q-DIFFERENTIAL EQUATIONS FOR q-CLASSICAL POLYNOMIALS AND q-JACOBI-STIRLING NUMBERS

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ABSTRACT. The q-classical polynomials are orthogonal polynomial sequences that are eigenfunctions of a second order q-differential operator of a certain type. We explicitly construct q-differential equations of arbitrary even order fulfilled by these polynomials, while giving explicit expressions for the integer composite powers of the aforementioned second order q-differential operator. The latter is accomplished through the introduction of a new set of numbers, the q-Jacobi Stilring numbers, whose properties along with a combinatorial interpretation are thoroughly worked out. The results here attained are the q-analogue of those given by Everitt *et al.* and the first author, whilst the combinatorics of this new set of numbers is a q-analogue version of the Jacobi-Stirling numbers given by Gelineau and the second author.

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1. INTRODUCTION

The present paper concerns with even order q-differential operators having the so-called q-classical polynomials as eigenfunctions. These are essentially sequences of

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orthogonal polynomial whose sequences of the corresponding q-derivatives is also orthogonal. So, at the centre is the q-derivative operator or *Jackson*-derivative, here denoted as D_q and defined for any polynomial f as follows

$$\big(D_q f\big)(x) := \frac{f(qx) - f(x)}{(q-1)x} \tag{1.1}$$

where q is a complex number such that $q \neq 0$ and $q^n \neq 1$ for any positive integer n. Collectively, the q-classical polynomials share a number of properties and, among them, we single out the fact they are eigenfunctions of a second-order q-differential operator of q-Sturm-Liouville type, $\mathscr{L}_q := \Phi(x) D_q \circ D_{q^{-1}} - \Psi(x) D_{q^{-1}}$, where Φ is a monic polynomial of degree two at most and Ψ a polynomial of degree one. Regarding their importance in various fields of mathematics, there is a considerable list of bibliography – without any attempt for completion, we refer to [9, 16, 18, 19, 21, 22, 30]. Worth to mention, that different perspectives have been taken to study q-polynomials: some based on the fact that they are (terminating) basic hypergeometric series (with no pretension for completess we refer to [4, 9, 15, 19, 21, 22]); or labelling them as eigenfunctions of the q-differential equation [22, 30]; other studies are grounded on an algebraic approach [18, 26] where the orthogonality information is embedded in a linear functional. The latter has the merit of gathering all the properties in coherent and consistent framework and we will follow this approach. The q-classical polynomials include the q-polynomials of Al-Salam and Carlitz, the Wall, the Stieltjes and Wigert; the Big q-Laguerre, the q-Laguerre, the Little q-Laguerre, the Little q-Jacobi, the Big q-Jacobi polynomials. Naturally, when q is set equal to 1, the q-classical polynomials we recover the classical ones of Hermite, Laguerre, Bessel or Jacobi.

Precisely, the classical polynomial sequences (Hermite, Laguerre, Bessel and Jacobi) are the only orthogonal polynomial sequences (OPS) whose elements are eigenfunctions of the second-order differential operator $\mathscr{L} := \mathscr{L}_1$, known as the Bochner differential operator, named after [2]. Later on, in 1938, Krall [20] showed that if the elements of a classical polynomial sequence are eigenfunctions of a differential operator, then it must be of even order. In fact, any even order differential operator having classical orthogonal polynomials as eigenfunctions must be a polynomial with constant coefficients in the Bochner differential operator $\mathscr{L}$. This has been addressed in [28] without proposing any method providing explicit expressions for the powers of the operator $\mathscr{L}$. Meanwhile, in [25] the classical polynomials were characterised as eigenfunctions of even differential operators with a certain structure. A result improved by the first author in [24], where an explicit expression for the integral composite powers of $\mathscr{L}$ was given in a coherent framework, for each of the four classical families. The goal was accomplished via the Stirling numbers to study the cases of Hermite and Laguerre polynomials, whereas the cases of Bessel and Jacobi polynomials required the introduction of a new set of numbers: the so-called A-modified Stirling numbers. The Stirling numbers of the first and second kind, $s(n,k)$ and $S(n,k)$ are defined respectively by

$$\prod_{i=0}^{n-1}(x-i) = \sum_{k=0}^{n} s(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n,k)\prod_{i=0}^{k-1}(x-i),$$

with $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\} = \{0,1,2,3,\ldots\}$, whereas, the A-modified Stirling numbers of the first and second kind, $\widehat{s}_A(n,k)$ and $\widehat{S}_A(n,k)$, are defined respectively by

$$\prod_{i=0}^{n-1}(x-i(A+i)) = \sum_{k=0}^{n} \widehat{s}_A(n,k)x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} \widehat{S}_A(n,k)\prod_{i=0}^{i-1}(x-i(A+i)),\ n\in\mathbb{N}_0,$$

where A represents a complex number. In fact, these A-modified Stirling numbers are the so-called Jacobi-Stirling numbers, introduced by Everitt *et al.* in [6] while dealing with the left-definite analysis associated with the Jacobi self-adjoint operator. The latter operator is, in fact, the operator $\mathscr{L}$ with $\Phi(x) = x^2-1$ and $\Psi(x) = (\alpha+\beta+2)x+(\alpha-\beta)$, where $\alpha,\beta > -1$. The Jacobi Stirling numbers, introduced with the purpose of determining the powers of the Jacobi operator, coincide with the A-modified Stirling numbers which were built on with the purpose of determining the powers of the Jacobi and the Bessel operator.

Since then, in analogy to the Stirling numbers, several properties of the Jacobi-Stirling numbers (or Legendre-Stirling numbers) and its companions including combinatorial interpretations have been established, [1, 5, 10, 11, 29].

The main aim of this work is to explicitly construct a q-differential operator of arbitrary even order, $\mathscr{L}_{k;q}$ say (where k is a positive integer), that has the q-classical polynomials as eigenfunctions while relating them to integral composite power of the aformentioned operator $\mathscr{L}_q$ (which equals $\mathscr{L}_{1;q}$). The key to link these two operators is via the q-Stirling numbers, when $\deg\Phi \leqslant 1$, whereas the case where $\deg\Phi = 2$ requires the introduction of a new pair of sets of numbers mimicking the behaviour of the q-Stirling numbers to which we call the *q-Jacobi-Stirling numbers*. In other words, these pairs of numbers are the connection coefficients between the kth composite power of $\mathscr{L}_q$ and an even order operator $\mathscr{L}_{k;q}$ that will be explicitly given. Regarding their importance, we provide a combinatorial interpretation to the new set of q-Jacobi-Stirling numbers, along with their explicit expression and generating functions. Consequently, by means of these q-Stirling numbers we are able to explicitly give an expression for any integral composite power of $\mathscr{L}_q$. We highlight, nonetheless, that whenever the orthogonality measure of the corresponding of a q-classical sequence $\{P_n\}_{n\geqslant 0}$ admits an integral representation via a weight function, U_q say, we will show that the aforementioned operator can be expressed as follows

$$\mathscr{L}_{k;q}[y](x) = q^{-k}\Big(U_q(x)\Big)^{-1} D^k_{q^{-1}}\left(\left(\prod_{\sigma=0}^{k-1} q^{-\sigma\deg\Phi}\Phi(q^\sigma x)\right)U_q(x)\Big(D^k_q y(x)\Big)\right), \quad (1.2)$$

for any polynomial y.

Inter alia, the present manuscript comprises a q-analog version of [24] along with combinatorial results on the q-Jacobi-Stirling numbers. Therefore, by setting $q=1$ in the results here obtained, we recover those in [24] and [10].

To come up with the explicit expressions of $\mathscr{L}_{k;q}$ we do not make use of the analytical properties of the measures of each of the q-classical sequences, nor of the properties of the basic hypergeometric corresponding series. (These two approaches would require to split the analysis for each of the q-classical sequences). Rather, we base our study on the algebraic approach developed by Maroni [26] and we make use of [18]. Here, the information on the orthogonality measures is cached in the corresponding linear functionals. The main advantage of this methodology is the unified and coherent analysis of all the possible q-classical cases: in fact, and loosely speaking, all the information can be extracted from the two polynomials Φ and Ψ, where Φ plays a guiding role. The q-classical sequences and the corresponding q-classical operators will be treated as whole. As it will soon become clear, there are mainly two cases requiring a separate analysis, depending on whether Φ has degree less than or equal to 2.

The outline of the manuscript reads as follows. After starting off with a revision in §2 on the q-differential operator and orthogonal polynomials, §3 is devoted to the q-classical polynomials. Here, from their algebraic properties, q-differential operators of arbitrary even order are thoroughly built on. In the mean time, arises a generalisation of the Rodrigues formula fulfilled by the q-classical polynomials. This section ends up with the question of relating these even order q-differential operators with the composite powers of that of 2nd order. A question that is addressed in §4 by means of the q-Stirling numbers together with a new set of numbers that we have introduced: the so-called q-Jacobi-Stirling numbers. The first part of §4 mainly concerns properties of this new set of numbers, which are are also set in comparison to those of the q-Stirling numbers and of the Jacobi-Stirling. For this reason, the proofs are either brief or omitted. The second and last part of §4 is devoted to a combinatorial interpretation for these new q-Jacobi-Stirling numbers. Finally, bringuing all the pieces together, in §5, we make use of the introduced q-Jacobi-Stirling numbers to come up with explicit expressions for the integer composite powers of $\mathscr{L}_q$ for each of the q-classical families. Among them, a special attention is driven to the Al-Salam-Carlitz, the Stieltjes-Wigert and the Little q-Jacobi polynomials.

2. Background

Throughout the text, $\mathbb{R}$ and $\mathbb{C}$ correspond to the field of the real and complex numbers, respectively. The symbols n and k will be allocated for integer values and we will write $n \geqslant k$ to mean all the integers n greater than or equal to k. The main tool in this work is the q-derivative operator D_q defined in (1.1). When $q \to 1$, it reduces to the derivative operator, $D_q \to D$. Its behaviour resembles that of the derivative operator, as for to start, we observe that its action on the sequence of monomials $\{x^n\}_{n\geqslant 0}$ leads to

$$\left(D_q\zeta^n\right)(x) = \begin{cases} [n]_q\, x^{n-1} & \text{if} \quad n \geqslant 1, \\ 0 & \text{if} \quad n = 0, \end{cases}$$

where $[a]_q := \frac{q^a-1}{q-1}$ for any number a (possibly complex). We will primarily use the classical q-notations of [9, 19]. Namely, the *q-shifted factorial* (also known as the *q-Pochhammer symbol*) is defined by $(x;q)_k := \prod_{i=0}^{k-1}(1-xq^i)$, for $k \geqslant 1$, and $(x;q)_0 := 1$. The q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \qquad \text{if} \quad 0 \leq k \leq n,$$

and $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ otherwise. In addition, for practical reasons, we introduce the supplementary notation

$$[a;q]_n := \prod_{k=0}^{n-1}[a+k]_q$$

and therefore $[n]_q! = [1;q]_n$.

The present investigation is primarily targeted at the analysis of sequences of polynomials of successive degrees starting at zero, which will be abbreviated to PS. Whenever the leading coefficient of each of its polynomials equals one, the PS is said to be a *monic polynomial sequence* (MPS). A PS or a MPS forms a basis of the vector space of polynomials with coefficients in $\mathbb{C}$, identified as $\mathscr{P}$, while its corresponding dual space will be denoted as $\mathscr{P}'$. The elements of $\mathscr{P}'$ are referred to either as *linear functionals* or *forms*. Further notations are introduced as needed. We adopt the standard notation of $\langle u, f\rangle$ to consider the action of $u \in \mathscr{P}'$ over $f \in \mathscr{P}$. The sequence of moments of $u \in \mathscr{P}'$ results from its action on the canonical sequence $\{x^n\}_{n\geqslant 0}$ and will be denoted as $(u)_n := \langle u, x^n\rangle, n \geqslant 0$.

Any linear operator $T : \mathscr{P} \to \mathscr{P}$ has a transpose ${}^tT : \mathscr{P}' \to \mathscr{P}'$ defined by

$$\langle {}^tT(u), f\rangle = \langle u, T(f)\rangle, \quad u \in \mathscr{P}',\, f \in \mathscr{P}. \tag{2.1}$$

For example, for any form u and any polynomial g, let $Du = u'$ and gu be the forms defined as usual by $\langle u', f\rangle := -\langle u, f'\rangle$, $\langle gu, f\rangle := \langle u, gf\rangle$, where D is the derivative operator [26]. Thus, with some abuse of notation, D acting on linear functionals is minus the transpose of the derivative operator D on polynomials. Within the same spirit, the dilation operator h_a, with $a \in \mathbb{C} - \{0\}$, defined in the $\mathscr{P}$ by $\big(h_a f\big)(x) := f(ax)$, for any $f \in \mathscr{P}$, and the q-derivative operator D_q have a transpose defined by duality according to

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle = \langle u, f(ax)\rangle, \quad \text{for any} \;\; f \in \mathscr{P},\, a \in \mathbb{C} - \{0\},$$

whereas

$$\langle D_q u, f\rangle := -\langle u, D_q f\rangle\,, \quad \text{for any} \;\; f \in \mathscr{P},\, u \in \mathscr{P}'. \tag{2.2}$$

Alike the usual derivative, the q-derivative is defined on $\mathscr{P}'$ as minus the transpose of the q-derivative operator D_q on $\mathscr{P}$: ${}^tD_q := -D_q$. In particular, this yields

$$\langle D_q u, x^n \rangle := -[n]_q \langle u, x^{n-1} \rangle,\ n \in \mathbb{N}_0.$$

We formally list some properties of this operator D_q, either on $\mathscr{P}$ or on $\mathscr{P}'$:

Lemma 2.1. *The following properties hold for any $f,g \in \mathscr{P}$ and $u \in \mathscr{P}'$* [18]

$$D_q\,(fg)\,(x) = f(qx)\left(D_q g\right)(x) + \left(D_q f\right)(x) g(x), \tag{2.3}$$

$$D_q\,(f(x)u) = f(q^{-1}x)\ \left(D_q u\right) + q^{-1}(D_{q^{-1}} f)(x)\ u, \tag{2.4}$$

$$\left(h_a fg\right)(x) = \left(h_a f\right)(x)\ \left(h_a g\right)(x), \quad (\textit{with } a \in \mathbb{C} - \{0\}), \tag{2.5}$$

$$D_q \circ h_{q^{-1}} = q^{-1} D_{q^{-1}} \quad \textit{in } \mathscr{P}, \tag{2.6}$$

$$D_q \circ h_a = a\, h_a \circ D_q \quad \textit{in } \mathscr{P} \quad (\textit{with } a \in \mathbb{C} - \{0\}), \tag{2.7}$$

$$D_q \circ D_{q^{-1}} = q^{-1} D_{q^{-1}} \circ D_q \quad \textit{in } \mathscr{P}. \tag{2.8}$$

The q-derivative formula for a product of two polynomials (2.3) induces the q-Leibniz formula [14] for the product of two polynomials

$$D_q^n\,(fg)\,(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(D_q^{n-k} f\right)(q^k x)\ \left(D_q^k g\right)(x)\ ,\ n \geqslant 0, \tag{2.9}$$

whereas, the nth order derivative of the product of a polynomial f by a linear functional u can be expanded from (2.4) to obtain

$$D_q^n\,(fu) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} q^{-(n-k)} \left(D_{q^{-1}}^{n-k} f\right)(q^{-k}x)\ \left(D_q^k u\right)\ ,\ n \geqslant 0. \tag{2.10}$$

Here, the symbol D_q^n denotes the nth composite power of the operator D_q and is defined by recurrence according to $D_q^n f = D_q\left(D_q^{n-1} f\right)$ for $n \geqslant 1$ and $D_q^0 f = 1$, for any $f \in \mathscr{P}$.

The dual sequence $\{u_n\}_{n\geqslant 0}$ of a given MPS $\{P_n\}_{n\geqslant 0}$ belongs to the dual space $\mathscr{P}'$ of $\mathscr{P}$ and whose elements are uniquely defined by

$$\langle u_n, P_k \rangle := \delta_{n,k},\ n,k \geqslant 0,$$

where $\delta_{n,k}$ represents the *Kronecker delta* function. Its first element, u_0, earns the special name of *canonical form* of the MPS. Any element u of $\mathscr{P}'$ can be written in a series of any dual sequence $\{\mathbf{u}_n\}_{n\geqslant 0}$ of a MPS $\{P_n\}_{n\geqslant 0}$ [26]:

$$u = \sum_{n\geqslant 0} \langle u, P_n \rangle\ u_n\ . \tag{2.11}$$

Whenever there is a form $u \in \mathscr{P}'$ such that $\langle u, P_n P_m \rangle = k_n \delta_{n,m}$ with $k_n \neq 0$ for all $n,m \in \mathbb{N}_0$ [26, 27] for some sequence $\{P_n\}_{n\geqslant 0}$, then u is called a *regular* form. The PS $\{P_n\}_{n\geqslant 0}$ is then said to be orthogonal with respect to u and we can assume the system (of orthogonal polynomials) to be monic and the original form u is proportional to u_0.

There is a unique MOPS $\{P_n(x)\}_{n\geqslant 0}$ with respect to the regular form u_0 and it can be characterised by the popular second order recurrence relation (see, for instance, [3])

$$\begin{cases} P_0(x)=1 \quad ; \quad P_1(x)=x-\beta_0 \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x)\ , \quad n\in\mathbb{N}_0, \end{cases} \tag{2.12}$$

where $\beta_n=\frac{\langle u_0,xP_n^2\rangle}{\langle u_0,P_n^2\rangle}$ and $\gamma_{n+1}=\frac{\langle u_0,P_{n+1}^2\rangle}{\langle u_0,P_n^2\rangle}$ for all $n\in\mathbb{N}_0$. In this case, the elements of the corresponding dual sequence $\{u_n\}_{n\geqslant 0}$ are given by

$$u_n=\big(\langle u_0,P_n^2\rangle\big)^{-1}P_nu_0\ ,\ n\geqslant 0. \tag{2.13}$$

When $u\in\mathscr{P}'$ is regular, let Φ be a polynomial such that $\Phi u=0$, then $\Phi=0$ [26, 27].

Given a MPS $\{P_n\}_{n\geqslant 0}$, for each positive integer k, we construct another MPS $\{P_n^{[k]}\}_{n\geqslant 0}$ by

$$P_n^{[k]}(x):=\frac{1}{[n+1;q]_k}D_q^kP_{n+k}(x)\ ,\ n\geqslant 0, \tag{2.14}$$

and their corresponding dual sequences $\{u_n\}_{n\geqslant 0}$ and $\{u_n^{[k]}\}_{n\geqslant 0}$, respectively, are related by [18]

$$D_q^k(u_n^{[k]})=(-1)^k[n+1;q]_k\,u_{n+k}\ ,\ n\geqslant 0. \tag{2.15}$$

Among all the orthogonal polynomial sequences we are primarily interested in the so-called q-classical sequences. The next section is devoted to these sequences, starting with a revision of their main properties (based on the survey [18]) to later on generalise the q-differential equation fulfilled by them.

3. On the q-differential properties of the q-classical polynomials

The q-classical are orthogonal polynomial sequences possessing the so-called Hahn's property (named after [14]), which goes as follows.

Definition 3.1. A MOPS $\{P_n\}_{n\geqslant 0}$ is said to be a D_q-classical sequence, or, in short, q-classical, whenever the sequence $\{P_n^{[1]}\}_{n\geqslant 0}$ defined in (2.14) is itself orthogonal.

Collectively, they share a number of properties. Among them, we formally state those needed for the sequel, which comprises the fact that their corresponding regular forms (or linear functionals) is solution of a first order differential equation; the polynomials fulfil a second order q-differential equation and they can be defined via a Rodrigues type formula. More formally, we have:

Proposition 3.2. [18] *Let $\{P_n\}_{n\geqslant 0}$ be a MOPS and u_0 be the corresponding regular form. The following statements are equivalent:*

(a) *$\{P_n\}_{n\geqslant 0}$ is a D_q-classical sequence.*
(b) *There are two polynomials Φ and Ψ with Φ monic,* $\deg\Phi\leqslant 2$ *and* $\deg\Psi=1$ *such that the regular form u_0 fulfills*

$$D_q\big(\Phi u_0\big)+\Psi u_0=0\ . \tag{3.1}$$

(c) *There are two polynomials* Φ *and* Ψ *with* Φ *monic,* $\deg\Phi \leqslant 2$ *and* $\deg\Psi = 1$ *such that the elements of* $\{P_n\}_{n\geqslant 0}$ *are solutions of the following second order q-differential equation*

$$\mathscr{L}_q[y](x) := \big(\Phi(x) D_q \circ D_{q^{-1}} - \Psi(x) D_{q^{-1}}\big) y(x) = \chi_n y(x)\ ,\ n \geqslant 0, \tag{3.2}$$

where $y(x) = P_n(x)$.

(d) *There is a monic polynomial* Φ *with* $\deg\Phi \leqslant 2$ *and a sequence of nonzero numbers* $\{\vartheta_n\}_{n\geqslant 0}$ *such that*

$$P_n u_0 = \vartheta_n D_q^n \left(\left(\prod_{\sigma=0}^{n-1} q^{-\sigma \deg\Phi} \Phi(q^\sigma x) \right) u_0 \right)\ ,\ n \geqslant 0. \tag{3.3}$$

The eigenvalues χ_n in (3.2) can be written as follows

$$\begin{aligned} \chi_n &= [n]_{q^{-1}} \left([n-1]_q \tfrac{\Phi''(0)}{2} - \Psi'(0) \right) \\ &= \begin{cases} -[n]_{q^{-1}} \Psi'(0) & \text{if } \deg\Phi = 0,1 \\ [n]_{q^{-1}} \left([n-1]_q - \Psi'(0) \right) & \text{if } \deg\Phi = 2 \end{cases}\ ,\ n \geqslant 0. \end{aligned} \tag{3.4}$$

Essentially, if $\{P_n\}_{n\geqslant 0}$ is D_q-classical, then so is the MOPS $\{P_n^{[k]}\}_{n\geqslant 0}$, for each integer $k \geqslant 0$, and this latter MOPS can be characterised in an analogous way.

Corollary 3.3. [18] *Let k be a positive integer. If $\{P_n\}_{n\geqslant 0}$ is D_q-classical, then so is $\{P_n^{[k]}\}_{n\geqslant 0}$ and the corresponding D_q-classical form $u_0^{[k]}$ fulfills*

$$D_q \left(\Phi_k u_0^{[k]} \right) + \Psi_k u_0^{[k]} = 0 \tag{3.5}$$

with

$$\Phi_k(x) = q^{-k \deg\Phi} \Phi(q^k x)\ , \tag{3.6}$$

$$\Psi_k(x) = q^{-k\deg\Phi} \left(\Psi(x) - \sum_{\nu=0}^{k-1} D_q\big(\Phi(q^\nu x)\big) \right) = q^{-k\deg\Phi} \left(\Psi(x) - [k]_q \left(D_{q^k}\Phi \right)(x) \right) \tag{3.7}$$

and the two D_q-classical forms u_0 and $u_0^{[k]}$ are related according to

$$u_0^{[k]} = \zeta_k \left(\prod_{\sigma=0}^{k-1} \Phi_\sigma(x) \right) u_0, \tag{3.8}$$

where ζ_k represents a nonzero constant such that $\left(u_0^{[k]}\right)_0 = 1$, with the convention that $\zeta_0 = 1$. Moreover, the D_q-classical sequence $\{P_n^{[k]}\}_{n\geqslant 0}$ fulfills

$$\Phi_k(x) D_q \circ D_{q^{-1}} \left(P_n^{[k]}(x) \right) - \Psi_k(x) D_{q^{-1}} \left(P_n^{[k]}(x) \right) = \chi_n^{[k]} P_n^{[k]}(x)\ ,\ n \geqslant 0, \tag{3.9}$$

where the eigenvalues $\chi_n^{[k]}$ *are given by*

$$\chi_n^{[k]} = [n]_{q^{-1}} \left(\left([n-1]_q + q^{-k\deg\Phi}[2k]_q \right) \frac{\Phi''(0)}{2} - q^{-k\deg\Phi}\Psi'(0) \right), \; n \geqslant 0.$$

Since the monic polynomial Φ has degree less than or equal to 2, we can write

$$\chi_n^{[k]} = \begin{cases} -[n]_{q^{-1}}\Psi'(0) & \text{if } \deg\Phi = 0 \\ -[n]_{q^{-1}}q^{-k}\,\Psi'(0) & \text{if } \deg\Phi = 1 \\ [n]_{q^{-1}}q^{-2k-1}\Big([n+2k]_q - (1+q\Psi'(0))\Big) & \text{if } \deg\Phi = 2 \end{cases}, \; n \geqslant 0. \quad (3.10)$$

The nonzero numbers ϑ_n, $n \geqslant 0$, presented in the Rodrigues type formula (3.3) are related to the coefficients ζ_k in (3.8) via $\vartheta_n = \dfrac{(-1)^n}{[n]_q!} < u_0, P_n^2 > \zeta_n$, $n \geqslant 0$. For further details we refer to [18], where we can also find (see p. 65 therein) an alternative expression for ϑ_n. More precisely, we can readily deduce from there the following expression:

$$\vartheta_n = q^{-\frac{n(n-1)}{2}\deg\Phi}[n]_q! \left(\prod_{\sigma=1}^{n} \chi_\sigma^{[n-\sigma]} \right)^{-1}, \; n \geqslant 0, \quad (3.11)$$

which obviously can be simplified according to the degree of Φ as follows

$$\vartheta_n = \begin{cases} q^{n(n-1)/2}(\Psi'(0))^n & \text{if } \deg\Phi \leqslant 1, \\ q^{n(n-1)/2} \displaystyle\prod_{\sigma=n-1}^{2n-2} \big([\sigma]_q - \Psi'(0)\big)^{-1} & \text{if } \deg\Phi = 2, \end{cases}, \; n \geqslant 0. \quad (3.12)$$

3.1. **Even order q-differential equations fulfilled by q-classical polynomials.** The main aim now is to explicitly obtain a q-differential operator of arbitrary even order having the q-classical polynomials as eigenfunctions. The methodology taken follows the principles of the algebraic approach, within the principles developed in [26] and in the survey [18]. The basic idea is to transfer the orthogonality properties of a q-classical polynomial sequence to the corresponding dual sequence. For this reason we begin by deriving q-differential relations between the dual sequences of any MPS (not necessarily orthogonal) and the MPS of its derivatives of any arbitrary order.

Lemma 3.4. *Let j and k be two positive integers with $j \leqslant k$ and let $\{P_n^{[k]}\}_{\geqslant 0}$ be the MPS defined in* (2.14). *If $\{u_n^{[k]}\}_{n\geqslant 0}$ and $\{u_n^{[k-j]}\}_{n\geqslant 0}$ are the corresponding dual sequences of $\{P_n^{[k]}\}_{\geqslant 0}$ and $\{P_n^{[k-j]}\}_{\geqslant 0}$, respectively, then they are related by*

$$D_q^j(u_n^{[k]}) = (-1)^j[n+1;q]_j\, u_{n+j}^{[k-j]} \;, \; n \geqslant 0. \quad (3.13)$$

Proof. According to the definition (2.14), the two polynomial sequences $\{P_n^{[k]}\}_{\geqslant 0}$ and $\{P_n^{[j]}\}_{\geqslant 0}$ are related by

$$P_n^{[k]}(x) = \frac{1}{[n+1;q]_j}\left(D_q^j P_{n+j}^{[k-j]}\right)(x)\ ,\ n \geqslant 0.$$

On the grounds of (2.2) , the action of the m^{th} element of the dual sequence $\{u_n^{[k]}\}_{n\geqslant 0}$ of the MPS $\{P_n^{[k]}\}_{\geqslant 0}$ on both sides of the latter identity leads to

$$\delta_{m,n} = \frac{(-1)^j}{[n+1;q]_j}\langle D_q^j(u_m^{[k]})\ ,\ P_{n+j}^{[k-j]}\rangle\ , \quad \text{for any } n,m \geqslant 0.$$

Therefore, for any $m \in \mathbb{N}_0$ we can successively write

$$\begin{aligned} D_q^j(u_m^{[k]}) &= \sum_{n\geqslant 0}\langle D_q^j(u_m^{[k]})\ ,\ P_n^{[k-j]}\rangle\, u_n^{[k-j]} = \sum_{n\geqslant 0}(-1)^j\left(\prod_{\sigma=0}^{j-1}[n-\sigma]_q\right)\delta_{m+j,n}\, u_n^{[k-j]} \\ &= (-1)^j[m+1;q]_j\, u_{m+j}^{[k-j]}, \end{aligned}$$

whence the result. □

When the orthogonality of both $\{P_n\}_{n\geqslant 0}$ and $\{P_n^{[k]}\}_{n\geqslant 0}$, for some fixed integer $k \geqslant 1$, is assumed (which amounts to the same as saying that $\{P_n\}_{n\geqslant 0}$ is q-classical), then, by virtue of the previous lemma, we are able to obtain the following result. In other words, we are able to deduce the explicit expression of a q-differential operator having the q-classical polynomials as eigenfunctions.

Theorem 3.5. *Let k be a positive integer. If $\{P_n\}_{n\geqslant 0}$ is D_q-classical MOPS, then there exist a monic polynomial Φ with $\deg\Phi \leqslant 2$ and a polynomial Ψ of degree one, such that the elements of $\{P_n\}_{n\geqslant 0}$ are solutions of the following $2k$-order q-differential equation*

$$\mathscr{L}_{k;q}[y](x) := \sum_{\nu=0}^{k}\Lambda_{k,\nu}(x;q)\left(D_{q^{-1}}^{k-\nu}\circ D_q^k\, y\right)(q^{-\nu}x) \;=\; \Xi_n(k;q)\, y(x) \tag{3.14}$$

with $y(x) = P_n(x)$ and where

$$\Lambda_{k,\nu}(x;q) = \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} q^{-(k-\nu)}\left(\prod_{\sigma=1}^{\nu}\chi_\sigma^{[k-\sigma]}\right)\left(\prod_{\sigma=0}^{k-\nu-1}q^{-\sigma\deg\Phi}\Phi(q^\sigma x)\right)P_\nu^{[k-\nu]}(x) \tag{3.15}$$

for $\nu = 0,1,\ldots,k$, and

$$\Xi_n(k;q) = \prod_{\sigma=0}^{k-1}\chi_{n-\sigma}^{[\sigma]}\ ,\ n \geqslant 0, \tag{3.16}$$

with $\chi_\mu^{[j]}$ given by (3.10).

Proof. In the light of Corollary 3.3, the q-classical character of the sequence $\{P_n\}_{n\geqslant 0}$ implies the q-classical character of $\{P_n^{[m]}\}_{n\geqslant 0}$, no matter the choice of the nonnegative integer m. This encloses the orthogonality of each of the sequences $\{P_n^{[m]}\}_{n\geqslant 0}$ with respect to $u_0^{[m]}$ and therefore the elements of the corresponding dual sequences, say $\{u_n^{[m]}\}_{n\geqslant 0}$, can be written as $u_n^{[m]} = \left(< u_0^{[m]}, \left(P_n^{[m]}\right)^2 >\right)^{-1} P_n^{[m]} u_0^{[m]}$ for any $n \in \mathbb{N}_0$. Consequently, recalling Lemma 3.4, the assumptions permit to obtain from relation (3.13) the following

$$D_q^{\nu}(P_n^{[k]} u_0^{[k]}) = \varpi_n(k, k-\nu) P_{n+\nu}^{[k-\nu]} u_0^{[k-\nu]} \ , \ n \geqslant 0, \tag{3.17}$$

where

$$\varpi_n(k, k-\nu) = (-1)^{\nu} [n+1; q]_{\nu} \frac{< u_0^{[k]}, \left(P_n^{[k]}\right)^2 >}{< u_0^{[k-\nu]}, \left(P_{n+\nu}^{[k-\nu]}\right)^2 >} \ , \ n \geqslant 0. \tag{3.18}$$

In this case, the corresponding regular forms $u_0^{[\nu]}$, with $\nu = 0, 1, \ldots, k$, satisfy the q-differential equation

$$D_q\left(\Phi_{\nu}(x)\, u_0^{[\nu]}\right) + \Psi_{\nu}(x)\, u_0^{[\nu]} = 0, \tag{3.19}$$

where Φ_{ν} and Ψ_{ν} are given in (3.6) and (3.7), respectively. By recalling (2.4), then, based on the fact that $u_0^{[k-1]}$ satisfy (3.19) with $\nu = k-1$, we successively obtain

$$\begin{aligned}
< \Phi_{k-1}(x)\, u_0^{[k-1]}, P_m^{[k-1]} P_n^{[k]} > \;&=\; -[n+1]_q^{-1} < D_q\left(P_m^{[k-1]}\, \Phi_{k-1}(x)\, u_0^{[k-1]}\right), P_{n+1}^{[k-1]} > \\
&=\; -[n+1]_q^{-1} < P_m^{[k-1]}(q^{-1}x)\, D_q\left(\Phi_{k-1}(x)\, u_0^{[k-1]}\right) \\
&\qquad + q^{-1} D_{q^{-1}}\left(P_m^{[k-1]}\right)(x) \Phi_{k-1}(x)\, u_0^{[k-1]}, P_{n+1}^{[k-1]} > \\
&=\; -[n+1]_q^{-1} < \Big(-\Psi_{k-1}(x) P_m^{[k-1]}(q^{-1}x) \\
&\qquad + q^{-1}\Phi_{k-1}(x)(D_{q^{-1}} P_m^{[k-1]})(x)\Big)\, u_0^{[k-1]}, P_{n+1}^{[k-1]} > \\
&=\; -[n+1]_q^{-1} < u_0^{[k-1]}, \Big(-\Psi_{k-1}(x) P_m^{[k-1]}(q^{-1}x) \\
&\qquad + q^{-1}\Phi_{k-1}(x)(D_{q^{-1}} P_m^{[k-1]})(x)\Big) P_{n+1}^{[k-1]} >, \ m, n \geqslant 0.
\end{aligned}$$

By virtue of the orthogonality of $\{P_n^{[k-1]}\}_{n\geqslant 0}$ with respect to $u_0^{[k-1]}$ and because of the degree of the polynomials Φ_{k-1} and Ψ_{k-1}, we deduce

$$\begin{aligned}
&< \Phi_{k-1}(x)\, u_0^{[k-1]}, P_m^{[k-1]} P_n^{[k]} > \\
&\quad = \begin{cases} 0 & , \quad m \leqslant n-1, \ \ n \geqslant 1, \\ -[n+1]_q^{-1} \lambda_{n,k-1} < u_0^{[k-1]}, (P_{n+1}^{[k-1]})^2 > & , \quad m = n, \ n \geqslant 0, \end{cases}
\end{aligned} \tag{3.20}$$

where

$$\lambda_{n,k-1} = q^{-n}\left([n]_q \frac{\Phi''_{k-1}(0)}{2} - \Psi'_{k-1}(0) \right) \ , \quad n \geqslant 0. \tag{3.21}$$

Now, bearing in mind that $u_0^{[\nu]} = \frac{\zeta_\nu}{\zeta_{\nu-1}} \Phi_{\nu-1}(x) u_0^{[\nu-1]}$ for $\nu = 1, \ldots, k$, where ζ_ν is such that $(u_0^{[\nu]})_0 = 1$, then from (3.20) it follows

$$< u_0^{[k]}, (P_n^{[k]})^2 > = -\frac{\zeta_k}{\zeta_{k-1}} \ [n+1]_q^{-1} \lambda_{n,k-1} < u_0^{[k-1]}, (P_{n+1}^{[k-1]})^2 > \quad , \quad n \geqslant 0.$$

Proceeding by finite induction, from the latter we deduce

$$\frac{< u_0^{[k]}, (P_n^{[k]})^2 >}{< u_0^{[k-\nu]}, (P_{n+\nu}^{[k-\nu]})^2 >} = (-1)^\nu \frac{\zeta_k}{\zeta_{k-\nu}} \Big(\prod_{\sigma=1}^{\nu} [n+\sigma]_q^{-1} \lambda_{n+\sigma-1,k-\sigma} \Big), \ n \geqslant 0,$$

and therefore the coefficients $\varpi_n(k,\nu)$ given in (3.18) can be expressed as well as

$$\varpi_n(k,k-\nu) = \frac{\zeta_k}{\zeta_{k-\nu}} \Big(\prod_{\sigma=1}^{\nu} \lambda_{n+\sigma-1,k-\sigma} \Big) \ , \ n \geqslant 0. \tag{3.22}$$

For this reason, (3.17) becomes

$$D_q^\nu \Big(\ P_n^{[k]} \ u_0^{[k]} \Big) = \frac{\zeta_k}{\zeta_{k-\nu}} \Big(\prod_{\sigma=1}^{\nu} \lambda_{n+\sigma-1,k-\sigma} \Big) P_{n+\nu}^{[k-\nu]} u_0^{[k-\nu]} \ , \ n \geqslant 0.$$

In the light of Corollary 3.3, for each $\nu = 0, 1, \ldots, k$, we have

$$u_0^{[k-\nu]} = \zeta_{k-\nu} \left\{ \prod_{\sigma=0}^{k-\nu-1} \Phi_\sigma(x) \right\} u_0 \ ,$$

allowing to transform the right-hand-side of the precedent equality into

$$D_q^\nu \left(\ P_n^{[k]} \ u_0^{[k]} \right) = \zeta_k \Big(\prod_{\sigma=1}^{\nu} \lambda_{n+\sigma-1,k-\sigma} \Big) \ \Big(\prod_{\sigma=0}^{k-\nu-1} \Phi_\sigma \Big) P_{n+\nu}^{[k-\nu]}(x) \ u_0 \ , \ n \geqslant 0. \tag{3.23}$$

The particular choice of $n = 0$ in (3.23) brings

$$D_q^\nu \left(u_0^{[k]} \right) = \zeta_k \Big(\prod_{\sigma=1}^{\nu} \chi_\sigma^{[k-\sigma]} \Big) \ \Big(\prod_{\sigma=0}^{k-\nu-1} \Phi_\sigma \Big) P_\nu^{[k-\nu]} \ u_0 \ , \quad n \geqslant 0. \tag{3.24}$$

Recalling the definition of $\{P_n^{[\mu]}\}_{n\geqslant 0}$ along with the expression of the eigenvalues $\chi_{n+\sigma}^{[k-\sigma]} = [n+\sigma]_q \lambda_{n+\sigma-1,k-\sigma}$ given in (3.10), the relation (3.23) can also be written as

$$D_q^\nu \Big(\ \big(D_q^k P_{n+k}\big)(x) \ u_0^{[k]} \Big) = \zeta_k \Big(\prod_{\sigma=1}^{\nu} \chi_{n+\sigma}^{[k-\sigma]} \Big) \ \Big(\prod_{\sigma=0}^{k-\nu-1} \Phi_\sigma(x) \Big) \big(D_q^{k-\nu} P_{n+k}\big)(x) \ u_0, \tag{3.25}$$

for any $n \geqslant 0$.

The functional version of the q-Leibniz rule (2.10), when $\nu = k$ the left-hand-side of (3.25) (the same as the one of (3.23)) can be rewritten as

$$D_q^k\left(D_q^k P_{n+k}(x)\, u_0^{[k]}\right) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} q^{-(k-\nu)} \left(D_{q^{-1}}^{k-\nu} \circ D_q^k\, P_{n+k}\right)(q^{-\nu}x)\ \left(D_q^{\nu} u_0^{[k]}\right)\ ,\ n \geqslant 0,$$

which, after (3.24), becomes

$$\begin{aligned} D_q^k\left(D_q^k P_{n+k}(x)\, u_0^{[k]}\right) &= \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} q^{-(k-\nu)} \zeta_k \Big(\prod_{\sigma=1}^{\nu} \chi_{\sigma}^{[k-\sigma]}\Big)\Big(\prod_{\sigma=0}^{k-\nu-1} \Phi_{\sigma}(x)\Big) \\ &\quad \times P_{\nu}^{[k-\nu]}(x)\left(D_{q^{-1}}^{k-\nu} \circ D_q^k\, P_{n+k}\right)(q^{-\nu}x)\, u_0\ ,\ n \geqslant 0. \end{aligned}$$

Incorporating this information into (3.25) with $\nu = k$, leads to

$$\begin{aligned} &\sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} q^{-(k-\nu)}\ \Big(\prod_{\sigma=1}^{\nu} \chi_{\sigma}^{[k-\sigma]}\Big)\ \Big(\prod_{\sigma=0}^{k-\nu-1} \Phi_{\sigma}(x)\Big) P_{\nu}^{[k-\nu]}\ \left(D_{q^{-1}}^{k-\nu} \circ D_q^k\, P_{n+k}\right)(q^{-\nu}x)\, u_0 \\ &= \Big(\prod_{\sigma=1}^{k} \chi_{n+\sigma}^{[k-\sigma]}\Big) P_{n+k}(x)\, u_0\ ,\ n \geqslant 0. \end{aligned}$$

Now, by virtue of the regularity of the form u_0 and after the substitution $n \to n-k$, the latter implies (3.14)-(3.16). □

From the proof of the Theorem 3.5, a generalisation of the Rodrigues-type formula previously recalled in (3.3) and whose nonzero coefficients were later on given in (3.11)-(3.12). More precisely, we have:

Corollary 3.6. *For any positive integer k, a q-classical polynomial sequence $\{P_n\}_{n\geqslant 0}$, orthogonal with respect to the q-classical linear functional u_0, fulfils the relation*

$$D_q^k\Big(\ \Big(\prod_{\sigma=0}^{k-1} \Phi_{\sigma}(x)\Big)\ \left(D_q^k P_n\right)(x)\, u_0\Big) = \Xi_n(k;q) P_n(x)\, u_0\ ,\ n \geqslant 0, \tag{3.26}$$

where $\Phi_{\sigma}(x)$ are the monic polynomials of degree at most 2 defined in (3.6) *and $\Xi_n(k;q)$ are the eigenvalues given by* (3.16).

Proof. The result is an immediate consequence of the relation (3.25) upon the choice of $\nu = k$. □

The aforementioned Rodrigues-type formula (3.3) can be recovered from (3.26) when we set $n = k$.

Despite (3.26) is actually a functional relation, in some cases it triggers a simply q-differential relation. This occurs, for instance, whenever u_0 admits an integral representation via a weight function. In other words, when there is a weight-function $U_q(x)$ as regular as necessary so that

$$\langle u_0, f\rangle = \int_{-\infty}^{+\infty} f(x) U_q(x)\, \mathrm{d}x \tag{3.27}$$

for any $f \in \mathscr{P}$. This weight-function U_q is such that the integral on the RHS of the precedent identity must exist for any polynomial f, and must represent the regular form u_0. For this reason U_q needs to be continuous at the origin or such that the integral

$$\lim_{\varepsilon\to 0+}\int_{\varepsilon}^{1}\frac{U_q(x)-U_q(-x)}{x}\mathrm{d}x$$

exists and, in addition, U_q must fulfill the q-differential equation

$$q^{-1}D_{q^{-1}}\big(\Phi(x)U_q(x)\big)+\Psi(x)U_q(x)=\lambda\ g(x),$$

where λ is an arbitrary complex number and $g\neq 0$ is a locally integrable function with rapid decay representing the null form. The example given by Stieltjes is, for instance, an example of such a function g:

$$g(x)=\begin{cases} 0 & ,\quad x\leqslant 0,\\ \exp(-x^{1/4})\sin(x^{1/4}) & ,\quad x>0.\end{cases}$$

Whenever the q-classical u_0 admits an integral representation of the type (3.27), then from (3.26) we deduce

$$q^{-k}D^k_{q^{-1}}\Bigg(\Big(\prod_{\sigma=0}^{k-1}\Phi_\sigma(x)\Big)\ \Big(D_q^kP_n\Big)(x)\ U_q(x)\Bigg)=\Xi_n(k;q)\ U_q(x)\ P_n(x)\ ,\ n\geqslant 0,\quad (3.28)$$

and it is basically an alternative version of the even order q-differential equation (3.14). In other words, the q-differential operator $\mathscr{L}_k$ given in (3.14) can be as well represented as

$$\mathscr{L}_{k;q}[y](x)=q^{-k}\big(U_q(x)\big)^{-1}D^k_{q^{-1}}\Big(\Big(\prod_{\sigma=0}^{k-1}\Phi_\sigma(x)\Big)\ U_q(x)\big(D_q^ky(x)\big)\Big)\qquad (3.29)$$

provided that $U_q(x)\neq 0$ for any admissible x.

Combining the two latter expressions, we can formally state the following result.

Corollary 3.7. *Let k be a positive integer. The q-classical sequence $\{P_n\}_{n\geqslant 0}$, orthogonal with respect to the q-classical linear functional u_0 admitting an integral representation of the type* (3.27)*, satisfies*

$$\mathscr{L}_{k;q}[P_n](x)=q^{-k}\big(U_q(x)\big)^{-1}D^k_{q^{-1}}\Bigg(\Big(\prod_{\sigma=0}^{k-1}\Phi_\sigma(x)\Big)\ U_q(x)\big(D_q^kP_n(x)\big)\Bigg)=\Xi_n(k;q)\ P_n(x),\qquad (3.30)$$

for any $n\geqslant 0$, where $\Xi_n(k;q)$ are given in (3.16) *and $\Phi_\sigma(x)$ are the monic polynomials of degree at most 2 defined in* (3.6).

Naturally, the choice of $n=k$ leads to the Rodrigues formula. In the last section, particular q-classical examples are considered. These include the Al-Salam-Carlitz, the Stiletjes-Wigert and the Little q-Jacobi polynomials.

3.2. **Integer powers of the second-order q-differential operator.** Any q-classical sequence $\{P_n\}_{n\geqslant 0}$ fulfills the second order q-differential equation (3.2). This amounts to the same as saying that the elements of $\{P_n\}_{n\geqslant 0}$ are eigenfunctions of the second order q-derivative operator $\mathscr{L}_q := \Phi(x)D_q \circ D_{q^{-1}} - \Psi(x)D_{q^{-1}}$.

As a consequence, the elements of $\{P_n\}_{n\geqslant 0}$ are necessarily eigenfunctions of any integer composite power of the aforementioned operator $\mathscr{L}_q$. In other words, the elements of $\{P_n\}_{n\geqslant 0}$ are solutions of the even order q-differential equation in y

$$\mathscr{L}_q^k[y](x) = (\chi_n)^k y(x)\ ,\ n \geqslant 0, \tag{3.31}$$

where $\mathscr{L}_q^k[y](x) := \mathscr{L}_q[\mathscr{L}_q^{k-1}[y](\zeta)](x)$.

This raises the natural question of obtaining an explicit expression for $\mathscr{L}_q^k$. Besides, it seems natural to investigate the existence of a linear relation between the two q-derivative operators $\mathscr{L}_q^k$ and $\mathscr{L}_{k;q}$, both of them of order $2k$.

As the elements of a q-classical polynomial sequence are eigenfunctions of both of those operators, these problems may be addressed through the comparison between the eigenvalues $(\chi_n)^k$ given by (3.4) and $\Xi_n(k)$ given by (3.16).

Depending on the degree of the monic polynomial Φ, the eigenvalues $\Xi_n(k;q) = \prod_{\sigma=0}^{k-1}\chi_{n-\sigma}^{[\sigma]}$ admit different expressions, exclusively depending on those of $\chi_n^{[k]}$ given by (3.10). More precisely, noticing the readily verified identities

$$[n-\sigma]_{q^{-1}} = q^{\sigma}\big([n]_{q^{-1}} - [\sigma]_{q^{-1}}\big) \text{ and } [n-\sigma]_{q^{-1}}\,[n+\sigma]_q = q^{\sigma}\big\{[n]_{q^{-1}}[n]_q - [\sigma]_{q^{-1}}[\sigma]_q\big\},$$

for any $\sigma \in \{0,1,\ldots,n-1\}$ and $n \in \mathbb{N}_0$, we can write

$$\Xi_n(k;q) = \begin{cases} \displaystyle\prod_{\sigma=0}^{k-1}\Big(-[n-\sigma]_{q^{-1}}\, q^{-\sigma\deg\Phi}\Psi'(0)\Big) & \text{if} \quad \deg\Phi < 2, \\ \displaystyle\prod_{\sigma=0}^{k-1}\Big([n-\sigma]_{q^{-1}}\, q^{-2\sigma-1}\big(z+[n+\sigma]_q\big)\Big) & \text{if} \quad \deg\Phi = 2, \end{cases}$$

as

$$\Xi_n(k;q) = \begin{cases} \Big(-q^{\frac{(k-1)}{2}(1-\deg\Phi)}\ \Psi'(0)\Big)^k \displaystyle\prod_{\sigma=0}^{k-1}\big([n]_{q^{-1}} - [\sigma]_{q^{-1}}\big) & \text{if } \deg\Phi < 2, \\ q^{-\frac{k(k+1)}{2}}\displaystyle\prod_{\sigma=0}^{k-1}\Big([n]_{q^{-1}}\big(z+[n]_q\big) - [\sigma]_{q^{-1}}\big(z+[\sigma]_q\big)\Big) & \text{if } \deg\Phi = 2, \end{cases} \tag{3.32}$$

where $z = -(1+q\Psi'(0))$.

Notice that $\Psi'(0) \neq [j]_q$ for any $j \in \mathbb{N}_0$, otherwise the q-classical form u_0 fulfilling (3.1) would have an undetermined sequence of moments, contradicting the assumptions. For this reason, $\Xi_n(k;q) \neq 0$ for any integer $n \geqslant k$.

Regarding the nature of the coefficients, the analysis should be split in two different cases, depending on whether the degree of the polynomial Φ is lower or equal than 1 or when it equals 2. While in the first case $(\chi_n)^k$ and $\Xi_n(k)$ are bridged by a set of coefficients mimicking the behaviour of the so-called q-Stirling numbers, the second case requires the introduction of the q-analogs of the Jacobi-Stirling numbers. For this reason, in the next section we introduce and analyse the so-called q-Jacobi-Stirling numbers, after revising the foremost important properties of the q-Stirling numbers.

4. BRIDGING THE q-FACTORIALS AND THE POWERS OF A NUMBER

The problem at issue encompasses the relation between powers of a (complex) number x and a corresponding q-factorial, and therefore the introduction of q-analogs of the Stirling numbers along with their modifications (also known as Jacobi-Stirling numbers).

The sequence of factorials $\left\{\prod_{i=0}^{n-1}\big(x-[i]_q\big)\right\}_{n\geqslant 0}$, alike the sequence of monomials $\{x^n\}_{n\geqslant 0}$, forms a basis of $\mathscr{P}$. For this reason, there exist two sets of coefficients performing the change of basis. In other words, there is a pair of sequences of numbers bridging each of these q-factorials of a complex number x and its corresponding powers – these are the so-called q-Stirling numbers, here denoted as $(c_q(n,k),S_q(n,k))_{0\leqslant k\leqslant n}$. Precisely, for each non-negative integer n, the pair $(c_q(n,k),S_q(n,k))_{0\leqslant k\leqslant n}$ (see [12]) is such that

$$\prod_{i=0}^{n-1}\big(x-[i]_q\big)=\sum_{k=0}^{n}(-1)^{n-k}c_q(n,k)x^k \quad \text{and} \quad x^n=\sum_{k=0}^{n}S_q(n,k)\prod_{i=0}^{k-1}\big(x-[i]_q\big). \tag{4.1}$$

A recurrence relation fulfilled by this pair of q-Stirling numbers can be readily deduced from the identity $x\prod_{i=0}^{n-1}\big(x-[i]_q\big)=\prod_{i=0}^{n}\big(x-[i]_q\big)+[n]_q\prod_{i=0}^{n-1}\big(x-[i]_q\big)$, which goes as follows:

$$c_q(n+1,k+1)=c_q(n,k)+[n]_q c_q(n,k+1), \qquad 0\leqslant k\leqslant n, \tag{4.2}$$

$$S_q(n+1,k+1)=S_q(n,k)+[k+1]_q S_q(n,k+1), \qquad 0\leqslant k\leqslant n, \tag{4.3}$$

with $c_q(n,k)=S_q(n,k)=0$, if $k\notin\{1,\ldots,n\}$, and $c_q(0,0)=S_q(0,0)=1$, $n\geqslant 0$. Naturally and as expected, the standard pair of Stirling numbers arise from the pair $(c_q(n,k),S_q(n,k))_{0\leqslant k\leqslant n}$ as $q\to 1$.

As it was previously explained, when Φ has degree 2, the bridge between $\Xi_n(k:q)$ and the powers of χ_n entails the problem of establishing inverse relations between the following sequence of factorials, depending on the prescribed parameter z (possibly complex),

$$\left\{\prod_{i=0}^{n-1}\Big(x-[i]_q\big(z+[i]_{q^{-1}}\big)\Big)\right\}_{n\geqslant 0} \tag{4.4}$$

and the sequence of monomials. Straightforwardly, the set (4.4) forms a basis of $\mathscr{P}$. For this reason, there exists a pair of coefficients, $(\mathrm{Jc}_n^k(z;q),\mathrm{JS}_n^k(z;q))_{0\leqslant k\leqslant n}$ say, performing the change of basis. This pair of coefficients to which we will refer as the *q-Jacobi-Stirling numbers* (that we could legitimately call as z-modified q-Stirling numbers)

$$\prod_{i=0}^{n-1}\Big(x-[i]_q\big(z+[i]_{q^{-1}}\big)\Big)=\sum_{k=0}^{n}(-1)^{n-k}\mathrm{Jc}_n^k(z;q)\,x^k, \tag{4.5}$$

$$x^n=\sum_{k=0}^{n}\mathrm{JS}_n^k(z;q)\prod_{i=0}^{k-1}\Big(x-[i]_q\big(z+[i]_{q^{-1}}\big)\Big). \tag{4.6}$$

Remark 4.1. A y-version of the q-Stirling numbers were introduced in [17, Definition 3] in order to compute the moments of some rescaled Al-Salam-Chihara polynomials. It is easy to see that the latter generalized Stirling numbers are a rescaled version of the Jacobi-Stirling numbers. For example, the y-version of the q-Stirling numbers of the second kind $S_q(n,k,y)$ are defined by

$$x^n=\sum_{k=0}^{n}S_q(n,k,y)\prod_{j=0}^{k-1}(x-[j]_q(1-yq^{-j})). \tag{4.7}$$

It follows from (4.6) and (4.7) that $\mathrm{JS}_n^k(z;q)=S_q(n,k,y^{-1})y^{n-k}(1-q^{-1})^k$ with $z=\frac{y-1}{1-q^{-1}}$.

Bearing in mind the identity fulfilled by the factorials in (4.4)

$$\begin{aligned} x\prod_{i=0}^{k-1}\Big(x-[i]_q\big(z+[i]_{q^{-1}}\big)\Big) &= \prod_{i=0}^{k}\Big(x-[i]_q\big(z+[i]_{q^{-1}}\big)\Big) \\ &\quad +[k]_q\Big(z+[k]_{q^{-1}}\Big)\prod_{i=0}^{n-1}\Big(x-[i]_q\big(z+[i]_{q^{-1}}\big)\Big), \end{aligned}$$

we readily deduce the following triangular recurrence relations fulfilled by these two sets of numbers

$$\mathrm{Jc}_{n+1}^{k+1}(z;q)=\mathrm{Jc}_n^k(z;q)+[n]_q\Big(z+[n]_{q^{-1}}\Big)\mathrm{Jc}_n^{k+1}(z;q),\ 0\leqslant k\leqslant n, \tag{4.8}$$

$$\mathrm{JS}_{n+1}^{k+1}(z;q)=\mathrm{JS}_n^k(z;q))+[k+1]_q\Big(z+[k+1]_{q^{-1}}\Big)\mathrm{JS}_n^{k+1}(z;q),\ 0\leqslant k\leqslant n, \tag{4.9}$$

with $\mathrm{Jc}_n^k(z;q)=\mathrm{JS}_n^k(z;q)=0$, if $k\notin\{1,\dots,n\}$, and $\mathrm{Jc}_0^0(z;q)=\mathrm{JS}_0^0(z;q)=1$, $n\geqslant 0$.

The pair (4.5)-(4.6) represents a pair of inverse relations and we have

$$\sum_{j\geqslant 0}\mathrm{Jc}_n^j(z;q)\mathrm{JS}_j^k(z;q)=\sum_{j\geqslant 0}\mathrm{JS}_n^j(z;q)\mathrm{Jc}_j^k(z;q)=\delta_{n,k},\ n,k\geqslant 0,$$

as it can be derived directly from their definition.

4.1. **Properties of the q-Jacobi-Stirling numbers.** An explicit expression for the q-Jacobi-Stilring numbers of second kind $\mathrm{Jc}_n^k(z;q)$ can be obtained directly from the Newton interpolation formula for a polynomial f that we next recall.

Lemma 4.2 (Newton's interpolation formula)**.** *Let $b_0, b_1, \dots, b_{n-1}$ be distinct numbers. Then, for any polynomial f of degree less than or equal to n we have*

$$f(x) = \sum_{j=0}^{n} \left(\sum_{r=0}^{j} \frac{f(b_r)}{\prod_{0\le k\le j,\, k\ne r}(b_r - b_k)} \right) \prod_{i=0}^{j-1}(x - b_i). \tag{4.10}$$

From (4.10), with $f(x) = x^n$ and $b_k = [k]_q([k]_{q^{-1}} + z)$, and (4.6) we derive immediately the following result:

Proposition 4.3. *For $0 \le j \le n$ we have*

$$JS_n^j(z;q) = \sum_{r=0}^{j} (-1)^{j-r} \frac{q^{-\binom{r}{2} - r(j-r)} \left([r]_q([r]_{q^{-1}} + z)\right)^n}{[r]_q![j-r]_q! \prod\limits_{0\le k\le j,\, k\ne r} (z + [k+r]_{q^{-1}})}. \tag{4.11}$$

Remark 4.4. By letting $q \to 1$, the pair $(\mathrm{Jc}_n^k(z;q), \mathrm{JS}_n^k(z;q))_{0\leqslant k\leqslant n}$ reduces to the Jacobi-Stirling numbers in [6, 10, 29] and the relation (4.11) reduces to [6, (4.4)].

For each integer $k \geqslant 1$, it follows from the recurrence relation (4.9) that

$$\sum_{n\ge k} \mathrm{JS}_n^k(z;q)x^n = \frac{x}{1 - [k]_q([k]_{q^{-1}} + z)} \sum_{n\ge k-1} \mathrm{JS}_n^{k-1}(z;q)x^n$$

and therefore

$$\prod_{i=1}^{k} \frac{x}{1 - [i]_q([i]_{q^{-1}} + z)x} = \sum_{n\ge k} \mathrm{JS}_n^k(z;q)x^n. \tag{4.12}$$

Theorem 4.5. *Let n, k be positive integers with $n \ge k$. The Jacobi-Stirling numbers $JS_n^k(z,q)$ and $Jc_n^k(z,q)$ are polynomials in z of degree $n-k$ with coefficients in $\mathbb{N}[q, q^{-1}]$. Moreover, if*

$$JS_n^k(z;q) = a_{n,k}^{(0)}(q) + a_{n,k}^{(1)}(q)z + \cdots + a_{n,k}^{(n-k)}(q)z^{n-k}, \tag{4.13}$$

$$Jc_n^k(z;q) = b_{n,k}^{(0)}(q) + b_{n,k}^{(1)}(q)z + \cdots + b_{n,k}^{(n-k)}(q)z^{n-k}, \tag{4.14}$$

then

$$a_{n,k}^{(n-k)} = S_q(n,k), \quad b_{n,k}^{(n-k)} = c_q(n,k).$$

Proof. This follows from (4.8) and (4.9) by induction on n. □

Remark 4.6. When $z = 0$, by analogy with the ordinary central factorial numbers (cf. [10]), we can define the corresponding q-Jacobi-Stirling numbers as q-central factorial numbers of even indices :

$$U_q(n,k) = \mathrm{JS}_n^k(0;q), \quad V_q(n,k) = \mathrm{Jc}_n^k(0;q). \tag{4.15}$$

Therefore, the following recurrences hold true :

$$U_q(n+1,k+1) = U_q(n,k) + [k+1]_q[k+1]_{q^{-1}} U_q(n,k+1),\ 0 \leqslant k \leqslant n, \tag{4.16}$$

$$V_q(n+1,k+1) = V_q(n,k) + [n]_q[n]_{q^{-1}} V_q(n,k+1),\ 0 \leqslant k \leqslant n, \tag{4.17}$$

with $U_q(n,k)=V_q(n,k)=0$, if $k\notin\{1,\dots,n\}$, and $U_q(0,0)=V_q(0,0)=1$, $n\geqslant 0$.

Some value of the q-Jacobi-Stirling numbers of first kind are as follows:

$$\begin{aligned}
\mathrm{Jc}_n^1(z;q) &= \prod_{k=1}^{n-1}[k]_q(z+[k]_{q^{-1}}), \quad Jc_n^{(n)}(z,q)=1,\\
\mathrm{Jc}_3^2(z;q) &= (3+q+q^{-1})+(2+q)z,\\
\mathrm{Jc}_4^2(z;q) &= (q^{-3}+5q^{-2}+11q^{-1}+q^3+11q+5q^2+15)\\
&\quad +\left(2q^3+14q+8q^2+2q^{-2}+7q^{-1}+15\right)z+\left(4q+q^3+3+3q^2\right)z^2,\\
\mathrm{Jc}_4^3(z;q) &= \left(3q^{-1}+6+q^2+3q+q^{-2}\right)+(3+2q+q^2)z.
\end{aligned}$$

Some values of the q-Jacobi-Stirling numbers of second kind are as follows:

$$\begin{aligned}
&\mathrm{JS}_n^1(z;q)=(1+z)^{n-1}, \quad JS_n^{(n)}(z;q)=1,\\
&\mathrm{JS}_3^2(z;q)=(3+q+q^{-1})+(2+q)z,\\
&\mathrm{JS}_4^2(z;q)=(9+q^{-2}+q^2+5q+5q^{-1})+(11+3q^{-1}+2q^2+8q)z+(3q+3+q^2)z^2.
\end{aligned}$$

4.2. **Combinatorial interpretation of the q-Jacobi-Stirling numbers.** For any positive integer n, we consider the set of two copies of the integers:

$$[n]_2=\{1_1,1_2,\dots,n_1,n_2\}.$$

Definition 4.7. A Jacobi-Stirling k-partition of $[n]_2$ is a partition of $[n]_2$ into $k+1$ subsets $B_0,B_1,\dots B_k$ of $[n]_2$ satisfying the following conditions:

(1) there is a zero block B_0, which may be empty and cannot contain both copies of any $i\in[n]$,
(2) $\forall j\in[k]$, each nonzero block B_j is not empty and contains the two copies of its smallest element and does not contain both copies of any other number.

We shall denote the zero block by $\{\dots\}_0$. For example, the partition

$$\pi=\{\{2_2,5_1\}_0,\{1_1,1_2,2_1\},\{3_1,3_2,4_2\},\{4_1,5_2\}\}$$

is not a Jacobi-Stirling 3-partition of $[5]_2$, while

$$\pi'=\{\{2_2,5_1\}_0,\{1_1,1_2,2_1\},\{3_1,3_2\},\{4_1,4_2,5_2\}\}$$

is a Jacobi-Stirling 3-partition of $[5]_2$. We order the blocks of a partition in increasing order of their minimal elements. By convention, the zero block is at the first position.

Definition 4.8. An inversion of type 1 of π is a pair (b_1,B_j), where $b_1\in B_i$ for some i $(1\le i<j)$ and $b_1>c_1$ for some $c_1\in B_j$. An inversion of type 2 of π is a pair (b_2,B_j), where $b_2\in B_i$ for some i $(0\le i<j)$ and $b_2>c_2$ for some $c_2\in B_j$ and $b_1\notin B_j$, where a_i means integer a with subscript $i=1,2$. Let $\mathrm{inv}_i(\pi)$ be the number of inversions of π of type $i=1,2$ and set $\mathrm{inv}(\pi)=\mathrm{inv}_2(\pi)-\mathrm{inv}_1(\pi)$. Let $\Pi(n,k,i)$ denote the set of Jacobi-Stirling k-partitions of $[n]_2$ such that the zero-block contains i numbers with subscript1.

For example, for the Jacobi-Stirling 2-partitions of $[3]_2$ we have the following tableau

JS 2-partitions of $[3]_2$	inv_1	inv_2	inv
$\{\}_0,\ \{1_1,1_2,3_2\},\ \{2_1,2_2,3_1\}$	0	0	0
$\{\}_0,\ \{1_1,1_2,3_1\},\ \{2_1,2_2,3_2\}$	1	0	-1
$\{3_2\}_0,\ \{1_1,1_2,3_1\},\ \{2_1,2_2\}$	1	1	0
$\{3_2\}_0,\ \{1_1,1_2\},\ \{2_1,2_2,3_1\}$	0	1	1
$\{2_2\}_0,\ \{1_1,1_2,2_1\},\ \{3_1,3_2\}$	0	0	0
$\{2_1\}_0,\ \{1_1,1_2,2_2\},\ \{3_1,3_2\}$	0	0	0
$\{3_1\}_0,\ \{1_1,1_2,3_2\},\ \{2_1,2_2\}$	0	1	1
$\{3_1\}_0,\ \{1_1,1_2\},\ \{2_1,2_2,3_2\}$	0	0	0

Thus,

$$\sum_{\pi\in\Pi(3,2,0)} q^{\mathrm{inv}(\pi)} = 3+q+q^{-1} \quad \text{and} \quad \sum_{\pi\in\Pi(3,2,1)} q^{\mathrm{inv}(\pi)} = 2+q.$$

Theorem 4.9. *For any positive integers n and k and* $0\le i\le n-k$ *we have*

$$a_{n,k}^{(i)}(q) = \sum_{\pi\in\Pi(n,k,i)} q^{inv(\pi)}.$$

Proof. We prove by induction on $n\ge 1$. The identity is clearly true for $n=1$. Assume $n>1$. We divide $\Pi(n,k,i)$ into three subsets as follows.

- $\{n_1,n_2\}$ forms a single block, the enumerative polynomial is $a_{n-1,k-1}^{(i)}(q)$.
- n_1 is in the zero-block and n_2 is in a non zero-block, the enumerative polynomial is $(1+q+\cdots+q^{k-1})a_{n-1,k}^{(i-1)}(q)$.
- n_1 is in a non zero-block B_j, so (n_1,B_i) is an inversion of type 1 for any $i=j+1,\dots,k$, and n_2 is in any other block B_l ($l\neq j$), so (n_2,B_i) is an inversion for any $i=l+1,\dots,k$ and $l\neq j$, so the enumerative polynomial is

$$\left(\sum_{j=1}^{k} q^{-(k-j)}\left(\sum_{l=0}^{j-1} q^{k-l-1} + \sum_{l=j+1}^{k} q^{k-l}\right)\right)a_{n-1,k}^{(i)}(q) = [k]_q[k]_{q^{-1}}a_{n-1,k}^{(i)}(q).$$

Summing up, we have

$$a_{n,k}^{(i)}(q) = a_{n-1,k-1}^{(i)}(q) + [k]_q a_{n-1,k}^{(i-1)}(q) + [k]_q[k]_{q^{-1}}a_{n-1,k}^{(i)}(q).$$

This is equivalent to (4.9). □

For a permutation σ of $[n]_0 := [n]\cup\{0\}$ (resp. $[n]$) and for $j\in[n]_0$ (resp. $[n]$), denote by $\mathrm{Orb}_\sigma(j) = \{\sigma^\ell(j) : \ell\ge 1\}$ the orbit of j and $\min(\sigma)$ the set of its positive cyclic minima, i.e.,

$$\min(\sigma) = \{j\in[n] : j = \min(\mathrm{Orb}_\sigma(j)\cap[n])\}.$$

Definition 4.10. Given a word $w = w(1)\dots w(\ell)$ on the finite alphabet $[n]$, a letter $w(j)$ is a *record* of w if $w(k) > w(j)$ for every $k\in\{1,\dots,j-1\}$. We define $\mathrm{rec}(w)$ to be the number of records of w and $\mathrm{rec}_0(w) = \mathrm{rec}(w)-1$.

For example, if $w = \mathbf{5}\mathbf{7}\mathbf{4}86\mathbf{2}3\mathbf{1}9$, then the records are $5,4,2,1$. Hence $\mathrm{rec}(w) = 4$.

Definition 4.11. Let $\mathscr{P}(n,k,i)$ be the set of all pairs (σ,τ) such that σ is a permutation of $[n]_0$, τ is a permutation of $[n]$, and both have k cycles.

i) 1 and 0 are in the same cycle in σ;
ii) among their nonzero entries, σ and τ have the same cycle minima;
iii) $\mathrm{rec}_0(w) = i$, where $w = \sigma(0)\sigma^2(0)\dots\sigma^l(0)$ with $\sigma^{l+1}(0) = 0$.

As Foata and Han [8], we define the B-code of a permutation σ of $[n]$ based on the decomposition of each permutation as a product of disjoint cycles. For a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ and each $i = 1,2,\dots,n$ let $k := k(i)$ be the smallest integer $k \geq 1$ such that $\sigma^{-k}(i) \leq i$. Then, define

$$\text{B-code}(\sigma) = (b_1,b_2,\dots,b_n) \quad \text{with} \quad b_i := \sigma^{-k(i)}(i) \quad (1 \leq i \leq n).$$

We define the sorting index for permutation σ of $[n]$ by $\mathrm{Sor}(\sigma) = \sum_{i=1}^n (i - b_i)$, while for a permutation σ of $[n]_0$ we define the modified sorting index by $\mathrm{Sor}_0(\sigma) = \sum_{i=1}^n (i - b'_i)$, where $b'_i = b_i$ if $\sigma^{-1}(i) \neq 0$ and $b'_i = i$ if $\sigma^{-1}(i) = 0$. Finally, for any pair (σ,τ) in $\mathscr{E}_{n,k}^{(i)}$ we define the statistic

$$\mathrm{Sor}(\sigma,\tau) = \mathrm{Sor}(\tau) - \mathrm{Sor}_0(\sigma).$$

Theorem 4.12. *We have*

$$b_{n,k}^{(i)}(q) = \sum_{(\sigma,\tau)\in\mathscr{P}(n,k,i)} q^{Sor(\sigma,\tau)}.$$

Proof. We proceed by induction on $n \geq 1$. The case $n = 1$ is clear. Assume that $n > 1$. We divide $\mathscr{P}(n,k,i)$ into three parts:

(i) the pairs (σ,τ) such that $\sigma^{-1}(n) = n$. Then n forms a cycle in both σ and τ and the enumerative polynomial is clearly $b_{n-1,k-1}^{(i)}(q)$.
(ii) the pairs (σ,τ) such that $\sigma^{-1}(n) = 0$. We can construct such pairs starting from a pair (σ',τ') in $\mathscr{P}(n-1,k,i-1)$ as follows: we insert n in σ' as image of 0, i.e., $\sigma(0) = n$, $\sigma(n) = \sigma'(0)$ and $\sigma(i) = \sigma'(i)$ for $i \neq 0,n$, and then we insert n in τ') by choosing an $j \in [n]$ and define $\tau(j) = n$, $\tau(n) = \tau(j)$ and $\tau(l) = \tau'(l)$ for $l \neq j,n$. Clearly, the enumerative polynomial is $[n-1]_q b_{n-1,k}^{(i-1)}(q)$.
(iii) the pairs (σ,τ) such that $\sigma^{-1}(n) \notin \{0,n\}$. We can construct such pairs by first choosing an ordered pair (σ',τ') in $\mathscr{P}(n-1,k,i)$ and then inserting n in σ' and τ', respectively. Clearly, the corresponding enumerative polynomial is given by $[n-1]_q[n-1]_{q^{-1}} b_{n-1,k}^{(i)}(q)$.

Summing up, we get the following equation:

$$b_{n,k}^{(i)}(q) = b_{n-1,k-1}^{(i)}(q) + [n-1]_q b_{n-1,k}^{(i-1)}(q) + [n-1]_q[n-1]_{q^{-1}} b_{n-1,k}^{(i)}(q). \tag{4.18}$$

By (4.5) and (4.14), it is easy to see that the coefficients $b_{n,k}^{(i)}(q)$ satisfy the same recurrence. $\square$

For example, for $n=3$ and $k=2$ the pairs (σ,τ) and the associated statistics are as follows:

(σ,τ)	$\mathrm{rec}_0(\sigma)$	B_0-code σ	B-code τ	$\mathrm{Sor}(\tau)$	$\mathrm{Sor}_0(\sigma)$	$\mathrm{Sor}(\sigma,\tau)$
$(01)(23)$, $(1)(23)$	0	(1,2,2)	(1,2,2)	1	1	0
$(01)(23)$, $(13)(2)$	0	(1,2,2)	(1,2,1)	2	1	1
$(012)(3)$, $(12)(3)$	0	(1,1,3)	(1,1,3)	1	1	0
$(013)(2)$, $(13)(2)$	0	(1,2,1)	(1,2,1)	2	2	0
$(013))(2)$, $(1)(23)$	0	(1,2,1)	(1,2,2)	1	2	-1
$(031)(2)$, $(1)(23)$	1	(0,2,3)	(1,2,2)	1	1	0
$(031)(2)$, $(1)(23)$	1	(0,2,3)	(1,1,3)	1	1	0
$(021)(3)$, $(1)(23)$	1	(0,2,3)	(1,2,1)	2	1	1

Thus,

$$\sum_{(\sigma,\tau)\in\mathscr{P}(3,2,0)} q^{\mathrm{Sor}(\sigma,\tau)} = 3+q+q^{-1}, \qquad \sum_{(\sigma,\tau)\in\mathscr{P}(3,2,1)} q^{\mathrm{Sor}(\sigma,\tau)} = 2+q.$$

4.3. **A symmetric generalization of Jacobi-Stirling numbers.** As suggested by Richard Askey (private communication in 2013), it is natural to consider the pair of connection coefficients $\{(S_{z,w}(n,k), s_{z,w}(n,k))\}_{n\geq k\geq 0}$ satisfying

$$x^n = \sum_{k=0}^{n} S_{z,w}(n,k) \prod_{i=0}^{k-1}(x-(i+z)(i+w)), \tag{4.19}$$

$$\prod_{i=0}^{n-1}(x-(i+z)(i+w)) = \sum_{k=0}^{n} s_{z,w}(n,k)x^k. \tag{4.20}$$

It is readily seen that we have the following recurrence relation

$$S_{z,w}(n+1,k+1) = S_{z,w}(n,k) + (z+k+1)(w+k+1)S_{z,w}(n,k+1), \tag{4.21}$$

$$s_{z,w}(n+1,k+1) = s_{z,w}(n,k) - (z+n)(w+n)s_{z,w}(n,k+1), \tag{4.22}$$

with $S_{z,w}(n,k) = s_{z,w}(n,k) = 0$, if $k \notin \{1,\ldots,n\}$, and $S_{z,w}(0,0) = s_{z,w}(0,0) = 1$, $n \geqslant 0$.

It is clear that $S_{z,w}(n,k)$ are symmetric polynomials in z and w with non negative integral coefficients and

$$\sum_{n\geq k} S_{z,w}(n,k)x^n = \prod_{i=1}^{k} \frac{x}{1-(i+z)(i+w)x}.$$

When $w=0$ these numbers reduce to Jacobi-Stirling numbers. Mimicking the arguments in [10], we can prove similar results for the symmetric variants. Hence, we will juste state the combinatorial interpretations of these numbers and omit the proofs.

Definition 4.13. A double signed k-partition of $[n]_2 = \{1_1, 1_2, \ldots, n_1, n_2\}$ is a partition of $[n]_2$ into $k+2$ subsets $(B_0, B_0', B_1, \ldots, B_k)$ such that

(1) there are two distinguishable zero blocks B_0 and B_0', any of which may be empty;

(2) there are k indistinguishable nonzero blocks, all nonempty, each of which contains both copies of its smallest element and does not contain both copies of any other number;
(3) each zero block does not contain both copies of any number and B_0' may contain only numbers with subscript 2.

Let $\Pi(n,k)$ be the set of double signed k-partitions of $[n]_2$. For $\pi \in \Pi(n,k)$ denote by $s(\pi)$ (resp. $t(\pi)$) the number of integers with subscript 1 (reps. 2) in B_0 (reps. B_0') of π.

Theorem 4.14. *The polynomial $S_{z,w}(n,k)$ is the enumerative polynomial of $\Pi(n,k)$ with z enumerating the numbers with subscript 1 in B_0 and w enumerating the numbers with subscript 2 in B_0', i.e.,*

$$S_{z,w}(n,k) = \sum_{\pi \in \Pi(n,k)} z^{s(\pi)} w^{t(\pi)}.$$

For example, all the double signed k-partitions of $[2]_2$ ($1 \leq k \leq 2$) with the corresponding weight are

$k=0$: $\pi = \{\{1_1,2_1\}_0, \{1_2,2_2\}_0'\}$; with weight z^2w^2;
$k=1$: $\pi_1 = \{\{2_2\}_0, \{\}_0', \{1_1,1_2,2_1\}\}$ with weight 1;
$\pi_2 = \{\{2_1\}_0, \{\}_0', \{1_1,1_2,2_2\}\}$ with weight z;
$\pi_3 = \{\}_0, \{2_2\}_0', \{1_1,1_2,2_1\}\}$ with weight w;
$\pi_4 = \{\{1_1\}_0, \{1_2\}_0', \{2_1,2_2\}\}$ with weight zw;
$\pi_5 = \{\{2_1\}_0, \{2_2\}_0', \{1_1,1_2\}\}$ with weight zw.
$k=2$: $\pi = \{\{\}_0, \{\}_0', \{1_1,1_2\}, \{2_1,2_2\}\}$ with weight 1.

Thus, by the above theorem, we have

$$x^2 = S_{z,w}(2,0) + S_{z,w}(2,1)(x-zw) + S_{z,w}(2,2)(x-zw)(x-(z+1)(w+1)),$$

where $S_{z,w}(2,0) = z^2w^2, \quad S_{z,w}(2,1) = 1+z+w+2zw, \quad S_{z,w}(2,2) = 1.$

Definition 4.15. Let $\mathscr{P}_0(n,k)$ be the set of all pairs (σ,τ) of permutations of $[n]_0$ such that σ and τ both have k cycles and

i) 1 and 0 are in the same cycle in σ and τ.
ii) Among their nonzero entries, σ and τ have the same set of cycle minima.

Let $\mathrm{rec}_0(w_\sigma)$ be the number of left-to right minima of the word $w_\sigma = \sigma(0)\sigma^2(0)\dots\sigma^l(0)$ with $\sigma^{l+1}(0) = 0$.

Theorem 4.16. *The polynomial $(-1)^{n-k}s_{z,w}(n,k)$ is the enumerative polynomial of pairs (σ,τ) in $\mathscr{P}_0(n,k)$ with z enumerating the $rec_0(w_\sigma)$ and w enumerating $rec_0(w_\tau)$, i.e.,*

$$(-1)^{n-k}s_{z,w}(n,k) = \sum_{(\sigma,\tau)\in\mathscr{P}_0(n,k)} z^{rec_0(w_\sigma)} w^{rec_0(w_\tau)}.$$

Let $(a)_n = a(a+1)\dots(a+n-1)$ for $n \geq 1$ and $(a)_0 = 1$. The Wilson polynomials (see [19]) are more natural and simply expanded in the basis of polynomials in x^2 given by $\{(a+\imath x)_n(a-\imath x)_n\}_{n\geqslant 0}$, where $\imath^2=-1$. So, it is legitimate to refer to the pair of set of numbers $\{(w(n,k),W(n,k))\}_{0\leqslant k\leqslant n}$ performing the following change of basis

$$(a+\imath x)_n(a-\imath x)_n=\sum_{k=0}^{n} w(n,k)x^{2k}, \tag{4.23}$$

$$x^{2n}=\sum_{k=0}^{n} W(n,k)(a+\imath x)_k(a-\imath x)_k, \tag{4.24}$$

as the *Wilson numbers of first and second kind*. Then, by Newton's interpolation formula we have

$$(-1)^{n-k}w(n,k)=e_k(a^2,(a+1)^2,\dots,(a+n-1)^2), \tag{4.25}$$

$$W(n,k)=\sum_{r=0}^{k}\frac{(-1)^{n-r}(a+r)^{2n}}{r!(k-r)!(2a+r)_r(2a+2r+1)_{k-r}}, \tag{4.26}$$

where $e_k(x_1,\dots,x_n)$ denotes the k-th elementary symmetric polynomial of $x_1,\dots,x_n$. It is readily seen that

$$W(n+1,k+1)=W(n,k)-(a+k+1)^2W(n,k+1), \tag{4.27}$$

$$w(n+1,k+1)=w(n,k)+(a+n)^2w(n,k+1), \tag{4.28}$$

with $W(n,k)=w(n,k)=0,\ $ if $k\notin\{1,\dots,n\}$, and $W(0,0)=w(0,0)=1$, $n\geqslant 0$.

It is easy to derive from Theorems 4.14 and 4.16 that

$$(-1)^{n-k}W(n,k)=\sum_{\pi\in\Pi(n,k)} a^{s(\pi)+t(\pi)}, \tag{4.29}$$

$$s_{z,w}(n,k)=\sum_{(\sigma,\tau)\in\mathscr{P}_0(n,k)} a^{\mathrm{rec}_0(w_\sigma)+\mathrm{rec}_0(w_\tau)}. \tag{4.30}$$

Remark 4.17. It is worth to notice that the change of basis between $\{(a+\imath x)_n(a-\imath x)_n\}_{n\geqslant 0}$ and $\{(x^2+(a-1)^2)^{2n}\}_{n\geqslant 0}$ is performed by the aforementioned pair of Jacobi-Stirling numbers $\{\big((-1)^{n+k}\mathrm{js}_{n+1}^{k+1}(2\alpha),\mathrm{JS}_{n+1}^{k+1}(2\alpha)\big)\}_{0\leqslant k\leqslant n}$ - see [23, Remark 3.9]. Indeed, the latter pair of numbers is associated to the linear change of basis in the vector space of polynomials in the variable x^2 performed by the Kontorovich-Lebedev integral transform after a slight modification of the kernel, as described in [23].

Of course, we can formally work out a q-version of the numbers $S_{z,w}(n,k)$ and $s_{z,w}(n,k)$ by mimicking our approach to the q-Jacobi-Stirling numbers. However, from analytical point of view, an appropriate q-analogue of Wilson numbers should be the connection coefficients between the basis $\{x^n\}_{n\geq 0}$ and the Askey-Wilson basis $\{(az,a/z;q)_n\}_{n\geq 0}$ with $x=(z+1/z)/2$ [16, Chap. 15 and 16]. More precisely, let $(a,b;q)_n=(a;q)_n(b;q)_n$, where $(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1})$ for $n \geq 1$ and $(a;q)_0 = 1$, then the Askey-Wilson numbers of the second and first kind are defined by

$$x^n = \sum_{k=0}^{n} W_q(n,k)(az, a/z;q)_k, \tag{4.31}$$

$$(az, a/z;q)_n = \sum_{k=0}^{n} w_q(n,k)x^k. \tag{4.32}$$

where $x = (z+1/z)/2$. By Newton's formula we have

$$W_q(n,k) = \frac{1}{2^n}\sum_{j=0}^{k} q^{k-j^2}a^{-2j}\frac{(q^j a + q^{-j}/a)^n}{(q, q^{-2j+1}/a^2;q)_j(q, q^{2j+1}a^2;q)_{k-j}}. \tag{4.33}$$

Since $(az, a/z;q)_n = \prod_{k=0}^{n-1}(1-2axq^k + a^2q^{2k})$, we derive from (4.31) and (4.32) that

$$W_q(n+1,k+1) = -\frac{1}{2aq^k}W_q(n,k) + \frac{1+a^2q^{2k}}{2aq^k}W_q(n,k+1), \tag{4.34}$$

$$w_q(n+1,k+1) = -2aq^n w_q(n,k) + (1+a^2q^{2n})w_q(n,k+1). \tag{4.35}$$

Let $T_q(n,k) = (-2a)^k q^{\binom{k}{2}} W_q(n,k)$. Then $T_q(n,k)$ are polynomials in a and q with non-negative coefficients satisfying the recurrence

$$T_q(n+1,k+1) = T_q(n,k) + (1+a^2q^{2k})T_q(n,k+1). \tag{4.36}$$

These numbers are related with the Askey-Wilson operator and their combinatorial model is quite different from that of Wilson numbers. We plan to investigate this topic in a future work.

5. POWERS OF THE SECOND ORDER q-DIFFERENTIAL OPERATOR ASSOCIATED WITH THE q-CLASSICAL POLYNOMIALS.

Having set the foremost important properties of the introduced q-Jacobi-Stirling numbers along with a combinatorial interpretation, we now turn back to the original problem: to give an explicit expression for the composite powers of the q-differential operator $\mathscr{L}_q$ associated to the q-classical polynomials and given in (3.2).

Proposition 5.1. *Let k be a positive integer. The kth composite power of the operator $\mathscr{L}_q$ of the q-classical differential expression* (3.2) *is given by*

$$\mathscr{L}_q^k[f](x) = \begin{cases} \displaystyle\sum_{j=0}^{k} S_{q^{-1}}(k,j)\, q^{(\deg\Phi-1)\frac{j(j-1)}{2}}\,(-\Psi'(0))^{k-j}\,\mathscr{L}_{j;q}[f](x) & \text{if} \quad \deg\Phi = 0,1, \\ \displaystyle\sum_{j=0}^{k} JS_k^j(z;q^{-1})\, q^{\frac{j(j+1)}{2}-k}\,\mathscr{L}_{j;q}[f](x) & \text{if} \quad \deg\Phi = 2, \end{cases} \tag{5.1}$$

which holds for any $f \in \mathscr{P}$, where $\mathscr{L}_{k;q}$ is the operator of the even order q-differential equation (3.14) *and $z = -(1+\Psi'(0))$.*

Proof. We begin by relating the kth power of the eigenvalues χ_n given by (3.4) and the eigenvalues $\Xi_n(k;q)$ given by (3.32) for any integer $n \geqslant 0$. In the light of equalities (4.1) with $x = [n]_{q^{-1}}$, when $\deg\Phi < 2$ we have

$$(\chi_n)^k = \sum_{j=0}^{k} S_{q^{-1}}(k,j)\, q^{(\deg\Phi-1)\frac{j(j-1)}{2}} \left(-\Psi'(0)\right)^{k-j}\, \Xi_n(j;q)\ ,\ n \geqslant 0,$$

whereas, based on the identity (4.6) with $x = [n]_{q^{-1}}\left(z + [n]_q\right)$, when $\deg\Phi = 2$ we obtain

$$(\chi_n)^k = \sum_{j=0}^{k} \mathrm{JS}_k^j(z;q^{-1}) q^{\frac{j(j+1)}{2}-k}\, \Xi_n(j;q)\ ,\ n \geqslant 0.$$

According to Proposition 3.2, the q-classical polynomials are solutions of the $2k$-order q-differential equation (3.31) where the operator $\mathscr{L}_q$ and the eigenvalues χ_n are those given by (3.2). On the other hand, Theorem 3.5 ensures the q-classical polynomials to be solutions of the $2k$-order q-differential equation (3.14), whose corresponding eigenvalues $\Xi_n(k;q)$ are given in (3.32). The result now follows from the fact that any q-classical sequence forms a basis of $\mathscr{P}$. $\square$

In a similar manner, for a given positive integer k we can express the eigenvalues $\Xi_n(k;q)$ in terms of $(\chi_n)^k$ for any integer $n \geqslant 0$. Indeed, based on (4.1) and (4.5) with $x = [n]_{q^{-1}}$ and $x = [n]_{q^{-1}}\left([n]_q - A\right)$, respectively, and after the substitution $q \to q^{-1}$, the equalities

$$\Xi_n(k;q) = \begin{cases} q^{(1-\deg\Phi)\frac{k(k-1)}{2}} \displaystyle\sum_{j=0}^{k} s_{q^{-1}}(k,j) \left(-\Psi'(0)\right)^{k-j} (\chi_n)^j & \text{if} \quad \deg\Phi = 0,1, \\ q^{-\frac{k(k+1)}{2}} \displaystyle\sum_{j=0}^{k} \mathrm{Jc}_k^j(z;q^{-1})(k,j)\, q^j\, (\chi_n)^j & \text{if} \quad \deg\Phi = 2, \end{cases}$$

hold for any integer $n \geqslant 0$.

Evoking the same arguments as those in the proof of Proposition 5.1, the fact that any q-classical sequence forms a basis of $\mathscr{P}$, allows to deduce from the latter equalities the following relation

$$\mathscr{L}_{k;q}[f](x) = \begin{cases} q^{(1-\deg\Phi)\frac{k(k-1)}{2}} \displaystyle\sum_{j=0}^{k} s_{q^{-1}}(k,j) \left(-\Psi'(0)\right)^{k-j} \mathscr{L}_q^j[f](x) & \text{if} \quad \deg\Phi = 0,1, \\ \displaystyle\sum_{j=0}^{k} \mathrm{Jc}_k^j(z;q^{-1})\, q^{j-\frac{k(k+1)}{2}}\, \mathscr{L}_q^j[f](x) & \text{if} \quad \deg\Phi = 2, \end{cases} \tag{5.2}$$

that is valid for any polynomial $f \in \mathscr{P}$.

All the q-classical polynomials were studied in [18]. The produced classification was leaded by the properties of the polynomial Φ in the q-differential equation (3.2) fullfilled by these polynomial sequences. The possible cases are listed below, according to the degree of the monic polynomial Φ:

TABLE 1. The q-classical polynomials.

deg Φ	q-**classical MOPS**
0	Al-Salam Carlitz polynomials · Discrete q-Hermite polynomials
1	Big q-Laguerre · q-Meixner · Wall q-polynomials q-Laguerre polynomials · Little q-Laguerre polynomials q-Charlier I polynomials
2 (with double root)	Alternative q-Charlier polynomials · Stieltjes-Wigert q-polynomials
2 (with 2 single roots)	Little q-Jacobi polynomials · q-Charlier II polynomials Generalized Stieltjes-Wigert q-polynomials · Big q-Jacobi Bi-generalized Stieltjes-Wigert q-polynomials

Among all the q-classical polynomials, we single out some of the families in order to illustrate the results here obtained. Namely, we make use of the q-Stirling numbers along with the q-Jacobi-Stirling numbers to obtain the expressions of the kth composite power of the operator $\mathscr{L}_q$ in (3.2). Among other things, it permits to obtain differently many expressions for even order q-differential equations fulfilled by the q-classical polynomials.

Example 5.2. The MOPS of the **Al Salam-Carlitz** q**-polynomials** $\{P_n(x;a;q)\}_{n\geqslant 0}$ satisfy a recurrence relation of the type (2.12) with $\beta_n=(1+a)q^n$ and $\gamma_{n+1}=aq^n(q^{n+1}-1)$. They are orthogonal with respect to the regular linear functional u_0 fulfilling (3.1) with $\Phi(x)=1$, $\Psi(x)=(a(q-1))^{-1}(x-(1+a))$, provided that $a\neq 0$. They can be written via a terminating basic hypergeometric series as follows

$$P_n(x;a;q)=(-a)^n q^{\binom{n}{2}}\,{}_2\phi_1\left(\begin{matrix} q^{-n},x^{-1} \\ 0\end{matrix};q,\frac{qx}{a}\right)=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q(-a)^{n-k}q^{\binom{n-k}{2}}(x^{-1};q)_k\,x^k$$

for any $n\in\mathbb{N}_0$. They are positive definite whenever $a<0$ and $0<q<1$ or when $a>0$ and $q>1$. The case where $a=-1$ and $0<q<1$ corresponds to the so-called Discrete q-Hermite polynomials. We further notice that this MOPS is also Appell with respect to the q-difference operator D_q, so is to say that $P_n^{[k]}(x;a;q)=P_n(x;a;q)$.

In the light of Theorem 3.5, for any positive integer k, the **Al Salam-Carlitz** q**-polynomials** are eigenfunctions of the q-differential operator

$$\mathscr{L}_{k;q}[y](x):=\sum_{\nu=0}^{k}\begin{bmatrix} k\\ \nu\end{bmatrix}_{q^{-1}}q^{-(k-\nu)}\left([\nu]_{q^{-1}}!\right)(a(1-q))^{-\nu}P_\nu^{[k-\nu]}(x)\left(D_{q^{-1}}^{k-\nu}\circ D_q^k\,y\right)(q^{-\nu}x).$$

They are also eigenfunctions of any integral composite power of the q-differential operator $\mathscr{L}_q=D_q\circ D_{q^{-1}}-(a(q-1))^{-1}(x-(1+a))D_{q^{-1}}$. Following Proposition 5.1, we can write for each $k \geqslant 1$,

$$\mathscr{L}_q^k[y](x) = \sum_{j=0}^{k} S_{q^{-1}}(k,j)\, q^{-\frac{j(j-1)}{2}}\, (a(q-1))^{-(k-j)}\, \mathscr{L}_{j;q}[y](x).$$

However, when $0<q<1$ and $a<0$, then u_0 admits an integral representation via a weight function U_q, as described in (3.27), where

$$U_q(x) = \begin{cases} 0 & \text{if} \quad x<aq^{-1} \text{ or } x>q^{-1}, \\ K(qx;q)_\infty\,(a^{-1}qx;q)_\infty & \text{if} \quad aq^{-1} \leqslant x \leqslant q^{-1}, \end{cases}$$

with $K = \left(\int_{aq^{-1}}^{q^{-1}} (qt;q)_\infty (a^{-1}qt;q)_\infty dt\right)^{-1}$. Thus, according to Corollary 3.7, in this case we may alternatively write

$$\mathscr{L}_{k;q}[y](x) := \left(q^k (qx;q)_\infty\,(a^{-1}qx;q)_\infty\right)^{-1} D_{q^{-1}}^k \Big((qx;q)_\infty\,(a^{-1}qx;q)_\infty \big(D_q^k y(x)\big)\Big)$$

which is valid for any $y \in \mathscr{P}$, provided that $0<q<1$ and $a<0$. When $k=1$, the latter reduces to the well known q-Sturm-Liouville equation for the Al Salam-Carlitz q-polynomials (see, for instance, [16, p.472]).

Example 5.3. The monic Stieltjes-Wigert q-polynomials $\{P_n(x;q)\}_{n\geqslant 0}$ are orthogonal with respect to u_0 fulfilling (3.1) with $\Phi(x)=x^2$ and $\Psi(x) = -(q-1)^{-1}\{x-q^{-3/2}\}$, and satisfying a recurrence relation of the type (2.12) with $\beta_n = (1+q-q^{n+1})q^{-2n-3/2}$ and $\gamma_{n+1} = (1-q^{n+1})q^{-4(n+1)}$, $n \geqslant 0$. They can be represented as well via a terminating basic hypergeometric function

$$P_n(x;q) = (-1)^n q^{-\frac{n}{2}(2n+1)}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ 0 \end{array};q,-q^{n+3/2}x\right) = \sum_{k=0}^{n} \frac{(-1)^{n+k} q^{k(k+\frac12)-n(n+\frac12)}}{(q;q)_{n-k}} x^k,$$

for any $n \in \mathbb{N}_0$. In addition, they are such that $P_n^{[k]}(x;q) = q^{-2nk}P_n(q^{2k}x;q), n \in \mathbb{N}_0$. When $0<q<1$, u_0 is positive definite, and admits an integral representation via a weight-function U_q, satisfying (3.27), which is given by

$$U_q(x) = \begin{cases} 0 & , \quad x \leqslant 0 \\ \sqrt{\frac{q}{2\pi \ln q^{-1}}}\exp\left(-\frac{\ln^2 x}{2\ln q^{-1}}\right) & , \quad x>0. \end{cases}$$

In the light of Corollary 3.7, for $0<q<1$, the Stieltjes-Wigert q-polynomials are eigenfunctions of

$$\mathscr{L}_{k;q}[y](x) = q^{-k}\exp\left(\frac{\ln^2 x}{2\ln q^{-1}}\right) D_{q^{-1}}^k\left(x^{2k}\ \exp\left(-\frac{\ln^2 x}{2\ln q^{-1}}\right)\big(D_q^k y(x)\big)\right).$$

According to Theorem 3.5, the latter operator can be written as follows

$$\mathscr{L}_{k;q}[y](x) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} \alpha_{k,\nu:q} x^{2k-2\nu} P_\nu^{[k-\nu]}(x) \left(D_{q^{-1}}^{k-\nu} \circ D_q^k\, y\right)(q^{-\nu}x)$$

with

$$\alpha_{k,\nu:q} = q^{-(k-\nu)} \ \big([\nu]_q!\big) \Big(\prod_{\sigma=1}^{\nu} q^{-2k+\sigma} \Big([2k-\sigma]_q + \frac{1}{q-1}\Big)\Big)$$

Finally, by means of the q-Jacobi-Stirling numbers of second kind, for any positive integer k, the kth composite power of the q-differential operator

$$\mathscr{L}_q := x^2 D_q \circ D_{q^{-1}} + (q-1)^{-1}\{x - q^{-3/2}\} D_{q^{-1}}$$

is given by the formula

$$\mathscr{L}_q^k[f](x) = \sum_{j=0}^{k} \mathrm{JS}_k^j((q-1)^{-1};q^{-1})\, q^{\frac{j(j+1)}{2}-k} \mathscr{L}_{j;q}[f](x)\ ,\ \forall f \in \mathscr{P}.$$

Example 5.4. The monic **Little q-Jacobi polynomials** $\{P_n(x;a,b|q)\}_{n\geqslant 0}$ satisfy the q-differential equation (3.2) with $\Phi(x) = x(x - b^{-1}q^{-1})$ and $\Psi(x) = (abq^2(q-1))^{-1}\{(1-abq^2)x + aq - 1\}$ and can be written as

$$\begin{aligned} P_n(x;a,b|q) &= \frac{(-1)^n q^{\binom{n}{2}}(aq;q)_n}{(abq^{n+1};q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} ; q, qx\right) \\ &= \frac{(aq;q)_n}{(abq^{n+1};q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^{n-k}(abq^{n+1};q)_k}{(aq;q)_k} q^{\binom{n-k}{2}} x^k, \ \ n \in \mathbb{N}_0. \end{aligned}$$

Hence, for each positive integer k, we have

$$P_n^{[k]}(x;a,b|q) = P_n(x;aq^k, bq^k|q),\ n \in \mathbb{N}_0.$$

They form a MOPS with respect to the regular linear functional u_0 fulfilling (3.1) provided that the parameters a,b satisfy $a,b,ab \neq q^{-(n+2)}$, for $n \in \mathbb{N}_0$. The linear functional u_0 is positive definite when one of the following two conditions is realised [18]

$$\begin{aligned} &0 < q < 1,\ 0 < a < q^{-1},\ b \in]-\infty, q^{-1}[- \{0\}, \\ &q > 1,\ a > q^{-1},\ b \in]-\infty, 0[\cup]q^{-1}, +\infty[. \end{aligned}$$

We refer to [18, 19] for further details, but we highlight some results from [18, pp. 100-102].

When $0 < q < 1$, $b \in]-\infty, 1[- \{0\}$ and $a := q^{\alpha-1}$ with $\alpha > 0$, the regular form u_0 admits an integral representation via weight-function $U_q(x)$ which is given by

$$U_q(x) = \begin{cases} 0 & ,\ \ x \leqslant 0 \ \text{ or } \ x \geqslant q^{-1} \\ Kx^{\alpha-1}\dfrac{(qx;q)_\infty}{(bqx;q)_\infty} & ,\ \ 0 < x < q^{-1}, \end{cases} \qquad K^{-1} = \int_0^{q^{-1}} t^{\alpha-1}\frac{(qt;q)_\infty}{(bqt;q)_\infty}\mathrm{d}t.$$

On the other hand, when $q > 1$, there are two situations to consider. In any of the cases $a := q^{\alpha-1}$ with $\alpha > 0$. The first one occurs when $b \geqslant 1$ where the regular form u_0 admits an integral representation via weight-function $U_q(x)$ which is given by

$$U_q(x) = \begin{cases} 0 & , \quad x \leqslant 0 \ \text{ or } \ x \geqslant b^{-1} \\ K x^{\alpha-1} \dfrac{(bx;q^{-1})_\infty}{(x;q^{-1})_\infty} & , \quad 0 < x < b^{-1}, \end{cases}$$

with

$$K = \left(\int_0^{b^{-1}} t^{\alpha-1} \frac{(bt;q^{-1})_\infty}{(t;q^{-1})_\infty} \mathrm{d}t \right)^{-1}.$$

The second one arises when $b < 0$, so that we have

$$U_q(x) = \begin{cases} 0 & , \quad x \leqslant b^{-1} \ \text{ or } \ x \geqslant 0 \\ K|x|^{\alpha-1} \dfrac{(bx;q^{-1})_\infty}{(-|x|;q^{-1})_\infty} & , \quad b^{-1} < x < 0, \end{cases}$$

with

$$K = \left(\int_0^{|b|^{-1}} t^{\alpha-1} \frac{(|b|t;q^{-1})_\infty}{(-t;q^{-1})_\infty} \mathrm{d}t \right)^{-1}.$$

Notice that u_0 can also be represented by a discrete measure. Namely, for $0 < q < 1$, we have

$$u_0 = \frac{(aq;q)_\infty}{(abq^2;q)_\infty} \sum_{k\in\mathbb{N}_0} \frac{(bq;q)_k}{(q;q)_k} (aq)^k \delta_{q^k}, \ \text{ with } \ |a| < q^{-1},\ ab \neq q^{-(m+2)},\ m \in \mathbb{N}_0,$$

whereas, when $q > 1$, we can write

$$u_0 = \frac{(a^{-1}q^{-1};q^{-1})_\infty}{(a^{-1}b^{-1}q^{-2};q^{-1})_\infty} \sum_{k\in\mathbb{N}_0} \frac{(b^{-1}q^{-1};q^{-1})_k}{(q^{-1};q^{-1})_k} (a^{-1}q^{-1})^k \delta_{b^{-1}q^{-(k+1)}},$$

with $|a| > q^{-1},\ ab \neq q^{m-2},\ m \in \mathbb{N}_0$.

In the light of Corollary 3.7, whenever the regular Little q-Jacobi linear functional u_0 admits an integral representation via a weight function $U_q(x)$, the Little q-Jacobi polynomials $\{P_n(x;a,b|q)\}_{n\geqslant 0}$ are eigenfunctions of

$$\mathscr{L}_{k;q}[y](x) = q^{-k} \big(U_q(x)\big)^{-1} D^k_{q^{-1}} \left(\left(\prod_{\sigma=0}^{k-1} x(x - b^{-1}q^{-(\sigma+1)}) \right) U_q(x) \big(D^k_q y(x)\big) \right).$$

In any case, following Theorem 3.5, the latter operator can be written as follows

$$\mathscr{L}_{k;q}[y](x) = \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q^{-1}} \alpha_{k,\nu:q} \left(\prod_{\sigma=0}^{k-1} x(x - b^{-1}q^{-(\sigma+1)}) \right) \left(D^{k-\nu}_{q^{-1}} \circ D^k_q\, y \right)(q^{-\nu}x)$$

with

$$\alpha_{k,\nu:q} = q^{-(k-\nu)} \left(\prod_{\sigma=1}^{\nu} [\sigma]_{q^{-1}} q^{-2k+\sigma} \left([2k-\sigma]_q + \frac{1}{q-1}\left(1 - (abq)^{-1}\right) \right) \right)$$

Finally, by means of the q-Jacobi-Stirling numbers of second kind, for any positive integer k, the kth composite power of the q-differential operator

$$\mathscr{L}_q := x(x-b^{-1}q^{-1})D_q \circ D_{q^{-1}} - \left((abq^2(q-1))^{-1}\{(1-abq^2)x+aq-1\}\right) D_{q^{-1}}$$

is given by the formula

$$\mathscr{L}_q^k[f](x) = \sum_{j=0}^{k} \mathrm{JS}_k^j(z;q^{-1})\, q^{\frac{j(j+1)}{2}-k}\, \mathscr{L}_{j;q}[f](x)$$

where $z = -(1+q\Psi'(0)) = \frac{1}{q-1}\left(1-(abq)^{-1}\right)$.

With regard to the other q-classical polynomials listed in Table 1, similar results to those in the previous examples can be obtained by taking into consideration the information in [18, 19].

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