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Discrete integrable systems and Poisson algebras from cluster maps

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Abstract

We consider nonlinear recurrences generated from cluster mutations applied to quivers that have the property of being cluster mutation-periodic with period 1. Such quivers were completely classified by Fordy and Marsh, who characterised them in terms of the skew-symmetric matrix that defines the quiver. The associated nonlinear recurrences are equivalent to birational maps, and we explain how these maps can be endowed with an invariant Poisson bracket and/or presymplectic structure.

Upon applying the algebraic entropy test, we are led to a series of conjectures which imply that the entropy of the cluster maps can be determined from their tropical analogues, which leads to a sharp classification result. Only four special families of these maps should have zero entropy. These families are examined in detail, with many explicit examples given, and we show how they lead to discrete dynamics that is integrable in the Liouville-Arnold sense.

Keywords: Integrable maps, Poisson algebra, cluster algebra, algebraic entropy, tropical, monodromy.

1 Introduction

Cluster algebras were first developed by Fomin and Zelevinsky more than a decade ago [10]. Their structure arises in diverse parts of mathematics and theoretical physics, including Lie theory, quantum algebras, Teichmüller theory, discrete integrable systems and T- and Y-systems. One of the original motivations for cluster algebras came from a series of observations made by Michael Somos and others (see [16]), concerning nonlinear recurrence relations of the form

$$x_{n+N}\, x_n = F(x_{n+1}, \ldots, x_{n+N-1}), \tag{1.1}$$

where F is a polynomial in $N-1$ variables. The original observation of Somos was that certain choices of F lead to integer sequences when all N initial values are chosen to be 1. This was explained by the further observation that for such special F the recurrence (1.1) exhibits the *Laurent phenomenon*, meaning that all iterates are Laurent polynomials in the initial data with integer coefficients. One of the most well-known examples is the Somos-4 recurrence given by

$$x_{n+4}\, x_n = x_{n+3}x_{n+1} + x_{n+2}^2, \tag{1.2}$$

which generates the sequence $1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots$[1] starting from four initial 1s, while if the initial data x_1, x_2, x_3, x_4 are viewed as variables then the iterates x_n belong to the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}]$.

In this paper we consider recurrences of the general form

$$x_{n+N}\, x_n = \prod_{a_j \geq 0} x_{n+j}^{a_j} + \prod_{a_j \leq 0} x_{n+j}^{-a_j}, \tag{1.3}$$

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[1] `http://oeis.org/A006720`

where the indices in each product lie in the range $1 \leq j \leq N-1$, with the exponents $(a_1, ..., a_{N-1})$ forming an integer $(N-1)$-tuple which is palindromic, so that $a_j = a_{N-j}$. The main purpose of this paper is to identify which recurrences of the form (1.3) can be regarded as finite-dimensional discrete integrable systems, in the sense of the standard Liouville-Arnold definition of integrability for maps [25, 37]. The latter requires that a map should preserve a Poisson bracket, as well as having sufficiently many first integrals that commute with respect to this bracket.

A quiver is a graph consisting of a number of nodes together with arrows between the nodes. In [13], Fordy and Marsh showed how recurrences of the form (1.3) are produced from sequences of mutations in cluster algebras defined by quivers with a special periodicity property with respect to mutations. They define a quiver Q with N nodes to be *cluster mutation-periodic with period* m if $\mu_m \cdots \mu_2 \cdot \mu_1 Q = \rho^m Q$, where μ_j denotes quiver mutation at node j and ρ denotes a cyclic permutation of the nodes. Associated with the quiver mutation there is a corresponding cluster mutation acting on a cluster $\mathbf{x} = (x_1, \ldots, x_N)$ in a (coefficient-free) cluster algebra. In the period 1 case, $m = 1$, the action of a suitable ordered sequence of cluster mutations on cluster variables is precisely equivalent to iteration of a recurrence of the form (1.3). A complete classification of period 1 quivers is given in [13]: any such quiver produces a recurrence, and conversely any recurrence of the form (1.3) determines a cluster mutation-periodic quiver with period 1.

1.1 Outline of the paper

In the next section, we briefly review how the recurrences (1.3), or the equivalent birational maps in dimension N, come about from cluster mutations. The main object is the $N \times N$ skew-symmetric integer matrix B corresponding to the quiver, which not only defines the exponents appearing in (1.3), but also produces a presymplectic form which is invariant under the map; this is the two-form introduced in [18]. When $\det B \neq 0$, the form is symplectic, so the map automatically has a nondegenerate Poisson bracket. The main result of section 2 is Theorem 2.6, which states that (even if $\det B = 0$) it is always possible to reduce (1.3) to a symplectic map, possibly on a space of lower dimension. This provides us with the appropriate setting in which to consider Liouville integrability in the rest of the paper.

In section 3 we consider the recurrences (1.3) in the light of the algebraic entropy test [2]. We give details of a series of conjectures which show that the algebraic entropy can be determined explicitly from the tropical version of (1.3), expressed in terms of the max-plus algebra. From the point of view of the rest of the paper, the main result is a corollary of these conjectures (Theorem 3.12), which classifies the cases with zero entropy into four families, labelled (i)-(iv). All of the maps in family (i) have periodic orbits, and it is a trivial task to show that they are Liouville integrable maps. The majority of the rest of the paper is devoted to families (ii), (iii) and (iv).

Section 4 is concerned with the family (ii), which arises from cluster mutations applied to the primitive quivers, denoted $P_N^{(q)}$, which are defined for each positive integer N and $q = 1, \ldots, \lfloor N/2 \rfloor$. These were introduced in [13], where they were shown to be the building blocks of all mutation-periodic quivers with period 1. They are also equivalent to affine A-type Dynkin quivers (or copies of such), and it was shown by Fordy and Marsh that the cluster variables in this case satisfy linear recurrence relations with constant coefficients. (This was subsequently shown for the general case of cluster algebras associated with affine Dynkin quivers, in [1] and [23].) Here we give a new proof of these linear recurrences, which relies on additional linear relations with periodic coefficients, and their associated monodromy matrices. These periodic quantities are the key to the Liouville integrability of the maps in family (ii). A large number of new examples of integrable maps arise in this construction and are explicitly presented.

Section 5 deals with the family (iii), each member of which arises from a quiver which is a deformation of a primitive $P_N^{(q)}$, for a particular q and N. The general properties of this family are very close to those of the primitives. In particular, the cluster variables satisfy linear recurrences with constant coefficients, and there are additional linear relations with periodic coefficients. Once again, associated monodromy arguments, and Poisson subalgebras defined by the periodic quantities, are the key to the Liouville integrability of members of this family. Again, many new examples of integrable maps arise and are explicitly presented.

The family (iv) consists of Somos-type recurrences with three terms, typified by (1.2). In section 6 we outline some different approaches to understanding the Liouville integrability of the maps in this family, such as making reductions of the Hirota-Miwa equation and its Lax pair and by deriving higher bilinear relations with constant coefficients.

Some of our results were announced previously in [15].

2 Symplectic maps from cluster recurrences

Given a recurrence, a major problem is to find an appropriate symplectic or Poisson structure which is invariant under the action of the corresponding finite-dimensional map. Remarkably, in the case of the cluster recurrences (1.3) this problem can be solved algorithmically.

2.1 Recurrences from periodic quivers

A quiver Q with N nodes and no 1-cycles or 2-cycles admits quiver mutation. The mutation μ_k at node k produces a new quiver $\tilde{Q} = \mu_k Q$ which is obtained as follows: (i) reverse all arrows in/out of node k; (ii) if there are p arrows from node j to node k, and q arrows from node k to node ℓ, then add pq arrows from node j to node ℓ; (iii) remove any 2-cycles created in step (ii).

An $N \times N$ skew-symmetric matrix B with integer matrix elements $b_{j\ell}$ defines a quiver Q with N nodes, without 1-cycles or 2-cycles. Matrix mutation applied at the vertex k is also denoted μ_k, and starting from B it produces a new matrix $\tilde{B} = \mu_k B = (\tilde{b}_{j\ell})$ defined by

$$\tilde{b}_{j\ell} = \begin{cases} -b_{j\ell} & \text{if} \quad j = k \quad \text{or} \quad \ell = k, \\ b_{j\ell} + \frac{1}{2}(|b_{jk}|b_{k\ell} + b_{jk}|b_{k\ell}|) & \text{otherwise.} \end{cases} \tag{2.1}$$

This matrix mutation is equivalent to the action of μ_k on Q, via quiver mutation, producing the new quiver $\tilde{Q} = \mu_k Q$. In addition to the matrix mutation, the cluster variables $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ are transformed by μ_k to a new cluster $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$ in such a way that all variables except x_k are left unaltered, so that $\tilde{x}_j = x_j,\ j \neq k$, and the exchange relation corresponding to the mutation μ_k is conveniently written as

$$\tilde{x}_k x_k = \prod_{j=1}^{N} x_j^{[b_{k,j}]_+} + \prod_{j=1}^{N} x_j^{[-b_{k,j}]_+}, \tag{2.2}$$

where $[b]_+ = \max(b, 0)$. Note the identity $\frac{1}{2}(a|b| - |a|b) = a[-b]_+ - b[-a]_+$.

In what follows, we require that the $N \times N$ skew-symmetric matrix B defines an N-node quiver Q that is cluster mutation-periodic with period 1. All such matrices were classified in [13]. Cluster mutation-periodicity, in the case that the period is 1, means that after applying a single step of mutation at one of the nodes, μ_1 say, the quiver $\tilde{Q}$ is the same as the quiver ρQ obtained from Q by applying the cyclic permutation ρ, given by $\rho:\ (1, 2, 3, \ldots, N) \mapsto (N, 1, 2, \ldots, N-1)$, such that the number of arrows from j to k in Q is the same as the number of arrows from $\rho^{-1}(j)$ to $\rho^{-1}(k)$ in ρQ. This periodicity requirement corresponds to explicit conditions on the matrix elements of B, namely that

$$\tilde{b}_{j+1,k+1} = b_{jk} \tag{2.3}$$

for all j, k, where the indices are read modulo N. Since $\rho^N = \mathrm{id}$, from the periodicity it is clear that Q is preserved by the composition of N mutations that cycle around its nodes, i.e. $\mu_N \cdots \mu_2 \cdot \mu_1 Q = Q$.

Starting from B, one constructs the Nth order recurrence relation

$$x_{n+N} x_n = \prod_{j=1}^{N-1} x_{n+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{n+j}^{[-b_{1,j+1}]_+}, \qquad n = 1, 2, 3, \ldots. \tag{2.4}$$

If B satisfies the conditions (2.3), then iterating the recurrence (2.4) starting from the initial data $(x_1, \ldots, x_N)$ is precisely equivalent to applying cluster mutation μ_1 at the vertex 1, followed by the subsequent mutations

$\mu_2, \mu_3, \ldots, \mu_j, \mu_{j+1}, \ldots$, and so on. The recurrence relation (2.4) is clearly reversible, in the sense that (as long as neither is zero) it can be solved both for x_{n+N}, to iterate forwards, and for x_n, to iterate backwards. This means that the index n in (2.4) is allowed to take all values $n \in \mathbb{Z}$, and also that iteration of the recurrence is equivalent to iteration of the birational map φ from $\mathbb{C}^N$ to itself, defined by

$$\varphi: \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_N \\ x_{N+1} \end{pmatrix}, \qquad \text{where} \qquad x_{N+1} = \frac{\prod_{j=1}^{N-1} x_{j+1}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{j+1}^{[-b_{1,j+1}]_+}}{x_1}. \tag{2.5}$$

One can decompose the map (2.5) as $\varphi = \rho^{-1} \cdot \mu_1$, where in terms of the cluster $\mathbf{x} = (x_j)$ the map μ_1 sends $(x_1, x_2, \ldots, x_N)$ to $(\tilde{x}_1, x_2, \ldots, x_N)$, with $\tilde{x}_1$ defined according to the exchange relation (2.2) with $k = 1$, and ρ^{-1} sends $(x_1, x_2, \ldots, x_N)$ to $(x_2, \ldots, x_N, x_1)$. Due to the periodicity requirement on B, we have $\rho^{-1} \cdot \mu_1 (B) = B$, so the action of φ on this matrix is trivial.

2.2 The Gekhtman-Shapiro-Vainshtein bracket

In [17] it was shown that very general cluster algebras admit a linear space of Poisson brackets of log-canonical type, compatible with the cluster maps generated by mutations, and having the form

$$\{x_j, x_k\} = c_{jk}\, x_j x_k \tag{2.6}$$

for some skew-symmetric constant coefficient matrix $C = (c_{jk})$. Compatibility of the Poisson structure means that the cluster transformations μ_i given by (2.2) correspond to a *change of coordinates*, $\tilde{\mathbf{x}} = \mu_i(\mathbf{x})$, with their bracket also being log-canonical,

$$\{\tilde{x}_j, \tilde{x}_k\} = \tilde{c}_{jk}\, \tilde{x}_j \tilde{x}_k, \tag{2.7}$$

for another skew-symmetric constant matrix $\tilde{C} = (\tilde{c}_{jk})$.

Our viewpoint is to regard the cluster transformation as a *birational map* $\mathbf{x} \mapsto \tilde{\mathbf{x}} = \varphi(\mathbf{x})$ in $\mathbb{C}^N$, and require a Poisson structure that is *invariant* with respect to φ (not just *covariant*). Therefore, in (2.7) we require $\tilde{C} = C$. However, there may not be a non-trivial log-canonical Poisson bracket that is covariant or invariant under cluster transformations.

Example 2.1. Corresponding to (2.5) with $N = 3$, the matrix B and birational map on $\mathbb{C}^3$ are given by

$$B = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_2 \\ x_3 \\ (x_2 x_3 + 1)/x_1 \end{pmatrix}. \tag{2.8}$$

Suppose that there is an invariant Poisson bracket of the form (2.6). The condition $\varphi^* x_j = x_{j+1}$, implies that $c_{j+\ell,k+\ell} = c_{jk}$ for all indices j, k, ℓ in the appropriate range, so C is a Toeplitz matrix:

$$C = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \alpha \\ -\beta & -\alpha & 0 \end{pmatrix}.$$

Upon taking the bracket of both sides of the relation $x_4 x_1 = x_2 x_3 + 1$ with x_2, one finds that $(\alpha - \beta) x_1 x_2 x_4 = -\alpha x_2^2 x_3$, so $(\alpha - \beta)(x_2^2 x_3 + x_2) = -\alpha x_2^2 x_3$, which gives $\alpha = \beta = 0$, so the bracket is trivial.

Remark 2.2 (Non-invariance of symplectic leaves)**.** Even in the case where the map φ does admit an invariant log-canonical Poisson bracket, it may be degenerate, and in that case φ need not preserve the symplectic leaves of the bracket. For instance, the Somos-4 recurrence (1.2) has the invariant Poisson bracket [21]

$$\{ x_j, x_k \} = (k-j)\, x_j x_k, \tag{2.9}$$

which has rank two, but the two independent Casimirs x_1x_3/x_2^2, x_2x_4/x_3^2 are not fixed by the action of φ (see Example 2.11 below), so the symplectic leaves are not preserved. The analogous observation for Somos-5 appears in [15].

In general, we shall see that it is more useful to start with a two-form in the variables x_j, rather than a Poisson bivector field corresponding to a bracket.

2.3 Symplectic forms for cluster maps

Given a skew-symmetric matrix B, one can define the log-canonical two-form

$$\omega = \sum_{j<k} \frac{b_{jk}}{x_j x_k} \mathrm{d}x_j \wedge \mathrm{d}x_k, \tag{2.10}$$

which is just the constant skew-form $\omega = \sum_{j<k} b_{jk}\, \mathrm{d}z_j \wedge \mathrm{d}z_k$, written in the logarithmic coordinates $z_j = \log x_j$, so it is evidently closed, but may be degenerate. In [18] (see also [9]) it was shown that for a cluster algebra defined by a skew-symmetric integer matrix B, this two-form is compatible with cluster transformations, in the sense that under a mutation map $\mu_i: \mathbf{x} \mapsto \tilde{\mathbf{x}}$, it transforms as $\mu_i^*\omega = \sum_{j<k} \tilde{b}_{jk} \mathrm{d}\log\tilde{x}_j \wedge \mathrm{d}\log\tilde{x}_k$.

In the case that the matrix B is nondegenerate, ω turns out to be a symplectic form for the map φ, but in general it is a presymplectic form. For the purposes of our discussion it is convenient to present some formulae from [13], which give the following result.

Lemma 2.3. *Let B be a skew-symmetric integer matrix. The following conditions are equivalent.*
(i) *The matrix B defines a cluster mutation-periodic quiver with period 1.*
(ii) *The matrix elements B satisfy the relations*

$$b_{j,N} = b_{1,j+1}, \qquad j = 1, \ldots, N-1, \tag{2.11}$$

and

$$b_{j+1,k+1} = b_{j,k} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+, \qquad 1 \leq j,k \leq N-1. \tag{2.12}$$

(iii) *The two-form ω is preserved by the map φ, i.e. $\varphi^*\omega = \omega$.*

Proof: The proof of (i) $\iff$ (ii) follows straight from the definition of periodicity (see the proof of Theorem 6.1 in [13]). The implication (ii) $\implies$ (iii) in Lemma 2.3 is a consequence of Theorem 2.1 in [18], after applying the permutation ρ^{-1} to the coordinates. The reverse implication follows from a direct calculation, which is omitted. $\square$

The formulae (2.11) and (2.12) entail that for a period 1 cluster mutation-periodic quiver, the matrix B is completely determined by the elements in its first row, so that each recurrence of the form (1.3) with palindromic exponents corresponds to a matrix B. Theorem 6.1 in [13] is equivalent to the following formula for b_{jk}:

$$b_{jk} = -b_{kj} = a_{k-j} + \sum_{\ell=1}^{\max(j-1,N-k)} a_\ell[-a_{\ell+k-j}]_+ - a_{\ell+k-j}[-a_\ell]_+, \qquad 1 \leq j < k \leq N, \tag{2.13}$$

where the $a_j = b_{1,j+1}$ for $j = 1, \ldots, N-1$ form a palindromic integer $(N-1)$-tuple, such that $a_j = a_{N-j}$. Apart from being skew-symmetric, B is also symmetric about the skew diagonal, i.e. $b_{jk} = b_{N-k+1,N-j+1}$.

Henceforth when we refer to a recurrence of the form (2.4), and the corresponding matrix $B = (b_{jk})$, we assume that the conditions (2.11) and (2.12) hold.

Example 2.4 (Corollary 2.2 in [21])**.** For integer values of $c \geq 0$, the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & -1 & c & -1 \\ 1 & 0 & -(c+1) & c \\ -c & (c+1) & 0 & -1 \\ 1 & -c & 1 & 0 \end{pmatrix}, \tag{2.14}$$

satisfies the conditions (2.11) and (2.12), which means that it defines a period 1 cluster mutation-periodic quiver Q, in the sense of [13]. Thus, by Lemma 2.3, for each c the map φ corresponding to the recurrence

$$x_{n+4}\, x_n = x_{n+3}\, x_{n+1} + x_{n+2}^c \tag{2.15}$$

preserves the two-form

$$\omega = -\left(\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{x_1 x_2} + \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_4}{x_1 x_4} + \frac{\mathrm{d}x_3 \wedge \mathrm{d}x_4}{x_3 x_4}\right) + c\left(\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_3}{x_1 x_3} + \frac{\mathrm{d}x_2 \wedge \mathrm{d}x_4}{x_2 x_4}\right) - (c+1)\frac{\mathrm{d}x_2 \wedge \mathrm{d}x_3}{x_2 x_3}. \tag{2.16}$$

Remark 2.5 (Invariant volume form)**.** For all maps φ of the form (2.5), or of the more general form (1.1), the volume N-form

$$\Omega = \frac{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_N}{\prod_{j=1}^N x_j}, \tag{2.17}$$

is invariant up to a sign, depending on the parity of N, i.e. $\varphi^*\Omega = (-1)^N \Omega$. In the case that ω is nondegenerate, which can only happen for even $N = 2K$, up to overall scale this volume form is the Poincaré invariant $\omega^K = \omega \wedge \ldots \wedge \omega$ (K terms).

If the matrix B is nondegenerate, then (up to rescaling by an overall constant) the associated log-canonical Poisson bracket (2.6) is given by the dual bivector field

$$\mathcal{J} = \sum_{j<k} c_{jk}\, x_j x_k \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} \qquad \text{with} \qquad C = (c_{jk}) = B^{-1}. \tag{2.18}$$

In the case that $\det B = 0$, it is necessary to consider a projection to a lower-dimensional space with a symplectic form, as follows.

Theorem 2.6. *The map φ is symplectic whenever B is nondegenerate. For* $\operatorname{rank} B = 2K \leq N$*, there is a rational map π and a symplectic birational map $\hat{\varphi}$ such that the diagram*

$$\begin{array}{ccc} \mathbb{C}^N & \xrightarrow{\varphi} & \mathbb{C}^N \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{C}^{2K} & \xrightarrow{\hat{\varphi}} & \mathbb{C}^{2K} \end{array} \tag{2.19}$$

is commutative, with a log-canonical symplectic form $\hat{\omega}$ on $\mathbb{C}^{2K}$ that satisfies $\pi^*\hat{\omega} = \omega$.

The proof of this theorem will occupy most of the rest of this subsection.

To begin with we consider the null distribution of ω, which (away from the hyperplanes $x_j = 0$) is generated by $N - 2K$ independent commuting vector fields, each of which is of the form

$$\frac{\partial}{\partial t} = \sum_{j=1}^N u_j x_j \frac{\partial}{\partial x_j} \qquad \text{for} \qquad \mathbf{u} = (u_j) \in \ker B. \tag{2.20}$$

Since this is an integrable distribution, Frobenius' theorem gives local coordinates $t_1, \ldots, t_{N-2K}, y_1, \ldots, y_{2K}$ such that the integral manifolds of the null distribution are given by y_j =constant, $j = 1, \ldots, 2K$. The coordinates y_j must be invariants for these commuting vector fields, and can be chosen as linear functions of the logarithmic coordinates $z_j = \log x_j$, but for our purposes it is more convenient to take functions of the form

$$y = \mathbf{x}^{\mathbf{v}} := \prod_j x_j^{v_j} \qquad \Longleftrightarrow \qquad (\mathbf{u}, \mathbf{v}) = 0 \quad \forall\, \mathbf{u} \in \ker B,$$

where $(\,,)$ denotes the standard scalar product. This yields a log-canonical symplectic form in terms of y_j, which is the generalised Weil-Petersson form in [18].

Alternatively, one can consider integral curves of the vector fields (2.20), each of which is the orbit of the point $\mathbf{x}$ under the action of the algebraic torus $\mathbb{C}^*$, which acts by scaling the coordinates x_j. We denote this one-parameter group action by

$$\mathbf{x} \to \lambda^{\mathbf{u}} \cdot \mathbf{x} = (\lambda^{u_j} x_j), \qquad \lambda \in \mathbb{C}^*. \tag{2.21}$$

Combining $N-2K$ independent vector fields of the form (2.20), which (without loss of generality) can be defined by choosing $N-2K$ independent vectors $\mathbf{u} = (u_j) \in \ker B$ with components $u_j \in \mathbb{Z}$, we see that the integral manifold through a generic point $\mathbf{x}$ is the same as its orbit under the scaling action of the algebraic torus $(\mathbb{C}^*)^{N-2K}$. The coordinates $y_j = \mathbf{x}^{\mathbf{v}_j}$ are the invariants under these scaling transformations, and, by choosing the $\mathbf{v}_j$ to be integer vectors, define a rational map π.

Lemma 2.7. *Suppose that the integer vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{2K}$ form a basis for $\operatorname{im} B$, and define the rational map*

$$\begin{array}{rccl} \pi: & \mathbb{C}^N & \longrightarrow & \mathbb{C}^{2K} \\ & \mathbf{x} & \longmapsto & \mathbf{y}, \qquad y_j = \mathbf{x}^{\mathbf{v}_j}, \quad j = 1, \ldots, 2K. \end{array}$$

Then the y_j are a complete set of Laurent monomial invariants for all of the scaling transformations (2.21) defined by integer vectors $\mathbf{u} \in \ker B$, and there is a log-canonical symplectic form

$$\hat{\omega} = \sum_{j<k} \frac{\hat{b}_{jk}}{y_j y_k} \mathrm{d}y_j \wedge \mathrm{d}y_k, \qquad \textit{with} \qquad \pi^* \hat{\omega} = \omega. \tag{2.22}$$

Proof: Since the integer matrix B is skew-symmetric, the vector space $\mathbb{Q}^N$ has an orthogonal direct sum decomposition $\mathbb{Q}^N = \operatorname{im} B \oplus \ker B$. The scaling action on Laurent monomials gives $\lambda^{\mathbf{u}} \cdot \mathbf{x}^{\mathbf{v}} = \prod_j \lambda^{u_j v_j} x_j^{v_j} = \lambda^{(\mathbf{u}, \mathbf{v})} \mathbf{x}^{\mathbf{v}}$, hence $\mathbf{x}^{\mathbf{v}}$ is invariant under the overall action of $(\mathbb{C}^*)^{N-2K}$ if and only if $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u} \in \ker B$, so $\mathbf{v} \in \operatorname{im} B$, and a basis of $\operatorname{im} B$ gives $2K$ independent monomial invariants. Now extend the basis of $\operatorname{im} B$ to a basis $\{\mathbf{v}_j\}_{j=1}^N$ for $\mathbb{Q}^N$, so that $\mathbf{v}_j \in \ker B$ for $2K+1 \le j \le N$, and let M be the matrix whose rows consist of the N basis vectors. Then one finds the block matrix

$$B^\natural = M^{-T} B M^{-1} = \begin{pmatrix} \hat{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

(with $M^{-T} = (M^{-1})^T$, and T denoting transpose) where $\hat{B} = (\hat{b}_{jk})$ is a nondegenerate skew-symmetric $2K \times 2K$ matrix. Defining the two-form (2.22) in terms of $\hat{B}$ gives $\pi^* \hat{\omega} = \omega$, as required. $\square$

The map π and the specific form of $\hat{\omega}$ depend on the choice of integer basis for $\operatorname{im} B$. Here we consider rational maps between fixed affine spaces, even if these are not defined everywhere. In the case where B is nondegenerate, the diagram (2.19) is trivial for $\pi = \mathrm{id}$, but there are other non-trivial choices of π, corresponding to different integer bases for $\operatorname{im} B$.

The following classical theorem (which is a special case of Theorem IV.1 in [29]) provides a canonical choice of π, and via Lemma 2.7 gives Darboux coordinates for the presymplectic form ω. The proof in [29] is constructive.

Theorem 2.8. *If B is a skew-symmetric matrix of rank $2K$ in $\mathrm{Mat}_N(\mathbb{Z})$ then there are integers $h_1, \ldots, h_K$ and a unit matrix $M \in \mathrm{Mat}_N(\mathbb{Z})$ such that $B = M^T B^\natural M$, where*

$$B^\natural = h_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus h_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \ldots \oplus h_K \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \mathbf{0},$$

with $h_j | h_{j+1}$ *for* $j = 1, \ldots, K-1$.

In the above result, the first $2K$ rows of the unit matrix M provide a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $\operatorname{im} B_{\mathbb{Z}} = \operatorname{im} B \cap \mathbb{Z}^N$ (a sublattice of $\mathbb{Z}^N$), and the remaining $N-2K$ rows provide a $\mathbb{Z}$-basis for $\ker B_{\mathbb{Z}}$. For our purposes, it is only the choice of integer vectors spanning $\operatorname{im} B$ that matters for the definition of the map π, and if the integers h_j are not all 1 then it is not essential to have a $\mathbb{Z}$-basis for $\operatorname{im} B_{\mathbb{Z}}$.

The second part of the proof of Theorem 2.6 involves showing that, when B is degenerate, it is possible to choose integer vectors $\mathbf{v}_j$ spanning $\operatorname{im} B$ in a way that is compatible with the map φ. In other words, φ should reduce to a symplectic map $\hat{\varphi}$ in the coordinates y_j, and we require that the latter map should also be birational. The next result provides two sets of sufficient conditions on the vectors $\mathbf{v}_j$ which ensure birationality of the map $\hat{\varphi}$.

Lemma 2.9. *Let* $\{\mathbf{v}_j\}_{j=1}^{2K}$ *be integer vectors that span* $\operatorname{im} B$*, and suppose that the columns of* B *belong to* $<\mathbf{v}_1, \ldots, \mathbf{v}_{2K}>_{\mathbb{Z}}$*. If either*

(a) $\operatorname{im} B_{\mathbb{Z}} =<\mathbf{v}_1, \ldots, \mathbf{v}_{2K}>_{\mathbb{Z}}$*, or*

(b) *each* $\mathbf{v}_j$ *belongs to the* $\mathbb{Z}$*-span of the columns of* B*,*

then for $y_j(\mathbf{x}) = \mathbf{x}^{\mathbf{v}_j}$*, there exist rational functions* $f_j(\mathbf{y}),\ f_j^\dagger(\mathbf{y}) \in \mathbb{Q}(\mathbf{y}) = \mathbb{Q}(y_1, \ldots, y_{2K})$ *such that* $\varphi^* y_j(\mathbf{x}) = \pi^* f_j(\mathbf{y})$*,* $(\varphi^{-1})^* y_j(\mathbf{x}) = \pi^* f_j^\dagger(\mathbf{y})$ *for* $j = 1, \ldots, 2K$*, and the map* $\hat{\phi} : \mathbf{y} \mapsto (f_j(\mathbf{y}))$ *is birational and symplectic.*

Proof: Let $y(\mathbf{x}) = \mathbf{x}^{\mathbf{v}}$ be one of the coordinate functions y_j in the image of the map π, with $\eta = \log y$, and denote the columns of B by $\mathbf{w}_1, \ldots, \mathbf{w}_N$. Then

$$\varphi^* \eta(\mathbf{x}) = \sum_{k=1}^{N} v_k \varphi^* z_k = \sum_{k=1}^{N-1} v_k z_{k+1} + v_N \Big(-z_1 + \sum_{j=2}^{N} [-b_{1j}]_+ z_j + \log(1 + \exp F) \Big),$$

with F given by $F := \sum_{k=1}^{N-1} b_{1,k+1} z_{k+1}$. By the initial assumption on the $\mathbf{v}_j$, we can write $\mathbf{w}_1 = \sum_{j=1}^{2K} c_j \mathbf{v}_j$ for $c_j \in \mathbb{Z}$, which implies that $\exp F = \pi^* \exp \sum_j c_j \eta_j = \pi^* \prod_j y_j^{c_j}$, the pullback of a rational function of $\mathbf{y}$. Then we have

$$\varphi^* \eta(\mathbf{x}) = (\mathbf{m}, \mathbf{z}) + \log(1 + \exp F)^{v_N}, \tag{2.23}$$

where $\mathbf{m}$ is an integer vector with components $m_1 = -v_N$, $m_j = v_{j-1} + [-b_{1j}]_+ v_N$ for $2 \leq j \leq N$.

By definition the $\mathbf{w}_j$ span $\operatorname{im} B$, so we can write $\mathbf{v} = \sum_{k=1}^{N} d_k \mathbf{w}_k$ for some $d_k \in \mathbb{Q}$, which in components gives $v_j = \sum_{k=1}^{N} d_k b_{jk}$ for $j = 1, \ldots, N$. By using (2.11) and (2.12) one obtains

$$m_j = \sum_{k=1}^{N} d_k (b_{j-1,k} + [-b_{1j}]_+ b_{Nk}) = \sum_{k=1}^{N-1} d_k (b_{j,k+1} + [-b_{1,k+1}]_+ b_{j1}) - d_N b_{j1}$$

for $j \geq 2$, while $m_1 = -\sum_{k=1}^{N} d_k b_{Nk} = \sum_{k=1}^{N-1} d_k b_{1,k+1}$, implying that

$$\mathbf{m} = \Big(\sum_{k=1}^{N-1} d_k [-b_{1,k+1}]_+ - d_N \Big) \mathbf{w}_1 + \sum_{j=2}^{N} d_{j-1} \mathbf{w}_j.$$

Hence we see that $\mathbf{m} \in \operatorname{im} B_{\mathbb{Z}}$. In case (a), $\mathbf{m} = \sum_{j=1}^{2K} \tilde{c}_j \mathbf{v}_j$ for $\tilde{c}_j \in \mathbb{Z}$, so substituting in (2.23) and exponentiating yields $\varphi^* y(\mathbf{x}) = \pi^* f$ with

$$f(\mathbf{y}) = \prod_j y_j^{\tilde{c}_j} \Big(1 + \prod_j y_j^{c_j} \Big)^{v_N} \in \mathbb{Q}(\mathbf{y}),$$

as required. In case (b), on the other hand, each $d_k \in \mathbb{Z}$ and the $\mathbf{w}_j$ are in $< \mathbf{v}_1, \ldots, \mathbf{v}_{2K} >_{\mathbb{Z}}$, so again one can expand $\mathbf{m}$ in this basis with coefficients $\tilde{c}_j \in \mathbb{Z}$, and the same formula holds for f. For the inverse map $\varphi^{-1} : (x_1, \ldots, x_N) \mapsto (x_0, x_1, \ldots, x_{N-1})$, with

$$x_0 = \frac{1}{x_N} \left(\prod_{j=1}^{N-1} x_j^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_j^{[-b_{1,j+1}]_+} \right),$$

the fact that $(\varphi^{-1})^* y(\mathbf{x}) = \pi^* f^\dagger$ with $f^\dagger(\mathbf{y}) \in \mathbb{Q}(\mathbf{y})$ follows from an almost identical calculation.

The rational map $\hat{\varphi}$ is defined by the $2K$ functions $f_j(\mathbf{y})$, with a rational inverse $\hat{\varphi}^{-1}$ given by the functions $f_j^\dagger(\mathbf{y})$, and by construction $\hat{\varphi} \cdot \pi = \pi \cdot \varphi$. Combining part (iii) of Lemma 2.3 together with the result of Lemma 2.7 gives $\varphi^*\omega - \omega = \varphi^*\pi^*\hat{\omega} - \pi^*\hat{\omega} = \pi^*(\hat{\varphi}^*\hat{\omega} - \hat{\omega}) = 0$, so $\hat{\varphi}^*\hat{\omega} = \hat{\omega}$ as required. Thus the proof of the lemma and the proof of Theorem 2.6 are complete. ☐

Remark 2.10. Theorem 2.8 provides a basis of $\mathrm{im}\, B$ which satisfies condition (a) above. To satisfy condition (b), one can choose any $2K$ independent columns (or rows) of B, corresponding to the τ-coordinates of [17, 18].

To illustrate Theorem 2.6 we now present a couple of examples.

Example 2.11 (Somos-4)**.** The Somos-4 recurrence (1.2) is the special case $c = 2$ of (2.15), in which case the skew-symmetric matrix B of (2.14) is degenerate, and $\ker B$ is spanned by the integer column vectors $\mathbf{u}_1 = (1,1,1,1)^T$, $\mathbf{u}_2 = (1,2,3,4)^T$. By (2.21), each of these vectors generates a scaling action on the phase space $\mathbb{C}^4$, with weights given by their components, so that the torus $(\mathbb{C}^*)^2$ acts via $(x_1, x_2, x_3, x_4) \to (\lambda_1\lambda_2\, x_1, \lambda_1\lambda_2^2\, x_2, \lambda_1\lambda_2^3\, x_3, \lambda_1\lambda_2^4\, x_4)$, for $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$. Then $\mathrm{im}\, B = (\ker B)^\perp =< \mathbf{v}_1, \mathbf{v}_2 >$, with $\mathbf{v}_1 = (1, -2, 1, 0)^T$, $\mathbf{v}_2 = (0, 1, -2, 1)^T$, whose components provide the exponents for the monomial invariants

$$y_1 = \frac{x_1 x_3}{x_2^2}, \qquad y_2 = \frac{x_2 x_4}{x_3^2}. \tag{2.24}$$

(These monomials also provide two independent Casimirs for the degenerate Poisson bracket (2.9) mentioned above. The coefficients of (2.9) are obtained from the bivector $\mathbf{u}_1 \wedge \mathbf{u}_2$.) This defines the rational map $\pi : \mathbb{C}^4 \to \mathbb{C}^2$ from the x_j to the y_j. Upon computing the pullback of φ on these monomials, one has the map

$$\hat{\varphi} : \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_2 \\ (y_2 + 1)/(y_1 y_2^2) \end{pmatrix}, \tag{2.25}$$

which is of QRT type [33], and preserves the symplectic form

$$\hat{\omega} = \frac{1}{y_1 y_2}\, \mathrm{d}y_2 \wedge \mathrm{d}y_1 \tag{2.26}$$

where $\omega = \pi^*\hat{\omega}$, with ω given by the formula (2.16) for $c = 2$.

Note that, in the preceding example, the chosen basis is such that every integer vector in $\mathrm{im}\, B$ can be written as a $\mathbb{Z}$-linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, so the condition (a) in Lemma 2.9 holds. At the same time, from the matrix (2.14) with $c = 2$ we see that the first column of B is $\mathbf{v}_2$, and the last column is $-\mathbf{v}_1$, so condition (b) holds as well. The next example shows that (a) and (b) can give inequivalent results.

Example 2.12. In [21], a singularity confinement pattern was used to obtain the sixth-order recurrence

$$x_{n+6}\, x_n = (x_{n+5}\, x_{n+1})^2 + x_{n+4}^2 x_{n+3}^4 x_{n+2}^2, \tag{2.27}$$

which is associated with mutations of a skew-symmetric matrix of rank 2, namely

$$B = \begin{pmatrix} 0 & -2 & 2 & 4 & 2 & -2 \\ 2 & 0 & -6 & -6 & 0 & 2 \\ -2 & 6 & 0 & -6 & -6 & 4 \\ -4 & 6 & 6 & 0 & -6 & 2 \\ -2 & 0 & 6 & 6 & 0 & -2 \\ 2 & -2 & -4 & -2 & 2 & 0 \end{pmatrix}.$$

A basis of $\operatorname{im} B$ corresponding to case (a) of Lemma 2.9 is given in terms of the first and last columns of B by $\mathbf{v}_1 = -\frac{1}{2}\mathbf{w}_6$, $\mathbf{v}_2 = \frac{1}{2}\mathbf{w}_1$, which gives the $\mathbf{y}$-coordinates $y_j = x_j x_{j+4}/(x_{j+1}x_{j+2}^2 x_{j+3})$, $j = 1, 2$, and (up to rescaling) produces the same symplectic form as in (2.26). The corresponding symplectic map is

$$\hat{\varphi}: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_2 \\ (y_2^2+1)/(y_1y_2) \end{pmatrix}, \tag{2.28}$$

whose singularity pattern under successive blowups was considered by Viallet [36]. The map (2.28) has positive algebraic entropy, indicating nonintegrability. (See Example 3.7 in the next section.)

However, one can take a different basis, corresponding to case (b) of Lemma 2.9, given by $\mathbf{v}_1' = -\mathbf{w}_6$, $\mathbf{v}_2' = \mathbf{w}_1$, which is not a $\mathbb{Z}$-basis for $\operatorname{im} B_{\mathbb{Z}}$. From this basis, one has the coordinates $(y_1', y_2') = (y_1^2, y_2^2)$, and the map becomes

$$\varphi': \qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} \longmapsto \begin{pmatrix} y_2' \\ (y_2'+1)^2/(y_1'y_2') \end{pmatrix}.$$

The map from (y_1, y_2) to (y_1', y_2') is ramified, as generically there are four pairs of values $(\pm y_1, \pm y_2)$ for each pair (y_1', y_2'), and so the two maps $\hat{\varphi}$ and φ' are not conjugate to each other.

3 Algebraic entropy and tropical recurrences

The deep connection between the integrability of maps and various weak growth properties of the iterates has been appreciated for some time (see [37] and references). In the case of rational maps, Bellon and Viallet [2] considered the growth of degrees of iterates, and used this to define a notion of algebraic entropy. Each component of a rational map φ in affine space is a rational function of the coordinates, and the degree of the map, $d = \deg \varphi$, is the maximum of the degrees of the components. By iterating the map n times one gets a sequence of rational functions whose degrees grow generically like d^n. At the nth step one can set $d_n = \deg \varphi^n$, and then the algebraic entropy $\mathcal{E}$ of the map is defined to be $\mathcal{E} = \lim_{n\to\infty} \frac{1}{n} \log d_n$. Generically, for a map of degree d, the entropy is $\log d > 0$, but for special maps there can be cancellations in the rational functions that appear under iteration, which means that the entropy is smaller than expected.

It is conjectured that Liouville-Arnold integrability corresponds to zero algebraic entropy. In an algebro-geometric setting, there are plausible arguments which indicate that zero entropy should be a necessary condition for integrability in the Liouville-Arnold sense [3]. In the latter setting, each iteration of the map corresponds to a translation on an Abelian variety (the level set of the first integrals), and the degree is a logarithmic height function, which necessarily grows like $d_n \sim \mathrm{C}n^2$.

Often the algebraic entropy of a map can only be guessed experimentally, by calculating the degree sequence (d_n) up to some large n and doing a numerical fit to a linear recurrence. This is increasingly impractical as the dimension increases, and provides no proof that the linear relation, with its corresponding entropy value, is correct. In dimension two, exact results are possible via intersection theory [35, 36].

In the rest of this section we seek to isolate those recurrences with $\mathcal{E} = 0$, by finding a condition on the exponents which should be necessary and sufficient for $\mathcal{E} > 0$. The main conjecture is the following.

Conjecture 3.1. *For a birational map given by (2.5), corresponding to a recurrence of the form (2.4), the algebraic entropy is* $\mathcal{E} = \log|\lambda_{max}|$, *where, of all the roots of the two polynomials*

$$P_{\pm}(\lambda) = \lambda^N + 1 - \sum_{j=1}^{N-1} [\pm b_{1,j+1}]_+ \lambda^j, \tag{3.1}$$

λ_{max} *is the one of largest magnitude. The entropy is positive if and only if*

$$\max\left(\sum_{j=1}^{N-1} [b_{1,j+1}]_+ \, , \, \sum_{j=1}^{N-1} [-b_{1,j+1}]_+ \right) \geq 3, \tag{3.2}$$

We now give very strong evidence for the above assertion, showing how it rests on a sequence of other conjectures. For the recurrences (2.4), the key to calculating their entropy is the Laurent phenomenon, which leads to an exact recursion for the degrees of the denominators. The Laurent property for the associated cluster algebra [11] implies that the iterates have the factorized form

$$x_n = \frac{\mathrm{N}_n(\mathbf{x})}{\mathrm{M}_n(\mathbf{x})}, \qquad \text{with} \quad \mathrm{N}_n \in \mathbb{Z}[\mathbf{x}] = \mathbb{Z}[x_1, \ldots, x_N], \quad \mathrm{M}_n = \prod_{j=1}^{N} x_j^{d_n^{(j)}},$$

where the polynomials N_n are not divisible by x_j for $1 \leq j \leq N$, and M_n are Laurent monomials. A lower bound for the entropy is provided by the growth of degrees of denominators, and if the exponents $d_n^{(j)}$ are all positive (for large enough n) then the monomial M_n is the denominator of the Laurent polynomial x_n, and N_n is the numerator.

In addition to being Laurent polynomials, the form of the exchange relations (2.2) in a cluster algebra means that all the cluster variables are given by subtraction-free rational expressions in terms of any initial cluster $\mathbf{x} = (x_1, \ldots, x_N)$. This implies that the dynamics of the M_n can be decoupled from the N_n.

Proposition 3.2. *For all* n, *the exponent* $d_n^{(j)}$ *of each variable in the Laurent monomial* M_n *satisfies the recurrence*

$$d_{n+N} + d_n = \max\left(\sum_{j=1}^{N-1} [b_{1,j+1}]_+ \, d_{n+j} \, , \, \sum_{j=1}^{N-1} [-b_{1,j+1}]_+ \, d_{n+j} \right), \tag{3.3}$$

with the initial conditions $d_1 = -1$, $d_2 = \ldots = d_N = 0$ *(up to shifting the index).*

Proof: Upon substituting $x_n = \mathrm{N}_n/\mathrm{M}_n$ into (2.4) and comparing monomial factors on both sides one has

$$\mathrm{M}_{n+N}\mathrm{M}_n = \mathrm{lcm}\left(\prod_{j=1}^{N-1} \mathrm{M}_{n+j}^{[b_{1,j+1}]_+} \, , \, \prod_{j=1}^{N-1} \mathrm{M}_{n+j}^{[-b_{1,j+1}]_+} \right), \tag{3.4}$$

where lcm denotes the lowest common multiple. To be more precise, for any sequence of Laurent polynomials (x_n) with positive or negative coefficients, the formula (3.4) certainly holds provided that the two products on the right are not of equal degree in any of the variables x_j, for $1 \leq j \leq N$. If it happens that $D := \sum_{\ell=1}^{N-1} [b_{1,\ell+1}]_+ \, d_{n+\ell}^{(j)} = \sum_{\ell=1}^{N-1} [-b_{1,\ell+1}]_+ \, d_{n+\ell}^{(j)}$ for some j, then the coefficient of x_j^{-D} in each of the two terms on the right hand side of (2.4) is a non-zero subtraction-free rational expression in the other variables x_k with $1 \leq k \leq N$, $k \neq j$, and the sum of these two coefficients cannot vanish. Hence no cancellations can occur between the two products of numerators N_n on the right, and the formula (3.4) always holds. Taking the degree of each variable on the left and right hand sides of (3.4) gives the same recurrence (3.3) in each case. From the initial data $x_1, \ldots, x_N$ for (2.4) it is clear that $d_1^{(1)} = -1$ and $d_n^{(1)} = 0$ for $n = 2, \ldots, N$, and the degrees $d_n^{(j)}$ for $2 \leq j \leq N$ have the same initial data but shifted along by an appropriate number of steps. $\square$

Remark 3.3. The recurrence (3.3) is the tropical (or ultradiscrete [30]) analogue of the original nonlinear recurrence (2.4), in terms of the max-plus algebra. It is a special case of the recursion for the denominator vectors in a general cluster algebra, which is stated without justification as equation (7.7) in [12].

Conjecture 3.4. *Suppose that the sequence d_n is not periodic and satisfies (3.3) with the initial conditions as in Proposition 3.2. Then*

(a) $d_n > 0$ *for all* $n > N$, *and*

(b) *the total degree of the numerators satisfies* $\deg \mathrm{N}_n \sim \tilde{\mathrm{C}}\, d_n$ *as* $n \to \infty$, *for some constant* $\tilde{\mathrm{C}} > 0$.

Remark 3.5. Part (a) above follows from the first part of Conjecture 7.4 in [12]. Part (b) implies that the growth of the denominators completely determines the algebraic entropy, since the numerators grow at the same rate; it should be a consequence of Proposition 6.1 in [12] (graded homogeneity of cluster variables).

Example 3.6 (Tropical Somos-4)**.** The tropical version of the Somos-4 recurrence is

$$d_{n+4} + d_n = \max(d_{n+3} + d_{n+1}, 2d_{n+2}). \tag{3.5}$$

With initial conditions $d_1 = -1$ and $d_2 = d_3 = d_4 = 0$ this generates a sequence that begins

$$-1, 0, 0, 0, 1, 1, 2, 3, 3, 5, 6, 7, 9, 10, 12, 14, 15, 18, 20, 22, 25, 27, 30, 33, \ldots,$$

which are the degrees (in each of the variables x_1, x_2, x_3, x_4) of the denominators of the Laurent polynomials generated by (1.2). The preceding sequence has quadratic growth, $d_n \sim \mathrm{C}n^2$ as $n \to \infty$ (consistent with the growth of logarithmic height on an elliptic curve), so that the algebraic entropy is zero, and this can be proved by considering the combination

$$Y_n = d_{n+2} - 2d_{n+1} + d_n, \tag{3.6}$$

whose coefficients are the exponents in (2.24). The sequence of quantities Y_n is generated by the tropical analogue of the map (2.25), that is

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \longmapsto \begin{pmatrix} Y_2 \\ [Y_2]_+ - 2Y_2 - Y_1 \end{pmatrix}, \tag{3.7}$$

and all of the orbits of the latter are periodic with period 8 (which is a special case of Nobe's results on tropical QRT maps [30]). Thus, if $\mathcal{S}$ denotes the shift operator such that $\mathcal{S} f_n = f_{n+1}$ for any function of the index n, then applying the operator $\mathcal{S}^8 - 1$ to both sides of (3.6) implies that the sequence of degrees d_n satisfies a linear relation of order 10, namely $(\mathcal{S}^8 - 1)(d_{n+2} + d_n - 2d_{n+1}) = 0$. All of the roots λ of the characteristic polynomial corresponding to this linear relation have modulus $|\lambda| = 1$, and $\lambda = 1$ is a triple root, which accounts for the growth rate of d_n; the fact that $d_n = O(n^2)$ also follows directly from (3.6), using $Y_n = O(1)$.

Example 3.7. The tropical version of the recurrence (2.27) in Example 2.12 is

$$d_{n+6} + d_n = \max(\,2d_{n+5} + 2d_{n+1}\,,\, 2d_{n+4} + 4d_{n+3} + 2d_{n+2}\,). \tag{3.8}$$

With the initial conditions $d_1 = -1$ and $d_2 = \ldots = d_5 = 0$, this generates a degree sequence beginning

$$-1, 0, 0, 0, 0, 0, 1, 2, 4, 8, 18, 38, 79, 164, 342, 712, 1482, 3084, 6417, 13356, \ldots,$$

that grows exponentially, such that the entropy is $\mathcal{E} = \log \lambda_{max}$, where $\lambda_{max} \approx 2.08$ is the root of largest magnitude of the polynomial $\lambda^4 - \lambda^3 - 2\lambda^2 - \lambda + 1$. To see this, note that the map (2.28), in recurrence form, is $y_{n+2}\, y_n = y_{n+1} + y_{n+1}^{-1}$, so that its tropical analogue is

$$Y_{n+2} + Y_n = |Y_{n+1}|. \tag{3.9}$$

From the tropical version of Lemma 2.9, an appropriate choice of basis for $\mathrm{im}\, B$ gives the reduction from (3.8) to (3.9), by setting $Y_n = d_{n+4} - d_{n+3} - 2d_{n+2} - d_{n+1} + d_n$. It can be shown directly that all the orbits of (3.9) are periodic with period 9 [5], and hence in this case the degrees d_n satisfy a linear recurrence of order 13, that is $(\mathcal{S}^9 - 1)Y_n = (\mathcal{S}^9 - 1)(\mathcal{S}^4 - \mathcal{S}^3 - 2\mathcal{S}^2 - \mathcal{S} + 1)d_n = 0$. From the periodicity of Y_n it is clear that $d_{n+4} - d_{n+3} - 2d_{n+2} - d_{n+1} + d_n = O(1)$, which implies that $d_n \sim \mathrm{C}\lambda_{max}^n$ for some $\mathrm{C} > 0$.

Observe that in the two examples above, the tropical recurrences (3.5) and (3.8) both exhibit periodic behaviour, in the sense that the maximum on the right hand side is achieved periodically by the first or the second entry. If one writes "+" in the case where $\sum_{j=2}^N [b_{1,j}]_+ > \sum_{j=2}^N [-b_{1,j}]_+$, and "−" otherwise, then for the tropical Somos-4 recurrence (3.5) one finds that the integer sequence in Example 3.6 repeats the block pattern "$+-+--+--$" of length 8, while for (3.8) the repeating symbolic sequence is the block "$++--+++--$" of length 9. The block length is in accord with the period of the associated periodic maps for the variables Y_n in each case; for other choices of initial data there are still repeating blocks of the appropriate length, but the patterns can be different. Based on a large amount of other numerical evidence, we are led to formulate the following.

Conjecture 3.8. *For each recurrence (3.3) there exists some $k \geq 1$ such that for all real solution sequences (d_n), the associated symbolic sequence corresponding to* max *is eventually periodic with minimal period k.*

Corollary 3.9. *For large enough n, every real solution of (3.3) satisfies a linear recurrence relation of order kN with constant coefficients.*

The above corollary results from the fact that, given Conjecture 3.8, for large enough n one can regard (3.3) as being equivalent to the iteration of a linear recurrence relation whose coefficients vary with period k, and then the following result can be applied.

Lemma 3.10. *Suppose that a sequence (s_n) satisfies a linear recurrence of order ℓ whose coefficients are periodic of period k, say*

$$s_{n+\ell} = \sum_{r=0}^{\ell-1} c_n^{(r)}\, s_{n+r}, \qquad c_{n+k}^{(r)} = c_n^{(r)}.$$

Then the terms of the sequence also satisfy a linear recurrence of order $k\ell$ with constant coefficients, of the form

$$s_{n+k\ell} = \sum_{r=0}^{\ell-1} \tilde{c}_r\, s_{n+rk}. \tag{3.10}$$

Proof: This result should be well known in the literature on linear recurrences, but for completeness we sketch the proof. It suffices to consider the $(\ell+1)\times(\ell+1)$ matrix

$$\tilde{\Phi}_n = \begin{pmatrix} s_n & s_{n+1} & \cdots & s_{n+\ell} \\ s_{n+k} & s_{n+k+1} & \cdots & s_{n+k+\ell} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n+k\ell} & s_{k\ell+1} & \cdots & s_{n+(k+1)\ell} \end{pmatrix} \tag{3.11}$$

which has vanishing determinant, by virtue of the fact that the vector $(c_n^{(0)}, \ldots, c_n^{(\ell-1)}, -1)^T$ is in the right kernel. The recurrence with constant coefficients is obtained from the left kernel (the kernel of $\tilde{\Phi}_n^T$). □

In general, it is not easy to determine the period of the symbolic sequence, and Conjecture 3.8 seems challenging. The real orbits of the recurrences (3.3) can be quite complicated.

A further refinement of Conjecture 3.8 is possible. Let the matrices

$$\mathbf{M}_\pm = \begin{pmatrix} \mathbf{0}^T & -1 \\ \mathbf{1}_{N-1} & \mathbf{a}_\pm \end{pmatrix}$$

be defined in terms of the palindromic vectors $\mathbf{a}_\pm$ of size $N-1$, which correspond to the exponents $[\pm a_j]_+ = [\pm b_{1,j+1}]_+$ (where $\mathbf{0}^T$ is a zero row vector of size $N-1$, and $\mathbf{1}_{N-1}$ is the corresponding identity matrix), so that their characteristic polynomials are $P_\pm$ as in (3.1), respectively. Also, let

$$\mathbf{\Pi} = \overrightarrow{\prod_{j=1,\ldots,k}} \mathbf{M}_{\epsilon_j} = \mathbf{M}_{\epsilon_1} \ldots \mathbf{M}_{\epsilon_k}$$

be the path-ordered product corresponding to the symbolic sequence of length k for a particular orbit of (3.3), defined by an appropriate choice of $\epsilon_j = \pm$, and let $\rho(\mathbf{M}_\pm)$ and $\rho(\mathbf{\Pi})$ denote the corresponding spectral radii.

Conjecture 3.11. *Let k be the period of Conjecture 3.8. There are two possibilities:*
(1) *If $|\lambda_{max}| = \rho(\mathbf{M}_\pm) > \rho(\mathbf{M}_\mp) \geq 1$ then $k=1$, with repeated symbol $\epsilon_1 = \pm$, respectively.*
(2) *If $|\lambda_{max}| = \rho(\mathbf{M}_+) = \rho(\mathbf{M}_-)$ then $\rho(\mathbf{\Pi}) = |\lambda_{max}|^k$ with $k \geq 1$.*

To see how Conjecture 3.1 now follows from all the rest, consider the matrix $\tilde{\Phi}_n$ in the proof of Lemma 3.10, for the case $s_n = d_n$ and $\ell = N$. If Φ_n is the $N \times N$ submatrix such that $\det \Phi_n$ is the upper left connected minor of size N in (3.11), then a single iteration of (3.3) gives $\Phi_{n+1} = \Phi_n \mathbf{M}_{\epsilon_1}$ for some choice of $\epsilon_1 = \pm$. After k iterations it follows from Conjecture 3.8 that $\Phi_{n+k} = \Phi_n \mathbf{\Pi}$, and if n is large enough then $\Phi_{n+rk} = \Phi_n \mathbf{\Pi}^r$ for all r, for some block of symbols ϵ_j of length k that is fixed up to a cyclic permutation (which depends on the choice of n mod k). By the Cayley-Hamilton theorem, if $\tilde{P}(\kappa) = \kappa^N - \sum_{r=0}^{N-1} \tilde{c}_r \kappa^r$ is the characteristic polynomial of $\mathbf{\Pi}$ (which is independent of cyclic permutations of the $\mathbf{M}_{\epsilon_j}$), then $\tilde{P}(\mathcal{S}^k)\Phi_n = \Phi_n \tilde{P}(\mathbf{\Pi}) = 0$, which shows that Φ_n, and hence d_n, satisfies the recurrence (3.10) with $\ell = N$, for all n large enough. Thus the characteristic roots λ for the growth of d_n satisfy $\tilde{P}(\lambda^k) = 0$, which implies that $d_n \sim \mathrm{C}|\lambda_{max}|^n$ for $|\lambda_{max}| = \rho(\mathbf{\Pi})^{1/k} > 1$, or d_n has polynomial growth for $|\lambda_{max}| = 1$. If $k=1$ holds, then in either case (1) or case (2) of Conjecture 3.11, $\mathbf{\Pi} = \mathbf{M}_+$ or $\mathbf{M}_-$, so $|\lambda_{max}|$ is given by one of the spectral radii of the matrices $\mathbf{M}_\pm$, whichever is the larger, and $\mathcal{E} = \log|\lambda_{max}|$. The condition in case (2) is required to ensure that first part of the statement of Conjecture 3.1 holds for $k > 1$ as well.

For the second statement in Conjecture 3.1, note that $P_\pm$ in (3.1) are reciprocal polynomials (their coefficients are palindromic), so that $P_+(\lambda^{-1}) = \lambda^{-N} P_+(\lambda)$, and similarly for P_-. Let

$$S_\pm = \sum_{j=1}^{N} [\pm b_{1,j+1}]_+,$$

respectively. By the symmetry of the matrices $B \to -B$ (or equivalently, the freedom to replace the quiver Q by its opposite), it can be assumed without loss of generality that S_+ is the larger of the two, so $S_+ \geq S_-$, and take $S_+ \geq 3$ so that condition (3.2) holds. Now $P_+(0) = 1$, and $P_+(1) = 2 - S_+ \leq -1$, so P_+ has a real root between 0 and 1. The reciprocal property implies that the reciprocal of a root is also a root, hence P_+ has a real root larger than 1, implying that $|\lambda_{max}| > 1$ and the entropy is positive.

The cases for which (3.2) does not hold are easily enumerated, and it can be checked directly that $|\lambda_{max}| = 1$ in these cases. The main conclusion of this section is the following.

Theorem 3.12. *Suppose that Conjecture 3.1 holds. Then up to the symmetry $S_+ \leftrightarrow S_-$, there are only four distinct choices of the pair (S_+, S_-) for which the algebraic entropy is zero, corresponding to four different families of recurrences:*
(i) $(S_+, S_-) = (1,0)$*: For even $N = 2m$ only, the recurrence is*

$$x_{n+2m}\, x_n = x_{n+m} + 1. \tag{3.12}$$

(ii) $(S_+, S_-) = (2,0)$*: For each $N \geq 2$ and $1 \leq q \leq \lfloor N/2 \rfloor$, the recurrence is*

$$x_{n+N}\, x_n = x_{n+N-q}\, x_{n+q} + 1. \tag{3.13}$$

(iii) $(S_+, S_-) = (2, 1)$*: For even* $N = 2m$ *only, and* $1 \leq q \leq m-1$*, the recurrence is*

$$x_{n+2m}\, x_n = x_{n+2m-q}\, x_{n+q} + x_{n+m}. \tag{3.14}$$

(iv) $(S_+, S_-) = (2, 2)$*: For each* $N \geq 2$ *and* $1 \leq p < q \leq \lfloor N/2 \rfloor$*, the recurrence is*

$$x_{n+N}\, x_n = x_{n+N-p}\, x_{n+p} + x_{n+N-q}\, x_{n+q}. \tag{3.15}$$

The simplest is case (i), where the recurrence (3.12) decouples into m independent copies of the Lyness map: all the orbits are periodic, and the overall period of the sequence of x_n is $5m$. For each n the function $F_n = x_n + x_{n+m} + x_{n+2m} + x_{n+3m} + x_{n+4m}$ is invariant, and can be written as a function of x_n and x_{n+m} only, using (3.12). Moreover, the functions $F_1, F_2, \ldots, F_m$ are independent and Poisson commute with respect to the bracket corresponding to the symplectic form ω, so trivially this system is also integrable in the Liouville-Arnold sense.

The integrability of the families (ii),(iii) and (iv) above is discussed in subsequent sections.

4 Linearisable recurrences from primitives

The primitives introduced in [13] are the simplest examples of cluster mutation-periodic quivers with period 1, and are the building blocks of all such quivers. The nonlinear recurrences that arise from the primitives have the form (3.13), corresponding to case (ii) in Theorem 3.12, and can be rewritten as

$$x_{n+N}\, x_n = x_{n+p}\, x_{n+q} + 1, \qquad p + q = N, \tag{4.1}$$

so that for each $q = 1, \ldots, \lfloor N/2 \rfloor$ there is a different recurrence corresponding to the primitive $P_N^{(q)}$. When p and q are coprime, the associated quivers are mutation-equivalent to the affine Dynkin quivers $\tilde{A}_{q,p}$, corresponding to the $A_{N-1}^{(1)}$ Dynkin diagram with q edges oriented clockwise and p oriented anticlockwise, while if $\gcd(p, q) = k > 1$ so that $p = k\hat{p}$, $q = k\hat{q}$, then the quiver is just the disjoint union of k copies of $\tilde{A}_{\hat{q},\hat{p}}$. In the latter case, it is clear that the recurrence (4.1) is also equivalent to k copies of the recurrence of order N/k corresponding to the coprime integers $\hat{p}, \hat{q}$. Hence it is sufficient to consider only the case where p, q are coprime, which we will assume from now on.

The cluster algebras generated by affine A-type Dynkin quivers arise from surfaces, and are of finite mutation type [8], meaning that only a finite number of distinct quivers is obtained under sequences of mutations (2.1). However, by the classification result in [10], these cluster algebras are not themselves finite: the recurrence (4.1) generates infinitely many cluster variables starting from the initial cluster $\mathbf{x} = (x_1, \ldots, x_N)$.

It was conjectured recently that the cluster variables in cluster algebras obtained from affine Dynkin quivers satisfy linear recurrence relations, and this was proved for all but the exceptional types [1]. A different proof using cluster categories, valid for all affine Dynkin types, was subsequently found by Keller and Scherotzke [23]. In the case of $\tilde{A}_{q,p}$ (with coprime p, q) a proof of the corresponding linear recurrence relations was already given in [13]. Here we present a more direct derivation of the linear recurrences arising from these affine $\tilde{A}_{q,p}$ quivers, before using the Poisson structures from section 2 to explain the Liouville integrability of the maps defined by (4.1).

4.1 Linear relations with periodic coefficients

The key to the properties of the nonlinear recurrence (4.1) is the fact it can be written in the form

$$\det \Psi_n = 1, \quad \text{where} \quad \Psi_n = \begin{pmatrix} x_n & x_{n+q} \\ x_{n+p} & x_{n+N} \end{pmatrix}. \tag{4.2}$$

The identity (4.2) is the frieze relation (see e.g. [1]); it implies that the iterates of (4.1) form an infinite frieze.

Upon forming the matrix

$$\tilde{\Psi}_n = \begin{pmatrix} x_n & x_{n+q} & x_{n+2q} \\ x_{n+p} & x_{n+N} & x_{n+N+q} \\ x_{n+2p} & x_{n+N+p} & x_{n+2N} \end{pmatrix}, \tag{4.3}$$

one can use the Dodgson condensation method [6] to expand the 3×3 determinant in terms of its 2×2 minors, as

$$\det\,\tilde{\Psi}_n = \frac{1}{x_{n+N}}\left(\det\Psi_n\,\det\Psi_{n+N} - \det\Psi_{n+q}\,\det\Psi_{n+p}\right) = 0,$$

by (4.2). By considering the right and left kernels of $\tilde{\Psi}_n$, we are led to the following result.

Lemma 4.1. *The iterates of the recurrence (4.1) satisfy the linear relations*

$$x_{n+2q} - J_n\,x_{n+q} + x_n = 0, \tag{4.4}$$

$$x_{n+2p} - K_n\,x_{n+p} + x_n = 0, \tag{4.5}$$

whose coefficients are periodic functions of period p, q respectively, that is

$$J_{n+p} = J_n, \qquad K_{n+q} = K_n, \qquad \text{for all } n.$$

Proof: A vector in the (right) kernel of $\tilde{\Psi}_n$, can be written as $\mathbf{k}_n = (\tilde{J}_n, -J_n, 1)^T$, where the third entry is normalised to 1 without loss of generality. From the first two rows of the equation $\tilde{\Psi}_n\mathbf{k}_n = 0$ we have the linear system

$$\Psi_n \begin{pmatrix} -\tilde{J}_n \\ J_n \end{pmatrix} = \begin{pmatrix} x_{n+2q} \\ x_{n+N+q} \end{pmatrix}.$$

From Cramer's rule it follows that

$$\tilde{J}_n = -(\det\Psi_n)^{-1} \begin{vmatrix} x_{n+2q} & x_{n+q} \\ x_{n+N+q} & x_{n+N} \end{vmatrix} = (\det\Psi_n)^{-1}\det\Psi_{n+q} = 1,$$

and

$$J_n = \begin{vmatrix} x_n & x_{n+2q} \\ x_{n+p} & x_{n+N+q} \end{vmatrix},$$

where we made use of (4.2). Hence the recurrence (4.4) is given by the first row of $\tilde{\Psi}_n\mathbf{k}_n = 0$. The second and third rows of the latter equation provide the 2×2 linear system

$$\Psi_{n+p} \begin{pmatrix} -\tilde{J}_n \\ J_n \end{pmatrix} = \begin{pmatrix} x_{n+N+q} \\ x_{n+2N} \end{pmatrix}.$$

whose solution gives an alternative formula for J_n. Noting that the coefficients in this linear system are the same as those in the first two rows of the equation $\tilde{\Psi}_{n+p}\mathbf{k}_{n+p} = 0$, it follows that $J_{n+p} = J_n$ for all n. If we consider the left kernel of $\tilde{\Psi}_n$ (the kernel of $\tilde{\Psi}_n^T$), then by symmetry we obtain the recurrence (4.5), whose coefficient K_n has period q. □

Remark 4.2. In the particular case $q = 1$, corresponding to $\tilde{A}_{1,N-1}$ Dynkin quivers, the coefficient K_n has period 1, so $K_{n+1} = K_n = \mathcal{K}$ (constant) for all n. In that case the recurrence (4.5) becomes

$$x_{n+2N-2} + x_n = \mathcal{K}\,x_{n+N-1}, \tag{4.6}$$

which is a *constant coefficient, linear difference equation* for x_n. This was first shown in [13] and is precisely of the form derived for affine A-type quivers in [1, 23].

The quantity $\mathcal{K}$ in (4.6) can be considered as a Laurent polynomial of the initial data $x_1, x_2, \ldots, x_N$ for the map φ associated with (4.1); from (4.6) this is obtained explicitly by repeatedly using the nonlinear recurrence in order to write $\mathcal{K}$ in terms of lower order iterates until it is given in terms of the N initial data. Moreover, the fact that it is independent of n means that $\mathcal{K}$ is a first integral of φ.

In the case of general coprime p, q, it is straightforward to obtain first integrals of the map φ corresponding to (4.1). Indeed, the orbit of J_1 under the action of φ generates p functions $J_1, J_2, \ldots, J_p$, which can be written as rational functions of the N initial data via back-substitution in the relation

$$J_n = \frac{x_n + x_{n+2q}}{x_{n+q}}. \tag{4.7}$$

Clearly any cyclically symmetric function of $J_1, J_2, \ldots, J_p$ is a first integral of φ and the first p elementary symmetric functions of these variables are independent functions of the J_n. Given that $J_1, J_2, \ldots, J_p$ are functionally independent as functions of $x_1, x_2, \ldots, x_N$, any p independent cyclically symmetric functions of the J_n can be picked as first integrals. Similarly, the action of φ generates the functions $K_1, K_2, \ldots, K_q$, and any q independent cyclically symmetric functions of them provide another set of independent first integrals of φ. This gives a total of $p+q = N$ first integrals. These cannot all be functionally independent, because that would imply that the map is periodic. However, the generic orbit generated by (4.1) is not periodic. Below we describe a single functional relation between these first integrals (equation (4.12) below), by considering the monodromy properties of the linear relations (4.4) and (4.5).

Remark 4.3. The existence of $N-1$ independent first integrals $\mathcal{I}_j$, together with the volume form Ω in (2.17), means that the map φ is Liouville integrable in a rather elementary sense. By applying a method in [4], one can pick, say, the first $N-2$ integrals and obtain a Poisson bivector field $\hat{\mathcal{J}}$ which is invariant (or anti-invariant, for odd N), namely

$$\hat{\mathcal{J}} = \hat{\Omega} \lrcorner \mathrm{d}\mathcal{I}_1 \lrcorner \ldots \lrcorner \mathrm{d}\mathcal{I}_{N-2},$$

where the N-vector field $\hat{\Omega}$ is defined by $\hat{\Omega} \lrcorner \Omega = 1$. By construction, this Poisson structure has Casimirs $\mathcal{I}_j$ for $j = 1, \ldots, N-2$, and restricting to the two-dimensional symplectic leaves one has an integrable system with one degree of freedom, with $\mathcal{I}_{N-1}$ being the independent first integral. However, we shall see that the Poisson structure coming from the two-form (2.10) leads to more interesting integrable systems.

4.2 Monodromy matrices and linear relations with constant coefficients

The relation (4.4) implies that the matrix Ψ_n satisfies

$$\Psi_{n+q} = \Psi_n \, \mathbf{L}_n, \qquad \mathbf{L}_n = \begin{pmatrix} 0 & -1 \\ 1 & J_n \end{pmatrix}. \tag{4.8}$$

Upon taking the ordered product of the $\mathbf{L}_n$ over p steps, shifting by q each time, we have the monodromy matrix

$$\mathbf{M}_n := \mathbf{L}_n \mathbf{L}_{n+q} \ldots \mathbf{L}_{n+(p-1)q} = \Psi_n^{-1} \, \Psi_{n+pq}. \tag{4.9}$$

On the other hand, the recurrence (4.5) yields

$$\Psi_{n+p} = \hat{\mathbf{L}}_n \, \Psi_n, \qquad \hat{\mathbf{L}}_n = \begin{pmatrix} 0 & 1 \\ -1 & K_n \end{pmatrix}, \tag{4.10}$$

which gives another monodromy matrix

$$\hat{\mathbf{M}}_n := \hat{\mathbf{L}}_{n+(q-1)p} \ldots \hat{\mathbf{L}}_{n+p} \hat{\mathbf{L}}_n = \Psi_{n+pq} \, \Psi_n^{-1}. \tag{4.11}$$

The cyclic property of the trace implies that

$$\mathcal{K}_n := \operatorname{tr} \mathbf{M}_n = \operatorname{tr} \hat{\mathbf{M}}_n. \tag{4.12}$$

Also, since $\mathbf{L}_n$ has period p, shifting $n \to n+p$ in (4.9) and taking the trace implies that $\mathcal{K}_{n+p} = \mathcal{K}_n$. Similarly, from (4.11) we have $\mathcal{K}_{n+q} = \mathcal{K}_n$. Now the periods p and q are coprime, and since $\mathcal{K}_n$ has both these periods it follows that $\mathcal{K}_n = \mathcal{K}$ =constant, for all n, hence $\mathcal{K}$ is an invariant of φ.

From the expression (4.9) it further follows that $\mathcal{K}$ is a cyclically symmetric function of the J_n, $n = 1, \ldots, p$, while from (4.11) it is also a cyclically symmetric function of the K_n, $n = 1, \ldots, q$. Thus we see that the equality of the traces in (4.12) provides the aforementioned functional relation between these two sets of functions.

We are now ready to show that in general the iterates of (4.1) satisfy a linear relation with constant coefficients. The existence of such a relation follows immediately from (4.4) or (4.5), by applying Lemma 3.10, but the monodromy matrices provide more detailed information about the coefficients.

Theorem 4.4. *The iterates of the nonlinear recurrence (4.1) satisfy the linear relation*

$$x_{n+2pq} + x_n = \mathcal{K}\, x_{n+pq}, \tag{4.13}$$

where $\mathcal{K}$ is the first integral defined by (4.12), with $\mathcal{K}_n = \mathcal{K}$ for all n.

Proof: Using (4.9) we see that $\Psi_{n+pq} = \Psi_n \mathbf{M}_n$, so $\Psi_{n+2pq} = \Psi_n \mathbf{M}_n \mathbf{M}_{n+pq} = \Psi_n \mathbf{M}_n^2$ by periodicity. Noting that $\mathbf{M}_n$ is a 2×2 matrix, with $\det \mathbf{M}_n = 1$ and $\operatorname{tr} \mathbf{M}_n = \mathcal{K}$, yields (by Cayley-Hamilton)

$$\Psi_{n+2pq} - \mathcal{K}\Psi_{n+pq} + \Psi_n = \Psi_n(\mathbf{M}_n^2 - \mathcal{K}\mathbf{M}_n + \mathbf{I}) = 0.$$

The $(1,1)$ component of this equation is just (4.13). $\square$

By making use of the Chebyshev polynomials of the first and second kind, defined by $T_n(\zeta) = \cos(n\theta)$ and $U_n(\zeta) = \sin((n+1)\theta)/\sin\theta$, respectively, for $\zeta = \cos\theta$, the linear equation (4.13) yields an exact expression for the iterates of the nonlinear recurrence.

Corollary 4.5 (Chebyshev polynomials)**.** *The recurrence (4.1) has the explicit solution*

$$x_{i+npq} = x_i\, T_n(\mathcal{K}/2) + \left(x_{i+pq} - \frac{x_i \mathcal{K}}{2}\right) U_{n-1}(\mathcal{K}/2), \qquad i = 1, \ldots, pq-1, \qquad \textit{for all } n. \tag{4.14}$$

4.3 The structure of monodromy matrices

Here we give some relations between the elements of $\mathbf{M}_n$, which provide properties of a natural Poisson tensor associated with the functions J_i. Analogous results regarding $\hat{\mathbf{M}}_n$ and K_i also hold, since the structure of $\mathbf{M}_n$ and $\hat{\mathbf{M}}_n$ is the same up to switching $p \leftrightarrow q$, $J_i \leftrightarrow K_i$ and taking the transpose. To simplify the presentation, we concentrate on the case of $P_N^{(1)}$. (Similar results hold for $P_{2m}^{(q)}$, with $p + q = 2m$ and p, q coprime.) The remarkable fact is that (when N is even) these properties of the Poisson bracket are derived directly from the monodromy matrix. In subsection 4.4 we proceed to show how the Poisson algebra of the J_i is derived from the Poisson bracket between the coordinates x_j.

For the case $q = 1$, we denote the matrix $\mathbf{M}_n = \mathbf{L}_n \mathbf{L}_{n+1} \cdots \mathbf{L}_{n+p-1}$ by $\mathbf{M}_n^{(2m)}$, when $p = 2m-1, m \geq 1$, and $\mathbf{M}_n^{(2m+1)}$, when $p = 2m, m \geq 1$.

It is important in the calculations below that $\mathbf{M}_n^{(p+1)}$ depends only upon the variables $J_n, \ldots, J_{n+p-1}$. For the moment, this is not really a monodromy matrix, if we do not assume any periodicity. The calculations below give a recursive procedure for building the matrices $\mathbf{M}_n^{(2m)}$ and $\mathbf{M}_n^{(2m+1)}$.

The recursion $\mathbf{M}_n^{(p+3)} = \mathbf{M}_n^{(p+1)} \mathbf{L}_{n+p} \mathbf{L}_{n+p+1}$, with the short-hand notation $\mathrm{A} = \mathrm{A}_n^{(p+1)}$, $\tilde{\mathrm{A}} = \mathrm{A}_n^{(p+3)}$, etc., leads to

$$\begin{aligned} &\tilde{\mathrm{A}} = J_{n+p}\mathrm{B} - \mathrm{A}, \quad \tilde{\mathrm{B}} = (J_{n+p}J_{n+p+1} - 1)\mathrm{B} - J_{n+p+1}\mathrm{A}, \\ &\tilde{\mathrm{C}} = J_{n+p}\mathrm{D} - \mathrm{C}, \quad \tilde{\mathrm{D}} = (J_{n+p}J_{n+p+1} - 1)\mathrm{D} - J_{n+p+1}\mathrm{C}, \end{aligned} \tag{4.15}$$

so that $\tilde{\mathcal{K}} = -\mathcal{K} + J_{n+p}\mathrm{B} - J_{n+p+1}\mathrm{C} + J_{n+p}J_{n+p+1}\mathrm{D}$.

Lemma 4.6 (Relations for $\tilde{\mathbf{M}}$)**.** *The components of* $\mathbf{M}_n^{(p+1)}$ *satisfy the relations*

$$\frac{\partial \mathrm{A}_n^{(p+1)}}{\partial J_n} = \frac{\partial \mathrm{B}_n^{(p+1)}}{\partial J_n} = 0, \quad \mathrm{A}_n^{(p+1)} = -\frac{\partial \mathrm{C}_n^{(p+1)}}{\partial J_n}, \quad \mathrm{B}_n^{(p+1)} = -\frac{\partial \mathcal{K}^{(p+1)}}{\partial J_n}, \quad \mathrm{C}_n^{(p+1)} = \frac{\partial \mathcal{K}^{(p+1)}}{\partial J_{n+p-1}},$$

where $\mathcal{K}^{(p+1)} = \mathrm{A}_n^{(p+1)} + \mathrm{D}_n^{(p+1)}$.

Proof: We just prove this for the even case. For $m = 1$, $\mathbf{M}_n^{(2)} = \mathbf{L}_n$, which clearly satisfies these relations. The recursion (4.15) provides us with an inductive step. We have

$$\frac{\partial \tilde{\mathrm{A}}}{\partial J_n} = J_{n+2m-1}\frac{\partial \mathrm{B}}{\partial J_n} - \frac{\partial \mathrm{A}}{\partial J_n} = 0, \quad \frac{\partial \tilde{\mathrm{B}}}{\partial J_n} = (J_{n+2m-1}J_{n+2m} - 1)\frac{\partial \mathrm{B}}{\partial J_n} + J_{n+2m}\frac{\partial \mathrm{A}}{\partial J_n} = 0.$$

Then

$$\frac{\partial \tilde{\mathrm{C}}}{\partial J_n} = J_{n+2m-1}\frac{\partial \mathcal{K}}{\partial J_n} - \frac{\partial \mathrm{C}}{\partial J_n} = \mathrm{A} - J_{n+2m-1}\mathrm{B} = -\tilde{\mathrm{A}},$$

where we have used $\frac{\partial \mathrm{D}}{\partial J_n} = \frac{\partial(\mathrm{A}+\mathrm{D})}{\partial J_n} = \frac{\partial \mathcal{K}}{\partial J_n}$, and $\frac{\partial \tilde{\mathcal{K}}}{\partial J_{n+2m}} = -\mathrm{C} + J_{n+2m-1}\mathrm{D} = \tilde{\mathrm{C}}$.
Finally

$$\begin{aligned}\frac{\partial \tilde{\mathcal{K}}}{\partial J_n} &= -\frac{\partial \mathcal{K}}{\partial J_n} + J_{n+2m-1}\frac{\partial \mathrm{B}}{\partial J_n} - J_{n+2m}\frac{\partial \mathrm{C}}{\partial J_n} + J_{n+2m-1}J_{n+2m}\frac{\partial \mathcal{K}}{\partial J_n} \\ &= (J_{n+2m-1}J_{n+2m} - 1)\frac{\partial \mathcal{K}}{\partial J_n} + J_{n+2m}\mathrm{A} = -(J_{n+2m-1}J_{n+2m} - 1)\mathrm{B} + J_{n+2m}\mathrm{A} = -\tilde{\mathrm{B}},\end{aligned}$$

where again we have used $\frac{\partial \mathrm{D}}{\partial J_n} = \frac{\partial \mathcal{K}}{\partial J_n}$. □

We can use the above relations to form *recursion operators* which can be used to build the functions $\mathcal{K}^{(p+1)}$. Starting with $\mathcal{K}^{(2)} = J_n$ and $\mathcal{K}^{(3)} = J_nJ_{n+1} - 2$, we can use $\mathcal{K}^{(p+3)} = \mathcal{R}^{(p)}\mathcal{K}^{(p+1)}$, where the *recursion operator* is

$$\mathcal{R}^{(p)} = J_{n+p}J_{n+p+1}\frac{\partial^2}{\partial J_n \partial J_{n+p-1}} - J_{n+p}\frac{\partial}{\partial J_n} - J_{n+p+1}\frac{\partial}{\partial J_{n+p-1}} + (J_{n+p}J_{n+p+1} - 1). \tag{4.16}$$

Remark 4.7. An alternative formula for $\mathcal{K}$ is given by a link with the dressing chain:

$$\mathcal{K} = \prod_{j=1}^{p}\left(1 - \frac{\partial^2}{\partial J_j \partial J_{j+1}}\right)\prod_{k=1}^{p} J_k.$$

When p is odd, this formula follows from the results in [38], by setting $\beta_i \to 0$ and $g_i \to J_i$.

4.3.1 Link with the Poisson structure

In the case $N = 2m$, we can use the monodromy matrix to build the Poisson bracket for the functions J_n, K_n. Staying within the context of one quiver $P_{2m}^{(1)}$ for fixed m, we now reinstate the periodicity $\mathbf{L}_{n+p} = \mathbf{L}_n$.

From (4.9), by periodicity, for all k and n we have

$$\mathbf{M}_{n+k}\mathbf{L}_{n+k} = \mathbf{L}_{n+k}\mathbf{M}_{n+k+1}.$$

When $k = 0$, this implies

$$\mathrm{A}_{n+1} = \mathrm{D}_n - J_n\mathrm{C}_{n+1}, \quad \mathrm{B}_{n+1} + \mathrm{C}_n = J_n(\mathrm{D}_n - \mathrm{D}_{n+1}), \quad \mathrm{C}_{n+1} = -\mathrm{B}_n, \quad \mathrm{D}_{n+1} = \mathrm{A}_n - J_n\mathrm{B}_n.$$

Shifting n, we can write $\mathrm{C}_n = -\mathrm{B}_{n-1} = -\mathrm{B}_{n+p-1}$. Only two of the remaining equations are independent, leading to

$$J_n\mathrm{B}_n = \mathrm{A}_n - \mathrm{D}_{n+1}, \quad \mathrm{B}_{n+p-1} - \mathrm{B}_{n+1} = J_n(\mathrm{D}_{n+1} - \mathrm{D}_n).$$

The equations for $k \neq 0$ are obtained by shifting the indices. The first equation leads to

$$\sum_{k=1}^{p-1}(-1)^{k+1}J_{n+k}\mathrm{B}_{n+k} = \sum_{k=1}^{p-1}(-1)^{k+1}(\mathrm{A}_{n+k} - \mathrm{D}_{n+k+1}) = \mathrm{A}_{n+1} - \mathrm{A}_{n+p} = \mathrm{A}_{n+1} - \mathrm{A}_n,$$

since the remaining sum consists of

$$\sum_{j=1}^{m-1}(\mathrm{A}_{n+2j+1} + \mathrm{D}_{n+2j+1}) - \sum_{j=1}^{m-1}(\mathrm{A}_{n+2j} + \mathrm{D}_{n+2j}) = (m-1)\mathcal{K} - (m-1)\mathcal{K} = 0.$$

On the other hand $\mathrm{B}_{n+p-1} - \mathrm{B}_{n+1} = J_n(\mathrm{D}_{n+1} - \mathrm{D}_n)$, so

$$J_n\sum_{k=1}^{p-1}(-1)^{k+1}J_{n+k}\mathrm{B}_{n+k} + \mathrm{B}_{n+p-1} - \mathrm{B}_{n+1} = J_n(\mathrm{A}_{n+1} - \mathrm{A}_{n+p} + \mathrm{D}_{n+1} - \mathrm{D}_n) = 0.$$

If we take cyclic permutations, we obtain a matrix equation of the form

$$(\mathbf{P}^{(2)} + \mathbf{P}^{(0)})\mathbf{b} = 0, \quad \text{with} \quad \mathbf{b} = (\mathrm{B}_n, \mathrm{B}_{n+1}, \ldots, \mathrm{B}_{n+p-1})^T = -\nabla\mathcal{K},$$

where the components of the matrix $\mathbf{P}^{(2)} + \mathbf{P}^{(0)}$ are those of the Poisson tensor in Lemma 4.9 below, and we have used the formula $\mathrm{B}_n = -\frac{\partial\mathcal{K}}{\partial J_n}$ for all n, coming from Lemma 4.6.

We can summarise these results in

Theorem 4.8. *The function $\mathcal{K}$, defined by (4.12) is the Casimir of the Poisson bracket of Lemma 4.9:*

$$(\mathbf{P}^{(2)} + \mathbf{P}^{(0)})\nabla\mathcal{K} = 0.$$

4.4 Poisson brackets and Liouville integrability for $P_{2m}^{(1)}$

It was proved in [14] that the linearisable maps coming from the primitives $P_N^{(1)}$ (the $\tilde{A}_{1,N-1}$ Dynkin quivers) are Liouville integrable when N is even. We give the proof here, since it is the basis for understanding the integrability of the maps for the other $P_N^{(q)}$ quivers.

The proof starts from the Poisson bracket for the cluster variables. The matrix B in this case is nondegenerate, having the form

$$B = \tau_N - \tau_N^T, \quad \text{with} \quad \tau_N = \sum_{r=1}^{N-1}\mathbf{E}_{r+1,r} - \mathbf{E}_{1,N}, \tag{4.17}$$

where $\mathbf{E}_{r,s}$ denotes an element of the standard basis for $gl(N)$. The "skew rotation" matrix τ_N plays an important role in the classification presented in [13]. The matrix (4.17) for $N = 4$ is obtained by setting $c = 0$ in (2.14).

By Theorem 2.6, the map is symplectic, and hence there is a nondegenerate Poisson bracket of the form (2.6), with $C = B^{-1}$, up to scaling. In accordance with [14], we take

$$C = \tau_N^T + (\tau_N^T)^3 + \cdots + (\tau_N^T)^{N-1} = \sum_{s=1}^{\frac{N}{2}}\sum_{r=1}^{N-2s+1}\left(\mathbf{E}_{r,r+2s-1} - \mathbf{E}_{r+2s-1,r}\right),$$

so that $CB = 2\,I$, which gives

$$\{\, x_j, x_k \,\} = \begin{cases} \operatorname{sgn}(k-j)\, x_j x_k, & k-j \quad \text{odd}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.18}$$

for $j, k = 1, \ldots, N$, with sgn denoting the sign function.

The key to the Liouville integrability of the $P_N^{(1)}$ maps is the expression of the Poisson bracket between the periodic functions J_n, which appeared in [21] (for $N = 4$) and [14] (for general even $N = 2m$).

Lemma 4.9. *For even $N = p + 1$ and $q = 1$, the functions J_n given by (4.7) define a Poisson subalgebra of codimension one in the algebra (4.18), with brackets*

$$\{\, J_j, J_k \,\} = 2\operatorname{sgn}(k-j)\,(-1)^{j+k+1} J_j J_k + 2(\delta_{j,k+1} - \delta_{j+1,k} + \delta_{j+p-1,k} - \delta_{j,k+p-1}), \tag{4.19}$$

for $j, k = 1, \ldots, p$.

The bracket for the J_n is clearly a sum $\{\,,\,\} = \{\,,\,\}_2 + \{\,,\,\}_0$, corresponding to the splitting of the Poisson tensor into homogeneous parts $\mathbf{P}^{(2)} + \mathbf{P}^{(0)}$, as in Theorem 4.8. The quadratic bracket $\{\,,\,\}_2$ is log-canonical, while the degree zero bracket can be defined simply by specifying its only non-zero terms as

$$\{\, J_{j+1}, J_j \,\}_0 = 2 = -\{\, J_j, J_{j+1} \,\}_0 \tag{4.20}$$

for all j mod $p = 2m - 1$. Since (4.19) is still a Poisson bracket after scaling $J_j \to \mu J_j$ for arbitrary μ, the brackets $\{,\}_0$ and $\{,\}_2$ are compatible, so define a bi-Hamiltonian structure. This means that one can use the standard bi-Hamiltonian chain [26], defining a sequence of functions $\mathcal{I}_j$ which satisfy

$$\{\mathcal{I}_j, \mathcal{I}_k\}_0 = 0 = \{\mathcal{I}_j, \mathcal{I}_k\}_2, \qquad \text{for all} \quad j, k,$$

where the sequence starts from $\mathcal{I}_0 = \sum_j J_j$, the Casimir of the bracket $\{,\}_0$ given by (4.20), and finishes with $\mathcal{I}_{m-1} = \prod_j J_j$, the Casimir of $\{,\}_2$. By Theorem 4.8, the function $\mathcal{K}$ defined by (4.12) is the generating function for these integrals, so that

$$\mathcal{K} = \sum_{j=0}^{m-1} (-1)^{m+j+1} \mathcal{I}_j, \tag{4.21}$$

where $\mathcal{I}_j$ is the term of degree $2j + 1$ in the variables J_i. Since these integrals commute with respect to both brackets, they commute with respect to the sum $\{\,,\,\}_2 + \{\,,\,\}_0$, and hence provide m commuting integrals for the map φ of the variables x_j in even dimension N, which implies that the map is Liouville integrable.

In summary, we have

Theorem 4.10. *For $N = 2m$ and $q = 1$ the map φ defined by (4.1) has m functionally independent Poisson commuting integrals, given by the terms of each odd homogeneous degree in the quantity $\mathcal{K}$, as given by equation (4.21). The map is also superintegrable, having a total of $N - 1$ independent first integrals.*

As discussed in subsection 4.2, extra first integrals are obtained by choosing $m - 1$ additional cyclically symmetric functions of $J_1, \ldots, J_p$.

4.5 Primitives of the form $P_{2m+1}^{(1)}$

The recurrences (4.1) for $p = 2m$ and $q = 1$ are given by

$$x_{n+2m+1}\, x_n = x_{n+2m}\, x_{n+1} + 1. \tag{4.22}$$

The formula (4.17) still holds, but now the matrix B is singular. It has a one-dimensional kernel, spanned by the vector $\mathbf{u} = (1, -1, \ldots, 1, -1, 1)^T$, which generates the scaling symmetry

$$x_j \to \lambda^{(-1)^{j+1}}\, x_j, \qquad \lambda \in \mathbb{C}^*, \tag{4.23}$$

and $\operatorname{im} B$ is spanned by

$$\mathbf{v}_j = \mathbf{e}_j + \mathbf{e}_{j+1}, \qquad j = 1, \ldots, 2m, \tag{4.24}$$

where $\mathbf{e}_j$ is the jth standard basis vector. Hence, by Lemma 2.7, the coordinates

$$y_j = x_j\, x_{j+1}, \qquad j = 1, \ldots, 2m, \tag{4.25}$$

are invariant under the scaling (4.23), and the degenerate form (2.10) pushes forward to a symplectic form (2.22) in dimension $2m$, whose coefficients $\hat{b}_{jk}$ are the matrix elements of

$$\hat{B} = \tau_{2m} - \tau_{2m}^2 + \tau_{2m}^3 - \cdots + \tau_{2m}^{2m-1},$$

where τ_{2m} is the $2m \times 2m$ version of τ_N. The inverse of this is the skew-symmetric matrix

$$\hat{B}^{-1} = \tau_{2m}^T + (\tau_{2m}^T)^2 + (\tau_{2m}^T)^3 + \cdots + (\tau_{2m}^T)^{2m-1},$$

with all components above the diagonal equal to 1, giving a nondegenerate Poisson bracket for the y_j, i.e.

$$\{\, y_j, y_k \,\} = \operatorname{sgn}(k-j)\, y_j y_k, \qquad 1 \le j, k \le 2m. \tag{4.26}$$

Upon applying the rest of Theorem 2.6, we see that (4.22) induces a symplectic map on the variables y_i, given (for $m \ge 2$) by

$$\hat{\varphi} : \, (y_1, y_2, \ldots, y_{2m-1}, y_{2m}) \mapsto (y_2, y_3, \ldots, y_{2m}, y_{2m+1}), \tag{4.27}$$

where

$$y_{2m+1} = y_2 y_4 \cdots y_{2m} (y_2 y_4 \cdots y_{2m} + y_3 \cdots y_{2m-1}) / (y_1 y_3^2 \cdots y_{2m-1}^2).$$

The map is simpler for the case $m = 1$, given in Example 4.11 below.

By the general discussion above, the iterates x_n satisfy the linear relation (4.6), where $\mathcal{K}$ is the trace of the monodromy matrix $\mathbf{M}_n$. The latter is given in terms of the quantities

$$J_n = \frac{x_n + x_{n+2}}{x_{n+1}}, \qquad n = 1, \ldots, 2m, \tag{4.28}$$

which cycle with period $2m$ under the action of the recurrence (4.22). The polynomial $\mathcal{K}$ can be expanded as

$$\mathcal{K} = \sum_{j=0}^{m} (-1)^{m+j} \mathcal{I}_j, \qquad \text{where} \qquad \mathcal{I}_0 = 2, \tag{4.29}$$

and each polynomial $\mathcal{I}_j$ is homogeneous of degree $2j$ in the variables J_n. The non-trivial homogeneous components $\mathcal{I}_1, \ldots, \mathcal{I}_m$ provide m first integrals for (4.22), and an additional m independent first integrals can be obtained by choosing cyclically symmetric functions of J_n for each odd degree $1, 3, \ldots, 2m-1$.

However, not all of these first integrals reduce to functions on the $2m$-dimensional symplectic manifold with coordinates y_j. Note that the scaling symmetry (4.23) acts on the variables (4.28) according to

$$J_n \longrightarrow \lambda^{2(-1)^{n+1}}\, J_n. \tag{4.30}$$

Applying this to the formula (4.16) shows that each component of $\mathcal{K}$ is invariant under scaling, which means that the m first integrals $\mathcal{I}_1, \ldots, \mathcal{I}_m$ can be rewritten as functions of the scale-invariant variables y_j. The Liouville integrability of the map (4.27) then follows, provided that these m functions are in involution with respect to the bracket (4.26). We shall not pursue the case of general m further here, but content ourselves with presenting some low-dimensional examples.

Example 4.11 (The primitive $P_3^{(1)}$)**.** For $m=1$ the map φ for the x_j variables is (2.8), which is associated with $P_3^{(1)}$, the affine $A_2^{(1)}$ Dynkin quiver. The matrix B has rank two, so in terms of the variables y_j given by (4.25) the symplectic form has the log-canonical form (2.26), and the induced map of the plane is

$$\hat{\varphi}: \qquad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ y_2(y_2+1)/y_1 \end{pmatrix}.$$

Symmetric functions of the period 2 quantities

$$J_1 = \frac{x_1+x_3}{x_2}, \quad J_2 = \frac{x_1x_2+x_2x_3+1}{x_1x_3}$$

give two first integrals for the map (2.8), namely

$$J_1+J_2 \qquad \text{and} \qquad \mathcal{I}_1 = J_1J_2 = \mathcal{K}+2 = \frac{(y_1+y_2)(y_1+y_2+1)}{y_1y_2}.$$

The latter is defined on the (y_1,y_2) plane, and $\hat{\varphi}^*\mathcal{I}_1 = \mathcal{I}_1$, so the symplectic map $\hat{\varphi}$ with one degree of freedom has an invariant function and hence is integrable.

Example 4.12 (The primitive $P_5^{(1)}$)**.** For $m=2$ the recurrence (4.22) has the first integrals $\mathcal{I}_1$ and $\mathcal{I}_2$ given by the homogeneous terms of degree 2 and 4, respectively, in the expression (4.29):

$$\mathcal{K} = 2-(J_1J_2+J_2J_3+J_3J_4+J_4J_1)+J_1J_2J_3J_4 = \mathcal{I}_0-\mathcal{I}_1+\mathcal{I}_2, \tag{4.31}$$

where the J_n are defined by (4.28). Picking another pair of cyclically symmetric functions of J_n, of degrees 1 and 3, say, adds two more independent first integrals.

Now defining y_j for $j=1,2,3,4$ by (4.25), these variables are endowed with the nondegenerate Poisson bracket (4.26), which is invariant under the map (4.27):

$$\hat{\varphi}: \quad (y_1,y_2,y_3,y_4) \mapsto \Big(y_2,y_3,y_4,y_2y_4(y_2y_4+y_3)/(y_1y_3^2)\Big).$$

To show that this map is Liouville integrable, it is necessary to verify that $\{\mathcal{I}_1,\mathcal{I}_2\}=0$ with respect to this bracket.

The terms in the formula (4.31) can all be expressed via the functions

$$w_i = J_i\,J_{i+1}, \tag{4.32}$$

so that $\mathcal{I}_1 = w_1+w_2+w_3+w_4$, $\mathcal{I}_2 = w_1w_3$. From (4.30) it is clear that these w_i are invariant under the action of the scaling symmetry (4.23). This means that they can be written as functions of y_j, viz:

$$w_1 = \frac{(y_1+y_2)(y_2+y_3)}{y_2^2}, \quad w_2 = \frac{(y_2+y_3)(y_3+y_4)}{y_3^2}, \quad w_3 = \frac{(y_3+y_4)(y_2y_3+y_1y_3^2+y_2^2y_4)}{y_1y_3^2y_4}.$$

Under the action of $\hat{\varphi}$, since the J_n cycle with period 4 under φ, the w_i transform as

$$\hat{\varphi}^*w_1 = w_2, \;\; \hat{\varphi}^*w_2 = w_3, \;\; \hat{\varphi}^*w_3 = w_4 = \frac{w_1w_3}{w_2}, \;\; \hat{\varphi}^*w_4 = w_1.$$

Although only the first three are independent, it is convenient to make use of w_4 as well.

The first three w_i form a three-dimensional Poisson subalgebra of the y_j, which is non-polynomial:

$$\{w_1,w_2\} = w_1w_2-w_1-w_2, \quad \{w_1,w_3\} = w_2-\frac{w_1w_3}{w_2}, \quad \{w_2,w_3\} = w_2w_3-w_2-w_3.$$

The Casimir of this algebra is $\mathcal{I}_1-\mathcal{I}_2 = w_1+w_2+w_3+\frac{w_1w_3}{w_2}-w_1w_3 = 2-\mathcal{K}$.

Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are both functions defined on this subalgebra, it follows that $\{\mathcal{I}_1,\mathcal{I}_2\} = \{\mathcal{I}_1,\mathcal{K}\}=0$, so the two first integrals are in involution, as required.

Remark 4.13. The Poisson bracket of the four functions w_i can be calculated in polynomial form as

$$\{w_1, w_2\} = w_1 w_2 - w_1 - w_2, \quad \{w_1, w_3\} = w_2 - w_4, \quad \{w_1, w_4\} = w_1 + w_4 - w_1 w_4,$$

with the remaining brackets following from the cyclic property. This bracket has the two Casimirs

$$\mathcal{C}_1 = w_1 w_3 - w_2 w_4 \quad \text{and} \quad \mathcal{C}_2 = w_1 + w_2 + w_3 + w_4 - w_1 w_3,$$

so that the three-dimensional algebra for w_1, w_2, w_3 arises from the constraint $\mathcal{C}_1 = 0$.

4.6 Primitives $P_N^{(q)}$ with $q > 1$

For general q, to obtain the quiver $P_N^{(q)}$ we must modify the formula (4.17) and write

$$B = \tau_N^q - (\tau_N^q)^T. \tag{4.33}$$

In this case, we have to take into account both sets of functions J_n, $n = 1, \ldots, p$ and K_n, $n = 1, \ldots, q$, defined through (4.4) and (4.5) respectively, with the function $\mathcal{K}$ being given by the two formulae of (4.12). It turns out that the essential properties of the quantities J_n with period p can be obtained by considering the J_n in the case of $P_{p+1}^{(1)}$ (the $\tilde{A}_{1,p}$ quiver), and applying a suitable permutation of indices; and similarly the properties of the K_n are the same as those of the J_n for $P_{q+1}^{(1)}$, up to a permutation of indices.

Concentrating for the moment on the functions J_n, we consider the formula (4.9) for $\mathbf{M}_n$. Each of the matrices $\mathbf{L}_{n+\ell q}$ (after using the cyclic property $J_{n+p} = J_n$) is just one of the matrices $\mathbf{L}_n$, $n = 1, \ldots, p$. For coprime p, q, each of the J_n appears exactly once in this product and in a specific order, which defines a permutation σ of the integers $(1, \ldots, p)$. For this discussion, for a general pair p, q (with $q < p$) let us use $\mathcal{K}_{p,q}$ to mean tr $\mathbf{M}_n$, considered as a function of the J_n. Then we have

$$\mathcal{K} = \mathcal{K}_{p,q}(J_1, \ldots, J_p) = \mathcal{K}_{p,1}(J_{\sigma(1)}, \ldots, J_{\sigma(p)}). \tag{4.34}$$

Similarly, in terms of the functions K_n, the formula (4.11) for $\hat{\mathbf{M}}_n$ defines a permutation $\hat{\sigma}$ of $(1, \ldots, q)$, and if we write $\mathcal{K}_{q,p}$ to denote $\mathrm{tr}\, \hat{\mathbf{M}}_n$, then we have

$$\mathcal{K} = \mathcal{K}_{q,p}(K_1, \ldots, K_q) = \mathcal{K}_{q,1}(K_{\hat{\sigma}(1)}, \ldots, K_{\hat{\sigma}(q)}). \tag{4.35}$$

(There is no risk of confusion between $\mathcal{K}_{q,p}$ and $\mathcal{K}_{p,q}$ once we have fixed $q < p$.)

For more detailed properties of the maps defined by (4.1), and the associated quantities J_n and K_n, it is necessary to consider even/odd N separately.

4.6.1 The even case

In the case that N is even, when q and p are coprime it can be shown by elementary row/column operations on the matrix (4.33) that det $B = 4$. The matrix B is of Toeplitz type, and invertible, with an inverse of the same type that defines a nondegenerate Poisson structure of the form (2.6). With the choice of scale $C = 2B^{-1}$, the Poisson bracket for the x_j is given explicitly by

$$\{\, x_j, x_k \,\} = \begin{cases} \mathrm{sgn}(k-j)\,(-1)^{r+s} x_j x_k, & k - j \quad \text{odd}, \\ 0 & \text{otherwise}, \end{cases} \tag{4.36}$$

where (by the Euclidean algorithm) the integers r, s are uniquely determined by writing

$$\frac{1}{2}\Big(p - |k-j|\Big) = sm - r\ell, \qquad \text{for} \quad 0 \le r < m, \quad 0 \le s \le \ell,$$

in terms of the coprime integers $\ell = (p-q)/2$, $m = (p+q)/2$. (Note that p and q are both odd.)

Consider the functions J_n once again, and for fixed n define the following sequences:

$$X_j = x_{n+(j-1)q}, \quad j=1,2,\ldots; \qquad J_k^\dagger = J_{n+(k-1)q} = \frac{X_k + X_{k+2}}{X_{k+1}}, \quad k=1,2,\ldots.$$

Note that the sequence of $J_k^\dagger$ is also periodic with the same period: $J_{k+p}^\dagger = J_k^\dagger$; in fact (up to the choice of n, which just gives an overall cyclic permutation) the ordering of this sequence corresponds to the permutation σ in (4.34), i.e. $J_k^\dagger = J_{\sigma(k)}$. Using the Poisson bracket (4.36), we compute $\{X_1, X_2\} = \{x_n, x_{n+q}\} = X_1 X_2$, $\{X_1, X_3\} = \{x_n, x_{n+2q}\} = 0$, and so on, and then in each case we find that, for a suitable range of indices, the X_j satisfy the same Poisson algebra (4.18) as the x_j in the $P_{p+1}^{(1)}$ case, so that the bracket is

$$\{\, X_j, X_k \,\} = \begin{cases} \operatorname{sgn}(k-j)\, X_j X_k, & k-j \quad \text{odd}, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for} \quad |j-k| \le p.$$

To verify the analogue of Lemma 4.9, it is sufficient to calculate the brackets $\{J_1^\dagger, J_j^\dagger\}$ for $j = 2, \ldots, (p+1)/2$ and then use the periodicity, which shows that the permuted quantities $J_k^\dagger$ satisfy

$$\{\, J_j^\dagger, J_k^\dagger \,\} = 2\operatorname{sgn}(k-j)\,(-1)^{j+k+1} J_j^\dagger J_k^\dagger + 2(\delta_{j,k+1} - \delta_{j+1,k} + \delta_{j+p-1,k} - \delta_{j,k+p-1}),$$

for $j,k = 1,\ldots,p$. This is the same as the Poisson algebra (4.19) for the J_n in the case of $P_{p+1}^{(1)}$. An identical argument implies that, up to the permutation $\hat\sigma$, the q functions K_n satisfy the algebra (4.19) corresponding to $P_{q+1}^{(1)}$.

Remark 4.14. Theorem 4.8, together with (4.12), implies that the same function $\mathcal{K}$ is simultaneously the Casimir of the J_n subalgebra and of the K_n subalgebra.

By further direct calculation using the bracket (4.36) one can verify that

$$\{J_i, K_i\} = 0 \qquad \forall\, i \qquad \Longrightarrow \qquad \{J_i, K_j\} = 0 \qquad \forall\, i,j, \tag{4.37}$$

where the second statement follows from the first by repeatedly shifting $i \to i+1$, and using periodicity and the coprimality of p and q.

Thus, from (4.37), we see that these two subalgebras Poisson commute with each other. The J_n subalgebra provides $(p+1)/2$ commuting integrals, which are found by applying the permutation σ to each homogeneous component of the sum (4.21) for the case of the primitive $P_{p+1}^{(1)}$. Similarly, by applying the permutation $\hat\sigma$ one obtains $(q+1)/2$ commuting integrals for the K_n subalgebra. These two sets of integrals also commute with each other, and the relation (4.12) provides a single constraint, which gives a total of $(p+1)/2+(q+1)/2-1 = N/2$ independent commuting integrals, as required for Liouville integrability.

Example 4.15 (The primitive $P_8^{(3)}$)**.** The matrix $C = 2B^{-1}$ is Toeplitz, with top row

$$(c_{1,j}) = (0,-1,0,1,0,1,0,-1),$$

which specifies the Poisson bracket (4.36) in this case. This Poisson bracket is invariant under the map

$$\varphi: \qquad (x_1,\ldots,x_8) \mapsto (x_2,\ldots,x_9), \qquad x_9 = \frac{x_4 x_6 + 1}{x_1}. \tag{4.38}$$

The functions J_n, which cycle with period 5, are given by

$$J_1 = \frac{x_1 + x_7}{x_4}, \quad J_2 = \frac{x_2 + x_8}{x_5}, \quad J_3 = \frac{x_1 x_3 + x_4 x_6 + 1}{x_1 x_6}, \quad J_4 = \frac{x_2 x_4 + x_5 x_7 + 1}{x_2 x_7}, \quad J_5 = \frac{x_3 x_5 + x_6 x_8 + 1}{x_3 x_8}.$$

These form a Poisson subalgebra with brackets $\{J_1, J_2\} = -2J_1J_2$, $\quad \{J_1, J_3\} = -2J_1J_3 + 2$, with all other brackets following from the cyclic property and skew-symmetry. This is the algebra of $P_6^{(1)}$, after the permutation $\sigma : (1, 2, 3, 4, 5) \mapsto (1, 4, 2, 5, 3)$. It provides three commuting functions,

$$\mathcal{I}_0 = J_1 + J_2 + J_3 + J_4 + J_5, \quad \mathcal{I}_1 = J_1J_4J_2 + J_2J_5J_3 + J_3J_1J_4 + J_4J_2J_5 + J_5J_3J_1, \quad \mathcal{I}_2 = J_1J_2J_3J_4J_5,$$

which (from Theorem 4.8) are the homogeneous components of the Casimir of this subalgebra, namely

$$\mathcal{K} = \operatorname{tr} \mathbf{M}_1 = \mathcal{I}_0 - \mathcal{I}_1 + \mathcal{I}_2.$$

The generators of the period 3 subalgebra are given by

$$K_1 = \frac{x_1x_3 + x_6x_8 + 1}{x_3x_6}, \quad K_2 = \frac{x_1 + x_7 + x_4(x_1x_2 + x_6x_7)}{x_1x_4x_7}, \quad K_3 = \frac{x_2 + x_8 + x_5(x_2x_3 + x_7x_8)}{x_2x_5x_8},$$

whose Poisson bracket relations are

$$\{K_1, K_2\} = -2K_1K_2 + 2, \quad \{K_2, K_3\} = -2K_2K_3 + 2, \quad \{K_1, K_3\} = 2K_1K_3 - 2.$$

Up to the permutation $\hat{\sigma} : (1, 2, 3) \mapsto (1, 3, 2)$, this is the algebra associated with $P_4^{(1)}$, with Casimir

$$\mathcal{K} = \operatorname{tr} \hat{\mathbf{M}}_1 = -\hat{\mathcal{I}}_0 + \hat{\mathcal{I}}_1, \qquad \text{where} \quad \hat{\mathcal{I}}_0 = K_1 + K_2 + K_3, \quad \hat{\mathcal{I}}_1 = K_1K_2K_3.$$

The latter two quantities commute with each other, and we have the relation $\mathcal{I}_0 - \mathcal{I}_1 + \mathcal{I}_2 + \hat{\mathcal{I}}_0 - \hat{\mathcal{I}}_1 = 0$.

Since (4.37) holds, any four of the five functions $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \hat{\mathcal{I}}_0, \hat{\mathcal{I}}_1$ provide the correct number of independent commuting first integrals to show Liouville integrability of the 8-dimensional map (4.38).

4.6.2 The odd case

When N is odd, then p is odd and q is even, or vice versa. With $N = 2m+1$ we find that, as for the primitives $P_{2m+1}^{(1)}$, the kernel of the matrix (4.33) is spanned by the same vector $\mathbf{u} = (1, -1, \ldots, 1, -1, 1)^T$, orthogonal to the vectors (4.24) providing the symplectic coordinates $y_j = \mathbf{x}^{\mathbf{v}_j}$, as in (4.25), which are invariant under the one-parameter scaling group (4.23). The symplectic form obtained via Lemma 2.7 gives a nondegenerate Poisson bracket for the y_j, of the form

$$\{\, y_j, y_k \,\} = \epsilon_{jk}\, y_j y_k, \qquad 1 \leq j, k \leq 2m. \tag{4.39}$$

In all examples we find that the Toeplitz matrix $\hat{B}^{-1} = (\epsilon_{jk})$ has only the entries $1, -1, 0$ above the diagonal, but a concise formula for these ϵ_{jk} in terms of the coprime integers p, q is presently unavailable.

By Theorem 2.6, we have an induced birational map $\hat{\varphi}$ in $2m$ dimensions, which is a Poisson map with respect to the nondegenerate bracket (4.39). We would like to assert that this is an integrable map.

Assume for the sake of argument that p is odd and q is even. Then the J_n can be written as functions of y_j, and from (4.39) they should satisfy the algebra (4.19) corresponding to $P_{p+1}^{(1)}$, hence providing $(p+1)/2$ commuting integrals. The scaling-invariant quantities $\hat{w}_n = K_nK_{n+q}$, for $1 \leq n \leq q-1$, can also be written in terms of the y_j, and (up to the permutation $\hat{\sigma}$) should give a Poisson subalgebra isomorphic to that of the functions (4.32) for $P_{q+1}^{(1)}$, providing another $q/2$ commuting integrals. With the constraint (4.12), this would give m independent commuting integrals in dimension $2m$, as required.

For the rest of this section, we present examples of primitives $P_N^{(q)}$ with odd N and $q > 1$.

Example 4.16 (The primitive $P_5^{(2)}$)**.** For $N = 5$, $q = 2$ the matrix (4.33) has null vector $\mathbf{u} = (1, -1, 1, -1, 1)^T$, and $\operatorname{im} B$ is spanned by the vectors

$$\mathbf{v}_1 = (1, 1, 0, 0, 0)^T,\ \mathbf{v}_2 = (0, 1, 1, 0, 0)^T,\ \mathbf{v}_3 = (0, 0, 1, 1, 0)^T,\ \mathbf{v}_4 = (0, 0, 0, 1, 1)^T.$$

Upon applying Lemma 2.7, the Toeplitz matrix $\hat{B}^{-1}$ is specified by its first row, namely $(\epsilon_{1,j}) = (0,0,1,1)$, which determines the components of the nondegenerate Poisson bracket (4.39) for the variables $y_i = x_i x_{i+1},\ i = 1,\ldots,4$. Explicitly (for indices $j<k$) this is just

$$\{y_1,y_3\} = y_1y_3, \quad \{y_1,y_4\} = y_1y_4, \quad \{y_2,y_4\} = y_2y_4,$$

all other brackets being zero. The Poisson bracket is invariant under the induced map

$$\hat{\varphi}: \qquad (y_1,y_2,y_3,y_4) \mapsto \Big(y_2,y_3,y_4,y_2y_4(y_3+1)/(y_1y_3)\Big). \tag{4.40}$$

The period 3 functions J_n take the form

$$J_1 = \frac{x_1+x_5}{x_3}, \quad J_2 = \frac{x_1x_2+x_3x_4+1}{x_1x_4}, \quad J_3 = \frac{x_2x_3+x_4x_5+1}{x_2x_5}.$$

Being invariant under the scaling symmetry (4.23), they can also be written in terms of the variables y_i, as

$$J_1 = \frac{y_1y_3+y_2y_4}{y_2y_3}, \quad J_2 = \frac{y_2(y_1+y_3+1)}{y_1y_3}, \quad J_3 = \frac{y_3(y_2+y_4+1)}{y_2y_4}.$$

The Poisson brackets between these functions follow by the cyclic property from $\{J_1,J_2\} = 1 - J_1J_2$, which, up to rescaling by a factor of 2 and applying the permutation $\sigma : (1,2,3) \mapsto (1,3,2)$, is the bracket (4.19) of the J_n for $P_4^{(1)}$. The J_n subalgebra provides a pair of first integrals in involution, namely

$$\mathcal{I}_0 = J_1 + J_2 + J_3, \qquad \mathcal{I}_1 = J_1J_2J_3,$$

which is sufficient for the map (4.40) to be Liouville integrable. (Since these functions are totally symmetric, not just cyclically symmetric, the permutation σ plays no role.)

The period 2 quantities K_n, which are not invariant under the scaling (4.23), are

$$K_1 = \frac{x_1x_2+x_4x_5+1}{x_2x_4}, \quad K_2 = \frac{x_1+x_5+x_3(x_1x_2+x_4x_5)}{x_1x_3x_5}.$$

As for the case of $P_3^{(1)}$ in Example 4.11, the product K_1K_2 is invariant under the scaling symmetry, so can be written in terms of the variables y_i. In fact, from (4.12) we have $\mathcal{K} = \mathcal{I}_1 - \mathcal{I}_0 = K_1K_2 - 2$, where (by Theorem 4.8) $\mathcal{K}$ is the Casimir of the bracket for the J_n.

Example 4.17 (The Case $P_7^{(2)}$)**.** In the case $N=7$, $q=2$, the Toeplitz matrix $\hat{B}^{-1}$ is specified by its first row, namely $(\epsilon_{1,j}) = (0,1,1,0,0,1)$, which determines the components of the nondegenerate Poisson bracket (4.39) for the variables $y_i = x_ix_{i+1},\ i=1,\ldots,6$. This bracket is preserved by the 6-dimensional map

$$\hat{\varphi}: \quad (y_1,\ldots,y_6) \mapsto (y_2,\ldots,y_7), \qquad y_7 = \frac{y_2y_6(y_3y_5+y_4)}{y_1y_3y_5},$$

which (by Theorem 2.6) is induced from the map φ defined by the recurrence (4.1) with $p=5$, $q=2$.

The functions J_n, which cycle with period 5 under the action of φ, can be written in terms of y_i thus:

$$J_1 = \frac{y_1y_3+y_2y_4}{y_2y_3}, \quad J_2 = \frac{y_2y_4+y_3y_5}{y_3y_4}, \quad J_3 = \frac{y_3y_5+y_4y_6}{y_4y_5},$$
$$J_4 = \frac{y_2y_4+y_1y_3y_4+y_2y_3y_5}{y_1y_3y_5}, \quad J_5 = \frac{y_3y_5+y_2y_4y_5+y_3y_4y_6}{y_2y_4y_6}.$$

The Poisson subalgebra generated by these functions is specified by

$$\{J_1,J_2\} = J_1J_2, \quad \{J_1,J_3\} = J_1J_3 - 1,$$

with all other brackets following from the cyclic property and skew-symmetry. By rescaling by a factor of 2 and applying a permutation to order these functions as $(J_1, J_3, J_5, J_2, J_4)$, this is seen to isomorphic to the algebra of the J_n for $P_6^{(1)}$, so we find the commuting functions

$$\mathcal{I}_0 = J_1 + J_3 + J_5 + J_2 + J_4, \quad \mathcal{I}_1 = J_1J_3J_5 + J_2J_4J_1 + J_3J_5J_2 + J_4J_1J_3 + J_5J_2J_4, \quad \mathcal{I}_2 = J_1J_3J_5J_2J_4.$$

Of course, the ordering is unimportant for the totally symmetric functions $\mathcal{I}_0$ and $\mathcal{I}_2$.

The period 2 quantities, which scale as $K_1 \to \lambda^2 K_1$, $K_2 \to \lambda^{-2} K_2$ under (4.23), are

$$K_1 = \frac{x_2 + x_6 + x_4(x_1x_2 + x_6x_7)}{x_2x_4x_6}, \quad K_2 = \frac{x_1x_3 + x_1x_7 + x_5x_7 + x_3x_5(x_1x_2 + x_6x_7)}{x_1x_3x_5x_7}.$$

From (4.12), the scaling-invariant combination K_1K_2 can be written in terms of y_i, via

$$K_1K_2 - 2 = \mathcal{K} = \mathcal{I}_0 - \mathcal{I}_1 + \mathcal{I}_2.$$

Example 4.18 (The Case $P_7^{(3)}$)**.** For $N = 7$, $q = 3$, the first row of the Toeplitz matrix $\hat{B}^{-1}$ is $(\epsilon_{1,j}) = (0, 0, 0, 1, 1, 0)$. This defines the nondegenerate Poisson bracket (4.39), which is invariant under the map

$$\hat{\varphi}: \quad (y_1, \ldots, y_6) \mapsto (y_2, \ldots, y_7), \qquad y_7 = \frac{y_2y_4y_6(y_4 + 1)}{y_1y_3y_5},$$

in terms of the variables $y_i = x_ix_{i+1}$, $i = 1, \ldots, 6$.

The functions J_n, with period 4, are not invariant under (4.23), but $w_n = J_nJ_{n+1}$, $n = 1, \ldots, 4$ are:

$$w_1 = \frac{(1 + y_1 + y_4)(y_1y_3y_5 + y_2y_4y_6)}{y_1y_3y_4y_5}, \quad w_2 = \frac{(1 + y_1 + y_4)(1 + y_2 + y_5)}{y_1y_5},$$
$$w_3 = \frac{(1 + y_2 + y_5)(1 + y_3 + y_6)}{y_2y_6}, \quad w_4 = \frac{w_1w_3}{w_2},$$

of which only three are independent (since $w_1w_3 = w_2w_4$). By periodicity and skew-symmetry, all of their Poisson brackets follow from

$$\{w_1, w_2\} = w_1 + w_2 - w_1w_2, \quad \{w_1, w_3\} = w_4 - w_2.$$

By applying the permutation $\sigma : (1, 2, 3, 4) \mapsto (1, 4, 3, 2)$, this is seen to be isomorphic to the algebra for $P_5^{(1)}$, as in Example 4.12. Hence we have two functions in involution, namely

$$\mathcal{I}_1 = w_1 + w_4 + w_3 + w_2, \qquad \mathcal{I}_2 = w_1w_3,$$

where the ordering is unimportant since both functions $\mathcal{I}_1$ and $\mathcal{I}_2$ are invariant under σ.

The necessary third function in involution is derived from the quantities K_n, which cycle with period 3. Being invariant under the scaling (4.23), they can be written in terms of the variables y_i:

$$K_1 = \frac{y_3(y_1 + y_5 + 1)}{y_2y_4}, \quad K_2 = \frac{y_4(y_2 + y_6 + 1)}{y_3y_5}, \quad K_3 = \frac{y_1y_3y_5(y_3 + 1) + y_2y_4y_6(y_4 + 1)}{y_1y_3y_4y_6}.$$

They generate a three-dimensional Poisson subalgebra with the same relations as for the subalgebra in $P_4^{(1)}$ (up to a factor of 2), i.e.

$$\{K_1, K_2\} = K_1K_2 - 1, \quad \{K_2, K_3\} = K_2K_3 - 1, \quad \{K_1, K_3\} = -K_1K_3 + 1.$$

There are two first integrals that are commuting functions defined on this subalgebra, which we denote by

$$\hat{\mathcal{I}}_0 = K_1 + K_2 + K_3, \qquad \hat{\mathcal{I}}_1 = K_1K_2K_3,$$

and the joint Casimir of the two subalgebras is given by $\mathcal{K} = 2 - \mathcal{I}_1 + \mathcal{I}_2 = \hat{\mathcal{I}}_1 - \hat{\mathcal{I}}_0$. Since $\{K_i, w_j\} = 0$ for all i, j, we may use any three of the functions $\mathcal{I}_1, \mathcal{I}_2, \hat{\mathcal{I}}_0, \hat{\mathcal{I}}_1$ to show Liouville integrability.

5 Linearisable recurrences from $P_{2m}^{(q)} - P_{2m}^{(m)} + P_{2(m-q)}^{(m-q)}$ quivers

In this section we consider the family of recurrences (3.14), as in case (iii) of Theorem 3.12, which come from the quivers of the form $P_{2m}^{(q)} - P_{2m}^{(m)} + P_{2(m-q)}^{(m-q)}$. It is convenient to rewrite each recurrence as

$$x_{n+N}\, x_n = x_{n+p}\, x_{n+q} + x_{n+m}, \qquad p+q = N = 2m, \tag{5.1}$$

which (for fixed m) gives a different recurrence for each $q = 1, \ldots, m-1$. The associated matrix B is given by

$$B = \tau_{2m}^{q} - (\tau_{2m}^{q})^T - \tau_{2m}^{m} + \hat{\tau}_{2(m-q)}^{m-q}, \tag{5.2}$$

where τ_N is defined in (4.17), and $\hat{\tau}_{2(m-q)}$ denotes the $N \times N$ matrix obtained by adding q left and right columns and upper and lower rows of zeros to the $2(m-q)\times 2(m-q)$ matrix $\tau_{2(m-q)}$. The simplest examples of the quivers corresponding to such B are shown in Figure 1.

Observe that if $\gcd(m,q) = r > 1$ then the quiver consists of r disjoint copies of the same type of quiver, but with the parameters q and m replaced by the coprime integers q/r and m/r, respectively, and similarly (5.1) decouples into r copies of the corresponding recurrence. Therefore we shall assume that $\gcd(m,q) = 1$ from now on. With this assumption it follows from $p+q = 2m$ that $\gcd(p,q) = 1$ or 2 only. The case $\gcd(p,q) = 1$ has a very similar structure to that of the primitives $P_N^{(q)}$ for even N, but the case $\gcd(p,q) = 2$ has several new features, so we will need to distinguish between these two cases in due course.

The family (5.1) has some basic properties that are analogous to those of the family in case (ii) of Theorem 3.12, as described in the previous section. In particular, all of the recurrences (5.1) are linearisable, and they have two sets of periodic functions, with periods p, q respectively, which lead to the construction of first integrals.

5.1 More linear relations with periodic coefficients

By analogy with (4.2), the recurrence (5.1) can be written in the form

$$\det\, \Psi_n = \begin{vmatrix} x_n & x_{n+q} \\ x_{n+p} & x_{n+N} \end{vmatrix} = x_{n+m}. \tag{5.3}$$

The above identity is the relation for a 2-frieze [28], and it implies that the iterates of (5.1) can be placed in the form an infinite 2-frieze. Using Dodgson condensation once again to condense a 3×3 determinant, we have

$$\det \tilde{\Psi}_n = (x_{n+m}x_{n+N+m} - x_{n+m+q}x_{n+m+p})/x_{n+N} = 1,$$

with $\tilde{\Psi}_n$ as in (4.3). Then condensing the appropriate 4×4 matrix Δ_n in terms of 3×3 minors yields

$$\det \Delta_n = \begin{vmatrix} x_n & x_{n+q} & x_{n+2q} & x_{n+3q} \\ x_{n+p} & x_{n+N} & x_{n+N+q} & x_{n+N+2q} \\ x_{n+2p} & x_{n+N+p} & x_{n+2N} & x_{n+2N+q} \\ x_{n+3p} & x_{n+N+2p} & x_{n+2N+p} & x_{n+3N} \end{vmatrix} = 0.$$

As in the case of the primitives considered before, the left and right kernels of the singular matrix Δ_n yield linear relations between the x_n.

Lemma 5.1. *The iterates of the recurrence (4.1) satisfy the linear relations*

$$x_{n+3q} - J_{n+m}\, x_{n+2q} + J_n\, x_{n+q} - x_n = 0, \tag{5.4}$$

$$x_{n+3p} - K_{n+m}\, x_{n+2p} + K_n\, x_{n+p} - x_n = 0, \tag{5.5}$$

whose coefficients are periodic functions of period p, q respectively, that is

$$J_{n+p} = J_n, \qquad K_{n+q} = K_n, \qquad \textit{for all } n.$$

Proof: Upon solving $\Delta_n \mathbf{k}_n = 0$, using the first three rows, it is convenient to normalise the first entry of $\mathbf{k}_n$ to be -1, and solve a 3×3 system to find the other three entries. Then from Cramer's rule and $\det \tilde{\Psi}_n = 1$ the fourth entry must be $+1$, so that this vector has the form $\mathbf{k}_n = (-1, J_n, \hat{J}_n, 1)^T$. The 3×3 linear system coming from the first three rows of the equation $\Delta_{n+p}\mathbf{k}_{n+p} = 0$ is the same as that coming from the last three rows of $\Delta_n \mathbf{k}_n = 0$, which implies that $J_{n+p} = J_n$ and $\hat{J}_{n+p} = \hat{J}_n$. Applying Cramer's rule in the first three rows of $\Delta_n \mathbf{k}_n = 0$ together with Dodgson condensation also implies that

$$J_n = \begin{vmatrix} x_n & x_{n+2q} & x_{n+3q} \\ x_{n+p} & x_{n+N+q} & x_{n+N+2q} \\ x_{n+2p} & x_{n+2N} & x_{n+2N+q} \end{vmatrix} = \frac{1}{x_{n+4m-p}} \begin{vmatrix} E_n & x_{n+5m-2p} \\ E_{n+p} & x_{n+5m-p} \end{vmatrix}, \tag{5.6}$$

and similarly

$$\hat{J}_n = \frac{1}{x_{n+2m}} \begin{vmatrix} E_{n+2m-p} & x_{n+m} \\ E_{n+2M} & x_{n+m+p} \end{vmatrix}, \qquad \text{where} \qquad E_n = \begin{vmatrix} x_n & x_{n+2q} \\ x_{n+p} & x_{n+N+q} \end{vmatrix}. \tag{5.7}$$

Permuting the first and second columns of the determinant in the identity $\det \tilde{\Psi}_n = 1$ and expanding in terms of 2×2 minors, and then doing the same thing after permuting the second and third columns instead, leads to the formulae

$$\begin{vmatrix} E_n & x_{n+m} \\ E_{n+p} & x_{n+m+p} \end{vmatrix} = -x_{n+p}, \qquad \begin{vmatrix} E_n & x_{n+3m-p} \\ E_{n+p} & x_{n+3m} \end{vmatrix} = x_{n+4m-p}, \tag{5.8}$$

respectively. Then the combination $x_{n+2m}x_{n+5m-p}(J_{n+m} + \hat{J}_n)$ can be rewritten as a sum of determinants, whose entries can be expanded using each of the identities (5.8), to yield

$$\begin{aligned}
&\begin{vmatrix} x_{n+2m}E_{n+m} & x_{n+6m-2p} \\ E_{n+m+p}x_{n+2m} & x_{n+6m-p} \end{vmatrix} + \begin{vmatrix} x_{n+5m-p}E_{n+2m-p} & x_{n+m} \\ x_{n+5m-p}E_{n+2m} & x_{n+m+p} \end{vmatrix} \\
&= \begin{vmatrix} x_{n+2m}E_{n+m} & x_{n+6m-2p} \\ x_{n+2m+p}E_{n+m} + x_{n+m+p} & x_{n+6m-p} \end{vmatrix} + \begin{vmatrix} x_{n+5m-2p}E_{n+2m} + x_{n+6m-2p} & x_{n+m} \\ x_{n+5m-p}E_{n+2m} & x_{n+m+p} \end{vmatrix} \\
&= E_{n+m} \begin{vmatrix} x_{n+2m} & x_{n+2m+2q} \\ x_{n+2m+p} & x_{n+2m+N+q} \end{vmatrix} + E_{n+2m} \begin{vmatrix} x_{n+m+2q} & x_{n+m} \\ x_{n+m+N+q} & x_{n+m+p} \end{vmatrix} = 0,
\end{aligned}$$

as required. This proves (5.4), and the relation (5.5) follows by symmetry, from considering the left kernel of Δ_n. $\square$

Remark 5.2. The four-term linear relations (5.4) and (5.5), together with $\det \tilde{\Psi}_n = 1$, should be compared with those of the pentagram map [31], but there the coefficients of the second and third terms are independent.

Remark 5.3. When $q = 1$ the coefficient K_n has period 1, so $K_{n+1} = K_n = \mathcal{K}$ for all n, and the recurrence (5.5) is just the *constant coefficient, linear difference equation*

$$x_{n+3N-3} - \mathcal{K}\, x_{n+2N-2} + \mathcal{K}\, x_{n+N-1} - x_n = 0. \tag{5.9}$$

An immediate consequence of the latter relation is an inhomogeneous version of (4.6), namely

$$x_{n+2N-2} - (\mathcal{K} - 1)\, x_{n+N-1} + x_n = F_n, \qquad \text{where} \qquad F_{n+N-1} = F_n,$$

for some quantity F_n (which has period $p = N - 1$ in this case). Thus, by a minor modification of Corollary 4.5, one can find explicit formulae for x_n in terms of Chebyshev polynomials.

Remark 5.4. When $q = 2$, the coefficients in the recurrence (5.5) have period 2, so $K_{n+2} = K_n$ for all n, and (since $\gcd(m, q) = 1$ implies that m is odd) we have $K_{n+m} = K_{n+1}$, whence

$$x_{n+3p} - K_{n+1}\, x_{n+2p} + K_n\, x_{n+p} - x_n = 0. \tag{5.10}$$

This is a four-term linear relation, whose coefficients alternate with the parity of the index n.

The existence of the periodic quantities J_n and K_n means that, as for the case of primitives considered in the last section, one can construct first integrals by taking cyclically symmetric functions of each of these sets of quantities. When $\gcd(p, q) = 1$ we find one relation between these two sets of quantities, and when $\gcd(p, q) = 2$ we find two relations, which will be discussed below.

5.2 Monodromy and linear relations with constant coefficients

This subsection follows closely the discussion of subsection 4.2 for the case of primitives. However, it will subsequently be necessary to refine the discussion further, depending on whether $\gcd(p, q) = 1$ or 2.

The relation (5.4) implies that the matrix $\tilde{\Psi}_n$ satisfies

$$\tilde{\Psi}_{n+q} = \tilde{\Psi}_n\, \mathbf{L}_n, \qquad \mathbf{L}_n = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -J_n \\ 0 & 1 & J_{n+m} \end{pmatrix}. \tag{5.11}$$

On the other hand, the recurrence (5.5) yields

$$\tilde{\Psi}_{n+p} = \hat{\mathbf{L}}_n\, \tilde{\Psi}_n, \qquad \hat{\mathbf{L}}_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -K_n & K_{n+m} \end{pmatrix}. \tag{5.12}$$

As before, upon taking the ordered product of the $\mathbf{L}_n$ over p steps, shifting by q each time, we have the monodromy matrix

$$\mathbf{M}_n := \mathbf{L}_n \mathbf{L}_{n+q} \ldots \mathbf{L}_{n+(p-1)q} = \tilde{\Psi}_n^{-1}\, \tilde{\Psi}_{n+pq}. \tag{5.13}$$

Taking the ordered product of the $\hat{\mathbf{L}}_n$ over q steps, shifting by p each time, gives another monodromy matrix

$$\hat{\mathbf{M}}_n := \hat{\mathbf{L}}_{n+(q-1)p} \ldots \hat{\mathbf{L}}_{n+p} \hat{\mathbf{L}}_n = \tilde{\Psi}_{n+pq}\, \tilde{\Psi}_n^{-1}. \tag{5.14}$$

From the cyclic property of the trace it follows that

$$\mathcal{K}_n := \operatorname{tr} \mathbf{M}_n = \operatorname{tr} \hat{\mathbf{M}}_n. \tag{5.15}$$

The periodicity of $\mathbf{L}_n$, together with (5.13), implies that $\mathcal{K}_{n+p} = \mathcal{K}_n$, and similarly, from (5.14), we have $\mathcal{K}_{n+q} = \mathcal{K}_n$. If the periods p and q are coprime, then $\mathcal{K}_n = \mathcal{K}$ =constant, for all n, hence $\mathcal{K}$ is a first integral for the map φ corresponding to (5.1). However, if $\gcd(p, q) = 2$ holds instead then we have $\mathcal{K}_{n+2} = \mathcal{K}_n$, so this quantity has period 2.

Once again the general result of Lemma 3.10 can be applied here, to show that the iterates of (5.1) satisfy a linear relation with constant coefficients.

Proposition 5.5. *The iterates of the nonlinear recurrence (5.1) satisfy a linear relation of order $3pq$ with constant coefficients. It is a four-term relation if $\gcd(p, q) = 1$, and a seven-term relation if $\gcd(p, q) = 2$.*

Proof: From (5.4), Lemma 3.10 implies that x_n satisfies a linear recurrence of order $3pq$ with constant coefficients, for which the gaps between indices of adjacent terms with nonzero coefficients are of size p. On the other hand, (5.5) implies a linear recurrence of the same order with gaps of size q. Thus the actual size of the gaps must be the lowest common multiple of p and q. Hence, when p and q are coprime, the gaps are of size pq, giving a four-term relation, while $\gcd(p, q) = 2$ gives a seven-term relation with gaps of size $pq/2$. □

Below we provide refined versions of the preceding result, with more precise details of the coefficients, by considering the cases $\gcd(p, q) = 1$ and $\gcd(p, q) = 2$ separately.

5.3 The case $\mathbf{gcd}(p, q) = 1$

The discussion of the case where p and q are coprime is almost identical to that for the primitives $P_N^{(q)}$ with N even, as in subsection 4.7 above. The integers p and q are both odd, and for the matrix (5.2) in each case we find that $\det B = 4$ whenever p or q is divisible by 3, and $\det B = 1$ otherwise. From such a nondegenerate matrix B we get an invariant Poisson bracket of the form (2.6), which is specified uniquely (up to scale) by the Toeplitz matrix $C = B^{-1}$. However, in this case a general closed-form expression for the entries of C, analogous to the formula (4.36) for the Poisson bracket of the even primitives, is not available to us at present.

We now consider the associated functions J_n and K_n, of periods p and q respectively. From the expression (5.13), the first integral $\mathcal{K}$ defined by (5.15) is a cyclically symmetric polynomial in the J_n, $n = 1, \ldots, p$, and from (5.14) it is also a cyclically symmetric polynomial in the K_n, $n = 1, \ldots, q$. Thus the equality of the traces in (5.15) provides a single functional relation between these two sets of functions. Note that we also have the same phenomenon as in (4.34) with regard to the different expressions for $\mathcal{K}$ as a function of the quantities J_n, when we compare the cases with $q = 1$ and $q > 1$, for the same value of p: up to the action of a suitable permutation σ of $(1, \ldots, p)$, the two expressions are identical; and the analogous statement applies to $\mathcal{K}$ considered as a function of the K_n.

Now observe that all of the preceding comments concerning $\mathcal{K}$ apply equally well to the quantity

$$\tilde{\mathcal{K}} := \operatorname{tr} \mathbf{M}_n^{-1} = \operatorname{tr} \hat{\mathbf{M}}_n^{-1},$$

which is also a first integral (so this definition holds for any n). We would like to assert that in fact $\tilde{\mathcal{K}} = \mathcal{K}$. To see this, note that $\operatorname{tr} \hat{\mathbf{L}}_n = K_{n+m}$, and $\operatorname{tr} \hat{\mathbf{L}}_n^{-1} = K_n$. Thus, in the case $q = 1$, when $K_n = \mathcal{K} = \text{constant}$, we have $\mathcal{K} = \operatorname{tr} \hat{\mathbf{M}}_n = \operatorname{tr} \hat{\mathbf{M}}_n^{-1} = \tilde{\mathcal{K}}$ by (5.14). This then implies that, in terms of functions of $J_1, \ldots, J_p$, we have

$$\operatorname{tr} \mathbf{M}_n = \operatorname{tr} \mathbf{M}_n^{-1} \tag{5.16}$$

for $q = 1$, and clearly this identity remains true when $q > 1$, since (for fixed p) the functions $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are obtained from the case $q = 1$ by applying the same permutation σ to both sides. This allows us to make a more precise statement than Proposition 5.5.

Theorem 5.6. *When gcd$(p, q) = 1$, the iterates of the nonlinear recurrence (5.1) satisfy the linear relation*

$$x_{n+3pq} - \mathcal{K}\, x_{n+2pq} + \mathcal{K}\, x_{n+pq} - x_n = 0, \tag{5.17}$$

where $\mathcal{K}$ is the first integral defined by (5.15).

Proof: Using (5.13) we see that $\tilde{\Psi}_{n+pq} = \tilde{\Psi}_n \mathbf{M}_n$, so $\tilde{\Psi}_{n+2pq} = \tilde{\Psi}_{n+pq} \mathbf{M}_n \mathbf{M}_{n+pq} = \tilde{\Psi}_n \mathbf{M}_n^2$, by periodicity of $\mathbf{M}_n$, and similarly $\tilde{\Psi}_{n+3pq} = \tilde{\Psi}_n \mathbf{M}_n^3$. Applying Cayley-Hamilton to both $\mathbf{M}_n$ and $\mathbf{M}_n^{-1}$, and noting that $\det \mathbf{M}_n = 1$ and $\operatorname{tr} \mathbf{M}_n = \mathcal{K}$, as well as (5.16), yields

$$\tilde{\Psi}_{n+3pq} - \mathcal{K}\tilde{\Psi}_{n+2pq} + \mathcal{K}\tilde{\Psi}_{n+pq} - \tilde{\Psi}_n = \tilde{\Psi}_n(\mathbf{M}_n^3 - \mathcal{K}\mathbf{M}_n^2 + \mathcal{K}\mathbf{M}_n - \mathbf{I}) = 0.$$

The $(1, 1)$ component of this equation is just (5.17). □

Remark 5.7 (Another construction of the integral for $P_3^{(1)}$)**.** We may think of the map $\hat{\varphi}$ in Example 4.11 as coming from the recurrence $y_{n+2} y_n = y_{n+1}^2 + y_{n+1}$, which is of the same form as (5.1), with $p = q = m = 1$, although not directly obtained from a cluster mutation. Replacing $x_n \to y_n$ and $\mathcal{K} \to \hat{\mathcal{K}}$ in (5.17), and solving, leads to the first integral

$$\hat{\mathcal{K}} = \frac{y_{n+3} - y_n}{y_{n+2} - y_{n+1}}.$$

When this is written purely in terms of y_n, y_{n+1}, using the map, we find the previously obtained quantity

$$\mathcal{I}_1 = \hat{\mathcal{K}} + 1 = \frac{(y_n + y_{n+1})(y_n + y_{n+1} + 1)}{y_n y_{n+1}}.$$

For the Liouville integrability of the map φ corresponding to (5.1), the counting of first integrals appears to be the same as for the even primitives $P^{(q)}_{2m}$. The two sets of quantities J_n and K_n generate Poisson subalgebras of dimensions p and q, which should contain $(p+1)/2$ and $(q+1)/2$ commuting integrals, respectively, including the Casimir $\mathcal{K}$, which is common to both subalgebras. Taking the constraint (5.15) into account, this produces $m = (p+q)/2$ independent commuting integrals in terms of the x_j, as required.

However, while the above counting argument is plausible, it rests on some unproven assumptions. Everything relies on the structure of the Poisson bracket for the J_n in the case $q = 1$, since all other subalgebras of J_n or K_n should be isomorphic to one with $q = 1$. Yet in general (excluding $m = 2$), this Poisson bracket consists of three homogeneous parts (of degrees $0, 1$ and 2), which (unlike the bracket (4.19) for the primitive $P^{(1)}_{2m}$) does not have an obvious splitting into a bi-Hamiltonian pair. Moreover, while every cyclically symmetric polynomial function of the J_n is a first integral of the map φ, we do not yet have an algorithm for selecting an involutive set of them.

Remark 5.8. In [31] a quadratic Poisson structure is presented for the coefficients of the four-term linear relations for twisted polygons, together with corresponding monodromy matrices and commuting first integrals for the pentagram map. However, a Dirac reduction of this bracket to the case of (5.4) or (5.5) gives only the trivial bracket. A general approach to Poisson structures related to twisted polygons is described in [27], which should shed some light on the situation here.

For want of more general statements, we illustrate the foregoing discussion with several examples of the integrable systems that arise in this case.

Example 5.9 (The quiver $P^{(1)}_4 - P^{(2)}_4 + P^{(1)}_2$)**.** This example was first studied in an ad hoc way in [21]. The appropriate matrix (5.2), which corresponds to the first quiver in Figure 1, is obtained by setting $c = 1$ in (2.14). The explicit form of the map is

$$\varphi: \qquad (x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, x_5), \qquad x_5 = \frac{x_2 x_4 + x_3}{x_1}. \tag{5.18}$$

The invariant Poisson bracket (2.6) for this map is given by the Toeplitz matrix $C = 2B^{-1}$, with top row $(c_{1,j}) = (0, 1, 1, 2)$. The period 3 functions J_i take the form

$$J_1 = \frac{x_3(x_2 + x_1 x_3) + x_4(x_1^2 + x_2^2)}{x_1 x_2 x_4}, \quad J_2 = \frac{x_1 x_4 + x_2^2 + x_3^2}{x_2 x_3}, \quad J_3 = \frac{x_3(x_2 + x_1 x_3) + x_4(x_1 x_4 + x_2^2)}{x_1 x_3 x_4}.$$

The Poisson brackets between these three functions follow by the cyclic property from

$$\{J_1, J_2\} = J_1 J_2 - 2J_3. \tag{5.19}$$

This example is exceptional, in that the bracket for the J_i is the sum of only *two* homogeneous terms:

$$\mathbf{P} = \mathbf{P}^{(2)} + \mathbf{P}^{(1)}, \quad \text{where} \quad \mathbf{P}^{(2)} = \begin{pmatrix} 0 & J_1 J_2 & -J_1 J_3 \\ -J_1 J_2 & 0 & J_2 J_3 \\ J_1 J_3 & -J_2 J_3 & 0 \end{pmatrix}, \quad \mathbf{P}^{(1)} = \begin{pmatrix} 0 & -2J_3 & 2J_2 \\ 2J_3 & 0 & -2J_1 \\ -2J_2 & 2J_1 & 0 \end{pmatrix}.$$

Each of the tensors specified by $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ satisfies the Jacobi identity, so (since their sum is a Poisson tensor) they define a compatible pair of Poisson brackets.

The first integrals

$$\mathcal{H}_1 = J_1^2 + J_2^2 + J_3^2, \qquad \mathcal{H}_2 = J_1 J_2 J_3$$

satisfy the bi-Hamiltonian ladder

$$\mathbf{P}_1 \nabla \mathcal{H}_1 = 0, \qquad \mathbf{P}_2 \nabla \mathcal{H}_1 = \mathbf{P}_1 \nabla \mathcal{H}_2, \qquad \mathbf{P}_2 \nabla \mathcal{H}_2 = 0,$$

so they commute with respect to the bracket defined by (5.19). The quantity

$$\mathcal{K} = 3 - \mathcal{H}_1 + \mathcal{H}_2 \tag{5.20}$$

provides the Casimir of this bracket; following (4.34), we will find it useful to denote this quantity by $\mathcal{K}_{3,1}$.

Example 5.10 (The quiver $P_6^{(1)} - P_6^{(3)} + P_4^{(2)}$)**.** Mutation of the second quiver in Figure 1 gives the map

$$\varphi: \qquad (x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_2, x_3, x_4, x_5, x_6, x_7), \qquad x_7 = \frac{x_2x_6 + x_4}{x_1}.$$

The invariant Poisson bracket for this map is given by (2.6), with the coefficients specified by the Toeplitz matrix $C = B^{-1}$ with top row $(c_{1,j}) = (0, 0, 1, 0, 1, 1)$. The functions J_i, which cycle with period 5 under the action of φ, take the form

$$J_1 = \frac{x_1x_5 + x_2x_6 + x_3x_4}{x_2x_5}, \quad J_2 = \frac{x_4(x_3 + x_1x_5) + (x_1 + x_3)x_2x_6}{x_1x_3x_6}, \quad J_3 = \frac{x_3(x_2 + x_4) + x_1x_5)}{x_2x_4},$$
$$J_4 = \frac{x_4(x_3 + x_5) + x_2x_6)}{x_3x_5}, \quad J_5 = \frac{x_4(x_3 + x_1x_5) + (x_1x_5 + x_2x_3)x_6}{x_1x_4x_6}.$$

The Poisson bracket between these five functions follow by the cyclic property from

$$\{J_1, J_2\} = -J_1J_2 - J_4 + 1, \quad \{J_1, J_3\} = 2J_1J_3.$$

This Poisson bracket is the sum of three homogeneous terms,

$$\mathbf{P} = \mathbf{P}^{(2)} + \mathbf{P}^{(1)} + \mathbf{P}^{(0)}, \quad \text{with} \quad \mathbf{P}^{(2)}_{ik} = c^{(2)}_{ik} J_iJ_k, \quad \mathbf{P}^{(1)}_{ik} = c^{(1)}_{ik} J_{k+2}, \quad \mathbf{P}^{(0)}_{ik} = c^{(0)}_{ik},$$

where $c^{(\ell)}_{ik}$ are the Toeplitz matrices with top rows given by

$$(c^{(2)}_{1,k}) = (0, -1, 2, -2, 1), \quad (c^{(1)}_{1,k}) = (0, -1, 0, 0, 1), \quad (c^{(0)}_{1,k}) = (0, 1, 0, 0, -1).$$

Both $\mathbf{P}^{(0)}$ and $\mathbf{P}^{(2)}$ satisfy the Jacobi identity, but $\mathbf{P}^{(1)}$ does not, so we cannot think of this sum as some sort of Poisson compatibility.

The Casimir for the 5-dimensional Poisson algebra generated by the J_i is the trace of the monodromy matrix, as in (5.15). It can be written as the sum

$$\mathcal{K} = \mathcal{H}_1 - \mathcal{H}_2 + \mathcal{H}_3, \tag{5.21}$$

where each of the components is a first integral:

$$\mathcal{H}_1 = \sum_{i=1}^{5}(J_i - J_iJ_{i+1}), \quad \mathcal{H}_2 = \sum_{i=1}^{5}(J_iJ_{i+1}J_{i+2} - J_iJ_{i+1}^2J_{i+2}), \quad \mathcal{H}_3 = \prod_{i=1}^{5} J_i.$$

We find that $\{\mathcal{H}_i, \mathcal{H}_j\} = 0$ for all i, j, so the 6-dimensional Poisson map φ has the correct number of first integrals in involution. For later comparison, we denote $\mathcal{K}$ in (5.21) by $\mathcal{K}_{5,1}$.

Example 5.11 (The quiver $P_8^{(1)} - P_8^{(4)} + P_6^{(3)}$)**.** The map obtained from mutation of this quiver is

$$\varphi: \qquad (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \mapsto (x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9), \qquad x_9 = \frac{x_2x_8 + x_5}{x_1}.$$

The corresponding non-singular matrix B defines an invariant Poisson bracket (2.6) with matrix $C = B^{-1}$. The top row of the Toeplitz matrix C is $(c_{1,j}) = (0, 1, 0, 0, 1, 1, 0, 1)$. The functions J_i with period 7 can be determined from J_1, which takes the form

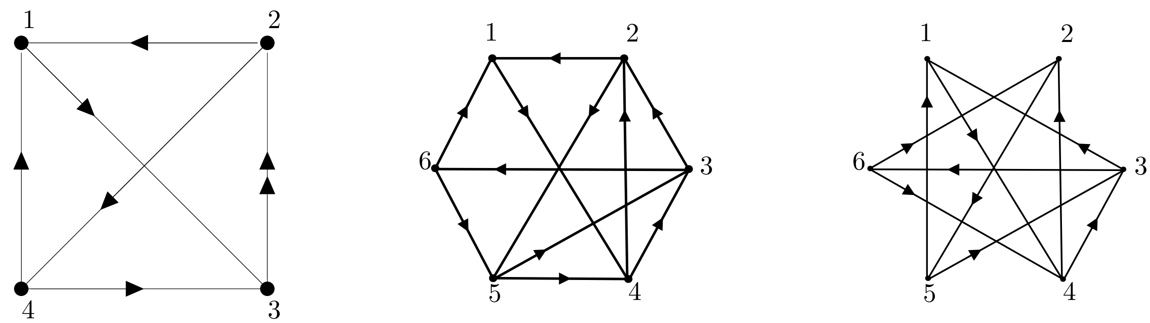
$$J_1 = \frac{x_1x_6 + x_2x_7 + x_3x_5}{x_2x_6},$$

The remaining six functions are obtained by applying $\varphi^*J_i = J_{i+1}$, with $(\varphi^*)^7 J_i = J_i$. The Poisson bracket relations between these functions follow by the cyclic property from

$$\{J_1, J_2\} = 2J_1J_2 - J_5, \quad \{J_1, J_3\} = -J_1J_3 + 1, \quad \{J_1, J_4\} = -J_1J_4.$$

Again, this Poisson bracket is the sum of three homogeneous terms, $\mathbf{P} = \mathbf{P}^{(2)} + \mathbf{P}^{(1)} + \mathbf{P}^{(0)}$, where $\mathbf{P}^{(0)}$ and $\mathbf{P}^{(2)}$ satisfy the Jacobi identity, but $\mathbf{P}^{(1)}$ does not.

The Casimir of the Poisson subalgebra generated by the J_i is again $\mathcal{K}$ (the trace of the monodromy matrix), but in this case it is not clear how to split the Casimir into four pieces that Poisson commute, as required for Liouville integrability. Of course, there are still seven functionally independent invariant functions (built from cyclically symmetric combinations of the J_i) and we expect that four commuting functions exist.



(a) Example 5.9. (b) Example 5.10. (c) Example 5.16.

Figure 1: The first three quivers in this class.

Example 5.12 (The quiver $P_8^{(3)} - P_8^{(4)} + P_2^{(1)}$)**.** For this quiver, with $p = 5$ and $q = 3$, the recurrence is

$$x_n x_{n+8} = x_{n+3}x_{n+5} + x_{n+4}.$$

The corresponding birational map φ is Poisson with respect to a log-canonical bracket (2.6) defined by the inverse of the associated matrix B. Upon setting $C = 2B^{-1}$, the bracket is given by the first row of this Toeplitz matrix: $(c_{1,j}) = (0, 1, 1, 0, -1, 1, 2, 1)$.

Since $\gcd(p, q) = 1$, we have the same phenomenon as in the discussion of (4.34). From (5.13), we have

$$\mathbf{M}_n := \mathbf{L}_n\mathbf{L}_{n+3}\mathbf{L}_{n+6}\mathbf{L}_{n+9}\mathbf{L}_{n+12} = \mathbf{L}_n\mathbf{L}_{n+3}\mathbf{L}_{n+1}\mathbf{L}_{n+4}\mathbf{L}_{n+2},$$

since $\mathbf{L}_n$ has period 5, so the permutation $\sigma : (1, 2, 3, 4, 5) \mapsto (1, 4, 2, 5, 3)$ gives the trace as

$$\mathcal{K} = \mathcal{K}_{5,3}(J_1, J_2, J_3, J_4, J_5) = \mathcal{K}_{5,1}(J_1, J_4, J_2, J_5, J_3),$$

where $\mathcal{K}_{5,1}$ is given by the formula (5.21) in Example 5.10. Similarly, from (5.14) we find

$$\hat{\mathbf{M}}_n := \hat{\mathbf{L}}_{n+10}\hat{\mathbf{L}}_{n+5}\hat{\mathbf{L}}_n = \hat{\mathbf{L}}_{n+1}\hat{\mathbf{L}}_{n+2}\hat{\mathbf{L}}_n,$$

where $\hat{\mathbf{L}}_n$ has period 3, so with the same notation as in (4.35) we also have

$$\mathcal{K} = \mathcal{K}_{3,5}(K_1, K_2, K_3) = \mathcal{K}_{3,1}(K_1, K_3, K_2),$$

where $\mathcal{K}_{3,1}$ is given by (5.20) in Example 5.9. Since $\mathcal{K}_{3,1}$ is totally symmetric, the permutation $\hat{\sigma} : (1, 2, 3) \mapsto (1, 3, 2)$ makes no difference to the result.

The explicit forms of the period 5 functions J_i, appearing as entries in the matrices $\mathbf{L}_n$, all derive from

$$J_1 = \frac{x_1x_3x_8 + x_3x_5x_7 + x_4x_6x_8 + x_4x_7}{x_3x_4x_8}$$

by acting with the map φ. From the above bracket for the x_j, determined by the matrix C, the Poisson brackets between the J_i can be calculated as

$$\{J_1, J_2\} = 4J_1J_2, \quad \{J_1, J_3\} = 2J_1J_3 + 2J_2 - 2,$$

with all other brackets being deduced through the cyclic property. The resulting 5-dimensional Poisson subalgebra is isomorphic to that for the J_i in Example 5.10, as can be seen by applying the permutation σ and rescaling by an overall factor of 2 (which depends on the choice of scale for c_{ij}). Therefore, subject to this permutation, it follows that the same three functions $\mathcal{H}_i$ are in involution:

$$\mathcal{H}_1 = \sum_{i=1}^{5}(J_i - J_iJ_{i+3}), \quad \mathcal{H}_2 = \sum_{i=1}^{5}(J_iJ_{i+1}J_{i+3} - J_iJ_{i+1}J_{i+3}^2), \quad \mathcal{H}_3 = \prod_{i=1}^{5} J_i.$$

The required fourth function that commutes with these $\mathcal{H}_i$ is derived from the algebra of the K_i, which appear as entries in the matrices $\hat{\mathbf{L}}_n$. In terms of x_j, we have

$$K_1 = \frac{x_1}{x_6} + \frac{x_6}{x_1} + \frac{x_2}{x_3x_6} + \frac{x_5}{x_1x_4} + \frac{x_8}{x_4x_7} + \frac{x_2x_8}{x_3x_7},$$

with $K_2 = \varphi^*K_1$ and $K_3 = (\varphi^*)^2K_1$ providing the other two functions which cycle with period 3 under the action of the map. The K_i generate a 3-dimensional Poisson subalgebra, whose brackets are all determined by acting with φ on the single relation

$$\{K_1, K_2\} = -K_1K_2 + 2K_3.$$

This algebra is isomorphic to that of the J_i in Example 5.9 (as is seen immediately by applying the permutation $\hat{\sigma}$), so it contains the two commuting quantities

$$\hat{\mathcal{H}}_1 = K_1^2 + K_2^2 + K_3^2, \qquad \hat{\mathcal{H}}_2 = K_1K_2K_3.$$

The subalgebras of J_i and K_i share the joint Casimir $\mathcal{K} = \mathcal{H}_1 - \mathcal{H}_2 + \mathcal{H}_3 = 3 - \hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2$, which gives a single relation between these two sets of functions. Since $\{K_i, J_j\} = 0$, for all i, j, we can take any four of the first integrals $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2$ as a commuting set, which demonstrates that this 8-dimensional map φ is integrable in the Liouville sense.

5.4 The case $\gcd(p, q) = 2$

The discussion of the case where p and q are both even involves some new features. Upon setting $p = 2\hat{p}$, $q = 2\hat{q}$ with $\gcd(\hat{p}, \hat{q}) = 1$, the fact that $\gcd(m, q) = 1$ implies that $m = \hat{p} + \hat{q}$ is odd, hence either $\hat{q}$ is odd and $\hat{p}$ is even, or vice versa. For the matrix (5.2) in each case we find that $\det B = 0$ whenever $\hat{p}$ or $\hat{q}$ is divisible by 3, and $\det B = 9$ otherwise. These two possibilities lead to quite different behaviour, so eventually we shall have to distinguish between them. For the time being we concentrate on the associated linear relations, which do not depend on whether B is degenerate or not.

Since q is even, observe that the ordered product of matrices $\mathbf{L}_n$ in (5.13) now cycles only through indices with the same parity, so the product cycles twice through $\hat{p} = p/2$ terms. Thus we see that $\mathbf{M}_n$ is a perfect square, and it is convenient to take the square root

$$\mathbf{M}_n^{1/2} = \mathbf{L}_n\mathbf{L}_{n+q} \ldots \mathbf{L}_{n+(\hat{p}-1)q},$$

and similarly for $\hat{\mathbf{M}}_n^{1/2}$. We know that in this situation the quantity $\mathcal{K}_n$ in (5.15) has period 2, but for our purposes it is more useful to consider the quantity

$$\mathcal{K}_n^* := \operatorname{tr} \mathbf{M}_n^{1/2} = \operatorname{tr} \hat{\mathbf{M}}_n^{1/2}, \tag{5.22}$$

which cycles with period 2 for the same reasons.

In this setting, with $\gcd(p,q)=2$, the algebraic structure in terms of the functions J_n and K_n is based on that for the case $q=2$ (up to suitable permutations), similarly to the way that for $\gcd(p,q)=1$, and for the primitives, this structure is based on the case $q=1$.

Now when $q=2$ we have the quantities K_n with period 2, giving

$$\operatorname{tr}\hat{\mathbf{M}}_n^{1/2}=\operatorname{tr}\hat{\mathbf{L}}_n=K_{n+1},\qquad \operatorname{tr}\hat{\mathbf{M}}_n^{-1/2}=\operatorname{tr}\hat{\mathbf{L}}_n^{-1}=K_n,$$

and hence the identity

$$\operatorname{tr}\mathbf{M}_n^{-1/2}=\operatorname{tr}\mathbf{M}_{n+1}^{1/2}=\mathcal{K}_{n+1}^* \tag{5.23}$$

holds. For even $q>2$, with p fixed, the formula for $\operatorname{tr}\mathbf{M}_n^{1/2}$ as a function of $J_1,\ldots,J_p$ is identical to that for the case $q=2$, up to a permutation, which means that the formula (5.23) holds in general.

Theorem 5.13. *When $\gcd(p,q)=2$, the iterates of the nonlinear recurrence (5.1) satisfy the linear relation*

$$x_{n+3pq/2}-\mathcal{K}_n^*\,x_{n+pq}+\mathcal{K}_{n+1}^*\,x_{n+pq/2}-x_n=0, \tag{5.24}$$

where $\mathcal{K}_n^*$ is the period 2 quantity defined by (5.22).

Proof: This follows by essentially the same argument as in the proof of Theorem 5.6, applying Cayley-Hamilton to $\mathbf{M}_n^{1/2}$ and $\mathbf{M}_n^{-1/2}$, and making use of (5.23). □

Remark 5.14. With respect to the functions J_n and K_n, the main new feature in this case, compared with the case $\gcd(p,q)=1$, is that here we have two quantities $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$, so the identity (5.22) for $n=1,2$ provides two independent relations between these two sets of functions. Given that these are the only relations, the cyclically symmetric functions of the J_n, and those of the K_n, together provide $N-2$ independent first integrals.

We are now ready to present a further refinement of Proposition 5.5.

Theorem 5.15. *When $\gcd(p,q)=2$, the iterates of the nonlinear recurrence (5.1) satisfy the constant coefficient, linear relation*

$$x_{n+3pq}-\mathcal{B}\,x_{n+5pq/2}+\mathcal{C}\,x_{n+2pq}-\mathcal{D}\,x_{n+3pq/2}+\mathcal{C}\,x_{n+pq}-\mathcal{B}\,x_{n+pq/2}+x_n=0, \tag{5.25}$$

where the non-trivial coefficients are first integrals, given by

$$\mathcal{B}=\mathcal{K}_1^*+\mathcal{K}_2^*,\qquad \mathcal{C}=\mathcal{K}_1^*\mathcal{K}_2^*+\mathcal{K}_1^*+\mathcal{K}_2^*,\qquad \mathcal{D}=(\mathcal{K}_1^*)^2+(\mathcal{K}_2^*)^2+2.$$

Proof: For the sake of argument, suppose that $\hat{q}$ is odd and $\hat{p}$ is even. To get the seven-term relation (5.25), we consider the 6×6 matrix that is specified in terms of its (j,k) entry by $\Phi_n=(x_{n+p\hat{q}(j-1)+\hat{q}(k-1)})$. The linear recurrence (5.4) implies that

$$\Phi_{n+\hat{q}}=\Phi_n\,\mathbf{L}_n^*,\qquad \text{where}\qquad \mathbf{L}_n^*=\begin{pmatrix} 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&-J_n\\ 0&0&1&0&0&0\\ 0&0&0&1&0&J_{n+m}\\ 0&0&0&0&1&0 \end{pmatrix}.$$

The monodromy matrix corresponding to p iterations of the latter equation is

$$\mathbf{M}_n^*=\mathbf{L}_n^*\,\mathbf{L}_{n+\hat{q}}^*\ldots\mathbf{L}_{n+(p-1)\hat{q}}^*,$$

so that

$$\Phi_{n+p\hat{q}} = \Phi_n \, \mathbf{M}_n^*. \tag{5.26}$$

We need to show that the characteristic polynomial of the 6×6 matrix $\mathbf{M}_n^*$ is given by

$$\chi(\zeta) = \zeta^6 - \mathcal{B}\,\zeta^5 + \mathcal{C}\,\zeta^4 - \mathcal{D}\,\zeta^3 + \mathcal{C}\,\zeta^2 - \mathcal{B}\,\zeta + 1, \tag{5.27}$$

as once we have determined this the recurrence (5.25) follows immediately, by applying the Cayley-Hamilton theorem to (5.26).

It is convenient to conjugate all of the 6×6 matrices by the permutation matrix corresponding to $(1,2,3,4,5,6)\to(1,3,5,2,4,6)$, which reduces everything to calculations with 3×3 blocks. This gives

$$\mathbf{L}_n^* \sim \begin{pmatrix} \mathbf{0} & \mathbf{L}_n \\ \mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \mathbf{L}_n^*\,\mathbf{L}_{n+\hat{q}}^* \sim \begin{pmatrix} \mathbf{L}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{n+\hat{q}} \end{pmatrix},$$

so that that for the monodromy matrix we have

$$\mathbf{M}_n^* \sim \begin{pmatrix} \mathbf{L}_n\mathbf{L}_{n+q}\ldots\mathbf{L}_{n+(\hat{p}-1)q} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{n+\hat{q}}\mathbf{L}_{n+\hat{q}+q}\ldots\mathbf{L}_{n+\hat{q}+(\hat{p}-1)q} \end{pmatrix} \sim \begin{pmatrix} \mathbf{M}_n^{1/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{n+1}^{1/2} \end{pmatrix}.$$

Thus the characteristic polynomial of $\mathbf{M}_n^*$ factors as the product of the characteristic polynomials of $\mathbf{M}_n^{1/2}$ and $\mathbf{M}_{n+1}^{1/2}$,

$$\chi(\zeta) = (\zeta^3 - \mathcal{K}_1^*\,\zeta^2 + \mathcal{K}_2^*\,\zeta - 1)\,(\zeta^3 - \mathcal{K}_2^*\,\zeta^2 + \mathcal{K}_1^*\,\zeta - 1),$$

which multiplies out to give (5.27) with the correct coefficients $\mathcal{B},\mathcal{C},\mathcal{D}$.

An analogous argument holds when $\hat{q}$ is even and $\hat{p}$ is odd. □

To discuss the Liouville integrability of the systems that appear when $\gcd(p,q)=2$, it is necessary to give a separate treatment according to whether the matrix B is invertible or not.

5.4.1 Nondegenerate B matrix

When matrix B is nondegenerate, it appears that the Liouville integrability of the symplectic map φ should follow by very similar arguments to those for the case $\gcd(p,q)=1$ (or for the primitives with even N). The map preserves a Poisson bracket of the form (2.6), which is specified uniquely (up to scale) by the Toeplitz matrix $C=B^{-1}$. There are the two sets of quantities, J_n and K_n, with periods p and q respectively, and by Remark 5.14 these provide $N-2$ independent first integrals, but it is necessary to find $m=N/2$ integrals in involution.

Given that the J_n generate a Poisson subalgebra of dimension p, and that both of the quantities $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ defined by (5.22) are Casimirs, with the symplectic leaves being of dimension $p-2=2(\hat{p}-1)$, a further $\hat{p}-1$ independent commuting functions of the J_n are required in order to define an integrable system on this subalgebra. Similarly, given that $\{J_i,K_j\}=0$ for all i,j, and the K_n produce a q-dimensional subalgebra, with the same functions $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ as Casimirs, it is necessary to have an additional $\hat{q}-1$ independent functions that define an integrable system in terms of K_n alone. The quantities $\mathcal{B}$ and $\mathcal{C}$ which appear as the coefficients in the linear relation (5.25) are first integrals, as well as being joint Casimirs for the two subalgebras, so combining these with the two sets of first integrals gives a total of $2+(\hat{p}-1)+(\hat{q}-1)=m$ independent functions in involution, as required.

Since we do not have a general proof of all the foregoing assertions, here we will merely illustrate the discussion with the simplest example of this kind, which arises for $p=4$, $q=2$.

Example 5.16 (The quiver $P_6^{(2)} - P_6^{(3)} + P_2^{(1)}$)**.** Mutation of the third quiver in Figure 1 gives the map

$$\varphi: \qquad (x_1,x_2,x_3,x_4,x_5,x_6)\mapsto(x_2,x_3,x_4,x_5,x_6,x_7), \qquad x_7=\frac{x_3x_5+x_4}{x_1}. \tag{5.28}$$

The inverse of the non-singular matrix B defines a Poisson bracket (2.6), invariant with respect to φ. Taking $C = 3B^{-1}$, the top row of this Toeplitz matrix is $(c_{1,j}) = (0,1,1,0,2,2)$.

The period 4 functions J_i, generated from J_1 by the action of the map, are

$$J_1 = \frac{x_1x_2x_6 + x_2x_4x_5 + x_3x_4x_6 + x_3x_5}{x_2x_3x_6}, \quad J_2 = \frac{x_1x_6 + x_2x_3 + x_4x_5}{x_3x_4},$$
$$J_3 = \frac{x_1x_3x_4 + x_2x_3x_5 + x_1x_5x_6}{x_1x_4x_5}, \quad J_4 = \frac{x_1x_3x_5 + x_2x_4x_6 + x_1x_2x_4x_5 + x_1x_3x_4x_6 + x_2x_3x_5x_6}{x_1x_2x_5x_6}.$$

The Poisson brackets between these functions all follow by the cyclic property from

$$\{J_1, J_2\} = 3J_1J_2 - 3, \quad \{J_1, J_3\} = 3(J_2 - J_4). \tag{5.29}$$

From the first equality in (5.22) we have the two functions

$$\mathcal{K}_1^* = J_2J_4 - J_1 - J_3, \qquad \mathcal{K}_2^* = J_1J_3 - J_2 - J_4,$$

which are both Casimir functions of the 4-dimensional algebra generated by the J_i, but cycle with period 2 under the map. Symmetric functions of $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are first integrals as well as being Casimirs, so we may take the quantities

$$\mathcal{B} = \mathcal{K}_1^* + \mathcal{K}_2^*, \qquad \mathcal{C} = \mathcal{K}_1^*\mathcal{K}_2^* + \mathcal{B},$$

as in Theorem 5.15. The homogeneous components of $\mathcal{B}$ are automatically first integrals, and they Poisson commute:

$$\mathcal{B} = -\mathcal{H}_1 + \mathcal{H}_2$$

where

$$\mathcal{H}_1 = J_1 + J_2 + J_3 + J_4, \quad \mathcal{H}_2 = J_1J_3 + J_2J_4, \quad \text{with} \quad \{\mathcal{H}_1, \mathcal{H}_2\} = 0.$$

Hence, we may take $\mathcal{H}_1, \mathcal{B}, \mathcal{C}$ as three independent first integrals in involution, which proves Liouville integrability of the 6-dimensional map (5.28). Clearly there are other choices, for instance $\mathcal{H}_1, \mathcal{H}_2, \mathcal{C}$, which would do just as well.

Remark 5.17. Note that in the preceding example, since $q = 2$ we have $K_1 = \mathcal{K}_2^*$ and $K_2 = \mathcal{K}_1^*$, which Poisson commute with each other, so the subalgebra of the K_i is trivial.

5.4.2 Degenerate B matrix

When B is degenerate, which only happens when either p or q is a multiple of 6, the maps that arise have features that make them much more like the odd primitives $P_{2m+1}^{(q)}$ than the other cases with even N.

For these particular cases, the matrix B has a two-dimensional kernel, which is spanned by two integer vectors of the form

$$\mathbf{u}_1 = (1,1,0,-1,-1,0,\ldots)^T, \qquad \mathbf{u}_2 = (0,1,1,0,-1,-1,\ldots)^T,$$

where in each vector the components continue to repeat the same blocks of six numbers, until the final block which is truncated (of length 2 or 4, since $N = 2m$ is even and not a multiple of 6). Hence there is a two-parameter scaling group, which acts by

$$(x_1, x_2, x_3, x_4, x_5, x_6, \ldots) \to (\lambda\, x_1, \lambda\,\mu\, x_2, \mu\, x_3, \lambda^{-1}\, x_4, \lambda^{-1}\mu^{-1}\, x_5, \mu^{-1}\, x_6, \ldots), \qquad (\lambda, \mu) \in (\mathbb{C}^*)^2. \tag{5.30}$$

This action extends to all the iterates x_n of (5.1); the pattern repeats itself on each successive block of six adjacent iterates.

Now $\operatorname{im} B$ is spanned by

$$\mathbf{v}_j = \mathbf{e}_j - \mathbf{e}_{j+1} + \mathbf{e}_{j+2}, \qquad j = 1, \ldots, 2m-2, \tag{5.31}$$

where $\mathbf{e}_j$ is the jth standard basis vector. Hence, by Lemma 2.7, the coordinates

$$y_j = \frac{x_j\, x_{j+2}}{x_{j+1}}, \qquad j = 1, \ldots, 2m-2, \tag{5.32}$$

are invariant under the scaling (5.30), and the degenerate form (2.10) pushes forward to a symplectic form (2.22) in dimension $2m-2$. The coefficients of the latter are obtained from a skew-symmetric matrix $\hat{B} = (\hat{b}_{jk})$, whose inverse provides a nondegenerate Poisson bracket for the y_j, i.e.

$$\{\, y_j, y_k \,\} = \epsilon_{jk}\, y_j y_k, \qquad 1 \leq j,k \leq 2m-2, \tag{5.33}$$

where $\hat{B}^{-1} = (\epsilon_{jk})$ must be a Toeplitz matrix, because the coordinates (5.32) transform as $\varphi^* y_j = y_{j+1}$ under the map corresponding to (5.1). Upon applying the rest of Theorem 2.6, we see that this induces a symplectic map $\hat{\varphi}$ on the variables y_j.

To prove the Liouville integrability of all of the maps $\hat{\varphi}$ that arise in this way, we require a general expression for the coefficients ϵ_{jk} in (5.33), which is presently lacking. However, it is possible to give a plausible argument for the counting of first integrals, which agrees with all examples we have checked so far.

Suppose, for the sake of argument, that $6|q$, hence $6 \not| p$. The scaling action of $(\mathbb{C}^*)^2$ on the x_j, as in (5.30), extends to an action on the coefficients J_n that appear in the linear equation (5.4). However, the indices of the terms x_{n+jq} for $j = 0,1,2,3$ differ by multiples of 6, which means that they all scale the same way, and hence the period p quantities J_n are invariant under this scaling, and can be expressed in terms of the y_j given by (5.32). Given that the J_n generate a p-dimensional Poisson subalgebra with respect to the bracket (5.33), with two Casimirs given by the quantities $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ as in (5.22), a further $\hat{p}-1$ independent commuting functions of the J_n are needed to have an integrable system defined on this subalgebra. Similarly, the scaling action (5.30) extends to an action on the coefficients K_n in (5.5), but now the terms x_{n+jp} for $j = 0,1,2,3$ do *not* all differ by multiples of 6, so they scale differently. This means that there is a non-trivial scaling action of the two-parameter group $(\mathbb{C}^*)^2$ on the quantities K_n, $n = 1, \ldots, q$, for which there should be $q-2$ invariant monomials. The invariant monomial functions of K_n, which we denote by w_i for $i = 1, \ldots, q-2$, can also be written in terms of the original variables x_j. The fact that they are invariant under (5.30) means that these w_i can be written as functions of the symplectic coordinates y_j as well. Given that the w_i generate a $(q-2)$-dimensional Poisson subalgebra in the $(2m-2)$-dimensional space with the bracket (5.33), and that the quantities $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ are Casimirs for this subalgebra too, an integrable system is defined on the $(q-4)$-dimensional symplectic leaves by an additional set of $\hat{q}-2$ commuting functions of the w_i. Supposing further that $\{J_i, w_j\} = 0$ for all i,j, we take the quantities $\mathcal{B}$ and $\mathcal{C}$ from (5.25), which are both first integrals and joint Casimirs for the two subalgebras, and combining these with the two sets of first integrals gives a total of $2 + (\hat{p}-1) + (\hat{q}-2) = m-1$ independent functions in involution, as required for the Liouville integrability of the symplectic map $\hat{\varphi}$ in dimension $2m-2$.

Since we do not have a general proof of all the preceding assertions, here we will only present the simplest example of this kind, which arises for $p = 6$, $q = 4$.

Example 5.18 (The quiver $P_{10}^{(4)} - P_{10}^{(5)} + P_2^{(1)}$)**.** The recurrence arising from mutation of this quiver is

$$x_{n+10}\, x_n = x_{n+6}\, x_{n+4} + x_{n+5}. \tag{5.34}$$

The matrix B has a two-dimensional kernel, spanned by the integer vectors

$$\mathbf{u}_1 = (1,1,0,-1,-1,0,1,1,0,-1)^T, \qquad \mathbf{u}_2 = (0,1,1,0,-1,-1,0,1,1,0)^T,$$

which generate the action of a two-parameter scaling group on the iterates of (5.34), as given in (5.30). Taking the scaling-invariant variables y_j, given by (5.32) for $j = 1, \ldots, 8$, we apply Lemma 2.7 to find the symplectic form $\hat{\omega}$ expressed in these coordinates, which leads to the nondegenerate Poisson bracket specified by

$$\{y_1, y_5\} = y_1 y_5, \qquad \{y_1, y_6\} = -y_1 y_6, \qquad \{y_1, y_7\} = y_1 y_7, \tag{5.35}$$

where all other brackets are either zero or follow from the Toeplitz property/skew-symmetry. From Theorem 2.6, this Poisson bracket is preserved by the induced map

$$\hat{\varphi}: \qquad (y_1, \ldots, y_8) \mapsto (y_2, \ldots, y_9), \qquad y_9 = \frac{y_4 y_5 y_6 (y_5 + 1)}{y_1 y_2 y_8}. \tag{5.36}$$

Note that (in contrast to the foregoing discussion) in this example q is *not* a multiple of 6, but rather p is. The period 6 functions J_n are given by the formula

$$J_n = \frac{x_n}{x_{n+4}} + \frac{x_{n+7}}{x_{n+3}} + \frac{x_{n-1} x_{n+8}}{x_{n+3} x_{n+4}}, \quad n = 1, \ldots, 6,$$

where each of the above quantities can be written as a function of $x_1, \ldots, x_{10}$ by iterating the recurrence (5.34) either forwards or backwards. Since the gaps between the indices of the terms x_j in the linear relation (5.4) are not all multiples of 6, these J_n are not invariant under the scaling action (5.30), but instead they transform as follows:

$$J_1 \to \lambda^2 \mu J_1, \quad J_2 \to \lambda \mu^2 J_2, \quad J_3 \to \lambda^{-1} \mu J_3, \quad J_4 \to \lambda^{-2} \mu^{-1} J_4, \quad J_5 \to \lambda^{-1} \mu^{-2} J_5, \quad J_6 \to \lambda \mu^{-1} J_6.$$

There are four independent scaling-invariant monomial functions of the J_n, but it is convenient to consider the five functions given by $w_1 = J_1 J_4$, $w_2 = J_2 J_5$, $w_3 = J_3 J_6$, $w_4 = J_1 J_3 J_5$, $w_5 = J_2 J_4 J_6$, which satisfy the single relation

$$w_1 w_2 w_3 = w_4 w_5. \tag{5.37}$$

Since these w_i can also be expressed in terms of $x_1, \ldots, x_{10}$, the scaling-invariance implies that they can be written as functions of the symplectic coordinates y_j as well. For instance, we have the formula

$$w_1 = \frac{(y_1 y_2 y_8 + y_4 y_5 y_6 + y_5 y_6 y_8 + y_5 y_6)(y_1 y_2 + y_1 y_6 + y_5 y_6 + y_1 + y_6)}{y_1 y_2 y_5 y_6 y_8},$$

and analogous formulae can be obtained for $w_2 = \hat{\varphi}^* w_1$, $w_3 = (\hat{\varphi}^*)^2 w_1$ using the map (5.36); there are different expressions for w_4 and $w_5 = \hat{\varphi}^* w_4$, but these are more unwieldy so they are omitted. Calculating the Poisson brackets between these functions, we have a subalgebra with four generators w_1, w_2, w_3, w_4, but this is more conveniently expressed with the extra function w_5 included. The brackets can be determined from the relations

$$\{w_1, w_2\} = w_1 w_2 - w_4 - w_5, \qquad \{w_1, w_4\} = w_1 (w_2 - w_3), \tag{5.38}$$

with all other brackets following by applying the map and noting that w_1 cycles with period 3, and w_4 has period 2, so $(\hat{\varphi}^*)^3 w_1 = w_1$ and so $(\hat{\varphi}^*)^2 w_4 = w_4$. The 5-dimensional algebra defined by (5.38) has three Casimirs, given by

$$\mathcal{K}_1^* = 3 - w_1 - w_2 - w_3 + w_5, \qquad \mathcal{K}_2^* = 3 - w_1 - w_2 - w_3 + w_4, \qquad \hat{\mathcal{C}} = w_1 w_2 w_3 - w_4 w_5,$$

where (as functions of J_n) the quantities $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ come from (5.22), by taking the trace of the monodromy matrix $\mathbf{M}_n^{1/2}$ for $n = 1, 2$, while fixing $\hat{\mathcal{C}} = 0$ corresponds to the constraint (5.37). The Casimir function $\mathcal{B} = \mathcal{K}_1^* + \mathcal{K}_2^* = 3 - 2\hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2$, is also a first integral, with the components

$$\hat{\mathcal{H}}_1 = J_1 J_4 + J_2 J_5 + J_3 J_6 = w_1 + w_2 + w_3, \qquad \hat{\mathcal{H}}_2 = J_1 J_3 J_5 + J_3 J_4 J_6 = w_4 + w_5,$$

which are homogeneous functions of the J_n, are also first integrals, and Poisson commute: $\{\hat{\mathcal{H}}_1, \hat{\mathcal{H}}_2\} = 0$. Either of the latter two functions defines an integrable system on the two-dimensional symplectic leaves of the algebra generated by the w_i. The function $\mathcal{C} = \mathcal{K}_1^* \mathcal{K}_2^* + \mathcal{B}$ is another first integral that is also a Casimir.

The functions K_n, which cycle with period 4, are specified by the formula

$$K_n = \frac{x_n}{x_{n+6}} + \frac{x_{n+9}}{x_{n+3}} + \frac{x_{n+1}}{x_{n+2} x_{n+6}} + \frac{x_{n+8}}{x_{n+3} x_{n+7}} + \frac{x_{n+1} x_{n+8}}{x_{n+2} x_{n+7}}, \qquad n = 1, \ldots, 4.$$

These quantities are invariant under the scaling (5.30), so they can also be written in terms of the iterates of (5.36):

$$K_n = \frac{y_{n+1} + y_{n+6} + y_n y_{n+1} + y_{n+1} y_{n+6} + y_{n+6} y_{n+7}}{y_{n+3} y_{n+4}}, \qquad n = 1, \dots, 4,$$

where $\hat{\varphi}^* y_n = y_{n+1}$ defines the sequence of y_n for all $n \in \mathbb{Z}$. Upon using (5.35) we find the relations

$$\{K_1, K_2\} = -K_1 K_2 + 1, \qquad \{K_1, K_3\} = -K_2 + K_4,$$

which provide all the Poisson brackets between the K_n by applying the cyclic property. Comparing with (5.29) and scaling by a factor of -3, this four-dimensional Poisson subalgebra is seen to be isomorphic to the algebra of the J_i in Example 5.16. Hence the Casimirs of this subalgebra are

$$\mathcal{K}_1^* = K_2 K_4 - K_1 - K_3, \qquad \mathcal{K}_2^* = K_1 K_3 - K_2 - K_4,$$

and, with $\mathcal{B} = -\mathcal{H}_1 + \mathcal{H}_2$, two first integrals in involution are

$$\mathcal{H}_1 = K_1 + K_2 + K_3 + K_4, \qquad \mathcal{H}_2 = K_1 K_3 + K_2 K_4.$$

From (5.22), these Casimirs are shared with the subalgebra generated by the J_n, and the two different formulae for $\mathcal{B}$ imply that the first integrals are related according to

$$3 - 2\hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2 + \mathcal{H}_1 - \mathcal{H}_2 = 0.$$

Since $\{w_i, K_j\} = 0$ for all i, j, the functions $\mathcal{B}, \mathcal{C}, \hat{\mathcal{H}}_1, \mathcal{H}_1$ provide four first integrals that Poisson commute, as required for Liouville integrability of the 8-dimensional symplectic map (5.36).

6 Integrable maps from Somos sequences

The quadratic recurrences (3.15) (case (iv) of Theorem 3.12), are referred to as three-term Gale-Robinson recurrences [16, 34]. We mow consider the slightly more general case where these recurrences have coefficients:

$$x_{n+N}\, x_n = \alpha\, x_{n+N-p}\, x_{n+p} + \beta\, x_{n+N-q}\, x_{n+q}. \tag{6.1}$$

These can be included by adding extra nodes to the quiver for the coefficient-free recurrence (see Section 10 in [13]). The coefficients are frozen variables, attached to these new nodes, which do not change under mutations. For three-term Gale-Robinson recurrences, one can add two extra nodes, corresponding to the parameters α, β, so a quiver with $N+2$ nodes is obtained (Proposition 10.4. in [13]). It is then straightforward to check that the presymplectic form ω in (2.10), defined by the same $N \times N$ skew-symmetric matrix (b_{jk}) as for the coefficient-free case, is preserved by (6.1). Hence Theorem 2.6 can be applied directly to the latter, to obtain a reduced symplectic map for suitable variables y_j.

Below we outline two different approaches to showing that the map for the variables y_j is integrable. The first way is to use the fact that all of the recurrences (6.1) are ordinary difference equations that arise as reductions of the Hirota-Miwa equation, which is an integrable partial difference equation with three independent variables. The Lax pair of the Hirota-Miwa equation allows one to obtain Lax pairs for its reductions, and the associated spectral curves provide first integrals in terms of the y_j. The second way is to find Somos-type recurrences of higher order that are satisfied by the iterates of (6.1). The coefficients of these Somos-k relations, for certain $k > N$, can also provide first integrals (analogous to the first integrals which appear as coefficients in *linear* recurrence relations for the iterates of the families (ii) and (iii)).

Recently, Goncharov and Kenyon have found Somos recurrences arising as discrete symmetries of classical integrable systems associated with dimer models on a torus [19]. A further connection with relativistic analogues of the Toda lattice appeared in [7].

6.1 Reductions of the Hirota-Miwa equation

The relations (6.1) all arise by reduction of the Hirota-Miwa (discrete KP) equation, which is the bilinear partial difference equation

$$T_1\, T_{-1} = T_2\, T_{-2} + T_3\, T_{-3}. \tag{6.2}$$

In the above, $T = T(n_1, n_2, n_3)$ is a function of three independent variables, and to denote shifts we have used $T_{\pm j} = T|_{n_j \to n_j \pm 1}$. If we set

$$T(n_1, n_2, n_3) = \exp\left(\sum_{i,j} S_{ij} n_i n_j\right)\, \tau(n), \tag{6.3}$$

where $S = (S_{ij})$ is a symmetric matrix and $n = n_0 + \delta_1 n_1 + \delta_2 n_2 + \delta_3 n_3$, then $\tau(n)$ satisfies the ordinary difference equation

$$\tau(n+\delta_1)\tau(n-\delta_1) = \alpha\, \tau(n+\delta_2)\tau(n-\delta_2) + \beta\, \tau(n+\delta_3)\tau(n-\delta_3)$$

where $\alpha = \exp(2(S_{22} - S_{11}))$, $\beta = \exp(2(S_{33} - S_{11}))$. Upon taking $x_n = \tau(n - \delta_1)$ with $\delta_1 = \frac{1}{2}N$, $\delta_2 = \frac{1}{2}(N - 2p)$, $\delta_3 = \frac{1}{2}(N - 2q)$, this becomes (6.1).

In the combinatorics literature, equation (6.2) is referred to as the octahedron recurrence, which has the Laurent property (shown in [11]). The Laurent property for three-term Somos (or Gale-Robinson) recurrences of the form (6.1) then follows by the reduction (6.3).

The Hirota-Miwa equation (6.2) has a scalar Lax pair (see equation (3.8) in [24], for instance): it is the compatibility condition for the linear system given by

$$\begin{array}{rcrcr} T_{-1,3}\, \psi_{1,2} & + & T\, \psi_{2,3} & = & T_{2,3}\, \psi, \\ T\, \psi_{-1,2} & + & T_{-1,3}\, \psi_{2,-3} & = & T_{-1,2}\, \psi, \end{array} \tag{6.4}$$

in terms of the scalar function $\psi = \psi(n_1, n_2, n_3)$, with the same notation for shifts as before. Using the latter, one can use the reduction (6.3) to obtain Lax pairs for all of the Somos recurrences (6.1), which leads directly to spectral curves whose coefficients are conserved quantities. Here we briefly illustrate how this works for the cases $N = 4$ and $N = 5$ only; reducing the Lax pair becomes more involved as N increases.

The general Somos-4 recurrence with coefficients is

$$x_{n+4}\, x_n = \alpha\, x_{n+3}\, x_{n+1} + \beta x_{n+2}^2. \tag{6.5}$$

By taking the same monomials as in (2.24), this reduces to the map

$$(y_1\, ,\, y_2) \mapsto \Big(y_2\, ,\, (\alpha y_2 + \beta)/(y_1 y_2^2)\Big), \tag{6.6}$$

which becomes (2.25) in the case $\alpha = \beta = 1$, and preserves the same symplectic from $\hat{\omega}$. This means that (6.6) has the invariant Poisson bracket

$$\{y_1\, ,\, y_2\} = y_1\, y_2. \tag{6.7}$$

The recurrence (6.5) arises from the reduction (6.3) with $\delta_1 = 2$, $\delta_2 = 1$, $\delta_3 = 0$. Upon taking $S_{jk} = 0$ for $j \neq k$, without loss of generality, and setting

$$\psi(n_1, n_2, n_3) = \exp\Big(\sum_{j=1}^{3} n_j \log \lambda_j + S_{jj} n_j^2\Big)\, \tau(n)\, \phi(n), \qquad \text{with} \qquad \lambda_1 = \frac{e^{-S_{11}}}{\sqrt{\zeta\xi}}, \quad \lambda_2 = \lambda_1 \zeta, \quad \lambda_3 = e^{2S_{22}}\, \lambda_2,$$

the scalar linear equations (6.4) reduce to a 2×2 linear system for the vector $\mathbf{w} = (\phi(n), \phi(n+1))^T$. Up to shifts of indices, the coefficients of the latter can all be written in terms of the y_j given in (2.24), as well as the spectral parameters ζ and ξ, as follows.

Example 6.1 (Somos-4 Lax pair)**.** The Lax pair for the map (6.6) takes the form

$$\mathbf{L}\,\mathbf{w} = \xi\mathbf{w}, \qquad \tilde{\mathbf{w}} = \mathbf{M}\,\mathbf{w}, \tag{6.8}$$

where the tilde denotes the shift $n \to n+1$. The matrices $\mathbf{L} = \mathbf{L}(\zeta)$, $\mathbf{M} = \mathbf{M}(\zeta)$ are functions of y_j and the spectral parameter ζ, given by

$$\mathbf{L} = \begin{pmatrix} -\frac{(\alpha y_1+\beta)}{y_1y_2}\,\zeta & -\alpha y_1\zeta + \frac{(\alpha y_1+\beta)}{y_1y_2} \\ \frac{\alpha}{y_1}\,\zeta^2 - \zeta & \left(-y_1y_2 - \frac{\alpha}{y_1}\right)\zeta + 1 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{y_1y_2}\,\zeta & \frac{1}{y_1y_2} \end{pmatrix}.$$

The discrete Lax equation $\tilde{\mathbf{L}}\mathbf{M} = \mathbf{M}\mathbf{L}$ holds if and only if the map (6.6) does. The spectral curve is

$$\det\left(\mathbf{L}(\zeta) - \xi\,\mathbf{1}\right) = \xi^2 + (H_1\,\zeta - 1)\xi + \alpha^2\zeta^3 + \beta\zeta^2 = 0,$$

in which the coefficient of $\zeta\xi$ is the first integral

$$H_1 = y_1y_2 + \frac{\alpha}{y_1} + \frac{\alpha}{y_2} + \frac{\beta}{y_1y_2}. \tag{6.9}$$

The level sets of H_1 are biquadratic curves of genus one in the (y_1, y_2) plane, and the map is a particular instance of the QRT family [33].

Applying the reduction (6.3) to the Hirota-Miwa equation (6.2) with $\delta_1 = 5/2$, $\delta_2 = 3/2$, $\delta_3 = 1/2$ leads to the general form of the Somos-5 recurrence with coefficients, which we denote by $\tilde{\alpha}$, $\tilde{\beta}$:

$$x_{n+5}\,x_n = \tilde{\alpha}, x_{n+4}\,x_{n+1} + \tilde{\beta}\,x_{n+3}\,x_{n+2}. \tag{6.10}$$

From the appropriate B matrix, one obtains $y_1 = x_1x_4/(x_2x_3)$, $y_2 = x_2x_5/(x_3x_4)$ as coordinates in the plane (see [15]), satisfying the Poisson bracket (6.7), and the corresponding map $\hat{\varphi}$ is also of QRT type, namely

$$\hat{\varphi}: \qquad (y_1\,,\, y_2) \mapsto \Big(y_2\,,\, (\tilde{\alpha}y_2 + \tilde{\beta})/(y_1y_2)\Big). \tag{6.11}$$

Similar calculations to those in the Somos-4 case yield the appropriate reduction of (6.4).

Example 6.2 (Somos-5 Lax pair)**.** The map (6.11) arises as the compatibility condition $\tilde{\mathbf{L}}\mathbf{M} = \mathbf{M}\mathbf{L}$ for a linear system of the form (6.8), where

$$\mathbf{L} = \mathbf{C}_0 + \mathbf{C}_1\,\zeta + \mathbf{C}_2\,\zeta^2, \qquad \mathbf{M} = \mathbf{C}_0 + \begin{pmatrix} 0 & 0 \\ -y_1 & 0 \end{pmatrix}\zeta, \tag{6.12}$$

$$\text{with} \qquad \mathbf{C}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{C}_1 = \begin{pmatrix} -y_1 & -\left(y_2 + \frac{\tilde{\alpha}}{y_1}\right) \\ -y_1 & -\left(y_2 + \frac{\tilde{\alpha}}{y_1} + \frac{\tilde{\alpha}}{y_2} + \frac{\tilde{\beta}}{y_1y_2}\right) \end{pmatrix}, \quad \mathbf{C}_2 = \begin{pmatrix} \tilde{\alpha} & 0 \\ \tilde{\alpha} + \frac{(\tilde{\alpha}y_1+\tilde{\beta})}{y_2} & \tilde{\alpha} \end{pmatrix}.$$

The coefficient of $\zeta\xi$ in the equation for the spectral curve, that is

$$\det\left(\mathbf{L}(\zeta) - \xi\,\mathbf{1}\right) = \xi^2 - (2\tilde{\alpha}\zeta^2 - \tilde{J}\,\zeta + 1)\xi + \tilde{\alpha}^2\zeta^4 + \tilde{\beta}\zeta^3 = 0,$$

gives a first integral whose level sets are cubic (also biquadratic) curves of genus one, that is

$$\tilde{J} = y_1 + y_2 + \tilde{\alpha}\left(\frac{1}{y_1} + \frac{1}{y_2}\right) + \frac{\tilde{\beta}}{y_1y_2}. \tag{6.13}$$

Remark 6.3. The first integral (6.9) for Somos-4 can be rewritten in terms of the cluster variables, so that it becomes a ratio of homogeneous polynomials of total degree 4 in x_1, x_2, x_3, x_4, with the denominator just being $x_1x_2x_3x_4$. Similarly, in the case of Somos-5 the first integral (6.13) can be rewritten as a ratio of homogeneous polynomials of degree 5, with the denominator $x_1x_2x_3x_4x_5$. It turns out that (6.10) has another rational first integral, also of degree 5 in terms of the cluster variables, which can be written as

$$\tilde{I} = f_1f_2f_3 + \tilde{\alpha}\left(\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}\right) + \frac{\tilde{\beta}}{f_1f_2f_3} \qquad \text{with} \quad f_j = \frac{x_jx_{j+2}}{x_{j+1}^2} \qquad \text{for} \quad j = 1, 2, 3$$

(see Proposition 2.3 in [20]). However, the quantity $\tilde{I}$ is not defined on the (y_1, y_2) plane, where there is only one first integral (as required for the Liouville-Arnold theorem).

6.2 Bilinear relations of higher order

In [32], Swart and van der Poorten proved that sequences generated by Somos-4 recurrences also satisfy quadratic (Somos-type) relations of order k, for all $k \geq 4$. They also noted that for Somos-5 sequences, there are Somos-k relations of all odd orders $k = 5, 7, 9, \ldots$. Moreover, the coefficients of the higher order relations are constant along orbits, which means that, as long as they are not trivially constant, they provide first integrals.

In this subsection we explain how to obtain first integrals for Somos-7 recurrences using associated quadratic (bilinear) relations of higher order. The analogous results for Somos-6 recurrences are in [22].

To present the results concisely, it is convenient to consider the most general form of a Somos-7 recurrence, which is the four-term Gale-Robinson relation

$$x_{n+7}\, x_n = \alpha\, x_{n+6}\, x_{n+1} + \beta\, x_{n+5}\, x_{n+2} + \gamma\, x_{n+4}\, x_{n+3}. \tag{6.14}$$

With all three terms on the right hand side, this does not arise from a cluster algebra. Nevertheless, the general Somos-7 recurrence can be obtained as a reduction of the cube recurrence (Miwa's equation), and in [11] this was used to prove the Laurent property for all four-term Gale-Robinson recurrences, including (6.14). In each of the cases where one of the parameters α, β, γ vanishes, (6.14) reduces to a bilinear relation with two terms on the right hand side, corresponding to a different cluster algebra in each case.

Our main result on the family of recurrences (6.14) is stated as follows and the remainder of this subsection is devoted to its proof.

Theorem 6.4. *The Somos-7 recurrence (6.14) has three independent first integrals, denoted $\mathcal{H}_1, \mathcal{H}_2, \hat{I}$, which are rational functions (in fact, Laurent polynomials) of degree 7 in $x_1, \ldots, x_7$. $\hat{I}$ can be written as*

$$\begin{aligned} \hat{I} \;=\; & \beta\, f_1f_2f_3f_4f_5 + \gamma(f_1f_2f_3 + f_2f_3f_4 + f_3f_4f_5) + \alpha\beta\left(\tfrac{1}{f_1} + \tfrac{1}{f_2} + \tfrac{1}{f_3} + \tfrac{1}{f_4} + \tfrac{1}{f_5}\right) \\ & + \alpha\gamma\left(\tfrac{1}{(f_1f_2f_3} + \tfrac{1}{f_2f_3f_4} + \tfrac{1}{f_3f_4f_5}\right) + \tfrac{\beta^2}{f_1f_2f_3f_4f_5} + \beta\gamma\left(\tfrac{1}{f_1f_2f_3^2f_4^2f_5} + \tfrac{1}{f_1f_2^2f_3^2f_4f_5}\right) + \tfrac{\gamma^2}{f_1f_2^2f_3^3f_4^2f_5}, \end{aligned} \tag{6.15}$$

in terms of the variables $f_j = x_jx_{j+2}/x_{j+1}^2$, $j = 1, 2, 3, 4, 5$, while $\mathcal{H}_1$ and $\mathcal{H}_2$ are given in terms of $y_j = x_jx_{j+3}/(x_{j+1}x_{j+2})$, as in Proposition 6.5 below. For $\alpha = 0$ or $\gamma = 0$, Somos-7 reduces to a Liouville integrable map $\hat{\varphi}$ in four dimensions, while for $\beta = 0$ it reduces to an integrable map of the plane.

The equation (6.14) admits the action of the two-parameter scaling group $(\mathbb{C}^*)^2$, via

$$x_n \to \lambda\, \mu^n\, x_n \tag{6.16}$$

for non-zero λ, μ. The variables f_j are invariants under this scaling symmetry. In terms of these variables, the Somos-7 recurrence (6.14) is transformed to a recurrence of fifth order, namely

$$f_{n+5}f_{n+4}^2f_{n+3}^3f_{n+2}^3f_{n+1}^2f_n = \alpha\, f_{n+4}f_{n+3}^2f_{n+2}^2f_{n+1} + \beta\, f_{n+3}f_{n+2} + \gamma. \tag{6.17}$$

Being of odd order, (6.14) has a further scaling symmetry depending on the parity of n:

$$x_n \to \nu^{(-1)^n} x_n, \qquad \nu \in \mathbb{C}^*. \tag{6.18}$$

The variables f_n have the symmetry $f_n \to \nu^{\pm 2} f_n$ for even/odd n respectively. The variables $y_j = f_j f_{j+1}$ are the invariants under the combined scaling group and lead to

$$y_{n+4}y_{n+3}y_{n+2}^2 y_{n+1}y_n = \alpha\, y_{n+3}y_{n+2}y_{n+1} + \beta\, y_{n+2} + \gamma. \tag{6.19}$$

Of the cluster algebra subcases, we shall see that the case $\alpha = 0$ and the case $\gamma = 0$, the 4D map defined by (6.19) is symplectic, while for $\beta = 0$, additional scaling symmetries allow reduction to the plane.

The quantities $\mathcal{H}_1, \mathcal{H}_2, \hat{I}$ in Theorem 6.4 can all be written in terms of f_j, so the recurrence (6.17) has three independent first integrals. The rational function $\hat{I}$ is not invariant under (6.18), which means that it does not provide a first integral for (6.19), but both $\mathcal{H}_1$ and $\mathcal{H}_2$ do.

The quantities $\mathcal{H}_1$ and $\mathcal{H}_2$ appear in coefficients of bilinear relations of higher order. Since the Somos-7 recurrence is invariant under the three-parameter family of scalings defined by (6.16) and (6.18), one expects there to be relations of odd order having the same symmetry. The first non-trivial relation is the Somos-11 recurrence (6.20) below. It can be seen that a combination of $\mathcal{H}_1$ and $\mathcal{H}_2$ appears in one of the coefficients. Since this coefficient remains constant along each orbit of (6.14), it provides a non-trivial first integral. A second independent integral is provided by the Somos-13 recurrence (6.21), given in the following.

Proposition 6.5. *The iterates of the Somos-7 recurrence (6.14) also satisfy the Somos-11 recurrence*

$$\begin{aligned} x_{n+11}\, x_n &= -\beta\, x_{n+10}\, x_{n+1} + \gamma(\gamma - \alpha^2) x_{n+8}\, x_{n+3} - \alpha\beta\gamma\, x_{n+7}\, x_{n+4} \\ &\quad + (\alpha^5 + 2\alpha\gamma^2 + 2\beta^3 + \beta\mathcal{H}_1 + \alpha^2\mathcal{H}_2)\, x_{n+6}\, x_{n+5}, \end{aligned} \tag{6.20}$$

as well as the Somos-13 recurrence

$$x_{n+13}\, x_n = -\beta\gamma\, x_{n+11}\, x_{n+2} + \Gamma\, x_{n+9}\, x_{n+4} + \Delta\, x_{n+8}\, x_{n+5} + \Theta\, x_{n+7}\, x_{n+6}, \tag{6.21}$$

where

$$\Gamma = \alpha^4\gamma + \alpha^2\gamma^2 - \alpha\beta^3 + \gamma^3 + \alpha\gamma\mathcal{H}_2, \quad \Delta = \alpha^5\beta + \alpha^3\beta\gamma + 4\alpha\beta\gamma^2 + \beta^4 + \beta^2\mathcal{H}_1 + \beta(\alpha^2+\gamma)\mathcal{H}_2,$$

$$\Theta = \alpha^7 + 3\alpha^3\gamma^2 + 2\alpha^2\beta^3 - \alpha\gamma^3 + \alpha^2\beta\mathcal{H}_1 + \alpha^4\mathcal{H}_2$$

and, in terms of $y_j = x_j x_{j+3}/(x_{j+1}x_{j+2})$ for $j = 1, 2, 3, 4$, the first integrals $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by

$$\begin{aligned} \mathcal{H}_1 &= \gamma y_1y_2y_3y_4 + \alpha\beta(y_1y_3 + y_1y_4 + y_2y_4) + \alpha\gamma\left(y_1 + y_2 + y_3 + y_4 + \tfrac{y_1y_3}{y_4} + \tfrac{y_2y_4}{y_1}\right) \\ &\quad + \alpha^2\beta\left(\tfrac{1}{y_1} + \tfrac{1}{y_2} + \tfrac{1}{y_3} + \tfrac{1}{y_4} + \tfrac{y_2}{y_1y_3} + \tfrac{y_3}{y_2y_4}\right) + \beta\gamma\left(\tfrac{1}{y_1} + \tfrac{1}{y_2} + \tfrac{1}{y_3} + \tfrac{1}{y_4}\right) \\ &\quad + \alpha^2\gamma\left(\tfrac{1}{y_1y_2} + \tfrac{1}{y_2y_3} + \tfrac{1}{y_3y_4} + \tfrac{1}{y_1y_3} + \tfrac{1}{y_2y_4}\right) + \gamma^2\left(\tfrac{1}{y_1y_3} + \tfrac{1}{y_2y_4}\right) + \alpha\beta^2\left(\tfrac{1}{y_1y_2y_4} + \tfrac{1}{y_1y_3y_4}\right) \\ &\quad + \alpha\beta\gamma\left(\tfrac{2}{y_1y_2y_3y_4} + \tfrac{1}{y_1y_2^2y_4} + \tfrac{1}{y_1y_3^2y_4}\right) + \alpha\gamma^2\left(\tfrac{1}{y_1y_2^2y_3y_4} + \tfrac{1}{y_1y_2y_3^2y_4}\right), \\ \mathcal{H}_2 &= \gamma(y_1y_2y_3 + y_2y_3y_4) + \alpha\beta(y_1 + y_2 + y_3 + y_4) + \alpha\gamma\left(\tfrac{y_1}{y_4} + \tfrac{y_4}{y_1}\right) + \alpha^2\beta\left(\tfrac{1}{y_1y_3} + \tfrac{1}{y_2y_4}\right) \\ &\quad + \beta\gamma\left(\tfrac{1}{y_1y_2} + \tfrac{1}{y_2y_3} + \tfrac{1}{y_3y_4}\right) + \gamma(\alpha^2+\gamma)\left(\tfrac{1}{y_1y_2y_3} + \tfrac{1}{y_2y_3y_4}\right) + \tfrac{\alpha\beta^2}{y_1y_2y_3y_4} \\ &\quad + \alpha\beta\gamma\left(\tfrac{1}{y_1y_2^2y_3y_4} + \tfrac{1}{y_1y_2y_3^2y_4}\right) + \tfrac{\alpha\gamma^2}{y_1y_2^2y_3^2y_4}. \end{aligned} \tag{6.22}$$

Having obtained the first integrals, we now consider each of the three cases corresponding to cluster algebras separately, and explain how the reduction result of Theorem 2.6 applies in each case.

The case $\alpha = 0$: When $\alpha = 0$, the recurrence (6.14) arises from a cluster algebra defined by a 7-node quiver. The latter comes from a 7×7 matrix, specified in terms of its columns by

$$B = (-\mathbf{v}_3, -\mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3 + \mathbf{v}_4, -\mathbf{v}_1 + \mathbf{v}_4, -\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_1, \mathbf{v}_2),$$

where $\mathrm{im}\, B$ is spanned by the vectors

$$\mathbf{v}_j = \mathbf{e}_j - \mathbf{e}_{j+1} - \mathbf{e}_{j+2} + \mathbf{e}_{j+3} \qquad \text{for} \qquad j = 1, 2, 3, 4, \tag{6.23}$$

where $\mathbf{e}_j$ is the jth standard basis vector. In this case, $\ker B$ is spanned by the three integer vectors

$$\mathbf{u}_1 = (1, 1, 1, 1, 1, 1, 1)^T, \qquad \mathbf{u}_2 = (1, 2, 3, 4, 5, 6, 7)^T, \qquad \mathbf{u}_3 = (1, -1, 1, -1, 1, -1, 1)^T. \tag{6.24}$$

These three integer vectors produce a three-dimensional group of scaling transformations,

$$\mathbf{x} \to \lambda^{\mathbf{u}_1} \cdot \mu^{\mathbf{u}_2} \cdot \nu^{\mathbf{u}_3} \cdot \mathbf{x},$$

which coincides with the scalings defined by (6.16) and (6.18). By Lemma 2.7 there is a symplectic form, given in terms of the scale-invariant monomials $y_j = \mathbf{x}^{\mathbf{v}_j}$ for $j = 1, 2, 3, 4$, as

$$\hat{\omega} = \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_3}{y_1 y_3} + \frac{\mathrm{d}y_2 \wedge \mathrm{d}y_3}{y_2 y_3} + \frac{\mathrm{d}y_2 \wedge \mathrm{d}y_4}{y_2 y_4}.$$

This yields the Poisson bracket

$$\{y_j, y_{j+1}\} = 0, \qquad \{y_j, y_{j+2}\} = -y_j y_{j+2}, \qquad \{y_j, y_{j+3}\} = y_j y_{j+3}. \tag{6.25}$$

In terms of y_j one finds the map

$$\hat{\varphi}: \qquad (y_1, y_2, y_3, y_4) \mapsto \Big(y_2, y_3, y_4, (\beta y_3 + \gamma)/(y_1 y_2 y_3^2 y_4)\Big),$$

which is equivalent to iteration of (6.19) with $\alpha = 0$, and preserves the nondegenerate Poisson bracket (6.25). Setting $\alpha = 0$ in the two first integrals in (6.22) and computing their bracket gives $\{\mathcal{H}_1, \mathcal{H}_2\}|_{\alpha=0} = 0$, so this is a Liouville integrable system in 4D.

The case $\beta = 0$: In this case the matrix B is specified by $B = (-\hat{\mathbf{v}}_2, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, -\hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2, -\hat{\mathbf{v}}_1, -\hat{\mathbf{v}}_2, \hat{\mathbf{v}}_1)$, where $\hat{\mathbf{v}}_j = \mathbf{v}_j + \mathbf{v}_{j+1} + \mathbf{v}_{j+2}$ for $j = 1, 2$. The kernel of B is 5-dimensional, being spanned by the integer vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ together with $\mathbf{u}_4 = (1, 0, 0, 1, 0, 0, 1)^T$, $\mathbf{u}_5 = (0, 1, 0, 0, 1, 0, 0)^T$, $\mathbf{u}_6 = (0, 0, 1, 0, 0, 1, 0)^T$. These five independent integer vectors give an action of the algebraic torus $(\mathbb{C}^*)^5$ on $\mathbf{x}$ by scaling transformations, and the scalings (6.16) and (6.18) form a three-parameter subgroup, since $\mathbf{u}_1 = \mathbf{u}_4 + \mathbf{u}_5 + \mathbf{u}_6$. The invariants under the full 5-parameter scaling group are

$$\hat{y}_j = \mathbf{x}^{\hat{\mathbf{v}}_j} = x_j x_{j+5}/(x_{j+2} x_{j+3}) = y_j y_{j+1} y_{j+2},$$

and in terms of these one obtains the general Lyness-2 recurrence with coefficients, that is

$$\hat{y}_{n+2}\, \hat{y}_n = \alpha\, \hat{y}_{n+1} + \gamma, \tag{6.26}$$

which is equivalent to iteration of a map in the $(\hat{y}_1, \hat{y}_2)$ plane and of QRT type [33], with invariant symplectic form $(\hat{y}_1 \hat{y}_2)^{-1} \mathrm{d}\hat{y}_1 \wedge \mathrm{d}\hat{y}_2$. The first integral $\mathcal{H}_1$ does not reduce to the plane, as it is not invariant under the full 5-dimensional scaling group, but $\mathcal{H}_2$ is fully invariant, and reduces to a first integral of (6.26):

$$\mathcal{H}_2|_{\beta=0} = \gamma\,(\hat{y}_1 + \hat{y}_2) + \alpha\gamma \left(\frac{\hat{y}_1}{\hat{y}_2} + \frac{\hat{y}_2}{\hat{y}_1}\right) + \gamma(\alpha^2 + \gamma) \left(\frac{1}{\hat{y}_1} + \frac{1}{\hat{y}_2}\right) + \frac{\alpha\gamma^2}{\hat{y}_1 \hat{y}_2}.$$

The level sets of the latter are cubic (and biquadratic) plane curves of genus one.

The case $\gamma = 0$: This is very similar to the case $\alpha = 0$, since the recurrence comes from the rank 4 matrix

$$B = (-\mathbf{v}_2 - \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4, -\mathbf{v}_1 + \mathbf{v}_2, -\mathbf{v}_2 + \mathbf{v}_3, -\mathbf{v}_3 + \mathbf{v}_4, -\mathbf{v}_1 - \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_1 + \mathbf{v}_3),$$

with the same vectors $\mathbf{v}_j$ as in (6.23). Since both $\ker B$ and $\operatorname{im} B$ are the same as for $\alpha = 0$, there is the same scaling group with invariants $y_j = \mathbf{x}^{\mathbf{v}_j}$. The symplectic form in this case is

$$\hat{\omega} = \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_2}{y_1 y_2} + \frac{\mathrm{d}y_1 \wedge \mathrm{d}y_4}{y_1 y_4} + \frac{\mathrm{d}y_3 \wedge \mathrm{d}y_4}{y_3 y_4},$$

and (up to an overall constant) this gives the unique log-canonical Poisson bracket

$$\{y_j, y_{j+1}\} = y_j y_{j+1}, \qquad \{y_j, y_{j+2}\} = 0 = \{y_j, y_{j+3}\} \tag{6.27}$$

that is preserved by the map

$$\hat{\varphi}: \qquad (y_1, y_2, y_3, y_4) \mapsto \Big(y_2, y_3, y_4, (\alpha y_2 y_4 + \beta)/(y_1 y_2 y_3 y_4)\Big).$$

The latter map in 4D corresponds to iteration of (6.19) with $\gamma = 0$. Setting $\gamma = 0$ in (6.22) and computing the bracket using (6.27) gives $\{\mathcal{H}_1, \mathcal{H}_2\}|_{\gamma=0} = 0$, so the two first integrals are involution, as required.

Remark 6.6. We have shown that for the two parameter subcases of (6.14), the reduced map is integrable in the Liouville sense, either in 4D or 2D. However, there is a different Poisson structure in each case. When $\alpha\beta\gamma \neq 0$, it is easy to check that there is no log-canonical Poisson bracket in the variables y_j that is compatible with (6.19). Nevertheless, we expect that there is a compatible Poisson structure for which the two first integrals in (6.22) are in involution, so that (6.19) defines a 4D map that is Liouville integrable.

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