A mean-variance optimisation approach to collectively pricing warranty policies

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Abstract

Warranty policy can influence the profit and cost of a product. In practice, a manufacturer commonly produces more than one product, or a portfolio of products, and provides warranty servicing for them. Many authors have attempted to optimise warranty policy to maximise the profit or minimise the cost of each individual product. Warranty claims of the products produced by the same manufacturer, however, may be due to common causes, since the products may be designed by the same engineer team or using the same type of components. This implies that the numbers of warranty claims of different products may be related, and optimisation of warranty policies for each individual product may therefore cause biased decisions. To overcome this disadvantage, this paper aims to collectively optimise a manufacturer's total profit for a portfolio of different products by using a mean-variance optimisation approach. A tool from the probability theory, *copulas*, is used to depict the dependence among the warranty claims of different products. Numerical examples are provided to illustrate the application of the proposed methods.

Keyword Modern portfolio theory, warranty, risk, mean-variance, optimisation.

1 Introduction

1.1 Background

Warranty plays an important role in consumer and commercial transactions. It is essentially offered with most durable products to promote product sales. Warranty is a tool to assure consumers' satisfaction with product performance over the warranty period (Liao, 2016). For a

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manufacturer, warranty is a contractual obligation: the manufacturer should compensate customers through offering repair or replacement service in the event of occurrence of premature failures of the warranted items or items' inability to perform its intended function (Karim & Suzuki, 2005; Wu, 2014b). There are several warranty acts enacted in the US over the last 100 years (UCC, Magnusson Moss Warranty Act, Tread Act, for example). The European Union (EU) also passed legislation requiring a two-year warranty for all products sold in Europe (Wu, 2014b; Murthy & Djamaludin, 2002).

Warranty expense is an important part of the manufacturer's operating expense. In the manufacturing industries, warranty incurs huge amount of cost, for example, the total cost of worldwide warranty claims of U.S.-based firms is $26.4 billion in 2015; and the balance of their total warranty reserves is $43.35 billion at the end of 2015 (WarrantyWeek, 2016).

A typical warranty transaction is: a consumer pays the warranty price when purchasing the product and the manufacturer (or dealer) provides repair or replacement service for product failures occurring during the warranty period. A good warranty policy will raise a firm's brand image and reputation among consumers. The assurance of warranty can reduce the costs associated with failure of the product purchased from a consumer's perspective. Warranties highlight product reliability and quality: a longer warranty can send out a strong signal about the products and service quality (Liao, 2016). However, warranty providers should be aware of two major uncertainties in warranty management:

Uncertainty of warranty cost. The demands for repairs or replacements always come out unexpectedly, and the number of warranty claims received during the warranty period normally appears only as probability-based forecasts (Yenipazarli, 2014).

Uncertainties of sales volume. The sales volume of most goods may increase with the decrease of warranty price. It is true that high sales price of a product may increase the profit per item; but it may become less attractive to its consumers and therefore reduce the product total demand. Meanwhile, from a consumer's perspective, longer warranty length, which implies higher warranty price for the manufacturer, indicates better product quality and reliability. It is reported that longer warranties bring more unnecessary cost to manufacturers (Aggrawal, Anand, Singh, & Singh, 2014) than short ones. A manufacturer should consider its total cost and profit comprehensively. The total profit is determined by the total revenue and cost, where the total revenue is product price multiplying sales volume, and the total cost is the sum of production cost, warranty cost and other operation costs multiplying sales volume.

Both product price and warranty length are normally set by the manufacturer to reflect the

current market. The goal of most manufacturers is to maximise profit, which may be achieved by optimally pricing the warranty policy of their products. Meanwhile, warranty cost can be treated as a random negative cash flow which depends on the sales volume and the length of warranty period. By *random*, it means uncertainty, presenting the manager the expected warranty cost, which is difficult to forecast. As such, precisely forecasting warranty cost is vitally important.

In existing publications relating optimisation of warranty policies, researchers have introduced many methods to maximise profit through optimising product price and warranty length in different scenarios. For example, Yazdian, Shahanaghi, and Makui (2016) jointly optimises the acquisition price, remanufacturing degree, selling price and warranty length of remanufacturing products, assuming there are linear and non-linear demand functions. Wei, Zhao, and Li (2015) investigates the optimal strategies on product price and warranty length of two complementary products from two manufacturers in a two-stage game theoretic perspective. Yeh and Fang (2015) introduces a model to optimise product price and warranty length considering the manufacturer's production capacity and preventive maintenance program. Aggrawal et al. (2014) present a method to optimise price and warranty length for a product based on a two dimensional innovation diffusion model. Wu, Chou, and Huang (2009) develop a decision model to determine the optimal price, warranty length and the production rate of a product to maximise profit based on the pre-determined life cycle in a static demand market. Similar to these articles, all of the other existing research, including Lin, Wang, and Chin (2009); Matis, Jayaraman, and Rangan (2008); Ladany and Shore (2007); Huang, Liu, and Murthy (2007), etc., only maximises the profit of each individual product for a manufacturer.

1.2 Motivation and novelty

The numbers of warranty claims of products produced by the same manufacturer may not be statistically independent because they may be designed by the same team of engineers, manufactured on the same production lines and share same types of components. As a result, they may have common causes. For example, Ford's turbocharged and direct injection gasoline engines, belonging to the EcoBoost family, are applied on many different types of Ford cars, including Focus, Fiesta, Mondeo, etc. If any design or quality problems happened on the EcoBoost, warranty claims from different products will crop up during a short period.

As can be seen from the above literature review, however, existing literature has been concentrated on warranty reserve optimisation for each individual product separately. Little attention has been paid to collectively optimising warranty policies and reserves for a portfolio of different products. To maximise the total profit for a manufacturer that sells a number of products, the

prices of all products should be optimised collectively, which is the aim of this paper.

This paper is the first attempt that collectively optimises the prices of a portfolio of products for a manufacturer, which creates novelty. It proposes to use an approach borrowed from the modern portfolio theory, which is widely applied in the financial sector. It develops a novel method that collectively optimises warranty reserves for a set of different products, considering the dependence of the numbers of warranty claims. The warranty prices of the set of products are then optimised under the mean-variance optimisation framework.

The paper uses a tool from the probability theory, *copulas*, to depict the dependence among warranty claims of different products, which can reduce the bias that may be caused in modelling a complicated dependence with a simple method such as covariance estimation. This is because the covariance matrix can only reflect the linear correlation whereas copulas can model more complicated nonlinear dependence

1.3 Overview

The other sections of the paper are structured as follows. Section 2 formulates the problem. Section 3 investigates the existence of the optimal values for different scenarios and uses copulas to depict the dependence among warranty claims of different products. Section 4 offers numerical examples to illustrate the proposed methods. Section 5 concludes the paper and proposes future work.

2 Formulation of the problem

Assume a manufacturer offers non-renewing free replacement warranty (NFRW) policies. Under a NFRW policy, the manufacturer provides its customers with repair or replacement at no cost within the warranty period, the original warranty is not altered upon a failed item, and the manufacturer only guarantees satisfactory service on the item within the original warranty period. We also assume the repair time is negligible. The items are new at $t = 0$ when they are sold. The number of claims follows the homogeneous Poisson process (HPP). The numbers of warranty claims and the claim cost are statistically independent.

In practice, warranty policies can be categorised into one- and two-dimensional (1-D and 2-D). A 1-D policy is characterised by an interval, such as age or usage, as warranty limit and a 2-D policy is characterised by a region in the 2-D plane (Ye & Murthy, 2016), such as age and usage. In the literature, many methods are used to deal with the 2-D siutations, for example, some authors use a so-called composite scale approach that integrates the two scales (age and usage)

to create a single composite scale and model the claim arrival process (Wu, 2012). In this paper, only one scale, the length of warranty, noted by T_k, is considered in modelling. Of course, one mat regard T_k as the usage in 1-D warranty or the composite scale in 2-D warranty.

The notations in Table 1 are used throughout this paper.

Table 1: Notation table

$X_{k,i}$	Cost of the ith warranty claim of product k
$N_k(t)$	Number of warranty claims of product k within time interval $(0,t)$
$S_k(t)$	Total cost of warranty claims of product k within time interval $(0,t)$
P_k	Price of product k
T_k	Warranty length of product k
λ_k	Parameter of the claim arrival process of product k
μ_k	Expected cost per claim of product k
c_k	Fixed manufacturing cost of product k
$M_k(P_k, T_k)$	Sales volume of product k when the warranty price is P_k and the warranty length is T_k
$\boldsymbol{M}$	Vector of all sales volume, $\mathbf{M} = [M_1, ..., M_n]'$, where $[.]'$ denotes the transpose of a matrix/vector,
$\omega_k(P_k, T_k)$	Profit of product k when the warranty price is P_k and the warranty length is T_k
$\Omega(\mathbf{P}, \mathbf{T})$	Total profit of the manufacturer when the warranty prices is $\mathbf{P} = [P_1, ..., P_n]'$ and the warranty length is $\mathbf{T} = [T_1, ..., T_n]'$.

2.1 The expected number of warranty claims and cost

Suppose a manufacturer produce n products. The aggregated warranty cost of product k follows a stochastic process $\{S_k(t)\}_{t\geq 0}$ over the time interval $(0,t)$, which is expressed by the following equation,

$$S_k(t) = \sum_{i=1}^{N_k(t)} X_{k,i}. \tag{1}$$

In Eq. (1), the cost of claim i of product k is described by the random variable $X_{k,i}$ and the counting process $N_k(t)$, which is the number of claims during $(0,t)$ and is assumed to take a form of a homogeneous Poisson process (HPP) with intensity $\lambda_k > 0$, $P(N_k(t) = i) = \frac{(\lambda_k t)^i e^{-\lambda_k t}}{i!}$. For a given k, $X_{k,i}$ are independent and identically distributed random variables which have finite values on the positive half-line $\mathbb{R}_{>0}$ with the probabilities $P(X_{k,i})$. The frequency $N_k(t)$ and severity $X_{k,i}$ are assumed to be independent.

Then, the expected value of $S_k(t)$ is given by

$$\mathrm{E}[S_k(t)] = \mathrm{E}[N_k(t)]\mathrm{E}[X_k] = \lambda_k \mu_k t, \tag{2}$$

and the variance of $S_k(t)$ is given by

$$\begin{aligned} \mathrm{Var}\left[S_k(t)\right] &= \mathrm{E}[N_k(t)]\mathrm{Var}[X_k] + \mathrm{Var}[N_k(t)]\mathrm{E}[X_k]^2 \\ &= \lambda_k t(\mathrm{Var}[X_k] + \mathrm{E}[X_k]^2) \\ &= \lambda_k t(\sigma_k^2 + \mu_k^2), \end{aligned} \tag{3}$$

where μ_k and σ_k are the mean and variance of X_k, respectively.

2.2 The sales volume and profit

In the literature, the sales volume of a product may be expressed in a linear (Yazdian et al., 2016; Lin et al., 2009) or non-linear form (Ladany & Shore, 2007; Xie, Liao, & Zhu, 2014; Huang et al., 2007), depending on different scenarios. For simplicity, this paper uses the linear form proposed by Yazdian et al. (2016) and assumes the following linear relationship among the length of warranty coverage, warranty price and sales amount of a product:

$$M_k = A_k - \beta_k P_k + \eta_k T_k, \tag{4}$$

where P_k is the warranty price, T_k is the warranty period of product k, M_k is the sales volume of product k, respectively. A_k,β_k and $\eta_k > 0$ are positive real numbers. The profit of one item of product k is

$$r_k = P_k - S_k(T_k) - c_k, \tag{5}$$

where c_k is the fixed cost of one item of product k, including manufacturing cost, management expenditures, etc.

Then, the profit of product k is

$$\omega_k(P_k, T_k) = M_k[P_k - S_k(T_k) - c_k], \tag{6}$$

the expected value of $\omega_k(P_k, T_k)$ is

$$\mathrm{E}[\omega_k(P_k, T_k)] = (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k \lambda_k \mu_k - c_k),$$

and the variance of $\omega_k(P_k, T_k)$ is

$$\mathrm{Var}[\omega_k(P_k, T_k)] = M_k^2 \mathrm{Var}[S_k(t)] = M_k^2 T_k \lambda_k (\sigma_k^2 + \mu_k^2).$$

For the n products, the total profit of the manufacturer is

$$\Omega(\boldsymbol{P}, \boldsymbol{T}) = \sum_{k=1}^{n} \omega_k(P_k, T_k). \tag{7}$$

$\Omega(\boldsymbol{P}, \boldsymbol{T})$ can be re-written as

$$\begin{aligned} \Omega(\boldsymbol{P}, \boldsymbol{T}) &= \sum_{k=1}^{n} \omega_k(P_k, T_k) \\ &= \sum_{k=1}^{n} (A_k - \beta_k P_k + \eta_k T_k) \left[P_k - S_k(T_k) \right]. \end{aligned} \tag{8}$$

The mean and variance of $\Omega(\boldsymbol{P}, \boldsymbol{T})$ are

$$\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] = \sum_{k=1}^{n} \mathrm{E}[\omega_k(P_k, T_k)] = \sum_{k=1}^{n} (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k \lambda_k \mu_k - c_k), \tag{9}$$

and

$$\mathrm{Var}(\Omega(\boldsymbol{P}, \boldsymbol{T})) = \boldsymbol{M}' \boldsymbol{V} \boldsymbol{M}, \tag{10}$$

respectively, where

$$\boldsymbol{M}' = \left[M_1, M_2, \ldots, M_n \right], \tag{11}$$

and

$$\boldsymbol{V} = \begin{bmatrix} \mathrm{Var}(S_1(T_1)) & \mathrm{Cov}(S_1(T_1), S_2(T_2)) & \ldots & \mathrm{Cov}(S_1(T_1), S_n(T_n)) \\ \mathrm{Cov}(S_2(T_2), S_1(T_1)) & \mathrm{Var}(S_2(T_2)) & \ldots & \mathrm{Cov}(S_2(T_2), S_n(T_n)) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Cov}(S_n(T_n), S_1(T_1)) & \mathrm{Cov}(S_n(T_n), S_2(T_2)) & \ldots & \mathrm{Var}(S_n(T_n)) \end{bmatrix}. \tag{12}$$

A manufacturer may wish to optimise $\boldsymbol{P}$ and $\boldsymbol{T}$ in the quantity $\Omega(\boldsymbol{P}, \boldsymbol{T})$ in Eq. (7) to achieve the maximum profit.

Each T_k in $\boldsymbol{T}$ may be either a commonly agreed quantity or an endogenous variable: If it is an endogenous variable, T_k may be determined by the warranty provider; if it is a commonly agreed quantity, T_k may be served as an exogenous variable. Since in reality, the length of warranty is pre-specified and it can be from a small number of discrete positive integers, 12 months, 24

months, 30 months, etc, for example. In this paper, each T_k in $\boldsymbol{T}$ is regarded as a commonly agreed quantity. In other words, we mainly focus on optimally pricing warranty policies for a given $\boldsymbol{T}$, that is, to seek $\boldsymbol{P}$ so that the expected profit $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ can be maximised.

3 Optimisation of the profit

For a manufacturer, $\Omega(\boldsymbol{P},\boldsymbol{T})$ in Eq. (7) can be maximised through seeking the optimal values of P. Another idea is to use the mean-variance approach: since the variance of $\Omega(\boldsymbol{P},\boldsymbol{T})$ can be regarded as a risk measure, one may integrate this risk when optimising $\Omega(\boldsymbol{P},\boldsymbol{T})$. These two ideas are discussed in the following two sub-sections, respectively.

3.1 Maximising the expected profit

For the convenience of further discussion, this subsection gives the optimisation objective function and does not consider the risk (i.e., the variance) of the portfolio of the warranty claim costs of the n products and merely aims to optimise $\Omega(\boldsymbol{P},\boldsymbol{T})$, which is a commonly used setting.

The profit of product k can then be derived and expressed by

$$\mathrm{E}[\omega_k(P_k,T_k)] = (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k\lambda_k\mu_k - c_k), \tag{13}$$

which is a bivariate quadratic function with respect to T_k and P_k.

Hence, we have the following proposition.

Proposition 1 *If both $\boldsymbol{P}$ and $\boldsymbol{T}$ are vectors of decision variables and there are no other constraints, the optimum solution that maximises the expected profit,* $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ *does not exist.*

The proofs of all the propositions in this paper can be found in the Appendix.

In practice, the warranty length $\boldsymbol{T}$ is normally a discrete variable in a finite range, such as 24 months, 30 months, etc. If $\boldsymbol{T}$ is given, it is easy to prove that there is a maxima in $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$.

Proposition 2 *If $\boldsymbol{T}$ is known and $\boldsymbol{P}$ is a vector of decision variables, there exists an optimum solution that maximises the expected profit* $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$.

In the following section, we assume $\boldsymbol{T}$ is known, and investigate the existence of the optimum solutions of $\boldsymbol{P}$ when considering the risk against profit.

3.2 Mean-variance approach to pricing warranty policies

In this section, we use the mean-variance optimisation approach to pricing warranty policies.

When both profits and risk are taken into consideration, manufacturers may have the following options:

Option 1. to maximise a combination of the profit and the risk of the estimated profit;

Option 2. to maximise the profit and meanwhile to limit the risk of the estimated profit; and

Option 3. to minimise the risk of the estimated profit subject to the constraint that the lower bound of the profit is greater than a pre-specified value.

The above three options are equivalent to the following three optimisation problems, respectively.

Option 1. Maximise $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$, subject to $\boldsymbol{M} \geq \mathbf{1}$, where $\mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is the standard deviation of total profit and $\boldsymbol{M} \geq \mathbf{1}$ means the sale volume of each product should not be less than 1 unit. $\mathbf{1}$ is the unit matrix with the same dimensions as those of $\boldsymbol{M}$.

Option 2. Maximise $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$, subject to $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq \varphi$, and $\boldsymbol{M} \geq \mathbf{1}$, which implies maximising the expected total profit at a given risk level.

Option 3. Minimise $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$, subject to $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq \psi$, and $\boldsymbol{M} \geq \mathbf{1}$. It implies minimising the risk at a given expected total profit.

On Option 1, we have the following Propositions 3 and 4.

Proposition 3 *For product k, if the warranty length T_k is known and the price P_k is a decision variable, there exists an optimum solution that maximises* $\mathrm{E}[\omega_k(P_k,T_k)] - \mathrm{SD}[\omega_k(P_k,T_k)]$.

As mentioned above, the warranty claims of different products could be dependent, which implies the variance-covariance matrix of the portfolio includes non-zero covariances.

The covariance between product k and l is:

$$\begin{aligned}
\mathrm{Cov}[S_k(T_k), S_l(T_l)] &= \mathrm{E}[S_k(T_k)S_l(T_l)] - \mathrm{E}[S_k(T_k)]\mathrm{E}[S_l(T_l)] \\
&= \mathrm{E}[\mathrm{E}[(\sum_{i=1}^{N_k(T_k)} X_{k,i})(\sum_{i=1}^{N_l(T_l)} X_{l,i})|N(T_k),N(T_l)]] - \lambda_k\mu_k T_k \lambda_k \mu_l T_l \\
&= \mathrm{E}[N_k(T_k)N_l(T_l)]\mathrm{E}[X_k X_l] - \lambda_k\mu_k T_k \lambda_k \mu_l T_l \\
&= \mathrm{Cov}[N_k(T_k), N_l(T_l)]\mu_k\mu_l
\end{aligned}$$

Then we have the following Propositions 4, 5 and 6.

Proposition 4 *If* $\boldsymbol{P}$ *is a vector of decision variables and* $\boldsymbol{T}$ *is known, there exists an optimum solution that maximises the combination of the profit and the risk of the estimated profit* $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$.

Proposition 5 *If* $\boldsymbol{P}$ *is a vector of decision variables and* $\boldsymbol{T}$ *is known, there exists an optimum solution that maximises* $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$*, subject to* $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T})) \leq \varphi$.

Proposition 6 *If* $\boldsymbol{P}$ *is a vector of decision variables and* $\boldsymbol{T}$ *is known, there exists an optimum solution that minimises* $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$*, subject to* $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq \psi$.

3.3 Methods of estimating $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$

For a given set of warranty claim data, one must estimate $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ for the three options discussed in Section 3.2 before any optimisation problems may be discussed. One may estimate them with two methods:

Method 1. The non-parametric method, for example, the method of moments estimation, with which there is no need to assume a probability distribution.

Method 2. The parametric method, for example, the maximum likelihood estimation method. With this method, one needs to estimate a joint distribution of the numbers of warranty claims and then derive $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ from the distribution.

Similar to the fact that the Pearson correlation coefficient can only describe the linear relationship between two random variables, the covariance can merely measure a linear correlation. As such, one should note, Method 1 may not be able to capture the nonlinear relationship among the numbers of warranty claims. In such cases, Method 2 may be advocated. Given the difficulty of estimating a joint multivariate probability distribution, a tool, *copula*, which is a widely studied topic in recent years and can be used to depict nonlinear relationships between random variables, is borrowed to model the joint distribution of the numbers of warranty claims.

Copulas are widely used in constructing multivariate distributions and formalising the dependence structures between random variables, whatever discrete or continuous. Abe Sklar first introduced the notion of copula in 1959, in recent years copula has attracted considerable attention in both theoretical and application aspects. The Theorem of Sklar states that any cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula that describes the dependence structure between the variables (Wu, 2014a).

Assume $(X_1, ..., X_d)$ is a given vector of random variables, its cumulative distribution function is $H(x_1, ..., x_d) = P(X_1 \leq x_1, ..., X_d \leq x_d))$, and its marginals are $F_k(x_k) = P(X_k \leq x_k)$, where

$k = 1, ...d$. Sklar proved that $H(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$, where $C(.)$ is defined as a copula. Copulas are useful in statistical applications because they allow one to estimate the marginals and the copula separately when modelling and estimating the distribution of a random vector (Wu, 2014a).

Similar to its work in modelling the dependence among continuous random variables, copula can also be used in constructing the joint probability distribution of discrete variables (Nikoloulopoulos & Karlis, 2010). In this area, Joe and Hu (1996) introduce multivariate parametric families of copulas which are mixtures of max-infinitely divisible (max-id) bivariate copulas. Nikoloulopoulos and Karlis (2010) show this class of copulas has superiority to others, because it allows flexible dependence among the random variables and has a closed form cdf (cumulative distribution function) and thus computations are rather easy.

In our case, the claim arrival process of product k follows a homogeneous Poisson process, i.e. $N_k \sim \text{Pois}(\lambda_k t_k)$. Assume that M_k items of product k have been sold to the market and the claim processes of a given product's items are mutual independent, then from time 0 to time t_k, the average number of claim per item also follows a homogeneous Poisson process, i.e. $\bar{N_k} \sim \text{Pois}(\lambda_k t_k)$.

Denote $u_k(n_k) = P(\bar{N_k} \leq n_k)$, the joint distribution of each products' average number of claim per item is

$$H(n_1, n_2, \ldots, n_k) = C(u_1(n_1), u_2(n_2), \ldots, u_k(n_k)), \tag{14}$$

where C is a copula. Then, the joint probability mass function is,

$$h(n_1, n_2, \ldots, n_k) = \sum_{\substack{y_j \in \{n_j, n_j - 1\} \\ 1 \leq j \leq k}} \left(H(y_1, y_2, \ldots, y_k) \prod_{i=1}^{k} \text{sgn}(y_i) \right), \tag{15}$$

where y_j is equal to n_j or $n_j - 1$, and

$$\text{sgn}(y_j) = \begin{cases} 1, y_j = n_j \\ -1, y_j = n_j - 1. \end{cases}$$

Apparently, $\sum_{\substack{y_j\in\{n_j,n_j-1\}\\1\le j\le k}}$ has 2^k elements in total. For example,

$$\begin{aligned} h(n_1,n_2,n_3) &= \sum_{\substack{y_j\in\{n_j,n_j-1\}\\1\le j\le 3}} [H(y_1,y_2,y_3)\prod_{i=1}^{3}\mathrm{sgn}(y_i)] \\ &= H(n_1,n_2,n_3)-H(n_1-1,n_2,n_3)-H(n_1,n_2-1,n_3)-H(n_1,n_2,n_3-1) \\ &\quad +H(n_1-1,n_2-1,n_3)+H(n_1-1,n_2,n_3-1)+H(n_1,n_2-1,n_3-1) \\ &\quad -H(n_1-1,n_2-1,n_3-1). \end{aligned}$$

The expected total number of warranty claims of the manufacturer is

$$\mathrm{E}[N]=\sum_{i=1}^{\infty}[h(n_{1,i},n_{2,i},\ldots,n_{k,i})(n_{1,i}+n_{2,i}+\cdots+n_{k,i})],$$

The expected total profit is

$$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]=\sum_{i=1}^{\infty}\{h(n_{1,i},n_{2,i},\ldots,n_{k,i})[M_1(P_1-n_{1,i}\mu_1-c_i)+\cdots+M_k(P_k-n_{k,i}\mu_k-c_k)]\}. \quad (16)$$

The variance of total expected profit is

$$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]=\sum_{k=1}^{n}(A_k-\beta_kP_k+\eta_kT_k)^2\lambda_kT_k(\sigma_k^2+\mu_k^2)+2\sum_{1\le k<l\le n}M_kM_l\mu_k\mu_l\mathrm{Cov}[N_k(T_k),N_l(T_l)], \quad (17)$$

where $\mathrm{Cov}[N_k(T_k),N_l(T_l)]$ can be calculated according to the joint probability mass function $h(n_1,n_2,\ldots,n_k)$.

Furthermore, the joint distribution of the total profit can also be estimated based on the joint probability mass function $h(n_1,n_2,\ldots,n_k)$,

$$\begin{aligned} &\mathrm{P}\left((A_1-\beta_1P_1+\eta_1T_1)\left[P_1-S_1(T_1)\right]=z_1,...,(A_n-\beta_nP_n+\eta_nT_n)\left[P_n-S_n(T_n)\right]=z_n\right)= \\ &h(n_1,n_2,\ldots,n_n)\prod_{i=1}^{n_1}P(X_{1,i}=\frac{z_1}{n_1})\prod_{i=1}^{n_2}P(X_{2,i}=\frac{z_2}{n_2})\cdots\prod_{i=1}^{n_n}P(X_{n,i}=\frac{z_n}{n_n}). \end{aligned} \quad (18)$$

4 Numeric example

Assume a manufacturer produces 3 products. The numbers of warranty claims follow homogeneous Poisson processes with intensity functions λ_1, λ_2, and λ_3, respectively. The warranty claim

costs of the three products follow normal distributions with means μ_1, μ_2, μ_3, respectively; and standard deviations σ_1, σ_2, and σ_3, respectively.

The related parameters of these products are presented in Table 2. $\boldsymbol{A}$, $\boldsymbol{\beta}$, and $\boldsymbol{\eta}$ are the parameter vectors of sales volume function; $\boldsymbol{\lambda}$ is the intensity vector of the HPPs; $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are the mean and standard deviation of claim cost; and $\boldsymbol{c}$ is the fixed cost vector.

Table 2: Parameters

	$\boldsymbol{A}$	$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$	$\boldsymbol{c}$
Product 1	$A_1 = 8000$	$\beta_1 = 80$	$\eta_1 = 5$	$\lambda_1 = 0.08$	$\mu_1 = 6$	$\sigma_1 = 1$	$c_1 = 40$
Product 2	$A_2 = 9000$	$\beta_2 = 85$	$\eta_2 = 6$	$\lambda_2 = 0.09$	$\mu_2 = 5$	$\sigma_2 = 0.8$	$c_2 = 50$
Product 3	$A_3 = 6000$	$\beta_3 = 60$	$\eta_3 = 10$	$\lambda_3 = 0.1$	$\mu_3 = 7$	$\sigma_3 = 1.5$	$c_3 = 30$

In this section, the unit of warranty length is assumed to be month and the unit of cost is assumed to be GBP (Great Britain Pound).

4.1 Maximizing the expected profit

The maximum expected total profit is the sum of all products' maximum expected profit. However, if P_k and T_k both are decision variables, the function, $\mathrm{E}[\omega_k(P_k, T_k)] = (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k\lambda_k\mu_k - c_k)$, does not have a maximum value.

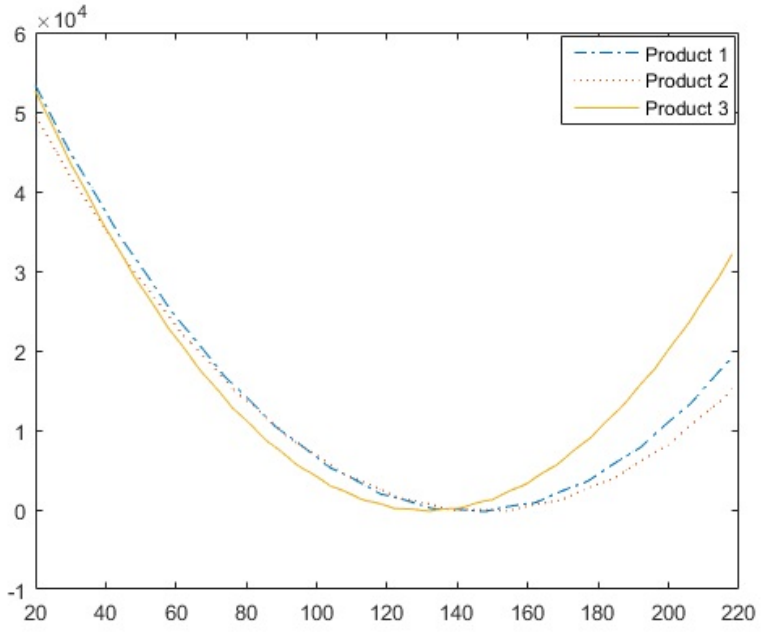

Figure 1: $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ (on the Y-axis) against T_k (on the X-axis).

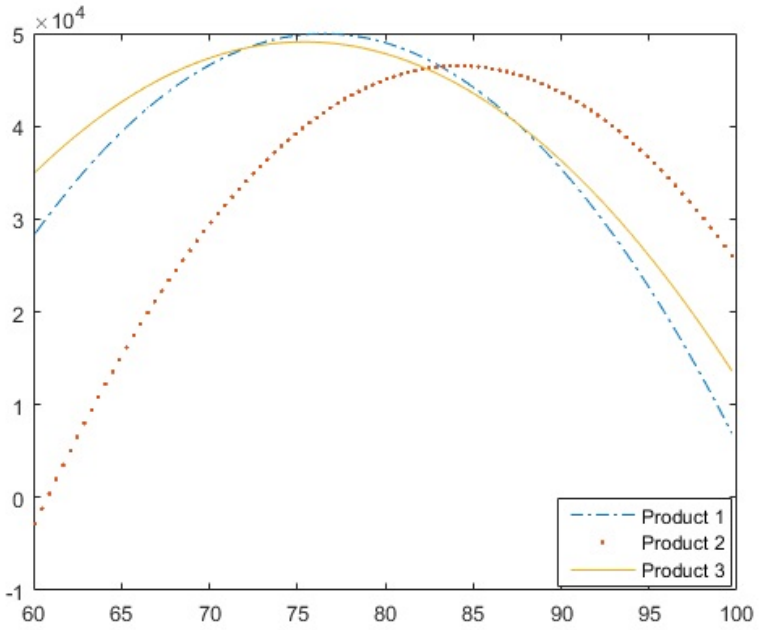

Figure 2: $\mathrm{E}[\omega_k(P_k, T_k)]$ (on the Y-axis) against P_k (on the X-axis).

Based on Proposition 1, Fig. 1 shows the relationship between the maximised expected profit $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ and the warranty length T_k of product k. For each given T_k, $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ exists. When T_k increases, $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ keeps changing.

Based on Proposition 2, Fig. 2 presents the case that when $T_k = 24$ $(k = 1, 2, 3)$, the expected-profit against price curves are parabolic. There exists an optimal value P_k to maximise $\mathrm{E}[\omega_k(P_k, T_k)]$ for each product. If the warranty length of the products is 2 years, i.e. 24 months, Table 3 presents the optimal prices and the maximised expected profits, i.e. the values at peak points of the curves in Fig. 2.

Table 3: Maximum profits of products with given warranty length, 2 years.

Warranty length	T_1=24	T_2=24	T_3=24
Optimal price	P_1=76.51	P_2=84.19	P_3=75.40
Maximised expected profit	49,960.01	46,495.81	49,077.60

4.2 Mean-variance optimisation approach

Based on Proposition 4, the manufacturer's profit can be optimised when the risk is taken into consideration. For product k, Fig. 3 presents the results of Option 1, as discussed in Section 3.2, in which the manufacturer tries to maximise $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ when $T_k = 24$

($k = 1, 2, 3$). This figure shows there exists an optimal P_k which maximise $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ for each product.

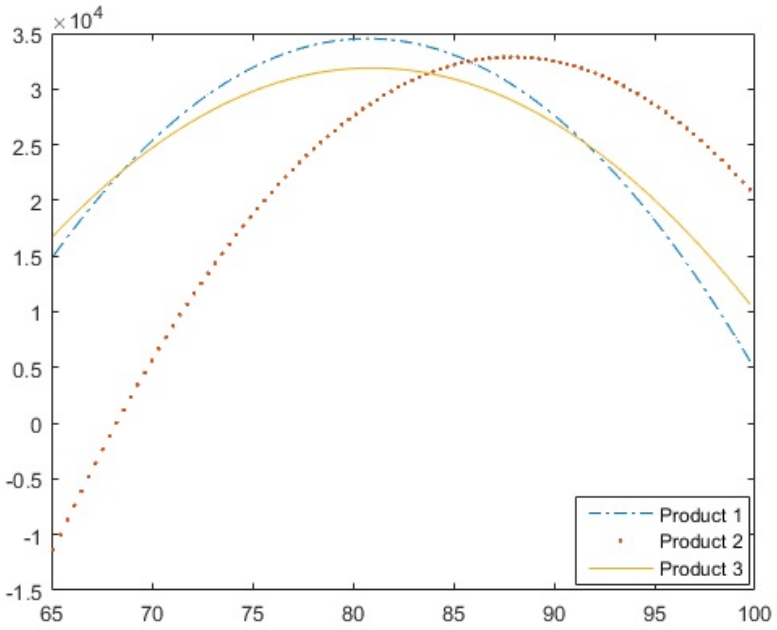

Figure 3: $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ (on the Y-axis) against P_k (on the X-axis).

For Option 2 mentioned in Section 3.2, the covariance $\boldsymbol{V}$ shown in Eq. (12) should be estimated. In this case, instead of estimating $\boldsymbol{V}$, we simply assume its correlation matrix, as shown in the following:

$$\boldsymbol{\rho} = \begin{Bmatrix} 1 & 0.3 & 0.4 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.5 & 1 \end{Bmatrix}. \tag{19}$$

Table 4 shows the result of Option 2, which agrees with Proposition 5 that there exists an optimal solution. Under the assumption that the manufacturer's goal is to maximise profit when the variance of the profit should be less than 100,000. Table 4 presents the optimal prices when $\boldsymbol{T}$ are varying within 24-month and 36-month. If the dependence among the products are ignored, i.e. use identity matrix $\boldsymbol{I}$ instead of the correlation matrix $\boldsymbol{\rho}$ in computing, the optimal results under the same assumption are presented in Table 5. The differences between the contents of Tables 4 and 5 indicate that if the dependence among products are ignored, the manufacturer may misprice individual products and overvalue the expected total profit.

Similarly, when the same $\boldsymbol{\rho}$ in Eq. (19) is used, Table 6 shows the result of Option 3, which

Table 4: Maximum profits with different $\boldsymbol{T}$ at given variance, where $\text{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \leq 100,000$.

Warranty length	T_1=24, T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=98.37 P_2=100.63 P_3=104.00	P_1=102.25 P_2=99.66 P_3=102.46	P_1=100.62 P_2=106.00 P_3=100.73	P_1=98.71 P_2=105.30 P_3=106.00
Maximised expected profit	35,235.27	31,303.88	24,449.68	22,106.10

Table 5: Maximum profits with different $\boldsymbol{T}$ at given variance ignoring dependence, where $\text{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \leq 100,000$.

Warranty length	T_1=24, T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=92.92 P_2=96.98 P_3=103.81	P_1=100.95 P_2=95.20 P_3=99.85	P_1=99.80 P_2=102.93 P_3=98.53	P_1=96.65 P_2=101.02 P_3=106.00
Maximised expected profit	61,632.49	53,951.25	42,455.64	39,544.61

agrees with Proposition 6 that there exists an optimal solution. Under the assumption that the manufacturer's goal is to minimise the variance of the profit when the expected profit should not be less than 50,000, Table 6 presents the optimal prices when $\boldsymbol{T}$ are varying within 24-month and 36-month. When the dependences among products are ignored, the optimal results under the same assumption are presented in Table 7. The results in Table 6 and 7 indicate if the manufacturer does not recognise the dependences among the products, the total risk level will be undervalued.

Table 6: Minimum variance with different $\boldsymbol{T}$ at given expected profit level, where $\text{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \geq 50,000$.

Warranty length	T_1=24,T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=94.33 P_2=100.36 P_3=103.65	P_1=100.22 P_2=98.14 P_3=99.89	P_1=97.85 P_2=103.52 P_3=97.36	P_1=94.70 P_2=100.86 P_3=104.26
Minimised variance	147,995.66	176,226.69	218,509.72	258,176.05

4.3 Measuring dependence with copulas

In case more complicated dependence is assumed, as discussed in Section 3.3, the copulas can be applied. Here, we assume that the multivariate Gumbel copula is applied. The multivariate

Table 7: Minimum variance with different $\boldsymbol{T}$ at given expected profit level ignoring dependence, where $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq 50,000$.

Warranty length	T_1=24,T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=95.29 P_2=98.83 P_3=104.00	P_1=101.45 P_2=95.46 P_3=100.43	P_1=98.89 P_2=102.22 P_3=97.48	P_1=94.43 P_2=98.75 P_3=106.00
Minimised variance	76,432.03	90,178.34	120,554.13	134,952.03

Gumbel copula can capture strong upper tail dependence and can be used in the case, for example, there is a design problem on a type engine used on different types of car, many similar failures, which cause warranty claims, may occur in the early life period, or the premature period. The multivariate Gumbel copula is,

$$C(u_1,u_2,u_3;\theta_1,\theta_2) = \exp\{-([(-\ln u_1)^{\theta_2} + (-\ln u_2)^{\theta_2}]^{\frac{\theta_1}{\theta_2}} + (-\ln u_3)^{\theta_1})^{\frac{1}{\theta_1}}\}. \tag{20}$$

Assume $\theta_1 = \theta_2 = 2$ in this case. In practice, these parameters may be estimated by empirical data. It is easy to obtain Eq.(18) by substituting $\theta_1 = \theta_2 = 2$ into Eq.(20). Then, the three options mentioned in Section 3.2 can be solved. The results are shown in Table 8 and interpreted below.

Table 8: Optimal solution based on trivariate Gumbel copula, where $\boldsymbol{T'} = \{24,24,24\}$.

	Option 1	Option 2	Option 3
Constraint	$\boldsymbol{M} \geq 1$	$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq 100,000$ $\boldsymbol{M} \geq 1$	$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq 1,000$ $\boldsymbol{M} \geq 1$
Optimal price	P_1=83.76 P_2= 113.53 P_3= 85.81	P_1=83.75 P_2= 107.56 P_3= 85.80	P_1=101.49 P_2= 107.57 P_3= 103.98
Optimised objective value	$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ =301,582.62	$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ =299,770	$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ =12,031

- In the 2nd column, the objective function $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is maximised with the constraints that $\boldsymbol{T'} = \{24,24,24\}$ and $\boldsymbol{M} = \boldsymbol{A} - \boldsymbol{\beta P} + \boldsymbol{\eta T} \geq 0$.
- In the 3rd column, the objective function $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is maximised with the constraints that $\boldsymbol{T'} = \{24,24,24\}$, $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq 100,000$ and $\boldsymbol{M} \geq 1$.
- In the 4th column, the objective function $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is minimised with the constraints that $\boldsymbol{T'} = \{24,24,24\}$, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq 1,000$ and $\boldsymbol{M} \geq 1$.

4.4 Remarks

In this section, the optimal prices of the three products with various warranty lengths are illustrated with numerical examples. Section 4.1 demonstrates the traditional warranty analysis method, which does not consider the risk caused by the uncertainty of warranty claim. Sections 4.2 and 4.3 demonstrate the optimisation problems with a consideration of risks represented by variance. Compared with the traditional methods, the proposed new methods have two advantages: firstly both the expected profit and corresponding risk are considered in warranty policy optimization, which can help the manufacturer to manage its operational risk in an efficient manner; secondly, the warranty risk of a manufacturer are estimated collectively, which means the warranty policies are optimized under more meaningful and accurate constraints than those of the traditional methods.

5 Conclusion

This paper optimally prices warranty policies when the dependence among warranty claims of different products is taken into consideration. Such optimisation is performed using the mean-variance optimisation approach, which considers the profit of a portfolio of different products with correlated numbers of warranty claims. The variance of the total profit is calculated based on a copula-based discrete joint distribution of the number of warranty claims of the products.

This paper provides a collective warranty policy optimization method when the expected profit and corresponding risk are taken into consideration. From a practical and applicable perspective, this method emphasizes the risk and potential dependence in warranty management and provides decision makers with a new approach to optimising the trade-off between the profit and risk in operation.

In the discussion in this paper, the process of warranty claims is assumed to be the homogeneous Poisson process (HPP). In addition to HPP, other processes, including the non-homogeneous Poisson process, the doubly Poisson process, and the like, may be used to model the numbers of warranty claims. In real application, one may analyse warranty claim data and then decide which a stochastic process should be used.

Our future research will aim to answer the following questions:

(1) the mean-variance optimisation approach to the situation that the compound non-homogeneous Poisson process is applied;

(2) the selection of a proper copula to construct the joint distribution;

(3) the modelling of the varying dependence in case that the dependences among the products are varying.

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Appendix

Proof of Proposition 1.

Proof. In this case, the expected profit of product k is

$$\begin{aligned}\mathrm{E}[\omega_k(P_k,T_k)] &= (A_k-\beta_k P_k+\eta_k T_k)(P_k-T_k\lambda_k\mu_k-c_k)\\ &= -\beta_k P_k^2-\eta_k\lambda_k\mu_k T_k^2+(A_k+\beta_k c_k)P_k-(A_k\lambda_k\mu_k+\eta_k c_k)T_k\\ &\quad +(\beta_k\lambda_k\mu_k+\eta_k)P_kT_k-A_kc_k,\end{aligned}$$

which is a bivariate quadratic function with respect to T_k and P_k, respectively and can be denoted as,

$$\mathrm{E}[\omega_k(P_k,T_k)]=AP_k^2+BT_k^2+CP_k+DT_k+EP_kT_k+F,$$

then, we have

$$\begin{aligned}\Delta &= 4AB-E^2\\ &= 4(-\beta_k)(-\eta_k\lambda_k\mu_k)-(\beta_k\lambda_k\mu_k+\eta_k)^2\\ &= -(\beta_k\lambda_k\mu_k-\eta_k)^2\le 0.\end{aligned}$$

When $\Delta<0$, i.e. $\beta_k\lambda_k\mu_k\neq\eta_k$, $\mathrm{E}[\omega_k(P_k,T_k)]$ dose not have maximum nor minimum. When $\Delta=0$, i.e. $\beta_k\lambda_k\mu_k=\eta_k$, because $4AB-E^2=0$, $DE-2CB=2AD-CE=0$, and $A<0$, $\mathrm{E}[\omega_k(P_k,T_k)]$ has a constant maximum with regardless of T_k. □

Proof of Proposition 2.

Proof. If $\boldsymbol{T}$ is known,

$$\mathrm{E}[\Omega(\boldsymbol{P})]=\sum_{k=1}^{n}\mathrm{E}[\omega_k(P_k)],$$

to optimise $\mathrm{E}[\Omega(\boldsymbol{P})]$ is equivalent to optimise $\mathrm{E}[\omega_k(P_k)]$.

By expanding $\mathrm{E}[\omega_k(P_k)]$, we have

$$\begin{aligned}\mathrm{E}[\omega_k(P_k)] &= (A_k-\beta_k P_k+\eta_k T_k)(P_k-T_k\lambda_k\mu_k-c_k)\\ &= -\beta_k P_k^2+(A_k+\beta_k c_k+\beta_k\lambda_k\mu_k T_k+\eta_k T_k)P_k-\eta_k\lambda_k\mu_k T_k^2-(A_k\lambda_k\mu_k+\eta_k c_k)T_k-A_kc_k.\end{aligned}$$

Let

$$\frac{d\mathrm{E}[\omega_k(P_k)]}{dP_k} = -2\beta_k P_k + A_k + \beta_k c_k + \beta_k \lambda_k \mu_k T_k + \eta_k T_k = 0,$$

then,

$$P_k = \frac{A_k + \beta_k c_k + \beta_k \lambda_k \mu_k T_k + \eta_k T_k}{2\beta_k}.$$

Because $-2\beta_k < 0$, then, when $P_k = \frac{A_k+\beta_k c_k+\beta_k \lambda_k \mu_k T_k+\eta_k T_k}{2\beta_k}$, $\mathrm{E}[\Omega(\boldsymbol{P})]$ is maximised. □

Proof of Proposition 3.

Proof. The optimisation problem is

$$\begin{aligned} \max \qquad & \mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)], \\ s.t. \qquad & A_k - \beta_k P_k + \eta_k T_k \geq 1, \\ & P_k > 0, \end{aligned}$$

where T_k is known.

The second-order derivatives of $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ respect to price P_k is

$$\frac{d^2\{\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]\}}{dP_k^2} = -2\beta_k < 0,$$

means $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ is concave. Hence, the optimisation problem is equivalent to:

$$\begin{aligned} \min \qquad & -\mathrm{E}[\omega_k(P_k, T_k)] + \mathrm{SD}[\omega_k(P_k, T_k)], \\ s.t. \qquad & -A_k + \beta_k P_k - \eta_k T_k + 1 \leq 0, \\ & -P_k < 0. \end{aligned}$$

Let

$$\begin{aligned} L &= -\mathrm{E}[\omega_k(P_k, T_k)] + \mathrm{SD}[\omega_k(P_k, T_k)] - h(A_k - \beta_k P_k + \eta_k T_k - 1) - uP_k \\ &= (A_k - \beta_k P_k + \eta_k T_k)\left[-(P_k - T_k \lambda_k \mu_k - c_k) + \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} - h\right] + h - uP_k, \end{aligned}$$

where, $h \geq 0$ and $u \geq 0$ are Lagrange multipliers.

The KKT (Karush-Kuhn-Tucker) conditions of this problem are

$$\frac{\partial L}{\partial P_k} = 2\beta_k P_k - (\beta_k \lambda_k \mu_k + \eta_k) T_k - \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} - \beta_k c_k - A_k + \beta_k h - u = 0,$$

$$h(A_k - \beta_k P_k + \eta_k T_k - 1) = 0,$$

$$u P_k = 0,$$

$$h \geq 0,$$

$$u \geq 0,$$

$$-A_k + \beta_k P_k - \eta_k T_k + 1 \leq 0,$$

$$-P_k < 0.$$

Then, when

$$\frac{(\beta_k \lambda_k \mu_k + \eta_k) T_k + \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} + \beta_k c_k + A_k}{2\beta_k} \leq \frac{A_k + \eta_k T_k - 1}{\beta_k},$$

the solution is

$$P_k = \frac{(\beta_k \lambda_k \mu_k + \eta_k) T_k + \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} + \beta_k c_k + A_k}{2\beta_k};$$

when

$$\frac{(\beta_k \lambda_k \mu_k + \eta_k) T_k + \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} + \beta_k c_k + A_k}{2\beta_k} > \frac{A_k + \eta_k T_k - 1}{\beta_k},$$

the solution is

$$P_k = \frac{A_k + \eta_k T_k - 1}{\beta_k}.$$

□

Proof of Proposition 4.

Proof. The optimisation problem is

$$\max \qquad F(\boldsymbol{P}, \boldsymbol{T}) = \mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] - \mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})],$$

$$s.t. \boldsymbol{M} \geq 1,$$

$$\boldsymbol{P} > 0,$$

where $\boldsymbol{T}$ is known.

Because $\boldsymbol{T}$ is known, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ can be denoted as

$$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k),$$

and

$$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\le k<l\le n}^{n-1} M_k M_l \sigma_{k,l}^2,$$

where $a_k = \eta_k T_k$, $b_k = T_k\lambda_k\mu_k + c_k$, $d_k = \lambda_k T_k(\sigma_k^2 + \mu_k^2)$ and $\sigma_{k,l}^2 = \mathrm{Cov}[S_k(T_k), S_l(T_l)]$ all are constants.

Then,

$$\begin{aligned} F(\boldsymbol{P},\boldsymbol{T}) &= \mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \\ &= \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k) - \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k - 2\sum_{1\le k<l\le n}^{n-1} M_k M_l \sigma_{k,l}^2. \end{aligned}$$

The partial derivative of $F(\boldsymbol{P},\boldsymbol{T})$ with respect to P_k is

$$\frac{\partial F}{\partial P_k} = -2\beta_k P_k + \beta_k b_k + A_k + a_k + 2\beta_k(A_k - \beta_k P_k + a_k)d_k + 2\beta_k \sum_{1\le k<l\le n}^{n-1} M_l \sigma_{k,l}^2,$$

the second order derivatives are

$$\frac{\partial^2 F}{\partial P_k^2} = -2\beta_k(1 + \beta_k d_k),$$

and

$$\frac{\partial^2 F}{\partial P_k \partial P_l} = -2\beta_k\beta_l\sigma_{k,l}^2.$$

The Hessian matrix is

$$
\begin{aligned}
\boldsymbol{H} &= \begin{bmatrix} \frac{\partial^2 F}{\partial P_1^2} & \frac{\partial^2 F}{\partial P_1 \partial P_2} & \cdots & \frac{\partial^2 F}{\partial P_1 \partial P_n} \\ \frac{\partial^2 F}{\partial P_2 \partial P_1} & \frac{\partial^2 F}{\partial P_2^2} & \cdots & \frac{\partial^2 F}{\partial P_2 \partial P_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 F}{\partial P_n \partial P_1} & \frac{\partial^2 F}{\partial P_n \partial P_2} & \cdots & \frac{\partial^2 F}{\partial P_n^2} \end{bmatrix} \\
&= -2 \begin{bmatrix} \beta_1(1+\beta_1 d_1) & \beta_1\beta_2\sigma_{1,2}^2 & \cdots & \beta_1\beta_n\sigma_{1,n}^2 \\ \beta_2\beta_1\sigma_{2,1}^2 & \beta_2(1+\beta_2 d_2) & \cdots & \beta_2\beta_n\sigma_{2,n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_n\beta_1\sigma_{n,1}^2 & \beta_2\beta_2\sigma_{n,2}^2 & \cdots & \beta_n(1+\beta_n d_n) \end{bmatrix} \\
&\to -2 \left(\begin{bmatrix} \mathrm{Var}[S_1(T_1)] & \mathrm{Cov}[S_1(T_1), S_2(T_2)] & \cdots & \mathrm{Cov}[S_1(T_1), S_n(T_n)] \\ \mathrm{Cov}[S_2(T_2), S_1(T_1)] & \mathrm{Var}[S_2(T_2)] & \cdots & \mathrm{Cov}[S_2(T_2), S_n(T_n)] \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Cov}[S_n(T_n), S_1(T_1)] & \mathrm{Cov}[S_n(T_n), S_2(T_2)] & \cdots & \mathrm{Var}[S_n(T_n)] \end{bmatrix} + \begin{bmatrix} \frac{1}{\beta_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\beta_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\beta_n} \end{bmatrix} \right) \\
&= -2(\boldsymbol{V} + \boldsymbol{B}),
\end{aligned}
$$

where $\boldsymbol{V}$ is a covariance matrix, and $\boldsymbol{B}$ is a positive definite matrix; hence $\boldsymbol{H}$ is negative definite and $F(\boldsymbol{P}, \boldsymbol{T})$ is concave.

Let $\frac{\partial F}{\partial P_k} = 0$, then, we have:

$$
-2\beta_k P_k + \beta_k b_k + A_k + a_k + 2\beta_k(A_k - \beta_k P_k + a_k)d_k + 2\beta_k \sum_{1 \le k < l \le n}^{n-1} M_l \sigma_{k,l}^2 = 0,
$$

$$
P_k = \frac{\beta_k b_k + A_k + a_k + 2\beta_k(A_k + a_k)d_k + 2\beta_k \sum_{1 \le k < l \le n}^{n-1} M_l \sigma_{k,l}^2}{2\beta_k(1+\beta_k)}
$$

It means the solution, which maximises the objective function $F(\boldsymbol{P}, \boldsymbol{T})$ exists. □

Proof of Proposition 5.

Proof. The optimisation problem is

$$
\max \qquad \mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})],
$$

$$
s.t. \qquad \mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \le \varphi,
$$

where $\boldsymbol{T}$ is known.

According to the proof of Proposition 2, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is a concave function on $\boldsymbol{P}$. Then, this optimisation problem is equivalent to

$$\begin{aligned} \min \qquad & -\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})], \\ s.t. \qquad & \mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq \varphi, \end{aligned}$$

where $\boldsymbol{T}$ is known.

Similar to the proof of Proposition 4, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ can be rewrote as

$$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k),$$

and

$$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\leq k<l\leq n}^{n-1} M_k M_l \sigma_{k,l}^2,$$

where $a_k = \eta_k T_k$, $b_k = T_k\lambda_k\mu_k + c_k$, $d_k = \lambda_k T_k(\sigma_k^2 + \mu_k^2)$ and $\sigma_{k,l}^2 = \mathrm{Cov}[S_k(T_k), S_l(T_l)]$ all are constants.

However,

$$\frac{\partial \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_k} = -2\beta_k(A_k - \beta_k P_k + a_k)d_k - 2\beta_k \sum_{1\leq k<l\leq n}^{n-1} M_l \sigma_{k,l}^2,$$

Then,

$$\begin{cases} \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_k^2} = 2\beta_k^2 d_k \\ \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_k \partial P_l} = 2\beta_k\beta_l\sigma_{k,l}^2. \end{cases}$$

The Hessian matrix,

$$\boldsymbol{H} = \begin{bmatrix} \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_1^2} & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_1 \partial P_2} & \cdots & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_1 \partial P_N} \\ \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_2 \partial P_1} & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_2^2} & \cdots & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_2 \partial P_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_N \partial P_1} & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_N \partial P_2} & \cdots & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_N^2} \end{bmatrix}$$

$$\rightarrow \quad 2 \begin{bmatrix} \sigma_1^2 & \sigma_{1,2}^2 & \sigma_{1,3}^2 & \cdots & \sigma_{1,n}^2 \\ \sigma_{2,1}^2 & \sigma_2^2 & \sigma_{2,3}^2 & \cdots & \sigma_{2,n}^2 \\ \sigma_{3,1}^2 & \sigma_{3,2}^2 & \sigma_3^2 & \cdots & \sigma_{3,n}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n,1}^2 & \sigma_{n,2}^2 & \sigma_{n,3}^2 & \cdots & \sigma_n^2 \end{bmatrix},$$

is positive-definite, means $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ is a convex function, and the $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T})) \leq \varphi^2$ is a convex set.

The Lagrange function is

$$L = -\sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k) + \rho[\sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\leq k<l\leq n}^{n-1} M_k M_l \sigma_{k,l}^2 - \varphi],$$

where, $\rho > 0$ is Lagrange multiplier φ is a positive constant.

Take the partial derivative respect to P_k and ρ, and let them equal 0, we have

$$\frac{\partial L}{\partial P_k} = 2\beta_k(1+\rho d_k)P_k - 2\rho\beta_k(A_k d_k + a_k d_k + \sum_{1\leq k<l\leq n}^{n-1} M_l \sigma_{k,l}^2) - \beta_k b_k - A_k - a_k = 0,$$

where $k = 1, 2, 3 \ldots n$, and

$$\frac{\partial L}{\partial \rho} = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\leq k<l\leq n}^{n-1} M_k M_l \sigma_{k,l}^2 - \varphi = 0.$$

Hence, the optimal solution is the solution of the following equation set,

$$\begin{cases} 2\beta_1(1+\rho d_1)P_1 - 2\rho\beta_1(A_1 d_1 + a_1 d_1 + \sum_{l\neq 1, l\leq n}^{n-1} M_l \sigma_{1,l}^2) - \beta_1 b_1 - A_1 - a_1 = 0 \\ 2\beta_2(1+\rho d_2)P_2 - 2\rho\beta_2(A_2 d_2 + a_2 d_2 + \sum_{l\neq 2, l\leq n}^{n-1} M_l \sigma_{2,l}^2) - \beta_2 b_2 - A_2 - a_2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2\beta_n(1+\rho d_n)P_n - 2\rho\beta_n(A_n d_n + a_n d_n + \sum_{l<n}^{n-1} M_l \sigma_{n,l}^2) - \beta_2 b_2 - A_2 - a_2 = 0 \\ \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\leq k<l\leq n}^{n-1} M_k M_l \sigma_{k,l}^2 - \varphi = 0 \end{cases}.$$

□

Proof of Proposition 6.

Proof. According to the proof of Proposition 5, when $\boldsymbol{P}$ is the decision variables, and $\boldsymbol{T}$ is known, the $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ is a convex function, and $-\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is a convex function. Hence, $-\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq -\psi$ is a convex set; and there exists an optimum solution that minimises $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$, subject to $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq \psi$. The optimum solution can be obtained through the method of Lagrange multipliers. □