Rational solutions of the Boussinesq equation and applications to rogue waves

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Abstract

We study rational solutions of the Boussinesq equation, which is a soliton equation solvable by the inverse scattering method. These rational solutions, which are algebraically decaying and depend on two arbitrary parameters, are expressed in terms of special polynomials that are derived through a bilinear equation, have a similar appearance to rogue-wave solutions of the focusing nonlinear Schrödinger (NLS) equation. Further the rational solutions have an interesting structure as they are comprised of a linear combination of four independent solutions of the bilinear equation. Rational solutions of the Kadomtsev-Petviashvili I (KPI) equation are derived in two ways, from rational solutions of the NLS equation and from rational solutions of the Boussinesq equation. It is shown that these two families of rational solutions of the KPI equation are fundamentally different and a unifying framework is found which incorporates both families of solutions.

1 Introduction

"Rogue waves", sometimes knows as "freak waves" or "monster waves", are waves appearing as extremely large, localized waves in the ocean which have been of considerable interest recently, cf. [40, 77, 88, 93]. The average height of rogue waves is at least twice the height of the surrounding waves, are very unpredictable and so they can be quite unexpected and mysterious. A feature of rogue waves is that they "come from nowhere and disappear with no trace" [13, 14]. In recent years, the concept of rogue waves has been extended beyond oceanic waves: to pulses emerging from optical fibres [38, 39, 76, 101]; waves in Bose-Einstein condensates [23]; in superfluids [63]; in optical cavities [82], in the atmosphere [102]; and in finance [114, 115]; for a comprehensive review of the different physical contexts rogue waves arise see [87]. The most commonly used mathematical model for rogue waves involves rational solutions of the focusing nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}|\psi|^2\psi = 0, \tag{1.1}$$

where subscripts denote partial derivatives, with ψ the wave envelope, t the temporal variable and x the spatial variable in the frame moving with the wave, see §2.

In this paper we are concerned with rational solutions of the Boussinesq equation

$$u_{tt} + u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0, \tag{1.2}$$

which are algebraically decaying and have a similar appearance to rogue-wave solutions of the NLS equation (1.1). Equation (1.2) was introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water [24, 25]; see, also [109, 112]. The Boussinesq equation (1.2) is also a soliton equation solvable by inverse scattering [4, 5, 8, 34, 118] which arises in several other physical applications including one-dimensional nonlinear lattice-waves [106, 116]; vibrations in a nonlinear string [118]; and ion sound waves in a plasma [70, 98]. We remark that equation (1.2) is sometimes referred to as the "bad" Boussinesq equation, i.e. when the ratio of the u_{tt} and u_{xxxx} terms is negative. If the sign of the u_{xxxx} term is reversed in (1.2), then the equation is sometimes called the "good" Boussinesq equation. The coefficients of the u_{xx} and $(u^2)_{xx}$ terms can be changed by scaling and translation of the dependent variable u. For example, letting $u \to u + 1$ in (1.2) gives

$$u_{tt} - u_{xx} - (u^2)_{xx} - \tfrac{1}{3}u_{xxxx} = 0, \tag{1.3}$$

which is the non-dimensionalised form of the equation originally written down by Boussinesq [24, 25].

There has been considerable interest in partial differential equations solvable by inverse scattering, the *soliton equations*, since the discovery in 1967 by Gardner, Greene, Kruskal and Miura [64] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.4}$$

During the past forty years or so there has been much interest in rational solutions of the soliton equations. For some soliton equations, solitons are given by rational solutions, e.g. for the Benjamin-Ono equation [81, 97] Further applications of rational solutions to soliton equations include: in the description of vortex dynamics [17–19]; vortex solutions of the complex sine-Gordon equation [20, 86]; and in the transition behaviour for the semi-classical sine-Gordon equation [26].

In §2, we discuss rational solutions of the focusing NLS equation (1.1), including some generalised rational solutions which involve two arbitrary parameters. In §3, we discuss rational solutions of the Boussinesq equation (1.2), also including some generalised rational solutions which involve two arbitrary parameters. Further the generalised rational solutions have an interesting structure as they are comprised of a linear combination of four independent solutions of an associated bilinear equation. In §4, we discuss rational solutions of the Kadomtsev-Petviashvili I (KPI) equation

$$(v_\tau + 6vv_\xi + v_{\xi\xi\xi})_\xi = 3v_{\eta\eta}, \tag{1.5}$$

which are derived in two ways, first from rational solutions of the focusing NLS equation (1.1) and second from rational solutions of the Boussinesq equation (1.2). In the simplest nontrivial case, it is shown that these two types of rational solutions are different. Further we derive a more general rational solution which has those related to the focusing NLS and Boussinesq equations as special cases and so provides a unifying framework. In §5 we discuss our results.

2 Rational solutions of the focusing nonlinear Schrödinger equation

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\psi_t + \psi_{xx} + \tfrac{1}{2}\sigma|\psi|^2\psi = 0, \qquad \sigma = \pm 1, \tag{2.1}$$

is one of the most important nonlinear partial differential equations. In 1972, Zakharov and Shabat [119] developed the inverse scattering method of solution for it. Prior to the discovery that the NLS equation (2.1) was solvable by inverse scattering, it had been considered by researchers in water waves [21, 22, 117] (see also [1, 7, 8]). In 1973, Hasegawa and Tappert [67, 68] discussed the relevance of the NLS equation (2.1) in optical fibres and their associated solitary wave solutions. Hasegawa and Tappert showed that optical fibres could sustain envelope solitons – both bright and dark solitons. Bright solitons, which decay as $|x| \to \infty$, arise with anomalous (positive) dispersion for (2.1) with $\sigma = 1$, the *focusing NLS equation*. Dark solitons, which do not decay as $|x| \to \infty$, arise with normal (negative) dispersion for (2.1) with $\sigma = -1$, the *de-focusing NLS equation*.

Rational solutions of the focusing NLS equation (1.1) have the general form

$$\psi_n(x,t) = \left\{1 - 4\frac{G_n(x,t) + \mathrm{i}tH_n(x,t)}{D_n(x,t)}\right\}\exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{2.2}$$

where $G_n(x,t)$ and $H_n(x,t)$ are polynomials of degree $\frac{1}{2}(n+2)(n-1)$ in both x^2 and t^2, with total degree $\frac{1}{2}(n+2)(n-1)$, and $D_n(x,t)$ is a polynomial of degree $\frac{1}{2}n(n+1)$ in both x^2 and t^2, with total degree $\frac{1}{2}n(n+1)$ and has no real zeros. The polynomials $D_n(x,t)$, $G_n(x,t)$ and $H_n(x,t)$ satisfy the Hirota equations

$$\begin{aligned}
&4(t\mathrm{D}_t + 1)H_n \bullet D_n + \mathrm{D}_x^2 D_n \bullet D_n - 4\mathrm{D}_x^2 D_n \bullet G_n = 0,\\
&\mathrm{D}_t G_n \bullet D_n + t\mathrm{D}_x^2 H_n \bullet D_n = 0,\\
&\mathrm{D}_x^2 D_n \bullet D_n = 8G_n^2 + 8t^2H_n^2 - 4D_nG_n,
\end{aligned}$$

with D_x and D_t the Hirota operators

$$\mathrm{D}_x^\ell\,\mathrm{D}_t^m F(x,t)\bullet F(x,t) = \left[\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^\ell \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m F(x,t)F(x',t')\right]_{x'=x,t'=t}. \tag{2.3}$$

The first two rational solutions of the focusing NLS equation (1.1) have the form [12, 15]

$$\psi_1(x,t) = \left\{1 - \frac{4(1+\mathrm{i}t)}{x^2+t^2+1}\right\} \exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{2.4}$$

$$\psi_2(x,t) = \left\{1 - 12\,\frac{G_2(x,t)+\mathrm{i}tH_2(x,t)}{D_2(x,t)}\right\} \exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{2.5}$$

where

$$G_2(x,t) = x^4 + 6(t^2+1)x^2 + 5t^4 + 18t^2 - 3, \tag{2.6a}$$

$$H_2(x,t) = x^4 + 2(t^2-3)x^2 + (t^2+5)(t^2-3), \tag{2.6b}$$

$$D_2(x,t) = x^6 + 3(t^2+1)x^4 + 3(t^2-3)^2x^2 + t^6 + 27t^4 + 99t^2 + 9, \tag{2.6c}$$

The solution $\psi_1(x,t)$ given by (2.4) is known as the "Peregrine solution" [94]. Further

$$|\psi_n(x,t)|^2 = 1 + 4\frac{\partial^2}{\partial x^2}\ln D_n(x,t).$$

Dubard *et al.* [35] show that the rational solutions of the focusing NLS equation (1.1) can be generalised to include some arbitrary parameters. The first of these generalized solutions has the form

$$\widehat{\psi}_2(x,t;\alpha,\beta) = \left\{1 - 12\frac{\widehat{G}_2(x,t;\alpha,\beta)+\mathrm{i}\widehat{H}_2(x,t;\alpha,\beta)}{\widehat{D}_2(x,t;\alpha,\beta)}\right\} \exp\left(\tfrac{1}{2}\mathrm{i}t\right), \tag{2.7}$$

where

$$\widehat{G}_2(x,t;\alpha,\beta) = G_2(x,t) - 2\alpha t + 2\beta x, \tag{2.8a}$$

$$\widehat{H}_2(x,t;\alpha,\beta) = tH_2(x,t) + \alpha(x^2-t^2+1) + 2\beta xt, \tag{2.8b}$$

$$\widehat{D}_2(x,t;\alpha,\beta) = D_2(x,t) + 2\alpha t(3x^2-t^2-9) - 2\beta x(x^2-3t^2-3) + \alpha^2 + \beta^2, \tag{2.8c}$$

with α and β arbitrary constants, see also [36, 37, 73–75]. These generalized solutions have now been expressed in terms of Wronskians, see Gaillard [42–56, 59–61], Guo, Ling and Liu [66], Ohta and Yang [85]. We note that the polynomial $\widehat{D}_2(x,t;\alpha,\beta)$ has the form

$$\widehat{D}_2(x,t;\alpha,\beta) = D_2(x,t) + 2\alpha t P_1(x,t) + 2\beta x Q_1(x,t) + \alpha^2 + \beta^2, \tag{2.9}$$

where $P_1(x,t)$ and $Q_1(x,t)$ are linear functions of x^2 and t^2. In Figure 2.1, plots of the generalised rational solution $|\widehat{\psi}_2(x,t;\alpha,\beta)|$ given by (2.7) of the focusing NLS equation for various values of the parameters α and β. The solution has a single peak when $\alpha=\beta=0$, which splits into three peaks as $|\alpha|$ and $|\beta|$ increase; this solution is called a "rogue wave triplet" in [16, 73] and the "three sisters" in [42].

3 The Boussinesq equation

3.1 Introduction

Clarkson and Kruskal [32] showed that Boussinesq equation (1.2) has symmetry reductions to the first, second and fourth Painlevé equations ($\mathrm{P_I}$, $\mathrm{P_{II}}$, $\mathrm{P_{IV}}$)

$$w'' = 6w^2 + z, \tag{3.1}$$

$$w'' = 2w^3 + zw + \alpha, \tag{3.2}$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}, \tag{3.3}$$

with $' = \mathrm{d}\mathrm{d}z$, and α, β arbitrary constants. Vorob'ev [111] and Yablonskii [113] expressed the rational solutions of $\mathrm{P_{II}}$ (3.2) in terms of polynomials, now known as the *Yablonskii–Vorob'ev polynomials* (see also [33]). Okamoto [84] derived analogous polynomials, now known as the *Okamoto polynomials*, related to some of the rational solutions of $\mathrm{P_{IV}}$ (3.3). Subsequently Okamoto's results were generalized by Noumi and Yamada [83] who showed that all

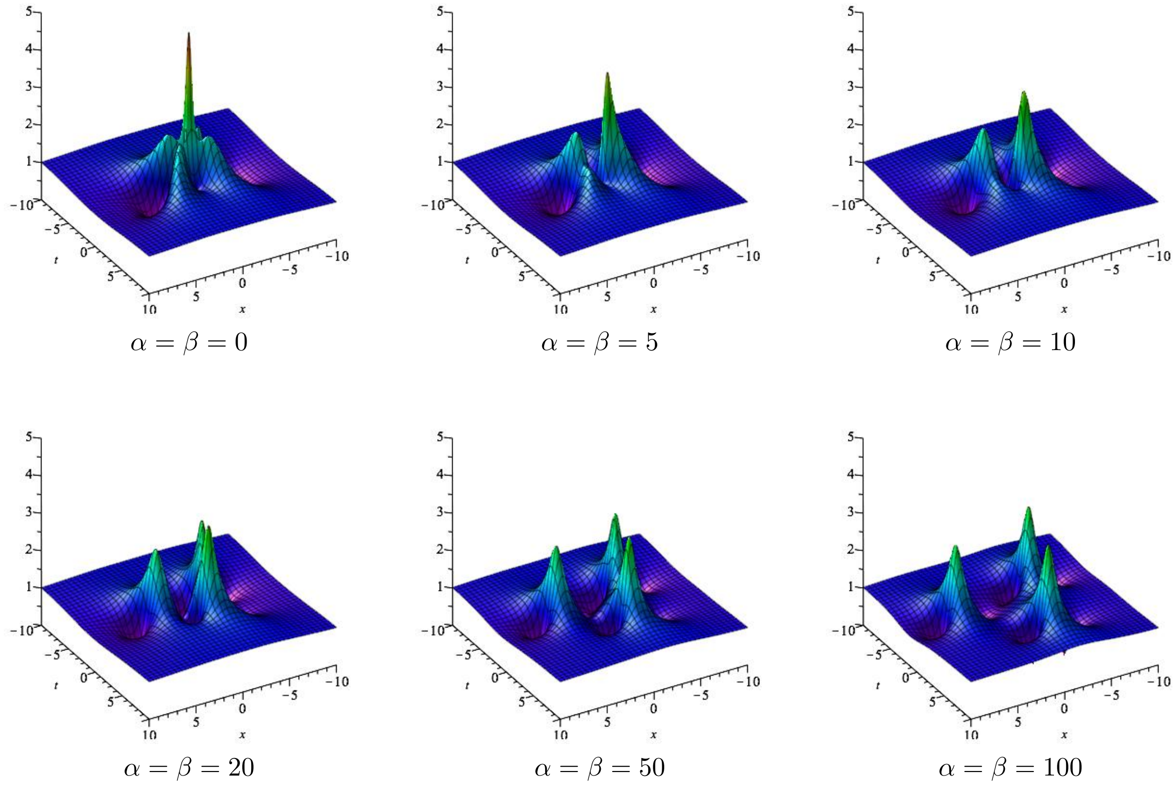


Figure 2.1: Plots of the generalised rational solution $|\widehat{\psi}_2(x, t; \alpha, \beta)|$ given by (2.7) of the focusing NLS equation for various values of the parameters α and β.

rational solutions of P_{IV} can be expressed in terms of logarithmic derivatives of two sets of special polynomials, called the *generalized Hermite polynomials* and the *generalized Okamoto polynomials* (see also [28]). Consequently rational solutions of (1.2) can be obtained in terms the Yablonskii–Vorob'ev, generalized Hermite and generalized Okamoto polynomials, cf. [31]. Some of the rational solutions that are expressed in terms of the generalized Okamoto polynomials are generalized to give the rational solutions of the Boussinesq equation (1.2) obtained in [31, 62, 90], which are analogs of the rational solutions of the KdV equation (1.4) [6, 10, 11, 27]. However none of these rational solutions of the Boussinesq equation (1.2) are bounded for all real x and t, so are unlikely to have any physical significance.

It is known that there are additional rational solutions of the Boussinesq equation (1.2) which don't arise from the above construction. For example, Ablowitz and Satsuma [6] derived the rational solution

$$u(x, t) = 2\frac{\partial^2}{\partial x^2} \ln(1 + x^2 + t^2) = \frac{4(1 - x^2 + t^2)}{(1 + x^2 + t^2)^2}, \tag{3.4}$$

by taking a long-wave limit of the two-soliton solution, see also [104, 105]. This solution is bounded for real x and t, and tends to zero algebraically as $|x|, |t| \to \infty$.

If in the Boussinesq equation (1.2), we make the transformation

$$u(x, t) = 2\frac{\partial^2}{\partial x^2} \ln F(x, t), \tag{3.5}$$

then we obtain the bilinear equation

$$FF_{tt} - F_t^2 + FF_{xx} - F_x^2 - \tfrac{1}{3}\left(FF_{xxxx} - 4F_xF_{xxx} + 3F_{xx}^2\right) = 0, \tag{3.6}$$

first derived by Hirota [69], which can be written in the form

$$(\mathrm{D}_t^2 + \mathrm{D}_x^2 - \tfrac{1}{3}\mathrm{D}_x^4)F \bullet F = 0, \tag{3.7}$$

where D_x and D_t are Hirota operators (2.3).

3.2 Rational solutions of the Boussinesq equation

Since the Boussinesq equation (1.2) has the rational solution (3.4) then we seek solutions in the form

$$u_n(x,t) = 2\frac{\partial^2}{\partial x^2}\ln F_n(x,t), \qquad n \geq 1, \tag{3.8}$$

where $F_n(x,t)$ is a polynomial of degree $\frac{1}{2}n(n+1)$ in x^2 and t^2, with total degree $\frac{1}{2}n(n+1)$, of the form

$$F_n(x,t) = \sum_{m=0}^{n(n+1)/2}\sum_{j=0}^{m} a_{j,m}x^{2j}t^{2(m-j)}, \tag{3.9}$$

with $a_{j,m}$ constants which are determined by equating powers of x and t. Using this procedure we obtain the following polynomials

$$F_1(x,t) = x^2+t^2+1, \tag{3.10a}$$

$$F_2(x,t) = x^6+\left(3t^2+\tfrac{25}{3}\right)x^4+\left(3t^4+30t^2-\tfrac{125}{9}\right)x^2+t^6+\tfrac{17}{3}t^4+\tfrac{475}{9}t^2+\tfrac{625}{9}, \tag{3.10b}$$

$$\begin{aligned} F_3(x,t) = x^{12} &+\left(6t^2+\tfrac{98}{3}\right)x^{10}+\left(15t^4+230t^2+\tfrac{245}{3}\right)x^8+\left(20t^6+\tfrac{1540}{3}t^4+\tfrac{18620}{9}t^2+\tfrac{75460}{81}\right)x^6\\ &+\left(15t^8+\tfrac{1460}{3}t^6+\tfrac{37450}{9}t^4+\tfrac{24500}{3}t^2-\tfrac{5187875}{243}\right)x^4\\ &+\left(6t^{10}+190t^8+\tfrac{35420}{9}t^6-\tfrac{4900}{9}t^4+\tfrac{188650}{27}t^2+\tfrac{159786550}{729}\right)x^2\\ &+t^{12}+\tfrac{58}{3}t^{10}+\tfrac{1445}{3}t^8+\tfrac{798980}{81}t^6+\tfrac{16391725}{243}t^4+\tfrac{300896750}{729}t^2+\tfrac{878826025}{6561}, \end{aligned} \tag{3.10c}$$

and the polynomials $F_4(x,t)$ and $F_5(x,t)$ are given in the Appendix. We note that these polynomials have the following form

$$F_n(x,t) = \left(x^2+t^2\right)^{n(n+1)/2} + G_n(x,t),$$

where $G_n(x,t)$ is a polynomial of degree $\frac{1}{2}(n+2)(n-1)$ in both x^2 and t^2. We remark that the polynomials $F_n(x,t)$ which arise in the rational solutions of the focusing NLS equation (1.1) have a similar structure, see for example (2.6c), though the coefficients in the polynomials $G_n(x,t)$ are different. The polynomials $F_j(x,t)$, for $j=2,3,4$, in scaled variables, are given by Pelinovsky and Stepanyants [91] – see their equations (6)–(8). However whilst they state that the polynomials are associated with solutions of their equation (2), which is a scaled variant of the Boussinesq equation (1.2), Pelinovsky and Stepanyants don't mention, or reference, the Boussinesq equation.

In Figure 3.1, plots of the rational solutions $u_n(x,t)$, for $n=1,2,\ldots,6$, of the Boussinesq equation. These show that the maxima of the solutions all lie on the line $t=0$, with n local maxima for the rational solution $u_n(x,t)$.

In Figure 3.2, plots of the complex roots of $F_n(x,t)$, for $3,4,5$, for $t=0$ and $t=3n$, i.e. $t=9$ for $n=3$, $t=12$ for $n=4$ and $t=15$ for $n=5$, are given. Each plot shows the complex x-plane with roots in x of $F_n(x,t)$ are shown at two different values of t. These show a "triangular" structure for both $t=0$ and $t=3n$, though with a different orientation. For $t=0$ the roots of the polynomials approximately form two isosceles triangles with curved sides. For $t=3n$ the roots of the polynomials again approximately form two isosceles triangles, though the values of the roots show that they actually also lie on curves rather than straight lines. An analogous situation arises for the Yablonskii–Vorob'ev polynomials [33] and generalized Okamoto polynomials [28].

In Figure 3.3, plots of the loci of the complex roots of $F_n(x,t)$, for $3,4,5$, as t varies, between between the "triangular" structures for $t=0$ and $t=3n$ are given. These show that as t increases the roots move away from the real axis.

In Figure 3.4, plots of the loci of the complex roots of $F_6(x,t)$ with the solution $u_6(x,t)$ superimposed, as t varies are given. The scale on the vertical axis relates to the complex x-plane for the roots of $F_6(x,t)$. These show that as the roots move away from the real axis, the solution decays to zero.

3.3 Generalised rational solutions of the Boussinesq equation

Since the focusing NLS equation (1.1) has generalised rational solutions, see (2.7), then a natural question is whether the Boussinesq equation (1.2) also has generalised rational solutions. To investigate this, we are concerned with the following theorem.

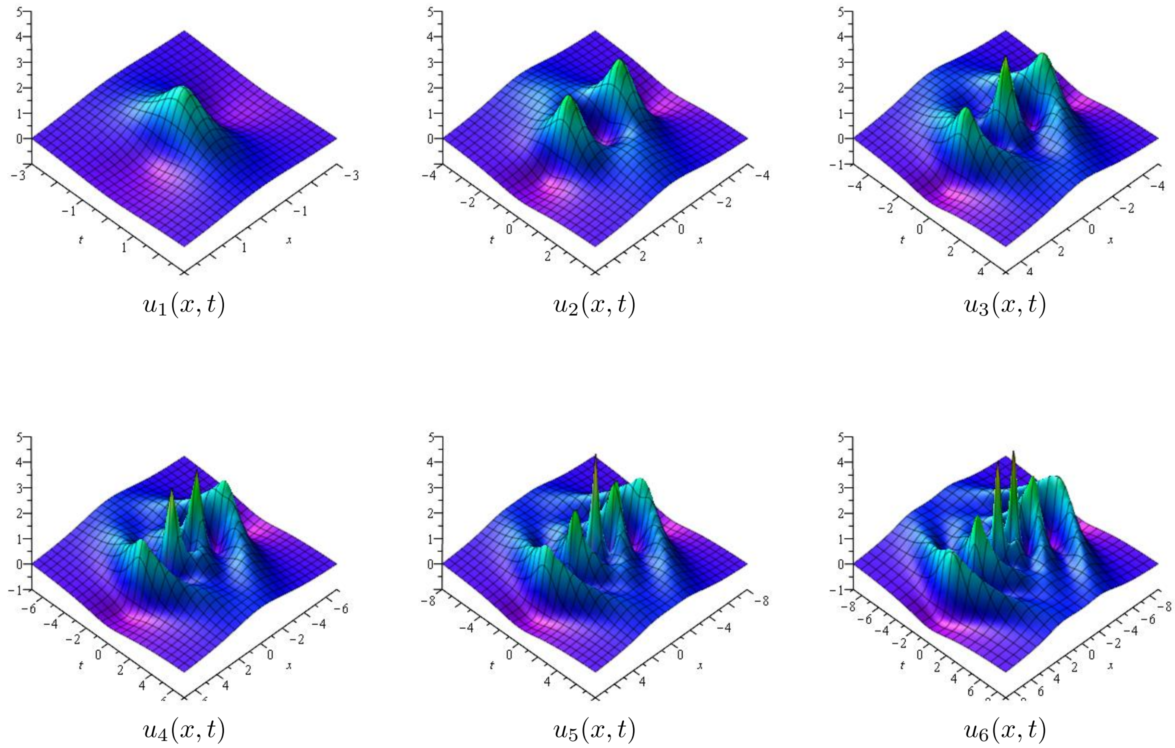


Figure 3.1: Plots of the rational solutions $u_n(x,t)$, for $n = 1, 2, \ldots, 6$, of the Boussinesq equation.

Theorem 3.1. *The Boussinesq equation* (1.2) *has generalised rational solutions in the form*

$$\widetilde{u}_n(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{F}_n(x,t;\alpha,\beta), \tag{3.11}$$

for $n \geq 1$, with

$$\widetilde{F}_{n+1}(x,t;\alpha,\beta) = F_{n+1}(x,t) + 2\alpha t P_n(x,t) + 2\beta x Q_n(x,t) + \left(\alpha^2+\beta^2\right)F_{n-1}(x,t), \tag{3.12}$$

where $F_n(x,t)$ is given by (3.10)*, $P_n(x,t)$ and $Q_n(x,t)$ are polynomials of degree $\frac{1}{2}n(n+1)$ in x^2 and t^2, and α and β are arbitrary constants.*

Since the generalised polynomial $\widehat{D}_2(x,t;\alpha,\beta)$ for the focusing NLS equation has the structure given by (2.9), we suppose that the Boussinesq equation (1.2) has a solution in the form (3.11), with $F_n(x,t)$ given by (3.10) and the polynomials $P_n(x,t)$ and $Q_n(x,t)$, which are of degree $\frac{1}{2}n(n+1)$ in x^2 and t^2, have the form

$$P_n(x,t) = \sum_{m=0}^{n(n+1)/2}\sum_{j=0}^{m} b_{j,m}x^{2j}t^{2(m-j)}, \qquad Q_n(x,t) = \sum_{m=0}^{n(n+1)/2}\sum_{j=0}^{m} c_{j,m}x^{2j}t^{2(m-j)}, \tag{3.13}$$

where the coefficients $b_{j,m}$ and $c_{j,m}$ are to be determined. Substituting (3.12) into the bilinear equation (3.6) with $F_1(x,t)$, $F_2(x,t)$, $F_3(x,t)$ and $F_4(x,t)$ given by (3.10), $P_n(x,t)$ and $Q_n(x,t)$ in the form (3.13), then by equating powers of x and t we find that

$$P_1(x,t) = 3x^2 - t^2 + \tfrac{5}{3}, \tag{3.14a}$$

$$Q_1(x,t) = x^2 - 3t^2 - \tfrac{1}{3}, \tag{3.14b}$$

$$P_2(x,t) = 5x^6 - \left(5t^2 - 35\right)x^4 - \left(9t^4 + \tfrac{190}{3}t^2 + \tfrac{665}{9}\right)x^2 + t^6 - \tfrac{7}{3}t^4 - \tfrac{245}{9}t^2 + \tfrac{18865}{81}, \tag{3.14c}$$

$$Q_2(x,t) = x^6 - \left(9t^2 - \tfrac{13}{3}\right)x^4 - \left(5t^4 + \tfrac{230}{3}t^2 + \tfrac{245}{9}\right)x^2 + 5t^6 + 15t^4 + \tfrac{535}{9}t^2 + \tfrac{12005}{81}, \tag{3.14d}$$

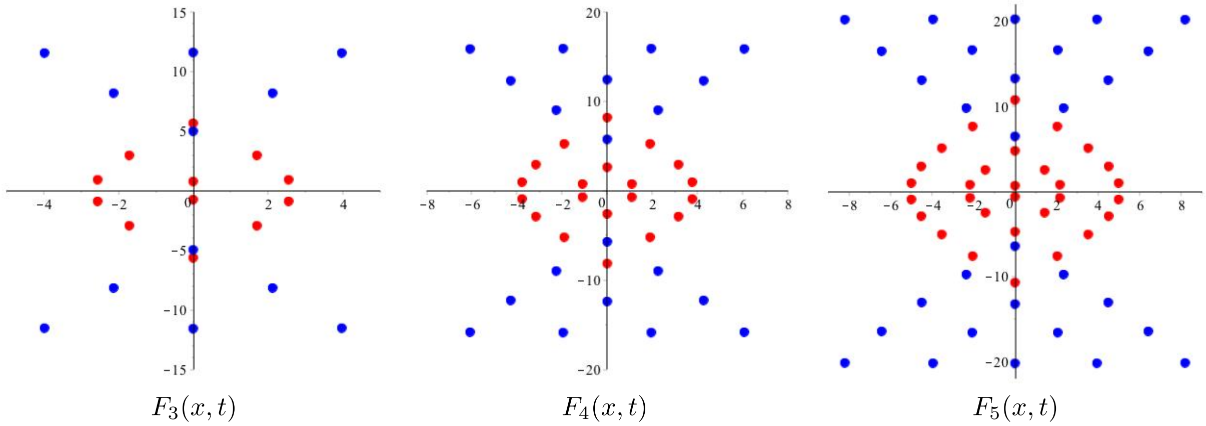


Figure 3.2: Plots of the complex roots of the polynomials $F_n(x,t)$, for $3, 4, 5$, for $t = 0$ (red) and $t = 3n$ (blue), i.e. $t = 9$ for $n = 3$, $t = 12$ for $n = 4$ and $t = 15$ for $n = 5$. Each plot shows the complex x-plane with roots in x of $F_n(x,t)$ are shown at two different values of t.

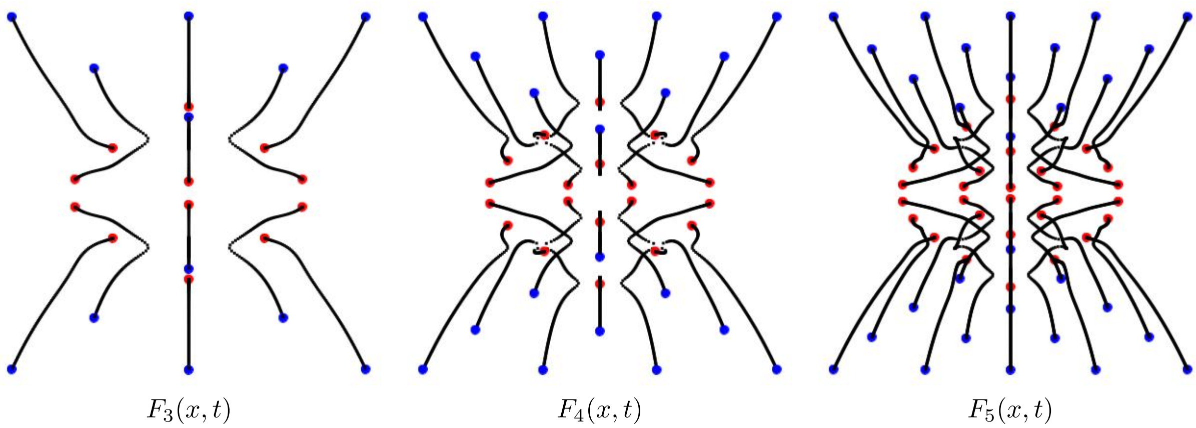


Figure 3.3: Plots of the loci of the complex roots of $F_n(x,t)$, for $3, 4, 5$, as t varies, with $t = 0$ (red) and $t = 3n$ (blue), i.e. $t = 9$ for $n = 3$, $t = 12$ for $n = 4$ and $t = 15$ for $n = 5$.

with α and β arbitrary constants; the polynomials $P_3(x,t)$, $Q_3(x,t)$, $P_4(x,t)$ and $Q_4(x,t)$ are given in the Appendix.

The first two generalised rational solutions are

$$\widetilde{u}_2(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{F}_2(x,t;\alpha,\beta), \tag{3.15}$$

$$\widetilde{u}_3(x,t;\alpha,\beta) = 2\frac{\partial^2}{\partial x^2}\ln \widetilde{F}_3(x,t;\alpha,\beta), \tag{3.16}$$

where

$$\begin{aligned}\widetilde{F}_2(x,t;\alpha,\beta) &= F_2(x,t) + 2\alpha t P_1(x,t) + 2\beta x Q_1(x,t) + \alpha^2 + \beta^2 \\ &= x^6 + \left(3t^2 + \tfrac{25}{3}\right)x^4 + \left(3t^4 + 30t^2 - \tfrac{125}{9}\right)x^2 + t^6 + \tfrac{17}{3}t^4 + \tfrac{475}{9}t^2 + \tfrac{625}{9} \\ &\qquad + 2\alpha t\left(3x^2 - t^2 + \tfrac{5}{3}\right) + 2\beta x\left(x^2 - 3t^2 - \tfrac{1}{3}\right) + \alpha^2 + \beta^2,\end{aligned} \tag{3.17}$$

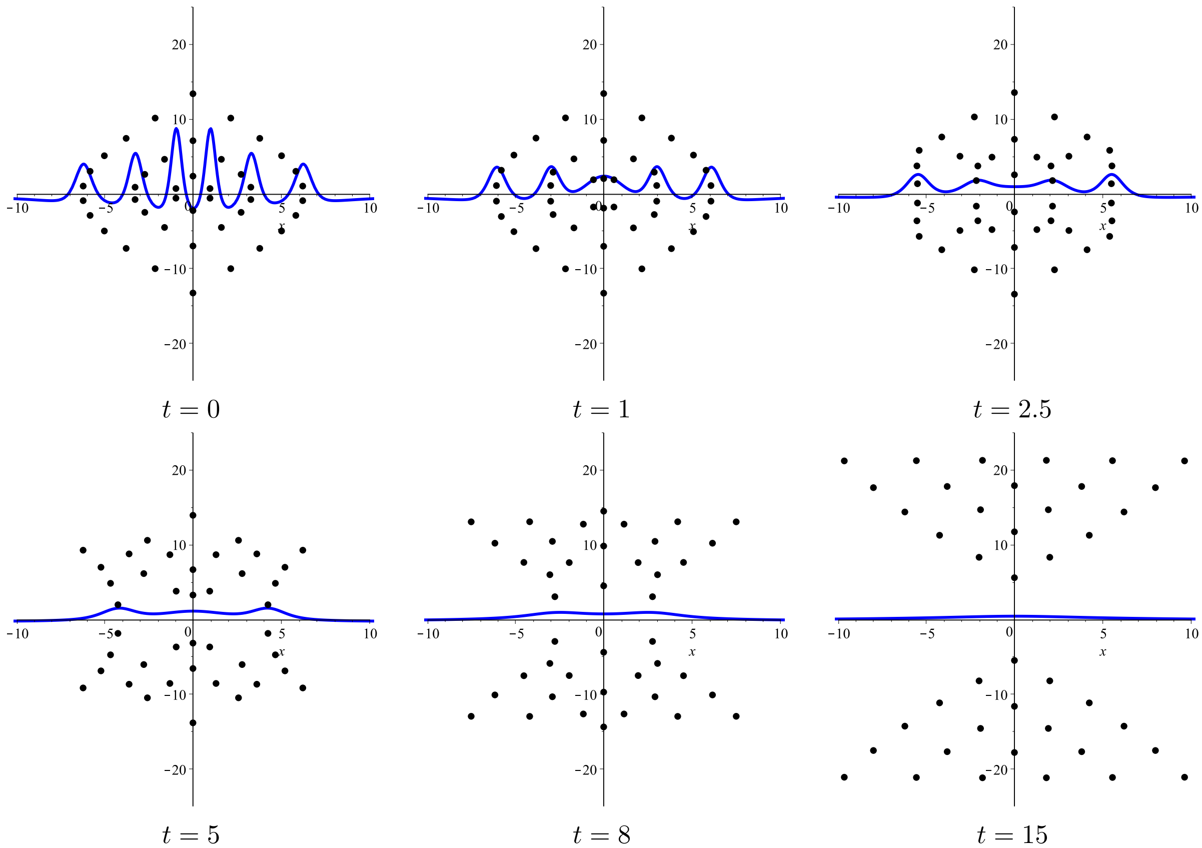


Figure 3.4: Plots of the loci of the complex roots of $F_6(x,t)$ with the solution $u_6(x,t)$ superimposed (blue), as t varies. The scale on the vertical axis relates to the complex x-plane for the roots of $F_6(x,t)$.

and

$$
\begin{aligned}
\widetilde{F}_3(x,t;\alpha,\beta) &= F_3(x,t) + 2\alpha t P_2(x,t) + 2\beta x Q_2(x,t) + (\alpha^2+\beta^2)F_1(x,t)\\
&= x^{12} + \left(6t^2 + \tfrac{98}{3}\right)x^{10} + \left(15t^4 + 230t^2 + \tfrac{245}{3}\right)x^8 + \left(20t^6 + \tfrac{1540}{3}t^4 + \tfrac{18620}{9}t^2 + \tfrac{75460}{81}\right)x^6\\
&\quad + \left(15t^8 + \tfrac{1460}{3}t^6 + \tfrac{37450}{9}t^4 + \tfrac{24500}{3}t^2 - \tfrac{5187875}{243}\right)x^4\\
&\quad + \left(6t^{10} + 190t^8 + \tfrac{35420}{9}t^6 - \tfrac{4900}{9}t^4 + \tfrac{188650}{27}t^2 + \tfrac{159786550}{729}\right)x^2\\
&\quad + t^{12} + \tfrac{58}{3}t^{10} + \tfrac{1445}{3}t^8 + \tfrac{798980}{81}t^6 + \tfrac{16391725}{243}t^4 + \tfrac{300896750}{729}t^2 + \tfrac{878826025}{6561}\\
&\quad + 2\alpha t\left\{5x^6 - \left(5t^2-35\right)x^4 - \left(9t^4 + \tfrac{190}{3}t^2 + \tfrac{665}{9}\right)x^2 + t^6 - \tfrac{7}{3}t^4 - \tfrac{245}{9}t^2 + \tfrac{18865}{81}\right\}\\
&\quad + 2\beta x\left\{x^6 - \left(9t^2 - \tfrac{13}{3}\right)x^4 - \left(5t^4 + \tfrac{230}{3}t^2 + \tfrac{245}{9}\right)x^2 + 5t^6 + 15t^4 + \tfrac{535}{9}t^2 + \tfrac{12005}{81}\right\}\\
&\quad + (\alpha^2+\beta^2)(x^2+t^2+1),
\end{aligned}
\tag{3.18}
$$

with α and β arbitrary constants. Plots of the solutions $\widetilde{u}_2(x,t;\alpha,\beta)$, $\widetilde{u}_3(x,t;\alpha,\beta)$ and $\widetilde{u}_4(x,t;\alpha,\beta)$ of the Boussinesq equation for various values of the parameters α and β are given in Figures 3.5, 3.6 and 3.7, respectively. Contour plots of the solutions $\widetilde{u}_2(x,t;10^4,10^4)$, $\widetilde{u}_3(x,t;10^7,10^7)$ and $\widetilde{u}_4(x,t;10^{10},10^{10})$ of the Boussinesq equation (1.2) illustrating this behaviour are given in Figure 3.8.

Figure 3.5 shows that the solution $\widetilde{u}_2(x,t;\alpha,\beta)$ has two peaks when $\alpha=\beta=0$, then as $|\alpha|$ and $|\beta|$ increase a third peak appears. Numerical evidence suggests that as $|\alpha|$ and $|\beta|$ increase then the three peaks all tend to the same height $\max(\widetilde{u}_2)=4$. For $|\alpha|$ and $|\beta|$ sufficiently large, then $\widetilde{u}_2(x,t;\alpha,\beta)$ has three lumps which are essentially copies of the lowest-order solution, i.e. $u_1(x,t)$, which equally spaced on a circle; an analogous situation arises for the second generalised rational solution of the NLS equation [74, 75].

Figure 3.6 shows that the solution $\widetilde{u}_3(x,t;\alpha,\beta)$ has three peaks when $\alpha=\beta=0$, then as $|\alpha|$ and $|\beta|$ increase three more peaks appear, for α and β sufficiently large with one central peak and five in a circle around it, so forming a pentagram. Again, numerical evidence suggests that as $|\alpha|$ and $|\beta|$ increase then the three peaks all tend to the same

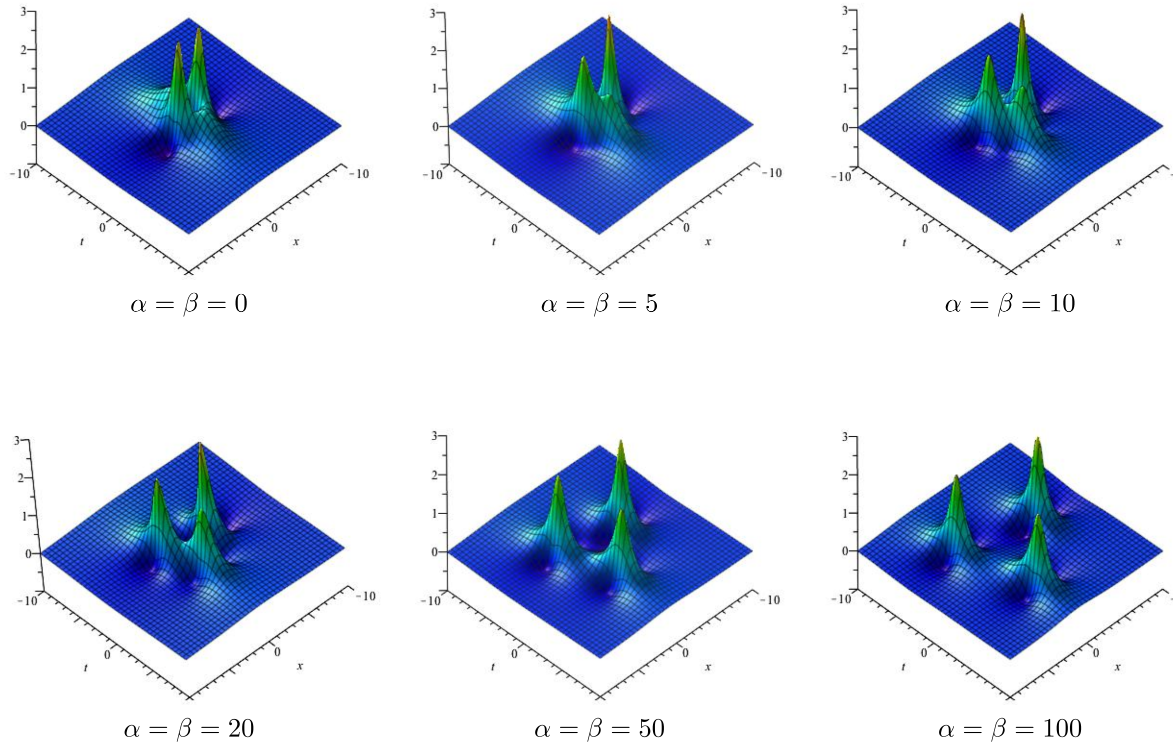


Figure 3.5: Plots of the generalised rational solution $\widetilde{u}_2(x,t;\alpha,\beta)$ of the Boussinesq equation for various values of the parameters α and β.

height $\max(\widetilde{u}_3)=4$. For α and β sufficiently large, the rational solution $\widetilde{u}_3(x,t;\alpha,\beta)$ has six lumps, again essentially copies of the lowest-order solution $u_1(x,t)$, with five equally spaced on a circle; an analogous situation arises for the third generalised rational solution of the NLS equation [73, 75].

Figure 3.7 shows that the solution $\widetilde{u}_4(x,t;\alpha,\beta)$ has four peaks when $\alpha=\beta=0$, then as $|\alpha|$ and $|\beta|$ increase five more peaks appear, with for α and β sufficiently large with two central peaks and seven in a ring around it, so forming a heptagram. As for $\widetilde{u}_2(x,t;\alpha,\beta)$ and $\widetilde{u}_3(x,t;\alpha,\beta)$, numerical evidence suggests that as $|\alpha|$ and $|\beta|$ increase then the peaks all tend to the same height $\max(\widetilde{u}_4)=4$. An analogous situation arises for the fourth generalised rational solution of the NLS equation [75].

Remark 3.2. Ohta and Yang [85, Figure 1] show that for focusing NLS equation (1.1), the generalised rational solution $\widehat{\psi}_2(x,t;\alpha,\beta)$ (2.7) has a single peak when $\alpha=\beta=0$, and three peaks otherwise. Ohta and Yang [85, Figure 2] also show that the generalised rational solution $\widehat{\psi}_3(x,t;\alpha,\beta)$ has a single peak when unperturbed, and six peaks otherwise.

Define the polynomials $\Theta_n^{\pm}(x,t)$, for $n\in\mathbb{N}$, by

$$\Theta_n^{\pm}(x,t)=xP_n(x,t)\pm \mathrm{i}tQ_n(x,t), \tag{3.19}$$

with $P_n(x,t)$ and $Q_n(x,t)$ the polynomials in the generalised rational solution (3.12). Then for $P_n(x,t)$ and $Q_n(x,t)$ given by (3.14), it is easily verified that $\Theta_n^{\pm}(x,t)$, for $n=1,2,3,4$, satisfy the bilinear equation (3.6). Hence in the general case we have the following conjecture.

Conjecture 3.3. *The polynomials $\Theta_n^{\pm}(x,t)$ given by* (3.19) *satisfy the bilinear equation* (3.6)*.*

Consequently, from this and Theorem 3.12 we have the following result.

Lemma 3.4. *Let $\Theta_n^{\pm}(x,t)$ be given by* (3.19)*, then the polynomial $\widetilde{F}_{n+1}(x,t;\alpha,\beta)$ given by* (3.12) *can be written as*

$$\widetilde{F}_{n+1}(x,t;\alpha,\beta)=F_{n+1}(x,t)+(\alpha+\mathrm{i}\beta)\Theta_n^{+}(x,t)+(\alpha-\mathrm{i}\beta)\Theta_n^{-}(x,t)+\big(\alpha^2+\beta^2\big)F_{n-1}(x,t), \tag{3.20}$$

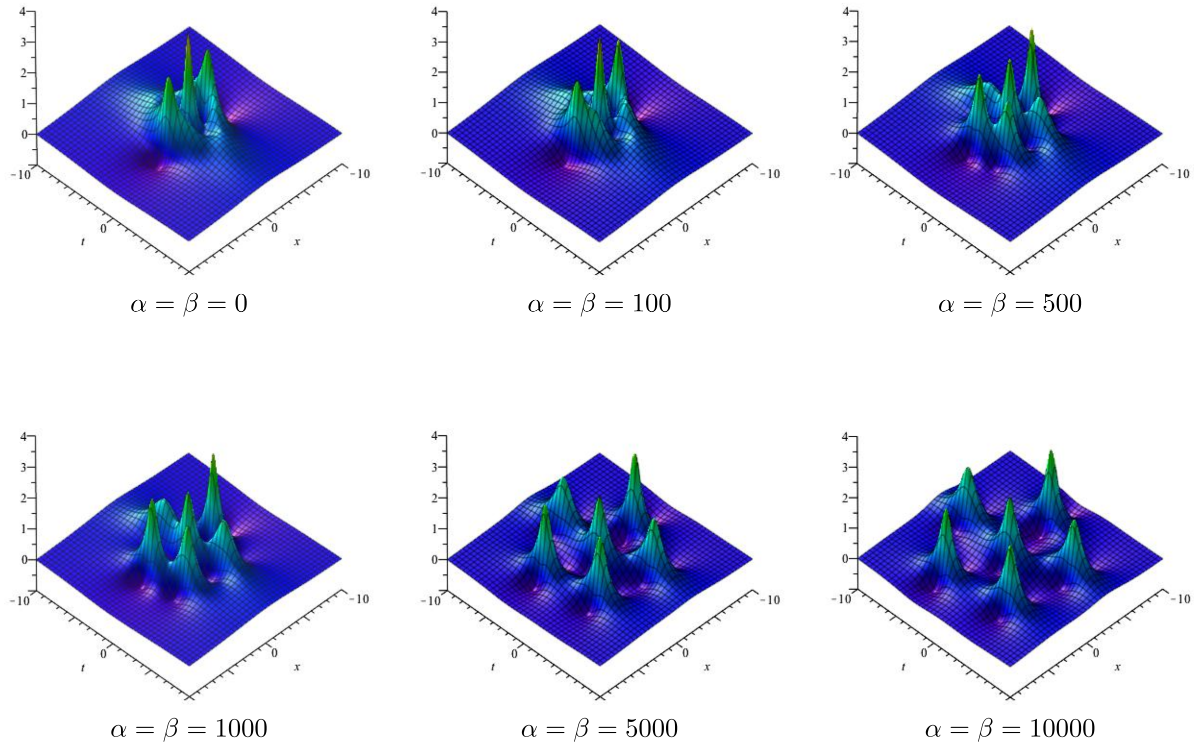


Figure 3.6: Plots of the generalised rational solution $\widetilde{u}_3(x,t;\alpha,\beta)$ of the Boussinesq equation for various values of the parameters α and β.

which is a linear combination of four solutions $F_{n+1}(x,t)$, $\Theta_n^{\pm}(x,t)$ and $F_{n-1}(x,t)$ of the bilinear equation (3.6)*.*

4 Rational solutions of the Kadomtsev-Petviashvili I equation

4.1 Introduction

The Kadomtsev-Petviashvili (KP) equation

$$(v_\tau + 6vv_\xi + v_{\xi\xi\xi})_\xi + 3\sigma^2 v_{\eta\eta} = 0, \qquad \sigma^2 = \pm 1, \tag{4.1}$$

which is known as KPI if $\sigma^2 = -1$, i.e. (1.5), and KPII if $\sigma^2 = 1$, was derived by Kadomtsev and Petviashvili [72] to model ion-acoustic waves of small amplitude propagating in plasmas and is a two-dimensional generalisation of the KdV equation (1.4). The KP equation arises in many physical applications including weakly two-dimensional long waves in shallow water [7, 99], where the sign of σ^2 depends upon the relevant magnitudes of gravity and surface tension, in nonlinear optics [92], ion-acoustic waves in plasmas [70], two-dimensional matter-wave pulses in Bose-Einstein condensates [107], and as a model for sound waves in ferromagnetic media [108]. The KP equation (4.1) is also a completely integrable soliton equation solvable by inverse scattering and again the sign of σ^2 is critical since if $\sigma^2 = -1$, then the inverse scattering problem is formulated in terms of a Riemann-Hilbert problem [41, 79], whereas for $\sigma^2 = 1$, it is formulated in terms of a $\overline{\partial}$ ("DBAR") problem [2].

The first rational solution of the KPI equation (1.5), is the so-called "lump solution"

$$v(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial\xi^2}\ln[(\xi-3\tau)^2+\eta^2+1] = -4\frac{(\xi-3\tau)^2-\eta^2-1}{[(\xi-3\tau)^2+\eta^2+1]^2}, \tag{4.2}$$

which was found by Manakov *et al.* [80]. Subsequent studies of rational solutions of the KPI equation (1.5) include Ablowitz *et al.* [3], Ablowitz and Villarroel [9, 110], Dubard and Matveev [36, 37], Gaillard [57, 58], Johnson and

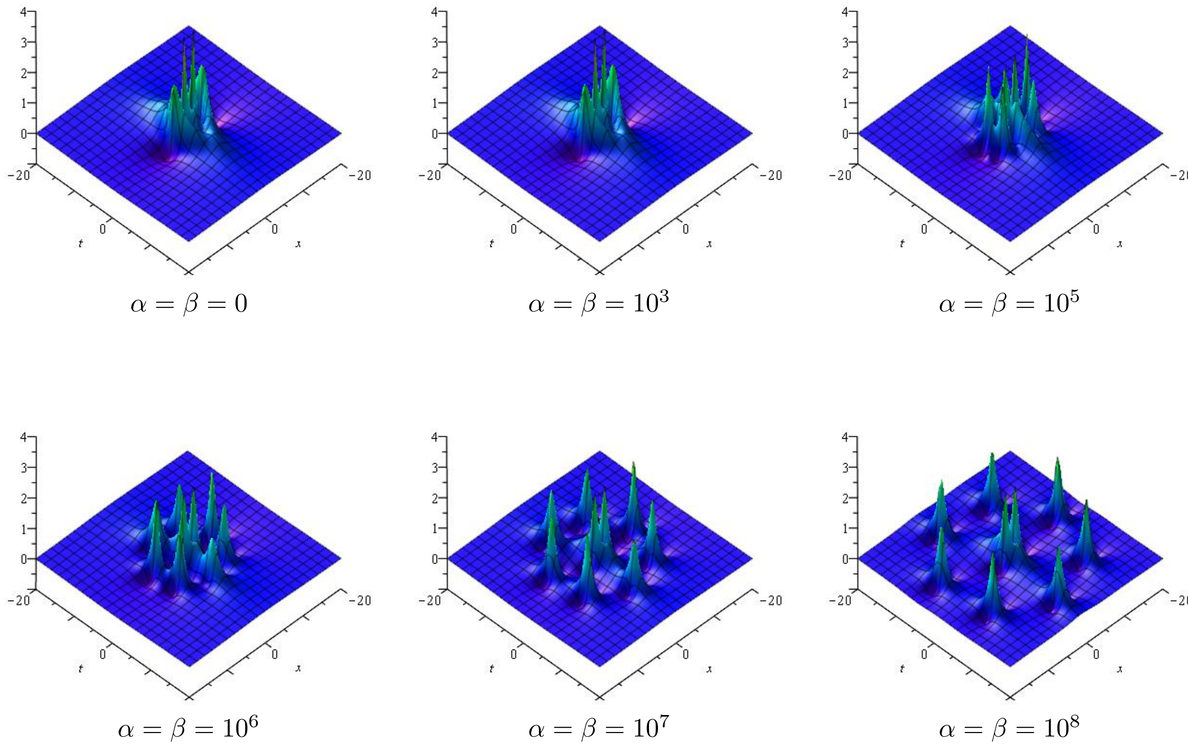


Figure 3.7: Plots of the generalised rational solution $\widetilde{u}_4(x,t;\alpha,\beta)$ of the Boussinesq equation for various values of the parameters α and β.

Thompson [71], Ma [78], Pelinovsky [89, 90], Pelinovsky and Stepanyants [91], Satsuma and Ablowitz [96], and Singh and Stepanyants [100].

We remark that the KP equation (4.1) is invariant under the *Galilean transformation*

$$(\xi,\eta,\tau,v)\mapsto(\xi+6\lambda,\eta,\tau,v+\lambda), \tag{4.3}$$

with λ an arbitrary constant. In fact the rational solutions of the KPI equation (1.5) derived by Dubard and Matveev [36, 37] and Gaillard [57, 58] are equivalent under the Galilean transformation (4.3).

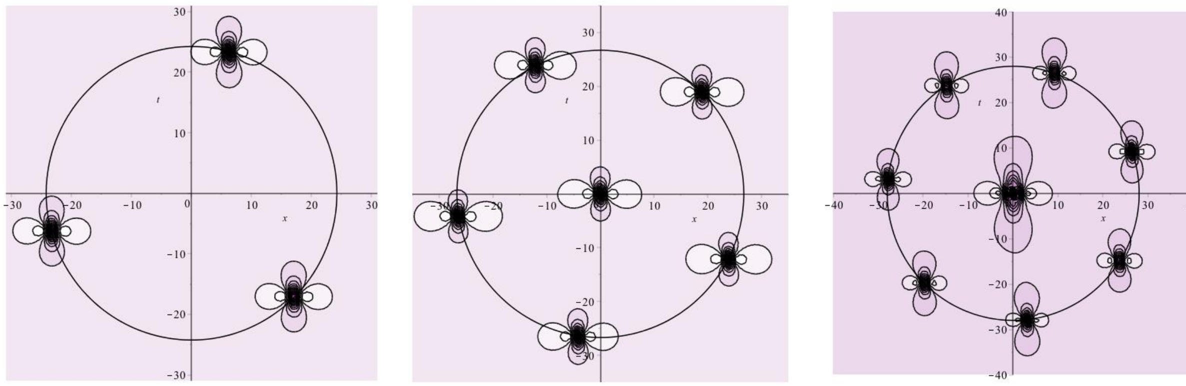
4.2 Rational solutions of KPI related to the focusing NLS equation

Dubard and Matveev [36, 37] derive rational solutions of the KPI equation (1.5) from the generalised rational solution $\widehat{\psi}_2(x,t;\alpha,\beta)$ (2.7) of the focusing NLS equation (1.1); see also [35, 57, 58]. Specifically Dubard and Matveev [36, 37] show that

$$v(\xi,\eta,\tau)=2\frac{\partial^2}{\partial\xi^2}\ln\widehat{D}_2(\xi-3\tau,\eta;\alpha,-48\tau)=\tfrac{1}{2}\left(|\widehat{\psi}_2(x,t;\alpha,\beta)|^2-1\right)\Big|_{x=\xi-3\tau,t=\eta,\beta=-48\tau}, \tag{4.4}$$

is a solution of the KPI equation (1.5). If we define $F_2^{\rm nls}(\xi,\eta,\tau;\alpha)=\widehat{D}_2(\xi-3\tau,\eta;\alpha,-48\tau)$, then

$$\begin{aligned} F_2^{\rm nls}(\xi,\tau;\alpha) = {} & \xi^6-18\tau\xi^5+3\left(45\tau^2+\eta^2+1\right)\xi^4-12\left(45\tau^2+3\eta^2-5\right)\tau\xi^3 \\ & +\left\{3\eta^4+18\left(9\tau^2-1\right)\eta^2+1215\tau^4-702\tau^2+27\right\}\xi^2 \\ & -\left\{18\tau\eta^4+36\left(9\tau^2+5\right)\tau\eta^2+1458\tau^5-2268\tau^3+450\tau\right\}\xi \\ & +\eta^6+27\left(\tau^2+1\right)\eta^4+9\left(27\tau^4+78\tau^2+11\right)\eta^2+729\tau^6-2349\tau^4+3411\tau^2+9. \end{aligned} \tag{4.5}$$



$\widetilde{u}_2(x,t;10^4,10^4)$ $\widetilde{u}_3(x,t;10^7,10^7)$ $\widetilde{u}_4(x,t;10^{10},10^{10})$

Figure 3.8: Contour plots of the generalised rational solutions $\widetilde{u}_2(x,t;10^4,10^4)$, $\widetilde{u}_3(x,t;10^7,10^7)$ and $\widetilde{u}_4(x,t;10^{10},10^{10})$ of the Boussinesq equation.

The polynomial $F_2^{\rm nls}(\xi,\tau;\alpha)$ satisfies

$$\left(\mathrm{D}_\xi^4+\mathrm{D}_\xi\mathrm{D}_\tau-3\mathrm{D}_\eta^2\right)F_2\bullet F_2=0, \tag{4.6}$$

which is the bilinear form of the KPI equation (1.5), and so

$$v_2^{\rm nls}(\xi,\eta,\tau;\alpha)=2\frac{\partial^2}{\partial\xi^2}\ln F_2^{\rm nls}(\xi,\eta,\tau;\alpha), \tag{4.7}$$

is a rational solution of the KPI equation (1.5).

4.3 Rational solutions of KPI related to the Boussinesq equation

The Boussinesq equation (1.2) is a symmetry reduction of the KPI equation (1.5) and so the generalised rational solutions $\widetilde{u}_n(x,t;\alpha,\beta)$ given by (3.11) of the Boussinesq equation can be used to generate rational solutions of the KPI equation. If in the KPI equation (1.5) we make the travelling wave reduction

$$v(\xi,\eta,\tau)=u(x,t),\qquad x=\xi-3\tau,\quad t=\eta,$$

then $u(x,t)$ satisfies the Boussinesq equation (1.2). Consequently given a solution of the Boussinesq equation (1.2), then we can derive a solution of the KPI equation (1.5). In particular, if

$$u(x,t)=2\frac{\partial^2}{\partial x^2}\ln F(x,t),$$

for some known $F(x,t)$, is a solution of the Boussinesq equation (1.2), then

$$v(\xi,\eta,\tau)=2\frac{\partial^2}{\partial\xi^2}\ln F(\xi-3\tau,\eta),$$

is a solution of the KPI equation (1.5). For example the choice $F(x,t)=x^2+t^2+1$ gives the lump solution (4.2) of KPI.

Using the generalised rational solution $\widetilde{u}_2(x,t;\alpha,\beta)$ (3.15) of the Boussinesq equation (1.2) we obtain the rational solution of the KPI equation (1.5) given by

$$v(\xi,\eta,\tau;\alpha,\beta)=2\frac{\partial^2}{\partial\xi^2}\ln F_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta), \tag{4.8}$$

where $F_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta)=\widetilde{F}_2(x,t;\alpha,\beta)$, i.e.

$$\begin{aligned}F_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta)=\xi^6&-18\tau\xi^5+3\left(45\tau^2+\eta^2+\tfrac{25}{9}\right)\xi^4-12\left(45\tau^2+3\eta^2+\tfrac{25}{3}\right)\tau\xi^3\\&+\left\{3\eta^4+18\left(9\tau^2+\tfrac{5}{3}\right)\eta^2+1215\tau^4+450\tau^2-\tfrac{125}{9}\right\}\xi^2\\&-\left\{18\eta^4+36\left(9\tau^2+5\right)\eta^2+1458\tau^4+900\tau^2+\tfrac{250}{3}\right\}\tau\xi\\&+\eta^6+27\left(\tau^2+\tfrac{17}{81}\right)\eta^4+9\left(27\tau^4+30\tau^2+\tfrac{475}{81}\right)\eta^2\\&+729\tau^6+675\tau^4-125\tau^2+\tfrac{625}{9}+2\alpha\left\{3\xi^2\eta-18\xi\tau\eta-\eta^3+\left(27\tau^2+\tfrac{5}{3}\right)\eta\right\}\\&+2\beta\left\{\xi^3-9\xi^2\tau-\left(3\eta^2-27\tau^2+\tfrac{1}{3}\right)\xi-27\tau^3+9\tau\eta^2+\tau\right\}+\alpha^2+\beta^2.\end{aligned}\tag{4.9}$$

We remark that this polynomial, in scaled coordinates, is given by Gorshkov, Pelinovsky and Stepanyants [65], see their equation (4.2), though the authors don't mention the Boussinesq equation.

4.4 A more general rational solution

If we compare the polynomials $F_2^{\rm nls}(\xi,\eta,\tau;\alpha)$ and $F_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta)$, respectively given by (4.5) and (4.9), then we see that they are fundamentally different. As we shall now demonstrate, they are special cases of a more general polynomial. Consider the polynomial $\mathcal{F}_2(\xi,\eta,\tau;\mu,\alpha,\beta)$, with parameters μ, α and β, given by

$$\begin{aligned}\mathcal{F}_2(\xi,\eta,\tau;\mu,\alpha,\beta)=\xi^6&-18\tau\xi^5+(3\eta^2+135\tau^2-6\mu^2+9)\xi^4-\left\{36\eta^2+540\tau^2-12(6\mu^2+6\mu-7)\right\}\tau\xi^3\\&+\left\{3\eta^4+18(9\tau^2-2\mu+1)\eta^2+1215\tau^4-54(6\mu^2+12\mu-5)\tau^2\right.\\&\qquad\left.+9\mu(\mu+2)(\mu^2-2\mu+2)\right\}\xi^2\\&-\left\{18\eta^4+36(9\tau^2+5)\eta^2+1458\tau^4-324(2\mu^2+6\mu-1)\tau^2\right.\\&\qquad\left.+18\mu(3\mu^3+12\mu^2-2\mu+12)\right\}\tau\xi+\eta^6+(27\tau^2+6\mu^2+12\mu+9)\eta^4\\&+\left\{243\tau^4+54(6\mu+7)\tau^2+9(\mu^4+4\mu^3+6\mu^2-4\mu+4)\right\}\eta^2\\&+729\tau^6-81(\mu^2+24\mu-1)\tau^4+9(9\mu^4+72\mu^3+150\mu^2+132\mu+16)\tau^2\\&+9(\mu^2-2\mu+2)^2+2\alpha\left\{3\eta\xi^2-18\tau\eta\xi-\eta^3+3\left[9\tau^2-\mu(\mu+2)\right]\eta\right\}\\&+2\beta\left\{\xi^3-9\tau\xi^2-6(\eta^2-9\tau^2+\mu^2)\xi+9\tau\eta^2-27\tau^3+3(3\mu^2+12\mu+4)\tau\right\}\\&+\alpha^2+\beta^2.\end{aligned}\tag{4.10}$$

This polynomial has both the polynomials $F_2^{\rm nls}(\xi,\eta,\tau;\alpha)$ and $F_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta)$ as special cases, specifically

$$F_2^{\rm nls}(\xi,\eta,\tau;\alpha)=\mathcal{F}_2(\xi,\eta,\tau;1,\alpha,0),\qquad F_2^{\rm bq}(\xi,\eta,\tau;\alpha,\beta)=\mathcal{F}_2(\xi,\eta,\tau;-\tfrac{1}{3},\alpha,\beta).$$

Furthermore

$$v(\xi,\eta,\tau;\mu,\alpha,\beta)=2\frac{\partial^2}{\partial\xi^2}\ln\mathcal{F}_2(\xi,\eta,\tau;\mu,\alpha,\beta),\tag{4.11}$$

with $\mathcal{F}_2(\xi,\eta,\tau;\mu,\alpha,\beta)$ given by (4.10), is a solution of the KPI equation (1.5), which includes as special cases the solutions (4.7), when $\mu=1$ and $\beta=0$, and (4.8), when $\mu=-\frac{1}{3}$, as is easily shown.

In Figure 4.1, the initial solution $v(\xi,\eta,0;\mu,0,0)$ given by (4.11) is plotted for various choices of the parameter μ. When $\mu=1$, then this arises from the solution (4.7) derived from the focusing NLS equation (1.1) whilst when $\mu=-\frac{1}{3}$, then this arises from the solution (4.8) derived from the Boussinesq equation (1.2). From Figure 4.1 we can see that for $\mu<\mu^*$, the solution $v(\xi,\eta,0;\mu,0,0)$ has two peaks on the line $\eta=0$, which coalesce when $\mu=\mu^*$ to form one peak at $\xi=\eta=0$. By considering when

$$\left.\frac{\partial^2}{\partial\xi^2}v(\xi,0,0;\mu,0,0)\right|_{\xi=0}=-\frac{8(3\mu^4+12\mu^3+16\mu^2-6)}{(\mu^2-2\mu+2)^2}=0,$$

then μ^* is the real positive root of

$$3\mu^4+12\mu^3+16\mu^2-6=3\left[\mu^2+2(1-\tfrac{1}{3}\sqrt{6})\mu+2-\sqrt{6}\right]\left[\mu^2+2(1+\tfrac{1}{3}\sqrt{6})\mu+2+\sqrt{6}\right]=0,$$

i.e. $\mu^*=-1+\frac{1}{3}\sqrt{6}+\frac{1}{3}\sqrt{-3+3\sqrt{6}}\approx0.5115960325$. For $\mu>\mu^*$, it can be shown that

$$v(0,0,0;\mu,0,0)=\frac{4\mu(\mu+2)}{\mu^2-2\mu+2},$$

increases until it reaches a maximum height of $4(2+\sqrt{5})$ when $\mu=\frac{1}{2}(1+\sqrt{5})$, which is the golden mean!

5 Discussion

In this paper we have derived a sequence of algebraically decaying rational solutions of the Boussinesq equation (1.2) which depend on two arbitrary parameters, have an interesting structure and have a similar appearance to rogue-wave solutions in the sense that they have isolated "lumps". The associated special polynomial, which has equal weight in x and t, satisfies a bilinear equation of Hirota type and comprises of a linear combination of four independent solutions of the bilinear equation, something remarkable for a solutions of a bilinear equation. The derivation of a representation of these special polynomials as determinants is currently under investigation and we do not pursue this further here. We remark that other types of exact solutions of the Boussinesq equation (1.2) can be derived using the bilinear equation (3.6) including breather solutions [103, 104] and rational-soliton solutions [95].

Using our rational solutions of the Boussinesq equation (1.2), we derived rational solutions of the the KPI equation (1.5) and compared them to those obtained from rational solutions of the focusing NLS equation (1.1) by Dubard and Matveev [36, 37]. It was shown that the two sets of solutions are fundamentally different and both are special cases of a more general rational solution. We remark that Ablowitz *et al.* [3, 9, 110] derived a hierarchy of algebraically decaying rational solutions of the KPI equation (1.5) which have the form

$$v_m(\xi,\eta,\tau) = 2\frac{\partial^2}{\partial\xi^2}\ln G_m(\xi,\eta,\tau), \tag{5.1}$$

where $G_m(\xi,\eta,\tau)$ is a polynomial of degree $2m$ in ξ, η and τ. These rational solutions are derived in terms of the eigenfunctions of the non-stationary Schrödinger equation

$$\mathrm{i}\varphi_\eta + \varphi_{\xi\xi} + v\varphi = 0, \tag{5.2}$$

with potential $v = v(\xi,\eta,\tau)$, which is used in the solution of KPI (1.5) by inverse scattering; equation (1.5) is obtained from the compatibility of (5.2) and

$$\varphi_\tau + 4\varphi_{\xi\xi\xi} + 6v\varphi_\xi + w\varphi = 0, \qquad w_\xi = v. \tag{5.3}$$

This is a fundamentally different hierarchy of solutions of the KPI equation (1.5) compared to those discussed in §4, not least because it involves polynomials of all even degrees, not just of degree $n(n+1)$, with $n \in \mathbb{N}$.

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Appendix

$$\begin{aligned}
F_4(x,t) = x^{20} &+ \left(10t^2+90\right)x^{18} + \left(45t^4+1010t^2+1845\right)x^{16}\\
&+ \left(120t^6+4600t^4+30600t^2+13000\right)x^{14}\\
&+ \left(210t^8+11480t^6+151900t^4+393400t^2-\tfrac{2097550}{9}\right)x^{12}\\
&+ \left(252t^{10}+17500t^8+367640t^6+2095800t^4+\tfrac{11948300}{9}t^2+\tfrac{232696100}{27}\right)x^{10}\\
&+ \left(210t^{12}+16940t^{10}+501550t^8+5010600t^6+\tfrac{39702250}{3}t^4+\tfrac{180407500}{9}t^2-\tfrac{6596112250}{27}\right)x^8\\
&+ \left(120t^{14}+10360t^{12}+400120t^{10}+5601400t^8+\tfrac{141659000}{3}t^6+\tfrac{23569000}{9}t^4\right.\\
&\qquad \left.-\tfrac{19319573000}{27}t^2+\tfrac{86014747000}{27}\right)x^6\\
&+ \left(45t^{16}+3800t^{14}+179900t^{12}+3504200t^{10}+\tfrac{98796250}{3}t^8+\tfrac{1675457000}{9}t^6\right.\\
&\qquad \left.-\tfrac{15031607500}{27}t^4+\tfrac{410944625000}{27}t^2+\tfrac{2352823598125}{81}\right)x^4\\
&+ \left(10t^{18}+730t^{16}+39400t^{14}+1320200t^{12}+\tfrac{74612300}{9}t^{10}+\tfrac{1165839500}{9}t^8\right.\\
&\qquad \left.+\tfrac{73409791000}{27}t^6+\tfrac{1122199715000}{27}t^4+\tfrac{10744980496250}{81}t^2-\tfrac{8594611821250}{243}\right)x^2\\
&+ t^{20}+50t^{18}+2565t^{16}+122200t^{14}+\tfrac{40078850}{9}t^{12}+\tfrac{2423740900}{27}t^{10}+\tfrac{44477105750}{27}t^8\\
&\qquad +\tfrac{177775871000}{9}t^6+\tfrac{4304738108125}{81}t^4+\tfrac{42895279813750}{243}t^2+\tfrac{73054200480625}{729}
\end{aligned}$$

$$
\begin{aligned}
F_5(x,t) = {} & x^{30} + \bigl(15t^2 + \tfrac{605}{3}\bigr)x^{28} + \bigl(105t^4 + 3290t^2 + 12705\bigr)x^{26} + \bigl(455t^6 + \tfrac{71575}{3}t^4 + \tfrac{2265725}{9}t^2 + \tfrac{25939375}{81}\bigr)x^{24} \\
& + \bigl(1365t^8 + \tfrac{309260}{3}t^6 + \tfrac{17897950}{9}t^4 + \tfrac{26849900}{3}t^2 + \tfrac{374564575}{243}\bigr)x^{22} \\
& + \bigl(3003t^{10} + 298375t^8 + \tfrac{79208990}{9}t^6 + \tfrac{725413150}{9}t^4 + \tfrac{1327947775}{9}t^2 + \tfrac{45146222275}{729}\bigr)x^{20} \\
& + \bigl(5005t^{12} + \tfrac{1842610}{3}t^{10} + \tfrac{74936225}{3}t^8 + \tfrac{30256387700}{81}t^6 + \tfrac{416681967625}{243}t^4 + \tfrac{1062878489750}{729}t^2 \\
& \qquad - \tfrac{29949453408875}{6561}\bigr)x^{18} \\
& + \bigl(6435t^{14} + 929005t^{12} + \tfrac{145887805}{3}t^{10} + \tfrac{9444440425}{9}t^8 + \tfrac{716701225625}{81}t^6 + \tfrac{4765327769125}{243}t^4 \\
& \qquad - \tfrac{16069741485875}{729}t^2 + \tfrac{1572487588700875}{6561}\bigr)x^{16} \\
& + \bigl(6435t^{16} + 1049400t^{14} + \tfrac{201729500}{3}t^{12} + \tfrac{17384033800}{9}t^{10} + \tfrac{679848919750}{27}t^8 + \tfrac{29329239247000}{243}t^6 \\
& \qquad + \tfrac{56763015732500}{729}t^4 + \tfrac{877079786275000}{729}t^2 - \tfrac{145319532381244375}{19683}\bigr)x^{14} \\
& + \bigl(5005t^{18} + 888965t^{16} + \tfrac{201107900}{3}t^{14} + \tfrac{7268596300}{3}t^{12} + \tfrac{397343633750}{9}t^{10} + \tfrac{3094794221750}{9}t^8 \\
& \qquad + \tfrac{877248309206500}{729}t^6 + \tfrac{7522818112617500}{2187}t^4 - \tfrac{338877246089256875}{6561}t^2 - \tfrac{1153508042510140625}{177147}\bigr)x^{12} \\
& + \bigl(3003t^{20} + \tfrac{1682450}{3}t^{18} + \tfrac{144448885}{3}t^{16} + \tfrac{18942077000}{9}t^{14} + 49769993350t^{12} + \tfrac{16141595185100}{27}t^{10} \\
& \qquad + \tfrac{2217737551163750}{729}t^8 + \tfrac{9963380300797000}{729}t^6 - \tfrac{1297656625261390625}{6561}t^4 \\
& \qquad + \tfrac{4533029626565151250}{19683}t^2 + \tfrac{4174111038326870361875}{531441}\bigr)x^{10} \\
& + \bigl(1365t^{22} + 258335t^{20} + \tfrac{73529225}{3}t^{18} + \tfrac{11361306425}{9}t^{16} + \tfrac{976840075750}{27}t^{14} + \tfrac{17752164295250}{27}t^{12} \\
& \qquad + \tfrac{3658725849605750}{729}t^{10} + \tfrac{3515840993183750}{243}t^8 - \tfrac{195785332934489375}{729}t^6 \\
& \qquad + \tfrac{24978207925819946875}{6561}t^4 + \tfrac{1159166663661630903125}{19683}t^2 + \tfrac{4904143764303914178125}{531441}\bigr)x^8 \\
& + \bigl(455t^{24} + \tfrac{251020}{3}t^{22} + \tfrac{76657070}{9}t^{20} + \tfrac{41289423700}{81}t^{18} + \tfrac{1362831787625}{81}t^{16} + \tfrac{98913216479000}{243}t^{14} \\
& \qquad + \tfrac{4366923310634500}{729}t^{12} + \tfrac{14325694558021000}{729}t^{10} + \tfrac{164980602695610625}{729}t^8 + \tfrac{38543006652688037500}{2187}t^6 \\
& \qquad + \tfrac{12142620899858806568750}{59049}t^4 + \tfrac{84368406785489229287500}{177147}t^2 + \tfrac{1033632925475218502809375}{4782969}\bigr)x^6 \\
& + \bigl(105t^{26} + \tfrac{53375}{3}t^{24} + \tfrac{16915150}{9}t^{22} + \tfrac{1171587550}{9}t^{20} + \tfrac{1229272389625}{243}t^{18} + \tfrac{32275315890125}{243}t^{16} \\
& \qquad + \tfrac{2128271542512500}{729}t^{14} + \tfrac{110365606933697500}{2187}t^{12} + \tfrac{6125181130562869375}{6561}t^{10} \\
& \qquad + \tfrac{184494438219511371875}{6561}t^8 + \tfrac{18829554428932184918750}{59049}t^6 + \tfrac{30319073658670395156250}{19683}t^4 \\
& \qquad + \tfrac{767901026020862022953125}{531441}t^2 - \tfrac{37763631956445485447328125}{14348907}\bigr)x^4 \\
& + \bigl(15t^{28} + 2170t^{26} + \tfrac{2043125}{9}t^{24} + \tfrac{163177700}{9}t^{22} + \tfrac{8631985775}{9}t^{20} + \tfrac{17793313441750}{729}t^{18} \\
& \qquad + \tfrac{584377965527125}{729}t^{16} + \tfrac{7043820768985000}{243}t^{14} + \tfrac{6235281337588043125}{6561}t^{12} \\
& \qquad + \tfrac{437562641832806971250}{19683}t^{10} + \tfrac{5034320101951909278125}{19683}t^8 + \tfrac{296816181647178511587500}{177147}t^6 \\
& \qquad - \tfrac{305501861525583991296875}{531441}t^4 + \tfrac{139014074702059270656250}{531441}t^2 + \tfrac{634083161524687235258734375}{43046721}\bigr)x^2 \\
& + t^{30} + \tfrac{325}{3}t^{28} + 10185t^{26} + \tfrac{71587775}{81}t^{24} + \tfrac{16294723375}{243}t^{22} + \tfrac{2934806885675}{729}t^{20} + \tfrac{1145785364618125}{6561}t^{18} \\
& \qquad + \tfrac{44166106891704875}{6561}t^{16} + \tfrac{4108707388089775625}{19683}t^{14} + \tfrac{774149365283245634375}{177147}t^{12} \\
& \qquad + \tfrac{24580063449195140376875}{531441}t^{10} + \tfrac{266920437967411700828125}{531441}t^8 + \tfrac{18940589955229082293759375}{4782969}t^6 \\
& \qquad + \tfrac{196432003698991651589796875}{14348907}t^4 + \tfrac{1654599020642266683930859375}{43046721}t^2 + \tfrac{293277952222570147203765625}{43046721}
\end{aligned}
$$

$$
\begin{aligned}
P_3(x,t) = {} & 7x^{12} - \bigl(14t^2 - 210\bigr)x^{10} - \bigl(63t^4 + 630t^2 - \tfrac{875}{3}\bigr)x^8 - \bigl(36t^6 + 2044t^4 + \tfrac{16100}{3}t^2 - \tfrac{16100}{3}\bigr)x^6 \\
& + \bigl(25t^8 + 260t^6 - \tfrac{39550}{3}t^4 - \tfrac{91700}{3}t^2 - \tfrac{1066975}{9}\bigr)x^4 \\
& + \bigl(18t^{10} + \tfrac{1310}{3}t^8 + \tfrac{26140}{3}t^6 + \tfrac{146300}{3}t^4 + \tfrac{1835050}{9}t^2 + \tfrac{32655350}{27}\bigr)x^2 \\
& - t^{12} - \tfrac{10}{3}t^{10} + 25t^8 - \tfrac{1900}{3}t^6 - \tfrac{1230775}{9}t^4 - \tfrac{2070250}{3}t^2 + \tfrac{32680375}{81}
\end{aligned}
$$

$$
\begin{aligned}
Q_3(x,t) = {} & x^{12} - \bigl(18t^2 - \tfrac{74}{3}\bigr)x^{10} - \bigl(25t^4 + \tfrac{1870}{3}t^2 + \tfrac{275}{3}\bigr)x^8 + \bigl(36t^6 - 580t^4 - \tfrac{8860}{3}t^2 + \tfrac{4700}{3}\bigr)x^6 \\
& + \bigl(63t^8 + 1820t^6 - \tfrac{2450}{3}t^4 - \tfrac{37100}{3}t^2 - \tfrac{247625}{9}\bigr)x^4 \\
& + \bigl(14t^{10} + 630t^8 + \tfrac{49700}{3}t^6 + 48300t^4 + \tfrac{1877750}{9}t^2 + \tfrac{2898350}{9}\bigr)x^2 \\
& - 7t^{12} - 98t^{10} - \tfrac{5075}{3}t^8 - 23100t^6 - \tfrac{2108225}{9}t^4 - \tfrac{43900150}{27}t^2 - \tfrac{4998175}{81}
\end{aligned}
$$

$$
\begin{aligned}
P_4(x,t) = 9x^{20} &- \left(30t^2-770\right)x^{18} - \left(243t^4+3390t^2-14245\right)x^{16} - \left(360t^6+24360t^4+107800t^2-\tfrac{754600}{9}\right)x^{14} \\
&+ \left(130t^8-23720t^6-\tfrac{2278220}{3}t^4-\tfrac{4419800}{3}t^2-\tfrac{51285850}{27}\right)x^{12} \\
&+ \left(780t^{10}+\tfrac{94820}{3}t^8-\tfrac{759640}{3}t^6-\tfrac{82510120}{9}t^4+\tfrac{16762900}{27}t^2+\tfrac{5563180700}{81}\right)x^{10} \\
&+ \left(690t^{12}+58700t^{10}+\tfrac{3917450}{3}t^8+\tfrac{79849000}{9}t^6-\tfrac{1064659750}{27}t^4-\tfrac{5795597500}{27}t^2-\tfrac{1367658734750}{729}\right)x^8 \\
&+ \left(152t^{14}+\tfrac{65800}{3}t^{12}+\tfrac{11986520}{9}t^{10}+\tfrac{1202215000}{81}t^8+\tfrac{8625185800}{81}t^6+\tfrac{69758781400}{243}t^4\right. \\
&\qquad \left.+\tfrac{1077743975000}{243}t^2+\tfrac{55941010279000}{2187}\right)x^6 \\
&- \left(75t^{16}+\tfrac{10360}{3}t^{14}+\tfrac{66500}{9}t^{12}-\tfrac{69057800}{9}t^{10}-\tfrac{20996610250}{243}t^8-\tfrac{275830555000}{243}t^6\right. \\
&\qquad \left.-\tfrac{5620866905500}{729}t^4+\tfrac{9843829765000}{729}t^2-\tfrac{404610075244375}{2187}\right)x^4 \\
&- \left(30t^{18}+1790t^{16}+\tfrac{803320}{9}t^{14}+\tfrac{28869400}{9}t^{12}+\tfrac{1473856300}{27}t^{10}+\tfrac{629478426500}{729}t^8\right. \\
&\qquad \left.+\tfrac{11032069279000}{729}t^6+\tfrac{133702667483000}{729}t^4+\tfrac{112127684226250}{243}t^2+\tfrac{5297582110686250}{19683}\right)x^2 \\
&+ t^{20}+\tfrac{70}{3}t^{18}+\tfrac{1855}{3}t^{16}+\tfrac{1899800}{81}t^{14}+\tfrac{438095350}{243}t^{12}+\tfrac{88186059500}{729}t^{10}+\tfrac{14094153477250}{6561}t^8 \\
&\qquad +\tfrac{138847640239000}{6561}t^6-\tfrac{823906531765625}{19683}t^4-\tfrac{20487539830546250}{177147}t^2+\tfrac{266883842659905625}{531441}
\end{aligned}
$$

$$
\begin{aligned}
Q_4(x,t) = x^{20} &- \left(30t^2-\tfrac{230}{3}\right)x^{18} - \left(75t^4+2830t^2-\tfrac{2695}{3}\right)x^{16} + \left(152t^6-\tfrac{18680}{3}t^4-\tfrac{615640}{9}t^2+\tfrac{237160}{81}\right)x^{14} \\
&+ \left(690t^8+\tfrac{63560}{3}t^6-\tfrac{1276100}{9}t^4-\tfrac{1832600}{3}t^2-\tfrac{60772250}{243}\right)x^{12} \\
&+ \left(780t^{10}+62860t^8+\tfrac{8213240}{9}t^6-586600t^4+\tfrac{116801300}{27}t^2+\tfrac{8356925500}{729}\right)x^{10} \\
&+ \left(130t^{12}+\tfrac{117700}{3}t^{10}+\tfrac{4962650}{3}t^8+\tfrac{1352661800}{81}t^6+\tfrac{179352250}{243}t^4-\tfrac{117373448500}{729}t^2-\tfrac{2249680490750}{6561}\right)x^8 \\
&- \left(360t^{14}+15400t^{12}-\tfrac{1286600}{3}t^{10}-\tfrac{82943000}{9}t^8-\tfrac{8897211400}{81}t^6-\tfrac{65165639000}{243}t^4\right. \\
&\qquad \left.-\tfrac{3029653781000}{729}t^2-\tfrac{31684368485000}{6561}\right)x^6 \\
&- \left(243t^{16}+19560t^{14}+\tfrac{2325260}{3}t^{12}+\tfrac{29280440}{9}t^{10}+\tfrac{59473750}{27}t^8-\tfrac{61024825400}{243}t^6\right. \\
&\qquad \left.+\tfrac{320631426500}{729}t^4+\tfrac{37020326189000}{729}t^2-\tfrac{425463980932375}{19683}\right)x^4 \\
&- \left(30t^{18}+2910t^{16}+191800t^{14}+\tfrac{23294600}{3}t^{12}+\tfrac{2501523500}{27}t^{10}+\tfrac{34775471500}{27}t^8\right. \\
&\qquad \left.+\tfrac{1233890119000}{81}t^6+\tfrac{9121143955000}{81}t^4+\tfrac{114976450146250}{243}t^2+\tfrac{15307611409956250}{177147}\right)x^2 \\
&+ 9t^{20}+370t^{18}+15325t^{16}+\tfrac{5192600}{9}t^{14}+\tfrac{475032950}{27}t^{12}+\tfrac{35488950700}{81}t^{10}+\tfrac{6433079133250}{729}t^8 \\
&\qquad +\tfrac{234408499325000}{2187}t^6+\tfrac{315145911994375}{2187}t^4+\tfrac{4908421805113750}{19683}t^2+\tfrac{140298620844930625}{531441}
\end{aligned}
$$

References

[1] M.J. Ablowitz, "*Nonlinear Dispersive Waves*," *Cambridge Texts Appl. Math.*, C.U.P., Cambridge (2011).

[2] M.J. Ablowitz, D. Bar Yaacov and A.S. Fokas, On the inverse scattering transform for the Kadomtsev-Petviashvili equation, *Stud. Appl. Math.*, **69** (1983) 135–143.

[3] M.J. Ablowitz, S. Chakravarty, A.D. Trubatch and J. Villarroel, A novel class of solutions of the non-stationary Schrödinger and the Kadomtsev-Petviashvili I equations, *Phys. Lett. A*, **267** (2000) 132–146.

[4] M.J. Ablowitz and P.A. Clarkson, "*Solitons, Nonlinear Evolution Equations and Inverse Scattering*," *L.M.S. Lect. Notes Math.*, vol. **149**, C.U.P., Cambridge (1991).

[5] M.J. Ablowitz and R. Haberman, Resonantly coupled nonlinear evolution equations, *J. Math. Phys.*, **16** (1975) 2301–2305.

[6] M.J. Ablowitz and J. Satsuma, Solitons and rational solutions of nonlinear evolution equations, *J. Math. Phys.*, **19** (1978) 2180–2186.

[7] M.J. Ablowitz and H. Segur, On the evolution of packets of water waves, *J. Fluid Mech.*, **92** (1979) 691–715.

[8] M.J. Ablowitz and H. Segur, "*Solitons and the Inverse Scattering Transform*," SIAM, Philadelphia (1981).

[9] M.J. Ablowitz and J. Villarroel, Solutions to the time dependent Schrödinger and the Kadomtsev-Petviashvili equations, *Phys. Rev. Lett.*, **78** (1997) 570–573.

[10] M. Adler and J. Moser, On a class of polynomials associated with the Korteweg-de Vries equation, *Commun. Math. Phys.*, **61** (1978) 1–30.

[11] H. Airault, H.P. McKean and J. Moser, Rational and elliptic solutions of the KdV equation and related many-body problems, *Commun. Pure Appl. Math.*, **30** (1977) 95–148.

[12] N. Akhmediev, A. Ankiewicz and J.M. Soto-Crespo, Rogue waves and rational solutions of the nonlinear Schrödinger equation, *Phys. Rev. E*, **80** (2009) 026601.

[13] N. Akhmediev, A. Ankiewicz and M. Taki, Waves that appear from nowhere and disappear without a trace, *Phys. Lett. A*, **373** (2009) 675–678.

[14] N. Akhmediev, J.M. Soto-Crespo and A. Ankiewicz, Extreme waves that appear from nowhere: on the nature of rogue waves, *Phys. Lett. A*, **373** (2009) 2137–2145.

[15] A. Ankiewicz, P.A. Clarkson and N. Akhmediev, Rogue waves, rational solutions, the patterns of their zeros and integral relations, *J. Phys. A*, **43** (2010) 122002.
[16] A. Ankiewicz, D.J. Kedziora and N. Akhmediev, Rogue wave triplets, *Phys. Lett. A*, **375** (2011) 2782–2785.
[17] H. Aref, Vortices and polynomials, *Fluid Dynam. Res.*, **39** (2007) 5–23.
[18] H. Aref, Point vortex dynamics: a classical Mathematics playground, *J. Math. Phys.*, **48** (2007) 065401.
[19] H. Aref, P.K. Newton, M.A. Stremler, T. Tokieda and D.L. Vainchtein, Vortices crystals, *Adv. Appl. Mech.*, **39** (2002) 1–79.
[20] I.V. Barashenkov and D.E. Pelinovsky, Exact vortex solutions of the complex sine-Gordon theory on the plane, *Phys. Lett.*, **436** (1998) 117–124.
[21] D.J. Benney and A.C. Newell, The propagation of nonlinear wave envelopes, *J. Math. & Phys. (Stud. Appl. Math.)*, **46** (1967) 133–139.
[22] D.J. Benney and G.J. Roskes, Waves instabilities, *Stud. Appl. Math.*, **48** (1969) 377–385.
[23] Y.V. Bludov, V.V. Konotop and N. Akhmediev, Matter rogue waves, *Phys. Rev. A*, **80** (2009) 033610.
[24] J. Boussinesq, Théorie de l'intumescence liquide, appelée onde solitaire ou de translation se propagente dans un canal rectangulaire, *Comptes Rendus*, **72** (1871) 755–759.
[25] J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblemant parielles de la surface au fond, *J. Pure Appl.*, **17** (1872) 55–108.
[26] R.J. Buckingham and P.D. Miller, The sine-Gordon equation in the semiclassical limit: critical behavior near a separatrix, *J. Anal. Math.*, **118** (2012) 397–492.
[27] D.V. Choodnovsky and G.V. Choodnovsky, Pole expansions of nonlinear partial differential equations, *Nuovo Cim.*, **40B** (1977) 339–353.
[28] P.A. Clarkson, The fourth Painlevé equation and associated special polynomials, *J. Math. Phys.*, **44** (2003) 5350–5374.
[29] P.A. Clarkson, Special polynomials associated with rational solutions of the Painlevé equations and applications to soliton equations, *Comp. Meth. Func. Theory*, **6** (2006) 329–401.
[30] P.A. Clarkson, Special polynomials associated with rational solutions of the defocusing non-linear Schrödinger equation and the fourth Painlevé equation, *Europ. J. Appl. Math.*, **17** (2006) 293–322.
[31] P.A. Clarkson, Rational solutions of the Boussinesq equation, *Anal. Appl.*, **6** (2008) 349–369.
[32] P.A. Clarkson and M.D. Kruskal, New similarity solutions of the Boussinesq equation, *J. Math. Phys.*, **30** (1989) 2201–2213.
[33] P.A. Clarkson and E.L. Mansfield, The second Painlevé equation, its hierarchy and associated special polynomials, *Nonlinearity*, **16** (2003) R1–R26.
[34] P. Deift, C. Tomei and E. Trubowitz, Inverse scattering and the Boussinesq equation, *Commun. Pure Appl. Math.*, **35** (1982) 567–628.
[35] P. Dubard, P. Gaillard, C. Klein and V.B. Matveev, On multi-rogue wave solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Spec. Top.*, **185** (2010) 247–258.
[36] P. Dubard and V.B. Matveev, Multi-rogue waves solutions to the focusing NLS equation and the KP-I equation, *Nat. Hazards Earth. Syst. Sci.*, **11** (2011) 667–672.
[37] P. Dubard and V.B. Matveev, Multi-rogue waves solutions: from the NLS to the KP-I equation, *Nonlinearity*, **26** (2013) R93–R125.
[38] J.M. Dudley, F. Dias, M. Erkintalo and G. Genty, Instabilities, breathers and rogue waves in optics, *Nature Photonics*, **8** (2014) 755–764.
[39] J.M. Dudley, G. Genty, F. Dias, B. Kibler and N. Akhmediev, Modulation instability, Akhmediev breathers and continuous wave supercontinuum generation, *Opt. Expr.*, **17** (2009) 21497–21508.
[40] K. Dysthe, H.E. Krogstad and P. Muller, Oceanic rogue waves, *Annu. Rev. Fluid Mech.*, **40** (2008) 287–310.
[41] A.S. Fokas and M.J. Ablowitz, On the inverse scattering of the time-dependent Schrödinger equation and the associated Kadomtsev-Petviashvili equation, *Stud. Appl. Math.*, **69** (1983) 211–228.
[42] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A*, **44** (2011) 435204.
[43] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *J. Math. Sci.: Adv. Appl.*, **13** (2012) 71–153.
[44] P. Gaillard, Degenerate determinant representation of solutions of the nonlinear Schrödinger equation, higher order Peregrine breathers and multi-rogue waves, *J. Math. Phys.*, **54** (2013) 013504.
[45] P. Gaillard, Six-parameters deformations of fourth order Peregrine breather solutions of the nonlinear Schrödinger equation, *J. Math. Phys.*, **54** (2013) 073519.
[46] P. Gaillard, Deformations of third-order Peregrine breather solutions of the nonlinear Schrödinger equation with four parameters, *Phys. Rev. E*, **88** (2013) 042903.
[47] P. Gaillard, Two parameters deformations of ninth Peregrine breather solution of the NLS equation and multi-rogue waves, *J. Math.*, **2013** (2013) 520214.
[48] P. Gaillard, Wronskian representation of solutions of NLS equation, and seventh order rogue waves, *J. Mod. Phys.*, **4** (2013) 246–266.
[49] P. Gaillard, Two-parameter determinant representation of seventh order rogue wave solutions of the NLS equation, *J. Theo. Appl. Phys.*, **7** (2013) 45.

[50] P. Gaillard, Ten-parameter deformations of the sixth-order Peregrine breather solutions of the NLS equation, *Phys. Scr.*, **89** (2014) 015004.
[51] P. Gaillard, Two parameters Wronskian representation of solutions of nonlinear Schrödinger equation, eighth Peregrine breather and multi-rogue wave, *J. Math. Phys.*, **55** (2014) 093506.
[52] P. Gaillard, The fifth order Peregrine breather and its eight-parameter deformations solutions of the NLS equation, *Commun. Theor. Phys.*, **61** (2014) 365–369.
[53] P. Gaillard, Higher order Peregrine breathers, their deformations and multi-rogue waves, *J. Phys. Conf. Ser.*, **482** (2014) 012016.
[54] P. Gaillard, Tenth Peregrine breather solution to the NLS equation, *Ann. Physics*, **355** (2015) 293–298.
[55] P. Gaillard, Other $2N-2$ parameters solutions of the NLS equation and $2N+1$ highest amplitude of the modulus of the Nth order AP breather, *J. Phys. A*, **48** (2015) 145203.
[56] P. Gaillard, Hierarchy of solutions to the NLS equation and multi-rogue waves, *J. Phys. Conf. Ser.*, **574** (2015) 012031.
[57] P. Gaillard, Rational solutions to the KPI equation and multi rogue waves, *Ann. Phys.*, **367** (2016) 1–5.
[58] P. Gaillard, Fredholm and Wronskian representations of solutions to the KPI equation and multi-rogue waves, *J. Math. Phys.*, **57** (2016) 063505.
[59] P. Gaillard and M. Gastineau, The Peregrine breather of order nine and its deformations with sixteen parameters solutions to the NLS equation, *Phys. Lett. A*, **379** (2015) 1309–1313.
[60] P. Gaillard and M. Gastineau, 18 parameter deformations of the Peregrine breather of order 10 solutions of the NLS equation, *Internat. J. Modern Phys. C*, **26** (2015) 1550016.
[61] P. Gaillard and M. Gastineau, Twenty parameters families of solutions to the NLS equation and the eleventh Peregrine breather, *Commun. Theor. Phys. (Beijing)*, **65** (2016) 136–144.
[62] V.M. Galkin, D.E. Pelinovsky and Yu.A. Stepanyants, The structure of the rational solutions to the Boussinesq equation, *Physica*, **D80** (1995) 246–255.
[63] A.N. Ganshin, V.B. Efimov, G.V. Kolmakov, L.P. Mezhov-Deglin and P.V.E. McClintock, Observation of an inverse energy cascade in developed acoustic turbulence in superfluid helium, *Phys. Rev. Lett.*, **101** (2008) 065303.
[64] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, Method for solving the Korteweg-de Vries equation, *Phys. Rev. Lett.*, **19** (1967) 1095–1097.
[65] K.A. Gorshkov, D.E. Pelinovsky and Yu.A. Stepanyants, Normal and anomalous scattering, formation and decay of bound states of two-dimensional solitons described by the Kadomtsev-Petviashvili equation, *JETP*, **77** (1993) 237–245.
[66] B. Guo, L. Ling and Q. P. Liu, Nonlinear Schrödinger equation: generalized Darboux transformation and rogue wave solutions, *Phys. Rev. E*, **85** (2012) 026607.
[67] A. Hasegawa and F.D. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres. I. Anomalous dispersion, *Appl. Phys. Lett.*, **23** (1973) 142–144.
[68] A. Hasegawa and F.D. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres. II. Normal dispersion, *Appl. Phys.Lett.*, **23** (1973) 171–172.
[69] R. Hirota, Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices, *J. Math. Phys.*, **14** (1973) 810–814.
[70] E. Infeld and G. Rowlands, "*Nonlinear Waves, Solitons and Chaos*," C.U.P., Cambridge (1990).
[71] R.S. Johnson and S. Thompson, A solution of the inverse scattering problem for the Kadomtsev-Petviashvili equation by the method of separation of variables, *Phys. Lett. A*, **66** (1978) 279–281.
[72] B.B. Kadomtsev and V.I. Petviashvili, On the stability of solitary waves in weakly dispersing media, *Sov. Phys. Dokl.*, **15** (1970) 539–541.
[73] D.J. Kedziora, A. Ankiewicz and N. Akhmediev, Circular rogue wave clusters, *Phys. Rev. E*, **84** (2011) 056611.
[74] D.J. Kedziora, A. Ankiewicz and N. Akhmediev, Triangular rogue wave cascades, *Phys. Rev. E*, **86** (2012) 056602.
[75] D.J. Kedziora, A. Ankiewicz and N. Akhmediev, Classifying the hierarchy of nonlinear-Schrödinger-equation rogue-wave solutions, *Phys. Rev. E*, **88** (2013) 132207.
[76] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev and J.M. Dudley, The Peregrine soliton in nonlinear fibre optics, *Nat. Phys.*, **6** (2010) 790–795.
[77] C. Kharif, E. Pelinovsky and A. Slunyaev, "*Rogue Waves in the Ocean*," Berlin: Springer (2009).
[78] W.-X. Ma, Lump solutions to the Kadomtsev-Petviashvili equation, *Phys. Lett. A*, **379** (2015) 1975–1978.
[79] S.V. Manakov, The inverse scattering transform for the time-dependent Schrödinger equation and Kadomtsev-Petviashvili equation, *Physica*, **3D** (1981) 420–427.
[80] S.V. Manakov, V.E. Zakharov, L.A. Bordag, A.R. Its and V.B. Matveev, Two-dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction, *Phys. Lett.*, **63A** (1977) 205–206.
[81] Y. Matsuno, Exact multi-soliton solution of the Benjamin-Ono equation, *J. Phys. A*, **12** (1979) 619–621.
[82] A. Montina, U. Bortolozzo, S. Residori and F.T. Arecchi, Non-Gaussian Statistics and extreme waves in a nonlinear optical cavity, *Phys. Rev. Lett.*, **103** (2009) 173901.
[83] M. Noumi and Y. Yamada, Symmetries in the fourth Painlevé equation and Okamoto polynomials, *Nagoya Math. J.*, **153** (1999) 53–86.
[84] K. Okamoto, Studies on the Painlevé equations III. Second and fourth Painlevé equations, $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$, *Math. Ann.*, **275** (1986) 221–255.

[85] Y. Ohta and J. Yang, General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, *Proc. R. Soc. London, Ser. A*, **468** (2012) 1716–1740.
[86] N. Olver and I.V. Barashenkov, Complex sine-Gordon-2: A new algorithm for multivortex solutions on the plane, *Theo. Math. Phys.*, **144** (2005) 1223–1226.
[87] M. Onorato, S. Residori, U. Bortolozzo, A. Montina and F.T. Arecchi, Rogue waves and their generating mechanisms in different physical contexts, *Phys. Rep.*, **528** (2013) 47–89.
[88] A.R. Osborne, "*Nonlinear Ocean Waves and the Inverse Scattering Transform*," *International Geophysics Series*, vol. **97**, Academic Press, Boston (2010).
[89] D.E. Pelinovsky, Rational solutions of the KP hierarchy and the dynamics of their poles. I. New form of a general rational solution, *J. Math. Phys.*, **35** (1994) 5820–5830.
[90] D.E. Pelinovsky, Rational solutions of the KP hierarchy and the dynamics of their poles. II. Construction of the degenerate polynomial solutions, *J. Math. Phys.*, **39** (1998) 5377–5395.
[91] D.E. Pelinovsky and Yu.A. Stepanyants, New multi-soliton solutions of the Kadomtsev-Petviashvili equation, *JETP Lett.*, **57** (1993) 24–28.
[92] D.E. Pelinovsky, Yu.A. Stepanyants and Y.A. Kivshar, Self-focusing of plane dark solitons in nonlinear defocusing media, *Phys. Rev. E*, **51** (1995) 5016–5026.
[93] E. Pelinovsky and C. Kharif (Editors), "*Extreme Ocean Waves*," Second Edition, Springer (2016).
[94] D.H. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, *J. Aust. Math. Soc. B*, **25** (1983) 16–43.
[95] J.G. Rao, Y.-B. Liu, C. Qian and J.-S. He, Rogue waves and hybrid solutions of the Boussinesq equation, *Z. Naturforsch. A*, **72** (2017) 307–314.
[96] J. Satsuma and M.J. Ablowitz, Two-dimensional lumps in non-linear dispersive systems, *J. Math. Phys.*, **20** (1979) 1496–1503.
[97] J. Satsuma and Y. Ishimori, Periodic wave and rational soliton solutions of the Benjamin-Ono equation, *J. Phys. Soc. Japan*, **46** (1979) 681–687.
[98] A.C. Scott, The application of Bäcklund transforms to physical problems, in "*Bäcklund Transformations*" [R.M. Miura, Editor], *Lect. Notes Math.*, vol. **515**, Springer-Verlag, Berlin (1975) pp. 80–105.
[99] H. Segur and A. Finkel, An analytical model of periodic waves in shallow water, *Stud. Appl. Math.*, **73** (1985) 183–220.
[100] N. Singh and Y. Stepanyants, Obliquely propagating skew KP lumps, *Wave Motion*, **64** (2016) 92–102.
[101] D.R. Solli, C. Ropers, P. Koonath and B. Jalali, Optical rogue waves, *Nature*, **450** (2007) 1054–1057.
[102] L. Stenflo and M. Marklund, Rogue waves in the atmosphere, *J. Plasma Physics*, **76** (2010) 293–295.
[103] M. Tajiri and Y. Murakami, On breather solutions to the Boussinesq equation, *J. Phys. Soc. Japan*, **58** (1989) 3585–3590.
[104] M. Tajiri and Y. Murakami, Rational growing mode: exact solutions to the Boussinesq equation, *J. Phys. Soc. Japan*, **60** (1991) 2791–2792.
[105] M. Tajiri and Y. Watanabe, Periodic wave solutions as imbricate series of rational growing modes: solutions to the Boussinesq equation, *J. Phys. Soc. Japan*, **66** (1997) 1943–1949.
[106] M. Toda, Studies of a nonlinear lattice, *Phys. Rep.*, **8** (1975) 1–125.
[107] S. Tsuchiya, F. Dalfovo and Lev Pitaevskii, Solitons in two-dimensional Bose-Einstein condensates, *Phys. Rev. A*, **77** (2008) 045601.
[108] S. Turitsyn and G. Falkovitch, Stability of magneto-elastic solitons and self-focusing of sound in antiferromagnet, *Soviet Phys. JETP*, **62** (1985) 146–152.
[109] F. Ursell, The long-wave paradox in the theory of gravity waves, *Proc. Camb. Phil. Soc.*, **49** (1953) 685–694.
[110] J. Villarroel and M.J. Ablowitz, On the discrete spectrum of the nonstationary Schrödinger equation and multipole lumps of the Kadomtsev-Petviashvili I equation, *Comm. Math. Phys.*, **207** (1999) 1–42.
[111] A.P. Vorob'ev, On rational solutions of the second Painlevé equation, *Diff. Eqns.*, **1** (1965) 58–59.
[112] G.B. Whitham, "*Linear and Nonlinear Waves*," Wiley, New York (1974).
[113] A.I. Yablonskii, On rational solutions of the second Painlevé equation, *Vesti Akad. Navuk. BSSR Ser. Fiz. Tkh. Nauk.*, **3** (1959) 30–35.
[114] Z.Y. Yan, Financial rogue waves, *Commun. Theor. Phys.*, **54** (2010) 947–949.
[115] Z.Y. Yan, Vector financial rogue waves, *Phys. Lett. A*, **375** (2011) 4274–4279.
[116] N.J. Zabusky, A synergetic approach to problems of nonlinear dispersive wave propagation and interaction, in "*Nonlinear Partial Differential Equations*" [W. F. Ames, Editor], Academic, New York (1967) pp. 233–258.
[117] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *Sov. Phys. J. Appl. Mech. Tech. Phys.*, **4** (1968) 190–194.
[118] V.E. Zakharov, On stocastization of one-dimensional chains of nonlinear oscillations, *Sov. Phys. JETP*, **38** (1974) 108–110.
[119] V.E. Zakharov and A.B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional of waves in nonlinear media, *Sov. Phys. JETP*, **34** (1972) 62–69.

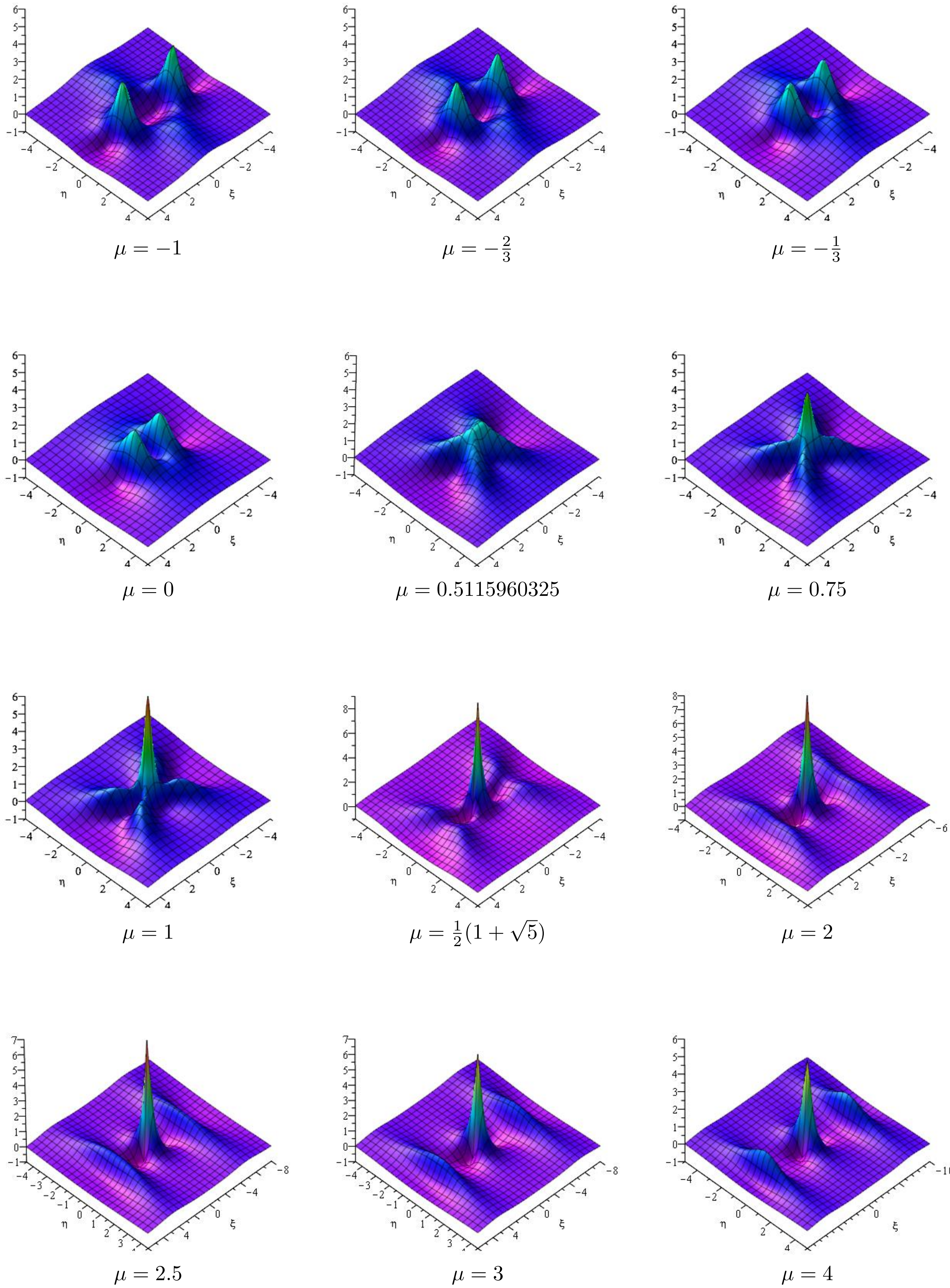

Figure 4.1: The initial solution $v(\xi, \eta, 0; \mu, 0, 0)$ given by (4.11) is plotted for various choices of the parameter μ. When $\mu = -\frac{1}{3}$ the initial solution corresponds to that arising from the Boussinesq equation (1.2) and when $\mu = 1$ to the initial solution from the focusing NLS equation (1.1).