Hadjiantoni, Stella and Kontoghiorghes, Erricos John (2017) *A recursive three-stage least squares method for large-scale systems of simultaneous equations.* Linear Algebra and its Applications, 536 . pp. 210-227. ISSN 0024-3795.

A Recursive Three-Stage Least Squares Method for Large-Scale Systems of Simultaneous Equations

Stella Hadjiantoni[1] and Erricos John Kontoghiorghes[2,3]

[1]University of Kent, UK
[2]Cyprus University of Technology
[3]Birkbeck University of London, UK

Abstract

A new numerical method is proposed that uses the QR decomposition (and its variants) to derive recursively the three-stage least squares (3SLS) estimator of large-scale simultaneous equations models (SEM). The 3SLS estimator is obtained sequentially, once the underlying model is modified, by adding or deleting rows of data. A new theoretical pseudo SEM is developed which has a non positive definite dispersion matrix and is proved to yield the 3SLS estimator that would be derived if the modified SEM was estimated afresh. In addition, the computation of the iterative 3SLS estimator of the updated observations SEM is considered. The new recursive method utilizes efficiently previous computations, exploits sparsity in the pseudo SEM and uses as main computational tool orthogonal and hyperbolic matrix factorizations. This allows the estimation of large-scale SEMs which previously could have been considered computationally infeasible to tackle. Numerical trials have confirmed the effectiveness of the new estimation procedures. The new method is illustrated through a macroeconomic application†.

Keywords: updating, QR decomposition, high dimensional data, matrix algebra

MSC: 15A23;15B10;62L12

1. Introduction

The simultaneous equations model (SEM) is a system of structural equations where some of the response variables also reappear in the system as explanatory variables. Let the SEM in compact form be

$$\mathrm{vec}\,(\boldsymbol{Y}) = (\boldsymbol{I}_G \otimes \boldsymbol{W})\,\boldsymbol{S}\boldsymbol{\delta} + \mathrm{vec}\,(\boldsymbol{E}),\ \ \mathrm{vec}\,(\boldsymbol{E}) \sim (\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_M)\,, \tag{1.1}$$

Corresponding author: S. Hadjiantoni, School of Mathematics, Statistics & Actuarial Science, University of Kent, Canterbury, Kent CT2 7NF, UK. Email address: s.hadjiantoni@kent.ac.uk

†The computational aspects of the strategies are included as a supplementary material (Appendix).

where $\boldsymbol{W} = (\boldsymbol{X} \quad \boldsymbol{Y})$, $\boldsymbol{X}$ is the $M \times K$ (full column rank) matrix of all exogenous (or predetermined) variables that satisfy the orthogonality condition $\mathbb{E}(\boldsymbol{X}^T\boldsymbol{E}) = \mathbf{0}$ and $\boldsymbol{Y}$ is the $M \times G$ matrix of all other explanatory variables that violate the orthogonality condition $\mathbb{E}(\boldsymbol{Y}^T\boldsymbol{E}) = \mathbf{0}$, herein referred to as endogenous variables. The value of an endogenous variable is determined within the system whereas the value of an exogenous variable is defined outside the system. Also $\boldsymbol{S} = \operatorname{diag}(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_G)$ is a selection matrix such that $\boldsymbol{W}\boldsymbol{S}_i = \boldsymbol{W}_i = (\boldsymbol{X}_i \quad \boldsymbol{Y}_i)$ and $\boldsymbol{\delta} = \operatorname{vec}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \ldots, \boldsymbol{\delta}_G)$. The notation $\operatorname{vec}(\boldsymbol{E}) \sim (\mathbf{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_M)$ implies that the error term $\operatorname{vec}(\boldsymbol{E})$ has zero mean and variance-covariance matrix $\boldsymbol{\Sigma} \otimes \boldsymbol{I}_M$, where $\boldsymbol{\Sigma} \in \mathbb{R}^{G \times G}$ is a symmetric non negative definite matrix and $\otimes$ denotes the Kronecker product [32]. In the ith equation, that is $\boldsymbol{y}_i = \boldsymbol{W}_i\boldsymbol{\delta}_i + \boldsymbol{\epsilon}_i$, where $\boldsymbol{W}_i = (\boldsymbol{X}_i \;\; \boldsymbol{Y}_i)$, $\boldsymbol{X}_i \in \mathbb{R}^{M \times k_i}$ is the full column rank matrix of exogenous variables, $\boldsymbol{Y}_i \in \mathbb{R}^{M \times g_i}$ is the matrix of endogenous variables for that equation, and where $\boldsymbol{\delta}_i = (\boldsymbol{\beta}_i^T \quad \boldsymbol{\gamma}_i^T)^T$, $\boldsymbol{\beta}_i \in \mathbb{R}^{k_i}$ and $\boldsymbol{\gamma}_i \in \mathbb{R}^{g_i}$ are the structural parameters to be estimated. It is assumed that $k_i + g_i \leq K$ so that the unknown parameters of the structural equations are uniquely identified [19].

The presence of the endogenous variables $\boldsymbol{Y}$ implies that the explanatory variables are not orthogonal to the error term, that is, $\mathbb{E}(\boldsymbol{W}^T\boldsymbol{E}) \neq \mathbf{0}$ since $\mathbb{E}(\boldsymbol{Y}^T\boldsymbol{E}) \neq \mathbf{0}$. The violation of the orthogonality condition due to the error term entering into the determination of the endogenous variable $\boldsymbol{y}_i$ is called endogeneity, and needs to be eliminated before generalized least squares (GLS) are applied to estimate (1.1). The effect of endogeneity is overcome by projecting $\boldsymbol{y}_i$ onto the $\operatorname{Span}(\boldsymbol{W})$ along $\operatorname{Span}^{\perp}(\boldsymbol{Z})$, where $\boldsymbol{Z}$ is a matrix of predetermined variables such that $\mathbb{E}(\boldsymbol{Z}^T\boldsymbol{E}) = \mathbf{0}$. For the 3SLS estimator, this is achieved by using the matrix of all exogenous variables $\boldsymbol{X}$ as an instrument where the projection matrix is $\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T$. Therefore, each structural equation is premultiplied by $\boldsymbol{X}^T$ [34] or equivalently, (1.1) is premultiplied by $\boldsymbol{I}_G \otimes \boldsymbol{X}^T$ which yields the transformed SEM (TSEM):

$$\operatorname{vec}\left(\boldsymbol{X}^T\boldsymbol{Y}\right) = \left(\boldsymbol{I}_G \otimes \boldsymbol{X}^T\boldsymbol{W}\right)\boldsymbol{S}\boldsymbol{\delta} + \operatorname{vec}\left(\boldsymbol{X}^T\boldsymbol{E}\right), \; \operatorname{vec}\left(\boldsymbol{X}^T\boldsymbol{E}\right) \sim \left(\mathbf{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{X}^T\boldsymbol{X}\right). \tag{1.2}$$

Applying GLS to (1.2) with $\boldsymbol{\Sigma}$ replaced by its consistent estimator [34], say $\hat{\boldsymbol{\Sigma}}$, gives the three-stage least squares (3SLS) estimator

$$\hat{\boldsymbol{\delta}}_{3SLS} = (\bar{\boldsymbol{W}}^T(\hat{\boldsymbol{\Sigma}} \otimes \boldsymbol{X}^T\boldsymbol{X})^{-1}\bar{\boldsymbol{W}})^{-1}\bar{\boldsymbol{W}}^T(\hat{\boldsymbol{\Sigma}} \otimes \boldsymbol{X}^T\boldsymbol{X})^{-1}\operatorname{vec}(\boldsymbol{X}^T\boldsymbol{Y}), \tag{1.3}$$

where $\bar{\boldsymbol{W}} = \left(\boldsymbol{I}_G \otimes \boldsymbol{X}^T\boldsymbol{W}\right)\boldsymbol{S}$ [34]. The 3SLS estimator (1.3) derives from the solution of the generalized linear least squares problem (GLLSP)

$$\underset{\boldsymbol{\delta},\boldsymbol{V}}{\text{argmin}} \ \|\boldsymbol{V}\|_F^2 \ \text{ subject to } \ \text{vec}\left(\boldsymbol{X}^T\boldsymbol{Y}\right) = \left(\boldsymbol{I}_G \otimes \boldsymbol{X}^T\boldsymbol{W}\right)\boldsymbol{S}\boldsymbol{\delta} + \left(\hat{\boldsymbol{C}} \otimes \boldsymbol{I}_K\right)\text{vec}\left(\boldsymbol{X}^T\boldsymbol{V}\right),$$

where $\hat{\boldsymbol{\Sigma}} = \hat{\boldsymbol{C}}\hat{\boldsymbol{C}}^T$ is the Cholesky decomposition, $\boldsymbol{V} \sim (\boldsymbol{0}, \boldsymbol{I}_K)$ is such that $\boldsymbol{E} = \boldsymbol{V}\hat{\boldsymbol{C}}^T$ and $\|\cdot\|_F$ denotes the Frobenius norm [21, 25].

Large-scale SEMs are intractable to employ due to their multivariate structure, whereas their implementation becomes further burdensome when they have to be estimated recursively. This is an essential procedure when dealing with big data sets, in window estimation and when there is structural change in the SEM [12, 15, 27, 30, 33]. The recursive estimation of the SEM entails the repeated updating of previous estimates, whereby they can absorb additional observations, while avoiding the use of the entire high dimensional data set. That is, when new data are acquired, a recursive procedure will obtain the 3SLS estimator of the augmented SEM

$$\text{vec}\begin{pmatrix}\boldsymbol{Y}\\ \boldsymbol{Y}_u\end{pmatrix} = \left(\boldsymbol{I}_G \otimes \begin{pmatrix}\boldsymbol{W}\\ \boldsymbol{W}_u\end{pmatrix}\right)\boldsymbol{S}\boldsymbol{\delta}^U + \text{vec}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{E}_u\end{pmatrix}, \ \text{vec}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{E}_u\end{pmatrix} \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M+M_u}\right), \tag{1.4}$$

without processing the entire system afresh but by utilizing previous computations. To derive the 3SLS estimator of the augmented SEM, requires premultiplying (1.4) by $\boldsymbol{I}_G \otimes (\boldsymbol{X}^T \ \ \boldsymbol{X}_u^T)$ and then solving the GLLSP

$$\begin{aligned}&\underset{\boldsymbol{\delta}^U,\boldsymbol{V},\boldsymbol{V}_u}{\text{argmin}} \ \|\boldsymbol{V}\|_F^2 + \|\boldsymbol{V}_u\|_F^2 \ \text{ subject to}\\ &\text{vec}\left(\boldsymbol{X}^T\boldsymbol{Y} + \boldsymbol{X}_u^T\boldsymbol{Y}_u\right) = \left(\boldsymbol{I}_G \otimes \left(\boldsymbol{X}^T\boldsymbol{W} + \boldsymbol{X}_u^T\boldsymbol{W}_u\right)\right)\boldsymbol{S}\boldsymbol{\delta}^U + \left(\hat{\boldsymbol{C}} \otimes \boldsymbol{I}_K\right)\text{vec}\left(\boldsymbol{X}^T\boldsymbol{V} + \boldsymbol{X}_u^T\boldsymbol{V}_u\right).\end{aligned}$$

The problem of re-estimating linear models after adding (updating) or removing (downdating) observations has already been addressed [6, 11, 13, 14, 16, 17, 26, 29, 33]. Methods had previously been proposed for the effective estimation of the SEM [1, 6, 8, 18, 20], however, the sequential derivation of the 3SLS estimator for large-scale SEMs has not, previously, been considered.

Herein, the problem of recursively estimating the SEM to add the effect of new or delete the effect of old (obsolete) data points is thoroughly investigated. A theoretical pseudo SEM is developed which has the same 3SLS solution as the modified SEM when estimated afresh. Specifically the proposed method entails a double updating of the original SEM. The first update incorporates the new observations and the second update eliminates the endogeneity that stems from these new observations. This is a challenging issue in the estimation of the SEM and is especially difficult when the model is estimated recursively. The new method removes the endogeneity by adding imaginary (complex) data. This creates a SEM that has a non positive definite dispersion matrix. Nonetheless, the estimation of this theoretical model does not use complex arithmetic. The advantages of the new method is numerical accuracy for the estimates and computational efficiency. They are achieved by implementing orthogonal and hyperbolic transformations, by exploiting the sparsity of the pseudo SEM and by utilizing the previous computations that have provided the estimates of the original model. However, hyperbolic transformations are known to encounter difficulties in terms of stability in the presence of ill conditioned problems. Prudent implementation of hyperbolic transformations can improve the stability of the downdating procedure [3, 4, 23, 24]. Also, applying a sequence of simultaneous updates and downdates has been shown to be relationally stable following careful application of hyperbolic transformations as discussed in [2, 31].

The next section provides a summary of how to derive the 3SLS estimator using the QR decomposition while avoiding the inversion of the large covariance matrix of the SEM. These preliminary results are needed for setting up the background of the recursive method. In Section 3 the new theoretical pseudo SEM for the recursive estimation of the SEM is proposed. The estimator of the model and the corresponding iterative 3SLS estimator are derived. In Section 4 the downdating problem of deleting observations from the SEM is solved. Section 5 employs the proposed recursive method for the estimation of a large-scale macroeconomic model. Finally, Section 6 concludes.

2. Numerical estimation of the SEM

In order to derive efficiently the 3SLS estimator of the SEM, orthogonal transformations are used [1]. Let the QR decomposition (QRD) of $\boldsymbol{X}$ be given by

$$\boldsymbol{Q}^T \begin{pmatrix} \boldsymbol{X} & \boldsymbol{Y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{0} & \boldsymbol{R}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_A \\ \boldsymbol{R}_B \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q}_A & \boldsymbol{Q}_B \end{pmatrix}, \tag{2.1}$$

where $\boldsymbol{Q} \in \mathbb{R}^{M\times M}$ is orthogonal and $\boldsymbol{R}_{11} \in \mathbb{R}^{K\times K}$ is upper triangular and non singular. Using the latter, the TSEM (1.2) is now written as

$$\mathrm{vec}\left(\boldsymbol{R}_{12}\right)=\left(\boldsymbol{I}_G\otimes\boldsymbol{R}_A\right)\boldsymbol{S}\boldsymbol{\delta}+\mathrm{vec}\left(\bar{\boldsymbol{E}}\right),\quad \mathrm{vec}\left(\bar{\boldsymbol{E}}\right)\sim\left(\boldsymbol{0},\boldsymbol{\Sigma}\otimes\boldsymbol{I}_K\right),\tag{2.2}$$

where $\bar{\boldsymbol{E}}=\boldsymbol{Q}_A^T\boldsymbol{E}$ [1, 20]. Observe that the dispersion matrix has been simplified to $\boldsymbol{\Sigma}\otimes\boldsymbol{I}_K$ and the dimensions of the model have been reduced. The 3SLS estimator in (1.3) is obtained if the method of GLS is applied to (2.2) and $\boldsymbol{\Sigma}$ is replaced by $\hat{\boldsymbol{\Sigma}}$, or equivalently from the solution of the GLLSP

$$\underset{\boldsymbol{\delta},\bar{\boldsymbol{V}}}{\mathrm{argmin}}\ \left\|\bar{\boldsymbol{V}}\right\|_F^2\ \text{ subject to }\ \mathrm{vec}\left(\boldsymbol{R}_{12}\right)=\left(\boldsymbol{I}_G\otimes\boldsymbol{R}_A\right)\boldsymbol{S}\boldsymbol{\delta}+\left(\hat{\boldsymbol{C}}\otimes\boldsymbol{I}_K\right)\mathrm{vec}\left(\bar{\boldsymbol{V}}\right),\tag{2.3}$$

where $\bar{\boldsymbol{V}}\sim(\boldsymbol{0},\boldsymbol{I}_K)$ is such that $\bar{\boldsymbol{E}}=\bar{\boldsymbol{V}}\hat{\boldsymbol{C}}^T$. In the case of singular or ill conditioned $\hat{\boldsymbol{\Sigma}}$, the method of GLLSP allows the estimation of the SEM and provides accurate results.

For simplicity, herein, it will be assumed that $\hat{\boldsymbol{\Sigma}}$ is non singular. For the solution of the GLLSP (2.3) compute the generalized QR decomposition (GQRD) of $(\boldsymbol{I}_G\otimes\boldsymbol{R}_A)\,\boldsymbol{S}$ and $(\hat{\boldsymbol{C}}\otimes\boldsymbol{I}_K)$, that is,

$$\tilde{\boldsymbol{Q}}^T\left(\left(\boldsymbol{I}_G\otimes\boldsymbol{R}_A\right)\boldsymbol{S}\quad\mathrm{vec}\left(\boldsymbol{R}_{12}\right)\right)=\begin{pmatrix}\oplus_i\boldsymbol{R}_i & \boldsymbol{y}_A\\ \boldsymbol{0} & \boldsymbol{y}_B\end{pmatrix}\tag{2.4a}$$

and

$$\tilde{\boldsymbol{Q}}^T\left(\hat{\boldsymbol{C}}\otimes\boldsymbol{I}_K\right)\boldsymbol{P}=\boldsymbol{U}=\begin{pmatrix}\boldsymbol{U}_{11} & \boldsymbol{U}_{12}\\ \boldsymbol{0} & \boldsymbol{U}_{22}\end{pmatrix}\begin{matrix}\kappa\\ GK-\kappa.\end{matrix}\tag{2.4b}$$

Here $\tilde{\boldsymbol{Q}},\boldsymbol{P}\in\mathbb{R}^{GK\times GK}$ are orthogonal matrices, $\boldsymbol{U}\in\mathbb{R}^{GK\times GK}$ and $\boldsymbol{R}_i\in\mathbb{R}^{(k_i+g_i)\times(k_i+g_i)}$ for $i=1,\ldots,G$ are upper triangular and non singular, $\kappa=\sum_{i=1}^{G}(k_i+g_i)$ and $\oplus_i$ denotes the direct sum for $i=1,\ldots,G$. Applying the GQRD in (2.4a)-(2.4b) to (2.3) will give the equivalent GLLSP

$$\underset{\tilde{v},\hat{v},\boldsymbol{\delta}}{\mathrm{argmin}}\ \left\|\begin{pmatrix}\boldsymbol{v}_A\\ \boldsymbol{v}_B\end{pmatrix}\right\|^2\ \text{ subject to }\ \begin{pmatrix}\boldsymbol{y}_A\\ \boldsymbol{y}_B\end{pmatrix}=\begin{pmatrix}\oplus_i\boldsymbol{R}_i\\ \boldsymbol{0}\end{pmatrix}\boldsymbol{\delta}+\begin{pmatrix}\boldsymbol{U}_{11} & \boldsymbol{U}_{12}\\ \boldsymbol{0} & \boldsymbol{U}_{22}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_A\\ \boldsymbol{v}_B\end{pmatrix},\tag{2.5}$$

where $(\boldsymbol{v}_A^T\ \ \boldsymbol{v}_B^T)=\mathrm{vec}(\bar{\boldsymbol{V}})^T\boldsymbol{P}$ and $\|\cdot\|$ denotes the Euclidean norm. Now observe that $\boldsymbol{v}_B=\boldsymbol{U}_{22}^{-1}\boldsymbol{y}_B$ and thus, $\boldsymbol{v}_A$ is set to zero to minimize the argument in (2.5). The 3SLS estimator is then given by $\hat{\boldsymbol{\delta}}_{3SLS}=\left(\oplus_i\boldsymbol{R}_i\right)^{-1}\hat{\boldsymbol{y}}_A$, where $\hat{\boldsymbol{y}}_A=\boldsymbol{y}_A-\boldsymbol{U}_{12}\boldsymbol{v}_B$.

3. Recursively estimating the SEM with new observations

The recursive estimation of a model is a procedure which is equivalent to the problem of updating a model consecutively when new observations become available. Similarly, when the data set is too large that cannot be accommodated within the computer's memory, then an out-of-core algorithm proceeds sequentially by updating at every step the current model with some extra observations. Assume that M_u new observations become available and their effect will be added to the model to update the 3SLS estimator. Let the system of structural equations of the new observations be denoted by

$$\operatorname{vec}(\boldsymbol{Y}_u) = (\boldsymbol{I}_G \otimes \boldsymbol{W}_u)\, \boldsymbol{S}\boldsymbol{\delta}_u + \operatorname{vec}(\boldsymbol{E}_u)\,, \ \operatorname{vec}(\boldsymbol{E}_u) \sim (\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_u})\,, \tag{3.1}$$

where $\boldsymbol{Y}_u, \boldsymbol{E}_u \in \mathbb{R}^{M_u \times G}$, $\boldsymbol{X}_u \in \mathbb{R}^{M_u \times K}$ and $\boldsymbol{W}_u = (\boldsymbol{X}_u \ \ \boldsymbol{Y}_u)$. Also define

$$\tilde{\boldsymbol{W}} = \begin{pmatrix}\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{Y}}\end{pmatrix} = \begin{pmatrix}\boldsymbol{W} \\ \boldsymbol{W}_u\end{pmatrix} \ \text{ and } \ \tilde{\boldsymbol{E}} = \begin{pmatrix}\boldsymbol{E} \\ \boldsymbol{E}_u\end{pmatrix}. \tag{3.2}$$

Then the updated SEM to be estimated is given by

$$\operatorname{vec}\left(\tilde{\boldsymbol{Y}}\right) = \left(\boldsymbol{I}_G \otimes \tilde{\boldsymbol{W}}\right) \boldsymbol{S}\boldsymbol{\delta}^U + \operatorname{vec}\left(\tilde{\boldsymbol{E}}\right),\ \operatorname{vec}\left(\tilde{\boldsymbol{E}}\right) \sim (\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M+M_u})\,. \tag{3.3}$$

In order to eliminate endogeneity, similarly to (1.2), premultiply each structural equation with $\tilde{\boldsymbol{X}}^T$, that is,

$$\operatorname{vec}\left(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}}\right) = \left(\boldsymbol{I}_G \otimes \tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{W}}\right) \boldsymbol{S}\boldsymbol{\delta}^U + \operatorname{vec}\left(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{E}}\right),\ \operatorname{vec}\left(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{E}}\right) \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}}\right), \tag{3.4}$$

where $\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{W}} = \boldsymbol{X}^T\boldsymbol{W} + \boldsymbol{X}_u^T\boldsymbol{W}_u$. Analogously to (1.3) for (1.2), the 3SLS estimator of the updated SEM (3.3) is obtained by applying GLS, that is,

$$\hat{\boldsymbol{\delta}}^U_{3SLS} = \left(\boldsymbol{S}^T\left(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \tilde{\boldsymbol{W}}^T\tilde{\boldsymbol{X}}(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}})^{-1}\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{W}}\right)\boldsymbol{S}\right)^{-1} \boldsymbol{S}^T\operatorname{vec}\left(\tilde{\boldsymbol{W}}^T\tilde{\boldsymbol{X}}(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}})^{-1}\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}}\hat{\boldsymbol{\Sigma}}^{-1}\right). \tag{3.5}$$

The new theoretical pseudo SEM, which yields the 3SLS estimator in (3.5) of the updated observations SEM (3.3), is shown in Theorem 1. This pseudo SEM is used to recursively derive the 3SLS estimator by exploiting the computations used in solving (2.3).

Theorem 1. *The updated observations 3SLS estimator in (3.5) is equivalent to the 3SLS estimator of the pseudo SEM*

$$\begin{pmatrix} \text{vec}(\boldsymbol{Y}) \\ \text{vec}(\boldsymbol{Y}_u) \\ \text{vec}\left(\imath\tilde{\boldsymbol{R}}_{22}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_G \otimes \boldsymbol{W} \\ \boldsymbol{I}_G \otimes \boldsymbol{W}_u \\ \boldsymbol{I}_G \otimes \imath\tilde{\boldsymbol{R}}_B \end{pmatrix} \boldsymbol{S}\boldsymbol{\delta}^U + \begin{pmatrix} \text{vec}(\boldsymbol{E}) \\ \text{vec}(\boldsymbol{E}_u) \\ \text{vec}\left(\imath\check{\tilde{\boldsymbol{E}}}\right) \end{pmatrix},$$
$$\begin{pmatrix} \text{vec}(\boldsymbol{E}) \\ \text{vec}(\boldsymbol{E}_u) \\ \text{vec}\left(\imath\check{\tilde{\boldsymbol{E}}}\right) \end{pmatrix} \sim \left(0, \begin{pmatrix} \boldsymbol{\Sigma} \otimes \boldsymbol{I}_M & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_u} \end{pmatrix}\right), \tag{3.6}$$

where the instruments to remove endogeneity of the first M and the M_u new observations are matrices $\boldsymbol{X}$ and $\imath\tilde{\boldsymbol{R}}_{22}$, respectively. Here $\imath$ is the imaginary unit ($\imath^2 = -1$) and $\tilde{\boldsymbol{R}}_{22}$, $\tilde{\boldsymbol{R}}_B$ are derived from the updating QRD (UQRD)

$$\begin{pmatrix} \boldsymbol{Q}_{u_A}^T \\ \boldsymbol{Q}_{u_B}^T \end{pmatrix} \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \boldsymbol{X}_u & \boldsymbol{Y}_u \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{R}}_{11} & \tilde{\boldsymbol{R}}_{12} \\ \boldsymbol{0} & \tilde{\boldsymbol{R}}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{R}}_A \\ \tilde{\boldsymbol{R}}_B \end{pmatrix}, \quad \boldsymbol{Q}_u = \overset{\qquad K \qquad\quad M_u}{\begin{pmatrix} \boldsymbol{Q}_{u_A} & \boldsymbol{Q}_{u_B} \end{pmatrix}}, \tag{3.7}$$

where $\boldsymbol{Q}_u$ is orthogonal of order $(K + M_u)$, $\boldsymbol{R}_{11}$, $\boldsymbol{R}_{12}$ are available from (2.1) and $\tilde{\boldsymbol{R}}_{11}$ is the upper triangular factor from the QRD of $\tilde{\boldsymbol{X}}$.

Proof. *Consider the QRD of $\boldsymbol{X}$ in (2.1) which gives*

$$\boldsymbol{X}^T\boldsymbol{X} = \boldsymbol{R}_{11}^T\boldsymbol{R}_{11} \quad \text{and} \quad \boldsymbol{X}^T\boldsymbol{Y} = \boldsymbol{R}_{11}^T\boldsymbol{R}_{12}. \tag{3.8}$$

Given that $\tilde{\boldsymbol{R}}_{11}$ is the upper triangular factor from the QRD of $\tilde{\boldsymbol{X}}$, it follows that

$$\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}} = \boldsymbol{R}_{11}^T\boldsymbol{R}_{11} + \boldsymbol{X}_u^T\boldsymbol{X}_u = \tilde{\boldsymbol{R}}_{11}^T\tilde{\boldsymbol{R}}_{11} \tag{3.9}$$

and also that

$$\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}} = \boldsymbol{R}_{11}^T\boldsymbol{R}_{12} + \boldsymbol{X}_u^T\boldsymbol{Y}_u = \tilde{\boldsymbol{R}}_{11}^T\tilde{\boldsymbol{R}}_{12}, \tag{3.10}$$

where $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Y}}$ are defined in (3.2). The latter imply the UQRD (3.7). Thus, from (3.9) and (3.10) it follows that the updated TSEM (3.4) is written as

$$\text{vec}\left(\boldsymbol{R}_{11}^T\boldsymbol{R}_{12} + \boldsymbol{X}_u^T\boldsymbol{Y}_u\right) = \left(\boldsymbol{I}_G \otimes \left(\boldsymbol{R}_{11}^T\boldsymbol{R}_A + \boldsymbol{X}_u^T\boldsymbol{W}_u\right)\right)\boldsymbol{S}\boldsymbol{\delta}^U + \text{vec}\left(\boldsymbol{R}_{11}^T\bar{\boldsymbol{E}} + \boldsymbol{X}_u^T\boldsymbol{E}_u\right),$$

or equivalently as

$$\text{vec}\left(\tilde{\boldsymbol{R}}_{12}\right) = \left(\boldsymbol{I}_G \otimes \tilde{\boldsymbol{R}}_A\right)\boldsymbol{S}\boldsymbol{\delta}^U + \text{vec}\left(\tilde{\tilde{\boldsymbol{E}}}\right), \quad \text{vec}\left(\tilde{\tilde{\boldsymbol{E}}}\right) \sim (\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_K), \tag{3.11}$$

where $\tilde{\tilde{\boldsymbol{E}}} = \boldsymbol{Q}_{u_A}^T\tilde{\boldsymbol{E}}$. The GLS estimator of (3.11) gives the 3SLS estimator (3.5) of the updated

TSEM (3.4), that is,

$$\hat{\boldsymbol{\delta}}^U_{3SLS} = \left(\boldsymbol{S}^T\left(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \tilde{\boldsymbol{R}}_A^T\tilde{\boldsymbol{R}}_A\right)\boldsymbol{S}\right)^{-1}\boldsymbol{S}^T\left(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \boldsymbol{I}_{K+G}\right)\operatorname{vec}\left(\tilde{\boldsymbol{R}}_A^T\tilde{\boldsymbol{R}}_{12}\right), \tag{3.12}$$

where $\hat{\boldsymbol{\Sigma}}$ is the consistent estimator of $\boldsymbol{\Sigma}$ obtained from the 2SLS residuals of the SEM (3.3).

Observe now from the 3SLS estimator in (3.12) that

$$\tilde{\boldsymbol{R}}_A^T\tilde{\boldsymbol{R}}_A = \begin{pmatrix} \tilde{\boldsymbol{R}}_{11}^T\tilde{\boldsymbol{R}}_{11} & \tilde{\boldsymbol{R}}_{11}^T\tilde{\boldsymbol{R}}_{12} \\ \tilde{\boldsymbol{R}}_{12}^T\tilde{\boldsymbol{R}}_{11} & \tilde{\boldsymbol{R}}_{12}^T\tilde{\boldsymbol{R}}_{12} \end{pmatrix}, \tag{3.13}$$

where $\tilde{\boldsymbol{R}}_{11}^T\tilde{\boldsymbol{R}}_{11}$ and $\tilde{\boldsymbol{R}}_{11}^T\tilde{\boldsymbol{R}}_{12}$ are known (see (3.9) and (3.10)), but $\tilde{\boldsymbol{R}}_{12}^T\tilde{\boldsymbol{R}}_{12}$ is unknown and it needs to be determined. From the UQRD (3.7), it holds that

$$\tilde{\boldsymbol{R}}_{12} = \boldsymbol{Q}_{u_A}^T\begin{pmatrix} \boldsymbol{R}_{12} \\ \boldsymbol{Y}_u \end{pmatrix}$$

and also that

$$\begin{pmatrix} \boldsymbol{R}_{12} \\ \boldsymbol{Y}_u \end{pmatrix} = \begin{pmatrix} \boldsymbol{Q}_{u_A} & \boldsymbol{Q}_{u_B} \end{pmatrix}\begin{pmatrix} \tilde{\boldsymbol{R}}_{12} \\ \tilde{\boldsymbol{R}}_{22} \end{pmatrix},$$

which imply that $\tilde{\boldsymbol{R}}_{12}^T\tilde{\boldsymbol{R}}_{12} = \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{X}}(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}})^{-1}\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}}$ and $\tilde{\boldsymbol{R}}_{12}^T\tilde{\boldsymbol{R}}_{12} = \boldsymbol{R}_{12}^T\boldsymbol{R}_{12} + \boldsymbol{Y}_u^T\boldsymbol{Y}_u - \tilde{\boldsymbol{R}}_{22}^T\tilde{\boldsymbol{R}}_{22}$, respectively. Now from the latter and (3.9) - (3.10) it follows that

$$\begin{aligned} \tilde{\boldsymbol{R}}_A^T\tilde{\boldsymbol{R}}_A &= \begin{pmatrix} \tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}} \\ \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{X}} & \tilde{\boldsymbol{Y}}^T\tilde{\boldsymbol{X}}\left(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}}\right)^{-1}\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}} \end{pmatrix} \\ &= \tilde{\boldsymbol{W}}^T\tilde{\boldsymbol{X}}(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}})^{-1}\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{W}} \\ &= \begin{pmatrix} \boldsymbol{R}_{11}^T\boldsymbol{R}_{11} + \boldsymbol{X}_u^T\boldsymbol{X}_u & \boldsymbol{R}_{11}^T\boldsymbol{R}_{12} + \boldsymbol{X}_u^T\boldsymbol{Y}_u \\ \boldsymbol{R}_{12}^T\boldsymbol{R}_{11} + \boldsymbol{Y}_u^T\boldsymbol{X}_u & \boldsymbol{R}_{12}^T\boldsymbol{R}_{12} + \boldsymbol{Y}_u^T\boldsymbol{Y}_u - \tilde{\boldsymbol{R}}_{22}^T\tilde{\boldsymbol{R}}_{22} \end{pmatrix} \\ &= \boldsymbol{R}_A^T\boldsymbol{R}_A + \boldsymbol{W}_u^T\boldsymbol{W}_u - \tilde{\boldsymbol{R}}_B^T\tilde{\boldsymbol{R}}_B \\ &= \begin{pmatrix} \boldsymbol{R}_A \\ \boldsymbol{W}_u \\ \imath\tilde{\boldsymbol{R}}_B \end{pmatrix}^H \boldsymbol{\Phi} \begin{pmatrix} \boldsymbol{R}_A \\ \boldsymbol{W}_u \\ \imath\tilde{\boldsymbol{R}}_B \end{pmatrix}, \end{aligned} \tag{3.14}$$

where $\boldsymbol{\Phi} = \operatorname{diag}(\boldsymbol{I}_K, \boldsymbol{I}_{M_u}, -\boldsymbol{I}_{M_u})$ and $(\cdot)^H$ denotes the conjugate transpose of a matrix. Similarly it can be shown that

$$\begin{aligned}\tilde{\boldsymbol{R}}_A^T\tilde{\boldsymbol{R}}_{12} &= \tilde{\boldsymbol{W}}^T\tilde{\boldsymbol{X}}(\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{X}})^{-1}\tilde{\boldsymbol{X}}^T\tilde{\boldsymbol{Y}} \\ &= \begin{pmatrix}\boldsymbol{R}_A \\ \boldsymbol{W}_u \\ \imath\tilde{\boldsymbol{R}}_B\end{pmatrix}^H \boldsymbol{\Phi} \begin{pmatrix}\boldsymbol{R}_{12} \\ \boldsymbol{Y}_u \\ \imath\tilde{\boldsymbol{R}}_{22}\end{pmatrix}.\end{aligned} \tag{3.15}$$

Then substituting (3.14) and (3.15) into (3.12), the 3SLS estimator in (3.5) is given by

$$\begin{aligned}\hat{\boldsymbol{\delta}}^U_{3SLS} = &\left(\boldsymbol{S}^T\begin{pmatrix}\boldsymbol{I}_G\otimes\boldsymbol{R}_A \\ \boldsymbol{I}_G\otimes\boldsymbol{W}_u \\ \boldsymbol{I}_G\otimes\imath\tilde{\boldsymbol{R}}_B\end{pmatrix}^H \begin{pmatrix}\hat{\boldsymbol{\Sigma}}\otimes\boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{\Sigma}}\otimes\boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\hat{\boldsymbol{\Sigma}}\otimes\boldsymbol{I}_{M_u}\end{pmatrix}^{-1} \begin{pmatrix}\boldsymbol{I}_G\otimes\boldsymbol{R}_A \\ \boldsymbol{I}_G\otimes\boldsymbol{W}_u \\ \boldsymbol{I}_G\otimes\imath\tilde{\boldsymbol{R}}_B\end{pmatrix}\boldsymbol{S}\right)^{-1} \\ &\cdot\boldsymbol{S}^T\begin{pmatrix}\boldsymbol{I}_G\otimes\boldsymbol{R}_A \\ \boldsymbol{I}_G\otimes\boldsymbol{W}_u \\ \boldsymbol{I}_G\otimes\imath\tilde{\boldsymbol{R}}_B\end{pmatrix}^H \begin{pmatrix}\hat{\boldsymbol{\Sigma}}\otimes\boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{\Sigma}}\otimes\boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\hat{\boldsymbol{\Sigma}}\otimes\boldsymbol{I}_{M_u}\end{pmatrix}^{-1} \begin{pmatrix}\mathrm{vec}\,(\boldsymbol{R}_{12}) \\ \mathrm{vec}\,(\boldsymbol{Y}_u) \\ \mathrm{vec}\left(\imath\tilde{\boldsymbol{R}}_{22}\right)\end{pmatrix},\end{aligned} \tag{3.16}$$

where $\hat{\boldsymbol{\delta}}^U_{3SLS}$ *is the GLS estimator of the TSEM*

$$\begin{aligned}\begin{pmatrix}\mathrm{vec}\,(\boldsymbol{R}_{12}) \\ \mathrm{vec}\,(\boldsymbol{Y}_u) \\ \mathrm{vec}\left(\imath\tilde{\boldsymbol{R}}_{22}\right)\end{pmatrix} &= \begin{pmatrix}\boldsymbol{I}_G\otimes\boldsymbol{R}_A \\ \boldsymbol{I}_G\otimes\boldsymbol{W}_u \\ \boldsymbol{I}_G\otimes\imath\tilde{\boldsymbol{R}}_B\end{pmatrix}\boldsymbol{S}\boldsymbol{\delta}^U + \begin{pmatrix}\mathrm{vec}\left(\bar{\boldsymbol{E}}\right) \\ \mathrm{vec}\,(\boldsymbol{E}_u) \\ \mathrm{vec}\left(\imath\check{\tilde{\boldsymbol{E}}}\right)\end{pmatrix}, \\ \begin{pmatrix}\mathrm{vec}\left(\bar{\boldsymbol{E}}\right) \\ \mathrm{vec}\,(\boldsymbol{E}_u) \\ \mathrm{vec}\left(\imath\check{\tilde{\boldsymbol{E}}}\right)\end{pmatrix} &\sim \left(\boldsymbol{0}, \begin{pmatrix}\boldsymbol{\Sigma}\otimes\boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}\otimes\boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{\Sigma}\otimes\boldsymbol{I}_{M_u}\end{pmatrix}\right).\end{aligned} \tag{3.17}$$

The TSEM (3.17) is the SEM (3.6) after it has been premultiplied by $\mathrm{diag}(\boldsymbol{I}_G\otimes\boldsymbol{X}^T, \boldsymbol{I}_{2M_u})$ *to eliminate the endogeneity of the first* M *observations. This concludes the proof.* ■

The relationships in (3.14) and (3.15) prove that the 3SLS estimators in (3.5) and (3.16) are identical and hence Theorem 1 guarantees the equivalence of the proposed pseudo SEM (3.6) with the updated SEM in (3.3). Equivalently the latter shows that the GLS estimators of the TSEM (3.4) and model (3.17), which are both free of endogeneity, are equivalent. Furthermore, note that the effect of the third block of rows in (3.6) is to eliminate the endogeneity arising from the observations added in the model. This means that once endogeneity has been eliminated in (3.6), that is, (3.17) is derived, its GLS estimator can be computed efficiently. Namely, the numerically accurate method of GLLSP (see (2.3)-(2.5)) is applied. Moreover, previous computations are utilized. Therefore, the computational cost is reduced.

3.1 Deriving the 3SLS estimator of the pseudo SEM

For the efficient computation of the 3SLS estimator in (3.16), the proposed transformed model in (3.17) is reformulated to the equivalent GLLSP, that is,

$$\underset{\bar{\boldsymbol{V}},\boldsymbol{V}_u,\hat{\boldsymbol{V}}_u,\boldsymbol{\delta}^U}{\operatorname{argmin}} \left(\left\|\operatorname{vec}(\bar{\boldsymbol{V}})\right\|_F^2 + \|\operatorname{vec}(\boldsymbol{V}_u)\|_F^2 - \left\|\operatorname{vec}(\hat{\boldsymbol{V}}_u)\right\|_F^2 \right) \text{ subject to}$$

$$\begin{pmatrix} \operatorname{vec}(\boldsymbol{R}_{12}) \\ \operatorname{vec}(\boldsymbol{Y}_u) \\ \operatorname{vec}(\imath\tilde{\boldsymbol{R}}_{22}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_G \otimes \boldsymbol{R}_A \\ \boldsymbol{I}_G \otimes \boldsymbol{W}_u \\ \boldsymbol{I}_G \otimes \imath\tilde{\boldsymbol{R}}_B \end{pmatrix} \boldsymbol{S}\boldsymbol{\delta}^U + \begin{pmatrix} \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} \end{pmatrix} \begin{pmatrix} \operatorname{vec}(\bar{\boldsymbol{V}}) \\ \operatorname{vec}(\boldsymbol{V}_u) \\ \operatorname{vec}(\imath\hat{\boldsymbol{V}}_u) \end{pmatrix}.$$

Assume that the solution of the GLLSP (2.3), for obtaining the 3SLS estimator of (1.1), is available. Employing the GQRD (2.4), yields the equivalent GLLSP

$$\underset{\boldsymbol{v}_A,\boldsymbol{v}_B,\boldsymbol{V}_u,\hat{\boldsymbol{V}}_u,\boldsymbol{\delta}^U}{\operatorname{argmin}} \left(\|\boldsymbol{v}_A\|^2 + \|\boldsymbol{v}_B\|^2 + \|\boldsymbol{V}_u\|_F^2 - \left\|\hat{\boldsymbol{V}}_u\right\|_F^2 \right) \text{ subject to}$$

$$\begin{pmatrix} \boldsymbol{y}_A \\ \boldsymbol{y}_B \\ \operatorname{vec}(\boldsymbol{Y}_u) \\ \operatorname{vec}(\imath\tilde{\boldsymbol{R}}_{22}) \end{pmatrix} = \begin{pmatrix} \oplus_i \boldsymbol{R}_i \\ \boldsymbol{0} \\ (\boldsymbol{I}_G \otimes \boldsymbol{W}_u)\boldsymbol{S} \\ (\boldsymbol{I}_G \otimes \imath\tilde{\boldsymbol{R}}_B)\boldsymbol{S} \end{pmatrix} \boldsymbol{\delta}^U + \begin{pmatrix} \boldsymbol{U}_{11} & \boldsymbol{U}_{12} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_{22} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes I_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_A \\ \boldsymbol{v}_B \\ \operatorname{vec}(\boldsymbol{V}_u) \\ \operatorname{vec}(\imath\hat{\boldsymbol{V}}_u) \end{pmatrix}, \tag{3.18}$$

where $\boldsymbol{v}_B = \boldsymbol{U}_{22}^{-1}\boldsymbol{y}_B$ and so the latter reduces to

$$\underset{\boldsymbol{v}_A,\boldsymbol{V}_u,\hat{\boldsymbol{V}}_u,\boldsymbol{\delta}^U}{\operatorname{argmin}} \left(\|\boldsymbol{v}_A\|^2 + \|\boldsymbol{V}_u\|_F^2 - \left\|\hat{\boldsymbol{V}}_u\right\|_F^2 \right) \text{ subject to}$$

$$\begin{pmatrix} \hat{\boldsymbol{y}}_A \\ \operatorname{vec}(\boldsymbol{Y}_u) \\ \operatorname{vec}(\imath\tilde{\boldsymbol{R}}_{22}) \end{pmatrix} = \begin{pmatrix} \oplus_i \boldsymbol{R}_i \\ (\boldsymbol{I}_G \otimes \boldsymbol{W}_u)\boldsymbol{S} \\ (\boldsymbol{I}_G \otimes \imath\tilde{\boldsymbol{R}}_B)\boldsymbol{S} \end{pmatrix} \boldsymbol{\delta}^U + \begin{pmatrix} \boldsymbol{U}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_A \\ \operatorname{vec}(\boldsymbol{V}_u) \\ \operatorname{vec}(\imath\hat{\boldsymbol{V}}_u) \end{pmatrix}, \tag{3.19}$$

where $\hat{\boldsymbol{y}}_A = \boldsymbol{y}_A - \boldsymbol{U}_{12}\boldsymbol{v}_B$. For the solution of (3.19) consider the hyperbolic QR decomposition (HQRD)

$$\tilde{\boldsymbol{Q}}_u^H \begin{pmatrix} \oplus_i \boldsymbol{R}_i & \hat{\boldsymbol{y}}_A \\ (\boldsymbol{I}_G \otimes \boldsymbol{W}_u)\boldsymbol{S} & \operatorname{vec}(\boldsymbol{Y}_u) \\ (\boldsymbol{I}_G \otimes \imath\tilde{\boldsymbol{R}}_B)\boldsymbol{S} & \operatorname{vec}(\imath\tilde{\boldsymbol{R}}_{22}) \end{pmatrix} = \begin{pmatrix} \oplus_i \tilde{\boldsymbol{R}}_i & \tilde{\boldsymbol{y}}_A \\ \boldsymbol{0} & \tilde{\boldsymbol{y}}_B \\ \boldsymbol{0} & \imath\tilde{\boldsymbol{y}}_C \end{pmatrix} \tag{3.20a}$$

and the RQ decomposition (RQD)

$$\tilde{\boldsymbol{Q}}_u^H \begin{pmatrix} \boldsymbol{U}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_u} \end{pmatrix} \tilde{\boldsymbol{P}} = \tilde{\boldsymbol{U}} = \begin{pmatrix} \tilde{\boldsymbol{U}}_{11} & \tilde{\boldsymbol{U}}_{12} & \imath\tilde{\boldsymbol{U}}_{13} \\ \boldsymbol{0} & \tilde{\boldsymbol{U}}_{22} & \imath\tilde{\boldsymbol{U}}_{23} \\ \boldsymbol{0} & \boldsymbol{0} & \tilde{\boldsymbol{U}}_{33} \end{pmatrix}. \tag{3.20b}$$

Here $\tilde{\boldsymbol{Q}}_u$ is a $\tilde{\boldsymbol{\Phi}}$-unitary matrix with respect to the signature matrix $\tilde{\boldsymbol{\Phi}} = \text{diag}(\boldsymbol{I}_{\kappa+GM_u}, -\boldsymbol{I}_{GM_u})$, that is, $\tilde{\boldsymbol{Q}}_u \tilde{\boldsymbol{\Phi}} \tilde{\boldsymbol{Q}}_u^H = \tilde{\boldsymbol{\Phi}}$ and is defined as the product of K hyperbolic Householder transformations [5, 22]. Also $\tilde{\boldsymbol{P}}$ is a unitary matrix of order $(\kappa + 2GM_u)$ and $\tilde{\boldsymbol{U}} \in \mathbb{C}^{(\kappa+2GM_u)\times(\kappa+2GM_u)}$, $\tilde{\boldsymbol{R}}_i \in \mathbb{R}^{(k_i+g_i)\times(k_i+g_i)}$ for $i = 1, \ldots, G$, are upper triangular and non singular. Then the GLLSP (3.19) becomes

$$\underset{\tilde{\boldsymbol{v}}_A, \tilde{\boldsymbol{v}}_B, \tilde{\boldsymbol{v}}_C, \boldsymbol{\delta}^U}{\text{argmin}} \left\| \begin{pmatrix} \tilde{\boldsymbol{v}}_A \\ \tilde{\boldsymbol{v}}_B \\ \imath\tilde{\boldsymbol{v}}_C \end{pmatrix} \right\|_h \text{ subject to } \begin{pmatrix} \tilde{\boldsymbol{y}}_A \\ \tilde{\boldsymbol{y}}_B \\ \imath\tilde{\boldsymbol{y}}_C \end{pmatrix} = \begin{pmatrix} \oplus_i \tilde{\boldsymbol{R}}_i \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \boldsymbol{\delta}^U + \begin{pmatrix} \tilde{\boldsymbol{U}}_{11} & \tilde{\boldsymbol{U}}_{12} & \imath\tilde{\boldsymbol{U}}_{13} \\ \boldsymbol{0} & \tilde{\boldsymbol{U}}_{22} & \imath\tilde{\boldsymbol{U}}_{23} \\ \boldsymbol{0} & \boldsymbol{0} & \tilde{\boldsymbol{U}}_{33} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{v}}_A \\ \tilde{\boldsymbol{v}}_B \\ \imath\tilde{\boldsymbol{v}}_C \end{pmatrix}, \quad (3.21)$$

where $\|\boldsymbol{x}\|_h = \boldsymbol{x}^H \boldsymbol{\Psi} \boldsymbol{x}$ is the hyperbolic norm of a complex column vector $\boldsymbol{x}$ with respect to the signature matrix $\boldsymbol{\Psi}$ [5, 28]. Also let

$$\tilde{\boldsymbol{P}}^H \begin{pmatrix} \boldsymbol{v}_A \\ \text{vec}(\boldsymbol{V}_u) \\ \text{vec}(\imath\hat{\boldsymbol{V}}_u) \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{v}}_A \\ \tilde{\boldsymbol{v}}_B \\ \imath\tilde{\boldsymbol{v}}_C \end{pmatrix}.$$

It follows that $\tilde{\boldsymbol{v}}_B$ and $\tilde{\boldsymbol{v}}_C$ can be obtained from the solution of the triangular system

$$\begin{pmatrix} \tilde{\boldsymbol{y}}_B \\ \imath\tilde{\boldsymbol{y}}_C \end{pmatrix} = \begin{pmatrix} \tilde{\boldsymbol{U}}_{22} & \imath\tilde{\boldsymbol{U}}_{23} \\ \boldsymbol{0} & \tilde{\boldsymbol{U}}_{33} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{v}}_B \\ \imath\tilde{\boldsymbol{v}}_C \end{pmatrix}, \quad (3.22)$$

and $\tilde{\boldsymbol{v}}_A$ is set to zero in order to minimize the argument in (3.21). Hence the updated 3SLS (U3SLS) estimator is given by

$$\hat{\boldsymbol{\delta}}^U_{3SLS} = \left(\oplus_i \tilde{\boldsymbol{R}}_i\right)^{-1} \hat{\tilde{\boldsymbol{y}}}_A, \quad (3.23)$$

where $\hat{\tilde{\boldsymbol{y}}}_A = \tilde{\boldsymbol{y}}_A - \tilde{\boldsymbol{U}}_{12}\tilde{\boldsymbol{v}}_B + \tilde{\boldsymbol{U}}_{13}\tilde{\boldsymbol{v}}_C$.

The main computational steps of the proposed numerical method for the recursive estimation of the SEM are illustrated in Algorithm 1. When the SEM (1.1) is updated with new observations for the first time, previous computations from the QRD (2.1) and the solution of the GLLSP (2.3) are utilized. If new observations become available and the SEM has already been updated by solving the GLLSP (3.19), previous computations from the UQRD (3.7) and the solution of the GLLSP (3.19) are utilized. When observations are sequentially added into the model, the input in the current updating of the model is the output obtained from the previous updating. Therefore, after the first update of the SEM, no data from the original SEM are required. Moreover, in practice, the special sparse structure of the matrices is exploited by employing the computational strategies presented in the Appendix.

Algorithm 1 Estimating the USEM (3.3) by obtaining the estimator of the pseudo SEM (3.17).

1. Given the SEM (1.1), estimate the USEM (3.3).

Input: The new data added to the model are $\boldsymbol{Y}_u,\ \boldsymbol{X}_u$ as defined in (3.1), $\boldsymbol{R}_{11},\ \boldsymbol{R}_{12}$ from the QRD (2.1), and also $\hat{\boldsymbol{y}}_A, \boldsymbol{R}_i,\ i=1,\ldots,G,\ \boldsymbol{U}_{11}$ from the solution of the GLLSP in (2.3).

Output: The 3SLS estimator $\hat{\boldsymbol{\delta}}^U_{3SLS}$ in (3.23), $\tilde{\hat{\boldsymbol{y}}}_A,\ \tilde{\boldsymbol{R}}_i,\ i=1,\ldots,G,\ \tilde{\boldsymbol{U}}_{11}$.

2. **Repeat updating**

Input: The new data added to the model are $\boldsymbol{Y}_u,\ \boldsymbol{X}_u$ as defined in (3.1), $\tilde{\boldsymbol{R}}_{11},\ \tilde{\boldsymbol{R}}_{12}$ from the UQRD (3.7), and also $\tilde{\hat{\boldsymbol{y}}}_A,\ \tilde{\boldsymbol{R}}_i,\ i=1,\ldots,G$ and $\tilde{\boldsymbol{U}}_{11}$ from the solution of the GLLSP in (3.19).

Output: The 3SLS estimator $\hat{\boldsymbol{\delta}}^U_{3SLS}$ in (3.23), $\tilde{\boldsymbol{R}}_{11},\ \tilde{\boldsymbol{R}}_{12}, \tilde{\hat{\boldsymbol{y}}}_A,\ \tilde{\boldsymbol{R}}_i,\ i=1,\ldots,G,\ \tilde{\boldsymbol{U}}_{11}$.

3. Compute the updating QRD in (3.7).
4. Compute the HQRD in (3.20a) and the RQD in (3.20b).
5. Solve the triangular system in (3.22).
6. Compute $\tilde{\hat{\boldsymbol{y}}}_A = \tilde{\boldsymbol{y}}_A - \tilde{\boldsymbol{U}}_{12}\tilde{\boldsymbol{v}}_B + \tilde{\boldsymbol{U}}_{13}\tilde{\boldsymbol{v}}_C$.
7. Solve the triangular system $\left(\oplus_i\tilde{\boldsymbol{R}}_i\right)\hat{\boldsymbol{\delta}}^U_{3SLS} = \tilde{\hat{\boldsymbol{y}}}_A$, for $\hat{\boldsymbol{\delta}}^U_{3SLS}$.
8. **End Repeat Updating**

3.2 Iterative recursive 3SLS

Assume that the U3SLS estimator in (3.23) has been obtained and the solution of (3.19) is available. Now the iterative estimator of the SEM (3.3) based on the solution of (3.19) needs to be computed so that the estimates for $\boldsymbol{\delta}^U$ are improved. In order to derive the iterative 3SLS (I3SLS) estimator, $\boldsymbol{\Sigma}$ is now estimated using the 3SLS residuals, that is,

$$\text{vec}(\tilde{\hat{\boldsymbol{E}}}) = \text{vec}(\tilde{\boldsymbol{Y}}) - (\boldsymbol{I}_G \otimes \tilde{\boldsymbol{W}})\boldsymbol{S}\hat{\boldsymbol{\delta}}^U_{3SLS}.$$

Now $\tilde{\hat{\boldsymbol{\Sigma}}} = \tilde{\hat{\boldsymbol{E}}}^T\tilde{\hat{\boldsymbol{E}}}/(M+M_u)$ is the updated variance-covariance matrix and $\tilde{\hat{\boldsymbol{\Sigma}}} = \tilde{\hat{\boldsymbol{C}}}^T\tilde{\hat{\boldsymbol{C}}}$ is the Cholesky decomposition. The GLLSP (3.19) is now updated with $\tilde{\hat{\boldsymbol{C}}}$ taken into account and hence the GLLSP to be solved is given by

$$\underset{\boldsymbol{v}_A,\boldsymbol{V}_u,\hat{\boldsymbol{V}}_u,\boldsymbol{\delta}^{IU}}{\text{argmin}} \left(\|\boldsymbol{v}_A\|^2 + \|\boldsymbol{V}_u\|_F^2 - \left\|\hat{\boldsymbol{V}}_u\right\|_F^2\right) \text{ subject to}$$

$$\begin{pmatrix} \hat{\boldsymbol{y}}_A \\ \text{vec}(\boldsymbol{Y}_u) \\ \text{vec}(\imath\tilde{\boldsymbol{R}}_{22}) \end{pmatrix} = \begin{pmatrix} \oplus_i\tilde{\boldsymbol{R}}_i \\ (\boldsymbol{I}_G \otimes \boldsymbol{W}_u)\boldsymbol{S} \\ (\boldsymbol{I}_G \otimes \imath\tilde{\boldsymbol{R}}_B)\boldsymbol{S} \end{pmatrix} \boldsymbol{\delta}^{IU} + \begin{pmatrix} \tilde{\boldsymbol{U}}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \tilde{\hat{\boldsymbol{C}}} \otimes I_{M_u} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \tilde{\hat{\boldsymbol{C}}} \otimes \boldsymbol{I}_{M_u} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_A \\ \text{vec}(\boldsymbol{V}_u) \\ \text{vec}(\imath\hat{\boldsymbol{V}}_u) \end{pmatrix}. \tag{3.24}$$

Given that the HQRD in (3.20a) is available, the RQD in (3.20b) will be re-computed, that is,

$$\tilde{\boldsymbol{Q}}_u^H \begin{pmatrix} \tilde{\boldsymbol{U}}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{\tilde{\boldsymbol{C}}} \otimes \boldsymbol{I}_{M_u} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\tilde{\boldsymbol{C}}} \otimes \boldsymbol{I}_{M_u} \end{pmatrix} \hat{\tilde{\boldsymbol{P}}} = \hat{\tilde{\boldsymbol{U}}} = \begin{pmatrix} \hat{\tilde{\boldsymbol{U}}}_{11} & \hat{\tilde{\boldsymbol{U}}}_{12} & \hat{\tilde{\boldsymbol{U}}}_{13} \\ \mathbf{0} & \hat{\tilde{\boldsymbol{U}}}_{22} & \hat{\tilde{\boldsymbol{U}}}_{23} \\ \mathbf{0} & \mathbf{0} & \hat{\tilde{\boldsymbol{U}}}_{33} \end{pmatrix}, \tag{3.25}$$

where $\hat{\tilde{\boldsymbol{P}}} \in \mathbb{R}^{(\kappa+2GM_u)\times(\kappa+2GM_u)}$ is a unitary matrix and $\hat{\tilde{\boldsymbol{U}}} \in \mathbb{C}^{(\kappa+2GM_u)\times(\kappa+2GM_u)}$ is upper triangular and non singular. The transformations in (3.25) are applied to (3.24). The resulting GLLSP is solved in a way similar to (3.21). The iterative procedure is repeated until the estimate for $\boldsymbol{\Sigma}$ of the previous and the current iteration converge.

4. Estimating the SEM after deleting observations

The downdating of the SEM is the problem of removing the effect of some observations from an existing estimator. Namely, this is the case where rows of data are excluded after the estimation procedure has been completed and hence a reduced observations model has to be estimated. Observations may have to be deleted when they are considered to be old and misleading, when they have been shown to be outliers or for the identification of influential data [1, 7, 11, 35].

Assume that the 3SLS estimator of the SEM (1.1) has been computed and then some observations, say M_d, will be deleted from each structural equation. This means that the 3SLS estimator will have to be re-computed. Without loss of generality, consider that the last M_d observations will be deleted from each equation and let

$$\boldsymbol{Y} = \begin{pmatrix} \check{\boldsymbol{Y}} \\ \boldsymbol{Y}_d \end{pmatrix}, \quad \boldsymbol{X} = \begin{pmatrix} \check{\boldsymbol{X}} \\ \boldsymbol{X}_d \end{pmatrix} \text{ and } \boldsymbol{E} = \begin{pmatrix} \check{\boldsymbol{E}} \\ \boldsymbol{E}_d \end{pmatrix},$$

where $\check{\boldsymbol{Y}} \in \mathbb{R}^{(M-M_d)\times G}$, $\boldsymbol{Y}_d \in \mathbb{R}^{M_d \times G}$ and $\boldsymbol{X}$, $\boldsymbol{E}$ are partitioned conformably. Also let $\check{\boldsymbol{W}} = (\check{\boldsymbol{X}} \quad \check{\boldsymbol{Y}})$. The downdated observations SEM to be estimated is given by

$$\text{vec}\left(\check{\boldsymbol{Y}}\right) = \left(\boldsymbol{I}_G \otimes \check{\boldsymbol{W}}\right) \boldsymbol{S}\boldsymbol{\delta}^D + \text{vec}\left(\check{\boldsymbol{E}}\right), \quad \text{vec}\left(\check{\boldsymbol{E}}\right) \sim \left(\mathbf{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M-M_d}\right), \tag{4.1}$$

which has the downdated 3SLS estimator

$$\hat{\boldsymbol{\delta}}_{3SLS}^D = \left(\boldsymbol{S}^T \left(\hat{\boldsymbol{\Sigma}}^{-1} \otimes \check{\boldsymbol{W}}^T \check{\boldsymbol{X}} (\check{\boldsymbol{X}}^T \check{\boldsymbol{X}})^{-1} \check{\boldsymbol{X}}^T \check{\boldsymbol{W}}\right) \boldsymbol{S}\right)^{-1} \boldsymbol{S}^T \text{vec}\left(\check{\boldsymbol{W}}^T \check{\boldsymbol{X}} (\check{\boldsymbol{X}}^T \check{\boldsymbol{X}})^{-1} \check{\boldsymbol{X}}^T \check{\boldsymbol{Y}} \hat{\boldsymbol{\Sigma}}^{-1}\right). \tag{4.2}$$

Theorem 2 presents the theoretical pseudo SEM which is proved to be equivalent to the downdated

SEM (4.1). The downdated observations 3SLS estimator in (4.2) is then derived recursively.

Theorem 2. *The downdated observations 3SLS estimator in (4.2) is equivalent to the 3SLS estimator of the pseudo SEM*

$$\begin{pmatrix} \text{vec}\,(\boldsymbol{X}) \\ \text{vec}\,(\imath \boldsymbol{Y}_d) \\ \text{vec}\left(\check{\boldsymbol{R}}_{22}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_G \otimes \boldsymbol{W} \\ \boldsymbol{I}_G \otimes \imath \boldsymbol{W}_d \\ \boldsymbol{I}_G \otimes \check{\boldsymbol{R}}_B \end{pmatrix} \boldsymbol{S}\boldsymbol{\delta}^D + \begin{pmatrix} \text{vec}\,(\boldsymbol{E}) \\ \text{vec}\,(\imath \boldsymbol{E}_d) \\ \text{vec}\left(\tilde{\check{\boldsymbol{E}}}_d\right) \end{pmatrix},$$
$$\begin{pmatrix} \text{vec}\,(\boldsymbol{E}) \\ \text{vec}\,(\imath \boldsymbol{E}_d) \\ \text{vec}\left(\tilde{\check{\boldsymbol{E}}}_d\right) \end{pmatrix} \sim \left(\boldsymbol{0}, \begin{pmatrix} \boldsymbol{\Sigma} \otimes \boldsymbol{I}_M & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_d} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_d} \end{pmatrix} \right), \tag{4.3}$$

where (4.3) $\check{\boldsymbol{R}}_{22}$ and $\check{\boldsymbol{R}}_B$ are defined from the HQRD

$$\check{\boldsymbol{Q}}^H \begin{pmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \\ \imath \boldsymbol{X}_d & \imath \boldsymbol{Y}_d \end{pmatrix} = \begin{pmatrix} \check{\boldsymbol{R}}_{11} & \check{\boldsymbol{R}}_{12} \\ \boldsymbol{0} & \check{\boldsymbol{R}}_{22} \end{pmatrix} = \begin{pmatrix} \check{\boldsymbol{R}}_A \\ \check{\boldsymbol{R}}_B \end{pmatrix}. \tag{4.4}$$

Here $\check{\boldsymbol{Q}}$ is $\check{\boldsymbol{\Phi}}$-unitary with respect to the signature matrix $\check{\boldsymbol{\Phi}} = \text{diag}(\boldsymbol{I}_K, -\boldsymbol{I}_{M_d})$, $\boldsymbol{R}_{11}$, $\boldsymbol{R}_{12}$ are available from (2.1) and $\check{\boldsymbol{R}}_{11} \in \mathbb{R}^{K\times K}$ is the upper triangular and non singular factor from the QRD of $\check{\boldsymbol{X}}$.

Proof. *The 3SLS estimator in (4.2) is the GLS estimator of the TSEM*

$$\text{vec}\left(\check{\boldsymbol{X}}^T\check{\boldsymbol{Y}}\right) = \left(\boldsymbol{I}_G \otimes \check{\boldsymbol{X}}^T\check{\boldsymbol{W}}\right)\boldsymbol{S}\boldsymbol{\delta}^D + \text{vec}\left(\check{\boldsymbol{X}}^T\check{\boldsymbol{E}}\right), \text{vec}\left(\check{\boldsymbol{X}}^T\check{\boldsymbol{E}}\right) \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \check{\boldsymbol{X}}^T\check{\boldsymbol{X}}\right). \tag{4.5}$$

after endogeneity has been eliminated in the downdated SEM (4.1). It holds that $\boldsymbol{X}^T\boldsymbol{X} = \check{\boldsymbol{X}}^T\check{\boldsymbol{X}} + \boldsymbol{X}_d^T\boldsymbol{X}_d = \boldsymbol{R}_{11}^T\boldsymbol{R}_{11}$ and also that $\boldsymbol{X}^T\boldsymbol{Y} = \check{\boldsymbol{X}}^T\check{\boldsymbol{Y}} + \boldsymbol{X}_d^T\boldsymbol{Y}_d = \boldsymbol{R}_{11}^T\boldsymbol{R}_{12}$ where they, respectively, give $\check{\boldsymbol{X}}^T\check{\boldsymbol{X}} = \boldsymbol{R}_{11}^T\boldsymbol{R}_{11} - \boldsymbol{X}_d^T\boldsymbol{X}_d = \check{\boldsymbol{R}}_{11}^T\check{\boldsymbol{R}}_{11}$ and $\check{\boldsymbol{X}}^T\check{\boldsymbol{Y}} = \boldsymbol{R}_{11}^T\boldsymbol{R}_{12} - \boldsymbol{X}_d^T\boldsymbol{Y}_d = \check{\boldsymbol{R}}_{11}^T\check{\boldsymbol{R}}_{12}$. The above imply the HQRD in (4.4). The downdated TSEM (4.5) is now written as

$$\text{vec}\left(\check{\boldsymbol{R}}_{12}\right) = \left(\boldsymbol{I}_G \otimes \check{\boldsymbol{R}}_A\right)\boldsymbol{S}\boldsymbol{\delta}^D + \text{vec}\left(\check{\check{\boldsymbol{E}}}\right), \quad \text{vec}\left(\check{\check{\boldsymbol{E}}}\right) \sim \left(\boldsymbol{0}, \boldsymbol{\Sigma} \otimes \boldsymbol{I}_K\right). \tag{4.6}$$

Based on the updating of the SEM (see (3.14)-(3.15)) it follows that

$$\check{\boldsymbol{R}}_A^T\check{\boldsymbol{R}}_A = \begin{pmatrix} \boldsymbol{R}_A \\ \imath \boldsymbol{W}_d \\ \check{\boldsymbol{R}}_B \end{pmatrix}^H \boldsymbol{\Psi} \begin{pmatrix} \boldsymbol{R}_A \\ \imath \boldsymbol{W}_d \\ \check{\boldsymbol{R}}_B \end{pmatrix} \quad \text{and} \quad \check{\boldsymbol{R}}_A^T\check{\boldsymbol{R}}_{12} = \begin{pmatrix} \boldsymbol{R}_A \\ \imath \boldsymbol{W}_d \\ \check{\boldsymbol{R}}_B \end{pmatrix}^H \boldsymbol{\Psi} \begin{pmatrix} \boldsymbol{R}_{12} \\ \imath \boldsymbol{Y}_d \\ \check{\boldsymbol{R}}_{22} \end{pmatrix}, \tag{4.7}$$

where $\boldsymbol{\Psi} = \text{diag}\left(\boldsymbol{I}_K, -\boldsymbol{I}_{M_d}, \boldsymbol{I}_{M_d}\right)$. The latter implies the TSEM

$$
\begin{pmatrix} \mathrm{vec}\,(\boldsymbol{R}_{12}) \\ \mathrm{vec}\,(\imath \boldsymbol{Y}_d) \\ \mathrm{vec}\,\left(\check{\boldsymbol{R}}_{22}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_G \otimes \boldsymbol{R}_A \\ \boldsymbol{I}_G \otimes \imath \boldsymbol{W}_d \\ \boldsymbol{I}_G \otimes \check{\boldsymbol{R}}_B \end{pmatrix} \boldsymbol{S}\boldsymbol{\delta}^D + \begin{pmatrix} \mathrm{vec}\,\left(\bar{\boldsymbol{E}}\right) \\ \mathrm{vec}\,(\imath \boldsymbol{E}_d) \\ \mathrm{vec}\,\left(\tilde{\check{\boldsymbol{E}}}_d\right) \end{pmatrix},
$$

$$
\begin{pmatrix} \mathrm{vec}\,\left(\bar{\boldsymbol{E}}\right) \\ \mathrm{vec}\,(\imath \boldsymbol{E}_d) \\ \mathrm{vec}\,\left(\tilde{\check{\boldsymbol{E}}}_d\right) \end{pmatrix} \sim \left(\boldsymbol{0}, \begin{pmatrix} \boldsymbol{\Sigma} \otimes \boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_d} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\Sigma} \otimes \boldsymbol{I}_{M_d} \end{pmatrix} \right) \tag{4.8}
$$

or equivalently the pseudo SEM in (4.3). ■

Analogously to Theorem 1, the relationships in Theorem 2 and (4.7) show the equivalence of models (4.8) and (4.6) when the method of GLS is applied. Hence, the equivalence of the 3SLS estimators of the pseudo SEM (4.3) and the downdated SEM (4.1) is proved.

To compute efficiently the 3SLS estimator of the proposed pseudo SEM in (4.8), consider the following GLLSP

$$
\underset{\bar{\boldsymbol{V}}, \boldsymbol{V}_d, \check{\boldsymbol{V}}_d, \boldsymbol{\delta}^D}{\mathrm{argmin}} \left(\left\|\bar{\boldsymbol{V}}\right\|_F^2 - \|\boldsymbol{V}_d\|_F^2 + \left\|\check{\boldsymbol{V}}_d\right\|_F^2 \right) \text{ subject to}
$$

$$
\begin{pmatrix} \mathrm{vec}\,(\boldsymbol{R}_{12}) \\ \mathrm{vec}\,(\imath \boldsymbol{Y}_d) \\ \mathrm{vec}\,\left(\check{\boldsymbol{R}}_{22}\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_G \otimes \boldsymbol{R}_A \\ \boldsymbol{I}_G \otimes \imath \boldsymbol{W}_d \\ \boldsymbol{I}_G \otimes \check{\boldsymbol{R}}_B \end{pmatrix} \boldsymbol{S}\boldsymbol{\delta}^D + \begin{pmatrix} \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_K & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_d} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_d} \end{pmatrix} \begin{pmatrix} \mathrm{vec}\,\left(\bar{\boldsymbol{V}}\right) \\ \mathrm{vec}\,(\imath \boldsymbol{V}_d) \\ \mathrm{vec}\,\left(\check{\boldsymbol{V}}_d\right) \end{pmatrix}. \tag{4.9}
$$

As in the case of the updating of the SEM, previous computations can be efficiently used for the solution of (4.9). That is, if the orthogonal transformations from the GQRD (2.4a) - (2.4b) are applied to the first block of rows in (4.9) and the solution of (2.5) is used, then the GLLSP (4.9) is equivalent to

$$
\underset{\boldsymbol{v}_A, \boldsymbol{V}_d, \check{\boldsymbol{V}}_d, \boldsymbol{\delta}^D}{\mathrm{argmin}} \left(\|\boldsymbol{v}_A\|^2 - \|\boldsymbol{V}_d\|_F^2 + \left\|\check{\boldsymbol{V}}_d\right\|_F^2 \right) \text{ subject to}
$$

$$
\begin{pmatrix} \mathrm{vec}\,(\boldsymbol{R}_{12}) \\ \mathrm{vec}\,(\imath \boldsymbol{Y}_d) \\ \mathrm{vec}\,\left(\check{\boldsymbol{R}}_{22}\right) \end{pmatrix} = \begin{pmatrix} \oplus_i \boldsymbol{R}_i \\ \boldsymbol{I}_G \otimes \imath \boldsymbol{W}_d \\ \boldsymbol{I}_G \otimes \check{\boldsymbol{R}}_B \end{pmatrix} \boldsymbol{S}\boldsymbol{\delta}^D + \begin{pmatrix} \boldsymbol{U}_{11} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_d} & \boldsymbol{0} \\ 0 & 0 & \hat{\boldsymbol{C}} \otimes \boldsymbol{I}_{M_d} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_A \\ \mathrm{vec}\,(\imath \boldsymbol{V}_d) \\ \mathrm{vec}\,\left(\check{\boldsymbol{V}}_d\right) \end{pmatrix}.
$$

The solution of the latter GLLSP is analogous to that of (3.19).

5. Numerical trials on a macroeconomic model

The effectiveness and practicability of the proposed method in estimating large-scale models is illustrated. A series of experiments has been conducted for the recursive estimation of the US macroeconomic model developed by [9, 10]. Herein, the specification of the most recent version of the US, the US+ and the Japan models is considered with 25, 116 and 10 equations, respectively[‡]. The method of 3SLS is employed for the recursive estimation of these macroeconometric models. The variables used are quarterly. For the purposes of investigating the effectiveness of the new methods, synthetic data is used. It is assumed that there are available data spanning the period 1952:Q1 to 2015:Q4 for all three models, resulting in 256 observations.

Two methods have been considered, herein referred to as *afresh* (see Section 2) and *recursive* (see Section 3.1). When new data arrive, the *afresh* method re-estimates the SEM using the full data set whereas the proposed *recursive* method estimates the model using previous estimates and the current data only. It is important to note that the *afresh* method is less computationally costly than the standard 3SLS method which requires the inversion of the large covariance matrix of the SEM [1, 20]. Figure 1 demonstrates the computational advantage of the *recursive* method when compared with the *afresh* method. Firstly, it is assumed that the Japan, the US and the US+ models have been estimated for the period 1952:Q1 up to 1994:Q4, giving 172 observations. Then as new data arrive, the estimates of the models are updated to incorporate the new available information. The times required by the two methods to update the model with the new data once they become available, starting from 1995:Q1 up to the last available observation in 2015:Q4, are compared in order to give the efficiency ratio shown in Figure 1. Moreover, leave-one-out experiments within the context of cross-validation analysis and for the identification of influential observations have been conducted. The conclusions reached are the same with those drawn from Figure 1.

The experimental results confirm that the new method for the recursive estimation of the SEM outperforms other methods that estimate the model afresh. The results shown vary in the number of structural equations (G) and the number of exogenous variables (K). It is shown that the efficiency of the proposed method is more significant when G and K increase. Further investigation demonstrates that the efficiency of the recursive method becomes more important as the number of observations in the model (M) increases. Therefore, the practicability of the new method arises when estimating multivariate models in high dimensions and when analysing big data sets. The

[‡]The latest version, as of writing this manuscript, is found in http://fairmodel.econ.yale.edu/mmm2.htm. The specification of the US+ model is the one in its original form [9].

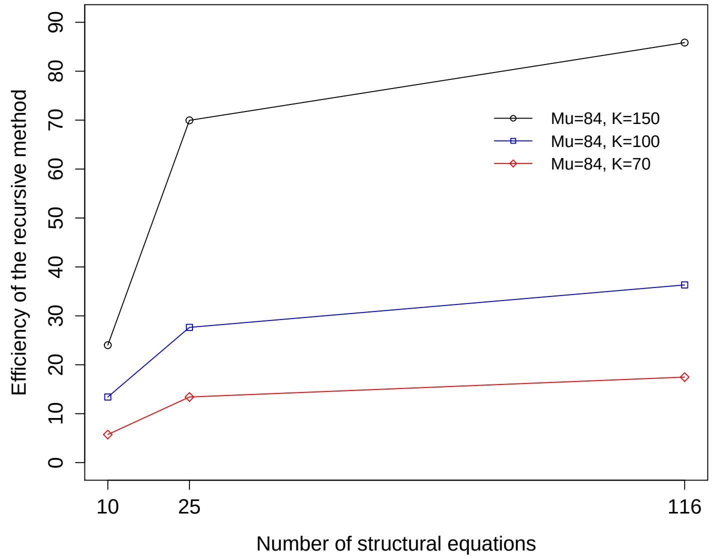

Figure 1: Effectiveness (execution times ratio) of the *afresh* and *recursive* methods for the sequential estimation of large-scale macroeconometric models which consist of 10, 25 and 116 stochastic equations. Results are presented when the number of exogenous variables included in the models is $K = 70, 100, 150$. It is assumed that the models are initially estimated using quarterly data for the period 1952:Q1 to 1994:Q4. Then the estimates are updated recurrently once new observations arrive. The last available data point is assumed to be 2015:Q4 so that a total of 84 new observations are included in the model sequentially.

strategies employed for the efficient execution of the new method and further experimental results are presented in the Appendix.

6. Conclusions

The aim has been to investigate thoroughly the recursive computation of the three-stage least squares (3SLS) estimator of the simultaneous equations model (SEM) using matrix factorizations. A novel method of updating the 3SLS estimator, when new observations are obtained, has been developed. The numerical solution derived an alternative SEM where the original SEM is updated with the extra observations and also with the factors that are required to purge the model of the endogeneity effects of the additional observations. The result is a pseudo transformed SEM which is free of endogeneity and can be estimated efficiently via the method of generalized least squares (GLS). The GLS estimator, of the pseudo transformed SEM, yields the 3SLS estimator that would be obtained if the original SEM was estimated afresh with all of the available data.

Within the context of developing numerically stable and computationally efficient algorithms, the new method derives the updated 3SLS estimator by considering the proposed pseudo model as a generalized linear least squares problem. In updating the 3SLS estimates, orthogonal, hyperbolic and unitary transformations are employed. This method not only solves the problem of recursively estimating the SEM when new data become available, it also enables an algorithm to be developed that can handle big data sets. The method has also been extended to allow observations to be deleted. In addition, an iterative algorithm has been developed that uses the 3SLS residuals in improving the initial estimates of the parameters. The proposed method can derive the 3SLS estimator even when the dispersion matrix is singular.

The designed algorithms have been implemented based on computationally efficient strategies that take advantage of the sparse structure of the SEM. Due to the structure of the SEM and the proposed pseudo SEM, the computational experiments show that the proposed algorithms are more efficient when the number of observations added to the model is smaller than the number of exogenous variables.

Acknowledgements

The authors are grateful to the Editor, the referee amd D.S.G. Pollock for their valuable comments and suggestions.

References

[1] David A. Belsley. Paring 3SLS calculations down to manageable proportions. *Computer Science in Economics and Management*, 5(3):157–169, 1992.

[2] A. W. Bojanczyk, R. P. Brent, P. van Dooren, and F. R. de Hoog. A note on downdating the Cholesky factorization. *SIAM Journal on Scientific and Statistical Computing*, 8(3):210–221, 1987.

[3] A. W. Bojanczyk and A. O. Steinhardt. Stabilized hyperbolic Householder transformations. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 37(8):1286–1288, 1989.

[4] Adam Bojanczyk, Nicholas J. Higham, and Harikrishna Patel. Solving the indefinite least squares problem by hyperbolic QR factorization. *SIAM Journal on Matrix Analysis and Applications*, 24(4):914–931, 2003.

[5] Adam Bojanczyk, Sanzheng Qiao, and Allan O. Steinhardt. Unifying unitary and hyperbolic transformations. *Linear Algebra and its Applications*, 316(13):183 – 197, 2000.

[6] Jean-Paul Chavas. Recursive estimation of simultaneous equation models. *Journal of Econometrics*, 18(2):207 – 217, 1982.

[7] R. Dennis Cook. Detection of influential observation in linear regression. *Technometrics*, 19(1):15–18, 1977.

[8] Warren Dent. Information and computation in simultaneous equations estimation. *Journal of Econometrics*, 4(1):89 – 95, 1976.

[9] Ray C. Fair. *Testing Macroeconometric Models*. Harvard University Press, Cambridge MA, 1994.

[10] Ray C. Fair. *Estimating How the Macroeconomy Works*. Harvard University Press, Cambridge MA, 2004.

[11] Stella Hadjiantoni and Erricos J. Kontoghiorghes. Estimating large-scale general linear and seemingly unrelated regressions models after deleting observations. *Statistics and Computing*, forthcoming, 2016.

[12] Andrew C. Harvey and Garry D. A. Phillips. *Statistical Analysis and Forecasting of Economic Structural Change*, chapter Testing for Structural Change in Simultaneous Equation Models, pages 25–36. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.

[13] John Haslett. A simple derivation of deletion diagnostic results for the general linear model with correlated errors. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 61(3):603–609, 1999.

[14] Stephen Haslett. Linear Algebra and Statistics: In Celebration of C. R. Rao's 75th Birthday (September 10, 1995) Updating linear models with dependent errors to include additional data and/or parameters. *Linear Algebra and its Applications*, 237:329 – 349, 1996.

[15] Jerry A. Hausman. Chapter 7 specification and estimation of simultaneous equation models. In Zvi Griliches and Michael D. Intriligator, editors, *Handbook of Econometrics*, volume 1 of *Handbook of Econometrics*, pages 391 – 448. Elsevier, 1983.

[16] S. R. Jammalamadaka and D. Sengupta. Inclusion and exclusion of data or parameters in the general linear model. *Statistics & Probability Letters*, 77(12):1235 – 1247, 2007.

[17] S.R. Jammalamadaka and D. Sengupta. Changes in the general linear model: A unified approach. *Linear Algebra and its Applications*, 289(1):225 – 242, 1999.

[18] L.S. Jennings. Simultaneous equations estimation: Computational aspects. *Journal of Econometrics*, 12(1):23 – 39, 1980.

[19] G. G. Judge, W. E. Griffiths, C. H. Hill, H. Lütkepohl, and T.-C. Lee. *The Theory and Practice of Econometrics, 2nd Edition*. Wiley, New York, 1985.

[20] Erricos J. Kontoghiorghes and Elias Dinenis. Computing 3SLS Solutions of simultaneous equation models with a possible singular variance-covariance matrix. *Computational Economics*, 10(3):231–250, August 1997.

[21] S. Kourouklis and C. C. Paige. A constrained least squares approach to the general Gauss-Markov linear model. *Journal of the American Statistical Association*, 76(375):620–625, 1981.

[22] Hanyu Li, Hu Yang, and Hua Shao. Two-sided generalized hyperbolic QR factorization and its perturbation analysis. *Linear Algebra and its Applications*, 438(3):1267 – 1292, 2013.

[23] Qiaohua Liu. Modified GramSchmidt-based methods for block downdating the Cholesky factorization. *Journal of Computational and Applied Mathematics*, 235(8):1897 – 1905, 2011.

[24] Serge J. Olszanskyj, James M. Lebak, and Adam W. Bojanczyk. Rank-k modification methods for recursive least squares problems. *Numerical Algorithms*, 7(2):325–354, 1994.

[25] C. C. Paige. Fast numerically stable computations for generalized linear least squares problems. *SIAM Journal on Numerical Analysis*, 16(1):165–171, 1979.

[26] R. L. Plackett. Some theorems in least squares. *Biometrika*, 37(1-2):149–157, 1950.

[27] D.S.G. Pollock. Recursive estimation in econometrics. *Computational Statistics & Data Analysis*, 44(12):37 – 75, 2003.

[28] C. Rader and A. Steinhardt. Hyperbolic Householder transformations. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 34(6):1589 – 1602, 12 1986.

[29] Craig Riddell. *Recursive Estimation Algorithms for Economic Research*, pages 397–406. NBER, July 1975.

[30] Barbara Rossi and Atsushi Inoue. Out-of-sample forecast tests robust to the choice of window size. *Journal of Business & Economic Statistics*, 30(3):432–453, 2012.

[31] Michael Stewart and G. W. Stewart. On hyperbolic triangularization: Stability and pivoting. *SIAM Journal on Matrix Analysis and Applications*, 19(4):847–860, 1998.

[32] Darrell A. Turkington. *Matrix Calculus and Zero-One Matrices: Statistical and Econometric Applications*. Cambridge University Press, 2002.

[33] Peter C. Young. *Recursive Estimation and Time-Series Analysis*. Springer-Verlag, Berlin Heidelberg, 2nd edition, 2011.

[34] Arnold Zellner and H. Theil. Three-stage least squares: Simultaneous estimation of simultaneous equations. *Econometrica*, 30(1):pp. 54–78, 1962.

[35] Junlong Zhao, Chenlei Leng, Lexin Li, and Hansheng Wang. High-dimensional influence measure. *Ann. Statist.*, 41(5):2639–2667, 10 2013.