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The cost of segregation in (social) networks ☆

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A B S T R A C T

This paper investigates the welfare effect of income redistribution in the private provision of public goods on networks. We first show that the welfare effect of income redistribution is determined by Bonacich centrality. Then we develop an index based on the network structure of interactions, which, roughly speaking, measures the welfare effect of income redistribution confined to a component of contributors. The proposed index vanishes when applied to large components of contributors that display special segregated interactions, which suggests an "asymptotic neutrality" of redistributive policies.

1. Introduction

Private contributions are important to the provision of many public goods. Examples include essential infrastructure, charitable education, and awareness campaigns helping individuals make better decisions over a wide spectrum of issues ranging from health and environment to voting. Access to these public goods, however, may often be constrained by geographical location or social interactions, benefiting nearby communities and acquaintances while effectively excluding others. A recent literature initiated by Bramoullé and Kranton (2007) studies a network model of public goods: each consumer can only access provisions available in his neighborhood, formed by himself and his direct neighbors in the network.[1] Other important contributions include Galeotti et al. (2010), Galeotti and Goyal (2010), Bramoullé et al. (2014), Elliott and Golub (2015),[2] and, for related empirical evidence, Acemoglu et al. (2015).

[1] Tiebout's seminal contribution provided an underpinning of the literature investigating local public goods in the absence of spillovers across communities and with a specific focus on the market forces underlying the movement of consumers to their preferred communities. An exception is Bloch and Zenginobuz (2006), who investigate local public goods with geographic spillovers among nearby communities.

[2] For related contributions, see also Acemoglu et al. (2016), Bervoets and Faure (2016), López-Pintado (2017), Günther and Hellmann (2017), Hiller (2017), and Kinateder and Merlino (2017).

Enhancing private provision of public goods has long been an important policy objective. Yet, for pure public goods, which correspond to a complete network structure of interactions, it is well known from Warr (1983) and Bergstrom et al. (1986) that private provision is subject to a strong neutrality result, whereby income redistribution among contributors that leaves the set of contributors unchanged has no effect either on the aggregate provision of public goods or the consumption of private goods.[3] The neutrality result, further analyzed in Bernheim (1986) and Andreoni (1989), is equivalent to complete crowding-out of tax-financed government provision, "dollar-for-dollar". Neutrality of income redistribution can be a serious problem for public goods that rely mostly on private provision, but these results for pure public goods may not hold when local interaction patterns are accounted for, as pointed out by Allouch (2015).[4]

In this paper, we investigate the welfare effects of income redistribution on the private provision of public goods in networks. The fact that neutrality often breaks down in general networks gives rise to welfare improvement possibilities. This is very much in the spirit of the Second Welfare Theorem, although, unlike competitive equilibrium, the Nash equilibrium outcomes of private provision will typically be inefficient. As a first step to understanding welfare improvement possibilities, we restrict our attention to particular preferences that yield affine Engel curves studied in Gorman (1961), of which Cobb–Douglas preferences are a special case. Our analysis shows that, under a standard utilitarian approach, the welfare effect of income redistribution confined to a component of contributors is determined by the Bonacich centrality. Bonacich centrality, due to Bonacich (1987), measures power and prestige in social networks and was first shown in the economics literature to be proportional to the Nash equilibrium outcomes of a game by Ballester et al. (2006). Bonacich centrality has been shown to be important in several other applications in economics, including Ghiglino and Goyal (2010) for conspicuous consumption, İlkiliç (2011) for the tragedy of the commons, Belhaj and Deroïan (2012) for risk sharing, Candogan et al. (2012) for monopoly pricing, and Acemoglu et al. (2012) for aggregate volatility.

In order to compare the welfare effect of income redistribution across components of contributors of different sizes and network structures, we introduce a new index, called the *Bonacich transfer index*, which measures the average welfare gain per unit of income redistribution. The proposed index is related to the standard deviation of Bonacich centralities. Intuitively, the higher is the heterogeneity of Bonacich centralities of a component of contributors, the more per-capita welfare gain can be achieved from a unit of income redistribution; in this regard, we show that the Bonacich transfer index may take a wide range of values. For instance, for a regular component of contributors the Bonacich transfer index is zero, whereas for a star component of contributors the Bonacich transfer index may be unbounded. Therefore, the Bonacich transfer index may be thought of as a summary statistic of the welfare effect of income redistribution that is determined by the network structure of interactions.

Next we analyze the welfare effect of income redistribution when components of contributors are comprised of several "segregated" groups. Segregated interactions of groups are represented by a network structure where, for each group, the density of inward ties is greater than the density of outward ties. We show that the Bonacich transfer index vanishes when applied to large components of contributors that display segregated interactions. This implies an "asymptotic neutrality" of income redistribution. Although our result mirrors the neutrality results for standard pure public goods, it is quite different in interpretation. More specifically, the asymptotic neutrality, unlike neutrality, allows for the possibility of income redistribution improving social welfare, but any resulting improvement becomes smaller as the component of contributors grows larger. The rest of the paper is organized as follows. Section 2 provides a review of related literature. Section 3 introduces the private provision of public goods in a network model. Section 4 relates the welfare effect of income redistribution to Bonacich centrality. Section 5 introduces the Bonacich transfer index. Section 6 applies the Bonacich transfer index to components of contributors that display special segregated interactions. We conclude the paper in Section 7 and prove our results in Section 8.

2. Related literature

The segregated network structure investigated in this paper could be the result of different processes. It could display physical or geographical linking constraints, or it could emerge as a result of processes and network formation dynamics, depending on the public goods under investigation. Schelling (1969) provides a simple, yet powerful, model showing that seemingly mild individual preferences for having neighbors of the same type may lead to full residential segregation, even though no individual prefers the final outcome.

The tendency of individuals to disproportionally form social ties with others similar to themselves is called the homophily principle in sociology. Homophily is a well-documented pattern of social networks and is often called upon in understanding various social interactions such as friendship and marriage, job market outcomes, speed of information diffusion, and even social mobility. There is an emerging literature in the economics of social networks that models a random process of network formation strongly influenced by homophily, including Currarini et al. (2009), Bramoullé et al. (2012), and Golub and Jackson (2012).

Our approach, although it is quite different in motivation, takes advantage of the insights of the above-mentioned literature, since we investigate components of contributors that already display segregated interactions, rather than the matching

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[3] The paper by Bergstrom et al. (1986) has been the focus of much attention, as evidenced by the special issue in the *Journal of Public Economics* celebrating its 20th anniversary.

[4] See Bourlès et al. (2017) for another investigation of the neutrality result in a network setting.

process or the network formation dynamics leading to them. As such, the Bonacich transfer index enables us to investigate the welfare effect of income redistribution in components of contributors that display segregated interactions, regardless of the process that has led to this pattern of segregation.

3. The model

We consider a set $N = \{1, \ldots, n\}$ of n agents embedded on a connected network $\mathbf{g}$ of interactions. Let $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that consumers $i \neq j$ are neighbors and $g_{ij} = 0$ otherwise. We denote by $\mathcal{N}_i = \{j \in N \mid g_{ij} = 1\}$ consumer i's neighbors in the network $\mathbf{g}$. The adjacency matrix of the network, $\mathbf{G}$, is symmetric with nonnegative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by $\lambda_{\max}(\mathbf{G}) = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n = \lambda_{\min}(\mathbf{G})$, where $\lambda_{\max}(\mathbf{G})$ is the largest eigenvalue and $\lambda_{\min}(\mathbf{G})$ is the lowest eigenvalue of $\mathbf{G}$. By the Perron–Frobenius Theorem, it holds that $\lambda_{\max}(\mathbf{G}) \geq -\lambda_{\min}(\mathbf{G}) > 0$.

Each consumer i's preferences are assumed to depend on his private good consumption x_i and public good consumption $Q_i = q_i + Q_{-i}$, where q_i is his public good provision and $Q_{-i} = \sum_{j \in \mathcal{N}_i} q_j$ is the sum of his neighbors' public good provisions. The preferences of each consumer i are represented by a twice continuously differentiable, strictly increasing, and strictly quasi-concave utility function u_i. Furthermore, the public good can be produced from the private good via a unit-linear production technology. At a Nash equilibrium $\mathbf{q}^*$ of private provision, it holds that for each consumer i, (x_i^*, q_i^*) solves

$$\max_{x_i, q_i} u_i(x_i, q_i + Q_{-i}^*)$$
$$\text{s.t.} \quad x_i + q_i = w_i \text{ and } q_i \geq 0,$$

where w_i is his income (exogenously fixed).

We will assume, throughout the paper, the following network-specific normality assumption, which amounts to both the normality of the private good and a strong normality of the public good:

Network normality. For each consumer $i = 1, \ldots, n$, the Engel curve[5] γ_i is differentiable and it holds that $1 + \frac{1}{\lambda_{\min}(\mathbf{G})} < \gamma_i'(\cdot) < 1$.

We will also focus our analysis on consumers' preferences that yield affine Engel curves, studied in Gorman (1961), of which Cobb–Douglas preferences are a special case. Although the assumption of affine Engel curves is restrictive, it is quite useful to gain some initial understanding of the welfare effect of income redistribution.

Affine Engel curves. There exists a real number a such that, for each consumer $i = 1, \ldots, n$, it holds that $\gamma_i'(\cdot) = 1 - a$.

The next result holds since affine Engel curves yield linear demand functions for local public goods.

Proposition 1 (Bramoullé et al., 2014). *Assume network normality and affine Engel curves. Then there exists a unique Nash equilibrium for the private provision.*

Bergstrom et al. (1986) show that the normality of the private good and the public good is sufficient to guarantee the existence and uniqueness of a Nash equilibrium in the private provision of pure public goods. Bramoullé et al. (2014) establish the existence and uniqueness of a Nash equilibrium in games of strategic substitutes in networks with linear best-reply functions.[6] Building on the above important contributions, Allouch (2015) introduces the assumption of network normality and establishes the existence and uniqueness of a Nash equilibrium in the private provision of public goods on networks, which simultaneously extends Bergstrom et al. (1986) to networks and Bramoullé et al. (2014) to nonlinear best-reply functions. More generally, Bramoullé et al. (2014) show that the lowest eigenvalue is also key to equilibrium analysis beyond the range of network normality: if there are multiple Nash equilibria, it is possible for a symmetric equilibrium to be unstable and for an asymmetric equilibrium to be stable.

4. Welfare analysis of income redistribution

This section investigates the effect of income redistribution on private provision in networks. Income redistribution as a policy instrument aims to achieve socially optimal outcomes and takes the form of lump-sum transfers, which are traditionally viewed as a reference point for other policy instruments. A policy of income redistribution is made up of a list of transfers $\mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n$. The transfers are budget-balanced—that is, $\sum_{i=1}^n t_i = 0$. Let $\mathbf{q^t} = (q_1^{\mathbf{t}}, \ldots, q_n^{\mathbf{t}})^T$ denote the

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[5] The Engel curve describes how a consumer's expenditure on a particular good relates to the consumer's wealth, holding prices fixed.

[6] Note that linear demand functions for local public goods also correspond to the case of imperfect substitutability of provisions in Bramoullé and Kranton (2007).

unique Nash equilibrium corresponding to the income distribution $\mathbf{w} + \mathbf{t} = (w_1 + t_1, \ldots, w_n + t_n)^T$. A subset S of contributors is called a component of contributors if $\mathbf{g}_S$ is a maximal connected subnetwork of $\mathbf{g}_C$, the network obtained by removing non-contributors as well as all the links emanating from them. Let $\mathbf{G}_S$ denote the adjacency matrix of the network $\mathbf{g}_S$.

Given a component of contributors S, a transfer $\mathbf{t}$ is called S-confined if $\mathbf{t}_{N \setminus S} = 0$. Similar to Warr (1983) and Bergstrom et al. (1986), we will focus our analysis on transfers confined to a component of contributors that do not change the composition of the set of contributors, and we will refer to such transfers as "relatively small".[7] For simplicity, we assume that the set of knife-edge non-contributors—that is, non-contributors on the verge of becoming contributors—is empty, which is typically the case as shown in Bramoullé et al. (2014).

4.1. The Bonacich centrality measure

In the social networks literature, a variety of network measures have been proposed to explore the potential importance, power, and influence of individuals in social interactions. The most intuitive network measure is degree centrality, defined as the number of direct neighbors in the network, which gives importance to individuals with more connections. Obviously, the fact that degree centrality overlooks indirect influences from distant neighbors gives rise to the use of global network measures. Ballester et al. (2006) were first to relate the Nash equilibrium outcomes of a game to a global network measure called Bonacich centrality, due to Bonacich (1987) and defined by

$$\mathbf{b}(\mathbf{G}, \delta) = (\mathbf{I} - \delta \mathbf{G})^{-1} \mathbf{1},$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is the vector with all components equal to one, and δ is the discount factor. Since for $|\delta| < \frac{1}{\lambda_{\max}(\mathbf{G})}$ it holds that[8]

$$\mathbf{b}(\mathbf{G}, \delta) = (\mathbf{I} - \delta \mathbf{G})^{-1} \mathbf{1} = \sum_{k=0}^{+\infty} \delta^k \mathbf{G}^k \mathbf{1}, \tag{4.1}$$

the Bonacich centrality of consumer i can be expressed as follows:

$$b_i(\mathbf{G}, \delta) = \sum_{k=0}^{+\infty} \delta^k \sum_{j=1}^{n} (\mathbf{G}^k)_{ij}.$$

Given that $(\mathbf{G}^k)_{ij}$ counts the number of walks of length k emanating from i and terminating at j, it follows that the Bonacich centrality of consumer i counts the number of walks emanating from i discounted by δ to the power of their length.

In order to investigate the welfare impact of income redistribution, we take a standard utilitarian approach. More specifically, given a component of contributors S, we consider the social welfare function

$$\mathcal{W}_S(\mathbf{w}) \overset{\text{def}}{=} \sum_{i \in S} u_i(x_i^*, Q_i^*),$$

which is the sum of utilities achieved by members of S at the unique Nash equilibrium with income distribution $\mathbf{w} = (w_1, \ldots, w_n)^T$.

Proposition 2. *Given a component of contributors S, for any relatively small S-confined transfer $\mathbf{t}$ it holds that*[9]

$$\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w}) = -\frac{(1-a)}{a} \mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S = -\frac{(1-a)}{a} \sum_{i \in S} b_i(\mathbf{G}_S, -a) \cdot t_i.$$

Proof. All proofs are provided in the Appendix. □

Proposition 2 shows that the welfare effect of income redistribution is determined by Bonacich centrality. In particular, it follows that a transfer from a high Bonacich central contributor to a low Bonacich central contributor will always raise social welfare. Recall that the intuition for high and low Bonacich centrality when the discount factor is negative is different from the case when the discount factor is positive. For instance, in a star network with three contributors (the subnetwork induced by (c_1, c_2, c_3) in Fig. 1), when the discount factor is negative, the least central node is contributor 2, at the center

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of the star. As a consequence, optimal transfers need to be made from the periphery of the star (high Bonacich central-ity)—that is, contributors 1 and 3—to the more connected contributor (low Bonacich centrality)—that is, contributor 2. The intuition behind this result is as follows. Contributors with low Bonacich centrality are those who have a strong ability to free-ride on the provision of their neighbors since they are more connected. By network position symmetry, if these contributors were to provide more public goods, their neighbors would benefit more from free-riding on them.

It is worth noting that there is significant empirical evidence that the network structure is key in determining the income distribution in society. For instance, social networks, being an important channel of information transmission and referral in the labor market, increase the labor income of more connected (low Bonacich central) individuals.[10] However, even starting with perfect income equality, a transfer from a less connected (high Bonacich central) contributor to a more connected (low Bonacich central) contributor that is optimal in the sense of overall welfare does not always increase utility inequality.[11]

In a related result, the following proposition shows that the impacts of income redistribution on the aggregate consump-tion of public goods, $\bar{Q}_S^* = \sum_{i \in S} Q_i^*$, and social welfare are proportional. Hence raising social welfare and raising aggregate consumption of public goods are identical policy objectives.

Proposition 3. *Given a component of contributors S, for any relatively small S-confined transfer* **t** *it holds that*

$$\bar{Q}_S^{\mathbf{t}} - \bar{Q}_S^* = (1-a)[\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w})] = -\frac{(1-a)^2}{a} \mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S.$$

In consequence, contributors with low Bonacich centrality, the biggest free-riders, are those who should be induced to contribute more to the public good, in order to raise the social welfare/aggregate consumption of public goods. Due to their network position, their increased contribution can be enjoyed by many other contributors.

From a policy perspective, income redistribution policies that can increase social welfare/aggregate consumption of public goods have desirable normative properties and, as such, are more interesting than policies that just increase aggregate pro-vision of public goods, $Q_S^* = \sum_{i \in S} q_i^*$. The following proposition reproduces a result on the impact of income redistribution on aggregate provision.

Proposition 4 (Allouch, 2015). *Given a component of contributors S, for any relatively small S-confined transfer* **t** *it holds that*

$$Q_S^{\mathbf{t}} - Q_S^* = (1-a) \mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S = (1-a) \sum_{i \in S} b_i(\mathbf{G}_S, -a) \cdot t_i.$$

Propositions 2, 3, and 4 show that the effects of a relatively small income redistribution on social welfare/aggregate consumption of public goods and aggregate provision of public goods of a component of contributors are determined by Bonacich centrality, although by pulling the income redistribution in opposite directions. As a consequence, we may con-clude that in the context of private provision of public goods in networks, raising social welfare/aggregate consumption of public goods and raising aggregate provision of public goods, through a relatively small income redistribution, are sharply conflicting policy objectives.

The following example illustrates this point.

Example 1. There are eight consumers with identical Cobb–Douglas preferences $u_i(x_i, q_i + Q_{-i}) = 2 \, x_i^{\frac{2}{5}} (q_i + Q_{-i})^{\frac{3}{5}}$ located on the network depicted in Fig. 1. We assume that consumers' incomes are as follows: $w_1 = w_2 = w_3 = w_4 = 1$ and $w_5 = w_6 = w_7 = w_8 = 2$. Since $\lambda_{\max}(\mathbf{G}) = -\lambda_{\min}(\mathbf{G}) = 2.34$ and the slope of the Engel curve is $\frac{3}{5}$, the network normality assumption holds and guarantees a unique Nash equilibrium. Let nodes in black represent contributors while nodes in white represent non-contributors. At the unique Nash equilibrium, there are two components of contributors, $S = \{c_1, c_2, c_3\}$ and $S' = \{c_5, c_6, c_7, c_8\}$, of different sizes (respectively, 3 and 4) and network structures (respectively, star and regular).

For the first component of contributors $S = \{c_1, c_2, c_3\}$, the Bonacich centrality is

$$\mathbf{b}(\mathbf{G}_S, -\frac{2}{5}) = (b_1(\mathbf{G}_S, -\frac{2}{5}), b_2(\mathbf{G}_S, -\frac{2}{5}), b_3(\mathbf{G}_S, -\frac{2}{5}))^T = (\frac{15}{17}, \frac{5}{17}, \frac{15}{17})^T.$$

Consider now a relatively small S-confined transfer $\mathbf{t}^\tau$ such that $\mathbf{t}_S^\tau = (-\tau, 2\tau, -\tau)^T$ with $\tau > 0$—that is, each of the two high Bonacich central contributors 1 and 3 transfer τ to the low Bonacich central contributor 2. The effect of the transfer $\mathbf{t}^\tau$ on social welfare is given from Proposition 2 as

$$\mathcal{W}_S(\mathbf{w} + \mathbf{t}^\tau) - \mathcal{W}_S(\mathbf{w}) = \frac{30}{17} \, \tau > 0,$$

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[10] See Saygin et al. (2014) and Brown et al. (2016), among others.

[11] For instance, this can be easily checked in the star network with three contributors (the subnetwork induced by (c_1, c_2, c_3) in Fig. 1). More generally, the contributions of Ghiglino and Goyal (2010) on social status in networks and Bourlès et al. (2017) on altruism in networks provide interesting channels for how network centrality, inequality, and redistribution impact each other.
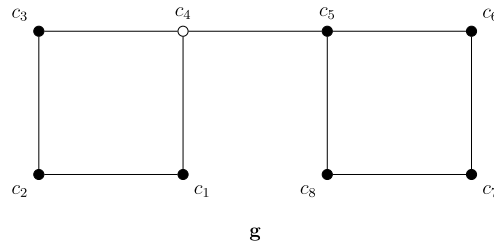
Fig. 1. Components of contributors.

on aggregate consumption of public goods is given from Proposition 3 as

$$\bar{Q}_S^{\mathbf{t}^\tau} - \bar{Q}_S^* = \frac{18}{17}\,\tau > 0,$$

and on aggregate provision of public goods is given from Proposition 4 as

$$Q_S^{\mathbf{t}^\tau} - Q_S^* = -\frac{12}{17}\,\tau < 0.$$

Hence the transfer $\mathbf{t}^\tau$ decreases aggregate provision of public goods and raises social welfare/aggregate consumption of public goods. This is intuitive because the center of the star, with the lowest Bonacich centrality in this component of contributors, is free-riding on the provision of contributors 1 and 3. The transfers from 1 and 3 to 2 will cause the center of the star to increase his provision and contributors 1 and 3 to reduce theirs. Because the center's provision benefits all three contributors, while 1 and 3's provision only benefits each one of them and the center, these transfers shift the provision of the public good to the consumer whose contribution can be enjoyed by more other consumers. Hence it brings higher social welfare/aggregate consumption of public goods while reducing the aggregate provision of public goods.

For the second component of contributors $S' = \{c_5, c_6, c_7, c_8\}$, the Bonacich centrality is

$$\mathbf{b}(\mathbf{G}_{S'}, -\frac{2}{5}) = (b_5(\mathbf{G}_{S'}, -\frac{2}{5}), b_6(\mathbf{G}_{S'}, -\frac{2}{5}), b_7(\mathbf{G}_{S'}, -\frac{2}{5}), b_8(\mathbf{G}_{S'}, -\frac{2}{5}))^T = \frac{5}{9}\mathbf{1}.$$

Since the Bonacich centrality vector has equal entries, it follows from Propositions 2, 3, and 4 that for any relatively small S'-confined transfer $\mathbf{t}$ it holds that

$$\mathcal{W}_{S'}(\mathbf{w} + \mathbf{t}) - \mathcal{W}_{S'}(\mathbf{w}) = \bar{Q}_{S'}^{\mathbf{t}} - \bar{Q}_{S'}^* = Q_{S'}^{\mathbf{t}} - Q_{S'}^* = 0.$$

Hence, a relatively small income redistribution confined to S' affects neither social welfare/aggregate consumption nor aggregate provision.

5. An economic index for income redistribution

Our investigation shows that the welfare effect of income redistribution confined to a component of contributors is determined by Bonacich centrality. In order to compare the welfare effect of income redistribution across components of contributors of different sizes and network structures, we develop an economic index that measures the potential welfare gains of income redistribution.

5.1. The Bonacich transfer index

Given a component of contributors S, we introduce a new measure, called the Bonacich transfer index, defined by

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) \overset{\text{def}}{=} \max_{\mathbf{t} \in \mathcal{T}_S} \frac{\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w})}{|S|\,\|\mathbf{t}\|},$$

where $\mathcal{T}_S$ denotes the set of relatively small S-confined transfers and $|S|$ denotes the size of the component S.

The Bonacich transfer index measures the maximum average welfare gain per unit of income redistribution.[12] This corresponds to an average utilitarian approach to welfare, which is appropriate for dealing with different sizes of components.

The following proposition relates the Bonacich transfer index to a network statistic: $\sigma(\mathbf{b}(\mathbf{G_S}, -a))$, the standard deviation of Bonacich centralities.

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[12] Note that the set $\mathcal{T}_S$ is nonempty since we assume away knife-edge non-contributors. Moreover, the fact that the Bonacich transfer index depends on the per-unit effect of income redistribution makes the index explicitly independent of the set $\mathcal{T}_S$, as will be shown later.

Proposition 5. *Given a component of contributors S, it holds that*

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) = \frac{1-a}{a\sqrt{|S|}}\, \sigma\left(\mathbf{b}(\mathbf{G_S}, -a)\right).$$

In interpretation, the Bonacich transfer index measures the dispersion of consumers' Bonacich centralities, relative to the size of the component of contributors. A large Bonacich transfer index indicates that consumers' Bonacich centralities are spread far from the mean Bonacich centrality and a small Bonacich transfer index indicates that they are clustered around the mean Bonacich centrality. As far as social welfare is concerned, when there is more heterogeneity in Bonacich centrality there is more scope for making transfers from less connected (high Bonacich central) consumers to more connected (low Bonacich central) consumers. These transfers improve the efficiency of public good provision and so enhance welfare. When there is little variation in Bonacich centrality there is little scope for the efficiency of provision to be substantially increased, as everyone is roughly equally efficient at providing public goods.

5.2. A spectral approach to income redistribution

In order to gain a further understanding of the relationship between the Bonacich transfer index and the network structure, we will follow a spectral approach. Given a component of contributors S, an eigenvalue μ of $\mathbf{G}_S$ that has an associated (unit) eigenvector $\mathbf{u}$ not orthogonal to the vector $\mathbf{1}$ is called a *main eigenvalue*. By the Perron–Frobenius Theorem, the largest eigenvalue has an associated eigenvector with all its entries positive and, therefore, is a main eigenvalue. Given a main eigenvalue, μ_i, let $\mathbf{u}_i$ denote the associated eigenvector such that any other associated eigenvector is orthogonal to $\mathbf{1}$. The cosine of the angle between $\mathbf{u}_i$ and the vector $\mathbf{1}$, denoted by β_i, is called a *main angle*.

The concept of main eigenvalues identifies the eigenvalues that contribute to the number of walks. In particular, Cvetković (1970) shows that $N_k = |S| \sum_{i=1}^{m} \beta_i^2 \mu_i^k$, where N_k is the number of walks of length k.[13] Hence, information on the structure of the network can be obtained from just the main eigenvalues.

The following result relates the Bonacich transfer index to the main eigenvalues.

Theorem 1. *Given a component of contributors S, it holds that*

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) = \frac{1-a}{a\sqrt{|S|}} \sqrt{\sum_{i=1}^{m} \frac{\beta_i^2}{(1+a\mu_i)^2} - \left(\sum_{i=1}^{m} \frac{\beta_i^2}{1+a\mu_i}\right)^2}.$$

Theorem 1 shows that variation in contributors' Bonacich centralities is completely captured by variation in the main eigenvalues of the network, providing an easier way for us to determine the scope for social-welfare-enhancing transfers. In particular, the Bonacich transfer index has, in view of Theorem 1, a natural geometric interpretation as the gap in Jensen's inequality for the convex function $f(x) = x^2$, applied to the convex combination of $\frac{1}{1+a\mu_1}, \frac{1}{1+a\mu_2}, \ldots, \frac{1}{1+a\mu_m}$ with weights $\beta_1^2, \beta_2^2, \ldots, \beta_m^2$. As shown below, this geometric interpretation of the Bonacich transfer index can be useful in characterizing two classes of networks.

Corollary 1. *Given a component of contributors S, $\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) = 0$ if and only if $\mathbf{g}_S$ is regular.*

Corollary 1 shows that a zero Bonacich transfer index characterizes regular networks. Actually, regular networks are the only instances where the two policy objectives of raising social welfare/aggregate consumption of public goods and raising aggregate provision of public goods coincide, as they are both redundant in regular networks.

Corollary 2. *Given a component of contributors S, $\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a)$ is bounded over the range of network normality if and only if $\lambda_{\min}(\mathbf{G}_S) \neq \mu_m$.*

Corollary 2 shows that a bounded Bonacich transfer index over the range of network normality characterizes networks where the lowest eigenvalue is not a main eigenvalue. This reinforces the recent findings in Bramoullé et al. (2014) on the importance of the lowest eigenvalue to economic outcomes.

Unlike the largest eigenvalue, the lowest eigenvalue is not always a main eigenvalue. For instance, in regular networks, only the largest eigenvalue is a main eigenvalue. The question of characterizing networks where the lowest eigenvalue is not a main eigenvalue was posed in Cardoso and Pinheiro (2009) and has only recently been answered in Abreu et al. (2016) for some canonical network structures: lines and trees with small diameters. For instance, it holds that for a line of length

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[13] Actually, the number of main eigenvalues can be thought of as the "degrees of freedom" when computing the number of walks, which in turn are linked to symmetry properties of the network. For instance, regular networks have one main eigenvalue and complete bipartite networks have at most two main eigenvalues.

n the lowest eigenvalue is not a main eigenvalue if and only if n is even. On the other hand, networks where the lowest eigenvalue is a main eigenvalue include complete bipartite networks with unequal sides−for instance, star networks−and the completely asymmetric networks, where all eigenvalues are distinct and main.

Unfortunately, relating networks where the lowest eigenvalue is a main eigenvalue to their link structure is beyond the scope of this paper. Nonetheless, similar to Bonacich centrality, the Bonacich transfer index is also related to the number of walks, which could be useful in explaining when the Bonacich transfer index is unbounded. Let $\overline{N}_k \stackrel{\text{def}}{=} \frac{N_k}{|S|}$ denote the average number of walks of length k. Then, for sufficiently small a, it follows from (4.1) that the average Bonacich centrality can be expressed as follows:

$$E\big[\mathbf{b}(\mathbf{G_S}, -a)\big] = \frac{1}{|S|}\mathbf{1} \cdot \mathbf{b}(\mathbf{G}_S, -a) = \frac{1}{|S|}\sum_{k=0}^{+\infty}(-a)^k\mathbf{1}^T\mathbf{G}_S^k\mathbf{1} = \sum_{k=0}^{+\infty}(-a)^k\overline{N}_k,$$

which implies that

$$E\big[\mathbf{b}(\mathbf{G_S}, -a)\big]^2 = \sum_{k=0}^{+\infty}(-a)^k\sum_{r=0}^{k}\overline{N}_r\overline{N}_{k-r}.$$

Similarly, it holds that

$$E\big[\mathbf{b}^2(\mathbf{G_S}, -a)\big] = \frac{1}{|S|}\Big(\sum_{k=0}^{+\infty}(-a)^k\mathbf{G}_S^k\mathbf{1}\Big) \cdot \Big(\sum_{k=0}^{+\infty}(-a)^k\mathbf{G}_S^k\mathbf{1}\Big) = \sum_{k=0}^{+\infty}(-a)^k\,(k+1)\overline{N}_k.$$

Therefore it holds that

$$\mathbf{b}^{\mathcal{T}\ell}(\mathbf{G_S}, -a) = \frac{1-a}{a\sqrt{|S|}}\sqrt{\sum_{k=0}^{+\infty}(-a)^k\sum_{r=0}^{k}\big(\overline{N}_k - \overline{N}_r\overline{N}_{k-r}\big)}.$$

Hence, the Bonacich transfer index can be understood by comparing term-by-term average number of walks, that is, $\overline{N}_k$ and $\overline{N}_r\overline{N}_{k-r}$. In particular, a high Bonacich transfer index can be related to heterogeneity in the average number of walks. For instance, in star networks $\overline{N}_k$ is strictly greater than $\overline{N}_r\overline{N}_{k-r}$ if both r and $k-r$ are odd numbers, possibly because there are no cycles of odd length; otherwise $\overline{N}_k$ and $\overline{N}_r\overline{N}_{k-r}$ are equal. On the other hand, in regular networks it holds that $\overline{N}_k$ and $\overline{N}_r\overline{N}_{k-r}$ are always equal.

The following example illustrates the Bonacich transfer index for a component of contributors.

Example 2. Consider again the component of contributors S in Example 1. The spectral decomposition shows that

$$\mathbf{G}_S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

has three distinct eigenvalues

$$\lambda_1 = \sqrt{2},\ \lambda_2 = 0,\ \text{and } \lambda_3 = -\sqrt{2},$$

with corresponding (unit) eigenvectors

$$\mathbf{v}_1 = (\tfrac{1}{2}, \tfrac{\sqrt{2}}{2}, \tfrac{1}{2})^T,\ \mathbf{v}_2 = (\tfrac{\sqrt{2}}{2}, 0, -\tfrac{\sqrt{2}}{2})^T,\ \text{and } \mathbf{v}_3 = (\tfrac{1}{2}, -\tfrac{\sqrt{2}}{2}, \tfrac{1}{2})^T.$$

Obviously, unlike $\mathbf{v}_1$ and $\mathbf{v}_3$, the sum of $\mathbf{v}_2$ coordinates is zero. Hence,

$$\mu_1 = \sqrt{2}\ \text{and}\ \mu_2 = -\sqrt{2}.$$

Fig. 2 provides the geometric interpretation of the Bonacich transfer index of S.

Since $\lambda_{\min}(\mathbf{G}_S) = \mu_2 = -\sqrt{2}$, it follows from Fig. 2 that $\lim_{a\uparrow\frac{\sqrt{2}}{2}}\mathbf{b}^{\mathcal{T}\ell}(\mathbf{G_S}, -a) = +\infty$. In this example, because the lowest eigenvalue is a main eigenvalue, the Bonacich transfer index is unbounded over the range of network normality.

Consider a relatively small S-confined (optimal) transfer $\mathbf{t}^\tau$ such that $\mathbf{t}_S^\tau = (-\tau, 2\tau, -\tau)^T$ with $\tau > 0$. Since for each $l \geq 0$

$$\mathbf{G}^{2l+1} = \begin{pmatrix} 0 & 2^l & 0 \\ 2^l & 0 & 2^l \\ 0 & 2^l & 0 \end{pmatrix}\ \text{and}\ \mathbf{G}^{2l+2} = \begin{pmatrix} 2^l & 0 & 2^l \\ 0 & 2^{l+1} & 0 \\ 2^l & 0 & 2^l \end{pmatrix},$$
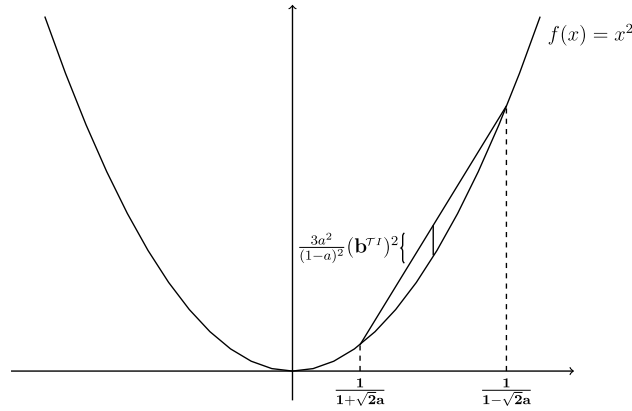
it follows from Proposition 2 that

Fig. 2. The Bonacich transfer index.

$$\mathcal{W}_S(\mathbf{w} + \mathbf{t}^\tau) - \mathcal{W}_S(\mathbf{w}) = -\frac{(1-a)}{a}\,\mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S^\tau = -\frac{(1-a)}{a}\sum_{k=0}^{+\infty}(-a)^k \mathbf{1}^T \mathbf{G}_S^k \mathbf{t}_S^\tau$$

$$= -\frac{(1-a)}{a}\Big[\sum_{l=0}^{+\infty}(-a)^{2l}\mathbf{1}^T \mathbf{G}_S^{2l}\mathbf{t}_S^\tau + \sum_{l=0}^{+\infty}(-a)^{2l+1}\mathbf{1}^T \mathbf{G}_S^{2l+1}\mathbf{t}_S^\tau\Big]$$

$$= -\frac{(1-a)\tau}{a}\Big[\sum_{l=0}^{+\infty}(-a)^{2l+1}2^{l+1}\Big] = \frac{2(1-a)\tau}{1-2a^2}.$$

In interpretation, $\sum_{k=0}^{+\infty}(-a)^k \mathbf{1}^T \mathbf{G}_S^k \mathbf{t}_S^\tau$ represents the sum of the welfare effects of the transfer $\mathbf{t}_S^\tau$ on neighbors located at distance k discounted by $(-a)^k$. In particular, it holds that the welfare effect on neighbors located at distance $2r$ (that is, $\mathbf{1}^T \mathbf{G}^{2r}\mathbf{t}_S^\tau$) is null and the welfare effect on neighbors located at distance $2r + 1$ (that is, $\mathbf{1}^T \mathbf{G}^{2r+1}\mathbf{t}_S^\tau$) is $\tau 2^{r+1}$. Hence the transfer $\mathbf{t}_S^\tau$ affects only neighbors at an odd distance.

Moreover, when a approaches $\frac{-1}{\lambda_{\min}(\mathbf{G}_S)} = \frac{\sqrt{2}}{2}$ from below, the higher a is, the greater is the importance of (odd) long walks, leading to a greater welfare effect from income redistribution, while a remains within the range of network normality, which ensures the uniqueness of a Nash equilibrium.

5.3. Direction of welfare-improving transfers

As far as the direction of welfare-improving transfers is concerned, networks where the lowest eigenvalue is a main eigenvalue may have a useful policy implication.

Proposition 6. *If $\lambda_{\min}(\mathbf{G}_S) = \mu_m$, then, when a approaches $\frac{-1}{\lambda_{\min}(\mathbf{G}_S)}$ from below, the (normalized) Bonacich centrality $\frac{\mathbf{b}(\mathbf{G}_S, -a)}{\|\mathbf{b}(\mathbf{G}_S, -a)\|}$ converges to the eigenvector $\mathbf{u}_m$.*

Corbo et al. (2007) explore the behavior of Bonacich centrality by taking a similar limit at the other end of the spectrum—that is, $\lambda_{\max}(\mathbf{G}_S)$—in order to investigate the problem of optimal network design. Recall that when a approaches $\frac{-1}{\lambda_{\min}(\mathbf{G}_S)}$ from below, the higher a is, the greater is the importance of long walks for Bonacich centrality, leading to greater heterogeneity of Bonacich centrality, while a remains within the range of network normality, which ensures the uniqueness of a Nash equilibrium.

Proposition 6 shows that the eigenvector $\mathbf{u}_m$ associated with the lowest eigenvalue could inform us about the welfare impact of income redistribution. In particular, since the eigenvector $\mathbf{u}_m$ is orthogonal to the (positive entries) principal eigenvector $\mathbf{u}_1$, it follows that the eigenvector $\mathbf{u}_m$ has both positive and negative entries. Thus, an income redistribution that follows the sign patterns of the eigenvector $\mathbf{u}_m$ will have a predictable welfare impact.

Furthermore, let $\hat{\mathbf{q}} \overset{\text{def}}{=} (1-a)\mathbf{w} + \mathbf{q}^s$, where $\mathbf{q}^s$ lists the vertical intercepts of consumers' Engel curves, denote the autarkic (or stand-alone) public good provisions of consumers. As highlighted in Allouch (2015), an autarkic public good provision of consumers $\hat{\mathbf{q}}$ almost proportional to the principal eigenvector $\mathbf{u}_1$ always yields a Nash equilibrium where all consumers are contributors. In this case, the autarkic public good provisions of consumers $\hat{\mathbf{q}}$ become almost orthogonal to Bonacich centrality, which determines the direction of welfare-improving transfers.

6. Segregated interactions

Now we show how the Bonacich transfer index can be used to investigate the welfare impact of an income redistribution that is confined to components of contributors with a special network structure. More specifically, we consider a component of contributors S composed of $H \geq 2$ nonempty and pairwise disjoint groups, $S_1, \ldots, S_H$. Given a consumer $i \in S$, let S_{h_i} denote i's own group.

We introduce the following two assumptions about the density of ties across groups.

Segregated interactions. For each $i \in S$ it holds that

$$|\mathcal{N}_i \cap S_{h_i}| > \sum_{h \neq h_i} |\mathcal{N}_i \cap S_h|.$$

The segregation assumption stipulates that the number of ties each group member has within his own group exceeds the number of ties he has to other groups.

Equitable partition. If $i, j \in S$ belong to the same group, then it holds that

$$|\mathcal{N}_i \cap S_h| = |\mathcal{N}_j \cap S_h|, \text{ for each } h = 1, \ldots, H.$$

The equitable partition assumption stipulates that interactions are determined by group membership—that is, the number of ties a group member has to any group is independent of the choice of the group member.[14]

Segregated interactions and equitable partition are arguably rather strong assumptions about the network structure, but we could simply think of the underlying network structure as a useful idealization of segregated interactions in a society.

The following theorem provides an upper bound for the Bonacich transfer index in a component of contributors that satisfies segregated interactions and equitable partition.

Theorem 2. *Given a component of contributors S that satisfies segregated interactions and equitable partition, it holds that* $\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) \leq \frac{1-a}{2a\sqrt{|S|}}$.

The economic intuition of Theorem 2 may be explained as follows. Recall that the Bonacich centrality of a consumer is related to the number of walks emanating from the consumer. From the segregated interactions assumption it follows that almost all long walks emanating from a consumer remain inside his group. This is simply due to the fact that adding an extra step to a walk inside a group will more likely reach a member of the same group. Therefore, the Bonacich centrality of a consumer can be approximated by his Bonacich centrality in the subnetwork induced by his group. Given that we assume that the subnetwork induced by each group is regular it follows that Bonacich centralities are always positive and bounded by 1. Therefore, the standard deviation of Bonacich centralities is bounded by $\frac{1}{2}$. The rest follows from Proposition 5.

The following corollary is an immediate consequence of Theorem 2.

Corollary 3. *Given a component of contributors S that satisfies segregated interactions and equitable partition, it holds that*

$$\lim_{|S| \to +\infty} \mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) = 0.$$

Corollary 3 shows that the Bonacich transfer index vanishes as the component of contributors grows large, which suggests an asymptotic neutrality of income redistribution. It is worth noting that when the limit is taken, it does not matter how the network is grown. For instance, it does not matter which groups increase in size, the rate at which different groups grow, or indeed whether new groups are added or removed, as long as the segregated interactions and equitable partition assumptions both hold.

The asymptotic neutrality result of Corollary 3 mirrors the standard neutrality result for pure public goods, although it allows for the possibility of small social welfare improvement relative to the size of the component of contributors. More specifically, it shows that the network structure of interactions in segregated components of contributors may limit the impact of redistributive policies and, by the same token, a wide range of other closely-related policies.

The following example illustrates our result in a segregated component of contributors comprising two groups.

Example 3. Consider a component of contributors S comprising two groups S_1 and S_2. Let the network $\mathbf{g}_S$ be defined as follows: for each member of S_1, the number of ties within S_1 is $c_{1,1}$ and the number of ties to S_2 is $c_{1,2}$; for each member of S_2, the number of ties within S_2 is $c_{2,2}$ and the number of ties to S_1 is $c_{2,1}$. Clearly, the equitable partition

---

[14] See Powers and Sulaiman (1982) for further discussion.

assumption holds and implies that each group's members have identical Bonacich centrality. We also assume that the segregated interactions assumption holds—that is, $c_{1,1} > c_{1,2}$ and $c_{2,2} > c_{2,1}$.

The intuition for asymptotic neutrality in this example may be explained as follows. From the segregated interactions assumption it follows that almost all long walks emanating from a consumer remain inside his group. Since the Bonacich centrality of a consumer is related to the number of walks emanating from the consumer, it can be approximated by his Bonacich centrality in the subnetwork induced by his group, that is,

$$b_i(\mathbf{G}_S, -a) \simeq \begin{cases} b_i(\mathbf{G}_{S_1}, -a) = \frac{1}{1+ac_{1,1}} & \text{if } i \in S_1, \\ b_i(\mathbf{G}_{S_2}, -a) = \frac{1}{1+ac_{2,2}} & \text{if } i \in S_2. \end{cases}$$

This provides a bound of 1 on the magnitude of the Bonacich centrality of each consumer, which in turn provides a bound of $\frac{1}{2}$ on the standard deviation of Bonacich centralities. Therefore, it holds that

$$\mathbf{b}^{\mathcal{TI}}(\mathbf{G}_S, -a) = \frac{1-a}{a\sqrt{|S|}} \, \sigma\left(\mathbf{b}(\mathbf{G}_S, -a)\right) \leq \frac{1-a}{2a\sqrt{|S|}}.$$

In particular, if we consider a component of contributors S_n with n-times the number of consumers in each group while keeping the number of ties of each group member constant, then it holds that

$$\mathbf{b}^{\mathcal{TI}}(\mathbf{G}_{S_n}, -a) \leq \frac{1-a}{2a\sqrt{|S|}} \, \frac{1}{\sqrt{n}}.$$

Hence, as n grows large the Bonacich transfer index goes to zero.

Finally, observe that if S_1 and S_2 are equally sized then it holds that the standard deviation of Bonacich centralities is bounded from below by $\theta = \frac{1}{2}|\frac{1}{1+ac_{1,1}} - \frac{1}{1+ac_{2,2}}|$. Hence it holds that

$$\frac{\theta(1-a)}{a\sqrt{|S|}} \leq \mathbf{b}^{\mathcal{TI}}(\mathbf{G}_S, -a).$$

Since the Bonacich transfer index measures the per-capita welfare gain, it follows that the aggregate welfare gain can grow at a rate of at least $\frac{\theta(1-a)}{a}\sqrt{|S|}$ as the size of the component of contributors $|S|$ increases.

7. Conclusion

This paper shows that understanding the network structure is very important when designing policies that could enhance the private provision of public goods in networks. In particular, our analysis shows that a neutrality-like result holds in large components of contributors that display special segregated interactions. In the literature, there is robust empirical evidence that segregation undermines public goods provision. The literature is also furnished with a variety of mechanisms to explore the channels through which segregation, and its main aspect of limited social interactions, operates on public goods provision.[15] Nevertheless, our approach is quite different since it investigates local public goods rather than global public goods and only sheds light on whether income redistribution is effective rather than comparing the optimality of public good provisions across components of contributors.

Finally, in view of the large body of research relating Bonacich centrality to various economic outcomes, the properties of the Bonacich transfer index developed in this paper provide network analysis tools, which could be used in a wide range of applications. In this regard, there are many stylized networks in the economic literature, some of which are empirically motivated such as core-periphery networks, for which our analysis of the Bonacich transfer index could provide a better understanding of the underlying economic interactions.

8. Appendix

Proof of Proposition 2. When consumers' preferences yield affine Engel curves—that is, $\gamma_i'(\cdot) = 1 - a$ for each consumer $i = 1, \ldots, n$—it follows from Gorman (1961) that, without loss of generality, the indirect utility function for each $i \in S$, at a given price $\mathbf{p} = (p_x, p_Q)$ and income w_i, can be written as

$$v_i(\mathbf{p}, w_i) = \alpha(\mathbf{p})w_i + \beta_i(\mathbf{p}),$$

where $\alpha(\mathbf{p}) = p_x^{-a} p_Q^{a-1}$ is common to all consumers. Since the price $\mathbf{p}$ is normalized to $(1, 1)$, it follows from applying Roy's identity to the indirect utility function that the public good provision of each consumer $i \in S$ can be expressed as

$$q_i = (1-a)w_i + q_i^s - aQ_{-i}, \tag{8.1}$$

---

[15] See Alesina et al. (1999) and Fernández and Levy (2008), among others.

where q_i^s denotes the vertical intercept of consumer i's Engel curve. Hence, it holds that

$$
\begin{aligned}
\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w}) &= \sum_{i \in S} (u_i(x_i^{\mathbf{t}}, q_i^{\mathbf{t}} + Q_{-i}^{\mathbf{t}}) - u_i(x_i^*, q_i^* + Q_{-i}^*)) \\
&= \sum_{i \in S} (w_i + t_i + Q_{-i}^{\mathbf{t}} + \beta_i(1,1) - w_i - Q_{-i}^* - \beta_i(1,1)) \\
&= \sum_{i \in S} (Q_{-i}^{\mathbf{t}} - Q_{-i}^*) = \mathbf{1}^T \mathbf{G}_S (\mathbf{q}_S^{\mathbf{t}} - \mathbf{q}_S^*) \\
&= (1 - a) \mathbf{1}^T \mathbf{G}_S (\mathbf{I} + a\mathbf{G}_S)^{-1} \mathbf{t}_S \\
&= (1 - a) \mathbf{1}^T (-\frac{1}{a} \mathbf{I} + \frac{1}{a} (\mathbf{I} + a\mathbf{G}_S))(\mathbf{I} + a\mathbf{G}_S)^{-1} \mathbf{t}_S \\
&= \frac{1 - a}{a} \mathbf{1}^T (-(\mathbf{I} + a\mathbf{G}_S)^{-1} + \mathbf{I}) \mathbf{t}_S \\
&= -\frac{1 - a}{a} \mathbf{1}^T (\mathbf{I} + a\mathbf{G}_S)^{-1} \mathbf{t}_S = -\frac{(1 - a)}{a} \mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S. \qquad \square
\end{aligned}
$$

Proof of Proposition 3. From (8.1) it holds that

$$
\begin{aligned}
\bar{Q}_S^{\mathbf{t}} - \bar{Q}_S^* &= \sum_{i \in S} (Q_i^{\mathbf{t}} - Q_i^*) = \mathbf{1}^T (\mathbf{I} + \mathbf{G}_S)(\mathbf{q}_S^{\mathbf{t}} - \mathbf{q}_S^*) \\
&= (1 - a) \, \mathbf{1}^T (\frac{(a - 1)}{a} \mathbf{I} + \frac{1}{a} (\mathbf{I} + a\mathbf{G}_S))(\mathbf{I} + a\mathbf{G}_S)^{-1} \mathbf{t}_S \\
&= -\frac{(1 - a)^2}{a} \mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S. \qquad \square
\end{aligned}
$$

Proof of Proposition 5. From Proposition 2 it follows that

$$
\begin{aligned}
\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w}) &= -\frac{1 - a}{a} \mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S \\
&= -\frac{1 - a}{a} (\mathrm{proj}_{\mathbf{1}} \mathbf{b}(\mathbf{G}_S, -a) + \mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)) \cdot \mathbf{t}_S \\
&= -\frac{1 - a}{a} (\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)) \cdot \mathbf{t}_S.
\end{aligned}
$$

If $\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a) = 0$ then the equality

$$
\mathbf{b}^{\mathcal{T}^I}(\mathbf{G}_S, -a) = \frac{1 - a}{a|S|} \|\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)\| = 0
$$

holds trivially. If $\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a) \neq 0$, the maximum of $\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w})$, for $\mathbf{t} \in \mathcal{T}_S$, occurs at $-\kappa \, \mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)$, for any $\kappa > 0$ such that $-\kappa \, \mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a) \in \mathcal{T}_S$. Hence

$$
\begin{aligned}
\mathbf{b}^{\mathcal{T}^I}(\mathbf{G}_S, -a) &= \max_{\mathbf{t} \in \mathcal{T}_S} \frac{\mathcal{W}_S(\mathbf{w} + \mathbf{t}) - \mathcal{W}_S(\mathbf{w})}{|S| \, \|\mathbf{t}\|} = \frac{(1 - a)}{a|S|} \mathbf{b}(\mathbf{G}_S, -a) \cdot \frac{\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)}{\|\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)\|} \\
&= \frac{1 - a}{a|S|} \frac{\|\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)\|^2}{\|\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)\|} = \frac{(1 - a)}{a|S|} \|\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)\|.
\end{aligned}
$$

From the Pythagorean theorem it holds that

$$
\|\mathrm{proj}_{\mathbf{1}\perp} \mathbf{b}(\mathbf{G}_S, -a)\|^2 = \|\mathbf{b}(\mathbf{G}_S, -a)\|^2 - \|\mathrm{proj}_{\mathbf{1}} \mathbf{b}(\mathbf{G}_S, -a)\|^2.
$$

Hence,

$$
\begin{aligned}
\mathbf{b}^{\mathcal{T}^I}(\mathbf{G}_S, -a) &= \frac{1 - a}{a|S|} \sqrt{\|\mathbf{b}(\mathbf{G}_S, -a)\|^2 - \left(\mathbf{b}(\mathbf{G}_S, -a) \cdot \frac{\mathbf{1}}{\sqrt{|S|}}\right)^2} \\
&= \frac{1 - a}{a|S|} \sqrt{|S| \left(\mathrm{E}[\mathbf{b}^2(\mathbf{G_S}, -a)] - \mathrm{E}[\mathbf{b}(\mathbf{G_S}, -a)]^2\right)} \\
&= \frac{1 - a}{a\sqrt{|S|}} \sigma(\mathbf{b}(\mathbf{G_S}, -a)). \qquad \square
\end{aligned}
$$

Proof of Theorem 1. Let $\mathbf{V}_S$ be a matrix whose columns, $\mathbf{v}_1, \ldots, \mathbf{v}_{|S|}$, are eigenvectors of $\mathbf{G}_S$ chosen to extend the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_m$ of $\mathbf{G}_S$ to an orthonormal basis of $\mathbb{R}^{|S|}$. Therefore, $\mathbf{G}_S = \mathbf{V}_S \mathbf{D}_S \mathbf{V}_S^T$, where $\mathbf{D}_S = \mathrm{diag}(\xi_1, \ldots, \xi_n)$. Therefore, it holds that

$$\mathbf{b}(\mathbf{G}_S, -a) = \mathbf{V}_S(\mathbf{I} + a\mathbf{D}_S)^{-1}\mathbf{V}_S^T \mathbf{1} = \sum_{i=1}^{m} \frac{\mathbf{1} \cdot \mathbf{u}_i}{1 + a\mu_i}\mathbf{u}_i = \sqrt{|S|} \sum_{i=1}^{m} \frac{\beta_i}{1 + a\mu_i}\mathbf{u}_i. \tag{8.2}$$

Hence, it holds that

$$\begin{aligned}
\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) &= \frac{1-a}{a\sqrt{|S|}} \sqrt{\mathrm{E}\big[\mathbf{b}^2(\mathbf{G_S}, -a)\big] - \mathrm{E}\big[\mathbf{b}(\mathbf{G_S}, -a)\big]^2} \\
&= \frac{1-a}{a\sqrt{|S|}} \sqrt{\|\sum_{i=1}^{m} \frac{\beta_i}{1+a\mu_i}\mathbf{u}_i\|^2 - \Big(\sum_{i=1}^{m} \frac{\beta_i}{1+a\mu_i}\mathbf{u}_i \cdot \frac{\mathbf{1}}{\sqrt{|S|}}\Big)^2} \\
&= \frac{1-a}{a\sqrt{|S|}} \sqrt{\sum_{i=1}^{m} \frac{\beta_i{}^2}{(1+a\mu_i)^2} - \Big(\sum_{i=1}^{m} \frac{\beta_i{}^2}{1+a\mu_i}\Big)^2}. \quad \square
\end{aligned}$$

Proof of Corollary 1. From the Jensen's gap interpretation of the Bonacich transfer index, it follows that $\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) = 0$ if and only if $m = 1$, which is equivalent to $\beta_1{}^2 = 1$. From the definition of main angles, it holds that $\beta_1{}^2 = 1$ is equivalent to $\mathbf{1}$ is an eigenvalue, which is also equivalent to $\mathbf{g}_S$ is a regular network. $\square$

Proof of Corollary 2. From the Jensen's gap interpretation of the Bonacich transfer index, it follows that $\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a)$ is bounded over the range of network normality if and only if $\frac{1}{1+a\mu_m}$ is bounded for all $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G}_S)}[$, which is equivalent to $\lambda_{\min}(\mathbf{G}_S) \neq \mu_m$. $\square$

Proof of Proposition 6. From (8.2) it holds that

$$\begin{aligned}
\lim_{a \uparrow \frac{-1}{\mu_m}} \frac{\mathbf{b}(\mathbf{G}_S, -a)}{\|\mathbf{b}(\mathbf{G}_S, -a)\|} &= \lim_{a \uparrow \frac{-1}{\mu_m}} \frac{1}{\sqrt{\sum_{i=1}^{m} \frac{\beta_i{}^2}{(1+a\mu_i)^2}}} \sum_{i=1}^{m} \frac{\beta_i}{1+a\mu_i}\mathbf{u}_i \\
&= \lim_{a \uparrow \frac{-1}{\mu_m}} \frac{1}{\sqrt{\sum_{i=1}^{m-1} \beta_i{}^2(\frac{1+a\mu_m}{1+a\mu_i})^2 + \beta_m^2}} \Big(\sum_{i=1}^{m-1} \beta_i \frac{1+a\mu_m}{1+a\mu_i}\mathbf{u}_i + \beta_m\mathbf{u}_m\Big) = \mathbf{u}_m. \quad \square
\end{aligned}$$

Proof of Theorem 2. The equitable partition $\pi = \{S_1, \ldots, S_H\}$ gives rise to a quotient network $\mathbf{g}_S/\pi$ characterized by the adjacency matrix $\mathbf{G}_S/\pi = [d_{lz}]_{1 \leq l,z \leq H}$, where d_{lz} denotes the number of links from a consumer in group S_l to consumers in group S_z. Notice that the adjacency matrix, $\mathbf{G}_S/\pi$, is not necessarily symmetric, since in general $d_{lz} \neq d_{zl}$.

The quotient network plays an important role in the study of the main part of spectrum $\mathcal{M} = \{\mu_1, \ldots, \mu_m\}$ since it holds that (see, for example, Cvetković et al., 1997, Theorems 2.4.3 and 2.4.5)

$$\mathcal{M} \subset \mathrm{spec}(\mathbf{G}_S/\pi) \subset \mathrm{spec}(\mathbf{G}_S). \tag{8.3}$$

Observe that all the eigenvalues of $\mathbf{G}_S$ are real and so the eigenvalues of $\mathbf{G}_S/\pi$ are also real. Moreover, the segregated interactions assumption implies that $\mathbf{G}_S/\pi$ is a diagonally dominant matrix. From the Geršgorin Circle Theorem (see Varga, 2004, Theorem 1.1), it follows that all eigenvalues of $\mathbf{G}_S/\pi$ are positive. From (8.3) one obtains $\mu_i > 0$ for each $\mu_i \in \mathcal{M}$. This implies that $0 < \frac{1}{1+a\mu_i} < 1$ for each $\mu_i \in \mathcal{M}$. Since the Jensen's gap is less than $\frac{1}{4}$ $(= \max_{x \in]0,1[} \{x - x^2\})$, the maximum possible Jensen's gap, it follows that

$$\mathbf{b}^{\mathcal{T}I}(\mathbf{G}_S, -a) = \frac{1-a}{a\sqrt{|S|}} \sqrt{\sum_{i=1}^{m} \frac{\beta_i{}^2}{(1+a\mu_i)^2} - \Big(\sum_{i=1}^{m} \frac{\beta_i{}^2}{1+a\mu_i}\Big)^2} \leq \frac{1-a}{2a\sqrt{|S|}},$$

which implies that as the size of the component of contributors $|S|$ grows large the Bonacich transfer index goes to zero. $\square$

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