) z_a \underline{T}^{\underline{i_a}} \in I$$

and hence, by [17, Proposition 1.4], each $(1 - \sigma(\underline{j_b}, \underline{i_a}))z_a$ must belong to I. Since $\sigma(\underline{j_b}, \underline{i_b}) \neq 1$, we must have $z_b \in I$. Because $b \in [\![1, n]\!]$ was chosen arbitrarily, we get $z_1, \ldots, z_n \in I$ and hence $x = z_0$ modulo I, completing the proof. ■

We are now ready to prove the main result of this section.

Theorem 3.3.4. *The uniparameter quantum tori $\mathcal{O}_{q,A}((\mathbb{K}^\times)^N)$ satisfy the strong Dixmier-Moeglin equivalence.*

Proof. Set $R = \mathcal{O}_{q,A}((\mathbb{K}^\times)^N)$ and choose any $P \in \operatorname{Spec} R$. By [6, Corollary II.6.10], P is completely prime.

Recall that $\mathcal{Z}(R)$ is a commutative Laurent polynomial ring; in particular, $\mathcal{Z}(R)$ is a commutative affine domain, so that it satisfies the strong Dixmier-Moeglin equivalence by Proposition 3.2.2. By Propositions 3.3.2 and 3.3.3, we have

$$\mathcal{Z}(\operatorname{Frac} R/P) \cong \operatorname{Frac} \mathcal{Z}(R/P) \cong \operatorname{Frac} \frac{\mathcal{Z}(R)}{\mathcal{Z}(R) \cap P}.$$

It follows that $\operatorname{rat.deg} P = \operatorname{rat.deg}(\mathcal{Z}(R) \cap P)$. Since $\mathcal{Z}(R)/(\mathcal{Z}(R) \cap P)$ is a commutative affine domain, Remark 3.2.4 gives $\operatorname{rat.deg} P = \operatorname{K.dim} \mathcal{Z}(R) - \operatorname{ht}(\mathcal{Z}(R) \cap P)$. Since extension and contraction are mutually inverse increasing homeomorphisms between $\operatorname{Spec} \mathcal{Z}(R)$ and $\operatorname{Spec} R$, we have $\operatorname{ht}(\mathcal{Z}(R) \cap P) = \operatorname{ht} P$, so that

$$\operatorname{rat.deg} P = \operatorname{K.dim} \mathcal{Z}(R) - \operatorname{ht} P.$$

Every maximal ideal of $\mathcal{Z}(R)$ has height $\operatorname{K.dim} \mathcal{Z}(R)$ and hence so does every maximal ideal of R. By [17, Corollary 1.5], the primitive ideals of R are exactly its maximal ideals, so that every primitive ideal of R has height $\operatorname{K.dim} \mathcal{Z}(R)$. Now the catenarity of R gives $\operatorname{prim.deg} P = \operatorname{K.dim} \mathcal{Z}(R) - \operatorname{ht} P$ and, in particular, $\operatorname{prim.deg} P = \operatorname{rat.deg} P$.

Let us set $d = \operatorname{prim.deg} P = \operatorname{rat.deg} P = \operatorname{K.dim} \mathcal{Z}(R) - \operatorname{ht} P$. Since all maximal (i.e. primitive) ideals of R have height $\operatorname{K.dim} \mathcal{Z}(R)$, all maximal (i.e. primitive) ideals of R/P have height d. Now $\operatorname{Spec}_{d+1} R/P$ is empty so that $\bigcap_{Q \in \operatorname{Spec}_{d+1} R/P} Q = R/P \neq 0$. Since R is a Jacobson ring, we get $\bigcap_{Q \in \operatorname{Spec}_d R/P} Q = 0$ and hence $\operatorname{loc.deg} P = d$, completing the proof. ■

Remark 3.3.5. *By Remark 3.2.3, Theorem 3.3.4 holds even when q is a root of unity (since in this case, the quantum torus satisfies a polynomial identity). As such, it seems likely that the strong Dixmier-Moeglin equivalence holds for all quantum tori, without restrictions on the parameters.*

3.4 Dimensions of H-strata

Our next aim is to show that uniparameter quantum affine spaces satisfy the strong Dixmier-Moeglin equivalence. For this, we will make use of the stratification theory of Goodearl and Letzter, which we discussed briefly in Section 2.2. Indeed, an examination of the stratification of a uniparameter quantum affine space reveals that every (prime homomorphic image of a) uniparameter quantum affine space localises to a (prime homomorphic image of a) uniparameter quantum torus. This allows us to transfer the quasi strong Dixmier-Moeglin equivalence from uniparameter quantum tori to uniparameter quantum affine spaces in Section 3.5. Further examination of the stratification of a uniparameter quantum affine space allows us to calculate the primitivity degrees of the prime ideals and hence, in the next section, complete the proof that uniparameter quantum affine spaces satisfy the strong Dixmier-Moeglin equivalence.

The material in this section shall be useful beyond quantum affine spaces, so we work in a more general setting. The aim of this section is to prove results on dimensions of strata which shall be utilised in our proof that quantum affine spaces, quantum Schubert cells, and some quantum groups satisfy the strong Dixmier-Moeglin equivalence.

Let us suppose that R is a noetherian $\mathbb{K}$-algebra and that $H = (\mathbb{K}^{\times})^r$ is an algebraic $\mathbb{K}$-torus acting rationally on R by automorphisms. Let us assume further that every H-prime ideal J of R is *strongly H-rational* in the sense that the fixed field $\mathcal{Z}(\mathrm{Frac}(R/J))^H$ is $\mathbb{K}$; in CGL extensions (including quantum affine spaces), [6, Theorem II.6.4] guarantees that every H-prime ideal is strongly H-rational. By [6, Theorem II.2.13], for each H-prime ideal J of R, there is a bi-increasing homeomorphism from $\mathrm{Spec}_J\, R$ to the prime spectrum of an appropriate commutative Laurent polynomial algebra over $\mathbb{K}$; the *Krull dimension* of the H-stratum $\mathrm{Spec}_J\, R$ is defined to be the Krull dimension of this commutative Laurent polynomial algebra.

Let us make a useful observation on the Krull dimensions of H-strata under localisation.

Lemma 3.4.1. *Let an algebraic $\mathbb{K}$-torus H act rationally by automorphisms on a noetherian $\mathbb{K}$-algebra R. Let $\mathcal{E}$ be a right Ore set in R consisting of regular H-eigenvectors with rational H-eigenvalues.*

(1) The action of H on R extends to a rational action by automorphisms on $R\mathcal{E}^{-1}$.

(2) Extension and contraction restrict to mutually inverse increasing bijections between the set of H-prime ideals of R which do not intersect $\mathcal{E}$ and the set of H-prime ideals of $R\mathcal{E}^{-1}$.

(3) For any H-prime ideal J of R which does not intersect $\mathcal{E}$, extension and contraction restrict to mutually inverse increasing bijections between $\operatorname{Spec}_J R$ *and* $\operatorname{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}$.

(4) If all H-prime ideals of R are strongly H-rational, then the same is true for $R\mathcal{E}^{-1}$.

Proof. (1) For all $r \in R$, $e \in \mathcal{E}$, $h \in H$, define $h \cdot (re^{-1}) = (h \cdot r)(h \cdot e)^{-1}$. If ν is the eigenvalue of e with respect to the action of h, then $h \cdot (re^{-1}) = \nu^{-1}(h \cdot r)e^{-1}$. It is routine to check that this gives a rational action of H by automorphisms on $R\mathcal{E}^{-1}$.

(2) Since the mutually inverse increasing bijections of extension ($P \mapsto P\mathcal{E}^{-1}$) and contraction ($Q \cap R \leftarrow\!\shortmid Q$) between $\{P \in \operatorname{Spec} R \mid P \cap \mathcal{E} = \emptyset\}$ and $\operatorname{Spec} R\mathcal{E}^{-1}$ clearly send H-invariant ideals to H-invariant ideals, they restrict to mutually inverse increasing bijections between the set of H-prime ideals of R which do not intersect $\mathcal{E}$ and the set of H-prime ideals of $R\mathcal{E}^{-1}$.

(3) Let J be an H-prime ideal of R such that $J \cap \mathcal{E} = \emptyset$.

Let $P \in \operatorname{Spec}_J R$ and notice that since J is the largest H-invariant ideal contained in P, P cannot intersect $\mathcal{E}$. Since $J \subseteq P$, we have $J\mathcal{E}^{-1} \subseteq P\mathcal{E}^{-1}$. We claim that $P\mathcal{E}^{-1} \in \operatorname{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}$. Indeed let K be an H-prime ideal of $R\mathcal{E}^{-1}$ which is contained in $P\mathcal{E}^{-1}$. By part (2) of this lemma, there exists $J' \in H - \operatorname{Spec} R$ such that $K = J'\mathcal{E}^{-1}$. Now since $J'\mathcal{E}^{-1} \subseteq P\mathcal{E}^{-1}$, we have $J' \subseteq P$, so that (since $P \in \operatorname{Spec}_J R$) $J' \subseteq J$. Hence $K \subseteq J\mathcal{E}^{-1}$, establishing the claim that $P\mathcal{E}^{-1} \in \operatorname{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}$.

Let $Q \in \operatorname{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}$. Since $J\mathcal{E}^{-1} \subseteq Q$, we have $J\mathcal{E}^{-1} \cap R \subseteq Q \cap R$ i.e. $J \subseteq Q \cap R$. We claim that $Q \cap R \in \operatorname{Spec}_J R$. Indeed let L be an H-prime ideal of R such that $L \subseteq Q \cap R$. Then $L\mathcal{E}^{-1}$ is an H-prime ideal of $R\mathcal{E}^{-1}$ which is contained in $(Q \cap R)\mathcal{E}^{-1} = Q$, so that (since $Q \in \operatorname{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}$) $L\mathcal{E}^{-1} \subseteq J\mathcal{E}^{-1}$. Hence $L \subseteq J$, establishing the claim that $Q \cap R \in \operatorname{Spec}_J R$.

(4) This follows immediately from part (2).

■

From Lemma 3.4.1 parts (3) and (4), we deduce

Corollary 3.4.2. *Let an algebraic $\mathbb{K}$-torus H act rationally by automorphisms on a noetherian $\mathbb{K}$-algebra R and suppose that all H-prime ideals of R are strongly H-rational. Let $\mathcal{E}$ be a right Ore set in R consisting of regular H-eigenvectors with rational H-eigenvalues. Then for any H-prime ideal J of R which does not intersect $\mathcal{E}$, we have*

$$\operatorname{K.dim} \operatorname{Spec}_J R = \operatorname{K.dim} \operatorname{Spec}_{J\mathcal{E}^{-1}} R\mathcal{E}^{-1}.$$

Under the further assumptions that R has finitely many H-prime ideals and that R satisfies the noncommutative Nullstellensatz over $\mathbb{K}$, [6, Theorem II.8.4] says that R satisfies the Dixmier-Moeglin equivalence and that the primitive ideals of R are exactly those prime ideals which are maximal in their H-strata. Assuming further that R is catenary and that the H-strata of R satisfy a technical condition (given in inequality (3.5)), we now show that if P is a prime ideal of R belonging to $\operatorname{Spec}_J R$ for an H-prime ideal J of R and if $M \supseteq P$ is a primitive (i.e. maximal) element of $\operatorname{Spec}_J R$, then $\operatorname{ht} M/P = \operatorname{prim.deg} P$ (and we compute these quantities in terms of the Krull dimension of $\operatorname{Spec}_J R$). Crucially, this allows us to look only at a single H-stratum of R in order to compute $\operatorname{prim.deg} P$.

Proposition 3.4.3. *Let R be a catenary noetherian $\mathbb{K}$-algebra satisfying the noncommutative Nullstellensatz over $\mathbb{K}$ and let H be an an algebraic $\mathbb{K}$-torus acting rationally by automorphisms on R. Suppose that H-$\operatorname{Spec} R$ is finite, that all H-prime ideals of R are strongly H-rational, and that for any pair of H-prime ideals $J \subseteq J'$ of R, we have*

$$\operatorname{K.dim} \operatorname{Spec}_J R + \operatorname{ht} J \leq \operatorname{K.dim} \operatorname{Spec}_{J'} R + \operatorname{ht} J'. \tag{3.5}$$

Then for any H-prime ideal J of R, any $P \in \operatorname{Spec}_J R$, and any primitive element $M \supseteq P$ of $\operatorname{Spec}_J R$, we have

$$\operatorname{prim.deg} P = \operatorname{ht} M/P = \operatorname{K.dim} \operatorname{Spec}_J R + \operatorname{ht} J - \operatorname{ht} P. \tag{3.6}$$

Proof. Let M be a primitive element of $\operatorname{Spec}_J R$ which contains P. Then M is maximal in $\operatorname{Spec}_J R$, so that $\operatorname{ht} M/J = \operatorname{K.dim} \operatorname{Spec}_J R$. It follows from the catenarity of R that

$$\operatorname{ht} M/P = \operatorname{K.dim} \operatorname{Spec}_J R + \operatorname{ht} J - \operatorname{ht} P. \tag{3.7}$$

Every primitive ideal of R/P corresponds to a primitive ideal of R which contains P. Choose any such primitive ideal N of R and say N belongs to $\operatorname{Spec}_{J'} R$ for an H-prime ideal J' of R. It is clear that $J \subseteq J'$.

Since N is maximal in $\operatorname{Spec}_{J'} R$, we have $\operatorname{ht} N/J' = \operatorname{K.dim} \operatorname{Spec}_{J'} R$. It follows from the catenarity of R that

$$\operatorname{ht} N/P = \operatorname{K.dim} \operatorname{Spec}_{J'} R + \operatorname{ht} J' - \operatorname{ht} P. \tag{3.8}$$

Equations (3.7) and (3.8), along with the assumption (3.5), show that the height of an arbitrary primitive ideal of R/P is at least $\operatorname{ht} M/P$. Since M/P is itself primitive, we get $\operatorname{ht} M/P = \operatorname{prim.deg} P$; combining this with equation (3.7) gives the result. ∎

Remark 3.4.4. *Except for the inequality* (3.5)*, the conditions of Proposition 3.4.3 are known to hold for many interesting algebras. Much of the rest of this chapter* *is concerned with verifying inequality* (3.5) *for uniparameter quantum affine spaces (Section 3.5), quantum Schubert cells (Section 3.7), and certain quantum groups (Section 3.8). Our proofs rely on knowledge of the dimensions of the H-strata [1, 2] and on knowledge of the posets of H-prime ideals [15, 17, 31].*

3.5 The SDME for uniparameter quantum affine spaces

In a further step towards proving the strong Dixmier-Moeglin equivalence for quantum Schubert cells, we prove it in this section for uniparameter quantum affine spaces.

Consider a uniparameter quantum affine space $\mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$. Recall from Example 2.2.1 that the algebraic $\mathbb{K}$-torus $H = (\mathbb{K}^\times)^N$ acts rationally on $\mathcal{O}_{q,A}(\mathbb{K}^N)$ by automorphisms as follows:

$$(a_1, \ldots, a_N) \cdot T_i = a_i T_i \text{ for all } i \in [\![1, N]\!] \text{ and all } (a_1, \ldots, a_N) \in H$$

and that, by [17, Proposition 2.11],

$$H - \operatorname{Spec} \mathcal{O}_{q,A}(\mathbb{K}^N) = \{K_\Delta \mid \Delta \subseteq [\![N]\!]\}, \tag{3.9}$$

where K_Δ is the ideal of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ generated by those T_i with $i \in \Delta$. For any $\Delta \subseteq \{1, \ldots, N\}$, recall that we denote by $\operatorname{Spec}_\Delta \mathcal{O}_{q,A}(\mathbb{K}^N)$ the H-stratum of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ associated to K_Δ.

Let us mention some properties of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ which will be relevant for us in this section. One checks easily that $\mathcal{O}_{q,A}(\mathbb{K}^N)$ is a domain. By [18, Theorem 2.6], $\mathcal{O}_{q,A}(\mathbb{K}^N)$ is noetherian. By [6, Corollary II.7.18], $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ and by [6, Theorem II.9.14], $\mathcal{O}_{q,A}(\mathbb{K}^N)$ is catenary and satisfies Tauvel's height formula. By (3.9), $H - \operatorname{Spec} \mathcal{O}_{q,A}(\mathbb{K}^N)$ has cardinality 2^N and is finite in particular. By [6, Theorem II.8.4], R satisfies the Dixmier-Moeglin equivalence and its primitive ideals are exactly those prime ideals which are maximal in their H-strata. By [6, Theorem II.6.4], all H-prime ideals of R are strongly H-rational and by [6, Corollary II.6.10], every prime ideal of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ is completely prime.

We use a transfer result from Section 3.1 to show that $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfies the quasi strong Dixmier-Moeglin equivalence.

Proposition 3.5.1. *The uniparameter quantum affine spaces $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfy the quasi strong Dixmier-Moeglin equivalence.*

Proof. Set $R = \mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$. Choose any $P \in \operatorname{Spec} R$ and say $P \in \operatorname{Spec}_\Delta R$ for a subset Δ of $\{1, \ldots, N\}$. Let $\mathcal{E}$ be the multiplicative system in R generated by those T_i for which $i \notin \Delta$. Then $\mathcal{E}$ satisfies the Ore condition on both sides in R and, denoting by $\overline{\mathcal{E}}$ and $\hat{\mathcal{E}}$ its images in R/P and R/K_Δ respectively, we have

$$(R/P)\overline{\mathcal{E}}^{-1} \cong ((R/K_\Delta)\hat{\mathcal{E}}^{-1})/((P/K_\Delta)\hat{\mathcal{E}}^{-1}).$$

The uniparameter quantum torus $(R/K_\Delta)\hat{\mathcal{E}}^{-1}$ satisfies the strong Dixmier-Moeglin equivalence by Theorem 3.3.4 and hence so does its homomorphic image $(R/P)\overline{\mathcal{E}}^{-1}$. The result now follows from Proposition 3.1.4. ■

Since we have proven that $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfies the quasi strong Dixmier-Moeglin equivalence, proving that $\operatorname{prim.deg} P = \operatorname{rat.deg} P$ holds for all prime ideals P of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ will establish the strong Dixmier-Moeglin equivalence for $\mathcal{O}_{q,A}(\mathbb{K}^N)$.

In order to invoke Proposition 3.4.3, which gives us an expression for the primitivity degree of any prime ideal P of $\mathcal{O}_{q,A}(\mathbb{K}^N)$ in terms of the dimension of the H-stratum to which P belongs, we must prove an inequality relating the dimensions of H-strata of $\mathcal{O}_{q,A}(\mathbb{K}^N)$. First we introduce some new notation:

Notation 3.5.2. *Let Δ be a subset of $\{1, \ldots, N\}$ and set $\{\ell_1 < \ldots < \ell_d\} = \{1, \ldots, N\} \setminus \Delta$. We define the* skew-adjacency matrix*, $A(\Delta)$, of Δ to be the $d \times d$ additively skew-symmetric submatrix of $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$ whose (s,t) entry $(s < t)$ is a_{ℓ_s,ℓ_t}.*

For any subset Δ of $\{1, \ldots, N\}$, it follows from [2, Theorem 3.1] that the dimension of the H-stratum $\mathrm{Spec}_\Delta(\mathcal{O}_{q,A}(\mathbb{K}^N))$ corresponding to the H-prime ideal $K_\Delta = \langle T_i \mid i \in \Delta \rangle$ is exactly $\dim_{\mathbb{Q}}(\ker A(\Delta))$. In fact, [2, Theorem 3.1] applies to the more general class of uniparameter CGL extensions (see Section 3.6).

Proposition 3.5.3. *For any pair of H-prime ideals $K_\Delta \subseteq K_{\Delta'}$ of $\mathcal{O}_{q,A}(\mathbb{K}^N)$, we have*

$$\mathrm{K.dim}\,\mathrm{Spec}_\Delta(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\, K_\Delta \leq \mathrm{K.dim}\,\mathrm{Spec}_{\Delta'}(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\, K_{\Delta'}.$$

Proof. Since $K_\Delta \subseteq K_{\Delta'}$, we clearly have $\Delta \subseteq \Delta'$. The matrix $A(\Delta')$ is an $(N - |\Delta'|)$-square submatrix of the $(N - |\Delta|)$-square matrix $A(\Delta)$, so that $\mathrm{rk}\, A(\Delta') \leq \mathrm{rk}\, A(\Delta)$ and

$$(N - |\Delta'|) - \dim_{\mathbb{Q}}(\ker A(\Delta')) \leq (N - |\Delta|) - \dim_{\mathbb{Q}}(\ker A(\Delta)).$$

Hence, we have

$$\dim_{\mathbb{Q}}(\ker A(\Delta)) + |\Delta| \leq \dim_{\mathbb{Q}}(\ker A(\Delta')) + |\Delta'|. \tag{3.10}$$

Tauvel's height formula holds in $\mathcal{O}_{q,A}(\mathbb{K}^N)$, so that

$$\mathrm{ht}\, K_\Delta = \mathrm{GK.dim}\, \mathcal{O}_{q,A}(\mathbb{K}^N) - \mathrm{GK.dim}(\mathcal{O}_{q,A}(\mathbb{K}^N)/K_\Delta) = N - (N - |\Delta|) = |\Delta|$$

and similarly $\mathrm{ht}\, K_{\Delta'} = |\Delta'|$. Now (3.10) and [2, Theorem 3.1] give

$$\mathrm{K.dim}\,\mathrm{Spec}_\Delta(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\, K_\Delta \leq \mathrm{K.dim}\,\mathrm{Spec}_{\Delta'}(\mathcal{O}_{q,A}(\mathbb{K}^N)) + \mathrm{ht}\, K_{\Delta'}.$$

∎

With Proposition 3.5.3 in hand, we can apply Proposition 3.4.3 to $\mathcal{O}_{q,A}(\mathbb{K}^N)$ in our proof of the main result of this section:

Theorem 3.5.4. *The uniparameter quantum affine spaces $\mathcal{O}_{q,A}(\mathbb{K}^N)$ satisfy the strong Dixmier-Moeglin equivalence.*

Proof. Set $R = \mathcal{O}_{q,A}(\mathbb{K}^N) = \mathbb{K}_{q,A}[T_1, \ldots, T_N]$. We showed in Proposition 3.5.1 that R satisfies the quasi strong Dixmier-Moeglin equivalence, so what remains is to prove that $\operatorname{prim.deg} P = \operatorname{rat.deg} P$ for all prime ideals P of R.

Let P be any prime ideal of R and say $P \in \operatorname{Spec}_\Delta R$ for a subset Δ of $\{1, \ldots, N\}$. In view of Proposition 3.5.3, Proposition 3.4.3 gives

$$\operatorname{prim.deg} P = \operatorname{K.dim} \operatorname{Spec}_\Delta R + \operatorname{ht} K_\Delta - \operatorname{ht} P. \tag{3.11}$$

Let $\mathcal{E}$ be the multiplicative system in R generated by those T_i for which $i \notin \Delta$. Then $\mathcal{E}$ satisfies the Ore condition on both sides in R and consists of regular H-eigenvectors with rational eigenvalues; denoting by $\hat{\mathcal{E}}$ the image of $\mathcal{E}$ in R/K_Δ, we have $R\mathcal{E}^{-1}/P\mathcal{E}^{-1} \cong ((R/K_\Delta)\hat{\mathcal{E}}^{-1})/((P/K_\Delta)\hat{\mathcal{E}}^{-1})$. Notice that $(R/K_\Delta)\hat{\mathcal{E}}^{-1}$ is a uniparameter quantum torus and that $P\mathcal{E}^{-1} \in \operatorname{Spec}_{K_\Delta\mathcal{E}^{-1}} R\mathcal{E}^{-1}$.

Since R is catenary and noetherian, so is $R\mathcal{E}^{-1}$. Moreover, $R\mathcal{E}^{-1}$ can be obtained from $\mathbb{K}$ by a finite number of skew-polynomial and skew-Laurent extensions; in particular, $R\mathcal{E}^{-1}$ is a constructible $\mathbb{K}$-algebra in the sense of [30, 9.4.12], so that $R\mathcal{E}^{-1}$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ by [30, Theorem 9.4.21]. From Lemma 3.4.1 and Corollary 3.4.2 (which deal with the effect of localisation on H-stratification), we deduce that $R\mathcal{E}^{-1}$ satisfies the conditions of Proposition 3.4.3. Now

$$\begin{aligned}
\operatorname{prim.deg} P\mathcal{E}^{-1} &= \operatorname{K.dim} \operatorname{Spec}_{K_\Delta\mathcal{E}^{-1}} R\mathcal{E}^{-1} + \operatorname{ht} K_\Delta\mathcal{E}^{-1} - \operatorname{ht} P\mathcal{E}^{-1} \qquad \text{(by Proposition 3.4.3)}\\
&= \operatorname{K.dim} \operatorname{Spec}_\Delta R + \operatorname{ht} K_\Delta - \operatorname{ht} P \qquad \text{(by Corollary 3.4.2)}\\
&= \operatorname{prim.deg} P \qquad \text{(by (3.11))}.
\end{aligned}$$

Since the uniparameter quantum torus $(R/K_\Delta)\hat{\mathcal{E}}^{-1}$ satisfies the strong Dixmier-Moeglin equivalence (Theorem 3.3.4), so does its homomorphic image $R\mathcal{E}^{-1}/P\mathcal{E}^{-1}$. So $\operatorname{prim.deg}\langle 0\rangle = \operatorname{rat.deg}\langle 0\rangle$ holds in $R\mathcal{E}^{-1}/P\mathcal{E}^{-1}$, which can be rephrased by saying that in $R\mathcal{E}^{-1}$, we have $\operatorname{prim.deg} P\mathcal{E}^{-1} = \operatorname{rat.deg} P\mathcal{E}^{-1}$. Since we have already shown that $\operatorname{prim.deg} P = \operatorname{prim.deg} P\mathcal{E}^{-1}$ and it is clear that $\operatorname{rat.deg} P\mathcal{E}^{-1} = \operatorname{rat.deg} P$, we have $\operatorname{prim.deg} P = \operatorname{rat.deg} P$, as required. ■

Remark 3.5.5. *By Remark 3.2.3, Theorem 3.5.4 holds even when q is a root of unity (since in this case, the quantum affine space satisfies a polynomial identity). As such, it seems likely that the strong Dixmier-Moeglin equivalence holds for all quantum affine spaces, without restrictions on the parameters.*

3.6 A sufficient condition for the SDME in CGL extensions

Let $R = \mathbb{K}[X_1][X_2; \sigma_2, \delta_2] \cdots [X_N; \sigma_N; \delta_N]$ be a uniparameter *Cauchon-Goodearl-Letzter (CGL)* extension with parameter q and associated additively Skew-symmetric matrix $A = (a_{i,j}) \in \mathcal{M}_N(\mathbb{Z})$, admitting a rational action of an algebraic $\mathbb{K}$-torus H (see Definition 2.3.1 and Remark 2.3.2).

Let us mention some properties of R which shall be relevant for us. It is easy to check that R is a domain. By [18, Theorem 2.6], R is noetherian. The Gelfand-Kirillov dimension of R is N by [6, Lemma II.9.7]. By [6, Theorem II.7.17], R satisfies the noncommutative Nullstellensatz over $\mathbb{K}$. Recall from Subsection 2.4.4 that $H - \operatorname{Spec} R$ is finite. By [6, Theorem II.8.4], R satisfies the Dixmier-Moeglin equivalence and its primitive ideals are exactly those prime ideals which are maximal in their H-strata. By [6, Theorem II.6.9], all prime ideals of R are completely prime and by [6, Theorem II.6.4], all H-prime ideals of R are strongly H-rational.

The algebra $R^{(2)}$ (which, loosely speaking, is obtained from R by "deleting" all the derivations $\delta_2, \ldots, \delta_N$) is a uniparameter quantum affine space in indeterminates $T_1, \ldots, T_N$ with commutation relations given by q and the matrix A, i.e. (in the notation of Subsection 2.1.1) we have

$$R^{(2)} = \mathbb{K}_{q,A}[T_1, \ldots, T_N] = \mathcal{O}_{q,A}(\mathbb{K}^N).$$

Before continuing, the reader might want to revisit Subsection 2.4.4 for some details and notation pertaining to Cauchon's deleting-derivations algorithm and to the canonical injection $\varphi\colon \operatorname{Spec} R \to \operatorname{Spec} R^{(2)}$ in particular.

Theorem 3.6.1. *Every uniparameter CGL extension satisfies the quasi strong Dixmier-Moeglin equivalence.*

Proof. Let R be a uniparameter CGL extension. Recall that both in R and in the uniparameter quantum affine space $R^{(2)}$, all prime ideals are completely prime.

Choose any $P \in \operatorname{Spec} R$ and say $P \in \operatorname{Spec}_\Delta R$ for a Cauchon diagram Δ of R. Let $\mathcal{E}$ be the image in $R^{(2)}/\varphi(P)$ of the multiplicative system in $R^{(2)}$ generated by those T_i for which $i \in \{1, \dots, N\} \setminus \Delta$. By [10, Théoremè 5.4.1], $\mathcal{E}$ satisfies the Ore condition on both sides in $R^{(2)}/\varphi(P)$ and there exists a finitely generated multiplicative system $\mathcal{F}$ in R/P satisfying the Ore condition on both sides such that

$$(R/P)\mathcal{F}^{-1} \cong (R^{(2)}/\varphi(P))\mathcal{E}^{-1}. \tag{3.12}$$

Since $R^{(2)}$ is a uniparameter quantum affine space, it satisfies the strong Dixmier-Moeglin equivalence (Theorem 3.5.4) and hence so does every homomorphic image of $R^{(2)}$. In particular, $R^{(2)}/\varphi(P)$ satisfies the strong Dixmier-Moeglin equivalence. Hence, by Lemma 3.1.3, $(R^{(2)}/\varphi(P))\mathcal{E}^{-1}$ satisfies the quasi strong Dixmier-Moeglin equivalence. The result now follows from (3.12) and Proposition 3.1.4. ■

Regarding the strong Dixmier-Moeglin equivalence, we can prove the following partial result.

Theorem 3.6.2. *If R is a catenary uniparameter CGL extension such that for any pair of H-prime ideals $J \subseteq J'$ of R, the following inequality holds:*

$$\operatorname{K.dim} \operatorname{Spec}_J R + \operatorname{ht} J \leq \operatorname{K.dim} \operatorname{Spec}_{J'} R + \operatorname{ht} J', \tag{3.13}$$

then R satisfies the strong Dixmier-Moeglin equivalence.

Proof. Since R satisfies the quasi strong Dixmier-Moeglin equivalence (Theorem 3.6.1), we need only show that for every prime ideal P of R, we have $\operatorname{prim.deg} P = \operatorname{rat.deg} P$. Recall that by [6, Theorem II.8.4], R and $R^{(2)}$ satisfy the Dixmier-Moeglin equivalence and, in each of these two algebras, the primitive ideals are exactly the prime ideals which are maximal in their H-strata.

Suppose that P is a prime ideal of R with $P \in \operatorname{Spec}_\Delta R$ for a Cauchon diagram Δ of R. Choose any primitive (i.e. maximal) element $M \supseteq P$ of $\operatorname{Spec}_\Delta R$. Since (by [10, Théorèmes 5.1.1 and 5.5.1]) the canonical injection $\varphi\colon \operatorname{Spec} R \to \operatorname{Spec} R^{(2)}$ restricts to a bi-increasing homeomorphism from $\operatorname{Spec}_\Delta R$ to $\operatorname{Spec}_\Delta R^{(2)}$, we get that $\varphi(M)$ is a maximal (i.e. primitive) element of $\operatorname{Spec}_\Delta R^{(2)}$ and that $\varphi(M)$ contains $\varphi(P)$. Proposition 3.5.3 and the assumption (3.13) allow us to invoke Proposition 3.4.3 to get

$$\operatorname{ht} M/P = \operatorname{prim.deg} P \text{ and } \operatorname{ht} \varphi(M)/\varphi(P) = \operatorname{prim.deg} \varphi(P). \tag{3.14}$$

Moreover, since φ restricts to a bi-increasing homeomorphism from $\operatorname{Spec}_\Delta R$ to $\operatorname{Spec}_\Delta R^{(2)}$, it induces a length-preserving one-to-one correspondence between the chains of prime ideals from P to M and the chains of prime ideals from $\varphi(P)$ to $\varphi(M)$. It follows that

$$\operatorname{ht} M/P = \operatorname{ht} \varphi(M)/\varphi(P). \tag{3.15}$$

We deduce from (3.14) and (3.15) that $\operatorname{prim.deg} P = \operatorname{prim.deg} \varphi(P)$. Now, recalling that the uniparameter quantum affine space $R^{(2)}$ satisfies the strong Dixmier-Moeglin equivalence (by Theorem 3.5.4) and that, by [10, Théoremè 5.4.1], $\operatorname{Frac}(R/P) \cong \operatorname{Frac}(R^{(2)}/\varphi(P))$, we have

$$\begin{aligned}\operatorname{prim.deg} P &= \operatorname{prim.deg} \varphi(P)\\ &= \operatorname{rat.deg} \varphi(P)\\ &= \operatorname{rat.deg} P,\end{aligned}$$

as required. ■

Remark 3.6.3. *The conditions on the uniparameter CGL extension R in the statement of Theorem 3.6.2 may turn out to be redundant: it is not known whether there are any non-catenary CGL extensions and we do not know whether there are any CGL extensions in which the inequality* (3.13) *fails.*

3.7 Quantum Schubert cells

Yakimov [35, Theorem 5.7] has shown that quantum Schubert cells are catenary and satisfy Tauvel's height formula. We show that they satisfy the inequality (3.13) so that, by Theorem 3.6.2, they satisfy the strong Dixmier-Moeglin equivalence. Our proofs exploit the CGL extension structure of quantum Schubert cells, which we now discuss.

3.7.1 Quantum Schubert cells as CGL extensions

It turns out that quantum Schubert cells are uniparameter CGL extensions. Let $\mathfrak{g}$ be a simple complex Lie algebra of rank n, choose a set $\{\alpha_1, \ldots, \alpha_n\}$ of simple roots and an element $w = s_{i_1} \cdots s_{i_N}$ of the Weyl group $\mathcal{W}$ of $\mathfrak{g}$. Recall from Subsection 2.1.5 the construction of the positive roots $\{\beta_1, \ldots, \beta_N\}$.

Recall from Subsection 2.1.6 the construction of elements $X_1, \ldots, X_N$ of $U_q^+(\mathfrak{g})$ which generate $U_q[w]$ such that $U_q[w]$ may be expressed as an iterated Ore extension

$$U_q[w] = k[X_1][X_2; \sigma_2; \delta_2] \cdots [X_N; \sigma_N, \delta_N]. \tag{3.16}$$

It is well known both that the algebraic torus $H = (\mathbb{K}^\times)^n$ acts rationally by automorphisms on $U_q^+(\mathfrak{g})$ as follows

$$(k_1, \ldots, k_n) \cdot E_i = k_i E_i \text{ for all } i \in [\![1, n]\!] \text{ and all } (k_1, \ldots, k_n) \in H$$

and that this restricts to a rational action of H by automorphisms on $U_q[w]$. Cauchon showed [10, Proposition 6.1.2 and Lemme 6.2.1] that with this action of H, the expression (3.16) of $U_q[w]$ as an iterated Ore extension is in fact a uniparameter CGL extension with parameter q and the following associated additively skew-symmetric matrix:

$$A := \begin{pmatrix} 0 & (\beta_1,\beta_2) & \cdots & \cdots & (\beta_1,\beta_N) \\ -(\beta_1,\beta_2) & 0 & (\beta_2,\beta_3) & & (\beta_2,\beta_N) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 & (\beta_{N-1},\beta_N) \\ -(\beta_1,\beta_N) & \cdots & \cdots & -(\beta_{N-1},\beta_N) & 0 \end{pmatrix}. \tag{3.17}$$

Theorem 3.6.1 immediately gives:

Proposition 3.7.1. *The quantum Schubert cell $U_q[w]$ satisfies the quasi strong Dixmier-Moeglin equivalence.*

3.7.2 The SDME for quantum Schubert cells

Considering $U_q[w]$ as a uniparameter CGL extension (3.16) in N indeterminates with associated additively skew-symmetric matrix A (see (3.17)), recall both that J_Δ denotes the H-prime ideal of $U_q[w]$ associated to a Cauchon diagram Δ of $U_q[w]$ and that $H{-}\mathrm{Spec}_\Delta\, U_q[w]$ denotes the H-stratum of $\mathrm{Spec}(U_q[w])$ associated to J_Δ. The remaining work lies in proving that for any pair of H-prime ideals $J_\Delta \subseteq J_{\Delta'}$ of $U_q[w]$, the following inequality holds:

$$\mathrm{K.dim}\,\mathrm{Spec}_\Delta\, U_q[w] + \mathrm{ht}\, J_\Delta \leq \mathrm{K.dim}\,\mathrm{Spec}_{\Delta'}\, U_q[w] + \mathrm{ht}\, J_{\Delta'}. \tag{3.18}$$

This will allow us to invoke Theorem 3.6.2 to show that $U_q[w]$ satisfies the strong Dixmier-Moeglin equivalence.

Remark 3.7.2. *Unlike in the case of uniparameter quantum affine spaces, the bijective map $\Delta \mapsto J_\Delta$ is not an isomorphism of posets between the Cauchon diagrams and the*

H-prime ideals of $U_q[w]$ (this map preserves inclusions but its inverse does not). This is the reason why (3.18) *is more difficult to verify than the corresponding inequality for a uniparameter quantum affine space.*

In contrast to that of most algebras supporting a torus action, the poset structure of the H-spectrum of $U_q[w]$ is known. Let us denote by $\leq$ the Bruhat order on $\mathcal{W}$ and let us set $\mathcal{W}^{\leq w} := \{u \in \mathcal{W} \mid u \leq w\}$. The posets $H\text{-Spec}\, U_q[w]$ and $\mathcal{W}^{\leq w}$ are isomorphic by results of Cauchon-Meriaux [31] and Geiger-Yakimov [15]. In order to describe the isomorphism, we introduce some notation:

Notation 3.7.3. *Recall that we have fixed a reduced expression $w = s_{i_1} \cdots s_{i_N}$ for w. Let $\Delta \subseteq \{1, \dots, N\}$ be any (not necessarily Cauchon) diagram.*

(i) For all $k = 1, \dots, N$, we set

$$s_{i_k}^{\Delta} := \begin{cases} s_{i_k} & \textit{if } k \in \Delta \\ id & \textit{otherwise.} \end{cases}$$

(ii) We set $\{l_1 < \cdots < l_d\} := \{1, \dots, N\} \setminus \Delta$ and $j_r = i_{l_r}$ for all $r = 1, \dots, d$.

(iii) We set $w^{\Delta} := s_{i_1}^{\Delta} \cdots s_{i_N}^{\Delta} \in \mathcal{W}$.

(iv) We set $A(w^{\Delta})$ to be the $d \times d$ additively skew-symmetric submatrix of A whose (s,t)-entry $(s < t)$ is $(\beta_{j_s}, \beta_{j_t})$.

Cauchon and Mériaux [31, Corollary 5.3.1] showed that the map

$$H\text{-Spec}\, U_q[w] \to \mathcal{W}^{\leq w};\ J_{\Delta} \mapsto w^{\Delta}, \tag{3.19}$$

where Δ runs over the set of Cauchon diagrams of $U_q[w]$, is a bijection; they asked whether or not this bijection is an isomorphism of posets and this question was answered affirmatively by Geiger and Yakimov [15, Theorem 4.4].

Lemma 3.7.4. *For any Cauchon diagram Δ of $U_q[w]$, we have $\operatorname{ht} J_{\Delta} = |\Delta|$.*

Proof. Set $R = U_q[w]$ and recall that $R^{(2)}$ denotes the uniparameter quantum affine space $\mathbb{K}_{q,A}[T_1, \ldots, T_N]$ which results from "deleting" the derivations in the expression (3.16) of R as a uniparameter CGL extension in N indeterminates. Recall that $K_\Delta = \langle T_i \mid i \in \Delta \rangle$ is the image of J_Δ under the canonical injection $\varphi\colon \operatorname{Spec} R \to \operatorname{Spec} R^{(2)}$.

Let $\mathcal{E}$ be the image in $R^{(2)}/K_\Delta$ of the multiplicative system in $R^{(2)}$ generated by those T_i for which $i \notin \Delta$. Then $\mathcal{E}$ satisfies the Ore condition on both sides in $R^{(2)}/K_\Delta$ and it follows from [10, Théoremè 5.4.1] both that R/J_Δ embeds in the uniparameter quantum torus $(R^{(2)}/K_\Delta)\mathcal{E}^{-1}$ and that $\operatorname{Frac}(R/J_\Delta) \cong \operatorname{Frac}((R^{(2)}/K_\Delta)\mathcal{E}^{-1})$. By [36, Proposition 7.2] (which is a special case of an earlier result of Lorenz - [28, Corollary 2.2]), the uniparameter quantum torus $(R^{(2)}/K_\Delta)\mathcal{E}^{-1}$ is Tdeg-stable in the sense of [36, Section 1]. Therefore, we can apply [36, Proposition 3.5(4)] to get $\operatorname{GK.dim} R/J_\Delta = \operatorname{GK.dim}(R^{(2)}/K_\Delta)\mathcal{E}^{-1} = N - |\Delta|$.

Since R satisfies Tauvel's height formula, we conclude that

$$N - |\Delta| = \operatorname{GK.dim} R/J_\Delta = \operatorname{GK.dim} R - \operatorname{ht} J_\Delta = N - \operatorname{ht} J_\Delta,$$

and so $\operatorname{ht} J_\Delta = |\Delta|$, as desired. ■

We are now in position to establish the crucial inequality required to prove that quantum Schubert cells satisfy the strong Dixmier-Moeglin equivalence.

Proposition 3.7.5. *For any pair of H-prime ideals $J_\Delta \subseteq J_{\Delta'}$ of $U_q[w]$, we have*

$$\operatorname{K.dim} \operatorname{Spec}_\Delta U_q[w] + \operatorname{ht} J_\Delta \leq \operatorname{K.dim} \operatorname{Spec}_{\Delta'} U_q[w] + \operatorname{ht} J_{\Delta'}.$$

Proof. As we have noted, $U_q[w]$ is a uniparameter CGL extension in N indeterminates with associated additively skew-symmetric matrix A. By [1, Theorems 2.3 and 3.1], we have

$$\operatorname{K.dim} \operatorname{Spec}_\Delta U_q[w] = \dim_{\mathbb{Q}} \ker(w^\Delta + w) \text{ and } \operatorname{K.dim} \operatorname{Spec}_{\Delta'} U_q[w] = \dim_{\mathbb{Q}} \ker A(w^{\Delta'}).$$

From the poset isomorphism (H-Spec $U_q[w] \to \mathcal{W}^{\leq w}$; $J_\Delta \mapsto w^\Delta$), we deduce that $w^\Delta \leq w^{\Delta'}$. Since the diagrams Δ and Δ' are Cauchon, the subexpressions w^Δ and $w^{\Delta'}$ of $w = s_{i_1} \cdots s_{i_N}$ are reduced by [31, Corollary 5.3.1(2)]. Since $w^\Delta \leq w^{\Delta'}$, [19, Corollary 5.8] allows us to choose a diagram (not necessarily Cauchon) $\widetilde{\Delta} \subseteq \Delta'$ such that $w^{\widetilde{\Delta}} = w^\Delta$ and the subexpression $w^{\widetilde{\Delta}}$ of $w = s_{i_1} \cdots s_{i_N}$ is reduced. Now $\mathrm{K.dim}\,\mathrm{Spec}_\Delta U_q[w] = \dim_{\mathbb{Q}} \ker(w^{\widetilde{\Delta}} + w)$, so that [1, Theorem 3.1] gives $\mathrm{K.dim}\,\mathrm{Spec}_\Delta U_q[w] = \dim_{\mathbb{Q}} \ker A(w^{\widetilde{\Delta}})$.

$A(w^{\Delta'})$ is an $(N - |\Delta'|)$-square submatrix of the $(N - |\widetilde{\Delta}|)$-square matrix $A(w^{\widetilde{\Delta}})$, so that $\mathrm{rk}\, A(w^{\Delta'}) \leq \mathrm{rk}\, A(w^{\widetilde{\Delta}})$ and hence $\dim_{\mathbb{Q}} \ker A(w^{\widetilde{\Delta}}) + |\widetilde{\Delta}| \leq \dim_{\mathbb{Q}} \ker A(w^{\Delta'}) + |\Delta'|$ and

$$\mathrm{K.dim}\,\mathrm{Spec}_\Delta U_q[w] + |\widetilde{\Delta}| \leq \mathrm{K.dim}\,\mathrm{Spec}_{\Delta'} U_q[w] + |\Delta'|. \tag{3.20}$$

By Lemma 3.7.4, we have $\mathrm{ht}\, J_\Delta = |\Delta|$ and $\mathrm{ht}\, J_{\Delta'} = |\Delta'|$. Since w^Δ and $w^{\widetilde{\Delta}}$ are equal as elements of $\mathcal{W}$, we have $\ell(w^\Delta) = \ell(w^{\widetilde{\Delta}})$. But since the subexpressions w^Δ and $w^{\widetilde{\Delta}}$ of $w = s_{i_1} \cdots s_{i_N}$ are reduced, we have $\ell(w^\Delta) = |\Delta|$ and $\ell(w^{\widetilde{\Delta}}) = |\widetilde{\Delta}|$; hence $|\Delta| = |\widetilde{\Delta}|$.

Now we have $|\widetilde{\Delta}| = \mathrm{ht}\, J_\Delta$ and $|\Delta'| = \mathrm{ht}\, J_{\Delta'}$, so that the result now follows from (3.20). ■

Yakimov has shown [35, Theorem 5.7] that $U_q[w]$ is catenary. We have discussed the uniparameter CGL extension structure of $U_q[w]$. Proposition 3.7.5 provides the final condition required for us to apply Theorem 3.6.2 to $U_q[w]$, giving one of our main results:

Theorem 3.7.6. *The quantum Schubert cell $U_q[w]$ satisfies the strong Dixmier-Moeglin equivalence.*

3.8 The SDME for two families of quantum groups

For a positive integer n, let us denote the algebra $\mathcal{O}_q(M_{n,n}(\mathbb{K}))$ of quantum $n \times n$ matrices over $\mathbb{K}$ by $\mathcal{O}_q(M_n(\mathbb{K}))$. The *quatum determinant* $\det_q$ of $\mathcal{O}_q(M_n(\mathbb{K}))$ is simply the $n \times n$ quantum minor $[1 \cdots n \mid 1 \cdots n]$. It is well known that $\det_q$ generates the centre of $\mathcal{O}_q(M_n(\mathbb{K}))$. The quantum special and general linear groups $\mathcal{O}_q(SL_n(\mathbb{K}))$ and

$\mathcal{O}_q(GL_n(\mathbb{K}))$ over $\mathbb{K}$ can be obtained from $\mathcal{O}_q(M_n(\mathbb{K}))$ by respectively setting $\det_q = 1$ and inverting $\det_q$. More precisely, we define

$$\mathcal{O}_q(SL_n(\mathbb{K})) := \mathcal{O}_q(M_n(\mathbb{K}))/\langle \det_q - 1\rangle \text{ and } \mathcal{O}_q(GL_n(\mathbb{K})) := \mathcal{O}_q(M_n(\mathbb{K}))[\det_q^{-1}].$$

Each of the algebras $\mathcal{O}_q(M_n(\mathbb{K}))$, $\mathcal{O}_q(SL_n(\mathbb{K}))$, and $\mathcal{O}_q(GL_n(\mathbb{K}))$ is catenary by [6, Corollary II.9.18]. Notice that Theorem 3.7.6 and the fact (due to Cauchon-Mériaux - see Example 2.1.3) that $\mathcal{O}_q(M_n(\mathbb{K}))$ is a quantum Schubert cell $U_q[w]$ (where w belongs to the Weyl group of $\mathfrak{sl}_{2n}(\mathbb{C})$) show immediately that $\mathcal{O}_q(M_n(\mathbb{K}))$ satisfies the strong Dixmier-Moeglin equivalence. Since the rank of $\mathfrak{sl}_{2n}(\mathbb{C})$ is $2n-1$, the algebraic torus $(\mathbb{K}^\times)^{2n-1}$ acts rationally on $\mathcal{O}_q(M_n(\mathbb{K}))$ as described in Subsection 3.7.1; it is known that this torus action on $\mathcal{O}_q(M_n(\mathbb{K}))$ has the same invariant prime ideals as the action of the torus $H = (\mathbb{K}^\times)^{2n}$ described in (2.2.2).

Theorem 3.8.1. *The quantum special and general linear groups $\mathcal{O}_q(SL_n(\mathbb{K}))$ and $\mathcal{O}_q(GL_n(\mathbb{K}))$ satisfy the strong Dixmier-Moeglin equivalence.*

Proof. Since the strong Dixmier-Moeglin equivalence clearly passes to homomorphic images, it is immediate that $\mathcal{O}_q(SL_n(\mathbb{K}))$ satisfies the strong Dixmier-Moeglin equivalence.

The Ore set $\{\det_q^n \mid n \in \mathbb{N}\}$ consists of regular H-eigenvectors with rational eigenvalues. From Lemma 3.4.1, one can easily deduce that the rational action of H by automorphisms on $\mathcal{O}_q(M_n(\mathbb{K}))$ extends to a rational action by automorphisms on $\mathcal{O}_q(GL_n(\mathbb{K}))$ such that all H-prime ideals are strongly H-rational and there are finitely many H-prime ideals. The algebra $\mathcal{O}_q(GL_n(\mathbb{K}))$ satisfies the noncommutative Nullstellensatz over $\mathbb{K}$ by [6, Corollary II.7.18] and all its prime ideals are completely prime by [6, Corollary II.6.10].

By Lemma 3.1.3, $\mathcal{O}_q(GL_n(\mathbb{K}))$ satisfies the quasi strong Dixmier-Moeglin equivalence since $\mathcal{O}_q(M_n(\mathbb{K}))$ does, so that we need only show that every $P \in \operatorname{Spec} \mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$ satisfies $\operatorname{prim.deg} P = \operatorname{rat.deg} P$. From the fact (Lemma 3.7.5) that the H-prime ideals of $\mathcal{O}_q(M_n(\mathbb{K}))$ satisfy the inequality (3.5) and from Lemma 3.4.1 and Corollary 3.4.2, we deduce that the H-prime ideals of $\mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$ satisfy the inequality (3.5).

Let us fix $P \in \operatorname{Spec} \mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$ and let $J \in H - \operatorname{Spec} \mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$ be such that $P \in \operatorname{Spec}_J \mathcal{O}_q(\mathrm{GL}_n(\mathbb{K}))$. By Lemma 3.4.1, there is an H-prime ideal J' of $\mathcal{O}_q(M_n(\mathbb{K}))$ and a prime ideal $P' \in \operatorname{Spec}_{J'} \mathcal{O}_q(M_n(\mathbb{K}))$ such that $J'[\det_q^{-1}] = J$ and $P'[\det_q^{-1}] = P$. It follows from Proposition 3.4.3 that

$$\operatorname{prim.deg} P = \operatorname{K.dim} \operatorname{Spec}_J \mathcal{O}_q(\mathrm{GL}_n(\mathbb{K})) + \operatorname{ht} J - \operatorname{ht} P \tag{3.21}$$

and that

$$\operatorname{prim.deg} P' = \operatorname{K.dim} \operatorname{Spec}_{J'} \mathcal{O}_q(M_n(\mathbb{K})) + \operatorname{ht} J' - \operatorname{ht} P'. \tag{3.22}$$

Since $P = P'[\det_q^{-1}]$ and $J = J'[\det_q^{-1}]$, it follows from Corollary 3.4.2, (3.21), and (3.22) that $\operatorname{prim.deg} P = \operatorname{prim.deg} P'$. Since it is clear that $\operatorname{rat.deg} P = \operatorname{rat.deg} P'$ and since $\mathcal{O}_q(M_n(\mathbb{K}))$ satisfies the strong Dixmier-Moeglin equivalence, we have

$$\operatorname{prim.deg} P = \operatorname{prim.deg} P' = \operatorname{rat.deg} P' = \operatorname{rat.deg} P.$$

This completes the proof. ∎

Chapter 4

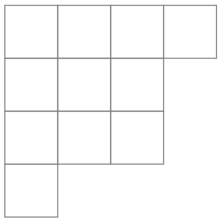
Partition subalgebras and Cauchon graphs

The material of this chapter comes from joint work with Prof. Stéphane Launois and Prof. Tom Lenagan. Sections 4.1, 4.2, and 4.3 consist of known results, some of which are rewritten in a fashion suitable for the purposes of the rest of the chapter; Sections 4.1, 4.2, and 4.3 are designed to set up Section 4.4, which consists of original results (most of which are generalisations of results of Karel Casteels from [7] and [8]).

4.1 Partition subalgebras of quantum matrices

Let us fix positive integers c, d, m, n with $c \leq m$ and $d \leq n$ and let us fix a partition $\lambda = (\lambda_1, \ldots, \lambda_c)$ with $d = \lambda_1 \geq \cdots \geq \lambda_c \geq 1$. Let Y_λ be the Young diagram corresponding to λ; Y_λ is formed by taking a $c \times d$ rectangular array of boxes and deleting the box in position (i, j) if and only if $j > \lambda_i$. In other words, Y_λ has a box in position (i, j) if and only if $j \leq \lambda_i$. We shall often abuse notation slightly by saying that $(i, j) \in Y_\lambda$ when we mean that Y_λ has a box in position (i, j).

Example 4.1.1. *Let $c = d = 4$ and consider the partition $\lambda = (4, 3, 3, 1)$. Then the Young diagram Y_λ is*



Definition 4.1.2. We associate a subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ to the partition λ: the *partition subalgebra* of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ associated to the partition λ is denoted by $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ and is defined to be the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ generated by all those $X_{i,j}$ such that $(i,j) \in Y_\lambda$.

Example 4.1.3. *Taking $c = d = 4$ and $\lambda = (4,3,3,1)$ as in Example 4.1.1, the partition subalgebra $\mathcal{O}_q(M^{\lambda}_{4,4}(k))$ is the subalgebra of $\mathcal{O}_q(M_{4,4}(k))$ generated by the following elements:*

$X_{1,1}$	$X_{1,2}$	$X_{1,3}$	$X_{1,4}$
$X_{2,1}$	$X_{2,2}$	$X_{2,3}$	
$X_{3,1}$	$X_{3,2}$	$X_{3,3}$	
$X_{4,1}$			

Definition 4.1.4. Let $I = \{i_1 < \ldots < i_t\} \subseteq [\![1,m]\!]$ and $J = \{j_1 < \ldots < j_t\} \subseteq [\![1,n]\!]$. The *pseudo quantum minor* $[I \mid J]$ of $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ is defined by

$$[I \mid J] = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_1,j_{\sigma(1)}} X_{i_2,j_{\sigma(2)}} \cdots X_{i_t,j_{\sigma(t)}},$$

with the convention that $X_{i,j} = 0$ if $(i,j) \notin Y_\lambda$.

Remark 4.1.5. *We use the term "pseudo quantum minor" as a reminder that we may not be dealing with the full algebra of quantum matrices but rather a partition subalgebra.*

Example 4.1.6. *Taking $c = d = 4$ and $\lambda = (4,3,3,1)$ as in Example 4.1.3, the pseudo quantum minor $[12 \mid 34]$ of $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ is $X_{1,3}X_{2,4} - qX_{1,4}X_{2,3}$, where $X_{2,4}$ is interpreted as zero since $(2,4) \notin Y_\lambda$. Hence we have $[12 \mid 34] = -qX_{1,4}X_{2,3}$.*

Although some of the standard quantum Laplace expansions for quantum minors of $\mathcal{O}_q(M_{m,n}(k))$ fail for pseudo quantum minors of $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$, it is evident immediately from the definition that we may expand with the first row on the left or the last row on the right:

Lemma 4.1.7 (quantum Laplace expansion with rows)**.** *Let* $I = \{i_1 < \ldots < i_t\} \subseteq [\![1, m]\!]$ *and* $J = \{j_1 < \ldots < j_t\} \subseteq [\![1, n]\!]$. *The pseudo quantum minor* $[I \mid J]$ *of* $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ *satisfies*

(1) $[I \mid J] = \sum_{l=1}^{t} (-q)^{l-1} X_{i_1,j_l} [i_2 \cdots i_t \mid j_1 \cdots \widehat{j_l} \cdots j_t]$;

(2) $[I \mid J] = \sum_{l=1}^{t} (-q)^{t-l} [i_1 \cdots i_{t-1} \mid j_1 \cdots \widehat{j_l} \cdots j_t] X_{i_t,j_l}$.

Useful for proving that we may also expand with the first column on the left or the last column on the right will be the following expression for a pseudo quantum minor:

Lemma 4.1.8. *If* $I = \{i_1 < \ldots < i_t\} \subseteq [\![1, m]\!]$ *and* $J = \{j_1 < \ldots < j_t\} \subseteq [\![1, n]\!]$, *then the pseudo quantum minor* $[I \mid J]$ *of* $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ *is given by*

$$[I \mid J] = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{i_{\sigma(1)},j_1} X_{i_{\sigma(2)},j_2} \cdots X_{i_{\sigma(t)},j_t}.$$

Proof. Let us assume for ease of notation that $I = J = [\![1, t]\!]$ (the proof for general I and J is the same but the notation is more unwieldy). Let us set $\{1 \cdots t \mid 1 \cdots t\} = \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} X_{\sigma(1),1} \cdots X_{\sigma(t),t}$. Our claim is that $\{1 \cdots t \mid 1 \cdots t\} = [1 \cdots t \mid 1 \cdots t]$. This claim clearly holds if $t = 1$ and we proceed by induction on t. We have

$$\begin{aligned}
[1 \cdots t \mid 1 \cdots t] &= \sum_{l=1}^{t} (-q)^{l-1} X_{1,l} [2 \cdots t \mid 1 \cdots \widehat{l} \cdots t] \quad \text{(Lemma 4.1.7(1))} \\
&= \sum_{l=1}^{t} (-q)^{l-1} X_{1,l} \{2 \cdots t \mid 1 \cdots \widehat{l} \cdots t\}. \quad \text{(induction hypothesis)}
\end{aligned}$$

Let us set $\{i_1 < \cdots < i_{t-1}\} = \{2 < \cdots < t\}$ and $\{j_1^l < \cdots < j_{t-1}^l\} = \{1 < \cdots < \widehat{l} < \cdots < t\}$, so that

$$\begin{aligned}[1\cdots t \mid 1\cdots t] &= \sum_{l=1}^{t}(-q)^{l-1}X_{1,l}\left(\sum_{\rho\in S_{t-1}}(-q)^{\ell(\rho)}X_{i_{\rho(1)},j_1^l}\cdots X_{i_{\rho(t-1)},j_{t-1}^l}\right)\\ &= \sum_{l=1}^{t}\sum_{\rho\in S_{t-1}}(-q)^{l-1+\ell(\rho)}X_{1,l}X_{i_{\rho(1)},j_1^l}\cdots X_{i_{\rho(t-1)},j_{t-1}^l}.\end{aligned}$$

For all $j < l$ and all $s = 1, \dots, t-1$, the relations in $\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))$ (which come from (2.6)) show that $X_{1,l}$ commutes with $X_{i_{\rho(s)},j}$ and hence

$$\begin{aligned}[1\cdots t \mid 1\cdots t] &= \sum_{l=1}^{t}\sum_{\rho\in S_{t-1}}(-q)^{l-1+\ell(\rho)}X_{i_{\rho(1)},1}\cdots X_{i_{\rho(l-1)},l-1}X_{1,l}X_{i_{\rho(l)},l+1}\cdots X_{i_{\rho(t-1)},t}\\ &= \sum_{l=1}^{t}\sum_{\substack{\sigma\,\in\, S_t\\ \sigma(l)\,=\,1}}(-q)^{\ell(\sigma)}X_{\sigma(1),1}\cdots X_{\sigma(t),t}\\ &= \sum_{\sigma\in S_t}(-q)^{\ell(\sigma)}X_{\sigma(1),1}\cdots X_{\sigma(t),t}\\ &= \{1\cdots t \mid 1\cdots t\},\end{aligned}$$

as required. ■

We get the following directly from Lemma 4.1.8:

Lemma 4.1.9 (quantum Laplace expansion with columns)**.** *Suppose that* $\{i_1 < \cdots < i_t\} \subseteq [\![1, m]\!]$ *and that* $\{j_1 < \cdots < j_t\} \subset [\![1, n]\!]$*. Then we may expand the pseudo quantum minor* $[i_1 \cdots i_t \mid j_1 \cdots j_t]$ *of* $\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))$ *with the first column on the left or the last column on the right:*

(1) $[i_1\cdots i_t \mid j_1\cdots j_t] = \sum_{l=1}^{t}(-q)^{l-1}X_{i_l,j_1}[\widehat{i_l} \mid \widehat{j_1}]$*;*

(2) $[i_1\cdots i_t \mid j_1\cdots j_t] = \sum_{l=1}^{t}(-q)^{t-l}[\widehat{i_l} \mid \widehat{j_t}]X_{i_l,j_t}$*.*

4.2 $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ as a CGL extension

The rational action of the algebraic torus $(\mathbb{K}^{\times})^{m+n}$ on $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ described in Example 2.2.2 clearly restricts to the partition subalgebra $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ so that $(\mathbb{K}^{\times})^{m+n}$ acts by automorphisms on $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ as follows:

$$(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \cdot X_{i,j} = \alpha_i \beta_j X_{i,j} \tag{4.1}$$

for all $(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) \in H$ and all $(i,j) \in Y_\lambda$; this action is clearly rational.

Because some details of the proof shall be useful to us, we check here the known fact (see [24, Proposition 3.2]) that $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ is a uniparameter CGL extension. Adding the generators $X_{i,j}$ $((i,j) \in Y_\lambda)$ in lexicographical order, we may write the partition subalgebra $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ of the algebra $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ of quantum $m \times n$ matrices as an iterated Ore extension

$$\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K})) = \mathbb{K}[X_{1,1}] \cdots [X_{i,j}; \sigma_{i,j}, \delta_{i,j}] \cdots [X_{c,\lambda_c}; \sigma_{c,\lambda_c}, \delta_{c,\lambda_c}], \tag{4.2}$$

where for each $(a,b) \in Y_\lambda$, the automorphism $\sigma_{a,b}$ and the left $\sigma_{a,b}$-derivation $\delta_{a,b}$ are defined such that for each $(i,j) \in Y_\lambda$ satisfying $(i,j) <_{\text{lex}} (a,b)$, we have

$$\sigma_{a,b}(X_{i,j}) = \begin{cases} q^{-1} X_{i,j} & \text{if } i = a \text{ or } j = b \\ X_{i,j} & \text{otherwise} \end{cases} \tag{4.3}$$

and

$$\delta_{a,b}(X_{i,j}) = \begin{cases} (q^{-1} - q) X_{i,b} X_{a,j} & i < a \text{ and } j < b \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

(As such, it is easy to check that $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ is a domain; it is noetherian by [18, Theorem 2.6]; all its prime ideals are completely prime by [6, Theorem II.6.9].)

Remark 4.2.1. *In order to show that the expression* (4.2) *of* $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ *as an iterated Ore extension (with the action of* $H = (\mathbb{K}^{\times})^{m+n}$ *described in* (4.1)*), is a uniparameter CGL extension with parameter* q*, it will suffice to show that*

(i) The elements $X_{i,j}$ *(*$(i,j) \in Y_{\lambda}$*) of* $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ *are* H*-eigenvectors with rational eigenvalues (it follows easily that the action of* H *is rational).*

(ii) For every $(a,b) \in Y_{\lambda} \setminus \{(1,1)\}$*,* $\delta_{a,b}$ *is locally nilpotent.*

(iii) For every $(a,b) \in Y_{\lambda} \setminus \{(1,1)\}$*, there exists* $q_{a,b} \in \mathbb{K}^{\times}$ *not a root of unity such that* $\sigma_{a,b} \circ \delta_{a,b} = q_{a,b}\delta_{a,b} \circ \sigma_{a,b}$*.*

(iv) For every $(a,b) \in Y_{\lambda} \setminus \{(1,1)\}$ *and every* $(i,j) \in Y_{\lambda}$ *such that* $(i,j) <_{lex} (a,b)$*, there exists* $\lambda_{(a,b),(i,j)} \in \mathbb{K}^{\times}$ *such that* $\sigma_{a,b}(X_{i,j}) = \lambda_{(a,b),(i,j)}X_{i,j}$ *.*

(v) The set $\{\alpha \in \mathbb{K}^{\times} \mid$ *there exists* $h \in H$ *such that* $h \cdot X_{1,1} = \alpha X_{1,1}\}$ *is infinite.*

(vi) For every $(a,b) \in Y_{\lambda} \setminus \{(1,1)\}$*, there exists* $h_{a,b} \in H$ *such that* $h_{a,b} \cdot X_{a,b} = q_{a,b}X_{a,b}$ *and for all* $(i,j) <_{lex} (a,b)$*,* $h_{a,b} \cdot X_{i,j} = \lambda_{(a,b),(i,j)}X_{i,j}$*.*

(vii) For every $(a,b) \in Y_{\lambda} \setminus \{(1,1)\}$ *and every* $(i,j) \in Y_{\lambda}$ *such that* $(i,j) <_{lex} (a,b)$*,* $\lambda_{(a,b),(i,j)}$ *is an integral power of* q*.*

Lemma 4.2.2. *The expression* (4.2) *of* $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ *as an iterated Ore extension, along with the action of* $H = (\mathbb{K}^{\times})^{m+n}$ *described in* (4.1)*, is a uniparameter CGL extension with parameter* q*.*

Proof. We show that the conditions of Remark 4.2.1 are satisfied.

(i) It is clear from (4.1) that each $X_{i,j}$ ($(i,j) \in Y_{\lambda}$) is an H-eigenvector with eigenvalue $H \to \mathbb{K}^{\times}; (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n) \mapsto \alpha_i\beta_j$. This eigenvalue is clearly rational, so that condition (i) of Remark 4.2.1 is satisfied.

(ii) Fix any $(a,b) \in Y_{\lambda} \setminus \{(1,1)\}$. One can check easily from (4.4) that $\delta^2_{a,b} = 0$, so that condition (ii) of Remark 4.2.1 is satisfied.

(iii) Fix any $(a,b) \in Y_\lambda \setminus \{(1,1)\}$. We claim that $\sigma_{a,b} \circ \delta_{a,b} = q^{-2}\delta_{a,b} \circ \sigma_{a,b}$. It suffices to check that these two morphisms agree on each $X_{i,j}$ with $(i,j) <_{\text{lex}} (a,b)$. Since $\delta_{a,b}(X_{i,j})$ is nonzero only if $(i,j) <_{\text{lex}} (a,b)$ and $X_{i,j}$ is an eigenvector for $\sigma_{a,b}$, we may assume that $(i,j) <_{\text{lex}} (a,b)$. In this case, we have

$$\sigma_{a,b} \circ \delta_{a,b}(X_{i,j}) = \sigma_{a,b}((q^{-1}-q)X_{a,j}X_{i,b}) = q^{-2}(q^{-1}-q)X_{a,j}X_{i,b} \tag{4.5}$$

and

$$\delta_{a,b} \circ \sigma_{a,b}(X_{i,j}) = \delta_{a,b}(X_{i,j}) = (q^{-1}-q)X_{a,j}X_{i,b}. \tag{4.6}$$

Comparing (4.5) and (4.6), we get $\sigma_{a,b} \circ \delta_{a,b} = q^{-2}\delta_{a,b} \circ \sigma_{a,b}$. Hence if we set $q_{a,b} = q^{-2}$, then condition (iii) of Remark 4.2.1 is satisfied.

(iv) Fix any $(a,b) \in Y_\lambda \setminus \{(1,1)\}$ and any $(i,j) \in Y_\lambda$ such that $(i,j) <_{\text{lex}} (a,b)$. It follows immediately from (4.3) that if we set

$$\lambda_{(a,b),(i,j)} = \begin{cases} q^{-1} & \text{if } i = a \text{ or } j = b \\ 1 & \text{otherwise,} \end{cases}$$

then condition (iv) of Remark 4.2.1 is satisfied.

(v) Condition (v) of Remark 4.2.1 follows immediately from the following observation: for every $\alpha \neq 0$ belonging to the infinite field $\mathbb{K}$, we have $(\alpha, 1, \ldots, 1) \cdot X_{1,1} = \alpha X_{1,1}$.

(vi) Fix any $(a,b) \in Y_\lambda \setminus \{(1,1)\}$ and define the element $h_{a,b} = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n)$ of H such that $\alpha_a = \beta_b = q^{-1}$ and all other entries are 1. Then $h_{a,b}X_{a,b} = q^{-2}X_{a,b} = q_{a,b}X_{a,b}$ and for any $(i,j) \in Y_\lambda$ such that $(i,j) <_{\text{lex}} (a,b)$, we have

$$h_{a,b} \cdot X_{i,j} = \begin{cases} q^{-1}X_{i,j} & \text{if } i = a \text{ or } j = b \\ X_{i,j} & \text{otherwise,} \end{cases}$$

so that $h_{a,b} \cdot X_{i,j} = \lambda_{(a,b),(i,j)}X_{i,j}$. This verifies condition (vi) of Remark 4.2.1.

(vii) Condition (vi) of Remark 4.2.1 is immediate from part (*iv*) of this proof.

■

4.3 Deleting derivations in $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$

We established in Lemma 4.2.2 that $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$ is a uniparameter CGL extension with parameter q. Let us denote by A_λ its associated skew-symmetric integral matrix (whose entries all belong to the set $\{0, 1, -1\}$ by part (iv) of the proof of Lemma 4.2.2).

We may apply the deleting-derivations algorithm (see Section 2.4) of Cauchon to $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$. Recall that $\lambda = (\lambda_1, \dots, \lambda_c)$ is a partition with $d = \lambda_1 \geq \cdots \geq \lambda_c \geq 1$, $c \leq m$, $d \leq n$. It follows that (c, λ_c) is the largest element of Y_λ with respect to the lexicographical ordering.

Notation 4.3.1. *Set* $E_\lambda = (Y_\lambda \backslash \{(1,1)\}) \sqcup \{(c, \lambda_c{+}1)\}$. *For* $(a,b) \in Y_\lambda$, *let* $(a,b)^+$ *be the smallest (with respect to the lexicographical order) element of* E_λ *satisfying* $(a,b)^+ >_{lex} (a,b)$. *Clearly* $(1,1)^+$ *and* $(c, \lambda_c + 1)$ *are respectively the smallest and largest elements of* E_λ *with respect to the lexicographical order. Moreover* $E_\lambda = \{(a,b)^+ \colon (a,b) \in Y_\lambda\}$. *For* $(a,b) \in E_\lambda$, *let* $(a,b)^-$ *be the largest (with respect to the lexicographical order) element of* Y_λ *satisfying* $(a,b)^- <_{lex} (a,b)$.

Set $X^{(c,\lambda_c+1)}_{i,j} := X_{i,j}$ for all $(i,j) \in Y_\lambda$. For each $(a,b) \in E_\lambda \setminus \{(c, \lambda_c + 1)\}$, Cauchon's deleting-derivations algorithm (see [10, Section 3] or Section 2.4 of this thesis) constructs from $(X^{(a,b)^+}_{i,j})_{(i,j)\in Y_\lambda}$ a family $(X^{(a,b)}_{i,j})_{(i,j)\in Y_\lambda}$ of elements of $\mathrm{Frac}(\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K})))$. For each $(a,b) \in E_\lambda$, the subalgebra of $\mathrm{Frac}(\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K})))$ generated by the family $(X^{(a,b)}_{i,j})_{(i,j)\in Y_\lambda}$ is denoted by $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))^{(a,b)}$; in particular, $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))^{(c,\lambda_c+1)} = \mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))$. By [10, Theorem 3.2.1], for each $(a,b) \in Y_\lambda$, there is an isomorphism

$$\begin{aligned} \mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))^{(a,b)^+} &\xrightarrow{\cong} k[X_{1,1}] \cdots [X_{a,b}; \sigma_{a,b}, \delta_{a,b}][X_{a',b'}; \tau_{a',b'}] \cdots [X_{c,\lambda_c}; \tau_{c,\lambda_c}] \\ X^{(a,b)^+}_{i,j} &\mapsto X_{i,j} \quad \text{for all } (i,j) \in Y_\lambda, \end{aligned} \tag{4.7}$$

where $(a',b') := (a,b)^+$ and where for each $(p,q) \in Y_\lambda$ such that $(p,q) \geq_{\text{lex}} (a',b')$, $\tau_{p,q}$ is the automorphism defined by

$$\tau_{p,q}(X_{i,j}) := \begin{cases} q^{-1}X_{i,j} & \text{if } i = p \text{ or } j = q \\ X_{i,j} & \text{otherwise} \end{cases}$$

for all $(i,j) \in Y_\lambda$ such that $(i,j) <_{\text{lex}} (p,q)$. In particular, by (4.7), there is an isomorphism

$$\begin{aligned} \mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(1,1)^+} &\xrightarrow{\cong} k[T_{1,1}]\cdots[T_{i,j};\tau_{i,j}]\cdots[T_{c,\lambda_c};\tau_{c,\lambda_c}] \\ X^{(1,1)^+}_{i,j} &\mapsto T_{i,j} \end{aligned} \tag{4.8}$$

where for each $(p,q) \in Y_\lambda \setminus \{(1,1)\}$, the automorphism $\tau_{p,q}$ is defined by

$$\tau_{p,q}(T_{i,j}) = \begin{cases} q^{-1}T_{i,j} & \text{if } i = p \text{ or } j = q \\ T_{i,j} & \text{otherwise} \end{cases} \tag{4.9}$$

for all $(i,j) \in Y_\lambda$ such that $(i,j) <_{\text{lex}} (p,q)$. The algebra $k[T_{1,1}]\cdots[T_{i,j};\tau_{i,j}]\cdots[T_{c,\lambda_c};\tau_{c,\lambda_c}]$ is the uniparameter quantum affine space $\mathbb{K}_{q,A_\lambda}[T_{1,1},\ldots,T_{c,\lambda_c}]$; let us identify $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(1,1)^+}$ with this quantum affine space via the isomorphism (4.8), so that $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(1,1)^+}$ is generated by $\{T_{i,j} \mid (i,j) \in Y_\lambda\}$ with relations

$$\begin{aligned} T_{i,j}T_{i,l} &= qT_{i,l}T_{i,j} && \text{if } (i,j),(i,l) \in Y_\lambda \text{ and } j < l; \\ T_{i,j}T_{k,j} &= qT_{k,j}T_{i,j} && \text{if } (i,j),(k,j) \in Y_\lambda \text{ and } i < k; \\ T_{i,j}T_{k,l} &= T_{k,l}T_{i,j} && \text{if } (i,j),(k,l) \in Y_\lambda,\ k \neq i, \text{ and } j \neq l. \end{aligned} \tag{4.10}$$

The deleting-derivations algorithm: Suppose that $(a,b) \in E_\lambda \setminus \{(c,\lambda_c+1)\}$ and that the family $(X^{(a,b)^+}_{i,j})_{(i,j)\in Y_\lambda}$ has been constructed. Notice that (4.7) shows in particular that $X^{(a,b)^+}_{a,b}$ is nonzero and hence invertible in $\mathrm{Frac}(\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K})))$. To construct the family $(X^{(a,b)}_{i,j})_{(i,j)\in Y_\lambda}$ from the family $(X^{(a,b)^+}_{i,j})_{(i,j)\in Y_\lambda}$, identify[1] $(X^{(a,b)^+}_{i,j})_{(i,j)\in Y_\lambda}$ with

[1]We make this identification in order that the term $\delta^n_{a,b} \circ \sigma^{-n}_{a,b}(X^{(a,b)^+}_{i,j})$ in (4.11) makes sense.

$(X_{i,j})_{(i,j)\in Y_\lambda}$ via the isomorphism (4.7) and for $(i,j) \in Y_\lambda$, set

$$X^{(a,b)}_{i,j} := \begin{cases} \sum_{n=0}^{+\infty} \frac{(1-q_{a,b})^{-n}}{[n]!_{q_{a,b}}} \delta^n_{a,b} \circ \sigma^{-n}_{a,b}(X^{(a,b)^+}_{i,j})(X^{(a,b)^+}_{a,b})^{-n} & \text{if } (i,j) <_{\text{lex}} (a,b); \\ X^{(a,b)^+}_{i,j} & \text{if } (i,j) \geq_{\text{lex}} (a,b). \end{cases} \tag{4.11}$$

Lemma 4.3.2. *Suppose that* $(a,b) \in E_\lambda \backslash \{(c, \lambda_c{+}1)\}$ *and that the family* $(X^{(a,b)^+}_{i,j})_{(i,j)\in Y_\lambda}$ *has been constructed. Then for* $(i,j) \in Y_\lambda$, *we have*

$$X^{(a,b)}_{i,j} = \begin{cases} X^{(a,b)^+}_{i,j} - X^{(a,b)^+}_{i,b}(X^{(a,b)^+}_{a,b})^{-1}X^{(a,b)^+}_{a,j} & \textit{if } i < a \textit{ and } j < b \\ X^{(a,b)^+}_{i,j} & \textit{otherwise.} \end{cases}$$

Proof. We may assume that $(i,j) <_{\text{lex}} (a,b)$. One can check easily from (4.4) that $\delta^2_{a,b} = 0$. Recall from Lemma 4.2.2 that $q_{a,b} = q^{-2}$. Now (4.11) gives $X^{(a,b)}_{i,j} = X^{(a,b)^+}_{i,j} + (1-q^{-2})^{-1}\delta_{a,b} \circ \sigma^{-1}_{a,b}(X^{(a,b)^+}_{i,j})(X^{(a,b)^+}_{a,b})^{-1}$. There are two cases to consider.

- Suppose that either $i = a$ or $j \geq b$. Then $\sigma^{-1}_{a,b}(X^{(a,b)^+}_{i,j})$ is a scalar multiple of $X^{(a,b)^+}_{i,j}$ by (4.3) and $\delta_{a,b}(X^{(a,b)^+}_{i,j}) = 0$ by (4.4). It follows immediately that $\delta_{a,b} \circ \sigma^{-1}_{a,b}(X^{(a,b)^+}_{i,j}) = 0$ and hence $X^{(a,b)}_{i,j} = X^{(a,b)^+}_{i,j}$.

- Suppose that $i < a$ and that $j < b$. Then

$$\begin{aligned} X^{(a,b)}_{i,j} &= X^{(a,b)^+}_{i,j} + (1-q^{-2})^{-1}\delta_{a,b} \circ \sigma^{-1}_{a,b}(X^{(a,b)^+}_{i,j})(X^{(a,b)^+}_{a,b})^{-1} \\ &= X^{(a,b)^+}_{i,j} + (1-q^{-2})^{-1}(q^{-1}-q)(X^{(a,b)^+}_{i,b}X^{(a,b)^+}_{a,j})(X^{(a,b)^+}_{a,b})^{-1} \qquad \text{(by (4.3) and (4.4))} \\ &= X^{(a,b)^+}_{i,j} + (1-q^{-2})^{-1}(q^{-1}-q)q^{-1}X^{(a,b)^+}_{i,b}(X^{(a,b)^+}_{a,b})^{-1}X^{(a,b)^+}_{a,j} \\ &= X^{(a,b)^+}_{i,j} - X^{(a,b)^+}_{i,b}(X^{(a,b)^+}_{a,b})^{-1}X^{(a,b)^+}_{a,j}. \end{aligned}$$

■

4.3.1 An injection of prime spectra

By [10, Section 4.3], for each $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$, there is an injection

$$\varphi_{a,b} \colon \operatorname{Spec}(\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)^+}) \to \operatorname{Spec}(\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}).$$

We shall not describe the construction of this injection but we shall describe some of its useful properties.

4.3.2 Identifying several total rings of fractions

Let $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$ and let Q be a prime ideal of $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)^+}$. The Lemmas [10, Lemme 5.3.1 and Lemme 5.3.2] give isomorphisms

$$\operatorname{Frac}(\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)^+}/Q) \xrightarrow{\cong} \operatorname{Frac}(\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}/\varphi_{a,b}(Q)). \tag{4.12}$$

4.3.3 Relationships between generators

Fix a prime ideal P of $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))$. For each $(a,b) \in E_\lambda$, set $P^{(a,b)} = \varphi_{a,b} \circ \cdots \circ \varphi_{c,\lambda_c}(P) \in \operatorname{Spec}(\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)})$ (which gives $P^{(c,\lambda_c+1)} = P$) and for each $(i,j) \in Y_\lambda$, let $\chi^{(a,b)}_{i,j}$ be the canonical image of $X^{(a,b)}_{i,j}$ in $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$. Let us denote by G the total ring of fractions of $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))/P$ (which is a division ring since all prime ideals of $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))$ are completely prime) and by varying (a,b) over $E_\lambda \setminus \{(c, \lambda_c+1)\}$ and Q over $P^{(1,1)^{++}}, \ldots, P^{(c,\lambda_c+1)}$ in the isomorphism (4.12), let us identify the total ring of fractions of each noetherian domain $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$ ($(a,b) \in E_\lambda$) with G.

Some immediate consequences of this setup (noted in [10, Proposition 5.4.1]) are that for each $(a,b) \in E_\lambda$,

- $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$ is the subalgebra of G generated by $(\chi^{(a,b)}_{i,j})_{(i,j)\in Y_\lambda}$;
- there is a morphism of algebras $f_{a,b} \colon \mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)} \to G$ which sends each $X^{(a,b)}_{i,j}$ ($(i,j) \in Y_\lambda$) to $\chi^{(a,b)}_{i,j}$;

- the kernel of $f_{a,b}$ is $P^{(a,b)}$ and its image is $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$.

Suppose that $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$. By [10, Proposition 5.4.2], we may construct the generators $\chi^{(a,b)}_{i,j}$ $((i,j) \in Y_\lambda)$ of the algebra $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$ from the generators $\chi^{(a,b)^+}_{i,j}$ $((i,j) \in Y_\lambda)$ of the algebra $\mathcal{O}_q(M^{\lambda}_{m,n}(\mathbb{K}))^{(a,b)^+}/P^{(a,b)^+}$ as follows: when we identify[2] $(X^{(a,b)^+}_{i,j})_{(i,j)\in Y_\lambda}$ with $(X_{i,j})_{(i,j)\in Y_\lambda}$ via the isomorphism (4.7), we get

$$\chi^{(a,b)}_{i,j} = \begin{cases} \displaystyle\sum_{n=0}^{+\infty} \frac{(1-q_{a,b})^{-n}}{[n]!_{q_{a,b}}} \lambda^{-n}_{(a,b),(i,j)} f_{(a,b)^+}(\delta^n_{a,b}(X^{(a,b)^+}_{i,j}))(\chi^{(a,b)^+}_{a,b})^{-n} & \text{if } *; \\ \chi^{(a,b)^+}_{i,j} & \text{otherwise,} \end{cases} \tag{4.13}$$

where, simply to reduce the length of the display, we denote by $*$ the conditions that $(i,j) <_{\mathrm{lex}} (a,b)$ and $\chi^{(a,b)^+}_{a,b} \neq 0$. Let us restate (4.13) in a form which will be more convenient for us:

Lemma 4.3.3. *Suppose that $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$. Then for all $(i,j) \in Y_\lambda$, we have*

$$\chi^{(a,b)^+}_{i,j} = \begin{cases} \chi^{(a,b)}_{i,j} + \chi^{(a,b)}_{i,b}(\chi^{(1,1)^+}_{a,b})^{-1}\chi^{(1,1)^+}_{a,j} & \textit{if } i < a,\ j < b,\ \textit{and } \chi^{(1,1)^+}_{a,b} \neq 0; \\ \chi^{(a,b)}_{i,j} & \textit{otherwise.} \end{cases}$$

Proof. Suppose that $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$.

- We claim first that for all $(i,j) \in Y_\lambda$, we have

$$\chi^{(a,b)}_{i,j} = \begin{cases} \chi^{(a,b)^+}_{i,j} - \chi^{(a,b)^+}_{i,b}(\chi^{(a,b)^+}_{a,b})^{-1}\chi^{(a,b)^+}_{a,j} & \text{if } i < a,\ j < b,\ \text{and } \chi^{(a,b)^+}_{a,b} \neq 0; \\ \chi^{(a,b)^+}_{i,j} & \text{otherwise.} \end{cases} \tag{4.14}$$

  We may assume that $\chi^{(a,b)^+}_{a,b} \neq 0$ and that $(i,j) <_{\mathrm{lex}} (a,b)$. If $i = a$ or $j \geq b$, then $\delta_{a,b}(X^{(a,b)^+}_{i,j}) = 0$ and hence $\chi^{(a,b)}_{i,j} = \chi^{(a,b)^+}_{i,j}$. On the other hand, if $i < a$ and $j < b$,

[2] We make this identification in order that the term $\delta^n_{a,b}(X^{(a,b)^+}_{i,j})$ in (4.13) makes sense.

then since $\delta_{a,b}^2 = 0$ (easily checked from (4.4)), we have

$$\begin{aligned}
\chi_{i,j}^{(a,b)} &= \chi_{i,j}^{(a,b)^+} + (1-q^{-2})^{-1} f_{(a,b)^+}(\delta_{a,b}(X_{i,j}^{(a,b)^+}))(\chi_{a,b}^{(a,b)^+})^{-1} && \text{(by (4.13))} \\
&= \chi_{i,j}^{(a,b)^+} + (1-q^{-2})^{-1} f_{(a,b)^+}((q^{-1}-q)X_{i,b}^{(a,b)^+} X_{a,j}^{(a,b)^+})(\chi_{a,b}^{(a,b)^+})^{-1} && \text{(by (4.4))} \\
&= \chi_{i,j}^{(a,b)^+} + (1-q^{-2})^{-1}(q^{-1}-q)\chi_{i,b}^{(a,b)^+}\chi_{a,j}^{(a,b)^+}(\chi_{a,b}^{(a,b)^+})^{-1} \\
&= \chi_{i,j}^{(a,b)^+} + (1-q^{-2})^{-1}(q^{-1}-q)q^{-1}\chi_{i,b}^{(a,b)^+}(\chi_{a,b}^{(a,b)^+})^{-1}\chi_{a,j}^{(a,b)^+} \\
&= \chi_{i,j}^{(a,b)^+} - \chi_{i,b}^{(a,b)^+}(\chi_{a,b}^{(a,b)^+})^{-1}\chi_{a,j}^{(a,b)^+},
\end{aligned}$$

establishing (4.14).

- We claim next that for all $(i,j) \in Y_\lambda$, we have

$$\chi_{i,j}^{(a,b)^+} = \begin{cases} \chi_{i,j}^{(a,b)} + \chi_{i,b}^{(a,b)}(\chi_{a,b}^{(a,b)})^{-1}\chi_{a,j}^{(a,b)} & \text{if } i<a,\ j<b, \text{ and } \chi_{a,b}^{(a,b)} \neq 0; \\ \chi_{i,j}^{(a,b)} & \text{otherwise.} \end{cases} \tag{4.15}$$

  By (4.14), for all $(i,j) \in Y_\lambda$, we have

$$\chi_{i,b}^{(a,b)} = \chi_{i,b}^{(a,b)^+},\ \chi_{a,b}^{(a,b)} = \chi_{a,b}^{(a,b)^+},\ \text{and } \chi_{a,j}^{(a,b)} = \chi_{a,j}^{(a,b)^+}.$$

  Substituting these identities back into (4.14) shows that for all $(i,j) \in Y_\lambda$, we have

$$\chi_{i,j}^{(a,b)} = \begin{cases} \chi_{i,j}^{(a,b)^+} - \chi_{i,b}^{(a,b)}(\chi_{a,b}^{(a,b)})^{-1}\chi_{a,j}^{(a,b)} & \text{if } i<a,\ j<b, \text{ and } \chi_{a,b}^{(a,b)} \neq 0; \\ \chi_{i,j}^{(a,b)^+} & \text{otherwise,} \end{cases} \tag{4.16}$$

  which immediately gives (4.15).

- Finally, from (4.15) it follows easily that for any $(a',b') \in E_\lambda$ satisfying $(a',b') <_{\text{lex}} (a,b)$, we have $\chi_{a,b}^{(a',b')^+} = \chi_{a,b}^{(a',b')}$ and $\chi_{a,j}^{(a',b')^+} = \chi_{a,j}^{(a',b')}$. Easy inductive arguments give

$$\chi_{a,b}^{(a,b)} = \chi_{a,b}^{(1,1)^+} \text{ and } \chi_{a,j}^{(a,b)} = \chi_{a,j}^{(1,1)^+}.$$

Substituting these identities back into (4.15) completes the proof.

■

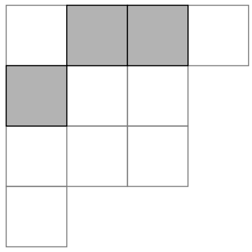
4.3.4 H-prime ideals of $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))$

Let us now assume that P is not just a prime ideal but an H-prime ideal of $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))$. The *canonical injection* $\varphi\colon \operatorname{Spec}(\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))) \to \operatorname{Spec}(\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))^{(1,1)^+})$ is defined by $\varphi = \varphi_{(1,1)^+} \circ \cdots \circ \varphi_{c,\lambda_c}$. By the results of Cauchon described in Subsection 2.4.4, the action of H on $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))$ induces an action of H on the quantum affine space $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))^{(1,1)^+} = \mathbb{K}_{q,A_\lambda}[T_{1,1},\dots,T_{c,\lambda_c}]$ such that φ sends P to an H-prime ideal $\varphi(P)$ $(= P^{(1,1)^+})$ of the quantum affine space $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))^{(1,1)^+} = \mathbb{K}_{q,A_\lambda}[T_{1,1},\dots,T_{c,\lambda_c}]$ and $\varphi(P)$ is generated by $\{T_{i,j} \mid (i,j) \in B\}$ for some subset B of Y_λ. Let us colour the squares of the Young diagram Y_λ in the following way: for $(i,j) \in Y_\lambda$, if $(i,j) \in B$, then assign colour black to the square of Y_λ in the (i,j)-position and if $(i,j) \notin B$, then assign colour white to the square of Y_λ in the (i,j)-position; call the resulting diagram $\mathcal{C}$. By [24, Theorem 3.5], the diagram $\mathcal{C}$ is a Cauchon diagram (see Definition 4.3.4 below) and all Cauchon diagrams on Y_λ arise from H-prime ideals of $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))$ in this way, giving us a one-to-one correspondence

$$H - \operatorname{Spec}\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K})) \longleftrightarrow \text{ Cauchon diagrams on the Young diagram } Y_\lambda. \quad (4.17)$$

Definition 4.3.4. Let Y be a Young diagram. A *Cauchon diagram* on Y is an assignment of the colours white and black to the squares of Y such that if a square S is black, then either every square above S is black or every square to the left of S is black. If $\mathcal{C}$ is a Cauchon diagram on Y, then we denote by $B_{\mathcal{C}}$ and $W_{\mathcal{C}}$ the set of squares of Y which are coloured black and white respectively in $\mathcal{C}$.

Example 4.3.5. *As in example 4.1.1, let* $c = d = 4$ *and consider the partition* $\lambda = (4,3,3,1)$. *Below is an example of a Cauchon diagram on* Y_λ. *If we call this Cauchon diagram* $\mathcal{C}$, *then we have* $B_{\mathcal{C}} = \{(2,1),(1,2),(1,3)\}$ *and*
$W_{\mathcal{C}} = \{(1,1),(1,4),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1)\}$.



Since we have identified the division ring $G = \mathrm{Frac}(\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))/P)$ with the total ring of fractions of each noetherian domain $\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$ $((a,b) \in E_\lambda)$, we have in particular identified G with the total ring of fractions of the algebra

$$\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))^{(1,1)^+}/\varphi(P) = \mathbb{K}_{q,A_\lambda}[T_{1,1},\dots,T_{c,\lambda_c}]/\langle T_{i,j} \mid (i,j) \in B_{\mathcal{C}}\rangle. \tag{4.18}$$

For $(i,j) \in Y_\lambda$, let $t_{i,j}$ denote the canonical image of $T_{i,j}$ in the algebra (4.18), so that $t_{i,j} = \chi_{i,j}^{(1,1)^+}$ and we may realise G as the total ring of fractions of the uniparameter quantum torus $\mathcal{B}$ which is generated by $\{t_{i,j}^{\pm 1} \mid (i,j) \in W_{\mathcal{C}}\}$ with relations

$$\begin{aligned}
t_{i,j}t_{i,l} &= q t_{i,l}t_{i,j} && \text{if } (i,j),(i,l) \in W_{\mathcal{C}} \text{ and } j < l;\\
t_{i,j}t_{k,j} &= q t_{k,j}t_{i,j} && \text{if } (i,j),(k,j) \in W_{\mathcal{C}} \text{ and } i < k;\\
t_{i,j}t_{k,l} &= t_{k,l}t_{i,j} && \text{if } (i,j),(k,l) \in W_{\mathcal{C}},\ k \neq i, \text{ and } j \neq l;\\
t_{i,j}t_{i,j}^{-1} &= 1 && \text{if } (i,j) \in W_{\mathcal{C}}.
\end{aligned} \tag{4.19}$$

Remark 4.3.6. *An easy way to understand these relations is as follows: Suppose that* a *and* b *are squares in* $W_{\mathcal{C}}$ *and that* $a <_{lex} b$. *Then* t_a *and* t_b *commute unless* a *and* b *are in the same row or column (i.e. unless* b *is east or south of* a*), in which case* $t_a t_b = q t_b t_a$.

Before stating the following very useful Corollary, we recap briefly on our setup and on some identifications which we will use implicitly:

Remark 4.3.7. *We have fixed an H-prime ideal P of $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))$ with corresponding Cauchon diagram $\mathcal{C}$ on the Young diagram Y_λ. For all $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$, we define $P^{(a,b)} := \varphi_{a,b} \circ \cdots \circ \varphi_{c,\lambda_c}(P)$ (which gives $P^{(c,\lambda_c+1)} = P$). For all $(a,b) \in E_\lambda$ and all $(i,j) \in Y_\lambda$, we denote by $\chi^{(a,b)}_{i,j}$ the canonical image of $X^{(a,b)}_{i,j}$ in $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$. We identify the total rings of fractions of the noetherian domains $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))^{(a,b)}/P^{(a,b)}$ ($(a,b) \in E_\lambda$) with the total ring of fractions G of $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))/P$ (which is a division ring) and we realise G as the total ring of fractions of the quantum torus $\mathcal{B}$ (whose relations are given in* (4.19)*).*

Corollary 4.3.8. *For all $(a,b) \in E_\lambda \setminus \{(c, \lambda_c+1)\}$ and all $(i,j) \in Y_\lambda$, we have*

$$\chi^{(a,b)^+}_{i,j} = \begin{cases} \chi^{(a,b)}_{i,j} + \chi^{(a,b)}_{i,b} t^{-1}_{a,b} t_{a,j} & \text{if } i < a,\ j < b, \text{ and } (a,b) \in W_{\mathcal{C}}; \\ \chi^{(a,b)}_{i,j} & \text{otherwise.} \end{cases}$$

Proof. Since $\chi^{(1,1)^+}_{a,b} = t_{a,b}$ is nonzero if and only if $(a,b) \in W_{\mathcal{C}}$ and since $\chi^{(1,1)^+}_{a,j} = t_{a,j}$, this result is an immediate consequence of Lemma 4.3.3. ∎

4.4 Cauchon graphs and path matrices

In this subsection, we generalise to partition subalgebras some results of Casteels [7] for quantum matrices, in particular his graph-theoretic method for deciding which quantum minors belong to a given H-prime ideal of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$. This section is based closely on the papers [7] and [8] of Casteels.

We continue with the setup and notation of the previous section. Recall in particular that $\lambda = (\lambda_1, \ldots, \lambda_c)$ is a partition (with $d = \lambda_1 \geq \cdots \geq \lambda_c \geq 1$, $c \leq m$, and $d \leq n$), that P is an H-prime ideal of the partition subalgebra $\mathcal{O}_q(M^\lambda_{m,n}(\mathbb{K}))$ of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$, and that $\mathcal{C}$ is the Cauchon diagram of P.

4.4.1 The Cauchon graph

Recall that the Cauchon diagram $\mathcal{C}$ is defined on the Young diagram Y_λ and that we denote by $W_\mathcal{C}$ and $B_\mathcal{C}$ the set of squares of Y_λ which are coloured white and black respectively in $\mathcal{C}$.

Notation 4.4.1. *If $(i,j) \in W_\mathcal{C}$ has at least one white square to its left, then denote by (i,j^-) the first white square to its left. If $(i,j) \in W_\mathcal{C}$ has at least one white square below it, then denote by (i^+,j) the first white square below it.*

The following is a generalisation of the notion of a Cauchon graph which appears in [7, Definition 3.1]; in [7], Casteels considers Cauchon diagrams on rectangular Young diagrams only, whereas we have a Cauchon diagram $\mathcal{C}$ on the not-necessarily-rectangular Young diagram Y_λ.

Definition 4.4.2 (cf. Definition 3.1 of [7])**.** We associate to $\mathcal{C}$ an edge-weighted directed graph with weights in the quantum torus $\mathcal{B}$ called the *Cauchon graph* of $\mathcal{C}$, which we denote by $\mathcal{G}_\mathcal{C}$ and which we define as follows: When $R = \{r_1, \dots, r_c\}$ and $C = \{c_1, \dots, c_d\}$, the set of vertices of $\mathcal{C}$ is $W_\mathcal{C} \sqcup R \sqcup C$. The set of weighted directed edges is constructed as follows:

(i) For every $i \in [\![1, c]\!]$ such that there is a white square in row i, put a directed edge from r_i to the right-most white square in row i, say (i,p). Give this edge weight $t_{i,p} \in \mathcal{B}$.

(ii) For every $j \in [\![1, d]\!]$ such that there is a white square in column j, put a directed edge from the bottom-most white square in column j to c_j and give this edge weight $1 \in \mathcal{B}$.

(iii) For every $(i,j) \in W_\mathcal{C}$ such that (i,j^-) exists, put a directed edge from (i,j) to (i,j^-) and give this edge weight $t_{i,j}^{-1} t_{i,j^-} \in \mathcal{B}$.

(iv) For every $(i,j) \in W_\mathcal{C}$ such that (i^+,j) exists, put a directed edge from (i,j) to (i^+,j) and give this edge weight $1 \in \mathcal{B}$.

Notation 4.4.3.

- *For us, "path" and "edge" shall always mean "directed path" and "directed edge" respectively.*
- *Let v and v' be vertices of $\mathcal{G}_{\mathcal{C}}$. There is clearly at most one edge from v to v' and if it exists, we denote it by (v, v').*
- *We denote the weight of an edge e of $\mathcal{G}_{\mathcal{C}}$ by $w(e)$.*
- *Suppose that $(i, j), (i, j') \in W_{\mathcal{C}}$ and that there is an edge $e = ((i, j), (i, j'))$ in $\mathcal{G}_{\mathcal{C}}$ (notice that this forces $j > j'$). Then we set* $\mathrm{row}(e) = i$, $\mathrm{col}_1(e) = j$, $\mathrm{col}_2(e) = j'$, *and* $\mathrm{col}(e) = \{j, j'\}$.
- *If $v_0, v_1, \ldots, v_k$ are vertices of $\mathcal{G}_{\mathcal{C}}$ such that the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ exist, then we write $(v_0, v_1, \ldots, v_k)$ for the path from v_0 to v_k given by the concatenation of the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$.*
- *For vertices v and v' of $\mathcal{G}_{\mathcal{C}}$, we write $P \colon v \implies v'$ to mean that P is a path from v to v'.*

Definition 4.4.4. By the *weight* of a path $(v_0, v_1, \ldots, v_k)$ in $\mathcal{G}_{\mathcal{C}}$, we mean the ordered product

$$w(v_0, v_1)w(v_1, v_2) \cdots w(v_{k-1}, v_k)$$

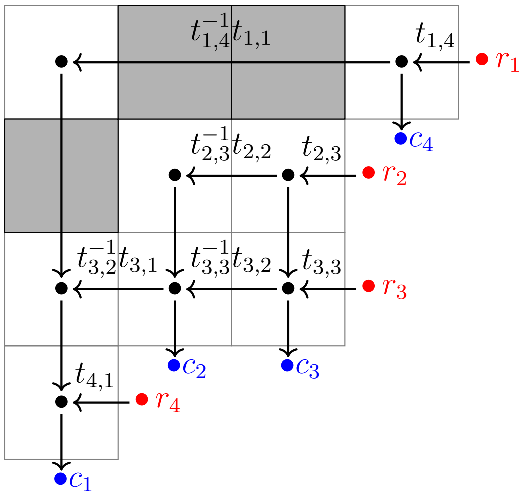
of the weights of its edges. We denote the weight of a path P in $\mathcal{G}_{\mathcal{C}}$ by $w(P)$.

Definition 4.4.5. An edge or path in $\mathcal{G}_{\mathcal{C}}$ is called *internal* if its beginning and end vertices belong to $W_{\mathcal{C}}$.

We embed the Cauchon graph $\mathcal{G}_{\mathcal{C}}$ in the plane in the following way: Place a vertex in each white square of $\mathcal{C}$, then place a vertex at the bottom of each column of $\mathcal{C}$ and to the right of each row of $\mathcal{C}$. For each $i \in [\![1, c]\!]$, assign the label of r_i to the vertex to the right of the i^{th} row of $\mathcal{C}$ and and for each $j \in [\![1, d]\!]$, assign the label c_j to the vertex at the bottom of the j^{th} column of $\mathcal{C}$. We shall always assume that Cauchon graphs are embedded in the plane in this way, which will allow us to use terms like horizontal, vertical, left, right, etc in reference to vertices and edges of a Cauchon graph, such as in the following remark:

Remark 4.4.6. *Since vertical edges in $\mathcal{G}_\mathcal{C}$ have weight* 1*, only horizontal edges contribute to the weight of any path in $\mathcal{G}_\mathcal{C}$. We shall often use this fact without explicit mention.*

Example 4.4.7. *Let $c = d = 4$, let $\lambda = (4, 3, 3, 1)$, and let $\mathcal{C}$ be the Cauchon diagram on Y_λ from Example* (4.3.5)*. Below is the Cauchon graph $\mathcal{G}_\mathcal{C}$ superimposed onto $\mathcal{C}$:*



Remark 4.4.8. *We shall always superimpose Cauchon graphs onto their Cauchon diagrams as in Example 4.4.7.*

Proposition 4.4.9 (cf. Proposition 3.3 of [7])**.** *The Cauchon graph $\mathcal{G}_C$ has the following properties:*

(1) $\mathcal{G}_\mathcal{C}$ is acyclic i.e. has no directed cycles.

(2) The embedding of the Cauchon graph $\mathcal{G}_\mathcal{C}$ in the plane described above is a planar embedding i.e. all edge crossings occur at vertices.

(3) An internal horizontal path $P : (i, j_2) \implies (i, j_1)$ has weight $t_{i,j_2}^{-1}t_{i,j_1}$.

(4) A path $r_i \implies (i,j)$ *beginning at a row vertex and consisting solely of horizontal edges has weight* $t_{i,j}$.

Proof. (1) Because all edges are directed leftwards or downwards, the graph $\mathcal{G}_{\mathcal{C}}$ cannot have a directed cycle.

(2) If two edges cross, then one edge must be vertical and the other horizontal. Let a vertical edge $e_1 = ((i_1, j), (i_2, j))$ cross a horizontal edge $e_2 = ((i, j_2), (i, j_1))$ at a black square (i, j) of the Cauchon diagram $\mathcal{C}$. The black square (i, j) has the white square (i_1, j) above it and the white square (i, j_1) to its left, contradicting the definition of a Cauchon diagram. It follows that the square (i, j) must be white and that the edges e_1 and e_2 cross at the vertex (i, j).

(3) If the path P consists of a single edge, then the result follows from the definition of the Cauchon graph $\mathcal{G}_{\mathcal{C}}$. Suppose that the path P consists of $n > 1$ edges and that the desired result holds for all internal horizontal paths in $\mathcal{G}_{\mathcal{C}}$ consisting of fewer than n edges. Let (i, k) be an internal vertex of P. When P' and P'' are the horizontal paths given by $P' \colon (i, j_2) \implies (i, k)$ and $P'' \colon (i, k) \implies (i, j_1)$, we have $P = P'P''$. Now the inductive hypothesis gives $w(P) = w(P')w(P'') = t_{i,j_2}^{-1} t_{i,k} t_{i,k}^{-1} t_{i,j_1} = t_{i,j_2}^{-1} t_{i,j_1}$.

(4) This follows from part (3) and the definition of the Cauchon graph.

■

4.4.2 Commutation relations between weights of paths

Lemma 4.4.10 (cf. Lemma 3.4 of [7])**.** *Let e and f be distinct internal horizontal edges in $\mathcal{G}_{\mathcal{C}}$ such that* $\mathrm{row}(f) \leq \mathrm{row}(e)$.

(1) If $\mathrm{col}(e) \cap \mathrm{col}(f) = \emptyset$, *then* $w(f)w(e) = w(e)w(f)$.

(2) Suppose that $|\mathrm{col}(e) \cap \mathrm{col}(f)| = 1$.

(i) if $\mathrm{col}_1(e) = \mathrm{col}_1(f)$ *or* $\mathrm{col}_2(e) = \mathrm{col}_2(f)$, *then* $w(f)w(e) = qw(e)w(f)$;

(ii) if $\mathrm{col}_1(e) = \mathrm{col}_2(f)$ *or* $\mathrm{col}_2(e) = \mathrm{col}_1(f)$ *and* $\mathrm{row}(e) \neq \mathrm{row}(f)$*, then* $w(f)w(e) = q^{-1}w(e)w(f)$*;*

(iii) if $\mathrm{col}_2(e) = \mathrm{col}_1(f)$ *and* $\mathrm{row}(e) = \mathrm{row}(f)$*, then* $w(f)w(e) = qw(e)w(f)$*.*

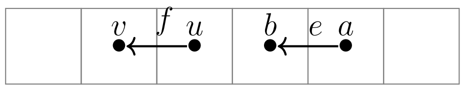
(3) If $|\,\mathrm{col}(e) \cap \mathrm{col}(f)| = 2$*, then* $w(f)w(e) = q^2w(e)w(f)$*.*

Proof. Notice that if d is any internal horizontal edge in the graph $\mathcal{G}_\mathcal{C}$ and $col_2(d) < j < \mathrm{col}_1(d)$, then the square $(\mathrm{row}(d), j)$ is a black square in $\mathcal{C}$ which has the white square $(\mathrm{row}(d), \mathrm{col}_2(d))$ to its left, so that for all $i \leq \mathrm{row}(d)$, the square (i, j) is black.

Let a, b, u, v be the vertices of $\mathcal{G}_\mathcal{C}$ such that $e = (a, b)$ and $f = (u, v)$.

(1) Suppose that $\mathrm{col}(e) \cap \mathrm{col}(f) = \emptyset$. If $\mathrm{row}(e) \neq \mathrm{row}(f)$, then the result follows immediately from the relations (4.19) because t_a and t_b commute with t_u and t_v.

Suppose that $\mathrm{row}(e) = \mathrm{row}(f)$ and notice that we may assume without loss of generality that u and v lie west of a and b; the following diagram illustrates the situation:

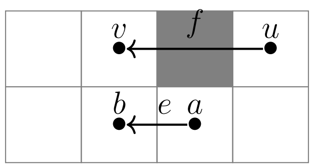


The relations (4.19) now give $t_ut_b = qt_bt_u$, $t_ut_a = qt_at_u$, $t_vt_b = qt_bt_v$, and $t_vt_a = qt_at_v$. Hence

$$w(e)w(f) = t_a^{-1}t_bt_u^{-1}t_v = qq^{-1}t_a^{-1}t_u^{-1}t_vt_b = qq^{-1}qq^{-1}t_u^{-1}t_vt_a^{-1}t_b = w(f)w(e).$$

(2) Suppose that $|\,\mathrm{col}(e) \cap \mathrm{col}(f)| = 1$.

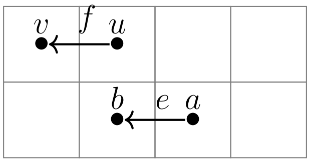
(i) Suppose that $\mathrm{col}_2(e) = \mathrm{col}_2(f)$ (the case where $\mathrm{col}_1(e) = \mathrm{col}_1(f)$ is similar). The following diagram illustrates the situation:



The relations (4.19) now give $t_a t_u = t_u t_a$, $t_a t_v = t_v t_a$, $t_b t_u = t_u t_b$, and $t_v t_b = q t_b t_v$. Hence

$$w(e)w(f) = t_a^{-1} t_b t_u^{-1} t_v = q^{-1} t_u^{-1} t_v t_a^{-1} t_b = q^{-1} w(f) w(e).$$

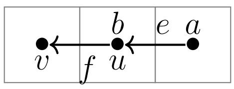
(ii) Suppose that $\operatorname{row}(e) \neq \operatorname{row}(f)$ and that $\operatorname{col}_1(f) = \operatorname{col}_2(e)$ (the case where $\operatorname{row}(e) \neq \operatorname{row}(f)$ and $\operatorname{col}_2(f) = \operatorname{col}_1(e)$ is similar). The following diagram illustrates the situation:



The relations (4.19) now give $t_a t_u = t_u t_a$, $t_a t_v = t_v t_a$, $t_b t_v = t_v t_b$, and $t_u t_b = q t_b t_u$. It follows that

$$w(e)w(f) = t_a^{-1} t_b t_u^{-1} t_v = q t_u^{-1} t_v t_a^{-1} t_b = q w(f) w(e).$$

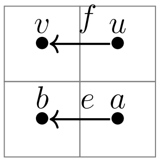
(iii) Suppose that $\operatorname{row}(e) = \operatorname{row}(f)$ and that $\operatorname{col}_2(e) = \operatorname{col}_1(f)$. Then the end vertex of e is the starting vertex of f i.e. $b = u$. The following diagram illustrates the situation:



Now the relations (4.19) give $t_v t_b = q t_b t_v$, $t_v t_a = q t_a t_v$, and $t_b t_a = q t_a t_b$. Hence

$$w(f)w(e) = t_b^{-1} t_v t_a^{-1} t_b = q^2 t_v t_a^{-1} = q t_a^{-1} t_v = q t_a^{-1} t_b t_b^{-1} t_v = q w(e) w(f).$$

(3) Suppose that $|\operatorname{col}(e) \cap \operatorname{col}(f)| = 2$. The following diagram illustrates the situation:



The relations (4.19) show that $t_a t_v = t_v t_a$, $t_b t_u = t_u t_b$, $t_v t_b = q t_b t_v$, and $t_u t_a = q t_a t_u$. It follows that

$$w(e)w(f) = t_a^{-1} t_b t_u^{-1} t_v = q^{-1} t_a^{-1} t_u^{-1} t_v t_b = q^{-2} t_u^{-1} t_v t_a^{-1} t_b = q^{-2} w(f)w(e).$$

■

Remark 4.4.11. *The reader may notice that part (2) of Lemma 4.4.10 differs from part 2 of [7, Lemma 3.4]. This is to clear up a slight ambiguity in part 2(ii) of [7, Lemma 3.4], namely that in the case where* $\mathrm{row}(e) = \mathrm{row}(f)$*, part 2(ii) of [7, Lemma 3.4] only holds if* e *begins where* f *ends.*

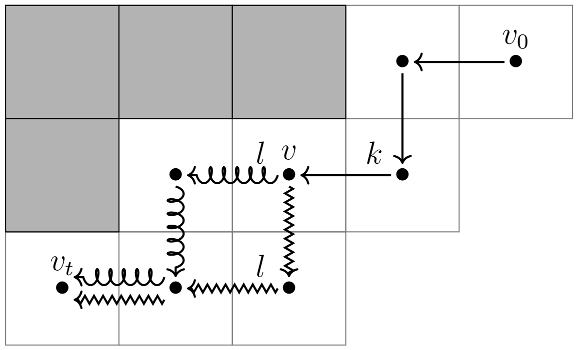
Lemma 4.4.12 (cf. Lemma 3.5 of [7])**.** *Let* $K\colon v_0 \implies v$ *and* $L\colon v \implies v_t$ *be internal paths in* $\mathcal{G}_{\mathcal{C}}$*.*

(1) If either K *or* L *contains only vertical edges, then* $w(K)w(L) = w(L)w(K)$*.*

(2) If K *contains a horizontal edge and* L *contains a horizontal edge, then* $w(K)w(L) = q^{-1}w(L)w(K)$*.*

Proof. (1) This follows immediately from the fact that vertical edges in $\mathcal{C}_{\mathcal{G}}$ have weight 1.

(2) Since all vertical edges have weight one, only the horizontal edges of K and L contribute to their weights. Let k be the last horizontal edge in K and let l be the first horizontal edge in L.



This diagram illustrates the situation. A possible example of K is drawn in straight lines and two possible examples of L are drawn in zigzagging and coiling lines.

The horizontal edges of L are always south-west of those in K. By Lemma 4.4.10(1), all edges in $K \setminus \{k\}$ commute with all edges in L and all edges in $L \setminus \{l\}$ commute with all edges in K. By Lemma 4.4.10(2), we have $w(k)w(l) = q^{-1}w(l)w(k)$. Now

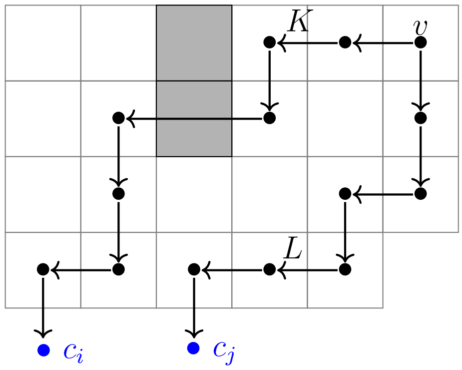
$$\begin{aligned} w(K)w(L) &= w(K \setminus \{k\})w(k)w(l)w(L \setminus \{l\}) \\ &= q^{-1}w(K \setminus \{k\})w(l)w(k)w(L \setminus \{l\}) \\ &= q^{-1}w(l)w(L \setminus \{l\})w(K \setminus \{k\})w(k) \\ &= q^{-1}w(L)w(K). \end{aligned}$$

■

Lemma 4.4.13 (cf. Lemma 3.6 of [7]). *Let $K\colon v \implies c_i$ and $L\colon v \implies c_j$ be two paths in $\mathcal{G}_\mathcal{C}$ which share their initial vertex and no other vertex. Let K be the path that starts with a horizontal edge and let L be the path that starts with a vertical edge.*

(1) If L consists only of vertical edges, then $w(K)w(L) = w(L)w(K)$.

(2) If L has a horizontal edge then $w(K)w(L) = qw(L)w(K)$.



Possible examples of K and L

Proof. (1) This follows immediately from the fact that vertical edges have weight 1.

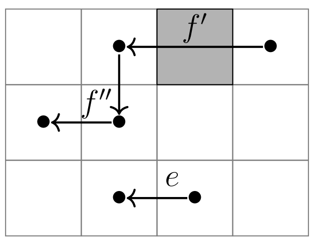
(2) Suppose that L has a horizontal edge. By the beginning of the proof of Lemma 4.4.10, no vertex of K lies (with respect to column coordinates) between the vertices of a horizontal edge of L.

Claim: If e is any horizontal edge of L except the first horizontal edge of L, then $w(e)w(K) = w(K)w(e)$.

There are four possibilities for e:

Case (i): No vertex in K shares a column coordinate with either vertex of e. In this case, Lemma 4.4.10(1) then shows that the weights of the horizontal edges of K commute with $w(e)$, so that $w(e)$ commutes with $w(K)$.

Case (ii): There are two distinct horizontal edges f' and f'' of K such that $|\operatorname{col}(e) \cap \operatorname{col}(f')| = |\operatorname{col}(e) \cap \operatorname{col}(f'')| = 1$, $\operatorname{col}_2(f') = \operatorname{col}_1(f'') = \operatorname{col}_2(e)$, and $\operatorname{col}(e) \cap \operatorname{col}(f) = \emptyset$ for all other edges f of K.

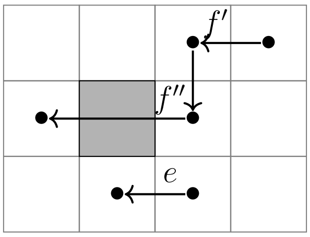


Possible examples of f', f'', e.

In this case, Lemma 4.4.10(2) shows that $w(f')w(f'')w(e) = qq^{-1}w(e)w(f')w(f'') = w(e)w(f')w(f'')$. Now with Lemma 4.4.10(1), we can conclude that $w(e)w(K) =$

$w(K)w(e)$.

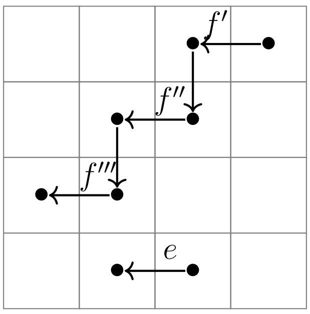
Case (iii): There are two distinct horizontal edges f' and f'' of K such that $|\operatorname{col}(e) \cap \operatorname{col}(f')| = |\operatorname{col}(e) \cap \operatorname{col}(f'')| = 1$, $\operatorname{col}_2(f') = \operatorname{col}_1(f'') = \operatorname{col}_1(e)$, and $\operatorname{col}(e) \cap \operatorname{col}(f) = \emptyset$ for all other edges of K.



Possible examples of f', f'', e.

This case is similar to Case (ii).

Case (iv): There are edges f', f'', f'' of K such that $|\operatorname{col}(e) \cap \operatorname{col}(f'')| = 2$, $\operatorname{col}_2(f') = \operatorname{col}_1(e)$, and $\operatorname{col}_1(f''') = \operatorname{col}_2(e)$.



Possible examples of f', f'', f''', e.

By Lemma 4.4.10 parts (2) and (3), we have

$$w(f')w(f'')w(f''')w(e) = q^{-1}q^2q^{-1}w(e)w(f')w(f'')w(f''') = w(e)w(f')w(f'')w(f''')$$

and now with Lemma 4.4.10(1), we can conclude that $w(e)w(K) = w(K)w(e)$. This establishes the claim that if e is any horizontal edge of L except the first horizontal edge of L, then $w(e)w(K) = w(K)w(e)$.

Let us turn now to the first horizontal edge e_1 of L.

Claim: $w(K)w(e_1) = qw(e_1)w(K)$.

Let us denote by f_1 the first horizontal edge of K. There are two cases to consider:

(i) Let us suppose that $\mathrm{col}_2(f_1) < \mathrm{col}_2(e_1)$. Then Lemma 4.4.10(2) gives $w(f_1)w(e_1) = qw(e_1)w(f_1)$ and Lemma 4.4.10(1) allows us to conclude that $w(K)w(e_1) = qw(e_1)w(K)$.

(ii) Let us suppose that $\mathrm{col}_2(f_1) = \mathrm{col}_2(e_1)$. Then the second horizontal edge f_2 of K satisfies $\mathrm{col}_1(f_2) = \mathrm{col}_2(f_1) = \mathrm{col}_2(e_1)$. Then by Lemma 4.4.10 parts (2) and (3), we have $w(f_1)w(f_2)w(e_1) = qw(e_1)w(f_1)w(f_2)$ and now with Lemma 4.4.10(1), we can conclude that $w(K)w(e_1) = qw(e_1)w(K)$. This completes the proof of the claim.

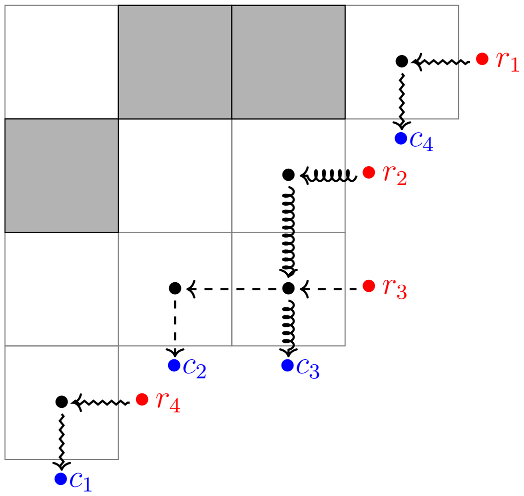
The result now follows from the two claims which we have proven, namely that if e is any horizontal edge of L except the first horizontal edge of L, then $w(K)w(e) = w(e)w(K)$ and that if e_1 is the first horizontal edge of L, then $w(K)w(e_1) = qw(e_1)w(K)$.

■

4.4.3 Path systems

Definition 4.4.14. Suppose that $I = \{i_1 < \cdots < i_t\} \subseteq [\![1, c]\!]$ and $J = \{j_1 < \cdots < j_t\} \subseteq [\![1, d]\!]$. An $R_{(I,J)}$*-path system* in $\mathcal{G}_{\mathcal{C}}$ is a collection $\mathcal{P} = (P_1, \ldots, P_t)$ of paths in $\mathcal{G}_{\mathcal{C}}$ starting respectively at the row vertices $r_{i_1}, \ldots, r_{i_t}$ and ending respectively at the column vertices $c_{j_{\sigma_{\mathcal{P}}(1)}}, \ldots, c_{j_{\sigma_{\mathcal{P}}(t)}}$ for some permutation $\sigma_{\mathcal{P}} \in S_t$ (called the permutation of the path system $\mathcal{P}$). The path system $\mathcal{P}$ is called *vertex-disjoint* if no two of its paths share a vertex. The weight of the path system $\mathcal{P} = (P_1, \ldots, P_t)$ is defined simply as the ordered product $w(P_1) \cdots w(P_t)$ of the weights of the paths $P_1, \ldots, P_t$.

Example 4.4.15. *Let $c = d = 4$ and let $\lambda = (4, 3, 3, 1)$. Below are the Cauchon diagram on Y_λ from Example 4.4.7, with a vertex-disjoint $R_{(\{1,4\},\{1,4\})}$-path system marked in zigzagging lines and a non vertex-disjoint $R_{(\{2,3\},\{2,3\})}$-path system marked in coiled and dashed lines; each path system has permutation $(1\ 2) \in S_2$.*



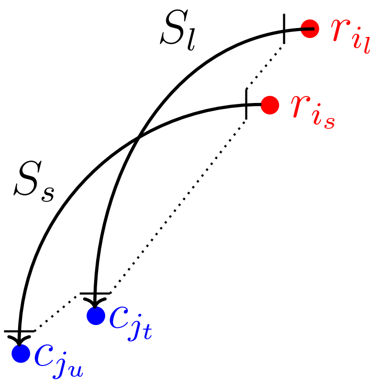
Because, unlike Casteels in [7], we deal here with Cauchon diagrams on not-necessarily-rectangular Young diagrams, we shall need the following lemma which does not appear in [7].

Lemma 4.4.16. *Suppose that $I = \{i_1 < \cdots < i_t\} \subseteq [\![1, c]\!]$ and $J = \{j_1 < \cdots < j_t\} \subseteq [\![1, d]\!]$. Then all vertex-disjoint $R_{(I,J)}$-path systems in $\mathcal{G}_{\mathcal{C}}$ have the same permutation.*

Proof. The proof is by induction on t. The case $t = 1$ is obvious; so, suppose that $t > 1$ and that the result holds for vertex disjoint path systems of smaller size than t.

Choose s as large as possible such that the Young diagram Y_λ has a square in the (i_s, j_t) position. Let $\mathcal{S}$ denote any vertex-disjoint (R_I, C_J)-path system. We claim that the path S_s in $\mathcal{S}$ starting at r_{i_s} must finish at c_{j_t}.

Suppose, for a contradiction, that the path S_s in $\mathcal{S}$ starting at r_{i_s} does not end at c_{j_t}. Then let l be such that the path S_l in $\mathcal{S}$ starting at r_{i_l} ends at c_{j_t}, forcing $l < s$. Suppose that S_s ends at c_{j_u}, and note that $u < t$. Then, the paths $S_s \colon r_{i_s} \implies c_{j_u}$ and $S_l \colon r_{i_l} \implies c_{j_t}$ must cross, as r_{i_s} is to the right of the path S_l and c_{j_u} is to the left of S_l; this crossing must occur at a vertex by Proposition 4.4.9(2). This gives the desired contradiction and proves the claim that the path S_s in $\mathcal{S}$ starting at r_{i_s} must finish at c_{j_t}.

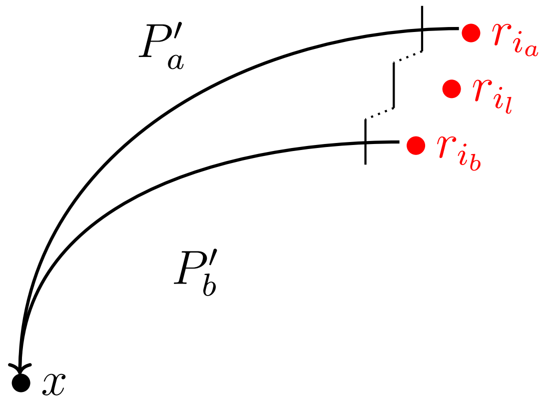


Consider any two vertex-disjoint (R_I, C_J)-path systems $\mathcal{P} = (P_1, \ldots, P_t)$ and $\mathcal{Q} = (Q_1, \ldots, Q_t)$ in $\mathcal{G}_{\mathcal{C}}$. The paths P_s and Q_s which start at r_{i_s} must finish at c_{j_t}. Now $\mathcal{P} \setminus \{P_s\}$ and $\mathcal{Q} \setminus \{Q_s\}$ are two vertex-disjoint $R_{(I \setminus \{i_s\}, J \setminus \{j_t\})}$-path systems and hence must have the same permutation by the induction hypothesis. The result follows immediately. ∎

Lemma 4.4.17 (cf. Lemma 4.2 of [7]). *Suppose that $I = \{i_1 < \cdots < i_t\} \subseteq [\![1, c]\!]$ and $J = \{j_1 < \cdots < j_t\} \subseteq [\![1, d]\!]$. If $\mathcal{P} = (P_1, \ldots, P_t)$ is a non-vertex-disjoint $R_{(I,J)}$-path system in $\mathcal{G}_{\mathcal{C}}$, then there exists $s \in [\![1, t-1]\!]$ such that P_s and P_{s+1} share a vertex.*

Proof. Let $d = \min\{|a - b| \mid a \neq b \text{ and } P_a \text{ and } P_b \text{ share a vertex}\}$ and suppose that $d > 1$. Let $a < b$ be such that $|a - b| = d$ and P_a shares a vertex with P_b. Let x be the first vertex which is common to P_a and P_b and consider the subpaths $P'_a \colon r_{i_a} \implies x$ of P_a and $P'_b \colon r_{i_b} \implies x$ of P_b.

Since $d > 1$, there exists $\ell \in [\![1, t]\!]$ such that $a < \ell < b$. The path $P_\ell \in \mathcal{P}$ which starts at r_{i_ℓ} must intersect either P'_a or P'_b and this intersection must be at a vertex of $\mathcal{G}_{\mathcal{C}}$ by Proposition 4.4.9(2), contradicting the minimality of d.



■

4.4.4 Path matrices and their quantum minors

Definition 4.4.18. Define the *path matrix* $M_{\mathcal{C}} = (M_{\mathcal{C}}[i,j])_{(i,j)\in[\![1,c]\!]\times[\![1,d]\!]}$ of $\mathcal{C}$ to be the $c \times d$ matrix[3] with entries from $\mathcal{B}$ such that for each $(i,j) \in [\![1,c]\!] \times [\![1,d]\!]$, $M_{\mathcal{C}}[i,j]$ is the sum of the weights of all paths from r_i to c_j in the Cauchon graph $\mathcal{G}_{\mathcal{C}}$ of $\mathcal{C}$. For $I = \{i_1 < \ldots < i_t\} \subseteq [\![1,c]\!]$ and $J = \{j_1 < \ldots < j_t\} \subseteq [\![1,d]\!]$, we define the quantum minor $[I \mid J]$ of $M_{\mathcal{C}}$ as follows

$$[I \mid J] = \sum_{\sigma\in S_t} (-q)^{\ell(\sigma)} M_{\mathcal{C}}[i_1, j_{\sigma(1)}] \cdots M_{\mathcal{C}}[i_t, j_{\sigma(t)}].$$

Naively adapting [7, Theorem 4.4] would suggest that for any $I \subseteq [\![1,c]\!]$ and $J \subseteq [\![1,d]\!]$ which have the same cardinality, the quantum minor $[I \mid J]$ of $M_{\mathcal{C}}$ is equal to the sum of the weights of all vertex-disjoint $R_{(I,J)}$-path systems in $\mathcal{G}_{\mathcal{C}}$. However our situation is slightly more complicated because (unlike when $\mathcal{C}$ is defined on a rectangular Young diagram) when $\mathcal{C}$ is defined on a generic Young diagram, there can be vertex-disjoint path systems whose permutation is not identity. Theorem 4.4.19 shows that the appropriate adjustment is to scale by $(-q)^{\ell(\sigma_{(I,J)})}$, where $\sigma_{(I,J)}$ is the permutation of every vertex-disjoint $R_{(I,J)}$-path system in $\mathcal{G}_{\mathcal{C}}$ (see Lemma 4.4.16).

[3]Recall that $\lambda = (\lambda_1, \ldots, \lambda_c)$ is a partition with $d = \lambda_1 \geq \cdots \geq \lambda_c \geq 1$, where $c \leq m$ and $d \leq n$.

Theorem 4.4.19 (cf. Theorem 4.4 of [7])**.** *Let $I \subseteq [\![1,c]\!]$ and $J \subseteq [\![1,d]\!]$ have the same cardinality and let $\sigma_{(I,J)}$ be the permutation of all vertex-disjoint (R_I, C_J)-path systems (see Lemma 4.4.16). Then the quantum minor $[I \mid J]$ of $M_{\mathcal{C}}$ is given by*

$$[I \mid J] = (-q)^{\ell(\sigma_{(I,J)})} \sum_{\mathcal{P}} w(\mathcal{P}), \tag{4.20}$$

where $\mathcal{P}$ runs over all vertex-disjoint (R_I, C_J)-path systems in $\mathcal{G}_{\mathcal{C}}$. In particular, if there are no vertex-disjoint (R_I, C_J)-path systems in $\mathcal{G}_{\mathcal{C}}$, then $[I \mid J] = 0$.

Proof. For ease of notation, let us take $I = J = \{1, \dots, t\}$ (the proof for general I and J is the same but the notation is more unwieldy). By the definition of the path matrix, we have

$$\begin{aligned}
[I \mid J] &= \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} M_{\mathcal{C}}[1, \sigma(1)] \cdots M_{\mathcal{C}}[t, \sigma(t)] \\
&= \sum_{\sigma \in S_t} (-q)^{\ell(\sigma)} \left(\sum_{P_1 \colon r_1 \implies c_{\sigma(1)}} w(P_1) \right) \left(\sum_{P_2 \colon r_2 \implies c_{\sigma(2)}} w(P_2) \right) \cdots \left(\sum_{P_t \colon r_t \implies c_{\sigma(t)}} w(P_t) \right) \\
&= \sum_{\mathcal{P}} (-q)^{\ell(\sigma_{\mathcal{P}})} w(\mathcal{P}),
\end{aligned}$$

where, in the final sum, $\mathcal{P}$ runs over all (R_I, C_J)-path systems.

When $\mathcal{N}$ is the set of non-vertex-disjoint (R_I, C_J)-path systems, we claim that

$$\sum_{\mathcal{P} \in \mathcal{N}} (-q)^{\ell(\sigma_{\mathcal{P}})} w(\mathcal{P}) = 0. \tag{4.21}$$

To show that (4.21) holds, we construct a fixed-point-free involution $\pi \colon \mathcal{N} \to \mathcal{N}$ which satisfies

$$(-q)^{\ell(\sigma_{\mathcal{P}})} w(\mathcal{P}) = -(-q)^{\ell(\sigma_{\pi(\mathcal{P})})} w(\pi(\mathcal{P})) \tag{4.22}$$

for every $\mathcal{P} \in \mathcal{N}$.

Let $\mathcal{P} = (P_1, \ldots, P_t) \in \mathcal{N}$ and let i be minimal such that P_i and P_{i+1} share a vertex (this i exists by Lemma 4.4.17). Let x be the last vertex shared by P_i and P_{i+1} and let $K_1 \colon r_i \implies x$ and $L_1 \colon x \implies c_{\sigma_{\mathcal{P}}(i)}$ be subpaths of P_i so that $P_i = K_1 L_1$; define K_2 and L_2 from P_{i+1} similarly. For any $j \in [\![1, t]\!]$, set

$$\pi(P_j) = \begin{cases} K_1 L_2 & j = i \\ K_2 L_1 & j = i+1 \\ P_j & \text{otherwise} \end{cases}$$

(see Example 4.4.20 for an example of the action of π). Define $\pi(\mathcal{P})$ to be the (R_I, C_J)-path system $(\pi(P_1), \ldots, \pi(P_t))$. This gives us a map $\pi \colon \mathcal{N} \to \mathcal{N}$ which is clearly an involution and which clearly has no fixed points. In order to prove (4.22), we may assume without loss of generality that $\sigma_{\mathcal{P}}(i) < \sigma_{\mathcal{P}}(i+1)$, so that $\sigma_{\pi(\mathcal{P})} = \sigma_{\mathcal{P}}(i\ i+1)$ satisfies $\ell(\sigma_{\pi(\mathcal{P})}) = \ell(\sigma_{\mathcal{P}}) + 1$. Notice that because x is the last vertex shared by P_i and P_{i+1}, the assumption $\sigma_{\mathcal{P}}(i) < \sigma_{\mathcal{P}}(i+1)$ forces L_1 to start with a horizontal edge.

We claim that $w(P_i)w(P_{i+1}) = qw(\pi(P_i))w(\pi(P_{i+1}))$. There are two cases to consider:

(i) Suppose that L_2 has a horizontal edge. Then

$$\begin{aligned} w(P_i)w(P_{i+1}) &= w(K_1)w(L_1)w(K_2)w(L_2) \\ &= qw(K_1)w(K_2)w(L_1)w(L_2) \qquad \text{(Lemma 4.4.12)} \\ &= q^2 w(K_1)w(K_2)w(L_2)w(L_1) \qquad \text{(Lemma 4.4.13)} \\ &= q^2 q^{-1} w(K_1)w(L_2)w(K_2)w(L_1) \qquad \text{(Lemma 4.4.12)} \\ &= qw(\pi(P_i))w(\pi(P_{i+1})). \end{aligned}$$

(ii) Suppose that L_2 consists of vertical edges. Then $w(L_2) = 1$ and we have

$$\begin{aligned} w(P_i)w(P_{i+1}) &= w(K_1)w(L_1)w(K_2)w(L_2) \\ &= w(K_1)w(L_2)w(L_1)w(K_2) \\ &= qw(K_1)w(L_2)w(K_2)w(L_1) \quad \text{(Lemma 4.4.12)} \\ &= qw(\pi(P_i))w(\pi(P_{i+1})). \end{aligned}$$

Hence we have

$$\begin{aligned} w(\mathcal{P}) &= \left(\prod_{j=1}^{i-1} w(P_j)\right) w(P_i)w(P_{i+1}) \left(\prod_{j=i+2}^{t} w(P_j)\right) \\ &= \left(\prod_{j=1}^{i-1} w(\pi(P_j))\right) qw(\pi(P_i))w(\pi(P_{i+1})) \left(\prod_{j=i+2}^{t} w(\pi(P_j))\right) \\ &= qw(\pi(\mathcal{P})). \end{aligned}$$

Now

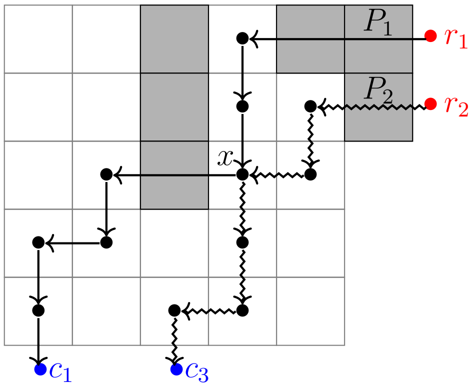
$$\begin{aligned} (-q)^{\ell(\sigma_{\mathcal{P}})}w(\mathcal{P}) &= (-q)^{\ell(\sigma_{\mathcal{P}})}qw(\pi(\mathcal{P})) \\ &= -(-q)^{\ell(\sigma_{\mathcal{P}})+1}w(\pi(\mathcal{P})) \\ &= -(-q)^{\ell(\sigma_{\pi(\mathcal{P})})}w(\pi(\mathcal{P})), \end{aligned}$$

proving that $\pi\colon \mathcal{N} \to \mathcal{N}$ satisfies (4.22); the claim (4.21) follows immediately. Moreover, the claim (4.21) immediately gives

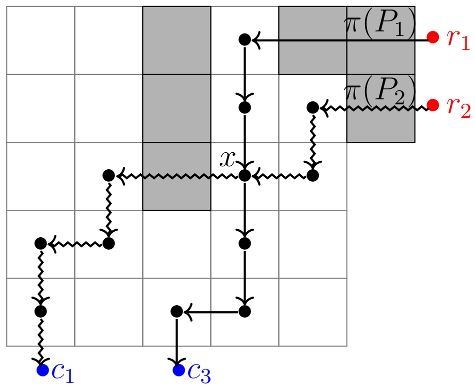
$$[I \mid J] = \sum_{\mathcal{P}} (-q)^{\ell(\sigma_{\mathcal{P}})}w(\mathcal{P}),$$

where $\mathcal{P}$ runs over all vertex-disjoint (R_I, C_J)-path systems in $\mathcal{G}_{\mathcal{C}}$. Lemma 4.4.16 shows that $\sigma_{\mathcal{P}} = \sigma_{(I,J)}$ for all such $\mathcal{P}$, giving the result. ∎

Example 4.4.20. *Below left is an example of a non-vertex-disjoint $R_{(\{1,2\},\{1,3\})}$-path system $\mathcal{P} = (P_1, P_2)$ on the Cauchon graph of a Cauchon diagram. Below right is the non-vertex-disjoint $R_{(\{1,2\},\{1,3\})}$-path system $\pi(\mathcal{P}) = (\pi(P_1), \pi(P_2))$.*

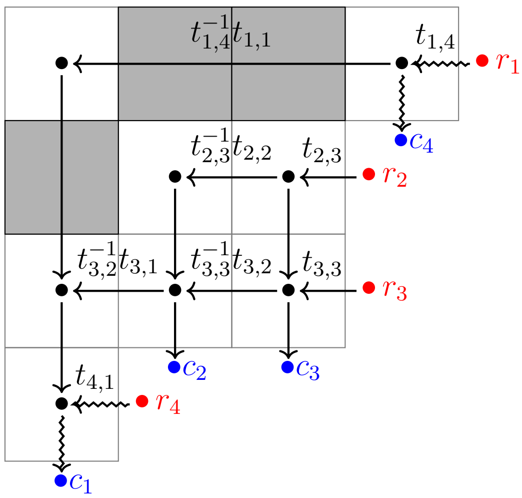


P_1 marked with straight lines. P_2 marked with zigzagging lines.



$\pi(P_1)$ marked with straight lines. $\pi(P_2)$ marked with zigzagging lines.

Example 4.4.21. *Let $c = d = 4$, let $\lambda = (4, 3, 3, 1)$, and let $\mathcal{C}$ be the Cauchon diagram on Y_λ from Example* (4.3.5)*. Below are $\mathcal{C}$ and $\mathcal{G}_\mathcal{C}$:*



The only vertex-disjoint $R_{(\{1,4\},\{1,4\})}$-path system in $\mathcal{G}_\mathcal{C}$ is that which is marked with zigzagging lines above; this path system has weight $t_{1,4}t_{4,1}$ and has permutation $(1, 2) \in S_2$, whose length is 1. Theorem 4.4.19 predicts that the quantum minor $[14 \mid 14]$ of $M_\mathcal{C}$ is $-qt_{1,4}t_{4,1}$. Computing the quantum minor $[14 \mid 14]$ of $M_\mathcal{C}$ directly, we indeed get

$$[14 \mid 14] = M_\mathcal{C}[1,1]M_\mathcal{C}[4,4] - qM_\mathcal{C}[4,1]M_\mathcal{C}[1,4] = 0 - qt_{4,1}t_{1,4} = -qt_{1,4}t_{4,1}.$$

There are no vertex-disjoint $R_{(\{2,3\},\{1,2\})}$-path systems in $\mathcal{G}_\mathcal{C}$, so that Theorem 4.4.19 predicts that the quantum minor $[23 \mid 12]$ of $M_\mathcal{C}$ is zero. Computing the quantum minor $[23 \mid 12]$ of $M_\mathcal{C}$ directly, we indeed get

$$\begin{aligned}
[23 \mid 12] &= M_\mathcal{C}[2,1]M_\mathcal{C}[3,2] - qM_\mathcal{C}[3,1]M_\mathcal{C}[2,2] \\
&= (t_{2,2}t_{3,2}^{-1}t_{3,1} + t_{2,3}t_{3,3}^{-1}t_{3,1})(t_{3,2}) - q(t_{3,1})(t_{2,3}t_{3,3}^{-1}t_{3,2} + t_{2,2}) \\
&= t_{2,2}t_{3,2}^{-1}t_{3,1}t_{3,2} + t_{2,3}t_{3,3}^{-1}t_{3,1}t_{3,2} - qt_{3,1}t_{2,3}t_{3,3}^{-1}t_{3,2} - qt_{3,1}t_{2,2} \\
&= qt_{2,2}t_{3,1} + qt_{3,1}t_{2,3}t_{3,3}^{-1}t_{3,2} - qt_{3,1}t_{2,3}t_{3,3}^{-1}t_{3,2} - qt_{3,1}t_{2,2} \\
&= 0.
\end{aligned}$$

4.4.5 Pseudo quantum minors in H-prime ideals

Definition 4.4.22 (cf. Definition 3.1.7 of [8])**.** Let $v \in W_\mathcal{C}$ be a vertex of a path $P\colon r_i \implies c_j$ in $\mathcal{G}_\mathcal{C}$. Let e be the edge of P which ends at v and let f be the edge of P which begins at v. Then we say that v is a Γ-*turn* of P (or that P has a Γ-*turn* at v) if e is horizontal and f is vertical and that v is a $⅃$-*turn* of P (or that P has a $⅃$-turn at v) if e is vertical and f is horizontal.

Proposition 4.4.23 (cf. Proposition 3.1.8 of [8])**.** *Let $P\colon r_i \implies c_j$ be a path in $\mathcal{G}_\mathcal{C}$. If $v_1, v_2, \ldots, v_t$ is the sequence of all Γ-turns and $⅃$-turns in P, then v_a is a Γ-turn for odd values of a and a $⅃$-turn for even values of a, t is odd, and*

$$w(P) = t_{v_1}t_{v_2}^{-1}t_{v_3}\cdots t_{v_{t-1}}^{-1}t_{v_t}.$$

Proof. It is clear that v_a is a Γ-turn for a odd and a $⅃$-turn for a even. Since P ends with a vertical edge, t must be odd. Consider the subpaths:

$$P_1\colon r_i \implies v_1,\ P_2\colon v_1 \implies v_2,\ \ldots,\ P_t\colon v_{t-1} \implies v_t,\ P_{t+1}\colon v_t \implies c_j$$

of P. For $a \in [\![1, t+1]\!]$ even (i.e. for $a = 2, 4, \dots, t-1, t+1$), the path P_a consists solely of vertical edges and hence $w(P_a) = 1$. It follows that

$$\begin{aligned} w(P) &= w(P_1)w(P_2)\cdots w(P_t)w(P_{t+1}) \\ &= w(P_1)w(P_3)\cdots w(P_{t-2})w(P_t) \\ &= t_{v_1}w(P_3)\cdots w(P_{t-2})w(P_t) \quad \text{(by Proposition 4.4.9(4))}. \end{aligned}$$

However, $P_3, \dots P_{t-2}, P_t$ are internal horizontal paths in $\mathcal{G}_{\mathcal{C}}$ and by Proposition 4.4.9(3), their respective weights are $t_{v_2}^{-1}t_{v_3}, \dots, t_{v_{t-3}}^{-1}t_{v_{t-2}}, t_{v_{t-1}}^{-1}t_{v_t}$. The result follows. ■

Notation 4.4.24. *Let M be any $c \times d$ matrix with entries from $\mathbb{Z}$. Then $\underline{t}^M$ denotes the element $\prod_{(i,j)\in W_{\mathcal{C}}} t_{i,j}^{M[i,j]}$ of $\mathcal{B}$, where the factors appear in lexicographical order.*

Theorem 4.4.25 (cf. Theorem 4.1.9 [8])**.** *Let $I \subseteq [\![1, c]\!]$ and $J \subseteq [\![1, d]\!]$ have the same cardinality. Then the quantum minor $[I \mid J]$ of $M_{\mathcal{C}}$ is zero if and only if there does not exist a vertex-disjoint (R_I, C_J)-path system in the Cauchon graph $\mathcal{G}_{\mathcal{C}}$.*

Proof. For ease of notation, let us take $I = J = \{1, \dots, t\}$ (the proof for general I and J is the same but notationally more unwieldy).

By Proposition 4.4.23, the weight of any vertex-disjoint (R_I, C_J)-path system $\mathcal{P}$ is equal to $q^{\alpha}\underline{t}^{M_{\mathcal{P}}}$ for some integer α, where the $c \times d$ matrix $M_{\mathcal{P}} = (M_{\mathcal{P}}[i,j])_{(i,j)\in[\![c]\!]\times[\![d]\!]}$ is defined as follows:

$$M_{\mathcal{P}}[i,j] = \begin{cases} 1 & \text{if there is a path in } \mathcal{P} \text{ with a } \Gamma\text{-turn at } (i,j); \\ -1 & \text{if there is a path in } \mathcal{P} \text{ with a } \lrcorner\text{-turn at } (i,j); \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{P} = (P_1, \dots, P_t)$ and $\mathcal{Q} = (Q_1, \dots, Q_t)$ be vertex-disjoint (R_I, C_J)-path systems satisfying $M_{\mathcal{P}} = M_{\mathcal{Q}}$. Fix any $i \in [\![1, t]\!]$ and let (i, ℓ) be the first vertex where P_i turns and (i, ℓ') be the first vertex where Q_i turns. Suppose that $\ell' > \ell$, so that P_i goes horizontally straight through (i, ℓ') and in particular, (i, ℓ') is a vertex of P_i but neither a Γ-turn nor a Ⅎ-turn of P_i. However, since (i, ℓ') is a Γ-turn of Q_i and $M_{\mathcal{P}} = M_{\mathcal{Q}}$, there must be a path $P \neq P_i$ in $\mathcal{P}$ which has a Γ-turn at (i, ℓ'), which is a contradiction since $\mathcal{P}$ is a vertex-disjoint path system. Hence $\ell' \not> \ell$. A similar argument shows that $\ell \not> \ell'$, so that $\ell = \ell'$ i.e. the first turning vertices of P_i and Q_i coincide. A similar argument can be applied to the remaining turning vertices (if any) of P_i and Q_i to show that P_i and Q_i have the same turning vertices and hence $P_i = Q_i$. Since $i \in [\![1, t]\!]$ was chosen arbitrarily, we conclude that $\mathcal{P} = \mathcal{Q}$.

We have shown that if $\mathcal{P} = (P_1, \ldots, P_t)$ and $\mathcal{Q} = (Q_1, \ldots, Q_t)$ are distinct vertex-disjoint (R_I, C_J)-path systems, then $M_{\mathcal{P}} \neq M_{\mathcal{Q}}$ and hence $M_{\mathcal{P}}[i, j] \neq M_{\mathcal{Q}}[i, j]$ for some $(i, j) \in W_{\mathcal{C}}$.

It follows easily that if there exists at least one vertex-disjoint (R_I, C_J)-path system in the Cauchon graph $\mathcal{G}_{\mathcal{C}}$, then $[I \mid J]$ is a nontrivial linear combination of pairwise distinct lex-ordered monomials in the $t_{i,j}^{\pm 1}$ $((i, j) \in W_{\mathcal{C}})$ and hence (since the lex-ordered monomials in the $t_{i,j}^{\pm 1}$ $((i, j) \in W_{\mathcal{C}})$ form a basis for $\mathcal{B}$) $[I \mid J] \neq 0$. Theorem 4.4.19 gives the converse. ■

Before reading the proof of the following theorem, the reader might want to review the notation and the result of Corollary 4.3.8.

Theorem 4.4.26 (cf. Lemma 5.4 [7])**.** *For each $(i, j) \in Y_\lambda$, $M_{\mathcal{C}}[i, j]$ is the canonical image in $\mathcal{O}_q(M_{m,n}^\lambda(\mathbb{K}))/P$ of $X_{i,j}$, namely $M_{\mathcal{C}}[i, j] = \chi_{i,j}^{(c, \lambda_c+1)}$. For each $(i, j) \in [\![1, c]\!] \times [\![1, d]\!] \setminus Y_\lambda$, $M_{\mathcal{C}}[i, j]$ is zero.*

Proof. It is obvious that for each $(i, j) \in [\![1, c]\!] \times [\![1, d]\!] \setminus Y_\lambda$, $M_{\mathcal{C}}[i, j]$ is zero, so for the rest of this proof all boxes shall be in Y_λ. For any $(a, b), (i, j) \in Y_\lambda$, let us define $M_{\mathcal{C}}^{(a,b)}[i, j]$ to be the sum of the weights of all paths $P \colon r_i \implies c_j$ in $\mathcal{G}_{\mathcal{C}}$ which have no Ⅎ-turn after (a, b) with respect to the lexicographical order (i.e. whose Ⅎ-turns v all satisfy $v \leq_{\text{lex}} (a, b)$). It will suffice to show that for any $(a, b), (i, j) \in Y_\lambda$, we have

$$M_{\mathcal{C}}^{(a,b)}[i, j] = \chi_{i,j}^{(a,b)^+}; \tag{4.23}$$

setting $(a,b)=(c,\lambda_c)$ in (4.23) gives the result. We prove the claim (4.23) by induction on (a,b). If $(i,j)\in B_{\mathcal{C}}$, then there is no path $P\colon r_i \implies c_j$ in $\mathcal{G}_{\mathcal{C}}$ which has no Ⅎ-turn after $(1,1)$ and we have $M^{(1,1)}[i,j]=0=t_{i,j}=\chi_{i,j}^{(1,1)^+}$. If $(i,j)\in W_{\mathcal{C}}$, then the only path in $\mathcal{G}_{\mathcal{C}}$ from r_i to c_j which has no Ⅎ-turn after $(1,1)$ is the path which runs horizontally from r_i to (i,j) and then vertically from (i,j) to c_j; this path has weight $t_{i,j}=\chi_{i,j}^{(1,1)^+}$ by Proposition 4.4.9(4), so that $M^{(1,1)}[i,j]=\chi_{i,j}^{(1,1)^+}$.

Let $(a,b)\in E_\lambda\setminus\{(c,\lambda_c+1)\}$ be such that

$$M_{\mathcal{C}}^{(a,b)^-}[i,j]=\chi_{i,j}^{(a,b)} \tag{4.24}$$

for all $(i,j)\in Y_\lambda$. For any $(i,j)\in Y_\lambda$, let us define $F_{i,j}$ to be the set of all paths in $\mathcal{G}_{\mathcal{C}}$ from r_i to c_j which have a Ⅎ-turn at (a,b) and no later Ⅎ-turn; it will suffice to show that for each $(i,j)\in Y_\lambda$, $\chi_{i,j}^{(a,b)^+}$ is obtained from $\chi_{i,j}^{(a,b)}$ by adding $\sum_{P\in F_{i,j}} w(P)$.

We may assume that $i<a$, $j<b$, and $(a,b)\in W_{\mathcal{C}}$ (since otherwise $F_{i,j}$ is empty and Corollary 4.3.8 gives $\chi_{i,j}^{(a,b)^+}=\chi_{i,j}^{(a,b)}$). By Corollary 4.3.8, we have $\chi_{i,j}^{(a,b)^+}=\chi_{i,j}^{(a,b)}+\chi_{i,b}^{(a,b)}t_{a,b}^{-1}t_{a,j}$, so that it will suffice to show that

$$\sum_{P\in F_{i,j}} w(P)=\chi_{i,b}^{(a,b)}t_{a,b}^{-1}t_{a,j}. \tag{4.25}$$

There are two cases to consider:

(a) Suppose that $(a,j)\in B_{\mathcal{C}}$. Then $F_{i,j}$ is empty and $t_{a,j}=0$; (4.25) follows immediately.

(b) Suppose that $(a,j)\in W_{\mathcal{C}}$. Let F_i be the set of all paths in $\mathcal{G}_{\mathcal{C}}$ from r_i to c_b which have no Ⅎ-turn after $(a,b)^-=(a,b-1)$, so that $\sum_{Q\in F_i} w(Q)=M_{\mathcal{C}}^{(a,b)^-}[i,b]$ and hence $\sum_{Q\in F_i} w(Q)=\chi_{i,b}^{(a,b)}$ by the induction hypothesis (4.24). Let the path $K_j\colon (a,b)\implies c_j$ be given by concatonating the horizontal path $(a,b)\implies(a,j)$ with the vertical path $(a,j)\implies c_j$. Proposition 4.4.9(3) gives $w(K_j)=t_{a,b}^{-1}t_{a,j}$. Let L_b be the vertical path from (a,b) to c_b. For any path $P\in F_{i,j}$, the subpath

$P' \colon r_i \implies (a,b)$ of P is such that $P = P'K_j$, $P'L_b \in F_i$, and $w(P'L_b) = w(P')$. Notice that each path in F_i has the form $P'L_b$ for a unique path $P \in F_{i,j}$.

We have

$$\begin{aligned}\sum_{P \in F_{i,j}} w(P) &= \sum_{P \in F_{i,j}} w(P')w(K_j) \\ &= \left(\sum_{P \in F_{i,j}} w(P'L_b)\right) w(K_j) \\ &= \left(\sum_{Q \in F_i} w(Q)\right) w(K_j) \\ &= \chi_{i,b}^{(a,b)} w(K_j) \\ &= \chi_{i,b}^{(a,b)} t_{a,b}^{-1} t_{a,j},\end{aligned}$$

establishing (4.25).

The proof is complete. ■

As an immediate corollary of Theorems 4.4.25 and 4.4.26, we get the main result of this section, which is a generalisation of the main result of [7]:

Theorem 4.4.27 (cf. Theorem 5.6 of [7])**.** *Let P be an H-prime ideal of the partition subalgebra $\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))$ corresponding to a Cauchon diagram $\mathcal{C}$ on the Young diagram Y_λ and let $I \subseteq [\![1, c]\!]$ and $J \subseteq [\![1, d]\!]$ have the same cardinality. Then the pseudo quantum minor $[I \mid J]$ of $\mathcal{O}_q(M_{m,n}^{\lambda}(\mathbb{K}))$ belongs to P if and only if there exists no vertex-disjoint (R_I, C_J)-path system in the Cauchon graph $\mathcal{G}_{\mathcal{C}}$ of $\mathcal{C}$.*

Chapter 5

Quantum Plücker coordinates in H-prime ideals of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$

The material of this chapter comes from joint work with Prof. Stéphane Launois and Prof. Tom Lenagan. Sections 5.1, 5.2, and 5.3 consist of known results, some of which are rewritten in a fashion suitable for the purposes of the rest of the chapter; Sections 5.1, 5.2, and 5.3 are designed to set up Section 5.4, which consists of original material. Sections 5.5 and 5.6 contextualise the results of Section 5.4.

5.1 The quantum Grassmannian $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$

Let us fix positive integers $m < n$. Consider the Grassmannian $G_{m,n}(\mathbb{K})$, which consists of the m-dimensional subspaces of $\mathbb{K}^n$; this is a projective variety whose homogeneous coordinate ring $\mathcal{O}(G_{m,n}(\mathbb{K}))$ can be constructed as follows: the coordinate ring $\mathcal{O}(M_{m,n}(\mathbb{K}))$ of the affine variety of $m \times n$ matrices (which is simply the affine space $\mathbb{K}^{mn}$) is the polynomial algebra in the mn indeterminates $X_{i,j}$ ($i = 1, \ldots, m$, $j =$

$1, \ldots, n$), which we can arrange in a matrix

$$\begin{pmatrix} X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,n} \end{pmatrix}$$

and the homogeneous coordinate ring $\mathcal{O}(G_{m,n}(\mathbb{K}))$ of the Grassmannian $G_{m,n}(\mathbb{K})$ is the subalgebra of $\mathcal{O}(M_{m,n}(\mathbb{K})))$ generated by the maximal minors (namely the $m \times m$ minors) of the matrix above. Analogously, the quantised homogeneous coordinate ring of the Grassmannian $G_{m,n}(\mathbb{K})$ (informally known as the $(m \times n)$ *quantum Grassmannian*), denoted by $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$, is defined as the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ generated by the maximal quantum minors of the matrix

$$\begin{pmatrix} X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,n} \end{pmatrix} \tag{5.1}$$

of canonical generators of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$. By [21, Theorem 1.1], the quantum Grassmannian $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is a noetherian domain.

Since an $m \times m$ quantum minor of the matrix (5.1) must involve each of the m rows of (5.1), specifying a maximal quantum quantum minor of (5.1) requires one only to specify m of the n columns. As such, the generators of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ are written as $[\gamma_1 \cdots \gamma_m]$ where $1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_m \leq n$; $[\gamma_1 \cdots \gamma_m]$ denotes the quantum minor $[1 \cdots m \mid \gamma_1 \cdots \gamma_m]$ of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$. These generators of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ are called its *quantum Plücker coordinates* and the set of quantum Plücker coordinates of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is denoted $\Pi_{m,n}$ (we shall simply write Π, since m and n are understood). We shall often identify Π with the set of all m-element subsets of $[\![1, n]\!]$ in the obvious way.

There is a natural partial order on Π given by

$$[\gamma_1 \cdots \gamma_m] \leq [\gamma'_1 \cdots \gamma'_m] \iff (\gamma_i \leq \gamma'_i \text{ for all } i \in [\![1, m]\!]). \tag{5.2}$$

Coming from the column action of $(\mathbb{K}^\times)^{m+n}$ by automorphisms on $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ (see (2.10)), there is an action of the algebraic torus $H = (\mathbb{K}^\times)^n$ by automorphisms on $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ defined as follows: for any $[\gamma_1 \cdots \gamma_m] \in \Pi$ and any $(\alpha_1, \ldots, \alpha_n) \in (\mathbb{K}^\times)^n$,

$$(\alpha_1, \ldots, \alpha_n) \cdot [\gamma_1 \cdots \gamma_m] = \alpha_{\gamma_1} \cdots \alpha_{\gamma_m} [\gamma_1 \cdots \gamma_m]. \tag{5.3}$$

By [21, Corollary 2.1], the algebra $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ has a $\mathbb{K}$-basis consisting of products of quantum Plücker coordinates. Since quantum Plücker coordinates are clearly H-eigenvectors with rational eigenvalues, it follows easily that the action of H on $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is rational.

The goal of this chapter is to develop a graph-theoretic method for deciding whether or not a given quantum Plücker coordinate belongs to a given H-prime ideal of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$. In fact, given an H-prime ideal J of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ and a quantum Plücker coordinate α, we shall show that the question of whether or not α belongs to J is equivalent to the question of whether or not a certain pseudo quantum minor belongs to a certain H-prime ideal of a certain partition subalgebra of $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\mathbb{K}))$; by Theorem 4.4.27, the latter is a question which we can answer.

5.2 Framing the question

For any $\gamma \in \Pi$, set $\Pi^\gamma = \{\alpha \in \Pi \mid \alpha \not\geq \gamma\}$. By [24, Theorem 5.1], for every $P \in \operatorname{Spec} \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ except the irrelevant ideal $\langle \Pi \rangle$, there is a unique $\gamma \in \Pi$ such that $\gamma \notin P$ and $\Pi^\gamma \subseteq P$. For any $\gamma \in \Pi$, let $(H-)\operatorname{Spec}_\gamma \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ denote the subspace of $\operatorname{Spec} \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ consisting of all those $(H-)$prime ideals J such that $\gamma \notin J$ and $\Pi^\gamma \subseteq J$; we have

$$(H-)\operatorname{Spec} \mathcal{O}_q(G_{m,n}(\mathbb{K})) = \bigsqcup_{\gamma \in \Pi} (H-)\operatorname{Spec}_\gamma \mathcal{O}_q(G_{m,n}(\mathbb{K})) \sqcup \langle \Pi \rangle. \tag{5.4}$$

Convention 5.2.1. *For the rest of this chapter, let us fix some $\gamma = [\gamma_1 \cdots \gamma_m] \in \Pi$.*

If $J \in H - \mathrm{Spec}_\gamma\, \mathcal{O}_q(G_{m,n}(\mathbb{K}))$, then by the definition of $H - \mathrm{Spec}_\gamma\, \mathcal{O}_q(G_{m,n}(\mathbb{K}))$, we know that $\gamma \notin J$ and that $\gamma' \in J$ for all $\gamma' \in \Pi$ such that $\gamma' \ngeq \gamma$. What remains is to decide which other quantum Plücker coordinates belong to J i.e. given $\alpha \in \Pi$ such that $\alpha > \gamma$, we seek to decide whether or not α belongs to J. The key to achieving this goal is to exploit the correspondence (established in [24]) between $H - \mathrm{Spec}_\gamma\, \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ and the H-spectrum of a certain partition subalgebra of $\mathcal{O}_{q^{-1}} M_{m,n-m}(\mathbb{K})$. We shall describe this correspondence in the next section.

5.3 The correspondence of Launois, Lenagan, and Rigal

5.3.1 Noncommutative dehomogenisation

The process of *noncommutative dehomogenisation*, introduced in [21, Section 3], is the foundation for the construction of Launois, Lenagan, and Rigal [24] of a bi-increasing one-to-one correspondence between $H - \mathrm{Spec}_\gamma(\mathcal{O}_q(G_{m,n}(\mathbb{K})))$ and $H - \mathrm{Spec}(\mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K})))$, where λ is a partition associated to γ.

We review here the general theory on noncommutative dehomogenisation from [21, Section 3]. Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be an $\mathbb{N}$-graded $\mathbb{K}$-algebra, let x be a homogeneous normal regular element of degree one, and set $S := R[x^{-1}]$. For $i < 0$, define $R_i := 0$. For $i \in \mathbb{Z}$ and $j \in \mathbb{N}$, define $R_i x^{-j}$ to be the $\mathbb{K}$-subspace of S consisting of all those elements of S which can be expressed in the form rx^{-j} with $r \in R_i$. For $l \in \mathbb{Z}$, set $S_l := \sum_{t=0}^{\infty} R_{l+t} x^{-t}$, so that since $R_i x^{-j} \subseteq R_{i+1} x^{-(j+1)}$ for all i and j, we have

$$S_l = \bigcup_{t=0}^{\infty} R_{l+t} x^{-t}. \tag{5.5}$$

We get a grading $S = \bigoplus_{l\in\mathbb{Z}} S_l$ on S.

Definition 5.3.1. Let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be an $\mathbb{N}$-graded $\mathbb{K}$-algebra and let x be a homogeneous regular normal element of R of degree one. The *noncommutatuve dehomogenisa-*

tion of R at x, written $\mathrm{Dhom}(R,x)$, is the subalgebra $S_0 = \sum_{t=0}^{\infty} R_t x^{-t} = \bigcup_{t=0}^{\infty} R_t x^{-t}$ of the $\mathbb{Z}$-graded algebra $R[x^{-1}] = S = \bigoplus_{l\in\mathbb{Z}} S_l$.

Denote by σ the conjugation automorphism of S given by $\sigma(s) = xsx^{-1}$ for all $s \in S$. It is easy to check that σ restricts to an automorphism of $\mathrm{Dhom}(R,x) = S_0$ (which we shall abusively denote by σ). By [21, Lemma 3.1], the inclusion $\mathrm{Dhom}(R,x) \hookrightarrow R[x^{-1}]$ extends to an isomorphism

$$\mathrm{Dhom}(R,x)[y^{\pm 1};\sigma] \xrightarrow{\cong} R[x^{-1}]$$

which sends y to x.

5.3.2 Quantum Schubert varieties and quantum Schubert cells

By [26, Corollary 3.1.7], the ideal $\langle \Pi^\gamma \rangle$ of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is completely prime, so that the noetherian algebra $S(\gamma) := \mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle$ is a domain. It is well known that $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ is an $\mathbb{N}$-graded $\mathbb{K}$-algebra with each quantum Plücker coordinate being homogeneous of degree 1; since the elements of Π^γ are homogeneous, there is an induced $\mathbb{N}$-grading on $S(\gamma)$. By [24, Remark 1.4], $\overline{\gamma} \in S(\gamma)$ is a homogeneous regular normal element of degree one, so that we may dehomogenise $S(\gamma)$ at $\overline{\gamma}$ (in fact this follows from a more general result of Lenagan and Rigal [25, Lemma 1.2.1]).

Definition 5.3.2. The algebra $S(\gamma) := \mathcal{O}_q G_{m,n}(\mathbb{K})/\langle \Pi^\gamma \rangle$ is called the *quantum Schubert variety* associated to γ. The algebra $S^o(\gamma) := \mathrm{Dhom}(S(\gamma), \overline{\gamma})$ is called the *quantum Schubert cell* associated to γ.

Remark 5.3.3. *We shall later describe an isomorphism (established in [24, Theorem 4.7]), between the quantum Schubert cell $S^o(\gamma)$ and a partition subalgebra of $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\mathbb{K}))$.*

Definition 5.3.4. The *ladder* associated to γ is denoted by $\mathcal{L}_\gamma$ and defined by

$$\mathcal{L}_\gamma = \{(i,j) \in [\![1,m]\!] \times [\![1,n]\!] \mid j > \gamma_{m+1-i} \text{ and } j \neq \gamma_l \ \text{ for all } l \in [\![1,m]\!]\}.$$

A generating set for the quantum Schubert cell $S^o(\gamma)$ was described in [24, Proposition 4.4]: if, for $(i,j) \in \mathcal{L}_\gamma$, one defines $m_{i,j} := [\{\gamma_1, \ldots, \gamma_m\} \setminus \{\gamma_{m+1-i}\} \sqcup \{j\}]$, then the quantum Schubert cell $S^o(\gamma)$ is generated by $\{\overline{m_{i,j}}\overline{\gamma}^{-1} \mid (i,j) \in \mathcal{L}_\gamma\}$. Let us set $\widetilde{m_{i,j}} := \overline{m_{i,j}}\overline{\gamma}^{-1}$ for all $(i,j) \in \mathcal{L}_\gamma$.

Since $\langle \Pi^\gamma \rangle$ is clearly an H-invariant ideal of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$, the action of H on $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ descends to $S(\gamma)$. Since $\overline{\gamma}$ is an H-eigenvector of $S(\gamma)$, the action of H on $S(\gamma)$ extends to $S(\gamma)[\overline{\gamma}^{-1}]$. This action of H on $S(\gamma)[\overline{\gamma}^{-1}]$ restricts to $S^o(\gamma)$; indeed for any $\widetilde{m_{i,j}}$ with $(i,j) \in \mathcal{L}_\gamma$, and any $(\alpha_1, \ldots, \alpha_n) \in H$, an elementary calculation shows that

$$(\alpha_1, \ldots, \alpha_n) \cdot \widetilde{m_{i,j}} = \alpha^{-1}_{\gamma_{m+1-i}} \alpha_j \widetilde{m_{i,j}}. \tag{5.6}$$

Recall from the general theory of noncommutative dehomogenisation that when σ is the restriction to $S^o(\gamma)$ of the automorphism of $S(\gamma)[\overline{\gamma}^{-1}]$ given by $s \mapsto \overline{\gamma} s \overline{\gamma}^{-1}$ for all $s \in S(\gamma)[\overline{\gamma}^{-1}]$, the inclusion $S^o(\gamma) \hookrightarrow S(\gamma)[\overline{\gamma}^{-1}]$ extends to an isomorphism

$$S^o(\gamma)[y^{\pm 1}; \sigma] \to (\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}] \tag{5.7}$$

which sends y to $\overline{\gamma}$. Notice here that by [26, Lemma 3.1.4(v)], the automorphism σ multiplies each $\widetilde{m_{i,j}}$ ($(i,j) \in \mathcal{L}_\gamma$) by q. The action of H on $(\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}]$ passes to $S^o(\gamma)[y^{\pm 1}; \sigma]$ via the isomorphism (5.7) and this action of H on $S^o(\gamma)[y^{\pm 1}; \sigma]$ restricts to the action of H on $S^o(\gamma)$ described in (5.6). In particular, the isomorphism (5.7) is H-equivariant where H acts on $S^o(\gamma)$ as in (5.6) and each $(\alpha_1, \ldots, \alpha_n) \in H$ acts on y as follows

$$(\alpha_1, \ldots, \alpha_n) \cdot y = \alpha_{\gamma_1} \cdots \alpha_{\gamma_m} y \tag{5.8}$$

(cf. (5.3)).

5.3.3 Quantum ladder matrix algebras

It was shown in [24] that the quantum Schubert cell $S^o(\gamma)$ can be identified with a well-behaved subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$, which can in turn be identified with a partition subalgebra of $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\mathbb{K}))$. We describe these isomorphisms in detail in this section.

Definition 5.3.5. The *quantum ladder matrix algebra* associated to γ is the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ generated by all those $X_{i,j}$ with $(i,j) \in \mathcal{L}_\gamma$; it is denoted by $\mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K}))$.

By [24, Lemma 4.6], there is an isomorphism

$$\begin{aligned} S^o(\gamma) &\xrightarrow{\cong} \mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K})) \\ \widetilde{m_{i,j}} &\mapsto X_{i,j}. \end{aligned} \tag{5.9}$$

One may obtain the generators of $\mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K}))$ as follows. Consider the matrix

$$\begin{pmatrix} X_{1,1} & \cdots & X_{1,n} \\ \vdots & \ddots & \vdots \\ X_{m,1} & \cdots & X_{m,n} \end{pmatrix} \tag{5.10}$$

of canonical generators of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ and recall that $\gamma = [\gamma_1 \cdots \gamma_m]$. For each $i \in [\![1, m]\!]$, remove the i^{th}-last entry of the γ_i^{th} column of (5.10) (namely the entry X_{m+1-i,γ_i}) and replace it with a bullet. For each bullet, replace all matrix entries which are to its left and all matrix entries which are below it with stars. Then the quantum ladder matrix algebra $\mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K}))$ is the subalgebra of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ which is generated by the entries of the matrix (5.10) which survive this process (i.e. which are not replaced by a bullet or a star).

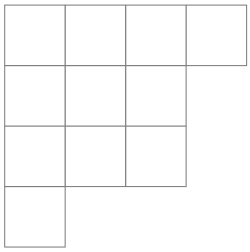
Example 5.3.6. *Let γ be the maximal quantum minor* $[1347]$ *of $\mathcal{O}_q(G_{4,8}(\mathbb{K}))$ and consider the matrix*

$$\begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} & X_{1,6} & X_{1,7} & X_{1,8} \\ X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} & X_{2,6} & X_{2,7} & X_{2,8} \\ X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} & X_{3,5} & X_{3,6} & X_{3,7} & X_{3,8} \\ X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4} & X_{4,5} & X_{4,6} & X_{4,7} & X_{4,8} \end{pmatrix}$$

of canonical generators of $\mathcal{O}_q(M_{4,8}(\mathbb{K}))$*. Applying the prescribed procedure, we are left with*

$$\begin{pmatrix} * & * & * & * & * & * & \bullet & X_{1,8} \\ * & * & * & \bullet & X_{2,5} & X_{2,6} & * & X_{2,8} \\ * & * & \bullet & * & X_{3,5} & X_{3,6} & * & X_{3,8} \\ \bullet & X_{4,2} & * & * & X_{4,5} & X_{4,6} & * & X_{4,8} \end{pmatrix} \tag{5.11}$$

The quantum ladder matrix algebra $\mathcal{O}_q(M_{4,8,\gamma}(\mathbb{K}))$ *is the subalgebra of* $\mathcal{O}_q(M_{4,8}(\mathbb{K}))$ *generated by those* $X_{i,j}$ *appearing in* (5.11)*. After rotating* (5.11) *through* 180° *and deleting the columns containing bullets, notice that the generators of* $\mathcal{O}_q(M_{4,8,\gamma}(\mathbb{K}))$ *lie in the Young diagram below*



(5.12)

In fact it turns out that the quantum ladder matrix algebra $\mathcal{O}_q(M_{4,8,\gamma}(\mathbb{K}))$ *is isomorphic to the partition subalgebra of* $\mathcal{O}_{q^{-1}}(M_{4,4}(\mathbb{K}))$ *corresponding to the partition whose Young diagram is* (5.12)*.*

Notation 5.3.7. *Notice that for each* $i \in [\![1, m]\!]$, $\gamma_i - i = |\{a \in [\![1, n]\!] \setminus \gamma \mid a < \gamma_i\}|$*. It follows easily that if we define* $\lambda_i = n - m - (\gamma_i - i)$ *for each* $i \in [\![1, m]\!]$*, then* $(\lambda_1, \ldots, \lambda_m)$ *is a partition with* $n - m \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$*. Let* c *be as large as possible such that* $\lambda_c \neq 0$ *and denote by* λ *the partition* $(\lambda_1, \ldots, \lambda_c)$*. Recall from Definition 4.1.2, that* $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ *denotes the partition subalgebra of* $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\mathbb{K}))$ *associated to the partition* λ*. Recall also that we denote by* Y_λ *the Young diagram corresponding to* λ*.*

Let $\{a_1 < \cdots < a_{n-m}\} = [\![1, n]\!] \setminus \gamma$ and notice that all elements of $\mathcal{L}_\gamma$ take the form (i, a_j) for some $i \in [\![1, m]\!]$ and some $j \in [\![1, n - m]\!]$. The following result appears in the proof of [24, Theorem 4.7]. We write down the maps explicitly here as we shall need them.

Lemma 5.3.8. *There is an isomorphism*

$$f\colon \mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K})) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$$

such that

- $f(X_{i,a_j}) = X_{m+1-i,n-m+1-j}$ *for each* $(i, a_j) \in \mathcal{L}_\gamma$*;*
- $f^{-1}(X_{i,j}) = X_{m+1-i,a_{n-m+1-j}}$ *for each* $(i,j) \in Y_\lambda$*.*

Proof. By the proof of [16, Corollary 5.9], there is an isomorphism $\mathcal{O}_q(M_n(\mathbb{K})) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M_n(\mathbb{K}))$ which sends each $X_{i,j}$ to $X_{n+1-i,n+1-j}$ (this isomorphism can be thought of as rotating the matrix of canonical generators for $\mathcal{O}_q(M_n(\mathbb{K}))$ through 180°).

There is an isomorphism $\delta\colon \mathcal{O}_q(M_{m,n}(\mathbb{K})) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M_{m,n}(\mathbb{K}))$ such that for each $(i,j) \in [\![1,m]\!] \times [\![1,n]\!]$, $\delta(X_{i,j}) = X_{m+1-i,n+1-j}$ (this isomorphism can be thought of as rotating the matrix of canonical generators for $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ through 180°). This isomorphism is constructed by identifying $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$ with the subalgebra of $\mathcal{O}_q(M_n(\mathbb{K}))$ generated by the last m rows of the matrix of canonical generators for $\mathcal{O}_q(M_n(\mathbb{K}))$, identifying $\mathcal{O}_{q^{-1}}(M_{m,n}(\mathbb{K}))$ with the subalgebra of $\mathcal{O}_{q^{-1}}(M_n(\mathbb{K}))$ generated by the first m rows of the matrix of canonical generators for $\mathcal{O}_{q^{-1}}(M_n(\mathbb{K}))$, and applying the isomorphism described in the previous paragraph.

There is an isomorphism $\delta(\mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K}))) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ which sends each $\delta(X_{i,a_j}) = X_{m+1-i,n+1-a_j}$ ($(i,a_j) \in \mathcal{L}_\gamma$) to $X_{m+1-i,n-m+1-j}$. Composing this isomorphism with δ (or rather the restriction of δ to $\mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K}))$) gives the desired isomorphism f.

∎

The isomorphism f is simpler than the notation of Lemma 5.3.8 might make it seem. The following example should illuminate the idea.

Example 5.3.9. *In the situation of Example 5.3.6, where* γ *is the quantum Plücker coordinate* [1347] *of* $\mathcal{O}_q(G_{4,8}(\mathbb{K}))$*, the generators of the quantum ladder matrix algebra*

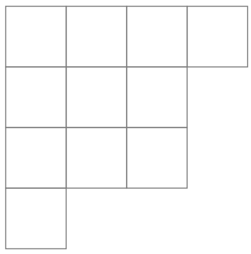
$\mathcal{O}_q(M_{4,8,\gamma}(\mathbb{K}))$ *are those appearing below*

$$\begin{pmatrix} \ast & \ast & \ast & \ast & \ast & \ast & \bullet & X_{1,8} \\ \ast & \ast & \ast & \bullet & X_{2,5} & X_{2,6} & \ast & X_{2,8} \\ \ast & \ast & \bullet & \ast & X_{3,5} & X_{3,6} & \ast & X_{3,8} \\ \bullet & X_{4,2} & \ast & \ast & X_{4,5} & X_{4,6} & \ast & X_{4,8} \end{pmatrix}.$$

The action of the isomorphism $\delta\colon \mathcal{O}_q(M_{4,8}(\mathbb{K})) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M_{4,8}(\mathbb{K}))$ *may be understood as rotating this picture through* $180°$*:*

$$\begin{pmatrix} X_{1,1} & \ast & X_{1,3} & X_{1,4} & \ast & \ast & X_{1,7} & \bullet \\ X_{2,1} & \ast & X_{2,3} & X_{2,4} & \ast & \bullet & \ast & \ast \\ X_{3,1} & \ast & X_{3,3} & X_{3,4} & \bullet & \ast & \ast & \ast \\ X_{4,1} & \bullet & \ast & \ast & \ast & \ast & \ast & \ast \end{pmatrix} \tag{5.13}$$

Let λ *be the partition associated to* γ *as in Notation 5.3.7, whose Young diagram is*



The subalgebra of $\mathcal{O}_{q^{-1}}(M_{4,8}(\mathbb{K}))$ *generated by the* $X_{i,j}$ *appearing in* (5.13) *is clearly isomorphic to the partition subalgebra* $\mathcal{O}_{q^{-1}}(M^{\lambda}_{4,4}(\mathbb{K}))$ *of* $\mathcal{O}_{q^{-1}}(M_{4,4}(\mathbb{K}))$.

The following is a more explicit statement of [24, Theorem 4.7].

Theorem 5.3.10. *There is an isomorphism* $\theta\colon S^o(\gamma) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ *such that*

- $\theta(\widetilde{m_{i,a_j}}) = X_{m+1-i,n-m+1-j}$ *for each* $(i, a_j) \in \mathcal{L}_\gamma$*;*
- $\theta^{-1}(X_{i,j}) = \widetilde{m_{m+1-i,a_{n-m+1-j}}}$ *for each* $(i, j) \in Y_\lambda$.

Proof. When g is the isomorphism $S^o(\gamma) \xrightarrow{\cong} \mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K}))$ given in (5.9) and f is the isomorphism $\mathcal{O}_q(M_{m,n,\gamma}(\mathbb{K})) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ given in Lemma 5.3.8, the desired isomorphism θ is given by $f \circ g$. ■

We may pass the action of H on $S^o(\gamma)$ through θ to get an action of H on $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ described by

$$(\alpha_1, \ldots, \alpha_n) \cdot X_{i,j} = \alpha_{\gamma_i}^{-1} \alpha_{a_{n-m+1-j}} X_{i,j} \tag{5.14}$$

for all $(\alpha_1, \ldots, \alpha_n) \in H$ and all $(i, j) \in Y_\lambda$. With this action of H on $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$, the isomorphism θ is H-equivariant.

⚠**WARNING**⚠ Because it allows the isomorphism θ to be H-equivariant, the H-action on $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ which we shall use is that given in (5.14); this is NOT the usual action of H on $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ (which is given in (4.1)).

In spite of the warning above, the following lemma shows that in fact we may use the term *H-prime ideal of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$* without ambiguity (cf. commentary in [24] before Theorem 4.8).

Lemma 5.3.11. *The same subsets (and in particular the same prime ideals) of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ are invariant under H whether one uses the action of H described in* (5.14) *or that described in* (4.1)*.*

Proof. Let us use "$\cdot$" to denote the action of H on $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ transferred through θ from the action on $S^o(\gamma)$ (described in (5.14)), let us use "#" to denote the standard action of H on $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ (described in (4.1)), and let us fix any $\alpha = (\alpha_1, \ldots, \alpha_n) \in H$.

Define $\alpha' = (\alpha'_1, \ldots, \alpha'_n)$, $\alpha'' = (\alpha''_1, \ldots, \alpha''_n) \in H$ by $\alpha'_i = \alpha_{\gamma_i}^{-1}$ for all $i \in [\![1, m]\!]$, $\alpha'_{m+j} = \alpha_{a_{n-m+1-j}}$ for all $j \in [\![1, n-m]\!]$, $\alpha''_{\gamma_i} = \alpha_i^{-1}$ for all $i \in [\![1, m]\!]$ and $\alpha''_{a_{n-m+1-j}} = \alpha_{m+j}$ for all $j \in [\![1, n-m]\!]$.

One checks easily that if $(i, j) \in Y_\lambda$, then $\alpha \cdot X_{i,j} = \alpha' \# X_{i,j}$ and $\alpha \# X_{i,j} = \alpha'' \cdot X_{i,j}$. Since these $X_{i,j}$ generate $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$, we have $\alpha \cdot x = \alpha' \# x$ and $\alpha \# x = \alpha'' \cdot x$ for all $x \in \mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$. The result follows.

■

5.3.4 The correspondence of Launois, Lenagan, and Rigal

Recall that we have set $\{a_1 < \cdots < a_{n-m}\} = [\![1, n]\!] \setminus \gamma$ and that all elements of $\mathcal{L}_\gamma$ take the form (i, a_j) for some $i \in [\![1, m]\!]$ and some $j \in [\![1, n-m]\!]$.

When σ is the automorphism of $\mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))$ which multiplies each $X_{i,j}$ ($(i,j) \in Y_\lambda$) by q, the H-equivariant isomorphism $\theta\colon S^o(\gamma) \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))$ (from Theorem 5.3.10) and the H-equivariant dehomogenisation isomorphism

$$S^o(\gamma)[y^{\pm 1}; \sigma] \xrightarrow{\cong} (\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}]$$

given in (5.7) induce an H-equivariant isomorphism

$$\begin{aligned} \Phi\colon \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))[y^{\pm 1}; \sigma] &\xrightarrow{\cong} (\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}] \\ X_{i,j} &\mapsto \widetilde{m_{m+1-i, a_{n-m+1-j}}} \qquad ((i,j) \in Y_\lambda) \\ y &\mapsto \overline{\gamma}. \end{aligned} \tag{5.15}$$

whose inverse we shall denote by Ψ.

Remark 5.3.12. *Recall that the dehomogenisation isomorphism $S^o(\gamma)[y^{\pm 1}; \sigma] \xrightarrow{\cong} (\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}]$ extends the inclusion $S^o(\gamma) \hookrightarrow (\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}]$, so that for any $x \in S^o(\gamma)$, we have $\Psi(x) = \theta(x) \in \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))$.*

By [24, Theorem 5.4], there is a bi-increasing bijection

$$\xi\colon H-\mathrm{Spec}_\gamma \mathcal{O}_q(G_{m,n}(\mathbb{K})) \xrightarrow{\cong} H-\mathrm{Spec}\,\mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K})) \tag{5.16}$$

such that for any $J \in H-\mathrm{Spec}_\gamma \mathcal{O}_q(G_{m,n}(\mathbb{K}))$,

$$\xi(J) = \Psi(\overline{J}[\overline{\gamma}^{-1}]) \cap \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))$$

and for any $K \in H - \operatorname{Spec} \mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$, $\xi^{-1}(K)$ is the preimage in $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ of

$$\Phi\left(\bigoplus_{i\in\mathbb{Z}} Ky^i\right) \cap \left(\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^{\gamma}\rangle\right).$$

Recall the one-to-one correspondence (4.17) (first established in [24, Theorem 3.5]) between the H-prime ideals of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ and the Cauchon diagrams on the Young diagram Y_λ. Composing this correspondence with ξ gives the one-to-one correspondence

$$H - \operatorname{Spec}_{\gamma} \mathcal{O}_q(G_{m,n}(\mathbb{K})) \longleftrightarrow \text{Cauchon diagrams on } Y_\lambda \tag{5.17}$$

which was established in [24, Corollary 5.5]: any $J \in H - \operatorname{Spec}_{\gamma} \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ corresponds to the Cauchon diagram of the H-prime ideal $\xi(J)$ of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ and any Cauchon diagram $\mathcal{C}$ on the Young diagram Y_λ corresponds to the image under ξ^{-1} of the H-prime ideal of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ which has Cauchon diagram $\mathcal{C}$.

5.4 Exploiting the correspondence of Launois, Lenagan, and Rigal

Let us fix any $J \in H - \operatorname{Spec}_{\gamma} \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ and let us denote by $\mathcal{C}$ the Cauchon diagram on Y_λ which corresponds to J under (5.17). Let us also fix any $\alpha \in \Pi$ which satisfies $\alpha > \gamma$. Notice that there exist $1 \le i_1 < \cdots < i_t \le m$ and $1 \le j_1 < \cdots < j_t \le n-m$ such that $a_{j_l} > \gamma_{i_l}$ for all $l = 1, \dots, t$ and $\alpha = [(\gamma \setminus \{\gamma_{i_1}, \dots, \gamma_{i_t}\}) \sqcup \{a_{j_1}, \dots, a_{j_t}\}]$.

Remark 5.4.1. *Recall that by the definition of $H - \operatorname{Spec}_{\gamma} \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ (see the beginning of Section 5.2), the question of whether or not a given quantum Plücker coordinate belongs to J is settled trivially unless that quantum Plücker coordinate is strictly greater than γ with respect to the partial order* (5.2).

Notice that when $h_0 = (\alpha_1, \dots, \alpha_n) \in H$ is such that $\alpha_i = q^2$ if $i \notin \{\gamma_1, \dots, \gamma_m\}$ and $\alpha_i = q$ otherwise, the isomorphism σ of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ (which multiplies each

$X_{i,j}$ $((i,j) \in Y_\lambda)$ by q) coincides with the action of h_0. Moreover $h_0 \cdot y = q^m y$ by (5.8). Hence the algebra $\mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))[y^{\pm 1};\sigma]$, along with its H action, satisfies [23, Hypothesis 2.1]. We shall use this fact in the proof of the following theorem.

Proposition 5.4.2. *The condition that α belongs to J is equivalent to the condition that $\Psi(\bar{\alpha}\bar{\gamma}^{-1})$ belongs to $\xi(J)$.*

Proof. By [23, Lemma 2.2], we have $\bigoplus_{i\in\mathbb{Z}} \xi(J)y^i = \Psi((J/\langle\Pi^\gamma\rangle)[\overline{\gamma}^{-1}])$, so that the isomorphism Ψ induces an isomorphism

$$\frac{(\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle\Pi^\gamma\rangle)[\overline{\gamma}^{-1}]}{(J/\langle\Pi^\gamma\rangle)[\overline{\gamma}^{-1}]} \xrightarrow{\cong} \frac{\mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))[y^{\pm 1};\sigma]}{\bigoplus_{i\in\mathbb{Z}} \xi(J)y^i},$$

which in turn induces an isomorphism

$$\overline{\Psi}\colon \frac{\mathcal{O}_q(G_{m,n}(\mathbb{K}))}{J}[\overline{\gamma}^{-1}] \xrightarrow{\cong} \frac{\mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))[y^{\pm 1};\sigma]}{\bigoplus_{i\in\mathbb{Z}} \xi(J)y^i}.$$

We have $\alpha \in J$ if and only if $\bar{\alpha}\bar{\gamma}^{-1} = 0$ in $(\mathcal{O}_q(G_{m,n}(\mathbb{K}))/J)[\overline{\gamma}^{-1}]$, which is true if and only if $\overline{\Psi}(\bar{\alpha}\bar{\gamma}^{-1}) = 0$. Since

$$\overline{\Psi}(\bar{\alpha}\bar{\gamma}^{-1}) = \overline{\Psi(\bar{\alpha}\bar{\gamma}^{-1})} \in \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))[y^{\pm 1};\sigma]/\bigoplus_{i\in\mathbb{Z}} \xi(J)y^i,$$

we have $\alpha \in J$ if and only if $\Psi(\bar{\alpha}\bar{\gamma}^{-1}) \in \bigoplus_{i\in\mathbb{Z}} \xi(J)y^i$. However the element $\bar{\alpha}\bar{\gamma}^{-1}$ of $(\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle\Pi^\gamma\rangle)[\overline{\gamma}^{-1}]$ in fact belongs to $S^o(\gamma)$, so that $\Psi(\bar{\alpha}\bar{\gamma}^{-1}) \in \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))$ by Remark 5.3.12; hence $\alpha \in J$ if and only if

$$\Psi(\bar{\alpha}\bar{\gamma}^{-1}) \in \left(\bigoplus_{i\in\mathbb{Z}} \xi(J)y^i\right) \cap \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K})) = \xi(J).$$

■

For the proof of the following theorem, we shall need a set of relations, known to hold in quantum Grassmannians, called the *generalised quantum Plücker relations.* We shall also need a version of the quantum Muir Law of extensible minors.

Quantum Muir Law (adapted from [26, Proposition 1.3]): *Let r be a positive integer. For $s \in [\![1, r]\!]$, let I_s, J_s be m-element subsets of $[\![1, n]\!]$ and let $c_s \in \mathbb{K}$ be such that $\sum_{s=1}^{r} c_s[I_s][J_s] = 0$ in $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$. Suppose that D is a subset of $[\![1, n]\!]$ such that $(\bigcup_{s=1}^{r} I_s) \cup (\bigcup_{s=1}^{r} J_s)$ does not intersect D. Then in $\mathcal{O}_q(G_{m+|D|,n}(\mathbb{K}))$, we have*

$$\sum_{s=1}^{r} c_s[I_s \sqcup D][J_s \sqcup D] = 0. \tag{5.18}$$

Generalised quantum Plücker relations [21, Theorem 2.1]: *Let $J_1, J_2, K \subseteq [\![1, n]\!]$ be such that $|J_1|, |J_2| \leq m$ and $|K| = 2m - |J_1| - |J_2| > m$. Then*

$$\sum_{K' \sqcup K'' = K} (-q)^{\ell(J_1;K') + \ell(K';K'') + \ell(K'';J_2)}[J_1 \sqcup K'][K'' \sqcup J_2] = 0, \tag{5.19}$$

where for any two sets I, J of integers, $\ell(I; J)$ denotes the cardinality of the set $\{(i, j) \in I \times J \mid i > j\}$.

Before reading the following proof, the reader might want to revisit the construction, given in Notation 5.3.7, of the partition λ from the quantum Plücker coordinate γ.

Theorem 5.4.3. *The isomorphism*

$$\Psi\colon (\mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle)[\overline{\gamma}^{-1}] \xrightarrow{\cong} \mathcal{O}_{q^{-1}}(M^\lambda_{m,n-m}(\mathbb{K}))[y^{\pm 1}; \sigma]$$

sends $\bar{\alpha}\bar{\gamma}^{-1}$ to

$$(-q)^{\ell(\{\gamma_{i_1},\ldots,\gamma_{i_t}\};\ \{a_{j_1},\ldots,a_{j_t}\})}[i_1 \cdots i_t \mid n-m+1-j_t \cdots n-m+1-j_1].$$

Proof. Suppose that $t = 1$. Then $\bar{\alpha}\bar{\gamma}^{-1} = [(\gamma \setminus \{\gamma_{i_1}\}) \sqcup \{a_{j_1}\}] = \overline{m}_{m+1-i_1,a_{j_1}}\bar{\gamma}^{-1} = \widetilde{m_{m+1-i_1,a_{j_1}}}$, which is sent by Ψ to $X_{i_1,n-m+1-j_1} = [i_1 \mid n-m+1-j_1]$. Since $\ell(\{\gamma_{i_1}\}; \{a_{j_1}\}) = 0$, the claim holds.

We proceed by induction on t. (In order to keep the notation managable here, we denote a singleton set by its element i.e. we write a singleton set $\{z\}$ simply as z.)

Let us set $\widetilde{a} = \{a_{j_2}, \dots, a_{j_t}\}$ and $\widetilde{\gamma} = \{\gamma_{i_1}, \dots, \gamma_{i_t}\}$. Applying the generalised quantum Plücker relations (5.19) with $J_1 = \widetilde{a}$, $J_2 = \emptyset$, $K = a_{j_1} \sqcup \widetilde{\gamma}$, and noticing that $\ell(\gamma_{i_l}; (a_{j_1} \sqcup \widetilde{\gamma}) \setminus \gamma_{i_l}) = \ell(\gamma_{i_l}; a_{j_1}) + l - 1$ (for all $l = 1, \dots, t$) and $\ell(\widetilde{a}; a_{j_1}) = t - 1$, we see that the following holds in $\mathcal{O}_q(G_{t,n}(\mathbb{K}))$

$$\sum_{l=1}^{t} (-q)^{\ell(\widetilde{a};\gamma_{i_l}) + \ell(\gamma_{i_l};a_{j_1}) + l - 1} [\widetilde{a} \sqcup \gamma_{i_l}][(a_{j_1} \sqcup \widetilde{\gamma}) \setminus \gamma_{i_l}] + (-q)^{t-1+\ell(a_{j_1};\widetilde{\gamma})} [a_{j_1} \cdots a_{j_t}][\widetilde{\gamma}] = 0.$$

Notice that no element of $\gamma \setminus \widetilde{\gamma}$ appears in any of the quantum Plücker coordinates in the above display, so that the quantum version of Muir's Law (5.18) with $D = \gamma \setminus \widetilde{\gamma}$ shows that in $\mathcal{O}_q(G_{t+|D|,n}(\mathbb{K})) = \mathcal{O}_q(G_{m,n}(\mathbb{K}))$, we have

$$\sum_{l=1}^{t} (-q)^{\ell(\widetilde{a};\gamma_{i_l}) + \ell(\gamma_{i_l};a_{j_1}) + l - 1} [(\widetilde{a} \sqcup \gamma_{i_l}) \sqcup (\gamma \setminus \widetilde{\gamma})][\overbrace{((a_{j_1} \sqcup \widetilde{\gamma}) \setminus \gamma_{i_l}) \sqcup (\gamma \setminus \widetilde{\gamma})}^{(\gamma \setminus \gamma_{i_l}) \sqcup a_{j_1}}]$$

$$+ (-q)^{t-1+\ell(a_{j_1};\widetilde{\gamma})} \overbrace{[(\gamma \setminus \widetilde{\gamma}) \sqcup \{a_{j_1}, \dots, a_{j_t}\}]}^{\alpha} \overbrace{[\widetilde{\gamma} \sqcup (\gamma \setminus \widetilde{\gamma})]}^{\gamma} = 0.$$

Let s be maximal such that $a_{j_1} > \gamma_{i_s}$, so that in $S(\gamma) = \mathcal{O}_q(G_{m,n}(\mathbb{K}))/\langle \Pi^\gamma \rangle$, we have $\overline{[(\gamma \setminus \gamma_{i_l}) \sqcup a_{j_1}]} = 0$ for $l > s$. Notice that if $l \leq s$, then $\ell(\widetilde{a}; \gamma_{i_l}) = t - 1$, $\ell(\gamma_{i_l}; a_{j_1}) = 0$, and $\ell(a_{j_1}; \widetilde{\gamma}) = s$, so that we may conclude from the above display that the following holds in $S(\gamma)$:

$$\bar{\alpha}\bar{\gamma} = -\sum_{l=1}^{s} (-q)^{l-1-s} \overline{[(\widetilde{a} \sqcup \gamma_{i_l}) \sqcup (\gamma \setminus \widetilde{\gamma})]}\, \overline{m}_{m+1-i_l, a_{j_1}}.$$

Now [26, Lemma 3.1.4 (v)] gives $\gamma m_{m+1-i_l,a_{j_1}} = q m_{m+1-i_l,a_{j_1}} \gamma$ for all $l = 1, \dots, s$, so that in $S^o(\gamma)$, we have

$$\bar{\alpha}\bar{\gamma}^{-1} = -q \sum_{l=1}^{s} (-q)^{l-1-s} \overline{[(\widetilde{a} \sqcup \gamma_{i_l}) \sqcup (\gamma \setminus \widetilde{\gamma})]}\, \overline{\gamma}^{-1} \overline{m}_{m+1-i_l, a_{j_1}} \overline{\gamma}^{-1}.$$

Now if we write $[\widehat{i_l} \mid n - \widehat{m+1} - j_1]$ for $[i_1 \cdots \widehat{i_l} \cdots i_t \mid n-m+1-j_t \cdots n-m+1-j_2]$ the induction hypothesis gives

$$\Psi(\bar{\alpha}\bar{\gamma}^{-1}) = \sum_{l=1}^{s}(-q)^{l-s}(-q)^{\ell(\{\gamma_{i_1},\ldots,\widehat{\gamma_{i_l}},\ldots,\gamma_{i_t}\},\widetilde{a})}[\widehat{i_l} \mid n - \widehat{m+1} - j_1]X_{i_l,n-m+1-j_1}.$$

For $l \leq s$, we have

$$\begin{aligned}
&\ell(\{\gamma_{i_1},\ldots,\gamma_{i_t}\};\ \{a_{j_1},\ldots,a_{j_t}\})\\
&= \ell(\{\gamma_{i_1},\ldots,\widehat{\gamma_{i_l}},\ldots,\gamma_{i_t}\},\widetilde{a}) + \ell(\gamma_{i_l},\{a_{j_1},\ldots,a_{j_t}\}) + \ell(\{\gamma_{i_1},\ldots,\gamma_{i_t}\};a_{j_1})\\
&= \ell(\{\gamma_{i_1},\ldots,\widehat{\gamma_{i_l}},\ldots,\gamma_{i_t}\},\widetilde{a}) + 0 + t - s.
\end{aligned}$$

Now

$$\Psi(\bar{\alpha}\bar{\gamma}^{-1}) = \sum_{l=1}^{s}(-q)^{\ell(\{\gamma_{i_1},\ldots,\gamma_{i_t}\};\ \{a_{j_1},\ldots,a_{j_t}\})+l-t}[\widehat{i_l} \mid n - \widehat{m+1} - j_1]X_{i_l,n-m+1-j_1}.$$

If $l > s$, then $a_{j_1} < \gamma_l$ and hence $|\{j \mid a_j < \gamma_{i_l}\}| \geq j_1$. Now since $|\{j \mid a_j < \gamma_{i_l}\}| = \gamma_{i_l} - i_l$ (see Notation 5.3.7), we have

$$\gamma_{i_l} - i_l \geq j_1 \qquad \text{for all } l > s. \tag{5.20}$$

If $l > s$, then (5.20) shows that $n - m + 1 - j_1 > n - m + i_l - \gamma_{i_l}$ and hence $(i_l, n-m+1-j_1) \notin Y_\lambda$ since the i_l^{th} row of the Young diagram Y_λ has only $\lambda_{i_l} = n-m+i_l-\gamma_{i_l}$ squares (again see Notation 5.3.7), so that our convention (see Definition 4.1.4) says that $X_{i_l,n-m+1-j_1} = 0$. Hence we get the following expression for $\Psi(\bar{\alpha}\bar{\gamma}^{-1})$:

$$(-q)^{\ell(\{\gamma_{i_1},\ldots,\gamma_{i_t}\};\ \{a_{j_1},\ldots,a_{j_t}\})}\sum_{l=1}^{t}(-q^{-1})^{t-l}[\widehat{i_l} \mid n - \widehat{m+1} - j_1]X_{i_l,n-m+1-j_1}.$$

Quantum Laplace expansion in $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ with the last column on the right (Lemma 4.1.9(2))[1] shows that, as required, we have

$$\Psi(\bar{\alpha}\bar{\gamma}^{-1}) = (-q)^{\ell(\{\gamma_{i_1},\ldots,\gamma_{i_t}\};\ \{a_{j_1},\ldots,a_{j_t}\})}[i_1 \cdots i_t \mid n-m+1-j_t \cdots n-m+1-j_1].$$

■

Recall the biincreasing bijection

$$\xi\colon H-\mathrm{Spec}_{\gamma}\mathcal{O}_q(G_{m,n}(\mathbb{K})) \xrightarrow{\cong} H-\mathrm{Spec}\,\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$$

given in (5.16). As in immediate consequence of Proposition 5.4.2 and Theorem 5.4.3, we get

Corollary 5.4.4. *The condition that α belongs to J is equivalent to the condition that the pseudo quantum minor $[i_1 \cdots i_t \mid n-m+1-j_t \cdots n-m+1-j_1]$ of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ belongs to the H-prime ideal $\xi(J)$ of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$.*

Recall from Notation 5.3.7 that we have set $\lambda_i = n-m-(\gamma_i - i)$ for each $i \in [\![1,m]\!]$, chosen c as large as possible such that $\lambda_c \neq 0$, and defined the partition λ by $\lambda = (\lambda_1,\ldots,\lambda_c)$. When $d = \lambda_1$, if we can show that $\{i_1,\ldots,i_t\} \subseteq [\![1,c]\!]$ and $\{n-m+1-j_t,\ldots,n-m+1-j_1\} \subseteq [\![1,d]\!]$, then the question of whether or not the pseudo quantum minor $[i_1 \cdots i_t \mid n-m+1-j_t \cdots n-m+1-j_1]$ of $\mathcal{O}_{q^{-1}}(M^{\lambda}_{m,n-m}(\mathbb{K}))$ is zero can be settled by the graph-theoretic method of Theorem 4.4.27.

[1] Care is needed with the parameters q and q^{-1} here because in the proof of Theorem 5.4.3, we are working with a partition subalgebra of $\mathcal{O}_{q^{-1}}(M_{m,n-m}(\mathbb{K}))$, whereas Lemma 4.1.9 is stated for partition subalgebras of $\mathcal{O}_q(M_{m,n}(\mathbb{K}))$.

Lemma 5.4.5. *We have*

(i) $\{i_1, \ldots, i_t\} \subseteq [\![1, c]\!]$ *and*

(ii) $\{n - m + 1 - j_t, \ldots, n - m + 1 - j_1\} \subseteq [\![1, d]\!]$.

Proof. (i) Clearly it will suffice to show that $\lambda_{i_t} > 0$. Recall from Notation 5.3.7 that

$$\begin{aligned} \lambda_{i_t} &= n - m - (\gamma_{i_t} - i_t) \\ &= n - m - |\{a \in [\![1, n]\!] \setminus \gamma \mid a < \gamma_{i_t}\}|. \end{aligned}$$

Now if $\lambda_{i_t} = 0$, then $|\{a \in [\![1, n]\!] \setminus \gamma \mid a < \gamma_{i_t}\}| = n - m$ so that every $a \in [\![1, n]\!] \setminus \gamma = \{a_1 < \cdots < a_{n-m}\}$ satisfies $a < \gamma_{i_t}$; this is impossible since $a_{j_t} > \gamma_{i_t}$.

(ii) Clearly it will suffice to show that $\lambda_1 \geq n - m + 1 - j_1$. Recall from Notation 5.3.7 that $\lambda_1 = n - m - (\gamma_1 - 1)$, so that it will suffice to show that $j_1 \geq \gamma_1$. Since $\alpha > \gamma$, γ cannot be the largest element $[n - m + 1 \cdots n]$ of Π with respect to the partial order on Π, so that $\gamma_1 \in [\![1, n - m]\!]$. Notice that $a_j = j$ for all $j < \gamma_1$ and $a_{\gamma_1} > \gamma_1$, so that $\inf\{j \in [\![1, n - m]\!] \mid a_j > \gamma_1\} = \gamma_1$. Since $a_{j_1} > \gamma_{i_1} \geq \gamma_1$, we must have $j_1 \geq \gamma_1$, as required.

∎

This brings us to the main result of this chapter, which tells us that α belongs to J if and only if there exists no vertex-disjoint $R_{\{i_1,\ldots,i_t\},\{n-m+1-j_t,\ldots,n-m+1-j_1\}}$-path system in the Cauchon graph of $\mathcal{C}$. For the sake of completeness, we include our conventions in the statement of the theorem.

Theorem 5.4.6. *Let $J \neq \langle \Pi \rangle$ be an H-prime ideal of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ and let $\gamma = [\gamma_1 < \cdots < \gamma_m]$ be the unique quantum Plücker coordinate such that $J \in H - \mathrm{Spec}_\gamma\, \mathcal{O}_q(G_{m,n}(\mathbb{K}))$ (see [24, Theorem 5.1]). Let λ be the partition corresponding to γ as in Notation 5.3.7 and let Y_λ be the Young diagram of λ. Let $\mathcal{C}$ be the Cauchon diagram on Y_λ corresponding to J as in* (5.17)*. Set $\{a_1 < \cdots < a_{n-m}\} = [\![1, n]\!] \setminus \gamma$. Let $\alpha \in \Pi$ be such that $\alpha > \gamma$ and let $1 \leq i_1 < \cdots < i_t \leq m$, $1 \leq j_1 < \cdots < j_t \leq n-m$ be such that $\alpha = [(\gamma \setminus \{\gamma_{i_1}, \ldots, \gamma_{i_t}\}) \sqcup \{a_{j_1}, \ldots, a_{j_t}\}]$ and $a_{j_l} > \gamma_{i_l}$ for all $l \in [\![1, t]\!]$. Then the quantum Plücker coordinate α belongs to J if and only if there does not exist a vertex-disjoint $R_{\{i_1,\ldots,i_t\},\{n-m+1-j_t,\ldots,n-m+1-j_1\}}$-path system in the Cauchon graph $\mathcal{G}_\mathcal{C}$ of $\mathcal{C}$.*

Proof. This follows immediately from Corollary 5.4.4, Lemma 5.4.5, and Theorem 4.4.27. ■

5.5 A link with totally nonnegative Grassmannians

Theorem 5.4.6 provides a link between the quantum and totally nonnegative Grassmannians. Let F be any family of m-element subsets of $[\![1, n]\!]$. Then F defines a nonempty cell in the totally nonnegative Grassmannian $\mathrm{Gr}^{\mathrm{tnn}}_{m,n}$ if and only if there is an H-prime ideal J of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ such that the quantum Plücker coordinates belonging to J are exactly those corresponding to the members of F (see a result of Postnikov appearing as Proposition 13 in [33]). Consequently, the main result of [33] gives alternative descriptions of the families of quantum Plücker coordinates which belong to H-prime ideals of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$.

5.6 A possible link with the strong Dixmier-Moeglin equivalence

Using the process of noncommutative dehomogenisation and some results from Chapter 3, it is easy to show that $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ satisfies the quasi strong Dixmier-Moeglin equivalence. We conjecture that $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ in fact satisfies the strong Dixmier-Moeglin equivalence. The key to proving this property for quantum Schubert cells was an understanding of the inclusions between the torus-invariant prime ideals and we believe that the same will be true for $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$. We believe that Theorem 5.4.6 may assist in a description of the inclusions between the torus-invariant prime ideals of $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$; this would both solve an open problem and, we believe, provide the key information for proving that $\mathcal{O}_q(G_{m,n}(\mathbb{K}))$ satisfies the strong Dixmier-Moeglin equivalence.

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