Bethe Ansatz equations for the classical $A_n^{(1)}$ affine Toda field theories

Panagiota Adamopoulou and Clare Dunning

SMSAS, University of Kent, Canterbury, CT2 7NF, United Kingdom

E-mails:
pma7@kent.ac.uk, t.c.dunning@kent.ac.uk

Abstract

We establish a correspondence between classical $A_n^{(1)}$ affine Toda field theories and A_n Bethe Ansatz systems. We show that the connection coefficients relating specific solutions of the associated classical linear problem satisfy functional relations of the type that appear in the context of the massive quantum integrable model.

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1 Introduction

In [1], a connection was established between a linear ordinary differential equation (ODE) defined in the complex plane, and the conformal field theory limit of a certain quantum integrable lattice model (IM). The spectral determinants of the ODE introduced in [1] satisfy Bethe ansatz equations of A_1-type, which are objects well known in the study of quantum integrable systems. Moreover the spectral determinants and the Stokes multipliers of the ODE are related by precisely the same functional relations as the vacuum eigenvalues of the Q-operators and transfer matrix operators of the relevant quantum integrable model [1–4].

Various instances of the correspondence are now known. Of particular relevance is the extension to $SU(n)$ Bethe ansatz systems [5–7]. Bethe ansatz systems corresponding to simple Lie algebras of $ABCD$-type have been related to certain pseudo-differential equations [8].

Until recently, this ODE/IM correspondence related the spectrum of the relevant ODEs to the Bethe systems of massless quantum field theories. However, work in supersymmetric gauge theory [9, 10] gave a clue as to how the massive quantum field theories may be brought into the correspondence. Subsequently, Lukyanov and Zamolodchikov [11] presented a way to obtain the vacuum eigenvalues of the T- and Q-operators of the massive quantum sine-Gordon model starting from an integrable partial differential equation related to the well-known classical sinh-Gordon model. The correspondence is also known for the Bullough-Dodd model or $A_2^{(2)}$ Toda field theory [12].

In this paper we establish the correspondence for Bethe ansatz systems associated with massive quantum field theories of type A_{n-1}, starting from the classical $A_{n-1}^{(1)}$ affine Toda field theories. In §2 we describe a set of partial differential equations obtained from the affine Toda field theories by a change of independent and dependent variables, and give the associated Lax pairs and linear problems. The explicit details of the correspondence for the A_2-model are presented in §3, the results for the general A_n case are summarised in §4 and the conclusions are found in §5.

2 $A_{n-1}^{(1)}$ Toda field theory

The two-dimensional $A_{n-1}^{(1)}$ Toda field theories are described by the Lagrangian [13, 14]

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}(\partial_t\eta_i)^2 - (\partial_x\eta_i)^2 - \sum_{i=1}^{n}\exp(2\eta_{i+1} - 2\eta_i) \tag{2.1}$$

with $\eta_{n+1}(x,t) = \eta_1(x,t)$ and $\sum_{i=1}^{n}\eta_i = 0$. In light-cone coordinates, $w = x+t\,, \bar{w} = x-t$, the corresponding equations of motion are

$$2\,\partial_{\bar{w}}\partial_w\eta_i = \exp(2\eta_i - 2\eta_{i-1}) - \exp(2\eta_{i+1} - 2\eta_i) \quad \text{with} \quad i = 1, \ldots, n\,. \tag{2.2}$$

However, these are not the partial differential equations through which we will establish a connection to Bethe ansatz systems of A_{n-1}-type. For the simplest case ($n = 2$) the relevant equation is found by modifiying the sinh-Gordon equation using the changes of variables $dw = p(z)^{1/2}\,dz$ and $d\bar{w} = p(\bar{z})^{1/2}\,d\bar{z}$ where $p(z) = z^{2M} - s^{2M}$ [11]. Sending $\eta = \eta - \log(p(z)p(\bar{z})/4$, the sinh-Gordon equation becomes

$$\partial_z\partial_{\bar{z}} - \exp(2\eta(z,\bar{z})) + p(z)p(\bar{z})\,\exp(-2\eta(z,\bar{z})) = 0\;. \tag{2.3}$$

The introduction of the function $p(z)$ brings two parameters M and s, which are related to the coupling and the mass scale of the quantum sine-Gordon model. For general n, motivated by the examples involving a single Toda field [11,12], we make the change of variables

$$\mathrm{d}w = p(z)^{\frac{1}{n}}\,\mathrm{d}z\,, \quad \mathrm{d}\bar{w} = p(\bar{z})^{\frac{1}{n}}\,\mathrm{d}\bar{z} \tag{2.4}$$

and introduce the function $p(t) = t^{nM} - s^{nM}$ where the real, positive parameter s is related to the mass scale in the associated quantum model and M is related to the coupling. Setting

$$\eta_i(z,\bar{z}) \to \eta_i(z,\bar{z}) + \frac{n-(2i-1)}{4n}\ln(p(z)p(\bar{z}))\,, \tag{2.5}$$

the modified affine Toda field equations are

$$\begin{aligned} 2\,\partial_{\bar{z}}\partial_z\eta_1 &= p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1-2\eta_n} - \mathrm{e}^{2\eta_2-2\eta_1}\,, \\ 2\,\partial_{\bar{z}}\partial_z\eta_i &= \mathrm{e}^{2\eta_i-2\eta_{i-1}} - \mathrm{e}^{2\eta_{i+1}-2\eta_i} \quad \text{for} \quad i=2,\ldots,n-1\,, \\ 2\,\partial_{\bar{z}}\partial_z\eta_n &= \mathrm{e}^{2\eta_{n-1}-2\eta_n} - p(z)p(\bar{z})\,\mathrm{e}^{2\eta_1-2\eta_n}\,. \end{aligned} \tag{2.6}$$

The equations (2.6) can alternatively be viewed as arising from the zero-curvature condition $V_z - U_{\bar{z}} + [U,V] = 0$ of the linear problem

$$\big(\partial_z + U(z,\bar{z},\lambda)\big)\boldsymbol{\Psi} = 0\,, \quad \big(\partial_{\bar{z}} + V(z,\bar{z},\lambda)\big)\boldsymbol{\Psi} = 0\,, \tag{2.7}$$

where $\lambda = \exp(\theta)$ is the spectral parameter and

$$(U(z,\bar{z},\lambda))_{ij} = \partial_z\eta_i\,\delta_{ij} + \lambda\,(C(z))_{ij}\,, \quad (V(z,\bar{z},\lambda))_{ij} = -\partial_{\bar{z}}\,\eta_i\,\delta_{ij} + \lambda^{-1}(C(\bar{z}))_{ji}\,, \tag{2.8}$$

$$(C(z))_{ij} = \begin{cases} \exp(\eta_{j+1}-\eta_j)\,\delta_{i-1,j} & , \quad j=1,\ldots,n-1 \\ p(z)\,\exp(\eta_{j+1}-\eta_j)\,\delta_{i-1,j} & , \quad j=n\,, \end{cases} \tag{2.9}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i \equiv j \mod n \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.10}$$

The Lax pair (2.7) for the modified Toda field equations (2.6) is related to that associated with the standard $A^{(1)}_{n-1}$ Toda equations. We start from the Lax pair for the $A^{(1)}_{n-1}$ Toda equations presented in [13], written in light-cone coordinates $(w,\bar{w})$ as

$$\big(\partial_w + \widehat{U}(w,\bar{w},\lambda)\big)\boldsymbol{\Phi} = 0\,, \quad \big(\partial_{\bar{w}} + \widehat{V}(w,\bar{w},\lambda)\big)\boldsymbol{\Phi} = 0 \tag{2.11}$$

noting that the zero-curvature condition $\widehat{V}_w - \widehat{U}_{\bar{w}} + [\widehat{U},\widehat{V}] = 0$ is equivalent to (2.2). We apply the change of variables (2.4) and the transformation (2.5) to (2.11) to obtain

$$\big(\partial_z + \widetilde{U}(z,\bar{z},\lambda)\big)\boldsymbol{\Phi} = 0\,, \quad \big(\partial_{\bar{z}} + \widetilde{V}(z,\bar{z},\lambda)\big)\boldsymbol{\Phi} = 0 \tag{2.12}$$

where

$$\widetilde{U}(z,\bar{z},\lambda) = \widehat{U}(w\to z,\bar{w}\to\bar{z},\lambda) \tag{2.13}$$

and the shift of the Toda fields (2.5) in $\widehat{U}$ is implied. The linear problem (2.12) is related to the modified linear problem (2.7) by the gauge transformation

$$U(z,\bar{z},\lambda) = g^{-1}\,g_z + g^{-1}\,\widetilde{U}(z,\bar{z},\lambda)\,g\,, \tag{2.14}$$

$$V(z,\bar{z},\lambda) = g^{-1}\,g_{\bar{z}} + g^{-1}\,\widetilde{V}(z,\bar{z},\lambda)\,g\,, \tag{2.15}$$

with $\mathbf{\Phi} = g\mathbf{\Psi}$ and the matrix g has entries

$$(g)_{ij} = \left(\frac{p(\bar{z})}{p(z)}\right)^{n-\frac{2i-1}{4n}} \delta_{ij} \ . \tag{2.16}$$

We observe that the function $p(t)$ appears in the entries of the Lax matrices U and V that are related to the generator associated to the affine root of the $A^{(1)}_{n-1}$ Lie algebra.

Since we will be concerned with specific solutions to the modified version of the $A^{(1)}_{n-1}$ Toda equations that are real-valued, it is convenient to introduce polar coordinates $z = \rho\, \mathrm{e}^{i\phi}$, $\bar{z} = \rho\, \mathrm{e}^{-i\phi}$ with $\rho, \phi \in \mathbb{R}$. However, $z, \bar{z}$ will sometimes be treated as independent complex variables.

The modified Toda equations of motion (2.6) are invariant under the discrete symmetry

$$z \to \exp\left(2\pi i/nM\right) z\,, \quad \bar{z} \to \exp\left(-2\pi i/nM\right) \bar{z}\,. \tag{2.17}$$

The linear problem is invariant under (2.17) if the spectral parameter is shifted as $\lambda \to \sigma^{-1/M}\lambda$ (or equivalently $\theta \to \theta - 2\pi i/nM$) where $\sigma = \exp(2\pi i/n)$. It will be useful to define the transformations

$$\widehat{\Omega}: \quad \phi \to \phi + \frac{2\pi}{nM}\,, \quad \theta \to \theta - \frac{2\pi i}{nM}\,, \tag{2.18}$$

and, for any $n \times n$ matrix $A(\lambda)$,

$$\widehat{S}: \quad A(\lambda) \to S\,A(\sigma^{-1}\lambda)\,S^{-1} \quad \text{where} \quad (S)_{jk} = \sigma^j\,\delta_{jk}\,. \tag{2.19}$$

Such groups of transformations (acting on a loop algebra) are known as reduction groups [14, 15]. Since $\widehat{S}^n = id$ the group generated by $\widehat{S}$ is isomorphic to $\mathbb{Z}_n$. The matrices U and V are invariant under the transformations $\widehat{\Omega}$ and $\widehat{S}$:

$$U\left(\rho, \phi + \tfrac{2\pi}{nM}, \theta - \tfrac{2\pi i}{nM}\right) = U\left(\rho, \phi, \theta\right) \quad , \quad V\left(\rho, \phi + \tfrac{2\pi}{nM}, \theta - \tfrac{2\pi i}{nM}\right) = V\left(\rho, \phi, \theta\right)\,, \tag{2.20}$$

$$S\,U\left(\rho, \phi, \theta - \tfrac{2\pi i}{n}\right) S^{-1} = U(\rho, \phi, \theta) \quad , \quad S\,V\left(\rho, \phi, \theta - \tfrac{2\pi i}{n}\right) S^{-1} = V(\rho, \phi, \theta)\,. \tag{2.21}$$

Applying $\widehat{S}$ to (2.7) we find $S\,\mathbf{\Psi}(\sigma^{-1}\lambda)$ satisfies the linear problem, and for each solution $\mathbf{\Psi}$ there is a constant c such that

$$S\,\mathbf{\Psi}(\sigma^{-1}\lambda) = \sigma^c\,\mathbf{\Psi}(\lambda)\,. \tag{2.22}$$

We will introduce a family of solutions to the linear problem which respects these symmetries.

3 The A_2 case

We start by explaining the approach for the simplest model involving two fields: the $A^{(1)}_2$ affine Toda field theory. Motivated by [11, 12], we specify a two-parameter family of solutions to the equations (2.6) that respects the discrete symmetry (2.17). These solutions are unique, real and finite everywhere except at $\rho = 0$ and have periodicity $\eta_i\big(\rho, \phi + 2\pi/3M\big) = \eta_i(\rho, \phi)$. The solutions are further specified in terms of the parameters g_0, g_2 via their asymptotic behaviour at small and large values of ρ as

$$\eta_1(\rho, \phi) = (2 - g_2)\ln\rho + O(1) \quad , \quad \eta_3(\rho, \phi) = -g_0 \ln\rho + O(1) \quad \text{as} \quad \rho \to 0\,, \tag{3.1}$$

$$\eta_1(\rho, \phi) = -M\ln\rho + o(1) \quad , \quad \eta_3(\rho, \phi) = M\ln\rho + o(1) \quad \text{as} \quad \rho \to \infty\ . \tag{3.2}$$

The coefficients have been chosen to ensure consistency with the notation in the massless limit [5].

The small-ρ asymptotic of $\eta_i(\rho,\phi)$ may be improved by noting that the relevant solutions $\eta_i(w,\bar{w})$ to the standard Toda field equations (2.2) depend only on $w\bar{w}$. Under the symmetry reduction $t=(2w\bar{w})^{1/2}$, the corresponding equations of motion become a coupled system of ODEs for $\eta_i(w,\bar{w})=y_i(t)$:

$$\begin{aligned} \frac{d^2}{dt^2}\,y_1+\frac{1}{t}\frac{d}{dt}\,y_1+\mathrm{e}^{-4y_1-2y_3}-\mathrm{e}^{2y_1-2y_3}&=0\,,\\ \frac{d^2}{dt^2}\,y_3+\frac{1}{t}\frac{d}{dt}\,y_3+\mathrm{e}^{2y_1-2y_3}-\mathrm{e}^{4y_3+2y_1}&=0\,. \end{aligned} \tag{3.3}$$

To determine the asymptotic behaviour of (3.3) we use the method of dominant balance (see, for example, [16]). The first two terms of each equation in (3.3) balance in the limit $t\to 0$ if

$$y_1(t)\sim(2-g_2)\ln t+b_1\,,\quad y_3(t)\sim -g_0\ln t+b_3\,, \tag{3.4}$$

where b_i are arbitrary constants and for dominance the real constants g_i must satisfy

$$g_0<g_1<g_2\quad,\quad 2g_0+g_1>0\,. \tag{3.5}$$

The additional parameter g_1 appears in the asymptotic of the eliminated field $\eta_2=y_2$ (recall $\eta_1+\eta_2+\eta_3=0$) and is such that $g_0+g_1+g_2=3$.

Extending the analysis, we find that the sub-leading contributions to (3.4) take the form of two power series in particular powers of t [17]. Using these results, we deduce the asymptotic behaviours as $\rho\to 0$ of the required solutions of the modified Toda equations (2.6) are

$$\begin{aligned} \eta_1(z,\bar{z}) \;\sim\; & (1-g_2/2)\ln(z\bar{z})+b_1+\sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{6k\,s^{3kM}}\left(z^{3kM}+\bar{z}^{3kM}\right)\\ &+\sum_{m=1}^{\infty}c_m\,(z\bar{z})^{m(g_0-g_2+3)}-\sum_{m=1}^{\infty}d_m\,(z\bar{z})^{m(g_0+2g_2-3)}\,, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \eta_3(z,\bar{z}) \;\sim\; & -(g_0/2)\ln(z\bar{z})+b_3-\sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{6k\,s^{3kM}}\left(z^{3kM}+\bar{z}^{3kM}\right)\\ &+\sum_{m=1}^{\infty}e_m\,(z\bar{z})^{m(3-2g_0-g_2)}-\sum_{m=1}^{\infty}f_m\,(z\bar{z})^{m(g_0-g_2+3)}\,. \end{aligned} \tag{3.7}$$

The series in single powers of $z,\bar{z}$ arises from the functions $p(z),p(\bar{z})$. We have redefined the constants b_i to incorporate all constant corrections, and the coefficients c_m,d_m,e_m and f_m may be determined recursively. For example

$$c_1=f_1=\frac{s^{6M}\mathrm{e}^{2b_1-2b_3}}{2(3{+}g_0{-}g_2)^2}\;,\;d_1=\frac{\mathrm{e}^{-4b_1-2b_3}}{2(g_0{+}2g_2{-}3)^2}\;,\;e_1=\frac{\mathrm{e}^{4b_3+2b_1}}{2(3{-}2g_0{-}g_2)^2}\,. \tag{3.8}$$

The leading logarithmic contribution to the large-ρ expansion of (3.2) arises from the term $p(z)p(\bar{z})$ because, arguing as above, in the large-t limit we obtain

$$y_1(t)=O(1)\,,\quad y_3(t)=O(1)\quad\text{as}\quad t\to\infty\,. \tag{3.9}$$

From the linear system (2.7) we may obtain a pair of third order linear ODEs for two of the components of $\mathbf{\Psi} = (\Psi_1, \Psi_2, \Psi_3)^{\mathrm{T}}$. Setting $\Psi_3 = \exp(\eta_3)\,\psi$ and $\Psi_1 = \exp(-\eta_1)\,\bar{\psi}$, we define a general solution to the linear system through

$$\begin{aligned}\mathbf{\Psi}(z,\bar{z},\lambda) &= \begin{pmatrix} \lambda^{-2}\,\mathrm{e}^{3\eta_1+2\eta_3}\,\partial_z\left(\mathrm{e}^{-2\eta_1-4\eta_3}\,\partial_z\left(\mathrm{e}^{2\eta_3}\,\psi\right)\right) \\ -\lambda^{-1}\,\mathrm{e}^{-\eta_1-3\eta_3}\,\partial_z\left(\mathrm{e}^{2\eta_3}\,\psi\right) \\ \mathrm{e}^{\eta_3}\,\psi \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-\eta_1}\,\bar{\psi} \\ -\lambda\,\mathrm{e}^{3\eta_1+\eta_3}\,\partial_{\bar{z}}\left(\mathrm{e}^{-2\eta_1}\,\bar{\psi}\right) \\ \lambda^2\,\mathrm{e}^{-2\eta_1-3\eta_3}\,\partial_{\bar{z}}\left(\mathrm{e}^{4\eta_1+2\eta_3}\,\partial_{\bar{z}}\left(\mathrm{e}^{-2\eta_1}\,\bar{\psi}\right)\right) \end{pmatrix}.\end{aligned} \tag{3.10}$$

Then applying $\partial_z + U$ and $\partial_{\bar{z}} + V$ to $\mathbf{\Psi}$ we deduce ψ and $\bar{\psi}$ satisfy third order ODES:

$$\partial_z^3\psi + u_1(z,\bar{z})\,\partial_z\psi + \left(u_0(z,\bar{z}) + \lambda^3 p(z)\right)\psi = 0\,, \tag{3.11}$$

$$\partial_{\bar{z}}^3\bar{\psi} + \bar{u}_1(z,\bar{z})\,\partial_{\bar{z}}\bar{\psi} + \left(\bar{u}_0(z,\bar{z}) + \lambda^{-3} p(\bar{z})\right)\bar{\psi} = 0\,, \tag{3.12}$$

with

$$\begin{aligned} u_1(z,\bar{z}) &= -2\left(2\left(\partial_z\eta_1\right)^2 + 2\partial_z\eta_1\partial_z\eta_3 + 2\left(\partial_z\eta_3\right)^2 + \partial_z^2\eta_1 - \partial_z^2\eta_3\right)\,, \\ u_0(z,\bar{z}) &= -4\partial_z\eta_3\left(2\partial_z\eta_1\partial_z(\eta_1+\eta_3) + \partial_z^2\eta_1 + 2\partial_z^2\eta_3\right) + 2\partial_z^3\eta_3\,, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \bar{u}_1(z,\bar{z}) &= -2\left(2\left(\partial_{\bar{z}}\eta_1\right)^2 + 2\partial_{\bar{z}}\eta_1\partial_{\bar{z}}\eta_3 + 2\left(\partial_{\bar{z}}\eta_3\right)^2 + \partial_{\bar{z}}^2\eta_1 - \partial_{\bar{z}}^2\eta_3\right)\,, \\ \bar{u}_0(z,\bar{z}) &= 4\partial_{\bar{z}}\eta_1\left(2\partial_{\bar{z}}\eta_3\partial_{\bar{z}}(\eta_1+\eta_3) - \partial_{\bar{z}}^2\eta_1 - \partial_{\bar{z}}^2\eta_3\right) - 2\partial_{\bar{z}}^3\eta_1\,. \end{aligned} \tag{3.14}$$

3.1 The Q functions

We now find the behaviour of $\mathbf{\Psi}$ for small and large values of ρ. Setting $\psi = z^\mu$ and treating $\bar{z}$ as a fixed parameter as $z \to 0$, the roots of the indicial polynomial of (3.11) in this limit provide three different solutions, defined by

$$\chi_0 \sim z^{g_0}, \quad \chi_1 \sim z^{g_1}, \quad \chi_2 \sim z^{g_2}\,, \qquad z \to 0. \tag{3.15}$$

Consequently the linear problem (2.7) has solutions for small ρ defined by the leading order behaviours as follows:

$$\mathbf{\Xi}_0 \sim \begin{pmatrix} 0 \\ 0 \\ \mathrm{e}^{g_0(\theta+i\phi)} \end{pmatrix},\ \mathbf{\Xi}_1 \sim \begin{pmatrix} 0 \\ \mathrm{e}^{(g_1-1)(\theta+i\phi)} \\ 0 \end{pmatrix},\ \mathbf{\Xi}_2 \sim \begin{pmatrix} \mathrm{e}^{(g_2-2)(\theta+i\phi)} \\ 0 \\ 0 \end{pmatrix} \tag{3.16}$$

where $\mathbf{\Xi}_j \equiv \mathbf{\Xi}_j(\rho,\phi,\theta,\mathbf{g})$ and $\mathbf{g} = \{g_0, g_1, g_2\}$. The θ-dependent constants have been introduced in (3.16) to ensure the solutions are invariant under the symmetry $\widehat{\Omega}$ (2.18). The matrix of the solutions is normalised as $\det\left(\mathbf{\Xi}_0, \mathbf{\Xi}_1, \mathbf{\Xi}_2\right) = -1$.

Concentrating instead on the large-ρ limit, the ODEs (3.11) and (3.12) reduce to

$$\partial_z^3\psi + \lambda^3 p(z)\,\psi = 0 \quad , \quad \partial_{\bar{z}}^3\bar{\psi} + \lambda^{-3} p(\bar{z})\,\bar{\psi} = 0\,. \tag{3.17}$$

Using a WKB-type of analysis (see [5]), there exist solutions with asymptotic behaviour for $M > 1/2$

$$\psi \sim z^{-M}\exp\left(-\lambda\frac{z^{M+1}}{M+1} + f_1(\bar{z})\right) \quad , \quad \bar{\psi} \sim \bar{z}^{-M}\exp\left(-\lambda^{-1}\frac{\bar{z}^{M+1}}{M+1} + f_2(z)\right) \tag{3.18}$$

as $\rho \to \infty$ with $|\phi| < 4\pi/(3M+3)$ where $z = \rho \exp(i\phi)$. In this sector of the complex plane, these solutions are the unique solutions for $\lambda \in \mathbb{R}$ that tend to zero fastest as $\rho \to \infty$. The functions $f_1(\bar{z})$ and $f_2(z)$ are fixed by equating any line of (3.10) in the limit $\rho \to \infty$:

$$f_1(\bar{z}) = -\lambda^{-1}\frac{\bar{z}^{M+1}}{M+1}\,, \quad f_2(z) = -\lambda\frac{z^{M+1}}{M+1}\,. \tag{3.19}$$

Hence the large-ρ asymptotic solution $\mathbf{\Psi}$ to the linear problem (2.7) is

$$\mathbf{\Psi} \sim \begin{pmatrix} \mathrm{e}^{i\phi M} \\ 1 \\ \mathrm{e}^{-i\phi M} \end{pmatrix} \exp\left(-2\,\frac{\rho^{M+1}}{M+1}\cosh(\theta + i\phi(M+1))\right) \quad \text{as} \quad \rho \to \infty\,. \tag{3.20}$$

Since the solutions $\mathbf{\Xi}_j$ defined at small-ρ by (3.16) are linearly independent, we take $\{\mathbf{\Xi}_j\}$ as a basis for the space of all solutions to the linear problem. Hence we can express $\mathbf{\Psi}$ in this basis as

$$\mathbf{\Psi} = Q_0(\theta,\mathbf{g})\,\mathbf{\Xi}_0 + Q_1(\theta,\mathbf{g})\,\mathbf{\Xi}_1 + Q_2(\theta,\mathbf{g})\,\mathbf{\Xi}_2\,, \tag{3.21}$$

where $\mathbf{\Psi}$ and $\mathbf{\Xi}_j$ depend implicitly on $(\rho,\phi,\theta,\mathbf{g})$ and

$$Q_0 = -\det(\mathbf{\Psi},\mathbf{\Xi}_1,\mathbf{\Xi}_2)\ ,\ Q_1 = -\det(\mathbf{\Xi}_0,\mathbf{\Psi},\mathbf{\Xi}_2)\ ,\ Q_2 = -\det(\mathbf{\Xi}_0,\mathbf{\Xi}_1,\mathbf{\Psi})\,. \tag{3.22}$$

We note that the functions $Q_j(\theta,\mathbf{g})$ are quasi-periodic functions of θ. We use the small-ρ asymptotics (3.16) to determine the constants c_j such that the relation $S\,\mathbf{\Xi}_j(\sigma^{-1}\lambda) = c_j\,\mathbf{\Xi}_j(\lambda)$ holds, then act with the symmetry $\widehat{\Omega}$ to obtain

$$S\,\mathbf{\Xi}_j\left(\rho,\phi+\tfrac{2\pi}{3M},\theta-\tfrac{2\pi i}{3M}-\tfrac{2\pi i}{3}\right) = \exp(-g_j\tfrac{2\pi i}{3})\,\mathbf{\Xi}_j(\rho,\phi,\theta)\,. \tag{3.23}$$

It is also straightforward to prove that

$$S\,\mathbf{\Psi}\left(\rho,\phi+\tfrac{2\pi}{3M},\theta-\tfrac{2\pi i}{3M}-\tfrac{2\pi i}{3}\right) = \exp\left(\tfrac{4\pi i}{3}\right)\mathbf{\Psi}(\rho,\phi,\theta)\ . \tag{3.24}$$

Inserting (3.23) and (3.24) into the expressions for Q_j given in (3.22), and recalling that $S^n = \mathbb{I}$, we obtain

$$Q_j(\theta,\mathbf{g}) = \exp\left(-\tfrac{2\pi i}{3}(g_j-1)\right)\,Q_j\left(\theta-\tfrac{2\pi i}{3}\tfrac{(M+1)}{M},\mathbf{g}\right)\,, \quad \text{with} \quad j=0,1,2\,. \tag{3.25}$$

Before we derive A_2-type functional relations for the Q_j, we show that in a specific limit the differential equations obtained from the modified A_2 linear system reduce to those of the massless $SU(3)$ ODE/IM correspondence [5].

3.2 Conformal limit

We initially focus on the ODE for ψ (3.11). We take the parameter $\bar{z}$ to zero and use the asymptotics (3.6) and (3.7) to simplify the contribution to (3.11) from the Toda fields η_i. If we now set

$$x = z\,\exp\left(\tfrac{\theta}{M+1}\right)\,, \quad E = s^{3M}\exp\left(\tfrac{3M\theta}{M+1}\right)\,, \tag{3.26}$$

then as $z \sim s \to 0$ and $\theta \to \infty$, the new variables x, E are finite. As a result, the ODE (3.11) reduces to

$$\left(\partial_x^3 + \frac{(g_0g_1+g_0g_2+g_1g_2-2)}{x^2}\,\partial_x - \frac{g_0g_1g_2}{x^3} + p(x,E)\right)\psi = 0 \tag{3.27}$$

with $p(x,E) = x^{3M} - E$, precisely matching the third-order ODE introduced in [5] in the context of the (massless) ODE/IM correspondence. In this massless (or conformal) limit the functions Q_j are related to the vacuum eigenvalues of certain Q-operators studied in [18] in the context of $\mathcal{W}_3$ conformal field theory. Therefore, we anticipate the Q_j in (3.21) are related to the integrable structures of the massive A_2 quantum field theory model.

Alternatively, sending the parameter $z \to 0$ in the ODE for $\bar{\psi}$ (3.12) while keeping $\bar{z}$ small but finite, then taking the limit $\bar{z} \sim s \to 0$ as $\theta \to -\infty$ with

$$\bar{x} = \bar{z}\,\exp\left(-\tfrac{\theta}{(M+1)}\right), \quad \bar{E} = s^{3M}\exp\left(-\tfrac{3M\theta}{(M+1)}\right), \tag{3.28}$$

yields

$$\left(\partial_{\bar{x}}^3 + \frac{(g_0^\dagger g_1^\dagger + g_0^\dagger g_2^\dagger + g_1^\dagger g_2^\dagger - 2)}{\bar{x}^2}\,\partial_{\bar{x}} - \frac{g_0^\dagger g_1^\dagger g_2^\dagger}{\bar{x}^3} + p(\bar{x},\bar{E})\right)\bar{\psi} = 0\,, \tag{3.29}$$

where $g_i^\dagger = 2 - g_{2-i}$. Note that (3.29) is the adjoint equation to (3.27).

3.3 Functional relations

We now return to the study of (3.11) and (3.12). We establish via a set of functional relations the Bethe ansatz equations matching those of the associated massive quantum integrable model. In terms of the variables (3.26), the ODE (3.11) is

$$\left(\partial_x^3 + u_1(x,\bar{x})\,\partial_x + (u_0(x,\bar{x}) + p(x,E))\right)\psi(x,\bar{x},E,\bar{E},\mathbf{g}) = 0\,, \tag{3.30}$$

where u_0 and u_1 are functions of the Toda fields η_1, η_3. We treat x as a complex variable, $\bar{x}$ as a parameter and define sectors in the complex x-plane as

$$\mathcal{S}_k : \left|\arg x - \frac{2k\pi}{3(M+1)}\right| < \frac{\pi}{3(M+1)}\,. \tag{3.31}$$

Then

$$\psi_k(x,\bar{x},E,\bar{E},\mathbf{g}) = \omega^k \psi(\omega^{-k}x, \omega^k\bar{x}, \omega^{-3kM}E, \omega^{3kM}\bar{E},\mathbf{g})\,, \quad \omega = \mathrm{e}^{\frac{2\pi i}{3(M+1)}}\,, \tag{3.32}$$

are all solutions to (3.30) for integer k. These solutions have large-x asymptotics

$$\psi_k \sim \omega^{k(M+1)}\frac{x^{-M}}{i\sqrt{3}}\exp\left(-\omega^{-k(M+1)}\frac{x^{M+1}}{M+1} - \omega^{k(M+1)}\frac{\bar{x}^{M+1}}{M+1}\right) \tag{3.33}$$

for $x \in \mathcal{S}_{k-\frac{3}{2}} \cup \mathcal{S}_{k-\frac{1}{2}} \cup \mathcal{S}_{k+\frac{1}{2}} \cup \mathcal{S}_{k+\frac{3}{2}}$. Moreover, each ψ_k is the unique solution that decays to zero fastest for large x within the Stokes sector $\mathcal{S}_k$. Defining the notation

$$W_{k_1,\ldots,k_m} = W[\psi_{k_1},\ldots,\psi_{k_m}] \tag{3.34}$$

where W denotes the Wronskian of m functions, it follows from the asymptotic (3.33) that the Wronskian $W_{k,k+1,k+2} = 1$. Thus any triplet $\{\psi_k, \psi_{k+1}, \psi_{k+2}\}$ is a basis of solutions to (3.30), and we write

$$\psi = C^{(1)}(E,\bar{E},\mathbf{g})\,\psi_1 + C^{(2)}(E,\bar{E},\mathbf{g})\,\psi_2 + C^{(3)}(E,\bar{E},\mathbf{g})\,\psi_3\,, \tag{3.35}$$

where the Stokes multipliers $C^{(j)}(E,\bar{E},\mathbf{g})$ are analytic functions and the dependence of ψ_k on $(x,\bar{x},E,\bar{E},\mathbf{g})$ has been omitted. Taking the Wronskian of (3.35) with ψ_2, using $C^{(3)} = 1$ and the determinant relation [5, 7]

$$W_1 W_{0,2} = W_0 W_{1,2} + W_2 W_{0,1}\,, \tag{3.36}$$

we obtain a functional relation between Wronskians of rotated solutions:

$$C^{(1)}(E,\bar{E},\mathbf{g})\,W_1W_{1,2} - W_0\,W_{1,2} - W_2W_{0,1} - W_1W_{2,3} = 0\,. \tag{3.37}$$

Alternatively, as in (3.21), we can express ψ in the basis $\{\chi_0,\chi_1,\chi_2\}$ where the χ_j are solutions to (3.30) defined by the behaviour $\chi_j \sim x^{g_j}$ as $x \to 0$:

$$\psi \equiv W_0 = Q_0(E,\bar{E},\mathbf{g})\,\chi_0 + Q_1(E,\bar{E},\mathbf{g})\,\chi_1 + Q_2(E,\bar{E},\mathbf{g})\,\chi_2\,. \tag{3.38}$$

Using (3.38), we will obtain from (3.37) a functional relation that is independent of $x,\bar{x}$. First we note, from the large- and small-x asymptotics of ψ_k and χ_j respectively, we have

$$\begin{aligned} W_{k,k+1,\dots,k+m}(x,\bar{x},E,\bar{E}) &= \omega^{m(3-m)k/2}\,W_{k,k+1,\dots,k+m}(\omega^{-k}x,\omega^{k}\bar{x},\omega^{-3Mk}E,\omega^{3Mk}\bar{E}) \\ \chi_j(x,\bar{x},E,\bar{E},\mathbf{g}) &= \omega^{kg_j}\,\chi_j(\omega^{-k}x,\omega^{k}\bar{x},\omega^{-3kM}E,\omega^{3kM}\bar{E},\mathbf{g})\;. \end{aligned} \tag{3.39}$$

We insert (3.38) into (3.37), then use (3.39) to write all Wronskians in terms of the basic Wronskians $W_{0,1,\dots,m-1}$. Since $g_0<g_1<g_2$, the basic Wronskian functions have leading order behaviour

$$W_{0,1,\dots,m-1}(x,\bar{x},E,\bar{E},\mathbf{g}) \sim Q^{(m)}(E,\bar{E},\mathbf{g})\,x^{\alpha}\,, \quad x \to 0 \tag{3.40}$$

where $\alpha = \frac{m(1-m)}{2} + \sum_{j=0}^{m-1} g_j$. Therefore, taking into account (3.40), the coefficient of the leading order term $x^{2g_0+g_1-1}$ for small x of (3.37) is

$$\begin{aligned} C^{(1)}(\omega^{3M}E,s)\,Q^{(1)}(E,s)\,Q^{(2)}(E,s) &= Q^{(1)}(\omega^{3M}E,s)\,Q^{(2)}(E,s)\,\omega^{g_0-1} \\ &+Q^{(1)}(\omega^{-3M}E,s)\,Q^{(2)}(\omega^{3M}E,s)\,\omega^{g_1-1}+Q^{(1)}(E,s)\,Q^{(2)}(\omega^{-3M}E,s)\,\omega^{g_2-1}\,, \end{aligned} \tag{3.41}$$

where $E\bar{E} = s^{6M}$. We should note that $Q^{(1)}(E,\bar{E},\mathbf{g}) = Q_0(E,\bar{E},\mathbf{g})$ and

$$Q^{(2)}(E,\bar{E},\mathbf{g}) = \omega^{1-g_1}\,Q_0(E,\bar{E},\mathbf{g})Q_1(\omega^{-3M}E,\omega^{3M}\bar{E},\mathbf{g}) - \omega^{1-g_0}\,Q_0(\omega^{-3M}E,\omega^{3M}\bar{E},\mathbf{g})Q_1(E,\bar{E},\mathbf{g})\,. \tag{3.42}$$

In terms of the functions

$$T^{(1)}(E,s) = C^{(1)}(\omega^{3M}E,s)\,,\;\; Q^{(m)}(E,s) = A^{(m)}(\omega^{-3M(m-1)/2}E,s)\,, \tag{3.43}$$

the functional relation (3.41) corresponds to the dressed vacuum form for the transfer matrix eigenvalue $T^{(1)}(E,s)$. The equivalent relation in the conformal case appears in (5.10) of [18].

Now suppose that the zeros of the functions $A^{(m)}(E)$ are at $E_k^{(m)}$ for $k = 1,2,\dots,\infty$. Evaluating (3.41) at $E = E_k^{(m)}$ we obtain a set of Bethe Ansatz equations of A_2-type for the Bethe roots $\{E_k^{(m)}\}$:

$$\frac{A^{(1)}(\omega^{3M}E_k^{(1)},s)\,A^{(2)}(\omega^{-\frac{3M}{2}}E_k^{(1)},s)}{A^{(1)}(\omega^{-3M}E_k^{(1)},s)\,A^{(2)}(\omega^{\frac{3M}{2}}E_k^{(1)},s)} = -\omega^{g_1-g_0}\,, \quad k = 1,2,\dots,\infty \tag{3.44}$$

$$\frac{A^{(1)}(\omega^{-\frac{3M}{2}}E_k^{(2)},s)\,A^{(2)}(\omega^{3M}E_k^{(2)},s)}{A^{(1)}(\omega^{\frac{3M}{2}}E_k^{(2)},s)\,A^{(2)}(\omega^{-3M}E_k^{(2)},s)} = -\omega^{g_2-g_1}\,, \quad k = 1,2,\dots,\infty\,. \tag{3.45}$$

4 The A_{n-1} cases

We now extend the results of the previous section to all $A_{n-1}^{(1)}$ affine Toda models. We require an $n{-}1$-parameter family of solutions to the Toda equations (2.6) that are real, have periodicity $\eta_i\big(\rho,\phi+2\pi/nM\big)=\eta_i(\rho,\phi)$, are finite everywhere except at $\rho=0$ and behave asymptotically as

$$\eta_i(\rho,\phi) = (n-i-g_{n-i})\ln\rho+O(1)\,, \qquad \rho\to 0\,, \tag{4.1}$$

$$\eta_i(\rho,\phi) = \tfrac{1}{2}(2i-1-n)M\,\ln\rho+o(1)\,, \qquad \rho\to\infty\,. \tag{4.2}$$

The parameters g_i are real constants satisfying $\sum_{i=0}^{n-1} g_i=n(n-1)/2$ and, to ensure (4.1) is the leading order contribution for small-ρ,

$$g_0<g_1<\cdots<g_{n-1}\quad,\quad g_0+n>g_{n-1}\,. \tag{4.3}$$

We now recast the linear problem (2.7) as two systems of n first order linear ODEs for the components of $\mathbf{\Psi}=(\Psi_1,\ldots,\Psi_n)^{\mathrm{T}}$ with solution of the form

$$\Psi_i(z,\bar z,\lambda) = \begin{cases} -\lambda^{-1}\mathrm{e}^{\eta_i-\eta_{i+1}}\big(\partial_z\Psi_{i+1}+\partial_z\eta_{i+1}\Psi_{i+1}\big) & ,\quad i=1,\ldots,n-1 \\ \mathrm{e}^{\eta_n}\psi & ,\quad i=n\,, \end{cases} \tag{4.4}$$

$$= \begin{cases} \mathrm{e}^{-\eta_1}\bar\psi & ,\quad i=1 \\ -\lambda\,\mathrm{e}^{\eta_{i-1}-\eta_i}\big(\partial_{\bar z}\Psi_{i-1}-\partial_{\bar z}\eta_{i-1}\Psi_{i-1}\big) & ,\quad i=1,\ldots,n-1\,. \end{cases} \tag{4.5}$$

Eliminating either the components $\Psi_1,\ldots,\Psi_{n-1}$ or $\Psi_2,\ldots,\Psi_n$, we obtain n^{th}-order differential equations for $\psi(z,\bar z,\lambda)$ or $\bar\psi(z,\bar z,\lambda)$ respectively:

$$\Big((-1)^{n+1}D_n(\eta)+\lambda^n p(z)\Big)\psi = 0\,, \tag{4.6}$$

$$\Big((-1)^{n+1}\bar D_n(\eta)+\lambda^{-n} p(\bar z)\Big)\bar\psi = 0\,, \tag{4.7}$$

where the n^{th}-order differential operators are defined by

$$D_n(\eta)=(\partial_z+2\,\partial_z\eta_1)\,(\partial_z+2\,\partial_z\eta_2)\cdots(\partial_z+2\,\partial_z\eta_n)\;, \tag{4.8}$$

$$\bar D_n(\eta)=(\partial_{\bar z}-2\,\partial_{\bar z}\eta_n)\cdots(\partial_{\bar z}-2\,\partial_{\bar z}\eta_2)\,(\partial_{\bar z}-2\,\partial_{\bar z}\eta_1)\;. \tag{4.9}$$

4.1 The Q functions

We focus on equation (4.6) for ψ and treat the variable $\bar z$ as a parameter. We set $\mathbf{g}=\{g_0,\ldots,g_{n-1}\}$. Given the asymptotic behaviour (4.1) of η_i, in the $\rho\to 0$ limit the differential operator $D_n(\eta)$ becomes

$$D_n(\mathbf{g})=\left(\partial_z-\frac{g_{n-1}-(n-1)}{z}\right)\left(\partial_z-\frac{g_{n-2}-(n-2)}{z}\right)\cdots\left(\partial_z-\frac{g_0}{z}\right)\;. \tag{4.10}$$

Hence we read off from (4.10) that the solutions to (4.6) behave as z^{g_j} as $z\to 0$. Therefore the linear problem (2.7) has n solutions defined as $\rho\to 0$ by

$$\mathbf{\Xi}_j\sim(\underbrace{0,\ldots,0,\mathrm{e}^{(g_j-j)(\theta+i\phi)},0\ldots,0}_{n\text{ components}})^{\mathrm{T}}\,, \tag{4.11}$$

where the $(n{-}j)^{\text{th}}$ component is non-zero. The solutions $\mathbf{\Xi}_j$ respect the symmetries of the linear problem induced by the transformations $\widehat\Omega$, $\widehat S$:

$$\mathbf{\Xi}_j\left(\rho,\phi+\tfrac{2\pi}{nM},\theta-\tfrac{2\pi i}{nM}\right)=\mathbf{\Xi}_j\left(\rho,\phi,\theta\right) \tag{4.12}$$

and

$$S\,\boldsymbol{\Xi}_j\left(\rho,\phi,\theta-\tfrac{2\pi i}{n}\right)=\exp\left(-g_j\tfrac{2\pi i}{n}\right)\boldsymbol{\Xi}_j\left(\rho,\phi,\theta\right)\,. \tag{4.13}$$

Linear independence of the set $\{\boldsymbol{\Xi}_0,\ldots,\boldsymbol{\Xi}_{n-1}\}$ follows from $\det\left(\boldsymbol{\Xi}_0,\ldots,\boldsymbol{\Xi}_{n-1}\right)=-1$.

On the other hand, in the large-ρ limit there exists a solution to the ODE (4.6) with asymptotic representation for $M>1/(n-1)$ given by

$$\psi\sim z^{-(n-1)M/2}\exp\left(-\lambda\,\frac{z^{M+1}}{M+1}-\lambda^{-1}\,\frac{\bar{z}^{M+1}}{M+1}\right)\,,\qquad\rho\to\infty\,. \tag{4.14}$$

This is the unique solution that decays fastest in the sector of the complex plane defined by $z=\rho\,\exp(i\phi)$ with $|\phi|<(n+1)\pi/n(M+1)$. Therefore in the limit $\rho\to\infty$ a solution to the linear problem (2.7) reads

$$\boldsymbol{\Psi}(\rho,\phi,\theta,\mathbf{g})\sim\left(\Psi_1,\ldots,\Psi_n\right)^{\mathrm{T}}\,\exp\left(-2\,\frac{\rho^{M+1}}{M+1}\cosh(\theta+i\phi(M+1))\right)\,, \tag{4.15}$$

where

$$\Psi_j=\exp\left(i\phi\,\frac{n-(2j-1)}{2}\,M\right)\,. \tag{4.16}$$

Expanding $\boldsymbol{\Psi}$ in the basis of solutions $\{\boldsymbol{\Xi}_0,\boldsymbol{\Xi}_1,\ldots,\boldsymbol{\Xi}_{n-1}\}$ yields

$$\boldsymbol{\Psi}(\rho,\phi,\theta,\mathbf{g})=\sum_{j=0}^{n-1}Q_j(\theta,\mathbf{g})\,\boldsymbol{\Xi}_j(\rho,\phi,\theta,\mathbf{g})\,. \tag{4.17}$$

4.2 Conformal limit

We now check that in the massless limit described below the differential equations (4.6) and (4.7) are consistent with the n^{th}-order differential equations of the relevant conformal quantum integrable models [6, 7]. It is convenient to define

$$x=z\,\mathrm{e}^{\frac{\theta}{M+1}}\,,\ \bar{x}=\bar{z}\,\mathrm{e}^{-\frac{\theta}{M+1}}\,,\ E=s^{nM}\mathrm{e}^{\frac{n\theta M}{M+1}}\,,\ \bar{E}=s^{nM}\mathrm{e}^{-\frac{n\theta M}{M+1}}\,. \tag{4.18}$$

The massless limit of the ODE (4.6) in terms of ψ is obtained by first taking $\bar{z}\to 0$ with z finite and small, then taking the limit $z\sim s\to 0$ while $\theta\to+\infty$. This process yields

$$\Big((-1)^{n+1}D_n(\mathbf{g})+p(x,E)\Big)\psi(x,E)=0\,,\quad p(x,E)=x^{nM}-E\,, \tag{4.19}$$

where the operator $D_n(\mathbf{g})$ defined in (4.10) is now a function of x. This is precisely the n^{th}-order ODE appearing in the massless $SU(n)$ ODE/IM correspondence [6, 7].

Similarly, sending the parameter $z\to 0$ in the ODE for $\bar{\psi}$ (4.7) while keeping $\bar{z}$ small but finite, then taking the limit $\bar{z}\sim s\to 0$ as $\theta\to-\infty$ we find

$$\Big((-1)^{n+1}D_n(\mathbf{g}^{\dagger})+p(\bar{x},\bar{E})\Big)\bar{\psi}(\bar{x},\bar{E})=0\,, \tag{4.20}$$

with $\mathbf{g}^{\dagger}=(g_0^{\dagger},g_1^{\dagger},\ldots,g_{n-1}^{\dagger})$ and $g_j^{\dagger}=n-1-g_{n-1-j}$. Equation (4.20) is the adjoint equation to (4.19). This equation also appears naturally in the $SU(n)$ ODE/IM correspondence [7].

4.3 Functional relations

By establishing a set of functional relations satisfied by functions of the $Q_j(\theta, \mathbf{g})$, we will obtain the A_{n-1} Bethe ansatz systems. Using the variables (4.18), we define solutions to (4.6) for integer k by

$$\psi_k(x, \bar{x}, E, \bar{E}, \mathbf{g}) = \omega^{(n-1)k/2} \psi(\omega^{-k}x, \omega^k \bar{x}, \omega^{-nkM}E, \omega^{nkM}\bar{E}, \mathbf{g}), \qquad \omega = \mathrm{e}^{\frac{2\pi i}{n(M+1)}} \tag{4.21}$$

with large-x asymptotics for fixed real $\bar{x}$

$$\psi_k \sim \omega^{k(n-1)(M+1)/2} \frac{x^{-M}}{i^{(n-1)/2}\sqrt{n}} \exp\left(-\omega^{-k(M+1)} \frac{x^{M+1}}{M+1} - \omega^{k(M+1)} \frac{\bar{x}^{M+1}}{M+1}\right). \tag{4.22}$$

These asymptotics are valid for $x \in \mathcal{S}_{k-\frac{n}{2}} \cup \cdots \cup \mathcal{S}_{k+\frac{n}{2}}$ where

$$\mathcal{S}_k : \left| \arg x - \frac{2k\pi}{n(M+1)} \right| < \frac{\pi}{n(M+1)}.$$

Moreover, each ψ_k is the unique solution that decays to zero fastest for large x within the Stokes sector $\mathcal{S}_k$. By construction $W[\psi_1, \dots, \psi_n] = 1$, so we have a basis of solutions to (4.6) and may write

$$\psi = \sum_{k=1}^{n} C^{(k)}(E, \bar{E}, \mathbf{g})\, \psi_k. \tag{4.23}$$

Since the next steps follow [7] closely we omit some of the details. Using the determinant relations given in [7], the general-n version of the functional relation (3.37) is

$$C^{(1)}(E, \bar{E}, \mathbf{g}) \prod_{j=0}^{n} W_1^{(j)} = \sum_{m=0}^{n-1} \left(\prod_{j=0}^{m-1} W_1^{(j)}\right) W_2^{(m)} W_0^{(m+1)} \left(\prod_{j=m+2}^{n} W_1^{(j)}\right), \tag{4.24}$$

where

$$W_k^{(m)} = W_{k,k+1,\dots,k+m}(x, \bar{x}, E, \bar{E}, \mathbf{g}). \tag{4.25}$$

The elimination of the dependence of (4.24) on $x, \bar{x}$ is achieved by expanding (4.24) in the alternative basis given by solutions to (4.6) that have small-x behaviour defined by the components of $\mathbf{\Xi}_j$ (4.11). The Wronskians $W_k^{(m)}$ may be written explicitly in terms of this basis (see (5.5) of [7]) using the bottom component of the expansion (4.17) of $\mathbf{\Psi}$. Let $Q^{(m)}(\omega^{-nMk}E, \omega^{-nMk}\bar{E}, \mathbf{g})$ denote the coefficient of the dominant term of $W_k^{(m)}$ as $x \to 0$. In particular, from (4.17) we have $Q_0 = Q^{(1)}$. Inserting the resulting expansions of $W_k^{(m)}$ into (4.24), we find the coefficient of the leading order term of (4.24) in the limit as $x \to 0$ reads:

$$C^{(1)}(\omega^{nM}E, s) \prod_{j=0}^{n} Q^{(j)}(E, s) =$$
$$\sum_{m=0}^{n-1} \left(\prod_{j=0}^{m-1} Q^{(j)}(E, s)\right) \omega^{\beta_{m+1}-\beta_m} Q^{(m)}(\omega^{-nM}E, s) Q^{(m+1)}(\omega^{nM}E, s) \left(\prod_{j=m+2}^{n} Q^{(j)}(E, s)\right) \tag{4.26}$$

where the dependence of all functions on $\mathbf{g}$ has been omitted, $\beta_m = \sum_{j=0}^{m-1} g_j - m(n-1)/2$ and $E\bar{E} = s^{2nM}$. The functional relation (4.26) is expected to coincide with the dressed vacuum form for one of the transfer matrix eigenvalues of the corresponding massive quantum integrable model. Related functional equations derived for the $\mathcal{W}_N$ conformal field theory appear in [19].

We make one more redefinition: $A^{(m)}(\omega^{-nM(n-1)/2}E,s) = Q^{(m)}(E,s)$. Finally, setting $E_k^{(m)}$ to denote a zero of $A^{(m)}(E,s)$ we obtain from (4.26) the A_{n-1} Bethe ansatz equations:

$$\frac{A^{(m-1)}(\omega^{-nM/2}E_k^{(m)},s)}{A^{(m-1)}(\omega^{nM/2}E_k^{(m)},s)}\,\frac{A^{(m)}(\omega^{nM}E_k^{(m)},s)}{A^{(m)}(\omega^{-nM}E_k^{(m)},s)}\,\frac{A^{(m+1)}(\omega^{-nM/2}E_k^{(m)},s)}{A^{(m+1)}(\omega^{nM/2}E_k^{(m)},s)} = -\omega^{-2\beta_m+\beta_{m-1}+\beta_{m+1}}\,. \tag{4.27}$$

5 Conclusions

In this paper we have demonstrated how the massive generalisation of the A_n ODE/IM correspondence of [7] can be constructed starting from the system of classical partial differential equations appearing in $A_n^{(1)}$ Toda field theory. Moreover, we have obtained functional relations satisfied by the massive Q functions and derived the corresponding Bethe ansatz systems.

Of immediate interest is to further study the analytic properties of the T and Q functions obtained in sections 3 and 4, and fully explore the integrable features. The Q functions considered here correspond to the vacuum eigenvalues of the corresponding quantum integrable model. It would be interesting to develop to all A_n models the recent work of Bazhanov and Lukyanov [20], which extends the massive sine Gordon/sinh Gordon correspondence to higher-level eigenvalues of the quantum integrable model.

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Note added – This paper is based on chapters 5 and 6 of [17], submitted by PA for the award of PhD in October 2013. As we were completing the article, we became aware of the preprint [21], which also reports results on the $A_n^{(1)}$ massive ODE/IM correspondence and extends the analysis to all simply laced affine Lie algebras and $A_{2n}^{(2)}, D_4^{(3)}$ and $G_2^{(1)}$.

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