Sliding mode control : a tutorial

Sarah Spurgeon[1]

***Abstract*— The fundamental nature of sliding mode control is described. Emphasis is placed upon presenting a constructive theoretical framework to facilitate practical design. The developments are illustrated with numerical examples throughout.**

I. INTRODUCTION

Sliding mode control evolved from pioneering work in the 1960's in the former Soviet Union [1], [2], [3], [4]. It is a particular type of Variable Structure System (VSS) which is characterised by a number of feedback control laws and a decision rule. The decision rule, termed the switching function, has as its input some measure of the current system behaviour and produces as an output the particular feedback controller which should be used at that instant in time. In sliding mode control, Variable Structure Control Systems (VSCS) are designed to drive and then constrain the system state to lie within a neighbourhood of the switching function. One advantage is that the dynamic behaviour of the system may be directly tailored by the choice of switching function - essentially the switching function is a measure of desired performance. Additionally, the closed-loop response becomes totally insensitive to a particular class of system uncertainty. This class of uncertainty is called *matched uncertainty* and is categorised by uncertainty that is implicit in the input channels. Large classes of problems of practical significance naturally contain matched uncertainty, for example, mechanical systems [5], [6], and this has fuelled the popularity of the domain.

A disadvantage of the method has been the necessity to implement a discontinuous control signal which, in theoretical terms, must switch with infinite frequency to provide total rejection of uncertainty. Control implementation via approximate, smooth strategies is widely reported [7], but in such cases total invariance is routinely lost. There are some important application domains where a switched control strategy is usual and desirable, for example, in power electronics, and many important applications and implementations have been developed [8],[9],[10]. More recent contributions have extended the sliding mode control paradigm and introduced the concept of *higher order sliding mode control* where one motivation is to seek a smooth control that will naturally and accurately encompass the benefits of the traditional approach to sliding mode control [11].

A simple example is the scaled pendulum

$$\ddot{y} = -a_1 \sin(y) + u \qquad (1)$$

[1]Sarah Spurgeon is with the School of Engineering and Digital Arts, University of Kent, Canterbury, UK S.K.Spurgeon@kent.ac.uk

where y denotes the angular position and u denotes the control, or torque, applied at the suspension point. The scalar a_1 is positive and when $a_1 = 0$ the dynamics (1) collapse to the case of a nominal double integrator. An alternative interpretation of equation (1) is that the case $a_1 = 0$ corresponds to a nominal system and the term $-a_1 \sin(y)$ corresponds to bounded uncertainty within the nominal dynamics. Define a switching function, s which represents idealised dynamics corresponding to a first order system with a pole at -1

$$s = \dot{y} + y \qquad (2)$$

In the sliding mode, when $s = 0$, the dynamics of the system are determined by the dynamics $\dot{y} = -y$, a free system where the initial condition is determined by $(y(t_s), \dot{y}(t_s))$, where t_s is the time at which the sliding mode condition, $s = 0$ is reached. Defining the sliding mode dynamics by selecting an appropriate sliding surface is termed as solving the *existence problem*. A control to ensure the desired sliding mode dynamics are attained and maintained is sought by means of solving the *reachability problem*. A fundamental requirement is that the sliding mode dynamics must be attractive to the system state and there are many *reachability conditions* defined in the literature [3], [4], [12]. Using the so called $\eta-$reachbility condition:

$$s\dot{s} < -\eta|s| \qquad (3)$$

it is straightforward to verify that the control

$$u = -\dot{y} - \rho \operatorname{sgn}(s) \qquad (4)$$

for $\rho > a_1 + \eta$ where η is a small positive design scalar ensures the reachability condition is satisfied. Figure 1 shows the response of the system (1) with the control (4) in the nominal case of the double integrator, when $a_1 = 0$ and in the case of the pendulum, when $a_1 = 1$. The transient onto the desired sliding mode dynamic is different in each

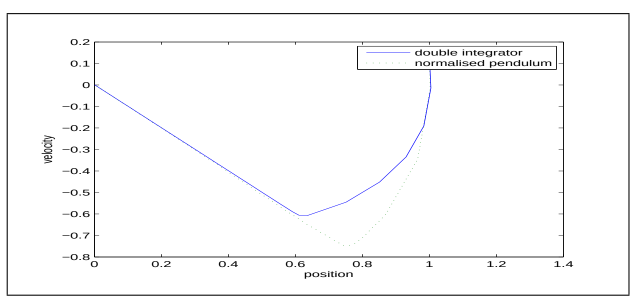

Fig. 1: Phase plane portrait showing the response of the double integrator ($a_1 = 0$) and the scaled pendulum system ($a_1 = 1$) with initial conditions $y(0) = 1, \dot{y}(0) = 0.1$

case, but once the sliding mode is reached both systems exhibit the dynamics of the free first order system with a pole at -1. Note that the trajectories appear smooth as the discontinuous $\text{sgn}(s)$ function in (4) is approximated by the smooth approximation $\frac{s}{|s|+\delta}$ where $\delta > 0$ is small and in the simulations was taken as $\delta = 0.01$.

This tutorial paper seeks to introduce sliding mode control. Particular emphasis is placed on describing constructive frameworks to facilitate sliding mode control design. The paper is structured as follows. Section II formulates the classical sliding mode control paradigm in a state space framework and introduces some of the defining characteristics of the approach. A framework for synthesis of classical sliding mode controllers is described in Section III and Section IV presents a tutorial design example. Section V introduces the concepts of higher order sliding mode control.

II. CLASSICAL SLIDING MODE CONTROL

Consider the following uncertain dynamical system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t,x,u) \\ y(t) &= Cx(t) \end{aligned} \tag{5}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m \leq p \leq n$ represent the usual state, input and output. The exposition is deliberately formulated as an output feedback problem in order to describe the constraints imposed by the availability of limited state information but the analysis collapses to state feedback when C is chosen as the identity matrix. Assume that the nominal linear system (A,B,C) is known and that the input and output matrices B and C are both of full rank. The system nonlinearities and model uncertainties are represented by the unknown function $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which is assumed to satisfy the matching condition whereby

$$f(t,x,u) = B\xi(t,x,u) \tag{6}$$

The bounded function $\xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\|\xi(t,x,u)\| < k_1\|u\| + \alpha(t,y) \tag{7}$$

for some known function $\alpha : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}_+$ and positive constant $k_1 < 1$. The intention is to develop a control law which induces an ideal sliding motion on the surface

$$\mathscr{S} = \{x \in \mathbb{R}^n \ : FCx = 0\} \tag{8}$$

for some selected matrix $F \in \mathbb{R}^{m\times p}$. A control law comprising linear and discontinuous feedback is sought

$$u(t) = -Gy(t) - \nu_y \tag{9}$$

where G is a fixed gain matrix and the discontinuous vector is given by

$$\nu_y = \begin{cases} \rho(t,y)\frac{Fy(t)}{\|Fy(t)\|} & \text{if } Fy \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $\rho(t,y)$ is some positive scalar function.

The motivating example presented in Section I clearly demonstrates that two systems with different dynamics, the double integrator and the scaled pendulum, exhibit the same first order dynamics when in the sliding mode. It is thus intuitively obvious that the effective control action experienced by what are two different plants must be different. The so-called *equivalent control* represents this effective control action which is necessary to maintain the ideal sliding motion on $\mathscr{S}$. The equivalent control action is not the control action applied to the plant but can be thought of as representing, on average, the effect of the applied discontinuous control.

To explore the concept of the equivalent control more formally, consider equation (5) and suppose at time t_s the systems states lie on the surface $\mathscr{S}$ defined in (8). It is assumed an ideal sliding motion takes place so that $FCx(t) = 0$ and $\dot{s}(t) = FC\dot{x}(t) = 0$ for all $t \geq t_s$. Substituting for $\dot{x}(t)$ from (5) gives

$$S\dot{x}(t) = FCAx(t) + FCBu(t) + FCf(t,x,u) = 0 \tag{11}$$

for all $t \geq t_s$. Suppose the matrix FC is such that the square matrix FCB is nonsingular. This does not present problems since by assumption B and C are full rank and F is a design parameter that can be selected. The corresponding *equivalent control* associated with (5) which will be denoted as u_{eq} to demonstrate that it is not the applied control signal, u, is defined to be the solution to equation (11):

$$u_{eq}(t) \quad = \quad -(FCB)^{-1}FCAx(t) - \xi(t,x,u) \tag{12}$$

from (6). The necessity for FCB to be nonsingular ensures the solution to (11), and therefore the equivalent control, is unique. The ideal sliding motion is then given by substituting the expression for the equivalent control into equation (5):

$$\dot{x}(t) = \left(I_n - B(FCB)^{-1}FC\right)Ax(t) \tag{13}$$

for all $t \geq t_s$ and $FCx(t_s) = 0$. The corresponding motion is independent of the control action and it is clear that the effect of the matched uncertainty present in the system has been nullified. The concept of the equivalent control enables the inherent robustness of the sliding mode control approach to be understood and it is clear from (12) why the total insensitivity to matched uncertainty holds when sliding motion is exhibited. Reconsider the perturbed double integrator in equation (1), this time it will be assumed that $a_1 = 0$ and that the system is subject to the persisting external perturbation $-0.1\sin(t)$. Figure 2 shows a plot of $0.1\sin(t)$ in relation to the smooth control signal applied to the plant. It is seen that the applied (smooth) control signal replicates very closely the applied perturbation, even though the control signal is not constructed with a priori knowledge of the perturbation. This property has resulted in great interest in the use of sliding mode approaches for condition monitoring and fault detection [13]. A key feature of the sliding mode control approach is the ability to specify desired plant dynamics by choice of the switching function. Whilst sliding $s = FCx = 0$ for all $t > t_s$ and it follows that exactly m of the states can be expressed in terms of the remaining $n-m$. It can be shown that the matrix (13) defining the equivalent system dynamics has at most $n-m$ nonzero eigenvalues and these are the poles of the reduced order dynamics in the sliding mode.

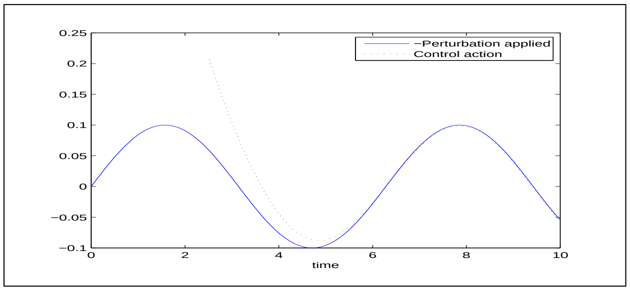

Fig. 2: The relationship between the smooth control signal applied and the external perturbation once the sliding mode is reached

An alternative interpretation is that the poles of the sliding motion are the invariant zeros of the triple A, B, FC.

III. CANONICAL FORM FOR DESIGN

This section will consider synthesis of a sliding mode control for the system in (5). It is assumed that $p \geq m$ and $\text{rank}(CB) = m$ where the rank restriction is required for existence of a unique equivalent control. The first problem which must be considered is how to choose F so that the associated sliding motion is stable. A control law will then be defined to guarantee the existence of a sliding motion.

A. Switching Function Design

In view of the fact that the outputs will be considered, it is first convenient to introduce a coordinate transformation to make the last p states of the system the outputs. Define

$$T_c = \begin{bmatrix} N_c^{\mathrm{T}} \\ C \end{bmatrix} \tag{14}$$

where $N_c \in \mathbb{R}^{n\times(n-p)}$ and its columns span the null space of C. The coordinate transformation $x \mapsto T_c x$ is nonsingular by construction and, as a result, in the new coordinate system

$$C = \begin{bmatrix} 0 & I_p \end{bmatrix}$$

From this starting point a special case of the so-called regular form defined for the state feedback case [14] will be established. Suppose

$$B = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix}$$

Then $CB = B_{c2}$ and so by assumption $\text{rank}(B_{c2}) = m$. Hence the left pseudo-inverse $B_{c2}^{\dagger} = (B_{c2}^{\mathrm{T}} B_{c2})^{-1} B_{c2}^{\mathrm{T}}$ is well defined and there exists an orthogonal matrix $T \in \mathbb{R}^{p\times p}$ such that

$$T^{\mathrm{T}} B_{c2} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{15}$$

where $B_2 \in \mathbb{R}^{m\times m}$ is nonsingular. Consequently, the coordinate transformation $x \mapsto T_b x$ where

$$T_b = \begin{bmatrix} I_{n-p} & -B_{c1} B_{c2}^{\dagger} \\ 0 & T^{\mathrm{T}} \end{bmatrix} \tag{16}$$

is nonsingular and the triple (A, B, C) is in the form

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ C &= \begin{bmatrix} 0 & T \end{bmatrix} \end{aligned} \tag{17}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly. Let

$$\begin{matrix} \overset{p-m}{\leftrightarrow} & \overset{m}{\leftrightarrow} \\ \end{matrix}$$
$$\begin{bmatrix} F_1 & F_2 \end{bmatrix} = FT$$

where T is the matrix from equation (15). As a result

$$FC = \begin{bmatrix} F_1 C_1 & F_2 \end{bmatrix} \tag{18}$$

where

$$C_1 \triangleq \begin{bmatrix} 0_{(p-m)\times(n-p)} & I_{(p-m)} \end{bmatrix} \tag{19}$$

Therefore $FCB = F_2 B_2$ and the square matrix F_2 is nonsingular. By assumption the uncertainty is matched and therefore the sliding motion is independent of the uncertainty. In addition, because the canonical form in (17) can be viewed as a special case of the regular form normally used in sliding mode controller design, the reduced-order sliding motion is governed by a free motion with system matrix $A_{11}^s \triangleq A_{11} - A_{12} F_2^{-1} F_1 C_1$ which must therefore be stable. If $K \in \mathbb{R}^{m\times(p-m)}$ is defined as $K = F_2^{-1} F_1$ then

$$A_{11}^s = A_{11} - A_{12} K C_1 \tag{20}$$

and the problem of hyperplane design is equivalent to a *static output feedback problem* for the triple (A_{11}, A_{12}, C_1). Appealing to established results from the wider control theory, it is necessary that the pair (A_{11}, A_{12}) is controllable and (A_{11}, C_1) is observable. It is straightforward to verify that (A_{11}, A_{12}) is controllable if the nominal matrix pair (A, B) is controllable. To investigate the observability of (A_{11}, C_1) partition the submatrix A_{11} so that

$$A_{11} = \begin{bmatrix} A_{1111} & A_{1112} \\ A_{1121} & A_{1122} \end{bmatrix} \tag{21}$$

where $A_{1111} \in \mathbb{R}^{(n-p)\times(n-p)}$ and suppose the matrix pair (A_{1111}, A_{1121}) is observable. It follows that

$$\begin{aligned} \text{rank} \begin{bmatrix} zI - A_{11} \\ C_1 \end{bmatrix} &= \text{rank} \begin{bmatrix} zI - A_{1111} & A_{1112} \\ A_{1121} & zI - A_{1122} \\ 0 & I_{p-m} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} zI - A_{1111} \\ A_{1121}^o \end{bmatrix} + (p-m) \end{aligned}$$

for all $z \in \mathbb{C}$ and hence from the Popov-Belevitch-Hautus (PBH) rank test and using the fact that (A_{1111}, A_{1121}) is observable, it follows that

$$\text{rank} \begin{bmatrix} zI - A_{11} \\ C_1 \end{bmatrix} = n - m$$

for all $z \in \mathbb{C}$ and hence (A_{11}, C_1) is observable. If the pair (A_{1111}, A_{1121}) is not observable then there exists a

$T_{obs} \in \mathbb{R}^{(n-p)\times(n-p)}$ which puts the pair into the following observability canonical form:

$$\begin{aligned} T_{obs}A_{1111}T_{obs}^{-1} &= \begin{bmatrix} A_{11}^o & A_{12}^o \\ 0 & A_{22}^o \end{bmatrix} \\ A_{1121}T_{obs}^{-1} &= \begin{bmatrix} 0 & A_{21}^o \end{bmatrix} \end{aligned}$$

where $A_{11}^o \in \mathbb{R}^{r\times r}$, $A_{21}^o \in \mathbb{R}^{(p-m)\times(n-p-r)}$, the pair (A_{22}^o, A_{21}^o) is completely observable and $r \geq 0$ represents the number of unobservable states of (A_{1111}, A_{1121}). The transformation T_{obs} can be embedded in a new state transformation matrix

$$T_a = \begin{bmatrix} T_{obs} & 0 \\ 0 & I_p \end{bmatrix} \tag{22}$$

which, when used in conjunction with T_c and T_b from equations (14) and (16), generates the required canonical form.

1) Canonical Form for Sliding Mode Control Design: Let (A,B,C) be a linear system with $p > m$ and $\mathrm{rank}(CB) = m$. Then a change of coordinates exists so that the system triple with respect to the new coordinates has the following structure:

- The system matrix can be written as

$$A_f = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{23}$$

  where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the sub-block A_{11} when partitioned has the structure

$$A_{11} = \left[\begin{array}{cc|c} A_{11}^o & A_{12}^o & \multirow{2}{*}{A_{12}^m} \\ 0 & A_{22}^o & \\ \hline 0 & A_{21}^o & A_{22}^m \end{array}\right] \tag{24}$$

  where $A_{11}^o \in \mathbb{R}^{r\times r}$, $A_{22}^o \in \mathbb{R}^{(n-p-r)\times(n-p-r)}$ and $A_{21}^o \in \mathbb{R}^{(p-m)\times(n-p-r)}$ for some $r \geq 0$ and the pair (A_{22}^o, A_{21}^o) is completely observable.
- The input distribution matrix has the form

$$B_f = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{25}$$

  where $B_2 \in \mathbb{R}^{m\times m}$ and is nonsingular.
- The output distribution matrix has the form

$$C_f = \begin{bmatrix} 0 & T \end{bmatrix} \tag{26}$$

  where $T \in \mathbb{R}^{p\times p}$ and is orthogonal.

In the case where $r > 0$, it is necessary to construct a new system $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ which is both controllable and observable with the property that

$$\lambda(A_{11}^s) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1 K \tilde{C}_1)$$

Partition the matrices A_{12} and A_{12}^m as follows

$$A_{12} = \begin{bmatrix} A_{121} \\ A_{122} \end{bmatrix} \quad \text{and} \quad A_{12}^m = \begin{bmatrix} A_{121}^m \\ A_{122}^m \end{bmatrix} \tag{27}$$

where $A_{122} \in \mathbb{R}^{(n-m-r)\times m}$ and $A_{122}^m \in \mathbb{R}^{(n-p-r)\times(p-m)}$. Form a new sub-system $(\tilde{A}_{11}, A_{122}, \tilde{C}_1)$ where

$$\begin{aligned} \tilde{A}_{11} &\triangleq \begin{bmatrix} A_{22}^o & A_{122}^m \\ A_{21}^o & A_{22}^m \end{bmatrix} \\ \tilde{C}_1 &\triangleq \begin{bmatrix} 0_{(p-m)\times(n-p-r)} & I_{(p-m)} \end{bmatrix} \end{aligned} \tag{28}$$

The spectrum of A_{11}^o represents the invariant zeros of the nominal system (A,B,C).

From discussion of the canonical form it follows that there exists a matrix F defining a surface $\mathscr{S}$ which provides a stable sliding motion with a unique equivalent control if and only if

- the invariant zeros of (A,B,C) lie in $\mathbb{C}_-$
- the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$ is stabilisable by output feedback.

The first condition ensures that any invariant zeros, which will appear within the poles of the reduced order sliding motion, are stable. The second condition ensures the reduced order sliding mode dynamics are rendered stable by the choice of sliding surface. Although these conditions may appear restrictive, techniques such as sliding mode differentiators [15], or other soft sensors, and the availability of inexpensive hardware sensors can be helpful in ensuring sufficient information is available to tailor the reduced order dynamics in the sliding mode.

B. Reachability of the Sliding Mode

Having established a desired sliding mode dynamics by selecting $K_1 \in \mathbb{R}^{m' \times (p-m)}$ such that $\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1$ is stable and providing any invariant zeros are stable, it follows that

$$\lambda(A_{11} - A_{12}KC_1) = \lambda(A_{11}^o) \cup \lambda(\tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_1)$$

and so the matrix $A_{11} - A_{12}KC_1$ determining the dynamics in the sliding mode is stable. Choose

$$F = F_2 \begin{bmatrix} K & I_m \end{bmatrix} T^{\mathrm{T}}$$

where nonsingular $F_2 \in \mathbb{R}^{m\times m}$ must be selected. Introduce a nonsingular state transformation $x \mapsto \bar{T}x$ where

$$\bar{T} = \begin{bmatrix} I_{(n-m)} & 0 \\ KC_1 & I_m \end{bmatrix} \tag{29}$$

and C_1 is defined in (19). In this new coordinate system the system triple $(\bar{A}, \bar{B}, F\bar{C})$ has the property that

$$\begin{aligned} \bar{A} &= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ F\bar{C} &= \begin{bmatrix} 0 & F_2 \end{bmatrix} \end{aligned} \tag{30}$$

where $\bar{A}_{11} = A_{11} - A_{12}KC_1$ and is therefore stable. An alternative description is that by an appropriate choice of F a new square system $(\bar{A}, \bar{B}, F\bar{C})$ has been synthesised which is minimum phase and relative degree 1.

Let P be a symmetric positive definite matrix partitioned conformably with the matrices in (30) so that

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tag{31}$$

where the symmetric positive definite sub-block P_2 is a design matrix and the symmetric positive definite sub-block P_1 satisfies the Lyapunov equation

$$P_1\bar{A}_{11} + \bar{A}_{11}^{\mathrm{T}}P_1 = -Q_1 \tag{32}$$

for some symmetric positive definite matrix Q_1. If $F \stackrel{\Delta}{=} B_2^{\mathrm{T}} P_2$ then the matrix P satisfies the structural constraint $P\bar{B} = \bar{C}^{\mathrm{T}} F^{\mathrm{T}}$ For notational convenience let

$$Q_2 \quad \stackrel{\Delta}{=} \quad P_1 \bar{A}_{12} + \bar{A}_{21}^{\mathrm{T}} P_2 \tag{33}$$

$$Q_3 \quad \stackrel{\Delta}{=} \quad P_2 \bar{A}_{22} + \bar{A}_{22}^{\mathrm{T}} P_2 \tag{34}$$

and define

$$\gamma_0 \stackrel{\Delta}{=} \tfrac{1}{2} \lambda_{max} \left((F^{-1})^{\mathrm{T}} (Q_3 + Q_2^{\mathrm{T}} Q_1^{-1} Q_2) F^{-1} \right) \tag{35}$$

The symmetric matrix $L(\gamma) \stackrel{\Delta}{=} PA_0 + A_0^{\mathrm{T}} P$ where $A_0 = \bar{A} - \gamma \bar{B} F \bar{C}$ is negative definite if and only if $\gamma > \gamma_0$. A control law to induce sliding on $\mathscr{S}$ is given by equations (9-10) with $G = \gamma F$ and $\gamma > \gamma_0$ where γ_0 is defined in (35). The uncertain system (5) is quadratically stable and an ideal sliding motion is induced on $\mathscr{S}$.

IV. NUMERICAL EXAMPLE

Consider the nominal linear system representing the longitudinal dynamics of a fixed wing Unmanned Aerial Vehicle (UAV) developed for medium-range autonomous flight missions including search and rescue, weather monitoring, aerial photography and reconnaissance [16]. The states represent deviations from nominal forward speed u (m/s), vertical velocity w (m/s), pitch rate q (rad/s) and pitch angle $\theta(rad)$ and the control signal is the elevator deflection η (rad) [16]:

$$A = \begin{bmatrix} -0.218 & -0.225 & 4.990 & -9.184 \\ -0.137 & -0.233 & 10.592 & -2.984 \\ 0.009 & -0.070 & -3.282 & -0.566 \\ 0 & -0.002 & 0.969 & -0.014 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1.754 & 2.301 & -4.741 & -0.063 \end{bmatrix} \tag{36}$$

The poles of the open-loop system are located at $-2.778, -0.044 \pm 0.1587j, -0.881$. Consider first the situation where the pitch angle, θ only is measured.

$$C_\theta = \begin{bmatrix} 0 & 0 & 0 & 10 \end{bmatrix} \tag{37}$$

In this case a single-input single-output system results and there are no degrees of freedom available to design the sliding surface, which in this case is wholly defined by the output equation. The dynamics in the sliding mode are given by the transmission zeros of the triple (A, B, C_θ) which can be computed to be $-76.331, -0.419, -0.056$.

Consider now the case where both u and θ are measured:

$$C_{u,\theta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \tag{38}$$

In this case the system has no transmission zeros, the number of measurements exceeds the number of controls and there is some design freedom available to tailor the sliding mode dynamics. In the canonical form (23-26)

$$A_f = \begin{bmatrix} 0.091 & -8.832 & -0.739 & -22.594 \\ 0.592 & 0.279 & -2.292 & 1.157 \\ 0. & 10.803 & -0.329 & 27.555 \\ -0.194 & -1.429 & 0.879 & -3.788 \end{bmatrix}$$

$$B_f^T = \begin{bmatrix} 0 & 0 & 0 & -1.864 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0 & 0 & 0.338 & -0.941 \\ 0 & 0 & 0.941 & 0.338 \end{bmatrix} \tag{39}$$

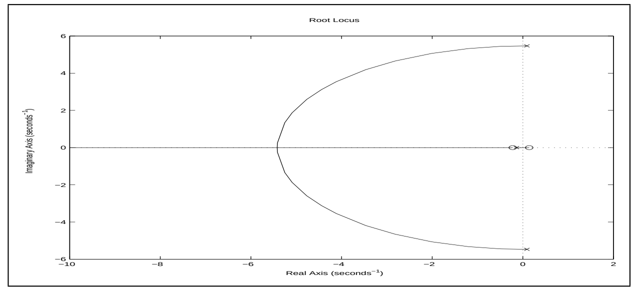

Fig. 3: Variation in the closed-loop poles with the switching surface parameter K

From (23) and (28)

$$A_{11} = \begin{bmatrix} 0.091 & -8.832 & -0.739 \\ 0.592 & 0.279 & -2.292 \\ 0 & 10.803 & -0.329 \end{bmatrix}$$

$$A_{12}^T = \begin{bmatrix} -11.401 & 27.719 & 3.106 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{40}$$

The design requirement is to determine K such that $A_{11} - A_{12} K C_1$ has desired dynamics. The root locus plot for the sub-system (40) is shown in Figure 3. The location of the open-loop pole at -0.140 which moves to the zero at 0.140 limits the acceptable dynamic performance; as gain is increased the system becomes unstable. Selecting $K = 1$ prescribes the poles of the sliding motion at $-0.1149, -1.1043, -26.2944$ and thus improves the stability margin of the initial design whilst reducing the speed of the fastest pole. In the original coordinates

$$F = \begin{bmatrix} 11.252 - 23.832 \end{bmatrix} \tag{41}$$

Figure 4 show the response of θ, η and s using the control (9-10) with $G = \gamma F$ and $\gamma = 123.0$.

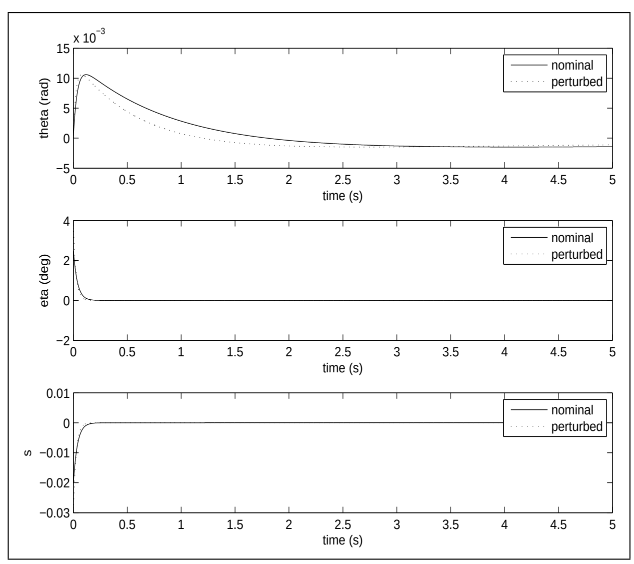

Fig. 4: Response of the perturbation in theta, elevator angle and switching function for both a nominal and perturbed model of the UAV with initial condition $x(0) = [0 \;\; 0 \;\; 0.5 \;\; 0]^T$

V. HIGHER ORDER SLIDING MODES

Thus far the focus has been on enforcing a sliding mode on $\mathscr{S}$ by applying a discontinuous injection to the $\dot{s}$ dynamics. As has been previously mentioned, a key disadvantage is the fundamentally discontinuous control signals result. The concept of Higher Order Sliding Modes (HOSM) generalise the sliding mode control concept so that the discontinuity acts on higher order derivatives of s and the applied control is smooth. In general, if the control appears on the rth derivative of s, the rth order ideal sliding mode is defined by:

$$s = \dot{s} = \ddot{s} = ... = s^{(r-1)} = 0 \tag{42}$$

In fact if the control is implemented with sample period T, $\mid s \mid = \mathscr{O}(T^r)$,...$\mid s^{(r-1)} \mid = \mathscr{O}(T^r)$. The total invariance to matched uncertainty exhibited by the traditional sliding mode control holds for higher order sliding modes as well as finite time convergence to the sliding surface. Perhaps the most commonly implemented higher order sliding mode control is the super-twisting algorithm which is a second order sliding mode control. Consider the sliding variable dynamics

$$\dot{s} = \phi(s,t) + \gamma(s,t)u_{st} \tag{43}$$

with $|\phi| \leq \Phi$ and $0 \leq \Gamma_m \leq \gamma(s,t) \leq \Gamma_M$. The super-twisting controller is defined by

$$\begin{aligned} u_{st} &= u_1 + u_2 \qquad (44) \\ \dot{u}_1 &= \begin{cases} -u_{st} & \text{if } |u_{st}| > U \\ -W\text{sgn}(s) & \text{if } |u_{st}| \leq U \end{cases} \\ u_2 &= \begin{cases} -\lambda|s_0|^{0.5}\text{sgn}(s) & \text{if } |s| > s_0 \\ -\lambda|s|^{0.5}\text{sgn}(s) & \text{if } |s| \leq s_0 \end{cases} \end{aligned}$$

The constants W and λ satisfy

$$\begin{aligned} W &> \frac{\Phi}{\Gamma_m} \\ \lambda^2 &\geq \frac{4\Phi}{\Gamma_m^2}\frac{\Gamma_M(W+\Phi)}{\Gamma_m(W-\Phi)} \end{aligned} \tag{45}$$

with U the maximum magnitude of the control and s_0 a boundary layer around the sliding surface s. For the scaled pendulum system (1) with sliding surface (2):

$$\dot{s} = \dot{y} - a_1 \sin y + u \tag{46}$$

Define

$$u = -\dot{y} + u_{st} \tag{47}$$

With $a_1 = 1$, $\Phi = 1.1$, $\Gamma_M = 1.1$ and $\Gamma_m = 0.9$, the parameters in the super-twisting algorithm are selected as $\lambda = 79.5$, $W = 1.3$, $U = 10$ and $s_0 = 0.01$ and the phase portrait is as shown in Figure 5.

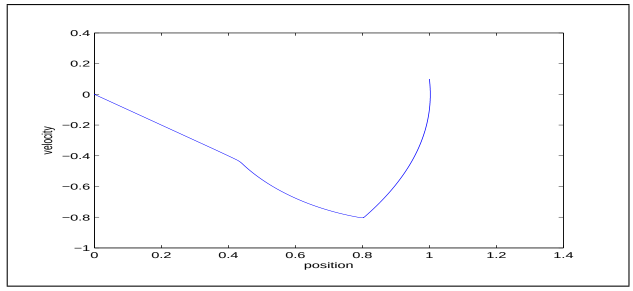

Fig. 5: Phase plane portrait of (1) with $a_1 = 1$, $y(0) = 1, \dot{y}(0) = 0.1$ and the super-twisting controller

VI. CONCLUSIONS

This paper has introduced sliding mode control. Canonical forms to facilitate design have been described and numerical examples have been presented to reinforce the theoretical discussions. Due to space available much has been omitted. Of particular importance is the case of digital implementation, or indeed digital design, of sliding mode controllers. In continuous time, discontinuous control strategies fundamentally rely upon very high frequency switching to ensure the sliding mode is attained and maintained. The introduction of sampling is disruptive. For example, switching of increasing amplitude can take place about the sliding surface. Reviews of the discrete time sliding mode control paradigm can be found in [17] [18].

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