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Iterated inflations of cellular algebras

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ABSTRACT

We present a result characterising iterated inflations of cellular algebras, derived from the work of König and Xi. This result is intended to replace an incorrect proposition in the literature, and gives explicit and readily checked conditions which establish that an algebra is an iterated inflation of cellular algebras, and hence is cellular, with cellular data directly related to the cellular data of the constituent cellular algebras.

The concept of a *cellular algebra* was introduced by Graham and Lehrer in [2], and has found wide application in the representation theory of finite groups and associative algebras; it has been especially useful in the study of *diagram algebras*. We refer the reader to the existing literature for the definition and basic properties of cellular algebras; we use the notation of [3]. We work over a field k, and all our k-algebras are finite-dimensional; since all tensor products are taken over k, we abbreviate $\otimes_k$ to $\otimes$.

In [4], König and Xi introduced the notion of an *iterated inflation* of cellular algebras. Informally, iterated inflation allows us to prove that an algebra is cellular by showing that it may be constructed using some family of existing cellular algebras in a certain

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way. There are, however, various technical details and conditions which must be satisfied in order for such a construction to yield a cellular algebra as desired. In [4] and [5], these conditions are not presented in a single definition or result, but rather are developed in the course of the text. In [7], Xi gave the following lemma to provide a precise characterisation of iterated inflations of cellular algebras:

Lemma. *(Xi — Lemma 3.3 in [7]) Let A be an algebra with an anti-involution σ. Suppose there is a vector space decomposition $A = \bigoplus_{j=1}^{m} V_j \otimes B_j \otimes V_j$ where V_j is a vector space and B_j is a cellular algebra with respect to an anti-involution σ_j and a cell chain $J_1^{(j)} \subseteq \cdots \subseteq J_{s_j}^{(j)} = B_j$ for each j. Define $J_t = \bigoplus_{j=1}^{t} V_j \otimes B_j \otimes V_j$. Assume that*

(i) the restriction of σ on $V_j \otimes B_j \otimes V_j$ is given by $w \otimes b \otimes v \longmapsto v \otimes \sigma_j(b) \otimes w$
(ii) for each j, there is a bilinear form $\phi_j : V_j \times V_j \to B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $v, w \in V_j$
(iii) for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have

$$(x \otimes b \otimes y)(u \otimes c \otimes v) = x \otimes b\phi_j(y, u)c \otimes v \qquad (\text{mod } J_{i-1})$$

(iv) $\left(V_j \otimes J_l^{(j)} \otimes V_j\right) + J_{j-1}$ is an ideal in A for all l and j.

Then A is a cellular algebra.

This lemma was used, for example, to prove the cellularity of the partition algebra in [7], and of the BMW-algebra in [6]; both of these papers have become standard references for these facts. The lemma is, however, incorrect, as the following example shows. Recall that the matrix algebra $M_n(k)$ is cellular with respect to the anti-involution which takes a matrix to its transpose. In particular, the field k is cellular with respect to the identity map. Let k be any field, and define A to be the k-algebra obtained by equipping $M_2(k) \oplus k \oplus k$ with the (commutative) multiplication

$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \alpha, \beta\right)\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \gamma, \delta\right) =$$
$$\left(\begin{pmatrix} \gamma a_{11} & \delta a_{12} \\ \delta a_{21} & \gamma a_{22} \end{pmatrix} + \begin{pmatrix} \alpha b_{11} & \beta b_{12} \\ \beta b_{21} & \alpha b_{22} \end{pmatrix}, \alpha\gamma, \beta\delta\right),$$

which has identity

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1, 1\right).$$

Let σ be the anti-involution $\sigma(M, \alpha, \beta) = \left(M^T, \alpha, \beta\right)$ on A. Let $V_1 = V_2 = V_3 = k$ and define cellular algebras $B_1 = M_2(k)$ and $B_2 = B_3 = k$. Thus we obtain an isomorphism of vector spaces

$$A \cong \big(V_1 \otimes B_1 \otimes V_1\big) \oplus \big(V_2 \otimes B_2 \otimes V_2\big) \oplus \big(V_3 \otimes B_3 \otimes V_3\big)$$

from the mapping $(M, \alpha, \beta) \longmapsto (1{\otimes}M{\otimes}1,\ 1{\otimes}\alpha{\otimes}1,\ 1{\otimes}\beta{\otimes}1)$. Define k-bilinear forms ϕ_1, ϕ_2, ϕ_3, where ϕ_i maps $V_i \times V_i$ to B_i, by taking ϕ_1 to be zero and $\phi_2(1,1) = \phi_3(1,1) = 1$. One may check that these definitions satisfy the hypotheses of Xi's lemma (note that B_j has cell chain $\{0\} \subseteq B_j$ for each j). Suppose for a contradiction that A is cellular, and let

$$e = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, 1\right).$$

Then e is an idempotent of A, and one may see by direct calculation that eAe is a 3-dimensional subspace of A upon which σ does not act as the identity map. Further, we have $\sigma(e) = e$ and so by Proposition 4.3 of [3], the algebra eAe must be cellular with respect to the restriction of σ. But it follows easily from the definition of a cellular algebra (and the fact that the only expression of 3 as a sum of squares is $1+1+1$) that the anti-involution on a 3-dimensional cellular algebra is the identity map; thus A is not cellular.[2]

We now present a result which can serve as a convenient replacement for the incorrect lemma above. This result characterises iterated inflations via two readily checked conditions; these two conditions were noted for the Brauer algebra by König and Xi in [5] (Lemmas 5.4 and 5.5). The proof of this theorem is a straightforward verification of the axioms for a cellular algebra given in [2], and thus is omitted.

Theorem 1. *Let A be a k-algebra, with an anti-involution σ. Suppose that we have, up to isomorphism of k-vector spaces, a decomposition*

$$A \cong \bigoplus_{i \in I} V_i \otimes B_i \otimes V_i$$

of A, where I is a finite partially ordered set, each V_i is a k-vector space, and each B_i is a cellular algebra over k with respect to an anti-involution σ_i and cellular data (Λ_i, M_i, C). We shall henceforth consider A to be identified with this direct sum of tensor products.

Suppose that for each $i \in I$, we have a basis $\mathcal{V}_i$ for V_i and a basis $\mathcal{B}_i$ for B_i, such that:

1. *For each $i \in I$, we have for any $u, v \in \mathcal{V}_i$ and any $b \in \mathcal{B}_i$ that*

$$\sigma(u \otimes b \otimes v) = v \otimes \sigma_i(b) \otimes u. \tag{1}$$

2. *Let $\mathcal{A}$ be the basis of A consisting of all elements $u \otimes b \otimes v$ for all $u, v \in \mathcal{V}_i$ and all $b \in \mathcal{B}_i$, as i ranges over I. Then for any $i \in I$ we have maps $\phi_i : \mathcal{A} \times \mathcal{V}_i \to V_i$ and*

[2] If k has characteristic 2, then A and σ may be shown to satisfy the alternative definition of cellularity given by Goodman and Graber in [1], demonstrating that this definition is indeed different from that given by Graham and Lehrer over fields of characteristic 2.

$\theta_i : \mathcal{A} \times \mathcal{V}_i \to B_i$ such that for any $u, v \in \mathcal{V}_i$ and any $b \in \mathcal{B}_i$, we have for any $a \in \mathcal{A}$ that

$$a \cdot (u \otimes b \otimes v) \equiv \phi_i(a, u) \otimes \theta_i(a, u) b \otimes v \mod J(< i) \tag{2}$$

where $J(< i) = \bigoplus_{l<i} V_l \otimes B_l \otimes V_l$.

Then A is cellular with respect to σ and the cellular data (Λ, M, C), where Λ is the set $\{(i, \lambda) : i \in I \text{ and } \lambda \in \Lambda_i\}$ with the lexicographic order, $M(i, \lambda)$ is $\mathcal{V}_i \times M_i(\lambda)$, and $C^{(i,\lambda)}_{(x,X),(y,Y)} = x \otimes C^{\lambda}_{X,Y} \otimes y$.

Note that we may use *any* bases of the cellular algebras B_i to check the conditions of Theorem 1: we need not use the cellular bases of the B_i. This is convenient as cellular bases are often awkward to work with in practice.

Proposition 2. *Let A be an algebra satisfying the hypotheses of Theorem 1. Then the multiplication in each "layer" of A is "governed" by a bilinear form as in Xi's lemma: for each $i \in I$ there is a unique B_i-valued k-bilinear form ψ_i on V_i such that for any $u, v, x, y \in V_i$ and $b, c \in B_i$, we have $\psi_i(y, u) = \sigma_i(\psi_i(u, y))$ and*

$$(x \otimes c \otimes y)(u \otimes b \otimes v) \equiv x \otimes c\,\psi_i(y, u) b \otimes v \mod J(< i). \tag{3}$$

Proof (outline). We shall show that there is a bilinear form ψ_i satisfying (3) by proving that we have a map $\psi_i : \mathcal{V}_i \times \mathcal{V}_i \longrightarrow B_i$ satisfying (3); the rest is then easy. First note that (1) holds even when u, v, b are not required to be basis elements, and likewise for v, b in (2); these equations are used extensively in what follows. By applying $\sigma^2 = \mathrm{id}_A$ we may show that

$$(x \otimes c \otimes y)(u \otimes \sigma_i(b) \otimes u) \equiv x \otimes c\,\sigma_i\big(\theta_i(u \otimes b \otimes u, y)\big) \otimes \phi_i(u \otimes b \otimes u, y) \tag{4}$$

modulo $J(< i)$, for $u, x, y \in \mathcal{V}_i$ and $b, c \in \mathcal{B}_i$. Now for $u, x, y \in \mathcal{V}_i$ and $c \in \mathcal{B}_i$, we may on the one hand apply (2) to $(x \otimes c \otimes y)(u \otimes 1_{B_i} \otimes u)$, and on the other hand we may expand 1_{B_i} in terms of the basis $\sigma(\mathcal{B}_i)$ and apply (4). By carefully comparing the results of these calculations, we may deduce that if $\theta_i(x \otimes c \otimes y, u) \neq 0$ then $\phi_i(x \otimes c \otimes y, u)$ is some scalar multiple of x, say $\alpha(x, c, y, u)x$, and further that

$$\phi_i(x \otimes c \otimes y, u) \otimes \theta_i(x \otimes c \otimes y, u) \otimes u = x \otimes c\,\psi_i(y, u) \otimes u$$

for a value $\psi_i(y, u) \in B_i$ depending only on y and u; hence we may deduce that $\alpha(x, c, y, u)\theta_i(x \otimes c \otimes y, u)$ is equal to $c\,\psi_i(y, u)$. Then for $u, v, x, y \in \mathcal{V}_i$ and $b, c \in \mathcal{B}_i$, we apply (2) to the left-hand side of (3) and then apply the above results, and hence obtain the right-hand side of (3). $\square$

The first part of our final result is implicit in [4] and [5], and a more explicit formulation was given in [7]; we believe that the second part has not previously appeared in the literature. The proof is easy and is omitted.

Proposition 3. *Let A be as in Theorem 1, let $(i,\lambda) \in \Lambda$, and let Δ^λ be the cell module of B_i corresponding to λ. The cell module $\Delta^{(i,\lambda)}$ of A may be obtained by equipping $V_i \otimes \Delta^\lambda$ with the action given, for $a \in \mathcal{A}$, $x \in \mathcal{V}_i$ and $z \in \Delta^\lambda$, by*

$$a(x \otimes z) = \phi_i(a,x) \otimes \theta_i(a,x)z.$$

Recall that the simple modules of a cellular algebra may be obtained as quotients of the cell modules by the radicals of certain bilinear forms; with the above construction of the cell module, if $\langle \cdot\,,\cdot \rangle$ is the bilinear form on $V_i \otimes \Delta^\lambda$ and $\langle \cdot\,,\cdot \rangle_\lambda$ is the bilinear form on Δ^λ, then for any $x, y \in V_i$ and any $z, w \in \Delta^\lambda$, we have

$$\langle x \otimes z,\, y \otimes w \rangle = \langle z,\, \psi_i(x,y)w \rangle_\lambda = \langle \psi_i(y,x)z,\, w \rangle_\lambda.$$

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References

[1] F. Goodman, J. Graber, Cellularity and the Jones basic construction, Adv. in Appl. Math. 46 (2011) 312–362.
[2] J.J. Graham, G.I. Lehrer, Cellular Algebras, Invent. Math. 123 (1) (1996) 1–34.
[3] S. König, C.C. Xi, On the structure of cellular algebras, Canad. Math. Soc. Conf. Proc., vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 365–386.
[4] S. König, C.C. Xi, Cellular algebras: inflations and Morita equivalences, J. Lond. Math. Soc. (2) 60 (1999) 700–722.
[5] S. König, C.C. Xi, A characteristic free approach to Brauer algebras, Trans. Amer. Math. Soc. 353 (2001) 1489–1505.
[6] C.C. Xi, On the quasi-heredity of Birman–Wenzl algebras, Adv. Math. 154 (2000) 280–298.
[7] C.C. Xi, Partition Algebras are Cellular, Compos. Math. 119 (1999) 99–109.