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Complete List of Authors:	Fleischmann, Peter; University of Kent , School of Mathematics, Statistics and Actuarial Science Horan, Katherine; Universitiy of Kent
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THE FINITE UNIPOTENT GROUPS CONSISTING OF BIREFLECTIONS

KATHERINE HORAN AND PETER FLEISCHMANN

ABSTRACT. Let k be a field of characteristic p and V a finite dimensional k-vector space. An element $g \in \mathrm{GL}(V)$ is called a *bireflection* if it centralizes a subspace of codimension less than or equal to 2. It is known by a result of Kemper, that if for a finite p-group $G \leq \mathrm{GL}(V)$ the ring of invariants $\mathrm{Sym}(V^*)^G$ is Cohen-Macaulay, G is generated by bireflections. Although the converse is false in general, it holds in special cases e.g. for particular families of groups *consisting* of bireflections. In this paper we give, for $p > 2$, a classification of all finite unipotent subgroups of $\mathrm{GL}(V)$ consisting of bireflections. Our description of the groups is given explicitly in terms useful for exploring the corresponding rings of invariants. This further analysis will be the topic of a forthcoming paper.

0. INTRODUCTION

Let k be a field and V be a finite dimensional k-vector space. An element $1 \neq g \in \mathrm{GL}(V)$ is called a *pseudo-reflection* if it centralizes a hyperplane and g is called a *transvection* if it is a unipotent pseudo-reflection. Generalizing this notion, g is called a *bireflection* if it centralizes a subspace of codimension less or equal to 2. A finite subgroup $G \leq \mathrm{GL}(V)$ will be called a *(pseudo)-reflection* group, if it is generated by pseudo-reflections and will be called a *bireflection* group if it is generated by bireflections.

Pseudo-reflection groups arise prominently in the invariant theory of finite groups: let $G \leq \mathrm{GL}(V)$ be a finite linear group and $S := \mathrm{Sym}(V^*)$ the symmetric algebra over the dual space. Then, due to a classical result of Serre, the ring of invariants $R := \mathrm{Sym}(V^*)^G$ can only be a polynomial ring if G is generated by pseudo-reflections. The converse is also true if the characteristic of k does not divide the group order $|G|$, which for characteristic zero is an earlier result by Chevalley and Shephard-Todd and for $\mathrm{char}(k) > 0$ is again due to Serre.

Similarly, a result of Gordeev, Kac and Watanabe ([6]) says that if S^G is a complete intersection ring, then G is a bireflection group. The following theorem by Kemper is even stronger, if one restricts to finite unipotent groups:

Theorem 0.1. [7, 3.7] *Let $0 < p = \mathrm{char}(k)$ and $G \leq \mathrm{GL}(V)$ be a finite p-group such that S^G is a Cohen–Macaulay ring, then G is generated by bireflections.*

The converses do not hold for any of the above results, in particular it is an open question for which finite unipotent bireflection groups the ring S^G is indeed Cohen-Macaulay. Intuitively one can expect that further restrictions on the groups may imply such a converse for special families of bireflection groups. In [8, 8.2] Smith looks at the modular groups consisting entirely of reflections.

Proposition 0.2. [8, 8.2.18] *Let k be a field of characteristic $p \neq 0$, $G \leq \mathrm{GL}(V)$ such that every non identity element of G is a reflection. Then either V^G has dimension $n-1$ or $(V^*)^G$ has dimension $n-1$.*

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Using this it can be shown that all groups consisting of reflections have polynomial rings of invariants. Similarly, there is a class of unipotent groups *consisting of bireflections*, where the invariant ring is always Cohen–Macaulay:

Theorem 0.3. [2, 3.9.1] *Let $G \leq \mathrm{GL}(V)$, $\dim_k(V) = n$ then:*

1. *if $\dim_k(V^G) = n-1$, then $k[V]^G$ is a polynomial ring.*
2. *if $\dim_k(V^G) = n-2$, then $k[V]^G$ is Cohen–Macaulay.*

On the other hand, there are examples of groups consisting of bireflections which do not have a Cohen–Macaulay invariant ring (see [3], where $p = 2$ and the group in question is an example of what will be called a "two–row group" below).
A classification of all *irreducible* bireflection groups has been given by Guralnick and Saxl ([5]). However, in the modular case and in the light of Kemper's theorem we are particularly interested in (reducible) unipotent bireflection groups.

Motivated by the above problems in modular invariant theory, this paper considers all finite unipotent subgroups $G \leq \mathrm{GL}(V)$ which entirely consist of bireflections, i.e. such that g is a bireflection for all elements $g \in G$. We shall call such a group a *pure bireflection group*. We now define certain classes of groups with this property.

Definition 0.4. *Let $\mathcal{B} := \{v_1, \cdots, v_{n-1}, v_n\}$ be an ordered basis of V and let $G \leq \mathrm{GL}(V)$ be a finite unipotent subgroup. Then:*

1. *If $\dim_k(V^G) \geq n-2$ then G is called a* **two–column group** *on V. If $V^G = \langle v_3, \cdots, v_n \rangle$, then for every $g \in G$, $M_{\mathcal{B}}(g)$ is of the form* $\begin{pmatrix} 1 & & & \\ * & 1 & & \mathbf{0} \\ * & * & 1 & \\ * & * & 0 & 1 \end{pmatrix}$.
2. *If $\dim_k([G, V]) \leq 2$ then G is called a* **two–row group** *on V. If $[G, V] = \langle v_{n-1}, v_n \rangle$, then for every $g \in G$, $M_{\mathcal{B}}(g)$ is of the form* $\begin{pmatrix} 1 & & & \\ 0 & 1 & & \mathbf{0} \\ * & * & 1 & \\ * & * & * & 1 \end{pmatrix}$.
3. *If there exists $U \subset V$ such that $\dim_k(U) = n-1$ and $[G, U] \leq kv$ for some $v \in U^G$, then G is called a* **hook group** *on V with hyperplane U and line $L := kv$. If $L = kv_n \leq U = \langle v_2, \cdots, v_n \rangle$, then for every $g \in G$, $M_{\mathcal{B}}(g)$ is of the form* $\begin{pmatrix} 1 & & & \\ * & 1 & & \mathbf{0} \\ * & 0 & 1 & \\ * & * & * & 1 \end{pmatrix}$.

Comparing with the unipotent groups consisting of reflections (see Proposition 0.2) we might expect these to be the only types of unipotent pure bireflection group. There are however two exceptional types, which we will briefly describe now and discuss later in more detail: **Exceptional groups of type 1** are formally introduced in Definition 6.1 as subgroups of $\mathrm{GL}(V)$ with $\dim_k V \geq 5$. It is shown in section 5 that they are isomorphic to subgroups of U_3, the group of upper unitriangular matrices in $\mathrm{SL}_3(k)$ (see Proposition 5.5). **Exceptional groups of type 2** are introduced in Definition 2.4. They are elementary abelian, isomorphic to subgroups of $(k, +)^3$ acting on V of dimension $n \geq 6$ as described in Definition 6.1 and Proposition 6.3.

Now we can describe the main result of this paper, namely the classification of finite unipotent pure bireflection groups in characteristic $p > 2$:

Theorem 0.5 (Main Theorem)**.** *Let $p = \mathrm{char}(k) > 2$ and let $G \leq \mathrm{GL}(V)$ be a p-group consisting of bireflections. Then one of the following holds:*

1. *G is a two–row group.*
2. *G is a two–column group.*
3. *G is a hook group.*

(4) *G is an exceptional group of type one.*
(5) *G is an exceptional group of type two.*

Proof. This will be proved at the end of section 2, after the proof of 2.14. □

The paper is organized as follows: In section 1 we define our notation and establish elementary properties and formulae for transvections and bireflections, which will be used throughout the paper. Section 2 analyzes pure bireflection groups generated by two (or three) elements. It is here where the exceptional cases mentioned above first appear. The end of section 2 also contains the proof of theorem 0.5. Sections 3 and 4 describe the structure of the "standard" unipotent pure bireflection groups, i.e. the two-row, two-column and hook-groups. Similarly Sections 5 and 6 give a detailed analysis of the structure of exceptional groups. Notice that for applications in invariant theory an explicit description of the linear groups, beyond the purely group theoretic structure is essential. Section 7 analyzes maximal pure unipotent bireflection groups over finite fields. These results are useful for the analysis of corresponding rings of invariants, which is the topic of a forthcoming paper.

1. Basic Definitions and formulae

Let V be a finite dimensional k-vector space. We start with a definition linking subspaces of V with subspaces of the dual space V^*. Note that $G \le \mathrm{GL}(V)$ acts naturally on V^* by the rule $g(\lambda) = \lambda \circ g^{-1}$ for $\lambda \in V^*$.

Definition 1.1. *Let V be a vector space, $U \subseteq V$. We define*

$$U^\perp = \{\lambda \in V^* \mid \lambda(u) = 0 \text{ for all } u \in U\}.$$

For $g \in \mathrm{GL}(V)$, we write $\delta_g \in \mathrm{End}_k(V)$ for the map which takes $v \in V$ to $(g-1)(v)$. For a unipotent element $g \in \mathrm{GL}(V)$ the ***index*** *of g, $\mathrm{ind}(g)$, is the nilpotence-index of δ_g, that is $c \in \mathbb{N}$ such that $\delta_g^c = 0$, $\delta_g^{c-1} \neq 0$. The index of a group $G \le \mathrm{GL}(V)$ is defined to be $\mathrm{ind}(G) := \max\{\mathrm{ind}(g) | g \in G\}$. For $G \le \mathrm{GL}(V)$, $v \in V$ or V^* we define the stabilizer (or isotropy) subgroup of v to be $G_v := \{g \in G \mid g(v) = v\}$.*

The next lemma will be used many times to move between groups and their dual representations. The proof is straightforward linear algebra and therefore omitted.

Lemma 1.2. *For $G \le \mathrm{GL}(V)$ the following hold:*

(1) $[G, V^*]^\perp = V^G$ *and* $[G, V]^\perp = (V^*)^G$.
(2) $\dim_k([G, V^*]) = \dim_k(V) - \dim_k(V^G) = \mathrm{codim}(V^G)$.
(3) $V^G \le [G, V]$ *if and only if* $(V^*)^G \le [G, V^*]$.
(4) *the canonical map* $V \to V/V^G$ *induces an isomorphism* $[G, V]/[G, V]^G \cong [G, V/V^G]$.

Definition 1.3. *For $u \in V$ and $\gamma \in u^\perp := (ku)^\perp$ we set $t_u^\gamma \in GL(V)$ to be the transvection mapping $s \in V$ to $s + \gamma(s)u$.*

From now on k denotes a field of characteristic $p > 0$. We start by proving some general results for transvections which are the reflections of order p in a field of characteristic p (for related results see also [4, Lemma 1.3]):

Lemma 1.4. *For $u_1, u_2 \in V$, $\gamma_1 \in u_1^\perp$, $\gamma_2 \in u_2^\perp$:*

(1) *If $u_2 \in \ker(\gamma_1)$ then* $|t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}| = \begin{cases} p^2 & \text{if } p = 2 \text{ and } u_1 \notin \ker(\gamma_2) \\ p & \text{otherwise} \end{cases}$
(2) *If $u_1 \in \ker(\gamma_2)$, then $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2} = t_{u_2'}^{\gamma_2} t_{u_1}^{\gamma_1}$ where $u_2' = t_{u_1}^{\gamma_1}(u_2)$*
(3) $|t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}|$ *is a power of p if and only if either $\gamma_1(u_2) = 0$ or $\gamma_2(u_1) = 0$*
(4) *If $u_1 \in \ker(\gamma_2)$ then $t_{u_1}^{\gamma_1} t_{u_1}^{\gamma_2} = t_{u_1}^{\gamma_1+\gamma_2}$.*
(5) *If $u_2 \in \ker(\gamma_1)$ then $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_1} = t_{u_1+u_2}^{\gamma_1}$.*
(6) $t_{u_1}^{c\gamma_1} = t_{cu_1}^{\gamma_1}$ *for all $c \in k$*

Proof. 1.: Let $t = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$. For $p \neq 2$ we will show that if $u_2 \in \ker(\gamma_1)$ then for $a \in \mathbb{N}$, $w \in V$: $t^a(w) = w + a\gamma_1(w)u_1 + a\gamma_2(w)u_2 + \frac{a(a-1)}{2}\gamma_1(w)\gamma_2(u_1)u_2$. We do this by induction- it is clear for $a = 1$, and then:

$$t^a(w) = tt^{a-1}(w) = t(w + (a-1)\gamma_1(w)u_1 + (a-1)\gamma_2(w)u_2 + \frac{(a-1)(a-2)}{2}\gamma_1(w)\gamma_2(u_1)u_2) =$$

$w + a\gamma_1(w)u_1 + a\gamma_2(w)u_2 + \frac{a(a-1)}{2}\gamma_1(w)\gamma_2(u_1)u_2$, so $|t| = p$. If $p = 2$ we can see that $t^2(w) = w + \gamma_1(w)\gamma_2(u_1)u_2$, so either $t^2 = 1$ or t^2 is a transvection with order 2, so $(t^2)^2 = t^4 = 1$ and t has order p^2.
2.: Let $w \in V$ so:

$$t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}(w) = t_{u_1}^{\gamma_1}(w + \gamma_2(w)u_2) = w + \gamma_1(w)u_1 + \gamma_2(w)u_2 + \gamma_2(w)\gamma_1(u_2)u_1 =$$

$(w + \gamma_1(w)u_1) + \gamma_2(w)(u_2 + \gamma_1(u_2)u_1) = t_{u_2'}^{\gamma_2} t_{u_1}^{\gamma_1}$, where u_2' is as given above.
3.: We can see using the first two parts that if $\gamma_1(u_2) = 0$ or $\gamma_2(u_1) = 0$, then $|t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}|$ is a power of p. Let $t = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$, if $|t|$ is power of p, then $[t, V]^t \neq \{0\}$. If $u_2 \in ku_1$ then we already know that $\gamma_1(u_2) = \gamma_2(u_1) = 0$, so assume u_1, u_2 linearly independent. We can see that $[t, V] \leq \langle u_1, u_2 \rangle$, so we can find $a_1, a_2 \in k$ not both zero such that $a_1u_1 + a_2u_2 \in [t, V]^t$: $a_1u_1 + a_2u_2 = t(a_1u_1 + a_2u_2) = (a_1 + a_2\gamma_1(u_2) + a_1\gamma_2(u_1)\gamma_1(u_2))u_1 + (a_2 + a_1\gamma_2(u_1))u_2$. Comparing u_2 terms we see that $a_1\gamma_2(u_1) = 0$, so either $\gamma_2(u_1) = 0$ or $a_1 = 0$. If $a_1 = 0$ then $a_2 \neq 0$ and comparing u_2 terms $a_2\gamma_1(u_2) = 0$ so $\gamma_1(u_2) = 0$.
4.+5.: For all $v \in V$,

$$t_{u_1}^{\gamma_1} t_{u_1}^{\gamma_2}(v) = t_{u_1}^{\gamma_1}(v + \gamma_2(v)u_1) = v + (\gamma_1(v) + \gamma_2(v))u_1 = v + (\gamma_1 + \gamma_2)(v)u_1 = t_{u_1}^{\gamma_1+\gamma_2}(v),$$

so $t_{u_1}^{\gamma_1} t_{u_1}^{\gamma_2} = t_{u_1}^{\gamma_1+\gamma_2}$. Similarly for all $v \in V$, $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_1}(v) = t_{u_1}^{\gamma_1}(v + \gamma_1(v)u_1) = v + \gamma_1(v)u_1 + \gamma_1(v)u_2 = v + \gamma_1(v)(u_1 + u_2) = t_{u_1+u_2}^{\gamma_1}$, so $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_1} = t_{u_1+u_2}^{\gamma_1}$.
6.: For any $c \in k$ and $v \in V$ $t_{u_1}^{c\gamma_1}(v) = v + c\gamma_1(v)(u_1) = v + \gamma_1(v)(cu_1) = t_{cu_1}^{\gamma_1}$. ☐

Later we will want to write bireflections as products of transvections. The next lemma will be useful when rewriting and comparing them.

Lemma 1.5. *Let $m \in \mathbb{N}$, $\gamma_1, \dots, \gamma_m \in V^*$ and $u_1, \dots, u_m$ such that $\gamma_i(u_j) = 0$ for $1 \leq i \leq j \leq m$. Let $g = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2} \dots t_{u_m}^{\gamma_m}$, then the following hold:*

1. *If $\gamma_1, \dots, \gamma_m$ are linearly independent and $h = t_{u_1'}^{\gamma_1} t_{u_2'}^{\gamma_2} \dots t_{u_m'}^{\gamma_m}$ such that $\gamma_i(u_j') = 0$ for $1 \leq i \leq j \leq m$, then $g = h$ if and only if $u_i' = u_i$ for $1 \leq i \leq m$.*
2. *If $u_1, \dots, u_m$ are linearly independent and $h = t_{u_1}^{\gamma_1'} t_{u_2}^{\gamma_2'} \dots t_{u_m}^{\gamma_m'}$ such that $\gamma_i'(u_j) = 0$ for $1 \leq i \leq j \leq m$. Then $g = h$ if and only if $\gamma_i' = \gamma$ for $1 \leq i \leq m$.*

Proof. 1.: As the γ_i's are linearly independent for $1 \leq i \leq n$ we can find $v_i \in V$ such that $V = \langle v_i \mid 1 \leq i \leq n \rangle$ and for $1 \leq j \leq m$ $\gamma_j(v_i) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$ We can see that $g = h$ if and only if $g(v_i) = h(v_i)$ for $1 \leq i \leq n$. For $m + 1 \leq i \leq n$ we have $v_i \in V^g \cap V^h$ so $g(v_i) = h(v_i)$. For $1 \leq i \leq m$ $g(v_i) = v_i + u_i$, $h(v_i) = v_i + u_i'$, so $g(v_i) = h(v_i)$ if and only if $u_i = u_i'$.
2.: Clearly if $\gamma_i = \gamma_i'$ for $1 \leq i \leq m$ then $g = h$. For any $v \in V$ $\delta_g(v) = \sum_{i=1}^m \gamma_i(v)u_i$ $\delta_h(v) = \sum_{i=1}^m \gamma_i'(v)u_i$. As the u_i are linearly independent if $g(v) = h(v)$ then we can equate coefficients and $\gamma_i(v) = \gamma_i'(v)$ for $1 \leq i \leq m$. If this holds for all $v \in V$ then $\gamma_i = \gamma_i'$ and $g = h$. ☐

The following can be used to check the commutator and fixed spaces of elements of $\mathrm{GL}(V)$ to see if they are bireflections.

Lemma 1.6. *Let $g, h \in \mathrm{GL}(V)$ be unipotent, $w \in V$. Then we have:*

1. $\delta_{gh}(w) = \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w)$.
2. $\delta_{g^i}(w) = \sum_{j=1}^i \binom{i}{j} \delta_g^j(w)$.
3. *If g is a bireflection then $\delta_{g^i}(w) = i\delta_g(w) + \frac{i(i-1)}{2}\delta_g^2(w)$.*

Proof. 1.: For $g, h \in G$, $w \in V$: $gh(w) = g(w + \delta_h(w)) = w + \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w)$, hence $\delta_{gh}(w) = \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w)$.
2.+3.: For $i = 1$ the second result is trivial, we now proceed by induction. If $\delta_{g^{i-1}}(w) = \sum_{j=1}^{i-1}\binom{i-1}{j}\delta_g^j(w)$, then using the first result: $\delta_{g^i}(w) = \delta_g(w) + \delta_{g^{i-1}}(w) + \delta_g\delta_{g^{i-1}}(w) = \delta_g(w) + \sum_{j=1}^{i-1}\binom{i-1}{j}\delta_g^j(w) + \delta_g\sum_{l=1}^{i-1}\binom{i-1}{l}\delta_g^l(w) = i\delta_g(w) + \sum_{j=2}^{i-1}\left(\binom{i}{j} + \binom{i}{j-1}\right)\delta_g^j + \delta_g^i(w) = \sum_{j=1}^{i}\binom{i}{j}\delta_g^j(w)$. If g is a bireflection then $\delta_g^j(v) = 0$ for any $j > 2$ and $v \in V$, so we get: $\delta_{g^ih}(w) = i\delta_g + \frac{i(i-1)}{2}\delta_g^2(w)$. □

From this we easily see that if G is a unipotent transvection group then G is abelian if and only if $[G, V] \leq V^G$ (see also [4, Lemma 3.5]). All reflections are bireflections hence reflection groups are bireflection groups and unipotent bireflections include transvections: $\begin{pmatrix} \mathbf{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-2} \end{pmatrix}$ as well as those elements of $\mathrm{GL}(V)$ congruent to one of $\begin{pmatrix} \mathbf{J}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-3} \end{pmatrix}$, an "*index* 3 *bireflection*, or $\begin{pmatrix} \mathbf{J}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n-4} \end{pmatrix}$, a "*double transvection*". Here $\mathbf{J}_2$, $\mathbf{J}_3$ are Jordan 2 and 3 blocks respectively.
If g is a unipotent bireflection it can be written as either t_u^γ for some $u \in V$ with $\gamma \in u^\perp$ in the case of a transvection, or as $t_{u_1}^{\gamma_1}t_{u_2}^{\gamma_2}$ for some $u_1, u_2 \in V$, $\gamma_1 \in u_1^\perp$, $\gamma_2 \in u_2^\perp$ with $\gamma_1(u_2) = 0$. If $\gamma_2(u_1) \neq 0$ it is an index 3 bireflection, if $\gamma_2(u_1) = 0$ then it is a double transvection.

2. Special configurations

Definition 2.1. *Let $n \geq 5$ and $G \leq \mathrm{GL}(V)$ a unipotent group. Let $g, h \in G$ be bireflections and $U = V^g + V^h$. We call g, h a* **special pair** *if we can find $r_1, r_2, v \in V$ linearly independent such that the following hold:*

$$\dim_k(U) = n-1, \quad \dim_k(V^g \cap V^h) = n-3, \ v \notin U, \quad r_1, r_2 \in V^g \cap V^h \text{ and}$$
$$\delta_g(U) = kv, \quad \delta_g(v) = r_2, \ \delta_h(U) = k(v + r_1), \quad \delta_h(v) = 2r_1 + r_2.$$

If $g, h \in G$ form a special pair and G is a pure bireflection group then we call G an **exceptional pure bireflection group** *(or exceptional group)* **of type one**, *and g, h an* **exceptional pair**.

Lemma 2.2. *For $g, h \in \mathrm{GL}(V)$ the following are equivalent:*

(1) *g, h form a special pair;*
(2) *$g = t_{\hat{v}}^{\zeta_1}t_{\hat{r}_2}^{\hat{v}^*}$ for $\zeta_1, \hat{v}^* \in V^*$ linearly independent, $\hat{r}_2, \hat{v} \in \ker(\zeta_1)$ such that*
$$\hat{v}^*(\hat{r}_2) = 0, \quad \hat{v}^*(\hat{v}) = 1$$
and we can find $a, b \in k$, $\hat{r}_1 \in \ker(\zeta_1) \cap \ker(\hat{v}^*)$ and some $\zeta_2 \in V^*$ linearly independent to ζ_1 and $\hat{v}^*$ such that $\zeta_2(\hat{r}_2) = \zeta_2(\hat{r}_1) = \zeta_2(\hat{v}) = \zeta_1(\hat{r}_1) = 0$ and $h = t_{\beta_1}^{\zeta_1}t_{\beta_2}^{\zeta_2}t_{\beta_3}^{\hat{v}^*}$ where
$$\beta_1 = b\hat{v} + (a - ab)\hat{r}_2 + (2a + b)\hat{r}_1, \ \beta_2 = v - a\hat{r}_2 + \hat{r}_1 \text{ and } \beta_3 = 2\hat{r}_1 + \hat{r}_2.$$
(3) *we can find some $\gamma_1, \gamma_2, v^* \in V^*$ linearly independent, and $r_1, r_2, v \in \ker(\gamma_1) \cap \ker(\gamma_2)$ linearly independent, with $v^*(r_1) = v^*(r_2) = 0$ and $v^*(v) = 1$, such that $g = t_v^{\gamma_1}t_{r_2}^{v^*}$ and $h = t_{v+r_1}^{\gamma_2}t_{2r_1+r_2}^{v^*}$.*

Proof. 1) ⇒ 2) Suppose g, h are a special pair. Then g is an index 3 bireflection so we can find $\zeta_1, \hat{v}^* \in V^*$ linearly independent, $\hat{r}_2, \hat{v} \in \ker(\gamma_1)$ such that $\hat{v}^*(\hat{r}_2) = 0$, $\hat{v}^*(\hat{v}) = 1$ and $g = t_{\hat{v}}^{\zeta_1}t_{\hat{r}_2}^{\hat{v}^*}$. Let $u_1 \in \ker(\hat{v}^*)$ such that $\zeta_1(u_1) = 1$. As g, h are a special pair if $U = V^g + V^h$ then $\dim_k(U) = n-1$. As V^g is $n-2$ dimensional and $\hat{v}, u_1 \notin V^g$ are linearly independent we can see that $V = ku_1 + k\hat{v} + V^g$. Therefore can find some $a', a \in k$ such that if $u = a'u_1 - a\hat{v}$ then $U = ku + V^g$. As $\delta_g(U) \not\leq V^g$ and $\delta_g(\hat{v}) \in V^g$, we see $a' \neq 0$, so we can assume $a' = 1$. We can now find $v = \delta_g(U)$, let $v = \delta_g(u) = \delta_g(u_1 - a\hat{v}) = \hat{v} - a\hat{r}_2$. By the definition of a special pair we can find $\hat{r}_1 \in V^g \cap V^h$ such that $\delta_h(U) = k(v + \hat{r}_1)$. Since $\delta_g(v) = \delta_g(\hat{v} - a\hat{r}_2) = \delta_g(\hat{v})$ we know that $\hat{r}_2 \in V^g \cap V^h$ and $\delta_h(v) = \delta_h(\hat{v} - a\hat{r}_2) = \delta_h(\hat{v}) = 2\hat{r}_1 + \hat{r}_2$. Let $u_2 \in \ker(\zeta_1) \cap \ker(\hat{v}^*)$ such that $\delta_h(u_2) = v + \hat{r}_1$. We know $u_2 \in U \backslash V^h$, so $U = ku_2 + V^h$. We know that $u = u_1 - a\hat{v} \in U$ so we can find some $b \in k$ such that $u_1 - a\hat{v} - bu_2 \in V^h$ and $\delta_h(u_1 - a\hat{v} - bu_2) = 0$. Therefore $\delta_h(u_1) = bv + (a - ab)r_2 + (2a + b)r_1$. Let $\zeta_2 \in V^*$ such that $\ker(\zeta_2) = V^g \cap V^h + ku_1 + k\hat{v}$ and $\zeta_2(u_2) = 1$. Let $\beta_1 = b\hat{v} + (a-ab)\hat{r}_2 + (2a+b)\hat{r}_1$, $\beta_2 = v - a\hat{r}_2 + \hat{r}_1$, $\beta_3 = 2\hat{r}_1 + \hat{r}_2$ and $\tilde{h} = t^{\zeta_1}_{\beta_1} t^{\zeta_2}_{\beta_2} t^{\hat{v}^*}_{\beta_3}$. We find $\delta_{\tilde{h}}(u_1) = u_1 + b\hat{v} + (a - ab)\hat{r}_2 + (2a + b)\hat{r}_1 = \delta_h(u_1)$, $\delta_{\tilde{h}}(u_2) = u_2 + \hat{v} - a\hat{r}_2 + \hat{r}_1 = \delta_h(u_2)$, $\delta_{\tilde{h}}(\hat{v}) = \hat{v} + 2\hat{r}_1 + \hat{r}_2 = \delta_h(\hat{v})$, and $\delta_{\tilde{h}}(V^g \cap V^h) = 0 = \delta_h(V^g \cap V^h)$, so $h = \tilde{h}$ as required.

2) $\Rightarrow$ 3) Let g, h be as described in part 2). Then we see that $\beta_1 = b\beta_2 + a\beta_3$ and $g = t^{\zeta_1}_{\hat{v}} t^{\hat{v}^*}_{\hat{r}_2} =$

$$t^{\zeta_1}_{\hat{v}} t^{\zeta_1}_{-a\hat{r}_2} t^{\zeta_1}_{a\hat{r}_2} t^{\hat{v}^*}_{\hat{r}_2} = t^{\zeta_1}_{\hat{v}-\hat{r}_2} t^{\hat{v}^*+a\zeta_1}_{\hat{r}_2}, \;\; h = t^{\zeta_1}_{b\beta_2+a\beta_3} t^{\zeta_2}_{\beta_2} t^{\hat{v}*}_{\beta_3} = t^{b\zeta_1}_{\beta_2} t^{a\zeta_1}_{\beta_3} t^{\zeta_2}_{\beta_2} t^{\hat{v}*}_{\beta_3} = t^{\zeta_2+b\zeta_1}_{\beta_2} t^{\hat{v}^*+a\zeta_1}_{\beta_3}$$

(using Lemma 1.4). Setting $\gamma_1 = \zeta_1, \gamma_2 = \zeta_2 + b\zeta_1, v^* = \hat{v}^* + a\zeta_1,\; r_1 = \hat{r}_1, r_2 = \hat{r}_2, v = \hat{v} - a\hat{r}_2$, we can write $g = t^{\gamma_1}_v t^{v^*}_{r_2}$, $h = t^{\gamma_2}_{v+r_1} t^{v^*}_{2r_1+r_2}$, so they are in the form required.

3) $\Rightarrow$ 1) Let g, h be as described in part 3). As v, r_2 and $v+r_1, 2r_1+r_2$ are linearly independent, $V^g = \ker(\gamma_1) \cap \ker(v^*)$ and $V^h = \ker(\gamma_2) \cap \ker(v^*)$ so $V^g + V^h = \ker(v^*) \cap \ker(\gamma_1) + \ker(v^*) \cap \ker(\gamma_2) =$

$$\ker(v^*) \cap (\ker(\gamma_1) + \ker(\gamma_2)) = \ker(v^*) \cap V = \ker(v^*),$$

which has dimension $n - 1$. We also find $V^g \cap V^h = \ker(\gamma_1) \cap \ker(\gamma_2) \cap \ker(v^*)$ and check it has the correct dimension

$$\dim_k(V^g \cap V^h) = \dim_k(V^g) + \dim_k(V^h) - \dim_k(V^g + V^h) = n - 2 + n - 2 - n + 1 = n - 3.$$

We can see that $\delta_g(U) = kv$, $\quad \delta_g(v) = r_2$, $\delta_h(U) = k(v + r_1)$, $\quad \delta_h(v) = 2r_1 + r_2$, so g, h are a special pair. $\square$

We now check that exceptional groups of type one do exist.

Lemma 2.3. *If G is generated by a special pair then G is an exceptional group of type one. Moreover, for $p \neq 2$, $G \cong M(p)$ is an extraspecial group of order p^3.*

Proof. Let $g, h \in \mathrm{GL}(V)$ be a special pair and $G = \langle g, h \rangle$. By Lemma 2.2 we can find some $\gamma_1, \gamma_2, v^* \in V^*$ linearly independent, and $r_1, r_2, v \in \ker(\gamma_1) \cap \ker(\gamma_2)$ linearly independent with $v^*(r_1) = v^*(r_2) = 0$, $v^*(v) = 1$, such that we can write: $g = t^{\gamma_1}_v t^{v^*}_{r_2}$, $h = t^{\gamma_2}_{v+r_1} t^{v^*}_{2r_1+r_2}$. Using this and Lemma 1.4 we can find the commutator $z = ghg^{-1}h^{-1}$: $ghg^{-1}h^{-1} =$

$$t^{\gamma_1}_v t^{v^*}_{r_2} t^{\gamma_2}_{v+r_1} t^{v^*}_{2r_1+r_2} t^{v^*}_{-r_2} t^{\gamma_1}_{-v} t^{v^*}_{-2r_1-r_2} t^{\gamma_2}_{-v-r_1} = t^{\gamma_1}_v t^{\gamma_1}_{-v-2r_1-r_2} t^{\gamma_2}_{v+r_1+r_2} t^{\gamma_2}_{-v-r_1} t^{v^*}_{r_2} t^{v^*}_{2r_1} t^{v^*}_{-2r_1-r_2} =$$

$t^{\gamma_1}_{-2r_1-r_2} t^{\gamma_2}_{r_2}$ and see that z commutes with g and h. As G is a p-group, $\Phi(G) = G^p[G, G]$. Suppose $p \neq 2$. As g and h are index 3 bireflections, $g^p = h^p = 1$, so $\langle z \rangle = Z(G) = \Phi(G)$. Knowing this and using Lemma 1.4 we see that any $\sigma \in G$ can be written as: $\sigma = g^l h^m z^n = (t^{\gamma_1}_v t^{v^*}_{r_2})^l (t^{\gamma_2}_{v+r_1} t^{v^*}_{2r_1+r_2})^m (t^{\gamma_1}_{-2r_1-r_2} t^{\gamma_2}_{r_2})^n = t^{\gamma_1}_{\alpha_1} t^{\gamma_2}_{\alpha_2} t^{v^*}_{\alpha_3}$ for some $0 \leq l, m, n \leq p - 1$, where: $\alpha_1 = lv - 2nr_1 + \frac{l(l-1)-2n}{2} r_2$, $\alpha_2 = mv + m^2 r_1 + \frac{m(m-1+2l)+2n}{2} r_2$, $\alpha_3 = 2mr_1 + (m + l)r_2$. We find that: $0 = 2m\alpha_1 - 2l\alpha_2 + (2n - lm)\alpha_3$ hence G is an extra special group consisting of bireflections with $|G| = p^3$. As all $\sigma \in G$ have order p, we see that $G \cong M(p)$. For $p = 2$ we find that $g = t^{\gamma_1}_v t^{v^*}_{r_2}$, $h = t^{\gamma_2}_{v+r_1} t^{v^*}_{r_2}$ are still index 3 bireflections, therefore $ghg^{-1}h^{-1} = g^2 h^2 = t^{\gamma_1}_{r_2} t^{\gamma_2}_{r_2} \in Z(G)$. Let $z_1 = g^2$ and $z_2 = h^2$ then $z_1, z_2 \in Z(G)$. Hence any $\sigma \in G$ can be written as $\sigma = g^{a_1} h^{a_2} z_1^{a_3} z_2^{a_4} = t^{\gamma_1}_{\alpha_1} t^{\gamma_2}_{\alpha_2} t^{v^*}_{\alpha_3}$, where $a_1, \ldots, a_4 \in \mathbb{F}_2$ and $\alpha_1 = a_1 v + a_3 r_2$, $\alpha_2 = a_2 v + a_2 r_1 + a_4 r_2$, $\alpha_3 = (a_1 + a_2) r_2$. We see that if $a_1 = a_2$ then $\alpha_3 = 0$, otherwise we must have $a_i = 0$, for either $i = 1$ or $i = 2$, in which case $\alpha_i \in k\alpha_3$. In any of these cases we see that σ is a bireflection, and G is a pure bireflection group. $\square$

We see from above that exceptional groups of type one look quite different when $p = 2$. This is not the case for our next family of exceptional pure bireflection groups.

Definition 2.4. *Let $G \leq \mathrm{GL}(V)$ be a unipotent group with $g_1, g_2, g_3 \in G$. We call g_1, g_2, g_3 a* **special triple** *if there exists $r_1, r_2, r_3 \in V$, $\gamma_1, \gamma_2, \gamma_3 \in r_1^{\perp} \cap r_2^{\perp} \cap r_3^{\perp}$ with $\dim_k(r_1, r_2, r_3) = \dim_k(\gamma_1, \gamma_2, \gamma_3) = 3$ and we can find $f \in k$ such that $g_1 = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2}$, $g_2 = t^{\gamma_1}_{r_3} t^{\gamma_3}_{r_2}$, $g_3 = t^{\gamma_2}_{fr_3} t^{\gamma_3}_{-fr_1}$.*

If G is a pure bireflection group then we call G an **exceptional pure bireflection group** *(or exceptional group)* **of type two**, *and g_1, g_2, g_3 an* **exceptional triple**.

Note that special triples g_1, g_2, g_3 have the property that the group generated by any pair $\langle g_i, g_j\rangle$ with $1 \leq i < j \leq 3$ is a hook group, so they are not an extension on exceptional groups of type one. Again with exceptional groups of type two we need to check that these groups exist.

Proposition 2.5. *A group G which is generated by a special triple is an exceptional group of type 2, moreover G is elementary abelian of order p^3.*

Proof. Let g_1, g_2, g_3 be a special triple, so for some $r_1, r_2, r_3 \in V$, $\gamma_1, \gamma_2, \gamma_3 \in r_1^\perp \cap r_2^\perp \cap r_3^\perp$, $f \in k$: $g_1 = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2}$, $g_2 = t_{r_3}^{\gamma_1} t_{r_2}^{\gamma_3}$, $g_3 = t_{fr_3}^{\gamma_2} t_{-fr_1}^{\gamma_3}$. From their definitions we can see g_1, g_2, g_3 commute, so for any $\sigma \in G$: $\sigma = g_1^a g_2^b g_3^c = t_{\alpha_1}^{\gamma_1} t_{\alpha_2}^{\gamma_2} t_{\alpha_3}^{\gamma_3}$ with $\alpha_1 = ar_1 + br_3$, $\alpha_2 = ar_2 + cfr_3$, $\alpha_3 = br_2 - cfr_1$. So $cf\alpha_1 = b\alpha_2 - a\alpha_3$ and for all $\sigma \in G$, σ is a bireflection. Therefore G is an exceptional group of type two, which is an abelian group of order p^3. □

If $G \leq \mathrm{GL}(V)$ is a pure bireflection group then the dual representation is also a pure bireflection group. Since the dual representation of a group is represented by the transpose inverse matrices (with regard to the dual basis), it follows that the dual representation of a hook group is also a hook group, and similarly for exceptional groups of types 1 and 2. However the dual of a two–row group is a two–column group (and visa versa).

We will show that the above are the only types of pure unipotent bireflection groups for $p \neq 2$, to do this we will make regular use of Proposition 1.6. First we show that an index 3 bireflection defines a unique hyperplane and line for any hook group containing it.

Lemma 2.6. *Let $G \leq \mathrm{GL}(V)$, and $g \in G$ an index three bireflection so we can find $\gamma_1, \gamma_2 \in V^*$ and $u_1, u_2 \in V$ linearly independent such that $\gamma_1(u_1) = \gamma_1(u_2) = \gamma_2(u_2) = 0$, $\gamma_2(u_1) \neq 0$ and $g = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$. If G is a hook group then it has hyperplane $U = \ker(\gamma_1)$ and line $ku_2 = [g, [g, V]]$.*

Proof. Let G be a hook group with hyperplane U. If $v \notin \ker(\gamma_1)$ then $\delta_g(v) \notin V^g$ so $v \notin U$, so $U \leq \ker(\gamma_1)$. As $\dim_k(U) = n - 1 = \dim_k \ker(\gamma_1)$, we see that $U = \ker(\gamma_1)$. As $\delta_g(U) = ku_2$ we see that the line of G must be ku_2. As $[g, V] = \langle u_1, u_2\rangle$ we see that $[g, [g, V]] = ku_2$. □

We now begin to look at pure bireflection groups generated by two elements.

Lemma 2.7. *Let $G = \langle g, h\rangle$ be a pure unipotent bireflection group and not a two–column or two–row group. Then $U = V^g + V^h < V$ is a hyperplane with $\dim_k(\delta_g(U)) = \dim_k(\delta_h(U)) = 1$. Furthermore G is a hook group if and only if $\delta_g(U) = \delta_h(U) \leq U$.*

Proof. As h is a bireflection $\dim_k(V^h) \geq n - 2$, however as G isn't a two–column group $\dim_k(V^g \cap V^h) < n - 2$. Hence $V^h \neq V^g \cap V^h$, so $V^h \not\leq V^g$. As g is also a bireflection $1 \leq \dim_k(\delta_g(V^h)) \leq 2$. Suppose $\dim_k(\delta_g(V^h)) = 2$ then $\delta_g(V^h) = [g, V] \leq [gh, V]$. For any $w \in V$ $\delta_{gh}(w) = \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w) \in [gh, V]$. As $\delta_g(w), \delta_g\delta_h(w) \in [g, V]$ this means that $\delta_h(w) \in [gh, V]$, hence $[h, V] \leq [gh, V]$. However this would mean that $\dim_k([gh, V]) = \dim_k([g, V] + [h, V]) > 2$ and gh is not a bireflection. So $\dim_k(\delta_g(V^h)) = \dim_k(\delta_h(V^g)) = 1$. Let $U = V^g + V^h$, then $(\delta_g(U) + \delta_h(U)) \leq 2$, so $U \neq V$. We can also see that: $\dim_k(V^g + V^h) = \dim_k(V^g) + \dim_k(V^h) - \dim_k(V^g \cap V^h) > (n-2) + (n-2) - (n-2) = n-2$, so $\dim_k(U) = n-1$. If $\delta_g(U) = \delta_h(U) \leq U$ then G is a hook group with hyperplane U and line $\delta_g(U)$. Suppose G is a hook group with hyperplane U' and line $kv \leq U'$. If $V^g \not\leq U'$ we can find some $u \in V^g \backslash U'$ such that $V = U' + ku$. Then $[g, V] = [g, U'] = kv$, $[h, V] = [h, ku] + [h, U'] = k\delta_h(u) + kv$. This would mean that $[G, V] = [g, V] + [h, V] = k\delta_h(u) + kv$ but G is not a two–row group; hence we must have $V^g \leq U'$. Similarly $V^h \leq U'$, so $V^g + V^h = U \leq U'$. As $\dim_k(U) = \dim_k(U')$ this means that $U = U'$ and $\delta_g(U) = \delta_h(U) = kv \leq U$. □.

The next two lemmas give conditions under which a group generated by two elements is either a two–row, two–column or hook group. It is here we start restricting to odd characteristic.

Lemma 2.8. *Let $p > 2$ and let $G = \langle g, h\rangle$ be a pure unipotent bireflection group with $U = V^g + V^h$. If $\delta_h(U) \leq U$ then G is either a hook, two–row or two–column group.*

Proof. Assume G is not a two–row or two–column group. We have shown in Lemma 2.7 that $U = V^g + V^h < V$ is a hyperplane and $\dim_k(\delta_g(U)) = \dim_k(\delta_h(U)) = 1$. Let $u_1, u_2 \in V$ such that: $\delta_g(U) = ku_1,\ \delta_h(U) = ku_2$ and choose some $v \in V\backslash U$ so $V = U + kv$. Assume $\delta_h(U) \subseteq U$, $u_2 \in U$. Therefore $\delta_h(u_2) = a_1 u_2$ for some $a_1 \in k$. Since δ_h is nilpotent $a_1 = 0$ and $u_2 \in V^h$. Similarly if $\delta_h(v) = a_2 v + r$ with $a_2 \in k$ and $r \in U$ then $a_2 = 0$, $\delta_h(v) \in U$ so $[h, V] \leq U$. We look at $gh \in G$. Let $u \in V^h\backslash V^g$ and $u' \in V^g\backslash V^h$, then: $\delta_{gh}(u) = \delta_g(u)$, $\delta_{gh}(u') = \delta_g\delta_h(u') + \delta_h(u')$. We can see that $\delta_g(u), \delta_g\delta_h(u') \in ku_1$ and $\delta_h(u') \in ku_2$. Since $\delta_g(u)$ and $\delta_h(u')$ are non-zero $ku_1 + ku_2 \subseteq [gh, V]$. Suppose that $\dim_k(ku_1 + ku_2) = 1$. Then $ku_1 = ku_2 \leq V^g \cap V^h$ and G is a hook group with hyperplane U and line ku_1. Assume $\dim_k(ku_1 + ku_2) = 2$. Then: $[gh, V] = ku_1 + ku_2$. From this we know: $\delta_{gh}(v) = \delta_g(v) + \delta_h(v) + \delta_g\delta_h(v) \in ku_1 + ku_2$. As $[h, V] \leq U$, we must have $\delta_g\delta_h(v) \leq ku_2$, so for some $c_1, c_2 \in k$: $\delta_g(v) + \delta_h(v) = c_1u_1 + c_2u_2$, $\delta_h(v) = c_1u_1 + c_2u_2 - \delta_g(v)$. As $[G, V] = [g, V] + [h, V] = \langle u_1, u_2, \delta_g(v), \delta_h(v)\rangle$ has dimension greater than two the set $\{u_1, u_2, \delta_g(v)\}$ must be linearly independent. Looking at the action of gh^i on U for $2 \leq i \leq p-1$ we find that: $\delta_{gh^i}(u) = \delta_g(u)$, $\delta_{gh^i}(u') = i\delta_h(u') + i\delta_g\delta_h(u')$. We see that $\delta_g(u), \delta_g\delta_h(u') \in ku_1$ and $\delta_h(u') \in ku_2$ so: $[g^ih, V] = ku_1 + ku_2$. Using Lemma 1.6 we find $\delta_{gh^i}(v) = \delta_g(v) + i\delta_h(v) + \frac{i(i-1)}{2}\delta_h^2(v) + i\delta_g\delta_h(v) + \frac{i(i-1)}{2}\delta_g\delta_h^2(v) \in ku_1 + ku_2$. As $[h, V] \leq U$ we can see that: $\delta_h^2(v) \in ku_2,\ \delta_g\delta_h(v), \delta_g\delta_h^2(v) \in ku_1$, so for some $b_1, b_2 \in k$ $\delta_g(v) + i\delta_h(v) = b_1u_1 + b_2u_2$. Substituting in $\delta_h(v) = c_1u_1 + c_2u_2 - \delta_g(v)$, $(i-1)\delta_g(v) = (b_1 - c_1)u_1 + (b_2 - c_2)u_2$ but then $\delta_g(v), u_1, u_2$ are not linearly independent and we have a contradiction. □

Now we see what happens if our group generated by two elements is not a two–row, two–column or hook group.

Lemma 2.9. *Let $p > 2$ and let $G = \langle g, h\rangle$ be a p-group consisting of bireflections, but not a two–row, two–column or hook group. Then $U = V^g + V^h$ has codimension one, $\delta_h(U), \delta_g(U) \not\subseteq U$ and $v \in V\backslash U$, $r \in U$ such that $\delta_g(U) = kv$, $\delta_h(U) = k(v + r)$. We can find $c \in k$ such that either: $\delta_g(v) = -cr + (c-1)\delta_h(v + r)$ or $\delta_h(v + r) = cr + (c-1)\delta_g(v)$.*

Proof. Using Lemma 2.7 we know that if G is not a two–row or two–column group then U has codimension one. By Lemma 2.8 if G is not a hook group then $\delta_h(U), \delta_g(U) \not\subseteq U$. Let $v \in V$ such that $kv = \delta_g(U)$. As $\delta_g(U) \not\subseteq U$ we can write $V = U + kv$. As $\delta_h(U) \not\subseteq U$ we can find some $r \in U$ such that $\delta_h(U) = k(v + r)$. We look at $gh \in G$. Let $u \in V^h\backslash V^g$, $u' \in V^g\backslash V^h$:
$\delta_{gh}(u) = \delta_g(u) \in kv$,
$\delta_{gh}(u') = \delta_h(u') + \delta_g\delta_h(u') \in k(v + r + \delta_g(v + r))$. As $r \in U$, $\delta_g(r) \in kv$ so: $kv + k(v + r + \delta_g(v)) \subseteq [gh, V]$. Suppose $\dim_k(kv + k(v + r + \delta_g(v)) = 1$, then: $k(v + r + \delta_g(v)) \leq kv$,which would mean that $r + \delta_g(v) \in kv$. As g is a bireflection and $v \in [g, V]$ we know that $\delta_g(v) \in \delta_g^2(V) \leq V^g$. Since $r, \delta_g(v) \in U$ this tells us $\delta_g(v) = -r$, so $\delta_g(v) = -cr + (c-1)\delta_h(v + r)$ for $c = 1$. Suppose $\dim_k(kv + k(v + r + \delta_g(v)) = 2$, then as gh is a bireflection $kv + k(v + r + \delta_g(v)) = [gh, V]$ and $\delta_{gh}(v + r) = \delta_g(v + r) + \delta_h(v + r) + \delta_g\delta_h(v + r) \in kv + k(r + \delta_g(v))$. As h is a bireflection, $\delta_h(v + r) \in V^h \subseteq U$, so $\delta_g\delta_h(v + r), \delta_g(r) \in kv$. We can find $c_1, c_2 \in k$ such that: $\delta_h(v + r) = c_1(r + \delta_g(v)) + c_2v - \delta_g(v)$. As v is the only term not in U, we can see $c_2 = 0$, so if $c = c_1$ we have: $\delta_h(v + r) = cr + (c-1)\delta_g(v)$. □

Lemma 2.10. *Let $p > 2$ and let $G = \langle g, h\rangle \leq \mathrm{GL}(V)$ be a p-group. Then G is a pure bireflection group if and only if one of the following holds:*

(1) *G is a hook group.*
(2) *G is a two–row group.*
(3) *G is a two–column group.*
(4) *G is an exceptional group of type one.*

Proof. If G is a two–column, two–row or hook group then we can easily check it consists of bireflections (see Lemmas 5.2, 4.2 and 3.3) and exceptional groups consist of bireflections by definition. Suppose $G = \langle g, h\rangle$ isn't a two–row, two–column, hook or exceptional group. Using Lemma 2.9 $U = V^g + V^h$ has codimension one, $\delta_h(U), \delta_g(U) \not\subseteq U$ and $v \in V\backslash U$, $r \in U$ such that $\delta_g(U) = kv$, $\delta_h(U) = k(v+r)$. We can choose g, h such that: $\delta_h(v+r) = cr + (c-1)\delta_g(v)$. As we are assuming our group is not a two–row group this means that $v, r, \delta_g(v)$ are linearly independent. As $r \in U$ we can find $s, t \in k$ such that: $\delta_h(r) = s(v+r)$, $\delta_g(r) = tv$. We will show that either:

$$(1)\ c \neq 0 \text{ and } s = t = 0,\ (2)\ c = 0 \text{ and } t = 0, \text{ or } (3)\ c = 0 \text{ and } s = 0.$$

We do this by looking at the descending commutator series. Firstly we find that $[G, V] = \langle v, r, \delta_g(v)\rangle$. We want to find $[G, [G, V]]$, so we look at $\delta_g(v) = \delta_g(v)$, $\delta_g(r) = tv$, $\delta_g^2(v) = 0$, $\delta_h(v) = (c-1)\delta_g(v) + cr - s(v+r)$, $\delta_h(r) = s(v+r)$. Hence $[G, [G, V]] \geq \langle \delta_g(v), tv, cr, s(v+r)\rangle$. As G is a p-group we know that $\dim_k([G, V]) > \dim_k([G, [G, V]]$, so two of c, s, t must equal 0. First assume $c \neq 0$, $s = t = 0$ so $r \in V^G$. We look at $g^i h \in G$ for $1 < i \leq p-1$. Let $u \in U\backslash V^g$, $u' \in U\backslash V^h$ then using Lemma 1.6: $\delta_{g^ih}(u) = i\delta_g(u) + \frac{i(i-1)}{2}\delta_g^2(u)$, $\delta_{g^ih}(u') = \delta_h(u') + i\delta_g\delta_h(u') + \frac{i(i-1)}{2}\delta_g^2\delta_h(u')$. We have already found $\delta_g(v), v, r$ to be linearly independent, and $i\delta_g(u) + \frac{i(i-1)}{2}\delta_g^2(u) \in k(2v + (i-1)\delta_g(v))$, $\delta_h(u') \in k(v+r)$, $i\delta_g(v+r) + \frac{i(i-1)}{2}\delta_g^2(v+r) = i\delta_g(v)$, so we can see that: $[g^ih, V] = k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v))$. Therefore $\delta_{g^ih}(v) =$

$$i\delta_g(v) + \delta_h(v) + i\delta_g\delta_h(v) + \tfrac{i(i-1)}{2}\delta_g^2(v) + \tfrac{i(i-1)}{2}\delta_g^2\delta_h(v) \in k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v)).$$

We know that: $\delta_h(v) = cr + (c-1)\delta_g(v)$, $\delta_g^2(v) = \delta_g(r) = 0$, so for some $\alpha_1, \alpha_2 \in k$:

$$(i + c - 1)\delta_g(v) + cr = \alpha_1(2v + (i-1)\delta_g(v)) + \alpha_2(v + r + i\delta_g(v)).$$

Comparing r terms $\alpha_2 = c$, then comparing v terms $\alpha_1 = -\frac{c}{2}$. Looking at the $\delta_g(v)$ terms: $i + c - 1 = -\frac{c}{2}(i-1) + ci$, $c(i-1) = 2(i-1)$, $c = 2$. Now we can see that: $\dim_k(U) = n-1$, $\dim_k(V^g \cap V^h) = n-3$, and if we let $r_1 = r$, $\delta_g(v) = r_2$ then: $\delta_g(U) = kv$, $\delta_g(v) = r_2$, $\delta_h(U) = k(v + r_1)$, $\delta_h(v) = 2r_1 + r_2$, so g, h are a special pair with G as described in Lemma 2.3 an exceptional group of type one. Now suppose $c = 0$. If $t = 0$ we have: $\delta_h(v+r) = -\delta_g(v+r)$. As above let $u \in U\backslash V^g$, $u' \in U\backslash V^h$ then: $\delta_{g^ih}(u) = i\delta_g(u) + \frac{i(i-1)}{2}\delta_g^2(u)$, $\delta_{g^ih}(u') = \delta_h(u') + i\delta_g\delta_h(u') + \frac{i(i-1)}{2}\delta_g^2\delta_h(u')$. We know that $i\delta_g(u) + \frac{i(i-1)}{2}\delta_g^2(u) \in k(2v + (i-1)\delta_g(v))$, $\delta_h(u') \in k(v+r)$, $\delta_g(r) = \delta_g^2(v) = 0$, so we find: $[g^ih, V] = k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v))$. Hence $\delta_{g^ih}(v) = i\delta_g(v) + \delta_h(v) + i\delta_g\delta_h(v) + \frac{i(i-1)}{2}\delta_g^2(v) + \frac{i(i-1)}{2}\delta_g^2(v) \in k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v))$. We know $\delta_h(v) = -\delta_g(v) - sv - sr$, $\delta_g^2(v) = 0$, $\delta_g(r) = 0$, so for some $\alpha_1, \alpha_2 \in k$: $(i - 1 - is)\delta_g(v) - sv - sr = \alpha_1(2v + (i-1)\delta_g(v)) + \alpha_2(v + r + i\delta_g(v))$. Comparing V and r terms $\alpha_2 = -s$, $\alpha_1 = 0$, but comparing $\delta_g(v)$ terms $i - 1 - is = -is$, which only holds for $i = 1$, so we have a contradiction. If $s = 0$ then we have: $\delta_h(v) = -\delta_g(v)$, which can be dealt with using the symmetric argument to the one above where $t = 0$. We need to exclude $p = 2$ in the above proposition as we can find additional groups, which don't exist in the odd p case. □

Examples 2.11. *Let $p = 2$ and $H := \langle g_1, g_2\rangle$ where:*

$$g_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},\quad g_2 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

We find that $H \cong C_2 \times C_2$ so it is an abelian group of order p^2. This just leaves one non-identity element not given explicitly. As $g_1g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$, *we see that H consists of bireflections but isn't a two–row, two–column or hook group.*

We now want to see what happens if our group has more than two generators. First we want to find non-transvection bireflections. We call G a one–row or one–column group, if in Definition 0.4 (1), "$n-2$" is replaced by "$n-1$" and in (2), "2" is replaced by "1", respectively.

Lemma 2.12. *Let $g_1, \dots, g_l$ be bireflections such that $G = \langle g_1, \dots, g_l\rangle$ is a p-group which isn't a one–row or one–column group. Then we can find $g \in G$ such that V^g has codimension two.*

Proof. By Proposition 8.2.12 of [8] G consists of transvections if and only if it is either a one–row or one–column group. It follows that either g_i is not a transvection for some $1 \le i \le l$ or there exists $g \in G$ not a transvection which is a product of two transvections. Suppose for some $\gamma_1, \gamma_2 \in V^*$, $u_1, u_2 \in V$, $g = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$. If either $\gamma_1 \in k\gamma_2$ or $u_1 \in ku_2$ we see that g is a transvection. Otherwise V^g has codimension 2. □

We want to be able to use Lemma 2.10 to help us with pure bireflection groups with more than two generators. The next lemma allows us to find a useful subgroup with two generators for groups which are not two–row or two–column groups.

Lemma 2.13. *Let G be a unipotent bireflection group which is not a two–row or two–column group. Then we can find $g_1, g_2 \in G$ such that $H = \langle g_1, g_2\rangle$ is not a two–row or two–column group.*

Proof. By the previous lemma we can pick $g \in G$ such that V^g has codimension 2. As G is not a 2–column group we can find $\sigma_1 \in G$ such that $V^g \not\le V^{\sigma_1}$. If also $[\sigma_1, V] \not\le [g, V]$ then choose $g_1 = g$, $g_2 = \sigma_1$ and we are done. Otherwise, as G is not a 2–row group, we can find $\sigma_2 \in G$ such that: $[\sigma_2, V] \not\le [g, V]$. Either:

(1) $V^g \not\le V^{\sigma_2}$, then pick $g_1 = g$, $g_2 = \sigma_2$,
(2) $V^g \le V^{\sigma_2}$ and $\dim_k(V^{\sigma_1}) = \dim_k(V^{\sigma_2}) = n-2$, then $V^g = V^{\sigma_2}$ so $V^{\sigma_2} \not\le V^{\sigma_1}$ so pick $g_1 = \sigma_1$, $g_2 = \sigma_2$, or
(3) $V^g \le V^{\sigma_2}$ and either $\dim_k(V^{\sigma_1}) > n-2$ or $\dim_k(V^{\sigma_2}) > n-2$.

In the third case, as $V^g \not\le V^{\sigma_1}$, $V^g \le V^{\sigma_2}$ we can find $u \in V^g \backslash V^{\sigma_1}$, so: $\sigma_1\sigma_2(u) = \sigma_1(u)$ and therefore $u \notin V^{\sigma_1\sigma_2}$. As $[\sigma_1, V] \le [g, V]$ and $[\sigma_2, V] \not\le [g, V]$ we can find some $v, r \in V$ such that $r \notin [g, V]$: $\sigma_2(v) = v+r$, $\sigma_1\sigma_2(v) = v+r+\delta_{\sigma_1}(v+r)$. We know that $\delta_{\sigma_1}(v+r) \in [\sigma_1, V] \le [g, V]$, so $r + \delta_{\sigma_1}(v+r) \notin [g, V]$. Therefore $V^g \not\le V^{\sigma_1\sigma_2}$ and $[\sigma_1\sigma_2, V] \not\le [g, V]$; so we choose $g_1 = g$, $g_2 = \sigma_1\sigma_2$. □

Now we are able to move up to looking at groups with three generators.

Lemma 2.14. *Let $p > 2$. Suppose $G = \langle g_1, g_2, h\rangle$ is a pure bireflection group such that $H = \langle g_1, g_2\rangle$ is a hook group with hyperplane U and line kv which isn't a two–row or two–column group. Then either:*

(1) *G is a hook group with hyperplane U and line kv,*
(2) *G is an exceptional group of type one and either g_1, h or g_2, h are a special pair,*
(3) *G is an exceptional group of type two and g_1, g_2, h are double transvections.*

Proof. Let $G_1 = \langle g_1, h\rangle$ and $G_2 = \langle g_2, h\rangle$. Suppose that neither g_1, h or g_2, h are special pairs (in which case G is an exceptional group of type one). As both G_1 and G_2 must consist of bireflections up to duality we only need to consider the following four cases:

(1) G_1 and G_2 two–column groups,

(2) G_1 a two–row group, G_2 a two–column group,
(3) G_1 a two–column group, G_2 a hook group but not a two–column or two–row group,
(4) G_1 and G_2 hook groups which aren't two–column or two–row groups.

We will use that as H is not a two–column or two–row group we can see by Lemma 2.7 that $U = V^{g_1} + V^{g_2}$, and we can find $u_1 \in V^{g_1} \backslash V^{g_2}$, $u_2 \in V^{g_2} \backslash V^{g_1}$ such that $\delta_{g_1}(u_2) = \delta_{g_2}(u_1) = v$. As U is of codimension one there exists some $w \notin U$ such that $\dim_k(k\delta_{g_1}(w) + k\delta_{g_2}(w)) = 2$.

Case 1 If G_1 and G_2 are two–column groups, then $V^{g_1} \leq V^h$ and $V^{g_2} \leq V^h$, so $V^{g_1} + V^{g_2} = U \leq V^h$. Therefore $\delta_h(U) = \{0\} < kv$ and G is a hook group with hyperplane U and line kv.

Case 2 If G_1 is a two–row group and G_2 is a two–column group then $[h, V] \leq [g, V] \leq U$ and $V^{g_2} \leq V^h$. We see that $g_1g_2h(w) = w + \delta_{g_1}(w) + \delta_{g_2}(w) + \delta_h(w) + c_1v$, $g_1g_2h(u_1) = u + \delta_h(u_1) + c_2v$, $g_1g_2h(u_2) = u' + v$ for some $c_1, c_2 \in k$. As G_1 is a two–row group: $\delta_h(u_1) \in \langle v, \delta_{g_1}(w)\rangle$. Suppose that G not a hook group. Then $v, \delta_h(u_1)$ are linearly independent, so in order for g_1g_2h to be a bireflection: $[g_1g_2h, V] = \langle v, \delta_h(u_1)\rangle = \langle v, \delta_{g_1}(w)\rangle$. This would mean that $\delta_{g_1}(w) + \delta_{g_2}(w) + \delta_h(w) + c_1v \in \langle v, \delta_{g_1}(w)\rangle$, $\delta_{g_2}(w) \in \langle v, \delta_{g_1}(w)\rangle$ and G (and therefore H) is a two–column group, which is a contradiction.

Case 3 If G_2 is a hook group but not a two–row or two–column group then by Lemma 2.7 it has hyperplane $U' = V^{g_2} + V^h$. Suppose the line of G_2 is kv'. If G_1 is a two–column group $V^{g_1} \leq V^h$, so $U = V^{g_1} + V^{g_2} \leq V^h + V^{g_2} = U'$. As $\dim_k(U) = \dim_k(U')$ this means that $U = U'$. As $\delta_{g_2}(U) = kv$ we see that $kv' = kv$, so G is a hook group with hyperplane U and line kv.

Case 4 Suppose G_1 and G_2 are both hook groups which are not two–row or two–column groups. Let $U_1 = V^{g_1} + V^h$, $U_2 = V^{g_2} + V^h$ be the hyperplanes of G_1 and G_2 with lines kv_1 and kv_2 respectively. If there exists $u \in (V^h \cap U) \backslash V^{g_1}$ then $u \in U_1$ so $U_1 = V^g + ku = U$, $kv_1 = kv$ and $\langle g_1, g_2, h\rangle$ is a hook group. Similarly if there exists $u \in (V^h \cap U) \backslash V^{g_2}$. Assume this is not the case. If we take u_1, u_2 as defined above then $u_1 \in U_1 \backslash U_2$ and $u_2 \in U_2 \backslash U_1$. We can see $U_1 + U_2 = U + U_1 = U + U_2 = V$ and by definition $\dim_k(U_1) = \dim_k(U_2) = \dim_k(U) = n-1$. From this we see that $\dim_k(U_1 \cap U_2) = n-2$, $\dim_k(U \cap U_1 \cap U_2) = n-3$. As $V^h \leq U_1$, $V^h \leq U_2$ and $\dim_k(V^h) \geq n-2$ we see that $U_1 \cap U_2 = V^h$. Similarly $V^{g_1} = U \cap U_1$, $V^{g_2} = U \cap U_2$. We can assume $w \in V^h \backslash U$, and as H not a two–row or two–column group, $\delta_{g_1}(w), \delta_{g_2}(w), v$ are linearly independent. Since $w \in U_1 \cap U_2$ and we can see that $kv_1 = k\delta_{g_1}(w) \in V^h$, $kv_2 = k\delta_{g_2}(w) \in V^h$. Let $a_1, a_2 \in k$ such that $\delta_h(u_1) = a_1\delta_{g_1}(w)$, $\delta_h(u_2) = a_2\delta_{g_2}(w)$. We now look at $G_3 = \langle g_1g_2, h\rangle$ and see that $g_1g_2(w) = w + \delta_{g_1}(w) + \delta_{g_2}(w) + \delta_{g_1}\delta_{g_2}(w)$ $g_1g_2(u_1) = u_1 + v$, $g_1g_2(u_2) = u_2 + v$. As $[g_1g_2, V] = \langle \delta_{g_1}(w) + \delta_{g_2}(w), v\rangle \neq \langle \delta_{g_1}(w), \delta_{g_2}(w)\rangle = [h, V]$ we know that G_3 is not a two–row group. As $\dim_k(V^{g_1g_2}) = \dim_k(V^h) = n-2$ and $w \in V^{g_1g_2} \backslash V^h$, we see that G_3 is not a two–column group either. By Lemma 2.7 this means that $U_3 = V^h + V^{g_1g_2}$ has codimension one. As $u_1 - u_2 \in V^{g_1g_2} \leq U$ and $\delta_h(u_1 - u_2) = \delta_h(u_1) - \delta_h(u_2) = a_1\delta_{g_1}(w) - a_2\delta_{g_2}(w) \in V^h$. By Lemma 2.8 we see that G_3 must be a hook group with line $k(a_1\delta_{g_1}(w) - a_2\delta_{g_2}(w))$. As $w \in V^h \leq U$ we find: $k(a_1\delta_{g_1}(w) - a_2\delta_{g_2}(w)) = k(\delta_{g_1}(w) + \delta_{g_2}(w) + \delta_{g_1}\delta_{g_2}(w))$. As $\delta_{g_1}(w), \delta_{g_2}(w), v$ are linearly independent and $\delta_{g_1}\delta_{g_2}(w) \in kv$ we see that $\delta_{g_1}\delta_{g_2}(w) = \delta_{g_2}\delta_{g_1}(w) = 0$ and $a_1 = -a_2$. Let $\gamma_0, \gamma_1, \gamma_2 \in V^*$ such that: $\gamma_0(w) = 1, \ker(\gamma_0) = U$, $\gamma_1(u_2) = 1$, $\ker(\gamma_1) = U_1$, $\gamma_2(u_1) = 1$, $\ker(\gamma_2) = U_2$. Let $G' = \langle \tilde{g}_1, \tilde{g}_2, \tilde{h}\rangle$ where $\tilde{g}_1 = t^{\gamma_0}_{\delta_{g_1}(w)} t^{\gamma_1}_v$, $\tilde{g}_2 = t^{\gamma_0}_{\delta_{g_2}(w)} t^{\gamma_2}_v$, $\tilde{h} = t^{\gamma_1}_{a_1\delta_{g_2}(w)} t^{\gamma_2}_{-a_1\delta_{g_1}(w)}$, then G' is an exceptional group of type two. We can see that for $i = 1, 2$ $V^{\tilde{g}_i} = U \cap U_i = V^{g_i}$ and $V^{\tilde{h}} = U_1 \cap U_2 = V^h$. We know that $V = V^{g_1} \oplus ku_2 \oplus kw = V^{g_2} \oplus ku_1 \oplus kw = V^h \oplus ku_1 \oplus ku_2$. From the definition of $\tilde{g}_1$ $\tilde{g}_1(u_2) = u_2 + v = g_1(u_2)$, $\tilde{g}_1(w) = w + \delta_{g_1}(w) = g_1(w)$, so $\tilde{g}_1 = g_1$, similarly $\tilde{g}_2 = g_2$ and $\tilde{h} = h$. Hence $G = G'$ is an exceptional group of type two, and g_1, g_2, h are double transvections.

□

Proof. (of Theorem 0.5) Suppose G is not a two–row, two–column group or exceptional group. By Lemma 2.13 we can find $g_1, g_2 \in G$ such that $\dim_k(V^{g_1} \cap V^{g_2}) < n-2$ and $\dim_k([g_1, V] + [g_2, V]) > 2$. Let $N := \langle g_1, g_2 \rangle$.

As G (and therefore N) consists of bireflections by Lemma 2.10 N must be a hook group with hyperplane U for some $U \subset V$, and line kv for some $v \in U^N$. As G is not an exceptional group by Lemma 2.14 for any $g \in G$, $\langle g, N \rangle$ is a hook group with hyperplane U and line kv. Hence $[G, U] \leq kv \leq V^G$, so G is a hook group with hyperplane U and line kv. □

3. Two–column and two–row groups

Now that we know the pure bireflection groups for $p \neq 2$ we can start to look at them in more detail. Although we don't have the same classification of pure bireflection groups, two–row, two–column and hook groups are still of interest for $p = 2$, so we don't restrict to $p \neq 2$ for these sections. We start by looking at two–column groups.

Definition 3.1. *Let $r_1, r_2 \in V$ linearly independent, $\zeta \in V$ such that $\zeta(r_1) = 1$, $\zeta(r_2) = 0$. Then for all $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp$, $c \in k$ set $\kappa^{r_1,r_2,\zeta}_{\gamma_1,\gamma_2,c} := t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta}_{cr_2}$ and define the sets*

$$K^{r_1,r_2,\zeta} := \{\kappa^{r_1,r_2,\zeta}_{\gamma_1,\gamma_2,c} \mid \gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp, c \in k\}, \ L^{r_1,r_2} := \{\kappa^{r_1,r_2,\zeta}_{0,\gamma,0} \mid \gamma \in r_1^\perp \cap r_2^\perp\}.$$

Whenever r_1, r_2, ζ are fixed in context we will write $\kappa_{\gamma_1,\gamma_2,c}$.

Lemma 3.2. *Let $r_1, r_2 \in V$ and $\zeta_1, \zeta_2 \in V^*$ such tha $\zeta_1(r_1) = \zeta_2(r_1) = 1$, $\zeta_1(r_2) = \zeta_2(r_2) = 0$. Then $K^{r_1,r_2,\zeta_1} = K^{r_1,r_2,\zeta_2}$.*

Proof. Let $g = \kappa^{r_1,r_2,\zeta_1}_{\gamma_1,\gamma_2,c} \in K^{r_1,r_2,\zeta_1}$. As ζ_1 and ζ_2 agree on r_1 and r_2 we can find $\gamma_3 \in r_1^\perp \cap r_2^\perp$ such that $\zeta_1 = \zeta_2 + \gamma_3$. Hence $g = \kappa^{r_1,r_2,\zeta_1}_{\gamma_1,\gamma_2,c} = \kappa^{r_1,r_2,\zeta_2}_{\gamma_1,\gamma_2+c\gamma_3,c} \in K^{r_1,r_2,\zeta_2}$. So $K^{r_1,r_2,\zeta_1} \leq K^{r_1,r_2,\zeta_2}$. A symmetric argument tells us that $K^{r_1,r_2,\zeta_1} \leq K^{r_1,r_2,\zeta_1}$, so $K^{r_1,r_2,\zeta_1} = K^{r_1,r_2,\zeta_2}$. □

From here on we shall write $K^{r_1,r_2} = K^{r_1,r_2,\zeta}$. we look at multiplication between the elements of this set.

Lemma 3.3. *If we fix r_1, r_2, ζ then*

1. *$\kappa_{\gamma_1,\gamma_2,c} = \kappa_{\gamma_1',\gamma_2',c'} \Leftrightarrow \gamma_1 = \gamma_1'$, $\gamma_2 = \gamma_2'$ and $c = c'$.*
2. *$\kappa_{\gamma_1,\gamma_2,c}\kappa_{\gamma_1',\gamma_2',c'} = \kappa_{\hat{\gamma}_1,\hat{\gamma}_2,\hat{c}}$, where: $\hat{\gamma}_1 = \gamma_1 + \gamma_1'$, $\hat{\gamma}_2 = \gamma_2 + \gamma_2' + c\gamma_1'$, and $\hat{c} = c + c'$.*
3. *$\kappa_{\gamma_1,\gamma_2,c}$ and $\kappa_{\gamma_1',\gamma_2',c'}$ commute iff $c\gamma_1' = c'\gamma_1$.*
4. *$\kappa^{-1}_{\gamma_1,\gamma_2,c} = \kappa_{-\gamma_1,c\gamma_1-\gamma_2,-c}$.*
5. *$\kappa_{\gamma_1,\gamma_2,c}\kappa_{\gamma_1',\gamma_2',c'}\kappa^{-1}_{\gamma_1,\gamma_2,c}\kappa^{-1}_{\gamma_1',\gamma_2',c'} = \kappa_{c\gamma_2'-c'\gamma_2,0,0} \in L^{r_1,r_2}$.*
6. *For any $\kappa_{\gamma_1,\gamma_2,c} \in \mathrm{GL}(V)$: $|\kappa_{\gamma_1,\gamma_2,c}| = \begin{cases} p^2, & \text{if } p = 2 \text{ and } c \neq 0, \\ p, & \text{otherwise.} \end{cases}$.*

Proof. (1): If $\kappa_{\gamma_1,\gamma_2,c} = \kappa_{\gamma_1',\gamma_2',c'}$ then: $t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta}_{cr_2} = t^{\gamma_1'}_{r_1} t^{\gamma_2'}_{r_2} t^{\zeta}_{c'}$. As r_1, r_2 are linearly independent by Lemma 1.5 $\gamma_1 = \gamma_1'$, $\gamma_2 + c\zeta = \gamma_2' + c'\zeta'$, so $\gamma_2 - \gamma_2' = (c - c')\zeta$. As $\gamma_2 - \gamma_2' \in r_2^\perp$, $\zeta \notin r_2^\perp$ we see $c = c'$ and $\gamma_2 = \gamma_2'$.
(2): If $g = \kappa_{\gamma_1,\gamma_2,c}$ and $h = \kappa_{\gamma_1',\gamma_2',c'}$ then $gh = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta}_{cr_2} t^{\gamma_1'}_{r_1} t^{\gamma_2'}_{r_2} = t^{\gamma_1}_{r_1} t^{\gamma_1'}_{r_1+cr_2} t^{\gamma_2+\gamma_2'}_{r_2} t^{\zeta}_{(c+c')r_2} = t^{\gamma_1+\gamma_1'}_{r_1} t^{\gamma_2+\gamma_2'+c\gamma_1'}_{r_2} t^{\zeta}_{(c+c')r_2}$.
(3),(4),(5) and (6) follow from (2). □

We now move from looking at a set to looking at a group and it's properties.

Proposition 3.4. *Let $G = \langle \kappa_{\gamma_1,\gamma_2,c} | \gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp, c \in k \rangle$. Then $G = K^{r_1,r_2}$ and $|G| = q^{2n-3}$, if $k = \mathbb{F}_q$.*

Proof. We know that $K^{r_1,r_2} \subseteq G$. By Proposition 3.3(2) all elements of the group can be written as $\kappa_{\gamma_1,\gamma_2,c}$ for some γ_1, γ_2, c, so $G = K^{r_1,r_2}$. By Proposition 3.3(1) $\kappa_{\gamma_1,\gamma_2,c} = \kappa_{\gamma_1',\gamma_2',c'}$ if and only if $\gamma_1 = \gamma_1'$, $\gamma_2 = \gamma_2'$ and $c = c'$. Therefore $|K^{r_1,r_2}| = |r_1^\perp \cap r_2^\perp|^2 \cdot |k| = (q^{n-2})^2 q = q^{2n-3}$, if $k = \mathbb{F}_q$. □

We want to see when different choices of r_1, r_2 determine different groups.

Lemma 3.5. *Let $r_1, r_2, u_1, u_2 \in V$, $G = K^{r_1,r_2}$ and $H = K^{u_1,u_2}$. Then $G = H$ if and only if $kr_2 = ku_2$ and $\langle r_1, r_2\rangle = \langle u_1, u_2\rangle$.*

Proof. Let $\zeta_1, \zeta_2 \in V^*$ such that $\zeta_1(r_2) = \zeta_2(u_2) = 0$ and $\zeta_1(r_1) = \zeta_2(u_1) = 1$. Suppose to start with that $kr_2 = ku_2$ and $\langle r_1, r_2\rangle = \langle u_1, u_2\rangle$. Firstly we note that this means $r_1 = a_1u_1 + a_2u_2$ and $r_2 = a_3u_2$ with $a_1, a_3 \neq 0$. Also $r_1^\perp \cap r_2^\perp = u_1^\perp \cap u_2^\perp$. For any $g \in G$ we can write $g = \kappa^{r_1,r_2,\zeta_1}_{\gamma_1,\gamma_2,c} \in K^{r_1,r_2}$ for some $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp$, $c \in k$. Using Lemma 1.4 this means that $g = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta_1}_{cr_2} = t^{\gamma_1}_{a_1u_1+a_2u_2} t^{\gamma_2}_{a_3r_2} t^{\zeta_1}_{cr_2} = t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_2\gamma_1}_{u_2} t^{\zeta_1}_{ca_3u_2}$. As $kr_2 = ku_2$ we can see that $\zeta_1(u_2) = 0$. Then $\zeta_1(a_1u_1 + a_2u_2) = a_1\zeta_1(u_1) = 1$ and $\zeta_1(u_1) = 1/a_1$. Let $b = 1/a_1$. As ζ_1 and $b\zeta_2$ agree on r_1, r_2 we can find some $\gamma_3 \in r_1^\perp \cap r_2^\perp$ such that $\zeta_1 = \gamma_3 + b\zeta_2$. Using Lemma 1.4 $g = t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_2\gamma_1}_{u_2} t^{\zeta_1}_{cr_2} = t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_2\gamma_1}_{u_2} t^{\gamma_3+b\zeta_2}_{ca_3u_2} = t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_1\gamma_1}_{u_2} t^{\gamma_3}_{ca_3u_2} t^{\zeta_2}_{bca_3u_2} = t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_1\gamma_1+ca_3\gamma_3}_{u_2} t^{\zeta_2}_{bca_3u_2} = \kappa^{u_1,u_2,\zeta_2}_{\gamma_1',\gamma_2',c'} \in H$, where $\gamma_1' = a_1\gamma_1$, $\gamma_2' = a_3\gamma_2 + a_1\gamma_1 + ca_3\gamma_3$, $c' = bca_3$. Hence $G \leq H$ and by symmetry, $H \leq G$, so $G = H$. Suppose that $\langle r_1, r_2\rangle \neq \langle u_1, u_2\rangle$. Then $[G, V] \neq [H, V]$, so $G \neq H$. Suppose that $\langle r_1, r_2\rangle = \langle u_1, u_2\rangle$ but $kr_2 \neq ku_2$. Then $kr_2 = [G, V]^G \neq [H, V]^H = ku_2$, so $G \neq H$. □

Clearly K^{r_1,r_2} is a two–column group for any $r_1, r_2 \in V$. We check that any two–column group can be written as a subgroup of K^{r_1,r_2} for some $r_1, r_2 \in V$.

Lemma 3.6. *Let H be a two–column group. If $[H, V]^H = [H, V]$ then $H \leq K^{r_1,r_2}$ for any $r_1, r_2 \in V$ such that $[H, V] = \langle r_1, r_2\rangle$. If $kv = [H, V]^H < [H, V]$, then $H \leq K^{r_1,r_2}$ for any $r_2 \in kv$ and $r_1 \in V$ such that $[H, V] = \langle r_1, r_2\rangle$.*

Proof. Suppose H is a two–column group with $[H, V]^H = [H, V]$. If we choose any $r_1, r_2 \in V$ such that $[H, V] \leq \langle r_1, r_2\rangle$, then any $h \in H$ can be written as $h = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2}$ for some $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp$. Then for any $\zeta \in V^*$ such that $\zeta(r_1) = 1$ and $\zeta(r_2) = 0$, $h = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta}_0 = \kappa^{r_1,r_2,\zeta}_{\gamma_1,\gamma_2,0} \in K^{r_1,r_2}$, hence $H \leq K^{r_1,r_2}$. Suppose that H is a two–column group with $kv = [H, V]^H < [H, V]$. If we choose any $r_2 \in kv$ and $r_1 \in V$ such that $[H, V] = \langle r_1, r_2\rangle$, then we can write and $h \in H$ as $h = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2}$ for some $\gamma_1 \in r_1^\perp \cap r_2^\perp$, $\gamma_2 \in V^*$. If $\gamma_2 \in r_1^\perp \cap r_2^\perp$ then $h \in K^{r_1,r_2}$ by the above argument. If $\gamma_2(r_1) = c \neq 0$ then let $\zeta = \frac{1}{c}\gamma_2$ and write $h = t^{\gamma_1}_{r_1} t^{0}_{r_2} t^{\zeta}_{cr_2} = \kappa^{r_1,r_2,\zeta}_{\gamma_1,0,c}$, so $h \in K^{r_1,r_2}$ and $H \leq K^{r_1,r_2}$. □

Proposition 3.7. *For $n \geq 3$ if $G = K^{r_1,r_2}$ then it is a special group with $Z(G) = \Phi(G) = [G, G] = L^{r_1,r_2}$.*

Proof. As G is a p-group we know that $\Phi(G) = G^p[G, G]$. We have shown in Proposition 3.3 that $[G, G] \leq L^{r_1,r_2}$, and that $G^p = \{e\}$ for p odd and $G^p \leq L^{r_1,r_2}$ for p even. Putting this together we find that $\Phi(G) \leq L^{r_1,r_2}$. For $\gamma \in r_1^\perp \cap r_2^\perp$ take $g_1 = \kappa_{\gamma,0,0}$ and $g_2 = \kappa_{0,0,1}$ then $g_1g_2g_1^{-1}g_2^{-1} = \kappa_{0,\gamma,0}$ so $[G, G] = L^{r_1,r_2} = \Phi(G)$. If $g \in L^{r_1,r_2}$ then it commutes with all elements $\kappa_{\gamma_1,\gamma_2,c}$ so $L^{r_1,r_2} \leq Z(G)$. If we choose $\kappa_{\gamma_1,\gamma_2,c} \in Z(G)$ then for any γ_1', c' we have that $c\gamma_1' = c'\gamma_1$ so $\gamma = 0$ and $c = 0$, so $\kappa_{\gamma_1,\gamma_2,c} \in L^{r_1,r_2}$ and $Z(G) = L^{r_1,r_2} = \Phi(G) = [G, G]$. G is a special p group. □

We can see that for any $\gamma_1, \gamma_2 \in V^*$, $G = (K^{\gamma_1,\gamma_2})^*$ is a two–row group. Results for two–row groups can be obtain by dualising the results of this section using Lemma 1.2.

4. Hook groups

We now move on to look at properties of hook groups. First we establish some notation.

Definition 4.1. *Let $U < V^*$ be a hyperplane and fix $0 \neq v \in U$ and $w \in V^* \backslash U$. For every $\lambda \in v^\perp \cap w^\perp$ and $u \in U$ define $b^{w,U,v}_{u,\lambda} \in \mathrm{GL}(V)$ by $b^{w,U,v}_{u,\lambda}(w) = w + u$, $b^{w,U,v}_{u,\lambda}|_U = t^{\lambda}_v$, so that $b^{w,U,v}_{u,\lambda}(u') = u' + u'(\lambda)v$ for any $u' \in U$.*
Choose $w^* \in V^*$ such that $w^*(w) = 1$ and $U = \ker(w^*)$. For $c \in k$ we can then define:

$\mathcal{B}_c^{w,U,v} := \{b_{u,\lambda}^{w,U,v} | \lambda \in v^\perp \cap w^\perp, \lambda(u) = c\}$, $B^{U,v} := \{b_{u,\lambda}^{w,U,v} | \lambda \in v^\perp \cap w^\perp, u \in U\}$, $R_{\hat{v},U} := \{t_{av}^{w^*} | a \in k\} = \{b_{av,0}^{w,U,v} | a \in k\}$. *For* $c \neq 0$ *the elements of* $\mathcal{B}_c^{w,U,v}$ *are index 3 bireflections. If* w, U, v *are fixed in context we will write* $b_{u,\lambda}$ *and* $\mathcal{B}_c$ *instead of* $b_{u,\lambda}^{w,U,v}$ *and* $\mathcal{B}_c^{w,U,v}$.

We look at multiplication of the elements of $B^{U,v}$.

Lemma 4.2. *If we fix* w, U, v *then*

(1) $b_{u,\lambda} = b_{u',\lambda'} \Leftrightarrow u = u'$ *and* $\lambda = \lambda'$,
(2) $b_{u,\lambda} b_{u',\lambda'} = b_{\hat{u},\hat{\lambda}}$ *where* $\hat{\lambda} = \lambda + \lambda'$ *and* $\hat{u} = u + u' + \lambda(u')v$,
(3) $b_{u,\lambda}$ *and* $b_{u',\lambda'}$ *commute iff* $\lambda(u') = \lambda'(u)$,
(4) $b_{u,\lambda}^{-1} = b_{-u+\lambda(u)v,\lambda}$,
(5) $b_{u',\lambda'} b_{u,\lambda} b_{u',\lambda'}^{-1} b_{h,\lambda}^{-1} = b_{(\lambda'(u)-\lambda(u'))v,0} \in R_{\hat{v},U}$,
(6) *For* $b_{u,\lambda} \in \mathcal{B}_c$ $|b_{u,\lambda}| = \begin{cases} p^2, & \text{if } p = 2 \text{ and } c \neq 0, \\ p, & \text{otherwise.} \end{cases}$

Proof. (1) If $b_{u,\lambda} = b_{u',\lambda'}$ then: $w + u = w + u'$, $u = u'$. For any $s \in U$ we find $s + \lambda(s)v = s + \lambda'(s)v$, $\lambda(s)v = \lambda'(s)v$, so $\lambda = \lambda'$.

(2) Let $\hat{\lambda} = \lambda + \lambda'$ and $\hat{u} = u + u' + \lambda(u')v$. We can look at the action of $b_{u,\lambda} b_{u',\lambda'}$ on w and on U. We start with w: $b_{u,\lambda} b_{u',\lambda'}(w) = b_{u,\lambda}(w + u') = w + (u + u' + \lambda(u')v) = b_{\hat{u},\hat{\lambda}}(w)$. Let $s \in U$ then $b_{u,\lambda} b_{u',\lambda'}(s) = b_{u,\lambda}(s + \lambda(s)v) = s + (\lambda(s) + \lambda'(s))v = b_{\hat{u},\hat{\lambda}}(s)$, so $b_{u,\lambda} b_{u',\lambda'} = b_{\hat{u},\hat{\lambda}}$.
(3),(4),(5,(6) follow from (2).

□

We see that $B^{U,v}$ is closed under multiplication, the next few propositions look at it's group structure.

Proposition 4.3. *Let* $G = \langle b_{u,\lambda} | u \in U, \lambda \in v^\perp \cap w^\perp \rangle$. *Then* $G = B^{U,v}$. *Morever,* $|G| = q^{2n-1}$ *if* $k = \mathbb{F}_q$.

Proof. From the definition of G, $\{b_{u,\lambda} | u \in U, \lambda \in v^\perp \cap w^\perp\} \subseteq G$. By Proposition 4.2(2) all elements of the group can be written as $b_{u,\lambda}$ for some u, λ, so $G = \{b_{u,\lambda} | u \in U, \lambda \in v^\perp \cap w^\perp\}$. By Proposition 4.2(1) $b_{u,\lambda} = b_{u',\lambda'}$ if and only if $u = u'$ and $\lambda = \lambda'$ so $|\{b_{u,\lambda} | u \in U, \lambda \in v^\perp \cap w^\perp\}| = |\{u \in U\}| \cdot |\{\lambda \in v^\perp \cap w^\perp\}| = q^{n-1} q^{n-2} = q^{2n-3}$. □

Proposition 4.4. *For* $n \geq 3$, *the group* $G = B^{U,v}$ *is a special group with:*

$$Z(G) = \Phi(G) = [G, G] = R_{\hat{v},U}$$

Proof. As G is a p-group we know that $\Phi(G) = G^p[G, G]$. We have shown in Proposition 4.2 that $[G, G] \leq R_{\hat{v},U}$, and that $G^p = \{e\}$ for p odd and $G^p \leq R_{\hat{v},U}$ for p even. So we have that $\Phi(G) \leq R_{\hat{v},U}$. Let $u \in V$, for any $d \in k$ we can choose $\lambda \in V^*$ such that $\lambda(u) = -d$. Then $b_{u,0}, b_{0,\lambda} \in G$ and: $b_{u,0} b_{0,\lambda} b_{u,0}^{-1} b_{0,\lambda}^{-1} = b_{dv,0} \in R_{\hat{v},U}$. so $[G, G] = R_{\hat{v},U} = \Phi(G)$. If $t \in R_{\hat{v},U}$ then it commutes with all elements $b_{u,\lambda}$ so $R_{\hat{v},U} \leq Z(G)$. If we choose $b_{u,\lambda} \in Z(G)$ then for any u', λ' we know $\lambda(u') = \lambda'(u)$. This can only happen if $u = cv$ and $\lambda = 0$, so $b_{u,\lambda} \in R_{\hat{v},U}$ and $Z(G) = R_{\hat{v},U} = \Phi(G) = [G, G]$. Hence G is a special p group. □

We know that $B^{U,v}$ is special, for $k = \mathbb{F}_p$ it is extra special so we know we can write it as a central product of copies of extraspecial groups of order p^3.

Lemma 4.5. *Let* $k = \mathbf{F}_p$, $n \geq 3$ *and* $G = B^{U,v}$. *If* $p = 2$ *then* $G \cong \underbrace{D_8 * D_8 * \ldots * D_8}_{n-2 \text{ copies}}$. *Otherwise* $G \cong \underbrace{M(p) * M(p) * \ldots * M(p)}_{n-2 \text{ copies}}$.

Proof. Let $\{e_1, \dots, e_n\}$ be a basis for V such that $e_1 = v$ and $\langle x_1, \dots, x_{n-1}\rangle = U$. For $1 \le i \le n-2$ let $H_i = \langle b_{e_{i+1},0}, b_{0,e^*_{i+1}}\rangle$. The H_i are groups of order p^3 and we can check using Lemma 4.2 that $[H_i, H_i] = Z(H_i) = R_{\hat{v},U}$. Hence H_i is extraspecial for $1 \le i \le n-2$. If $p = 2$ then as $|b_{u,0}| = 2$ and $b_{u,0} \notin \Phi(H_i)$ we see that $H_i \cong D_8$. If p is odd then as all elements have order p, $H_i \cong M(p)$. Let $H = H_1 H_2 \dots H_{n-2}$. For any $2 \le i, j \le n-1$, $i \ne j$ we see $Z(H_i) = Z(H_j)$, and H_i centralises H_j so for p even $H \cong \underbrace{D_8 * D_8 * \dots * D_8}_{n-2 \text{ copies}}$, and for p odd $H \cong \underbrace{M(p) * M(p) * \dots * M(p)}_{n-2 \text{ copies}}$. Clearly $H \le G$ and $|H| = p^{2n-1} = |G|$ so $H = G$. □

The next Proposition relates $\mathcal{B}_c$ and $B^{U,v}$. It is useful when looking for generators of $B^{U,v}$.

Proposition 4.6. *Let $n > 3$. For $c \in k$ let $G_c = \langle \mathcal{B}_c\rangle$. Then $G_c = B^{U,v}$.*

Proof. We know $G_c \le B^{U,v}$. We will show that for any element $b_{u,\lambda} \in B^{U,v}$, $b_{u,\lambda} \in G_c$, so $B^{U,v} \le G_c$. Since $\dim_k(V) = n > 3$ we can choose u', λ' such that: $\lambda'(u') = c$, $\lambda'(u) = 0$, $\lambda(u') = 0$, so $b_{u',\lambda'}, b_{u'+u,\lambda'}, b_{u',\lambda+\lambda'} \in \mathcal{B}_c$. Then: $b_{u'+u,\lambda'} b^{-1}_{u',\lambda'} = b_{u,0} \in G_c$, $b_{u',\lambda'+\lambda} b^{-1}_{u',\lambda'} = b_{0,\lambda} \in G_c$, $b_{u,0} b_{0,\lambda} = b_{u,\lambda} \in G_c$. □

We now look at some subgroups of $B^{U,v}$.

Proposition 4.7. *Let $G = \langle b_1, \dots, b_l\rangle$, where $b_i = b_{u_i,\lambda_i} \in B^{U,v}$ for $1 \le i \le l$ minimally generate G. Then $p^l \le |G| \le p^{l+r}$, if $k = \mathbb{F}_q$ with $q = p^r$.*

Proof. As G is a p-group $\Phi(G) = G^p[G, G]$. We know that $G^p \le R_{\hat{v},U}$ and $[G, G] \le R_{\hat{v},U}$ so $\Phi(G) \le R_{\hat{v},U}$ and $1 \le |\Phi(G)| \le |k|$. By [1, Theorem 23.1] $\langle X\rangle = G$ if and only if $\langle X, \Phi(G)\rangle = G$. As $G/\Phi(G)$ is elementary abelian this means if l is the minimal number of generators then $|G/\Phi(G)| = p^l$, so $p^l \le |G| \le qp^l = p^{l+r}$. □

Proposition 4.8. (1) *Let $G = \langle b_1, \dots, b_l, R_{\hat{v},U}\rangle$ where the set of $b_i = b_{h_i,\lambda_i} \in B^{U,v}$ for $1 \le i \le l$, and the b_i's and $R_{\hat{v},U}$ minimally generate G. Then $|G| = p^{l+r}$.*
(2) *Let $G = \langle b_1, \dots, b_l\rangle$ where the set of $b_i = b_{h_i,\lambda_i} \in B^{U,v}$ for $1 \le i \le l$, and the b_i's minimally generate G. Suppose $[G, [G, V]] = 0$, then $|G| = p^l$.*

Proof. (1) Since $\Phi(G) \le R_{\hat{v},U} \le Z(G)$, $G/R_{\hat{v},U}$ is elementary abelian of order p^l and $|R_{\hat{v},U}| = q = p^r$, so $|G| = p^{l+r}$.
(2) If $[G, [G, V]] = 0$ it can be seen from Proposition 4.2 that G is elementary abelian and if it is minimally generated by l elements it has order p^l.
□

5. Exceptional groups of type one

We now look at the exceptional groups of type one. In Lemma 2.3 we see that a group generated by a exceptional pair $G = \langle g, h\rangle$ for $p = 2$ is quite different to a group generated by a special pair for odd p. To start with we note that g and h have order p^2 and not order p. The centre of G also has order p^2 rather than p, and G is not an extra-special group. The types of bireflection we find are also quite different:

$$ghg^{-1}h^{-1} = t^{\gamma_1}_{r_2} t^{\gamma_2}_{r_2}$$

is a transvection and not a double transvection for $p = 2$. We will see in the odd case that exceptional groups do not contain any transvections (Lemma 5.9). For even p exceptional groups of type one are part of a larger family of pure bireflection groups containing a pair of elements

$$g = t^{\zeta_1}_{u_1} t^{\zeta_3}_{u_3}, \; h = t^{\zeta_2}_{u_2} t^{\zeta_3}_{u_3}$$

for $\zeta_1, \zeta_2, \zeta_3 \in V^*$, $u_1, u_2, u_3 \in V$. We have already seen another one of these groups in Example 2.11 but we will not look at them in any detail.

We will restrict to $p > 2$ for this section, we also need $n \geq 5$ for our definition of an exceptional group of type one to make sense. We start by defining some groups containing a special pair, and then show that these are the only possible exceptional groups of type one.

Definition 5.1. *Define linearly independent sets* $\mathbf{r} = \{r_1, r_2, v\}$ *and* $\gamma = \{\gamma_1, \gamma_2, v^*\}$ *with* $r_1, r_2, v \in V$, $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp \cap v^\perp$ *such that* $v^* \in r_1^\perp \cap r_2^\perp$ *and* $v^*(v) = 1$. *For all* $l, m, n \in k$ *set* $\chi_{l,m,n}^{\mathbf{r},\gamma} := t_{\alpha_1}^{\gamma_1} t_{\alpha_3}^{\gamma_2} t_{\alpha_2}^{v^*}$, *where* $\alpha_1 = lv - 2nr_1 + \frac{l(l-1)-2n}{2} r_2$, $\alpha_2 = mv + m^2 r_1 + \frac{m(m-1+2l)+2n}{2} r_2$, $\alpha_3 = 2mr_1 + (m+l)r_2$ *and define the sets* $X^{\mathbf{r},\gamma} := \{\chi_{l,m,n}^{\mathbf{r},\gamma} \mid l, m, n \in k\}$, $J_{\mathbf{r},\gamma} := \{\chi_{0,0,n}^{\mathbf{r},\gamma} \mid n \in k\}$. *If* $\mathbf{r}, \gamma$ *are fixed in context we will write* $\chi_{l,m,n}$.

Note that for all $l, m, n \in k$, $2m\alpha_1 - 2l\alpha_2 + (2n + ml)\alpha_3 = 0$, so $\chi_{l,m,n}$ is a bireflection.

Lemma 5.2. *For fixed* $\mathbf{r}, \gamma$ *we have:*

(1) $\chi_{l,m,n} = \chi_{l',m',n'} \Leftrightarrow l = l', m = m', n = n'$,
(2) $\chi_{l,m,n}\chi_{l',m',n'} = \chi_{l+l',m+m',n+n'-ml'}$,
(3) $\chi_{l,m,n}$ *and* $\chi_{l',m',n'}$ *commute iff* $ml' = m'l$,
(4) $\chi_{l,m,n}^{-1} = \chi_{-l,-m,-n-ml}$,
(5) $\chi_{l,m,n}\chi_{l',m',n'}\chi_{l,m,n}^{-1}\chi_{l',m',n'}^{-1} = \chi_{0,0,lm'-l'm}$.

Proof. (1): This is a direct application of Lemma 1.5.
(2): Let $l, m, n, l', m', n' \in k$ then

$$\chi_{l,m,n}\chi_{l',m',n'} = t_{\alpha_1}^{\gamma_1} t_{\alpha_2}^{\gamma_2} t_{\alpha_3}^{v^*} t_{\alpha_1'}^{\gamma_1} t_{\alpha_2'}^{\gamma_2} t_{\alpha_3'}^{v^*} = t_{\alpha_1+\alpha_1'+l'\alpha_3}^{\gamma_1} t_{\alpha_2+\alpha_2'+m'\alpha_3}^{\gamma_2} t_{\alpha_3+\alpha_3'}^{v^*},$$

where: $\alpha_1 = lv - 2nr_1 + \frac{l(l-1)-2n}{2} r_2$, $\alpha_2 = mv + m^2 r_1 + \frac{m(m-1+2l)+2n}{2} r_2$, $\alpha_3 = 2mr_1 + (m+l)r_2$, $\alpha_1' = l'v - 2n'r_1 + \frac{l(l-1)-2n}{2} r_2$, $\alpha_2' = m'v + (m')^2 r_1 + \frac{m'(m'-1+2l')+2'n}{2} r_2$ and

$$\alpha_3' = 2m'r_1 + (m' + l')r_2.$$

We find that $\alpha_1 + \alpha_1' + l'\alpha_3 = (l + l')v - 2(n + n' - ml')r_1 + \frac{(l+l')(l+l'-1)-2(n+n'-ml')}{2} r_2$,

$$\alpha_2 + \alpha_2' + m'\alpha_3 = (m + m')v + (m + m')^2 r_1 + \frac{(m+m')(m+m'-1+2(l+l'))+2(n+n'-ml')}{2} r_2,$$

$\alpha_3 + \alpha_3' = 2(m + m')r_1 + (m + m' + l + l')r_2$, so $\chi_{l,m,n}\chi_{l',m',n'} = \chi_{l+l',m+m',n+n'-ml'}$.
(3),(4),(5) and (6) follow from (2). □

We know that $X^{\mathbf{x},\gamma}$ is closed under multiplication, we can now start to look at it's group properties.

Proposition 5.3. $G := \langle \chi_{l,m,n} \mid l, m, n \in k \rangle = X^{\mathbf{r},\gamma}$, *with* $|G| = q^3$ *if* $k = \mathbb{F}_q$.

Proof. By Proposition 5.2(2) all elements of the group can be written as $\chi_{l,m,n}$ for some $l, m, n \in k$, so $G = \{\chi_{l,m,n} | l, m, n \in k\} = X^{\mathbf{r},\gamma}$. By Proposition 5.2(1) $\chi_{l,m,n} = \chi_{l',m',n'}$ if and only if $l = l', m = m', n = n'$ so $|X^{\mathbf{r},\gamma}| = |k|^3 = q^3$. □

Proposition 5.4. *Let* $G = X^{\mathbf{r},\gamma}$. *Then* G *is a special group with* $Z(G) = \Phi(G) = [G, G] = J_{\mathbf{r},\gamma}$.

Proof. As G is a p-group we know that $\Phi(G) = G^p[G, G]$. We have shown in Proposition 5.2 that $[G, G] \leq J_{\mathbf{r},\gamma}$. As G is a pure bireflection group with $p \neq 2$, $G^p = \{e\}$. So we see that $\Phi(G) \leq J_{\mathbf{r},\gamma}$. For any $l \in k$ if we let $b_1 = \chi_{l,0,0}, b_2 = \chi_{0,1,0}$ then $b_1 b_2 b_1^{-1} b_2^{-1} = \chi_{0,0,l}$, so $[G, G] = J_{\mathbf{r},\gamma} = \Phi(G)$. If $t \in J_{\mathbf{r},\gamma}$ then it commutes with all elements $\chi_{l,m,n}$ so $J_{\mathbf{r},\gamma} \leq Z(G)$. If we choose $\chi_{l,m,n} \in Z(G)$ then for any l', m' we have that $ml' = m'l$ so $m = l = 0$, so $\chi_{l,m,n} \in J_{\mathbf{r},\gamma}$. It follows that $Z(G) = \Phi(G) = [G, G]$ and G is a special p group. □

Using the above we see that $X^{\mathbf{r},\gamma}$ is isomorphic to a group we recognise.

Proposition 5.5. $X^{\mathbf{r},\gamma} \cong U_3 := \left\langle \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in k \right\rangle \leq \mathrm{SL}_3(k).$

Proof. It is easily seen that the map $\phi : G \to U_3$, $\chi_{l,m,n} \mapsto \begin{pmatrix} 1 & m & -n \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix}$ is a isomorphism. $\square$

Note that if $k = \mathbb{F}_q$, then U_3 is a Sylow p-subgroup of $\mathrm{SL}_3(k)$. The next couple of lemmas will help us towards our goal of showing that all exceptional groups of type one are isomorphic to subgroups of U_3.

Lemma 5.6. *Let $G_1, G_2 \leq \mathrm{GL}(V)$ be hook groups with hyperplanes U_1, U_2 and lines v_1, v_2 respectively. Let $\gamma_1, \gamma_2 \in V^*$ such that $\ker(\gamma_1) = U_1$ and $\ker(\gamma_2) = U_2$. If $U_1 \neq U_2$, $kv_1 \neq kv_2$ then for any $t \in G_1 \cap G_2$ we can find $a, b \in k$ such that $t = t^{\gamma_1}_{av_2} t^{\gamma_2}_{bv_1}$. Let $\gamma_3 \in V^*$ and $v_3 \in V$ such that $\dim_k\langle v_1, v_2, v_3\rangle = \dim_k\langle \gamma_1, \gamma_2, \gamma_3\rangle = 3$. If $G_3 \leq \mathrm{GL}(V)$ is also a hook group with hyperplane $U_3 = \ker(\gamma_3)$ and line v_3 and $t \in G_1 \cap G_2 \cap G_3$ then $t = 1$.*

Proof. For any $u \in \ker(\gamma_1) \cap \ker(\gamma_2)$ we see that $\delta_t(u) \in kr_2 \cap k(2r_1 + r_2) = \{0\}$ so we can find $r_3, r_4 \in V$ such that $t = t^{\gamma_1}_{r_3} t^{\gamma_2}_{r_4}$. As $\ker(\gamma_2) \not\leq \ker(\gamma_1)$ we see that $r_3 \in kv_2$ and similarly $r_4 \in kv_1$, so we can find some $a, b \in k$ such that: $t = t^{\gamma_1}_{av_2} t^{\gamma_2}_{bv_1}$. If $t \in G_1 \cap G_2 \cap G_3$ as above then we see that for some $c_1, c_2, c_3, c_4 \in k$ $t = t^{\gamma_1}_{c_1 v_2} t^{\gamma_2}_{c_2 v_1} = t^{\gamma_1}_{c_3 v_3} t^{\gamma_3}_{c_4 v_1} = t^{\gamma_1}_{c_1 v_2} t^{\gamma_2}_{c_2 v_1} t^{\gamma_3}_{0} = t^{\gamma_1}_{c_3 v_3} t^{\gamma_2}_{0} t^{\gamma_3}_{c_4 v_1}$, so using Lemma 1.5 $c_1 = c_2 = c_3 = c_4 = 0$. $\square$

Lemma 5.7. *Let $g_1 = \chi_{1,0,0}$, $g_2 = \chi_{0,1,0}$ and $\sigma \in \mathrm{GL}(V)$. If $G = \langle g_1, g_2, \sigma\rangle$ is a pure bireflection group then either σ is a double transvection and for some $a \in k$, $\sigma = \chi_{0,0,a}$ or σ is an index 3 bireflection and g_1, σ or g_2, σ are a special pair.*

Proof. Let $z = \chi_{0,0,1}$, $G_1 = \langle g_1, z, \sigma\rangle$ and $G_2 = \langle g_2, z, \sigma\rangle$. As G_1, G_2 are not two–row or two–column groups, by Lemma 2.14 each could be a hook group, exceptional group of type one or exceptional group of type two. As g_1 is an index 3 bireflection G_1 is not an exceptional group of type two. As z is a double transvection if G_1 is an exceptional group of type one then g_1, σ are an exceptional pair. Similarly either G_2 is a hook group or g_2, σ are an exceptional pair. Suppose both G_1 and G_2 are hook groups. As g_1 is an index 3 bireflection we see by Lemma 2.6 that G_1 has hyperplane $\ker(\gamma_1)$ and line kr_2. Similarly G_2 has hyperplane $\ker(\gamma_2)$ and line $k(2r_1 + r_2)$. Using Lemma 5.6 we can find some $a, b \in k$ such that: $\sigma = t^{\gamma_1}_{a(2r_1+r_2)} t^{\gamma_2}_{br_2}$. As $r_1, r_2 \in \ker(\gamma_1) \cap \ker(\gamma_2)$ we see σ is a double transvection.
Let $G_3 = \langle g_1 g_2, z, \sigma\rangle$. Using Lemma 2.14 again, G_3 is either a hook or an exceptional group. As $g_1 g_2$ is an index 3 bireflection it isn't an exceptional group of type two, and as σ, z are double transvections G_3 isn't a exceptional group of type one. This implies that G_3 is a hook group. We know that $g_1 g_2 = t^{\gamma_1}_{v+2r_1+r_2} t^{\gamma_2}_{v+r_1} t^{v^*}_{2r_1+2r_2} = t^{\gamma_1+\gamma_2}_{v+r_1} t^{2v^*+\gamma_1}_{r_1+r_2}$ and $[g_1 g_2, [g_1 g_2, V]] = k(r_1 + r_2)$. By Lemma 2.6 $k(r_1+r_2)$ is the line of G_3, and $U = \ker(\gamma_1+\gamma_2)$ is the hyperplane. If $u_1, u_2 \in \ker(v^*)$ such that for $i, j \in \{1, 2\}$, $\gamma_i(u_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise} \end{cases}$. We can see that $u_1 - u_2 \in \ker(\gamma_1 + \gamma_2) = U$, so $\delta_\sigma(u_1 - u_2) \in k(r_1 + r_2)$, $2ar_1 + (a - b)r_2 \in k(r_1 + r_2)$. For this to happen we must have $b = -a$ and then $\sigma = \chi_{0,0,a}$. $\square$

We can now prove that all exceptional groups of type one are as described above.

Proposition 5.8. *If G is an exceptional group of type one then $G \leq X^{\mathbf{r},\gamma}$ for some $\mathbf{r}, \gamma$.*

Proof. If G is an exceptional group of type one then we can find $\mathbf{r} = \{r_1, r_2, v\}$ and $\gamma = \{\gamma_1, \gamma_2, v^*\}$ such that $\chi^{\mathbf{r},\gamma}_{1,0,0}, \chi^{\mathbf{r},\gamma}_{0,1,0} \in G$. Let $g_1 = \chi^{\mathbf{r},\gamma}_{1,0,0}$, $g_2 = \chi^{\mathbf{r},\gamma}_{0,1,0}$. If $G \not\leq X^{\mathbf{r},\gamma}$ then we can find $\sigma \in G \backslash X^{\mathbf{r},\gamma}$. If G consists of bireflections then $\langle g_1, g_2, \sigma\rangle$ consists of bireflections so by Lemma 5.7 if $\sigma \notin X^{\mathbf{r},\gamma}$ then either g_1, σ or g_2, σ are an exceptional pair. Without loss of generality we can assume g_1, σ are an exceptional pair. By Lemma 2.2 we can find $a, b \in k$, $r_3 \in \ker(\gamma_1) \cap \ker(v^*)$ and $\gamma_3 \in V^*$ linearly independent to γ_1 and v^* such that

$$\gamma_3(r_2) = \gamma_2(r_3) = \gamma_3(v) = \gamma_1(r_3) = 0$$

and $\sigma = t^{\gamma_1}_{\beta_1} t^{\gamma_3}_{\beta_2} t^{v^*}_{\beta_3}$ where $\beta_1 = bv + (a - ab)r_2 + (2a + b)r_3$, $\beta_2 = v - ar'_2 + r_3$, $\beta_3 = 2r_3 + r_2$. Using Lemma 1.4 we can find $z' := \sigma g_1 \sigma^{-1} g_1^{-1} = t^{\gamma_1}_{2r_3+(1-b)r_2} t^{\gamma_3}_{-r_2}$. As z' is not an index 3 bireflection it can't be part of an exceptional pair so because $\langle g_1, g_2, z'\rangle$ must be a pure bireflection group by Lemma 5.7 $z' = \chi_{0,0,c}$ for some $c \in k$. Then $z' = t^{\gamma_1}_{c(2r_1+r_2)} t^{\gamma_2}_{-cr_2} = t^{\gamma_1}_{2r_3+(1-b)r_2} t^{\gamma_3}_{-r_2}$. As γ_1 is linearly independent to γ_2 and γ_3 we can find some $u_1 \in V$ such that $\gamma_1(u_1) = 1$ and $\gamma_2(u_1) = \gamma_3(u_1) = 0$. We find $\delta_{z'}(u_1) = 2r_3 + (1 - b)r_2 = 2cr_1 + cr_2$ so $r_3 = cr_1 + \frac{(c-1+b)r_2}{2}$. By multiplying z' on the right by $(t^{\gamma_1}_{2cr_1+cr_2})^{-1}$ and using Lemma 1.4 we get:

$$t^{\gamma_1}_{c(2r_1+r_2)} t^{\gamma_3}_{-r_2} = t^{\gamma_1}_{c(2r_1+r_2)} t^{\gamma_2}_{-cr_2},\ t^{\gamma_3}_{-r_2} = t^{\gamma_2}_{-cr_2},\ t^{\gamma_3}_{-r_2} = t^{c\gamma_2}_{-r_2}.$$

Using Lemma 1.5 we see that $\gamma_3 = c\gamma_2$. Now we see that $\sigma = t^{\gamma_1}_{\beta_1} t^{\gamma_2}_{c\beta_2} t^{v^*}_{\beta_3}$ for

$$\beta_1 = bv + (a - ab)r_2 + (2a + b)(cr_1 + \frac{(c-1+b)}{2} r_2) = bv + (2ac + bc)r_1 + \frac{b(b-1) + 2ac + bc}{2} r_2,$$

$$c\beta_2 = cv - car_2 + c^2 r_1 + \frac{c(c-1+b)r_2}{2} = cv + c^2 r_1 + \frac{c(c-1+2b) - 2ac - bc}{2},$$

$$\beta_3 = 2(cr_1 + \frac{(c-1+b)r_2}{2}) + r_2 = 2cr_1 + (c+b)r_2.$$

If $L = b$, $M = c$ and $L = -\frac{2ac+bc}{2}$ then $\sigma = \chi^{\mathbf{r},\gamma}_{L,M,N}$, so $\sigma \in X^{\mathbf{r},\gamma}$. □

This allows us to say more about exceptional groups of type one.

Corollary 5.9. *If G is an exceptional group of type one then it contains no transvections and any double transvections in G are contained within $J_{\mathbf{r},\gamma}$, which is a two–row and two–column group.*

Corollary 5.10. *If $k = \mathbb{F}_p$, for fixed $\mathbf{r}, \gamma$, there is only one exceptional group of type one which is an extra special group of order p^3 which is isomorphic to $M(p)$.*

Proof. If G is an exceptional group of type one then by the above proposition $G \le X^{\mathbf{r},\gamma}$, for some $\mathbf{r}, \gamma$, however G has no non-trivial subgroups which contain a special pair, so $G = X^{\mathbf{r},\gamma}$. We can see that $\Phi(G) = [G, G] = Z(G) = J_{\mathbf{r},\gamma}$ so G is extraspecial, and the order of G is p^3. As G has no elements of order greater than p, $G \cong M(p)$. □

6. Exceptional groups of type two

In this section we will treat exceptional groups of type two, as we have with exceptional groups of type one above. Unlike exceptional groups of type one, many of our results for exceptional groups of type two still hold for $p = 2$, so we do not restrict to odd characteristic when we define some groups containing a special triple. We cannot, however, use our earlier classification results for even characteristic, so we restrict to $p \neq 2$ when we show that these are all possible exceptional groups of type two in Proposition 6.4. To be able to find $G \le \mathrm{GL}(V)$ an exceptional group of type two we need $n \ge 6$.

Definition 6.1. *Let $\mathbf{r} = \{r_1, r_2, r_3\}$ with $r_1, r_2, r_3 \in V$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ with $\gamma_1, \gamma_2, \gamma_3 \in r_1^\perp \cap r_2^\perp \cap r_3^\perp$ and $\dim_k\langle r_1, r_2, r_3\rangle = \dim_k\langle \gamma_1, \gamma_2, \gamma_3\rangle = 3$. For all $a, b, c \in k$ define $w^{\mathbf{r},\gamma}_{a,b,c} = t^{\gamma_1}_{\alpha_1} t^{\gamma_2}_{\alpha_2} t^{\gamma_3}_{\alpha_3}$ where: $\alpha_1 = ar_1 + br_3$, $\alpha_2 = ar_2 + cr_3$, $\alpha_3 = br_2 - cr_1$, and $V^{*\mathbf{r},\gamma} = \{w^{\mathbf{r},\gamma}_{a,b,c} | a, b, c \in k\}$. Whenever $\mathbf{r}, \gamma$ are fixed in context we shall write $w^{\mathbf{r},\gamma}_{a,b,c} = w_{a,b,c}$.*

Lemma 6.2. *For fixed $\mathbf{r}, \gamma$ we have:*

(1) $w_{a,b,c} = w_{a',b',b'} \Leftrightarrow l = l', m = m', n = n'$.
(2) $w_{a,b,c} w_{a',b',c'} = w_{a+a',b+b',c+c'}$.
(3) $w_{a,b,c}$ *and* $w_{a',b',c'}$ *commute for all* $a, b, c, a', b', c' \in k$.
(4) $w^{-1}_{a,b,c} = w_{-a,-b,-c}$.

Proof. (1): We can see by using Lemma 1.5.. (2): For $a, b, c, a', b', c' \in k$ we see that

$$w_{a,b,c}w_{a'b',c'} = t^{\gamma_1}_{ar_1+br_3}t^{\gamma_2}_{ar_2+cr_3}t^{\gamma_3}_{br_2-cr_3}t^{\gamma_1}_{a'r_1+b'r_3}t^{\gamma_2}_{a'r_2+c'r_3}t^{\gamma_3}_{b'r_2-c'r_3} =$$

$t^{\gamma_1}_{(a+a')r_1+(b+b')r_3}t^{\gamma_2}_{(a+a')r_2+(c+c')r_3}t^{\gamma_3}_{(b+b')r_2-(c+c')r_3}$.

(3),(4),(5),(6) follow from (2). □

Proposition 6.3. *$G := \langle w^{\mathbf{r},\gamma}_{a,b,c} | a, b, c \in k\rangle = W^{\mathbf{r},\gamma}$ is an elementary abelian group, with $|G| = q^3$ if $k = \mathbb{F}_q$.*

Proof. By Proposition 6.2(2) all elements of the group can be written as $w_{a,b,c}$ for some $a, b, c \in k$, so $G = \{w_{a,b,c} | a, b, c \in k\} = W^{\mathbf{r},\gamma}$. As all elements commute and have order p we see that G is elementary abelian. By Proposition 6.2(1) $w_{a,b,c} = w_{a',b',c'}$ if and only if $a = a', b = b', c = c'$ so $|W^{\mathbf{r},\gamma}| = q^3$. □

Proposition 6.4. *Let $p \neq 2$. If $G \in \mathrm{GL}(V)$ is an exceptional group of type two then there exists some $\mathbf{r}, \gamma$ such that for all $h \in G$, $h = w^{\mathbf{r},\gamma}_{a,b,c}$ for some $a, b, c \in k$.*

Proof. As G is an exceptional group of type two we can find a subgroup $H = \langle g_1, g_2, g_3\rangle$ such that g_1, g_2, g_3 are a special triple. This means that for some $\mathbf{r} = \{r_1, r_2, r_3\}$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $s \in k$: $g_1 = w^{\mathbf{r},\gamma}_{1,0,0}$, $g_2 = w^{\mathbf{r},\gamma}_{0,1,0}$, $g_3 = w^{\mathbf{r},\gamma}_{0,0,s}$. We will show that for all $h \in G$ we can find some $a, b, c \in k$ such that $h = w^{\mathbf{r},\gamma}_{a,b,c}$. From Proposition 5.9 we can see that G is not an exceptional group of type one: for any exceptional group of type one all elements which are not index three bireflections are contained within the centre which is a two–column (and two–row group). The special triple g_1, g_2, g_3 are all double transvections which are not contained in any single two–row or two–column group. For all $h \in G$ the subgroups $\langle g_i, g_j, h\rangle$ for $1 \leq i < j \leq 3$ consist of bireflections so by Lemma 2.14 they are either hook groups or exceptional groups of type two. Suppose $\langle g_1, g_2, h\rangle$ is a hook group then it has hyperplane $\ker(\gamma_1)$ and line kr_2. Similarly if $\langle g_1, g_3, h\rangle$ is a hook group then it has hyperplane $\ker(\gamma_2)$ and line kr_1, and if $\langle g_2, g_3, h\rangle$ then it has hyperplane γ_3 and line r_3. As $\gamma_1, \gamma_2, \gamma_3$ and r_1, r_2, r_3 are linearly independent this means if all three groups are hook groups by Lemma 5.6, $h = 1$. For h not the identity we know that for some $1 \leq i < j \leq 3$ that $\langle g_i, g_j, h\rangle$ is not a hook group, we can assume $i = 1$ and $j = 2$ without loss of generality. We can find $\mathbf{r}' = \{r'_1, r'_2, r'_3\}$, $\gamma' = \{\gamma'_1, \gamma'_2, \gamma'_3\}$ such that for some $n \in k$: $g_1 = w^{\mathbf{r}',\gamma'}_{1,0,0}$, $g_2 = w^{\mathbf{r}',\gamma'}_{0,1,0}$, $h = w^{\mathbf{r}',\gamma'}_{0,0,n}$. Then: $t^{\gamma_1}_{r_1}t^{\gamma_2}_{r_2} = t^{\gamma'_1}_{r'_1}t^{\gamma'_2}_{r'_2}$, $t^{\gamma_1}_{r_3}t^{\gamma_3}_{r_2} = t^{\gamma'_1}_{r'_3}t^{\gamma'_3}_{r'_2}$. As $(k\gamma_1 + k\gamma_2) \cap (k\gamma_1 + k\gamma_3) = k\gamma_1$, $(kr_1 + kr_2) \cap (kr_2 + kr_3) = kr_2$ for some $l, m \in k$, $\gamma_1 = \gamma'_1$, $\gamma'_2 = \gamma_2 + l\gamma_1$, $\gamma'_3 = \gamma_3 + m\gamma_1$, $r'_1 = r_1 - lr_2$, $r_2 = r'_2$, $r'_3 = r_3 - mr_2$. Using this we find that $h = t^{\gamma'_2}_{nr'_3}t^{\gamma'_3}_{-nr'_1} = t^{l\gamma_1}_{n(r_3-mr_2)}t^{m\gamma_1}_{-n(r_1-lr_2)}t^{\gamma_2}_{n(r_3-mr_2)}t^{\gamma_3}_{-n(r_1-lr_2)} = t^{\gamma_1}_{-mnr_1+lnr_2}t^{\gamma_2}_{nr_3-mnr_2}t^{\gamma_3}_{-nr_1+lnr_2} = w^{\mathbf{r},\gamma}_{-mn,ln,n}$ as required. □

Corollary 6.5. *If G is an exceptional group of type two then it contains no transvections or index 3 bireflections.*

Corollary 6.6. *If $k = \mathbb{F}_p$ then for fixed $\mathbf{r}, \gamma$ there is only one exceptional group of type two which is an elementary abelian group of order p^3.*

7. Maximal pure bireflection groups over finite fields

For later applications in invariant theory it is useful to know more about "maximal" pure bireflection group over finite fields. Throughout this section, k is a finite field of order $q = p^r$.

Definition 7.1. *A subgroup $G \leq \mathrm{GL}(V)$ is called a maximal pure bireflection group if it is a pure bireflection group, and for all $G \leq H \leq \mathrm{GL}(V)$ either $H = G$ or H is not a pure bireflection group.*

Lemma 7.2. *Let $p \neq 2$, $n \geq 3$. If G is a maximal pure unipotent bireflection group then it is a special group and one of the following holds:*

(1) *$G = B^{U,v}$ for some hyperplane $U < V$, $v \in U$. $|G| = q^{2n-3}$.*

(2) *$G = K^{r_1,r_2}$ or $G = (K^{\gamma_1,\gamma_2})^*$ for some $r_1, r_2 \in V$ or $\gamma_1, \gamma_2 \in V^*$. Then $|G| = q^{2n-3}$.*
(3) *$G = X^{\mathbf{r},\gamma}$ for some $\mathbf{r} = \{r_1, r_2, v\}$, $\gamma = \{\gamma_1, \gamma_2, v^*\}$. Then $|G| = q^3$.*
(4) *$G = W^{\mathbf{r},\gamma}$ for some $\mathbf{r} = \{r_1, r_2, r_3\}$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Then $|G| = q^3$.*

If $k = \mathbb{F}_p$, then G is extra special or abelian if and only if it is self-dual.

Proof. We show in Proposition 0.5 that if G is a pure bireflection group then it is either a hook, two–row, two–column or exceptional group. Suppose it is a hook group. Then we can find some U, v such that $[G, V] \leq kv$, so $G \leq B^{U,v}$, as G is maximal $G = B^{U,v}$, similarly for G a two–row, two–column and exceptional group. Let $k = \mathbb{F}_p$. By Proposition 3.7 if $G = K^{r_1,r_2}$ then $|\Phi(G)| = |L^{r_1,r_2}| = p^{n-2} > p$ for $n \geq 3$, so G is not extra special if it is a two–row or two–column group. If G is not a two–row or two–column group then it is either a hook group or an exceptional group and is self dual. If G is a hook group then $|\Phi(G)| = |R_{v,U}| = p$, so G is extraspecial. If G is exceptional of type one then $|\Phi(G)| = |J_{\mathbf{r},\gamma}| = p$, so it is extra special. If it is exceptional of type two then it is abelian. □

Corollary 7.3. *If G is a pure unipotent bireflection group, $p \neq 2$, $n \geq 3$, then it is a subgroup of one of the groups in Lemma 7.2 and it has class less than or equal to two.*

Proof. If G is a pure unipotent bireflection group then it must be either a maximal pure bireflection group or contained in a maximal pure bireflection group. Above gives the list of all possible pure bireflection groups which are all special, so each of their subgroups must have class less than or equal to two. □

The following Proposition summarises the results of this Chapter and contains the proof of the Main Theorem 0.5.

Proposition 7.4. *Let $p > 2$, $n \geq 3$ and $g \in G$, a unipotent pure bireflection group.*

(1) *If $g = t_u^\zeta$ is a transvection then G is one of the following*
  (a) *A subgroup of K^{r_1,r_2} with $u \in \langle r_1, r_2\rangle$,*
  (b) *A subgroup of $(K^{\gamma_1,\gamma_2})^*$ with $\zeta \in \langle \gamma_1, \gamma_2\rangle$,*
  (c) *A subgroup of $B^{U,v}$ with either $U = \ker(\zeta)$ or $u \in kv$.*
(2) *If $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ is a double transvection so $u_1, u_2 \in \ker(\zeta_1) \cap \ker(\zeta_2)$ then G is one of the following*
  (a) *A subgroup of K^{r_1,r_2} with $\langle r_1, r_2\rangle = \langle u_1, u_2\rangle$,*
  (b) *A subgroup of $(K^{\gamma_1,\gamma_2})^*$ with $\langle \gamma_1, \gamma_2\rangle = \langle \zeta_1, \zeta_2\rangle$,*
  (c) *A subgroup of $B^{U,v}$ such that $v \in \langle u_1, u_2\rangle$*
  (d) *A subgroup of $G \leq X^{\mathbf{r},\gamma}$ where $\langle r_1, r_2\rangle = \langle u_1, u_2\rangle$ and $\langle \gamma_1, \gamma_2\rangle = \langle \zeta_1, \zeta_2\rangle$,*
  (e) *A subgroup of $G \leq W^{\mathbf{r},\gamma}$ where $\langle r_1, r_2, r_3\rangle > \langle u_1, u_2\rangle$ and $\langle \gamma_1, \gamma_2, \gamma_3\rangle > \langle \zeta_1, \zeta_2\rangle$.*
(3) *If $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ is an index 3 bireflection so $u_1 \notin \ker(\zeta_2)$ and $u_2 \in \ker(\zeta_1)$ then G is one of the following*
  (a) *A subgroup of K^{u_1,u_2},*
  (b) *A subgroup of $(K^{\zeta_1,\zeta_2})^*$,*
  (c) *A subgroup of $B^{U,v}$ where $U = \ker(\zeta_2)$ and $v \in ku_2$,*
  (d) *A subgroup of $G \leq X^{\mathbf{r},\gamma}$ where $\langle r_1, r_2, v\rangle > \langle u_1, u_2\rangle$ and $\langle \gamma_1, \gamma_2, v^*\rangle > \langle \zeta_1, \zeta_2\rangle$.*

Proof. (1): Suppose $g = t_u^\zeta$ is a transvection. By the above corollary we know that G must be a subgroup of one of the groups in Lemma 7.2. If G is a two–column group then by Lemma 3.6 we can find $r_1, r_2 \in V$ such that $ku = [g, V] \leq [G, V] \leq \langle r_1, r_2\rangle$ and $G \leq K^{r_1,r_2}$. If G is a two–row group then G^* is a two–column group so we can find $\gamma_1, \gamma_2 \in V^*$ such that $k\zeta \leq [G, V^*] \leq \langle \gamma_1, \gamma_2\rangle$ and $G \leq (K^{\gamma_1,\gamma_2})^*$. Suppose G is a hook group with line kv and hyperplane U. Either $u \in kv$ or $U = \ker(\zeta)$. By Corollaries 5.9 and 6.5 we know that G is not contained in an exceptional group of type one or type two.
(2): Suppose $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ is a double transvection. If G is a two–row group then by Lemma 3.6 we can find $r_1, r_2 \in V$ with $\langle u_1, u_2\rangle = [G, V] = \langle r_1, r_2\rangle$ such that $G \leq K^{r_1,r_2}$. If G is two–column group then G^* is a two–row group and by Lemma 3.6 we can find γ_1, γ_2 with

$\langle \zeta_1, \zeta_2 \rangle = [G, V^*] = \langle \zeta_1, \zeta_2 \rangle$ such that $G \leq (K^{\gamma_1,\gamma_2})^*$. If $G \leq B^{U,v}$ is a hook group then as V^g has codimension two $U \neq V^g$, hence $kv \leq \langle u_1, u_2 \rangle$. If G is an exceptional group of type one then by Corollary 5.9 $g \in J_{\mathbf{r},\gamma} = \{\chi^{\mathbf{r},\gamma}_{0,0,n} \mid n \in k\}$. This means that $\langle u_1, u_2 \rangle = [J_{\mathbf{r},\gamma}, V] = \langle r_1, r_2 \rangle$ and $\langle \gamma_1, \gamma_2 \rangle = \langle \zeta_1, \zeta_2 \rangle$. Let $H = W^{\mathbf{r},\gamma}$, if $G \leq H$ then $[g, V] = \langle u_1, u_2 \rangle \leq [H, V] = \langle r_1, r_2, r_3 \rangle$ and similarly $V^H \leq V^g$ so $\langle \zeta_1, \zeta_2 \rangle < \langle \gamma_1, \gamma_2, \gamma_3 \rangle$.
(3): Suppose $g = t^{\zeta_1}_{u_1} t^{\zeta_2}_{u_2}$ is an index 3 bireflection. If G is a two row group then we can again use Lemma 3.6 to see that $G = K^{u_1,u_2}$. Similarly by looking at the dual space we see that if G is a two–row group then $G \leq (K^{\zeta_1,\zeta_2})^*$. If G is a hook group we just apply Lemma 2.6. If $G \leq X^{\mathbf{r},\gamma}$ then $[g, V] \leq [X^{\mathbf{r},\gamma}, V] = \langle r_1, r_2, v \rangle$, so $\langle r_1, r_2, v \rangle > \langle u_1, u_2 \rangle$. By looking at the fixed space (or by looking at the duals of both groups) we see that $\langle \gamma_1, \gamma_2, v^* \rangle > \langle \zeta_1, \zeta_2 \rangle$. By Corollary 6.5 we know that G is not contained in an exceptional group of type two. □

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