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# Warranty return policies for products with unknown claim causes and their optimisation

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## Abstract

In practical warranty services management, faults may not always be found in claimed items by warranty service agents, which is the well-known no-fault-found phenomenon (for example, caused by a loose connection between parts, or simply human error). This phenomenon can contribute more than 40% of reported service faults in electronic products and it can be due to faults of manufacturers or product users. Little research, however, considers this phenomenon in warranty management since faults are normally assumed to be found in the claimed items. On the basis of different levels of testing, this paper proposes three warranty return policies, which decide whether new items should be sent to warranty claimants or not. It then derives and compares the expected costs of the policies, and obtains the optimal warranty periods under supply chain environments. The paper illustrates the results with artificially generated data.

*Keywords:* supply chain, optimisation, game theory, cost benefit analysis, warranty management

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## 1. Introduction

Product warranty is a contractual obligation incurred by a manufacturer (or retailer) in connection with the sale of a product. It has become increasingly more important in consumer and commercial transactions and is widely used to serve many different purposes (Karim and Suzuki, 2005; Wu, 2012; Wu, 2013). The US Congress has enacted several warranty acts (UCC, Magnusson Moss Warranty Act, Tread Act, etc.) over the last

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Suggested citation: S. Wu, Warranty return policies for products with unknown claim causes and their optimization, *International Journal of Production Economics*. 156. pp 52--61. DOI: 10.1016/j.ijpe.2014.05.016.

26 100 years. The European Union (EU) passed legislation requiring a two-year warranty for  
27 all products sold in Europe (Murthy and Djameludin, 2002).

28 Warranty expense is one of the operating expenses for manufacturers. A product  
29 might be sold with a warranty agreement and the manufacturer needs to cover labour  
30 and parts needed for repairs or replacement within the warranty period. As a  
31 consequence, warranty incurs tremendous cost in the manufacturing industries. For  
32 example, the automotive industry spends roughly \$10–\$13 billion per year in the U.S. on  
33 warranty claims and up to \$40 billion globally (MSX International Inc, 2010).

34 Although warranty only covers items that have failed, it has been noted that faults  
35 may not always be found in claimed items, which is also referred to as *no-fault-found*  
36 (NFF) (Prakash et al., 2009; Wu, 2011; Huang et al., 2011). Brombacher (1999) showed  
37 that the observed categories of reliability problems were distributed as: components  
38 21%; customers 17%, apparatus 24% and no fault found 38%. On these statistics, the  
39 author further interpreted that *the reliability failures in products were split into problems*  
40 *on a component level, problems on “internal product level” (e.g. interaction problems) and*  
41 *problems on a customer/application level. This analysis showed the largest single group*  
42 *where the cause of the failure remained unknown. The no-fault-found (NFF) phenomenon*  
43 *is a big problem when dealing with multipart products. For example, the NFF contributes*  
44 *on average to 45% of reported service faults in electronic products (Jones and Hayes,*  
45 *2001), and the problem of NFFs in aircraft electronic equipment has long plagued*  
46 *operators (Ramsey, 2005). The problem is not new, but many believe it is getting worse,*  
47 *in part because today's highly complex products are equipped with more and more*  
48 *electronic sensors, computers, control functions and wires (Ramsey, 2005).*

49 Our literature review shows, however, that the following assumption has been  
50 imposed with no explanation in most of the existing research on warranty management:

51           *Fault can always be found in claimed items by warranty service agents. That is, all*  
52           *claimed items are failed ones.*

53           Following the above assumption, research in the literature normally takes one of the  
54 following two assumptions: (1) for repairable products, claimed items are returned to the  
55 claimants after repair; or (2) for non-repairable products, new items are returned to the  
56 claimants. Such assumptions may simplify the calculation process. However, as  
57 mentioned above, in practice, fault might not always be found in claimed items, for which  
58 two methods can therefore be used to handle warranty claims. (1) A new item is returned  
59 to a claimant if fault is found in her claimed item, and (2) the original claimed item  
60 (without any maintenance conducted on it) is returned to the claimant if no fault is found  
61 in her claimed item. This will of course raise another question, which is the ability to  
62 diagnose the real fault in the claimed items.

63           A couple of authors have conducted cost-benefit analysis for product returns with the  
64 NFF phenomenon (see, Prakash et al., 2009; Wu, 2011; Huang et al., 2011, for example).  
65 Prakash et al. (2009) presented a manufacturing process adjustment to eliminate  
66 warranty related NFF product failures in the field when all key product characteristics  
67 measured are within design tolerances. Huang et al. (2011) suggested using a  
68 coordination mechanism to resolve the profit conflict in a reverse supply chain in the  
69 presence of false failure returns. Wu (2011) derived the expected warranty costs for  
70 repairable products when the NFF phenomenon is considered and found that the  
71 expected claim cost per individual product incurred by NFF is sensitive to the total  
72 number of products sold.

73           It's widely accepted that reducing NFF has the potential for dramatic cost savings  
74 across the industry, particularly in terms of additional spares, logistics, workshop time,  
75 test equipment and training (Burchell, 2007).

76 NFF is also referred to as intermittent failures, which is the loss of some functions or  
77 performance characteristics of a product for a limited period of time until subsequent  
78 recovery of the function. Users may experience a failure and restart the item (for  
79 example, computers) and it runs OK. When the item is taken to a service agent, the  
80 repairman might not experience this failure when the item is being inspected. As a  
81 consequence, the warranty service agent may develop different product return policies:  
82 they may either return the claimed item to the claimant, or may send a new item to her.  
83 Different return policies can apparently incur different cost. For example, misdiagnosing  
84 a failed item to be non-failed and then returning it to the claimant can cause losses  
85 directly relating to the manufacturer. Such losses can be: cost of repairing or replacing,  
86 cost of customer dissatisfaction, loss of customer good will, and loss of market share, for  
87 example. However, misdiagnosing a non-failed item to be failed and sending a new item  
88 to the claimant may only incur the cost of the new product. Analysing such return policies  
89 is therefore crucially important for service suppliers. This motivates the authors to write  
90 this paper, which analyses and further derives the expected costs of three return policies.

91 Under different return policies, the following interesting questions can emerge:

92 (a) What is the expected cost of each return policy?

93 (b) Which return policy should be adopted under a given cost setting?

94 (c) What are the optimal warranty periods under a supply chain environment?

95 This paper answers the above three questions. It proposes three product return  
96 policies, derives their expected cost, and optimises warranty periods under two supply  
97 chain environments. As little research on those issues exists in the literature, the paper  
98 develops novelty.

99 The rest of this paper is structured as follows. Section 2 includes assumptions and  
100 notation. Section 3 derives the expected costs of three return policies. Section 4 compares

101 the costs derived from Section 3 and derives optimal warranty periods for base warranty  
102 and extended warranty, considering supply chain environments. Section 5 offers  
103 discussion on estimation of the parameters assumed in the paper. Section 6 gives  
104 numerical examples, and Section 7 concludes the paper.

## 105 **2. Settings and notation**

106 Suppose that the following general assumptions hold.

- 107 • **Causes of claims.** A claim can be reported to the warranty provider  
108 (manufacturer/retailer) due to one of the following three causes: known faults,  
109 unknown faults, and human error. To avoid ambiguity in writing, we refer to the  
110 claims due to known faults, unknown faults, and human error as claim causes 1, 2 and  
111 3, respectively. That is, claim cause 1 is due to known faults, with which an item is not  
112 repaired and a new item should be sent to the claimant. Claim cause 2 is due to  
113 unknown faults that are caused by the manufacturing side, but it may not be detected.  
114 Human error, ie., human error, can also cause a claim and it can be an intended or an  
115 unintended human error, and it is caused by the product users. Either claim cause 2  
116 or claim cause 3 might be diagnosed correctly or incorrectly: the real cause is  
117 revealed if diagnosed correctly, and they are classified as NFF if diagnosed  
118 incorrectly. That is, NFF can be due to claim cause 2 or claim cause 3.
- 119 • **Testing techniques.** There are two types of testing techniques available.
  - 120 (a) Type I testing  $T_1$ : it is an initial testing and aims to identify claim cause 1. This  
121 type can only identify known faults, or claim cause 1, and it cannot detect claim  
122 causes 2 or 3.
  - 123 (b) Type II testing  $T_2$ : which is a more sophisticated testing than Type I testing and it  
124 aims to take a further diagnosis on those items in which no fault has been found

125 with Type I testing. The probability that claim causes 2 and 3 can be detected and  
126 confirmed with Type II testing is  $\rho$  ( $0 \leq \rho \leq 1$ ).

127 • **Return policies.** Once a claimed item is received, one of the following three return  
128 policies is applied.

129 (a) Return Policy 1. Once a claimed item is received, a new and identical item will be  
130 sent to the claimant.

131 (b) Return Policy 2. Once a claimed item is received, it will be tested with Type I  
132 testing.

133 ○ if claim cause 1 is confirmed in the claimed item, a new item will be sent to the  
134 claimant,

135 ○ if no fault is confirmed in the claimed item, the original claimed item will be  
136 returned to the claimant.

137 (c) Return Policy 3. Once a claimed item is received, it will be tested with Type I  
138 testing. Then

139 ○ if claim cause 1 is confirmed in the claimed item, a new item will be sent to the  
140 claimant;

141 ○ if no fault can be confirmed in the claimed item, the claimed item will be tested  
142 with Type II testing. If claim cause 2 can be confirmed with Type II testing, then  
143 a new and identical item is be sent to the claimant. Otherwise, the claimed item  
144 is returned to the claimant.

145 • **Independence.** The occurrences of the three claim causes are statistically  
146 independent. Each failure mechanism leading to a particular type of failure (i.e.,  
147 failure cause) proceeds independently of every other one, at least until a failure  
148 occurs.

- 149 • **Maintenance.** No maintenance, neither corrective maintenance nor preventive  
 150 maintenance, is conducted on the product. If no fault is found in Return Policy 2 or  
 151 Return Policy 3, the claimed item is returned to the claimant and the hazard rate  
 152 function of the item is not altered.
- 153 • **Warranty policy.** Only non-renewing warranty policy is considered, that is, under  
 154 this policy, the manufacturer/retailer offers a satisfactory service only within the  
 155 original warranty period, and an item with a confirmed failure is replaced by the  
 156 manufacturer at no cost to the buyer or at a pre-specified cost to the buyer within the  
 157 original warranty period, and the original warranty is not renewable.
- 158 • **Warranty processing time.** Assume that time on processing a claimed item is  
 159 negligible.

160 In this paper, we use the following notation.

161 **Notation**

$F_i(t)$	Cumulative distribution function (cdf) of time to failure due to claim cause $i$ , where $i=1,2,3$ .
$f_i(t)$	$f_i(t) = dF_i(t)/dt$ with $i=1,2,3$ .
$\lambda_i(u)$	Failure intensity function corresponding to $F_i(t)$ , $i=2,3$ .
$\Lambda_i(t)$	$\Lambda_i(t) = \int_0^t \lambda_i(u)du$ , $i=2,3$ .
$m_i(t)$	Renewal function corresponding to the cdf $F_i(t)$ , where $i=1,2,3$ .
$c_{32}$	Expected cost of diagnosing claim cause 3 to claim cause 2
$c_{23}$	Expected cost of diagnosing claim cause 2 to claim cause 3
$c_a$	Expected administration cost per claim
$c_n$	Cost of returning a new item
$c_{t1}$	Expected cost of Type I testing per item
$c_{t2}$	Expected cost of Type II testing per item
$\rho$	Probability of correctly diagnosing claim causes 2 and 3
$C_k(t)$	Expected cost of return policy $k$ per an item, within time interval $(0,t)$ , where $k=1,2,3$
$w$	Length of a warranty period

162 **3. Expected costs of return policies**

163 All of the three Return Policies can correctly detect claim cause 1, which results in  
 164 returning new items.



165 However, items with claim causes 2 or 3 may be misdiagnosed. As a result, items with  
 166 claim cause 2 may be returned to the claimants, although new items should be sent to  
 167 claimants. A new item may be sent to the claimant although her claim was reported due  
 168 to claim cause 3.

169 From the assumptions in the preceding section, the cost distribution of diagnosing  
 170 claimed items can be illustrated in Table 1. In Table 1, for example, the values in the cell  
 171 in the 2nd column and the 2nd row means that the cost of implementing Return Policy 1  
 172 when the claim cause 2 is correctly identified is  $c_n + c_a$ , and the cost of implementing  
 173 Return Policies 2 and 3 when the claim cause 2 is correctly identified is  $c_n + c_a + c_{t1}$  and  
 174  $c_n + c_a + c_{t1} + c_{t2}$ , respectively. The values in the cell in the 2nd column and the 3rd row  
 175 means that the cost of implementing Return Policy 1 when the claim cause 3 is incorrectly  
 176 identified to be claim cause 2 is  $c_n + c_a + c_{32}$ , but Return Policies 2 and 3 do not  
 177 mistakenly diagnose claim cause 3 to claim cause 2 and therefore does not incur any  
 178 costs.

179  
 180 **Table 1. Cost distribution**

Actual Diagnosed	Claim cause 2 (Actual)	Claim cause 3 (Actual)
Claim cause 2 (Diagnosed)	Return Policy 1: $c_n + c_a$ Return Policy 2: $c_n + c_a + c_{t1}$ Return Policy 3: $c_n + c_a + c_{t1} + c_{t2}$	Return Policy 1: $c_n + c_a + c_{32}$ Return Policy 2: not applicable Return Policy 3: not applicable
Claim cause 3 (Diagnosed)	Return Policy 1: not applicable Return Policy 2: $c_a + c_{t1} + c_{23}$ Return Policy 3: $c_a + c_{t1} + c_{t2} + c_{23}$	Return Policy 1: not applicable Return Policy 2: $c_a + c_{t1}$ Return Policy 3: $c_a + c_{t1} + c_{t2}$

181  
 182 Return Policy 1 is quite simply. Return Policy 2 and Return Policy 3 are also illustrated  
 183 in Figure 1 (a) and Figure 1 (b), respectively.

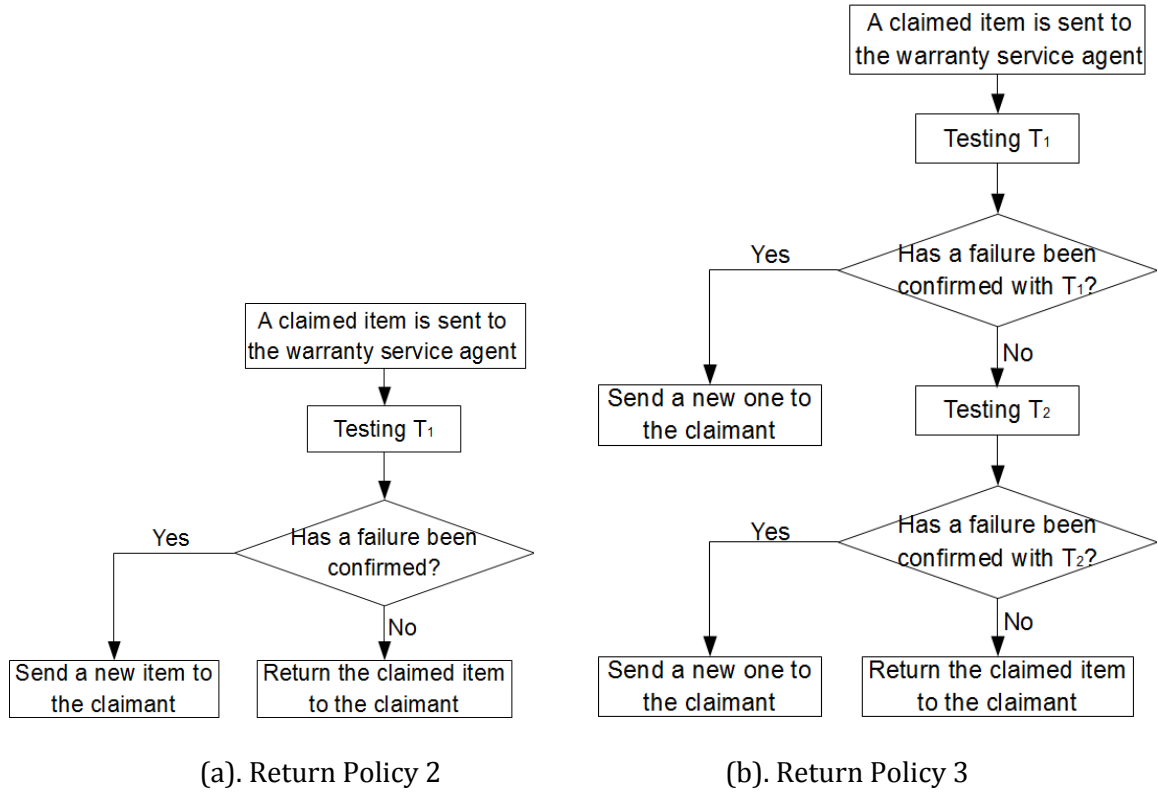


Figure 1. Warranty claim handling procedure in Return Policy 2 and Return Policy 3

This following derives the expected cost of each return policy.

### 3.1. Expected Costs of the Three Return Policies

#### 3.1.1 Expected Cost of Return Policy 1

Under Return Policy 1, new items are sent to warranty claimants regardless of the causes of the claims. A potential loss incurred with this Policy is to send new items to those claimants whose claims are due to claim cause 3, although the original claimed items should be returned to the claimants. We therefore have the following proposition.

**Proposition 1.** The expected cost of Return Policy 1 is given by

$$C_1(w) = (c_n + c_a) m_{123}(w) + c_{32}(1 - q_{X_{12} < X_3}) m_{123}(w) \quad (1)$$

where  $m_{123}(w) (= H_{123}(w) + \int_0^w m_{123}(w-t) dH_{123}(t))$  that is the expected number of renewals within time interval  $(0, w)$ ,  $H_{123}(t) (= 1 - (1 - F_1(t))(1 - F_2(t))(1 - F_3(t)))$  that is the probability distribution of time to receive a claim due to one of the three claim

199 causes,  $q_{X_{12} < X_3} (= \int_0^w H_{12}(t) dF_3(t))$  that is the probability of the occurrence of claim  
 200 causes 1 and 2, and  $H_{12}(t) (= 1 - (1 - F_1(t))(1 - F_2(t)))$  that is the probability  
 201 distribution of time to receive a claim due to either of the claim causes 2 and 3.

202 **Proof.** Under Return Policy 1, claims due to one of the three claim causes result in  
 203 renewals, hence, the three causes are three competing risks. As such, the probability  
 204 distribution of time-to-renewal is  $H_{123}(t)$ . The expected number of warranty claims  
 205 during period  $(0, w)$  is  $m_{123}(w)$ , or the renewal function corresponding to the cumulative  
 206 distribution function  $H_{123}(t)$ .  $c_n + c_a$  is the sum of cost of sending a new item and  
 207 administration cost per item. Hence, the total returns incurred due to returning new  
 208 items upon any claim causes is  $(c_n + c_a) m_{123}(w)$ .

209 Under Return Policy 1, denote time to return a new item upon claim due to cause 3 by  $X_3$   
 210 and time to return a new item upon claim due to causes 1 or 2 by  $X_{12}$ .

211 Apparently,  $m_{123}(w)$  can be re-written as

$$212 \quad m_{123}(w) = m_{123}(w) \Pr(X_{12} < X_3) + m_{123}(w)(1 - \Pr(X_{12} < X_3))$$

$$213 \quad = m_{123}(w) q_{X_{12} < X_3} + m_{123}(w)(1 - q_{X_{12} < X_3}),$$

214 where  $q_{X_{12} < X_3} = \Pr(X_{12} < X_3) = \int_0^\infty H_{12}(t) dF_3(t)$ .

215 In the above equation,  $m_{123}(w)(1 - q_{X_{12} < X_3})$  is the number of warranty claims due to  
 216 claim cause 3, which incurs cost  $c_{32}(1 - q_{X_{12} < X_3})m_{123}(w)$  of incorrectly classifying claim  
 217 cause 3 to claim cause 2.

218 Hence, the total cost incurred in Return Policy 1 is  $(c_n + c_a) m_{123}(w) + c_{32}(1 -$   
 219  $q_{X_{12} < X_3}) m_{123}(w)$ . This completes the proof. ■

220 The expected cost  $C_1(w)$  of Return Policy 1 is the cost of returning new items upon claims  
 221 due to any of the three claim causes. As claim cause 3 is the human error that is caused by  
 222 the product users and that the warranty provider should not be responsible for, any

223 additional cost relating to claim cause 3 should be considered. As such,  $C_1(w)$  includes  
 224 two elements: (1) cost of returning items due to all the claim causes, and (2) cost of  
 225 wrongly sending a new item to the customer, resulting from misclassifying claim cause 3  
 226 to claim causes 1 or 2.

### 227 3.1.2 Expected Cost of Return Policy 2

228 Under Return Policy 2, Type I testing is carried out to detect known faults. New items are  
 229 sent to the claimants whose claim causes are confirmed known faults. Otherwise, the  
 230 original claimed items are returned to the claimants.

231 **Proposition 2.** The expected cost of Return Policy 2 is given by

$$\begin{aligned}
 232 \quad C_2(w) &= (c_n + c_a + c_{t1})m_1(w) + (c_a + c_{t1} + c_{23})m_1(w) \int_0^\infty \Lambda_2(t)dF_1(t) \\
 233 \quad &\quad (c_a + c_{t1})m_1(w) \int_0^\infty \Lambda_3(t)dF_1(t). \quad (2)
 \end{aligned}$$

234 *Proof.*

- 235 • Under Return Policy 2, the causes of any claimed items are diagnosed with Type I  
 236 testing. New items will be sent to warranty claimants if claim cause 1 is confirmed,  
 237 which incurs cost  $(c_n + c_a + c_{t1})m_1(w)$ , where  $m_1(w)$  is the renewal function  
 238 corresponding to the cumulative distribution function  $F_1(t)$ .
- 239 • If the causes of warranty claims are not detected or confirmed, the original claimed  
 240 items will be returned. This essentially forms a renewal-reward process: claimed  
 241 items due to claim cause 1 are renewed and the process is a renewal process, and  
 242 within each inter-arrival period, the number of claimed items whose causes are not  
 243 confirmed can be seen as a reward function depending on the length of the inter-  
 244 arrival time. Since the occurrences of claim cause 1 and claim cause 2 are assumed to  
 245 be statistically independent, according to Gallager (1995), the total expected number

246 of warranty claims due to claim cause 2 is  $m_1(w) \int_0^\infty \Lambda_2(t) dF_1(t)$ . Hence, the cost on  
 247 returns, including administration cost and cost of Type I testing, due to claim cause 2  
 248 is given by  $(c_a + c_{t1})m_1(w) \int_0^\infty \Lambda_2(t) dF_1(t)$ .

249 • Claimed items may be due to cause 2, under which new items should be sent but the  
 250 original claimed items are incorrectly returned to the claimants. Returning such  
 251 items can cause potential or latent problems such as damaging manufacturer's  
 252 reputation, and therefore incur cost  $c_{23}m_1(w) \int_0^\infty \Lambda_2(t) dF_1(t)$ .

253 • The original claimed items due to cause 3 are correctly returned to the claimants.  
 254 Returning such products can incur cost  $(c_a + c_{t1})m_1(w) \int_0^\infty \Lambda_3(t) dF_1(t)$ , which  
 255 includes administration cost and cost of Type I testing.

256 This completes the proof. ■

### 257 3.1.3 Expected Cost of Return Policy 3

258 Under Return Policy 3, a further testing, Type II testing, is conducted on those claims  
 259 whose causes have not been identified with Type I testing.

260 Denote  $F_T(t) = 1 - (1 - F_1(t))e^{-\rho \int_0^t \lambda_2(u) du}$ ,  $m_T(t) = F_T(t) + \int_0^t m_T(t - u) dF_T(u)$ ,  
 261  $F_{T2}(t) = 1 - e^{-\rho \int_0^t \lambda_2(u) du}$ , and  $q_{X_1 < X_{T2}} = \Pr(X_1 < X_{T2}) = \int_0^\infty F_1(y) dF_{T2}(y)$ . Then we have  
 262 the following proposition.

263 **Proposition 3.** The expected warranty cost of Return Policy 3 is given by

$$\begin{aligned}
 264 \quad C_3(w) &= (c_n + c_a + c_{t1})m_T(w) + (1 - q_{X_1 < X_{T2}})c_{t2}m_T(w) \\
 265 \quad &+ (c_a + c_{t1} + c_{t2} + c_{23})m_T(w) \int_0^\infty (1 - \rho)\Lambda_2(t) dF_T(t) + (c_a + c_{t1} \\
 266 \quad &+ c_{t2})m_T(w) \int_0^\infty (1 - \rho)\Lambda_3(t) dF_T(t). \tag{3}
 \end{aligned}$$

267 *Proof.*

268 • An item is put in operation at time 0. If warranty on this item is claimed, the cause of  
269 this claim is checked with Type I testing. If either claim cause 1 or claim cause 2 is  
270 confirmed, then a new item will be returned to the customer. Otherwise, the original  
271 claimed item will be returned. Claim cause 1 can be detected and identified by Type I  
272 testing, whereas claim cause 2 can be correctly detected and identified with a  
273 probability  $\rho$ . That is, claim cause 2 may not be detected with a probability of  $1 - \rho$ . If  
274 only the returns due to claim cause 2 is considered, according to (Block et al., 1985),  
275 the successive times on returning new items forms a renewal process with an inter-  
276 arrival distribution  $1 - e^{-\rho \int_0^t \lambda_2(u) du}$ . Hence, if both claim causes 1 and 2 are  
277 considered, the successive times on returning new items forms a renewal process  
278 with an inter-arrival distribution  $F_T(t)$  (ie.,  $1 - (1 - F_1(t))e^{-\rho \int_0^t \lambda_2(u) du}$ ). The  
279 number of new items returned to the customers is  $m_T(w)$ . Hence, the cost is  
280  $(c_n + c_a + c_{t1})m_T(w)$ .

281 • On the other hand, those items whose claim causes are not identified are returned to  
282 the customers. They may be diagnosed correctly (reveal the real claim cause  
283 correctly) or incorrectly (diagnosed claim causes 2 to claim cause 3, or claim cause 3  
284 to claim cause 2). Among those items,  
285 (a) the number of items with claim cause 2, which are diagnosed correctly, is  
286  $(1 - q_{X_1 < X_{T_2}})m_T(w)$  and they incur cost  $(1 - q_{X_1 < X_{T_2}})c_{t2}m_T(w)$  on Type II  
287 testing (the cost due to Type I testing on those items has already been included in  
288 the first term in Eq (3)),  
289 (b) the number of items with claim cause 2, which are incorrectly diagnosed as claim  
290 cause 3, is  $m_T(w) \int_0^\infty (1 - \rho)\Lambda_2(t)dF_1(t)$ , which incurs a total cost of  
291  $(c_a + c_{t1} + c_{t2} + c_{23})m_T(w) \int_0^\infty (1 - \rho)\Lambda_2(t)dF_1(t)$ ,

292 (c) the number of items with claim cause 3, which are correctly diagnosed as claim  
 293 cause 3, is  $m_T(w) \int_0^\infty (1 - \rho) \Lambda_3(t) dF_1(t)$ , which incurs a total cost of  $(c_a + c_{t1} +$   
 294  $c_{t2})m_T(w) \int_0^\infty (1 - \rho) \Lambda_3(t) dF_1(t)$ .

295 To sum up the different costs, one can obtain  $C_{r3}(w, T)$ , as shown in Eq. (3). ■

296 **Remarks.** In Eq. (3),

- 297 •  $\rho = 0$  implies that the probability of correctly diagnosing claim causes 2 and 3 is 0  
 298 and there is therefore no need to conduct Type II testing,
- 299 •  $\rho = 1$  implies that that each of claim causes 2 and 3 can be correctly diagnosed and  
 300 new items are sent to the claimants who deserve the treatment, and
- 301 • if  $\rho = 0$  and  $c_{t2} = 0$ , then  $C_2(w) = C_3(w)$ . Due to the following reason, both  $\rho = 0$  and  
 302  $c_{t2} = 0$  should hold to ensure that the expected costs of Policy 2 and Policy 3 are  
 303 equal.

304 (a) In the case when  $\rho = 0$  and  $c_{t2} \neq 0$ , time on Type II testing still incurs cost  
 305 although the probability of correctly diagnosing claim causes 2 and 3 is 0.

306 (b) In the case when  $c_{t2} = 0$  and  $\rho \neq 0$ , correctly diagnosing claim causes 2 and 3 is  
 307 possible. Consequently, some items are handled correctly (ie., correctly returning  
 308 new items or old items), which impacts cost.

### 309 ***3.2. Comparison of the expected costs on special cases***

310 The preceding section derived the expected costs of the three return policies.

311 Implementing Return Policy 1 is quite simply and straightforward, but it may incur the  
 312 largest losses if new items are expensive. Implementing Return Policy 2 requires Type I  
 313 testing and it can potentially damage the reputation of both the manufacturer and the  
 314 retailer due to the fact that the original claimed items with claim causes 2 may be  
 315 returned. Implementing Return Policy 3 is the most complicated but it can potentially

316 benefit the manufacturer and/or the retailer as it maximises the chance to correctly  
 317 respond the warranty claimants. An interesting question is to compare these costs and  
 318 optimise the warranty periods, which are investigated below.

319 Denote

320 •  $\theta_1 = \left( c_n + c_a + \left( \frac{\lambda_3}{\lambda_1 + \lambda_3} + \frac{\lambda_3}{\lambda_2 + \lambda_3} - \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \right) c_{32} \right) (\lambda_1 + \lambda_2 + \lambda_3),$

321 •  $\theta_2 = (c_n + c_a + c_{t1})\lambda_1 + (c_a + c_{t1} + c_{23})\lambda_2 + (c_a + c_{t1})\lambda_3,$

322 and

323 •  $\theta_3 = (c_a + c_{t1})(\lambda_1 + \lambda_2) + c_n(\lambda_1 + \rho\lambda_2) + c_{t2}\lambda_2 + (1 - \rho)c_{23}\lambda_2 + (1 - \rho)(c_a + c_{t1} +$   
 324  $+ c_{t2})\lambda_3.$

325 The following Lemma can be derived from Propositions 1, 2, and 3.

326 **Lemma 1.** Assume  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1,2,3$ ). The expected costs of Return Policy  $k$  is  
 327 given by

328  $C_k(w) = \theta_k w,$  (4)

329 where  $k=1,2,3$ .

330 **Proof.** Since  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1,2,3$ ), we have  $H_{123}(w) = 1 - e^{-(\lambda_1 + \lambda_2 + \lambda_3)w}$ , and

331  $m_{123}(w) = (\lambda_1 + \lambda_2 + \lambda_3)w.$

332 Hence,

333 
$$C_1(w) = (c_n + c_a) m_{123}(w) + c_{nf}(1 - q_{X_{12} < X_3}) m_{123}(w)$$
  
 334 
$$= \left( c_n + c_a + c_{nf}(1 - q_{X_{12} < X_3}) \right) (\lambda_1 + \lambda_2 + \lambda_3)w$$

335 Since

336 
$$q_{X_{12} < X_3} = \int_0^\infty H_{12}(u) dF_3(u) = \left( 1 - \frac{\lambda_3}{\lambda_1 + \lambda_3} - \frac{\lambda_3}{\lambda_2 + \lambda_3} + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} \right)$$

337 Hence

338 
$$C_1(w) = \theta_1 w.$$



339 Since  $m_1(w) = \lambda_1 w$ ,  $\Lambda_2(t) = \lambda_2 t$ ,  $\Lambda_3(t) = \lambda_3 t$ ,  $\int_0^\infty \Lambda_2(t) dF_1(t) = \frac{\lambda_2}{\lambda_1}$ , and  $\int_0^\infty \Lambda_3(t) dF_1(t) =$

340  $\frac{\lambda_3}{\lambda_1}$ , from Eq. (2), we have

$$\begin{aligned}
341 \quad C_2(w) &= (c_n + c_a + c_{t1})m_1(w) + (c_a + c_{t1} + c_{23})m_1(w) \int_0^\infty \Lambda_2(t) dF_1(t) \\
342 \quad &\quad + (c_a + c_{t1})m_1(w) \int_0^\infty \Lambda_3(t) dF_1(t) \\
343 \quad &= \theta_2 w.
\end{aligned}$$

344 Since  $m_T(w) = \lambda_1 + \rho\lambda_2$ ,  $\Lambda_2(t) = \lambda_2 t$ ,  $\int_0^\infty \Lambda_2(t) dF_T(t) = \frac{\lambda_2}{\lambda_1 + \rho\lambda_2}$ ,  $\int_0^\infty \Lambda_3(t) dF_T(t) = \frac{\lambda_3}{\lambda_1 + \rho\lambda_2}$ ,

345 and

$$346 \quad q_{X_1 < X_{T2}} = \Pr(X_1 < X_{T2}) = \int_0^\infty F_1(y) dF_{T2}(y) = 1 - \frac{\rho\lambda_2}{\lambda_1 + \rho\lambda_2}.$$

347 From Eq. (3), we have

$$\begin{aligned}
348 \quad C_3(w) &= (c_n + c_a + c_{t1})(\lambda_1 + \rho\lambda_2) + \frac{\rho\lambda_2}{\lambda_1 + \rho\lambda_2} c_{t2}(\lambda_1 + \rho\lambda_2) + (c_a + c_{t1} + c_{t2} + c_{23})(\lambda_1 \\
349 \quad &\quad + \rho\lambda_2) \frac{(1 - \rho)\lambda_2}{\lambda_1 + \rho\lambda_2} + (c_a + c_{t1} + c_{t2})(\lambda_1 + \rho\lambda_2) \frac{(1 - \rho)\lambda_3}{\lambda_1 + \rho\lambda_2} \\
350 \quad &= \theta_3 w.
\end{aligned}$$

351 This completes the proof. ■

352 Lemma 1 implies that the cost of each Return Policy is proportional to the length of  
353 warranty, which is evident.

354 As mentioned above, an interesting question is, among the three return policies,  
355 which policy is the cheapest? For general distributions  $F_1(t)$  and  $F_2(t)$ , however, to derive  
356 simple close forms of  $m_1(w)$ ,  $m_{12}(w)$ ,  $F_{12}^*(w, T)$ , and  $m_{12}^*(w, T)$  is not possible. Even if  
357  $F_1(t)$  is the Weibull distribution, for example, only approximation of its renewal function  
358 can be derived (see, Cui and Xie, 2003; Jiang, 2010, for example). Hence, we will only  
359 compare the three return policies for special cases of  $F_i(t)$  ( $i=1,2,3$ ).

360 **Lemma 2.** If  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1,2,3$ ), then we have

361 (a) If  $\rho = 1, \lambda_3 = 0, c_{t1} = c_{t2} = 0$ , then  $C_1(w) = C_3(w)$ ;

362 (b) If  $\rho = 1, \lambda_3 = 0, c_{t1} = c_{t2} = 0$ , and  $c_{23} > c_n$ , then  $C_2(w) > C_1(w)$  and

363  $C_2(w) > C_3(w)$ ; and

364 (c) If  $\rho c_{23} - \rho c_n - c_{t2}(1 + \frac{\lambda_3}{\lambda_2}) + \rho c_a \frac{\lambda_3}{\lambda_2} + \rho(c_{t1} + c_{t2}) \frac{\lambda_3}{\lambda_2} > 0$ , then  $C_2(w) > C_3(w)$ .

365 **Proof.** The proof can be easily completed based on the results of Lemma 1. ■

366 **Remarks.** From Lemma 2, we make the following remarks.

367 • From (a) and (b) of Lemma 2,  $\lambda_3 = 0$  implies that there is no claim cause 3,  $c_{t1} =$   
368  $c_{t2} = 0$  implies that neither Type I testing nor Type II testing incurs cost, and  $\rho = 0$   
369 implies that Type II testing can correctly reveal the claim cause, then we have the  
370 following results.

371 ○ The expected cost incurred in Return Policy 1 equals to that in Return Policy 3.

372 This is evident as there are only claim causes 1 and 2, both of which are caused  
373 due to the manufacturer and new items should therefore be sent on any claims.

374 With either Return Policy 1 or Return Policy 3, new items are sent upon claims

375 due to claim cause 1. If claims due to claim cause 2 are reported, with Return

376 Policy 1, a new item will be sent to the claimant; with Return Policy 3, the

377 claimed item will be tested with Type I testing and then Type II testing. Since the

378 Type II testing can correctly reveal the claim cause, the problem that was

379 diagnosed as NFF by Type I testing can be correctly detected. Consequently, a

380 new item will be sent to the claimant. In other words, claims with either Return

381 Policy 1 or Return Policy 3 will end up with returning new items to the claimants

382 and the costs will only include administration cost and cost of returning new

383 items.

- 384      ○ if  $c_{23} > c_n$  also holds, Return Policy 2 incurs more cost than both Return Policy 1  
385              and Return Policy 3. Return Policy 2 returns a claimed item back to the claimant  
386              although the claim cause may be due to claim cause 2. If this may cause more cost  
387              than sending a new item to the claimant, then Return Policy 2 is more expensive  
388              than Return Policy 1 and Return Policy 3, which sends new items to the  
389              claimants.
- 390      • From (c), whether Return Policy 2 is more costly than Return Policy 3 is independent  
391              of  $\lambda_1$  and of the actual values of  $\lambda_2$  and  $\lambda_3$ , but depends on the ratio of  $\lambda_3$  to  $\lambda_2$ .
- 392      • From (c), it can also be seen that  $C_2(w) < C_3(w)$  if  $\rho=0$ . As  $\rho=0$  indicates the  
393              probability of correctly detecting claim cause 2 is 0, spending time and cost on claim  
394              cause 2 is not necessary.

### 395 **3.3. Sensitivity analysis**

396 The preceding section 3.2 investigates the roles of some parameters for special cases. In  
397 this section, we conduct sensitivity analyses on different cost parameters without the  
398 assumption of the exponential distributions.

399 It can easily come to the following results.

- 400      • The costs of all the three return policies are increasing in  $c_n$  and  $c_a$ .
- 401      • The costs of Return Policies 2 and 3 are increasing in  $c_{t1}$  and  $c_{23}$ .
- 402      • The cost of Return Policy 1 is increasing in  $c_{32}$ , the costs of Return Policy 3 is  
403              increasing in  $c_{t2}$ .

404 As the major difference between the return policies lies in whether new items should be  
405 sent to the claimants, we further analyse the impact of  $C_n$  on the costs of return policies.

406 Since  $\frac{\partial C_1(w)}{\partial c_n} = m_{123}(w)$ ,  $\frac{\partial C_2(w)}{\partial c_n} = m_1(w)$ ,  $\frac{\partial C_3(w)}{\partial c_n} = m_T(w)$ , and  $m_{123}(w) \geq m_T(w) \geq$

407  $m_1(w)$ , we have  $\frac{\partial C_1(w)}{\partial c_n} \geq \frac{\partial C_3(w)}{\partial c_n} \geq \frac{\partial C_2(w)}{\partial c_n}$ . This implies that the expected cost of Return

408 Policy 1 is more sensitive to the change of  $c_n$  than the other two Return Policies, while  
 409 the expected cost of Return Policy 2 is less sensitive to the change of  $c_n$  than the other  
 410 two Return Policies.

411 Section 6 uses numerical examples to investigate the roles of  $\rho$ ,  $C_{23}$ , and  $C_{t1}$ .

## 412 **4. Optimisation of warranty periods under supply chain** 413 **environments**

414 In this section, we derive optimal warranty periods for the base warranty and the  
 415 extended warranty, respectively. The following derivation is needed in this subsection.

416 From Eqs. (1)---(3), we have

$$417 \quad \frac{\partial C_1(w_i)}{\partial w_i} = (c_n + c_a + c_{32}(1 - q_{X_{12} < X_3}))\pi_1(w_i), \quad (5)$$

$$418 \quad \frac{\partial C_2(w_i)}{\partial w_i} = \left( c_n + c_a + c_{t1} + (c_a + c_{t1}) \int_0^\infty (\Lambda_2(t) + \Lambda_3(t)) dF_1(t) + c_{23} \int_0^\infty \Lambda_2(t) dF_1(t) \right) \pi_2(w_i)$$

419 (6)

$$420 \quad \frac{\partial C_3(w_i)}{\partial w_i} = (c_n + c_a + c_{t1} + (1 - q_{X_1 < X_{T_2}})c_{t2} + (c_a + c_{t1} + c_{t2} + c_{23}) \int_0^\infty (1 - \rho)\Lambda_2(t) dF_T(t)$$

$$421 \quad + c_a \int_0^\infty (1 - \rho)\Lambda_3(t) dF_T(t))\pi_3(w_i) \quad (7)$$

422 where

$$423 \quad \pi_1(w_i) = f_{123}(w_i) + \int_0^\infty \pi_1(w_i - t) f_{123}(t) dt,$$

$$424 \quad \pi_2(w_i) = f_1(w_i) + \int_0^\infty \pi_2(w_i - t) f_1(t) dt,$$

425 and

$$426 \quad \pi_3(w_i) = f_T(w_i) + \int_0^\infty \pi_3(w_i - t) f_T(t) dt.$$

### 427 **4.1. The supply chain context**

428 We assume the following supply chain context. We take the assumptions used in (Chen et  
 429 al., 2012), which assumed the manufacturer, as a Stackelberg leader, specified wholesale

430 prices to two competing retailers, retailer 1 and retailer 2, who faced warranty period-  
 431 dependent demand and had different sales costs and then analysed different strategies  
 432 from both the manufacturer's and the retailers' perspective. They considered demands  
 433 primarily influenced by extended warranty offered by retailers, provided the price  
 434 differentiation between the retailers becomes insignificant to their customers at the time  
 435 of purchase decision (Chen et al., 2012).

#### 436 **4.2. Period of the base warranty**

437 Assume, under a supply chain environment, that the primary demand of a product is  
 438 sensitive to the period of the base warranty. One can then define warranty period  
 439 dependent demand as following:

$$440 \quad D_1(w) = \alpha_0 + \alpha_1 w, \quad (8)$$

441 where  $\alpha_0 (> 0)$  is the primary demand, and  $\alpha_1 (> 0)$  is the consumers' sensitivity to  
 442 warranty period.

443 The warranty provider's profit with Return Policy  $k$  is defined as

$$444 \quad Q_{1,k}(w) = \beta_0 D_1(w) - C_k(w) D_1(w), \quad (9)$$

445 where  $k = 1, 2, 3$ ,  $\beta_0 D_1(w)$  = sales revenue – purchasing cost – sales cost, and  $C_k(w) D_1(w)$   
 446 is the cost incurred due to warranty period service. Then, combine both Eqs. (8) and (9),  
 447 we obtain

$$448 \quad Q_{1,k}(w) = (\beta_0 - C_k(w))(\alpha_0 + \alpha_1 w). \quad (10)$$

449 **Proposition 4.** If  $\frac{\partial^2 C_k(w)}{\partial w^2} > 0$ , the optimal warranty period  $w^*$  for Return Policy  $k$  satisfies

$$450 \quad (\alpha_0 + \alpha_1 w^*) \frac{\partial C_k(w)}{\partial w} \Big|_{w=w^*} + \alpha_1 C_k(w^*) - \alpha_1 \beta_0 = 0. \quad (11)$$

451 Assume  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1,2,3$ ). The optimal warranty period for Return Policy  $k$  is  
 452 given by

453 
$$w^* = \frac{\beta_0}{2\theta_k} - \frac{\alpha_0}{2\alpha_1} \quad (12)$$

454 where  $k=1,2,3$ , respectively.

455 Proof. From Eq. (10), we have

456 
$$\frac{\partial Q_{1,k}(w)}{\partial w} = \alpha_1(\beta_0 - C_k(w)) - \frac{\partial C_k(w)}{\partial w}(\alpha_0 + \alpha_1 w),$$

457 and

458 
$$\frac{\partial^2 Q_{1,k}(w)}{\partial w^2} = -\frac{\partial^2 C_k(w)}{\partial w^2}(\alpha_0 + \alpha_1 w) - 2\alpha_1 \frac{\partial C_k(w)}{\partial w}.$$

459 From Eqs. (1)—(3),  $\frac{\partial C_k(w)}{\partial w}$  is the derivative of a renewal function within time interval

460  $(0, w)$ . As any renewal function increases in  $w$ ,  $\frac{\partial C_k(w)}{\partial w} > 0$ . Hence  $\frac{\partial^2 Q_{1,k}(w)}{\partial w^2} < 0$  if  $\frac{\partial^2 C_k(w)}{\partial w^2} >$

461  $0$ . That is,  $Q_{1,k}(w)$  is concave in  $w$ .

462 Let  $\frac{\partial Q_{1,k}(w)}{\partial w} = 0$ , one has

463 
$$(\alpha_0 + \alpha_1 w) \frac{\partial C_k(w)}{\partial w} + \alpha_1 C_k(w) - \alpha_1 \beta_0 = 0. \quad (13)$$

464 If  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1,2,3$ ), substitute  $C_1(w)$ ,  $C_2(w)$ , and  $C_3(w)$  to the above Eq. (13),

465 one can derive the optimal warranty periods shown in Eq. (12).

466 This completes the proof. ■

467 **Lemma 3.** Assume  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1,2,3$ ). Then the minimum expected cost of

468  $Q_{1,k}(w^*)$  is given by

469 
$$Q_{1,k}(w^*) = \frac{(\alpha_0 \theta_k - \alpha_1 \beta_0)^2}{4\alpha_1 \theta_k} + \alpha_0 \beta_0. \quad (14)$$

470 Proof. Substitute  $w^*$  in Eq. (12) into Eq. (10), we can obtain  $Q_{1,k}(w^*)$  in Eq (14). ■

### 471 **4.3. Period of extended warranty**

472 In this paper, we consider the following pricing strategy:

473 *Manufacturer negotiates with both retailers simultaneously considering their sales*  
 474 *cost and specifies the same wholesale price for both retailers.*

475 One can then define warranty period dependent demand for retailer  $j$  as following

$$476 \quad D_2(w_j) = \alpha_2 + \alpha_3 w_j - \alpha_4 w_{3-j} \quad (15)$$

477 where  $j = 1, 2$ ,  $\alpha_2 (> 0)$  is the primary demand,  $\alpha_3 (> 0)$  represents the consumers'  
 478 sensitivity to warranty period, and  $\alpha_4 (> 0)$  denotes the competitive factors, and  $\alpha_4 < \alpha_3$ .

479 The retailer  $j$ 's profit with Return Policy  $k$  is defined as

$$480 \quad Q_{2,k}(w_j) = \beta_1 D_2(w_j) - \delta_j D_2(w_j) - C_k(w_j) D_2(w_j) \quad (16)$$

481 where  $j = 1, 2$ ,  $k = 1, 2, 3$ ,  $\beta_1 D_2(w_j)$  is the difference between the retailer  $j$ 's sales revenue  
 482 and purchasing cost,  $\delta_j D_2(w_j)$  is the sales cost, and  $C_k(w_j) D_2(w_j)$  is the costs incurred by  
 483 warranty period service.

484 Then we have the following Proposition.

485 **Proposition 5.** The retailer  $j$ 's optimal warranty period  $w_j^*$  in Return Policy  $k$  satisfies

$$486 \quad \alpha_3(\beta_1 - \delta_j) - (\alpha_2 + \alpha_3 w_j^* - \alpha_4 w_{3-j}^*) \frac{\partial C_k(w_j)}{\partial w_j} \Big|_{w_j=w_j^*} - \alpha_3 C_k(w_j^*) = 0, \quad (17)$$

487 where  $\frac{\partial C_k(w_j)}{\partial w_j}$  ( $k=1, 2, 3$ ) are given in Eqs. (5)---(7).

488 Assume  $F_i(t) = 1 - e^{-\lambda_i t}$  ( $i=1, 2, 3$ ), then the retailer  $j$ 's optimal extended warranty period

489  $w_j^*$  with Return Policy  $k$  is given by

$$490 \quad w_j^* = \left( \frac{\beta_1}{\theta_k} - \frac{\alpha_2}{\alpha_3} - \frac{2\alpha_3 \delta_j + \alpha_4 \delta_{3-j}}{(2\alpha_3 + \alpha_4)\theta_k} \right) \frac{\alpha_3}{2\alpha_3 - \alpha_4}, \quad (18)$$

491 where  $k=1, 2, 3$ , respectively.

492 **Proof.** By mimicking the proof of Proposition 4, one can easily complete the proof. ■

## 493 **5. Discussion**

494 To derive the expected costs expressed in Eqs. (1)—(3), we need to obtain distribution  
495 functions  $F_k(t)$ , probability  $\rho$ , different costs  $c_{32}$ ,  $c_{23}$ ,  $c_a$ ,  $c_n$ ,  $c_{t1}$ , and  $c_{t2}$ . Their estimations  
496 are discussed below, respectively.

### 497 **5.1. Estimation of the probability functions $F_k(t)$**

498 In the preceding sections, we assume that  $F_k(t)$  ( $k = 1,2,3$ ) can be obtained, which is  
499 possible in practice. One may estimate them based on warranty data, which are  
500 comprised of claims data and supplementary data. Warranty claims data are the data  
501 collected during the servicing of items under warranty and supplementary data are  
502 additional data (such production and marketing related, items with no claims, etc.) that  
503 are needed for effective warranty management (Wu, 2013).

### 504 **5.2. Estimation of the probability $\rho$**

505  $\rho$  is the probability of correctly diagnosing claim causes 2 and 3. Because time to  
506 detect unknown claim cause is uncertain, such a fault detection process can be regarded  
507 as a time-dependent stochastic process  $\{X(t), t \geq 0\}$ , where  $X(t)$  is a random variable  
508 and is the time to successfully detect claim cause 2 at time ( $t \geq 0$ ). One may regard the  
509 process of the ability to detect claim causes as a gamma process for the following reason.  
510 Time to successfully detecting the real cause is always positive; and it may become stochastically  
511 shorter over time. The learning process is monotonic in the sense that the probability of correctly  
512 detecting the real causes becomes larger with time. As such, a Gamma process can be used for  
513 modelling the learning process in which the detection of the real causes is supposed to take place  
514 gradually over time in a sequence of positive increments. In theory, a Gamma process  $\{X(t), t \geq 0\}$  has  
515 the following three properties.

516 (1) The increment  $X(t_i) - X(t_{i-1})$  for a given time interval  $\Delta = t_i - t_{i-1}$  follows the Gamma distribution,



- 517 (2) The increments for any set of disjoint time intervals are independent random variables having the  
518 distributions described in property (1), and  
519 (3)  $X(0) = 0$  almost surely.

520 Let the probability density function of  $X(t)$  in conformity with the definition of the  
521 gamma process, be given by  $f_{X(t)}(x) = GA(x|v(t), u)$ , with  $GA(x|v(t), u) =$   
522  $\frac{u^{v(t)}}{\Gamma(v(t))} x^{v(t)-1} \exp\{-ux\} I_{(0,\infty)}(x)$ ,  $E(X(t)) = \frac{v(t)}{u}$ , and  $\text{Var}(X(t)) = \frac{v(t)}{u^2}$ , where  $I_A(x) = 1$  for  
523  $x \in A$  and  $I_A(x) = 0$  otherwise.

524 At time  $t$ , denote the time when claim cause 2 is detected by time point  $T$ ,

$$525 \quad \Pr\{X(t) \leq T\} = \int_0^T f_{X(t)}(x) dx = \frac{\gamma(v(t), Tu)}{\Gamma(v(t))}$$

526 where  $\Gamma(v(t)) = \int_0^\infty \tau^{v(t)-1} e^{-\tau} d\tau$  and  $\gamma(v(t), Tu) = \int_0^{Tu} \tau^{v(t)-1} e^{-\tau} d\tau$ .

527 Then

$$528 \quad p(t, T) = \frac{\partial \Pr\{X(t) \leq T\}}{\partial t}. \quad (19)$$

529 The above method has been used in reliability engineering to model the deterioration  
530 process of reliability systems. Of course, one needs to collect historical data for estimating  
531  $\Pr\{X(t) \leq T\}$ . Again, supplementary data can be used for this purpose.

### 532 **5.3. Estimation of $c_{23}$ and $c_{t2}$**

533 Estimating  $c_{32}$ ,  $c_a$  and  $c_n$  is not difficult. Below we discuss methods of estimating  $c_{23}$   
534 and  $c_{t2}$ , respectively.

535  $c_{23}$  is the cost of returning faulty items to users, which can result in profit losses. The  
536 losses can be larger if more claimed items with claim cause 2 are returned to the  
537 claimants, which is essentially similar to the situation that product costs are associated  
538 with its reliability, as the relationship proposed in Mettas (2000).

539  $c_{t2}$  is the cost incurred in Type II testing. Type II testing might start from the first  
540 claim with claim causes 2 or 3, and then such effort might continue until all of the claim  
541 causes 2 and 3 are eventually detected and fixed or until a new model of products is  
542 launched to replace the old ones. In this case, the probability of successfully detecting and  
543 then fixing the causes depends on time. If we can set the time instant after the  $n$  products  
544 were sold to be 0, then the cumulative distribution function of time to the first failure  
545 (and then claim) is  $F_{23}^{(n)}(t) = 1 - ((1 - F_2(t))(1 - F_3(t)))^n$ . The probability that claim  
546 cause 2 or claim cause 3 occurs during the warranty period is given by  $\int_0^w dF_{23}^{(n)}(t)$ .

547 **Proposition 6.** The expected cost on detecting and fixing the cause of NFFs per unit time  
548 is given by

$$549 \quad c_{t2} = \frac{C_{t2}}{n} \int_0^w \int_t^{T_n} \tau p(\tau, T) d\tau dF_{23}^{(n)}(t) \quad (20)$$

550 where  $T_n$  is an estimated time when the manufacturer might give up trying to diagnose  
551 the cause (or the time when a new model of products is launched),  $p(\tau, T)$  can be  
552 estimated from Eq. (19), and  $C_{t2}$  is the total cost on diagnosing claim causes 2 and 3.

#### 553 **5.4. The expected number of warranty claims**

554 The expected number of warranty claims of each return policy is another interesting  
555 quantity that can be required from time to time in practice. As can be seen, the  
556 expected numbers of warranty claims of the return policies have already been derived in  
557 the process of proving the first three Propositions.

### 558 **6. Numerical examples**

559 Section 4 discusses the three return policies for some special cases. In this section, we  
560 consider more complicated parameter settings and investigate the changes of the costs  
561 derived from the three policies, as we mentioned that it is unlikely to derive closed

562 explicit forms for the renewal functions used in the expected costs for general inter-  
563 arrival distribution functions. As such, we use Monte Carlo simulation to generate  
564 random numbers with the parameters in Table 2 to estimate the expected cost values  
565 derived from the preceding sections. That is, we generate random numbers  $S_i$  as the time  
566 elapsed before an item fails (or is reported) for the “ith” time since the last time it failed  
567 (or was reported), and then count  $\sup\{n: \sum_{i=1}^n S_i\}$  as the renewal functions in  $C_1(w)$ ,  
568  $C_2(w)$ , and  $C_3(w)$ . For each renewal function, we iterate this procedure for 5000 times  
569 and calculate the average of values  $\sup\{n: \sum_{i=1}^n S_i\}$  to obtain a robust estimate of the  
570 renewal function.

571 Table 2. The distribution functions and the warranty period

$F_1(t) = 1 - \exp\left(-\left(\frac{t}{20}\right)^{1.1}\right)$	$F_3(t) = 1 - \exp\left(-\left(\frac{t}{28}\right)^{1.3}\right)$
$F_2(t) = 1 - \exp\left(-\left(\frac{t}{24}\right)^{1.2}\right)$	$w = 24$

### 572 **6.1. The role of the probability $\rho$**

573  $\rho$  is the probability of correctly diagnosing claim causes 2 and 3. It is important to  
574 understand its role in Return Policy 3. In Figures 2, 3, and 4,  $\rho$  changes from 0.01 to 1, as  
575 shown in the X-axis, and the Y-axis represents the expected costs.

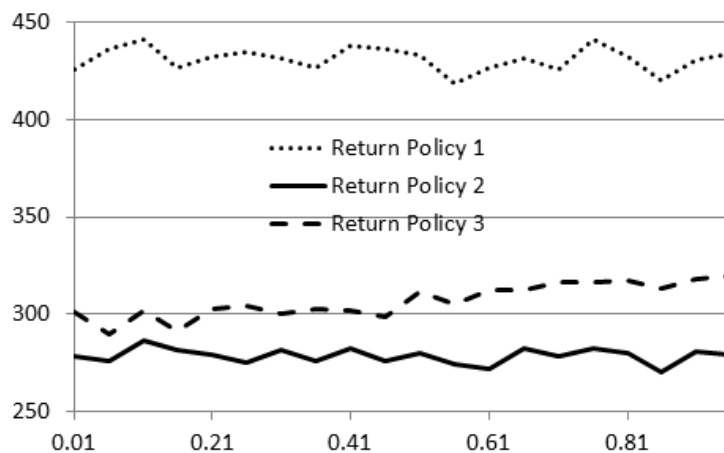
- 576 • If  $\rho$  changes from 0.01 to 1 with step 0.05 and let  $c_a=1$ ,  $c_n=100$ ,  $c_{32}=80$ ,  $c_{23}=20$ ,  
577  $c_{t1}=4$  and  $c_{t2}=5$ , then the changes of the expected costs  $C_1(w)$ ,  $C_2(w)$ , and  $C_3(w)$   
578 are shown in Figure 2. From the figure, it can be found that Return Policy 2 is the  
579 cheapest one whereas Return Policy 1 is the most expensive one. The expected  
580 cost of Return Policy 3 increases slowly, and the other two return policies have  
581 stable costs. The increase of the expected cost of Return Policy 3 is due to the fact  
582 that more new items are required as a result of the correct diagnosis of claim

583 cause 2, comparing to the small cost of misdiagnosing claim cause 2 to claim cause  
 584 3.

- 585 • If  $\rho$  changes from 0.01 to 1 with step 0.05 and let  $c_a=1, c_n=100, c_{32}=80, c_{23}=120,$   
 586  $c_{t1}=4$  and  $c_{t2}=5$ , then the changes of the expected costs  $C_1(w), C_2(w),$  and  $C_3(w)$   
 587 are shown in Figure 3. From the figure, it can be found that Return Policy 1 is the  
 588 cheapest before  $\rho$  changes to 0.21. The expected cost of Return Policy 3  
 589 dramatically decreases when  $\rho$  is larger than 0.25 and it then keeps the smallest  
 590 one, whereas Return Policy 2 is the most expensive one when  $\rho$  is larger than 0.06.
- 591 • If  $\rho$  changes from 0.01 to 0.96 with step 0.05, let  $c_a=1, c_n=10, c_{32}=8, c_{23}=20, c_{t1}=5$   
 592 and  $c_{t2}=10$ , then the changes of the expected costs  $C_1(w), C_2(w),$  and  $C_3(w)$  are  
 593 shown in Figure 4. In this case, Return Policy 1 remains the cheapest one whatever  
 594  $\rho$  is. The expected cost of Return Policy 3 decreases.

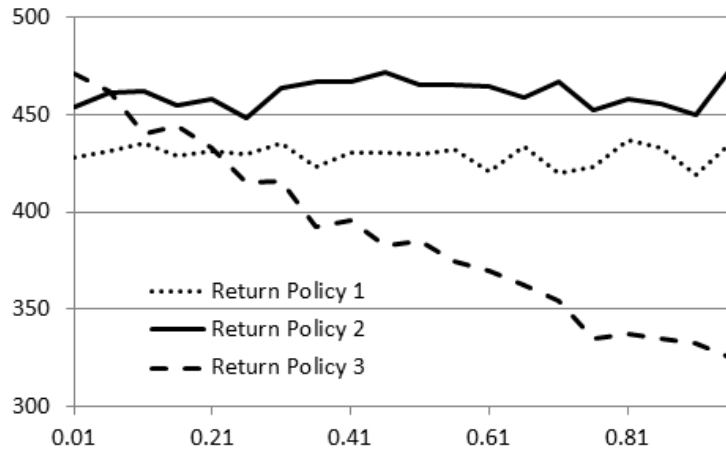
595 From the three examples, we can find that the expected cost of Return Policy 3 can  
 596 increase or decrease if the probability of correctly diagnosing claim causes 2 and 3  
 597 increases. Each of the three return policies can be the cheapest one or the most expensive  
 598 one, it depends on different costs of  $c_n, c_{32},$  and  $c_{23}$ .

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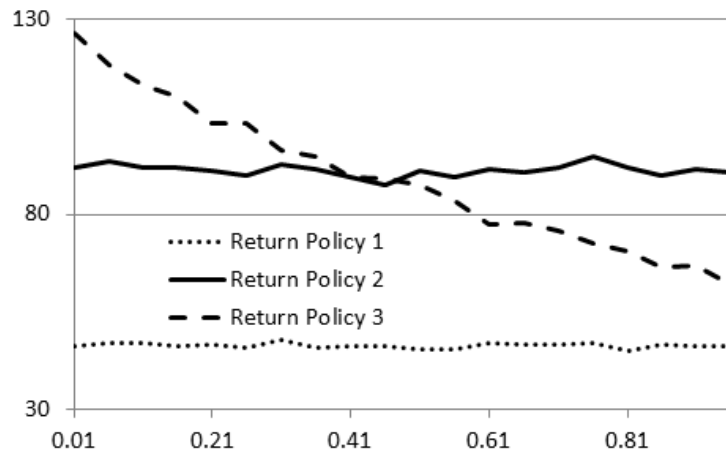


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Figure 2. The expected costs of the three return policies when  $c_a=1, c_n=100, c_{32}=80,$   
 $c_{23}=120, c_{t1}=4,$  and  $c_{t2}=5$ .



604 Figure 3. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=100$ ,  $c_{32}=80$ ,  $c_{23}=20$ ,  
 605  $c_{t1}=4$ , and  $c_{t2}=5$ .  
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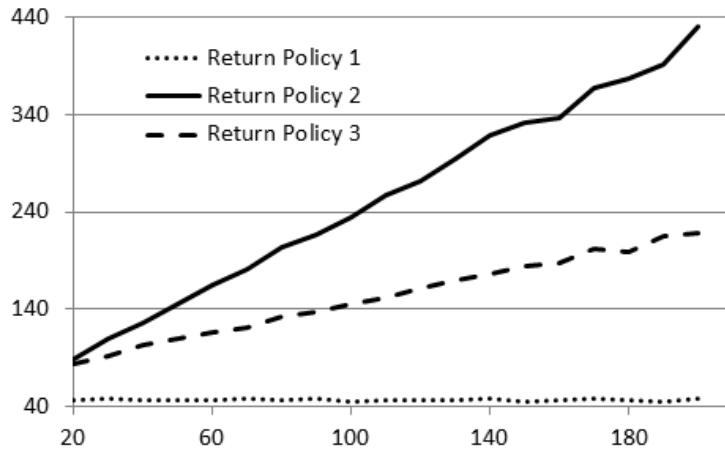


608 Figure 4. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=10$ ,  $c_{32}=8$ ,  $c_{23}=20$ ,  
 609  $c_{t1}=5$  and  $c_{t2}=10$ .  
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## 6.2. Dependence of the return policies on $c_{23}$ and $c_{t1}$

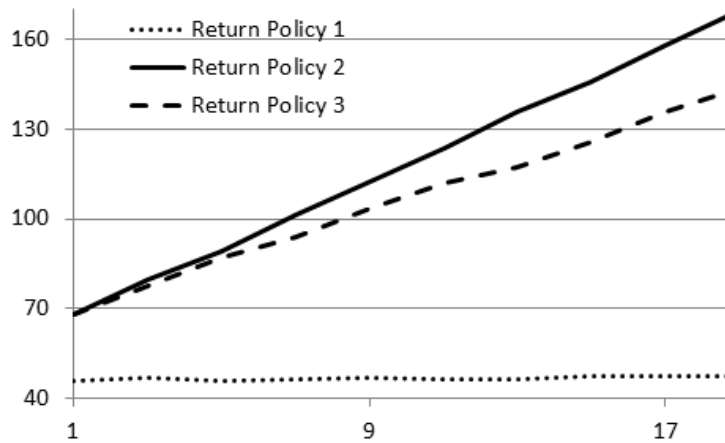
612  $c_{23}$  and  $c_{t1}$  are parameters used in the expected costs of Return Policy 2 and Return Policy  
 613 3, respectively. We therefore investigate their roles in the policies.  
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615 Figures 5 and 6 show the expected costs of the three return policies when  $c_{23}$  and  $c_{t1}$   
 616 increase. It can be seen that both the expected costs of Return Policy 2 and Return Policy  
 617 3 increase: the expected cost of Return Policy 2 increases faster than that of Return Policy  
 618 3.



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Figure 5. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=10$ ,  $c_{32}=8$ ,  $c_{t1}=5$ , and  $c_{t2}=10$ .  $c_{23}$  changes from 20 to 200, as shown in the X-axis. The Y-axis represents the expected costs.



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Figure 6. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=10$ ,  $c_{32}=8$ , and  $c_{t2}=10$ ,  $c_{t1}$  changes from 1 to 20, as shown in the X-axis. The Y-axis represents the expected costs.

## 629 7. Conclusions

630 This paper considered the fact that product returns can be due to other factors in  
631 addition to product failures. It proposed three warranty return policies, derived the  
632 expected costs of the policies and a testing method, respectively. It then compared the  
633 expected costs and derived optimal warranty periods under supply chain environments.

634 In estimating the number of warranty claims, traditionally, the renewal process is  
635 applied in the scenario when claimed items are not repairable and the nonhomogeneous  
636 Poisson process is used when the claimed items are repairable. This is the first paper that  
637 used the renewal-reward process to estimate the number of warranty claims. It is noted  
638 that this is the first paper that systematically studies and compares different solutions for

639 warranty claims with the no-fault-found phenomenon. The paper also offers alternates  
640 for the industrialists to design different warrant policies.

641 Our future work will focus on developing new warranty policies. For example, if a  
642 customer continually returns an item whose failure mechanism has not been detected  
643 and confirmed, it may not wise to return the same item back to him/her. Instead, a new  
644 item should be returned. This can lead to develop a new return policy.

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## Captions of the Figures

- Figure 1. Warranty claim handling procedure in Return Policy 2 and Return Policy 3
- Figure 2. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=100$ ,  $c_{32}=80$ ,  $c_{23}=120$ ,  $c_{t1}=4$ , and  $c_{t2}=5$ .
- Figure 3. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=100$ ,  $c_{32}=80$ ,  $c_{23}=20$ ,  $c_{t1}=4$ , and  $c_{t2}=5$
- Figure 4. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=10$ ,  $c_{32}=8$ ,  $c_{23}=20$ ,  $c_{t1}=5$  and  $c_{t2}=10$ .
- Figure 5. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=10$ ,  $c_{32}=8$ ,  $c_{t1}=5$ , and  $c_{t2}=10$ .  $c_{23}$  changes from 20 to 200, as shown in the X-axis. The Y-axis represents the expected costs.
- Figure 6. The expected costs of the three return policies when  $c_a=1$ ,  $c_n=10$ ,  $c_{32}=8$ , and  $c_{t2}=10$ ,  $c_{t1}$  changes from 1 to 20, as shown in the X-axis. The Y-axis represents the expected costs.