Learning from actuarial science: approaches to collectively optimising warranty policies

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Abstract

This research starts from the comparison between warranty and insurance in their coverages, policies, data and, particularly, policy optimisation techniques. Based on abundant literature in related areas, the result of this comparison indicates that warranty policy optimisation can be improved by considering the application of the portfolio theory, dependence modelling and risk measures that are widely used in the actuarial science and the financial discipline. In the following chapters, Chapter 1 introduces the Importance of this research and lists its aim and objectives. Chapter 2 mainly conducts a critical and comprehensive literature review relating to warranty management and actuarial science and summarised the knowledge gaps identified. Chapter 3 establishes a collective warranty policy optimisation framework, with the benefits of the modern portfolio theory borrowed from the actuarial and financial disciplines and copulas from the probability and statistics. With progressing of this research, the disadvantage of the symmetric risk measure, variance, is uncovered in dealing with the extreme events. Chapter 4 proposes using two of the downside risk measures used in the financial discipline, Value-at-Risk and Conditional Value-at-Risk, into the optimisation of warranty policy and a new portfolio optimisation framework of warranty optimisation based on copulas. Chapter 5 investigates the interplay among the hardware, software and users of individual products under different scenarios relating to the warranty claims. Considering such an interplay, it then develops a more comprehensive framework for warranty policy optimisation. This fits the trend that that more and more products can be considered as a system composed of three subsystems: hardware, software and user subsystems and considers that the existing warranty policy optimisation methods in the literature merely focus on products composed of hardware systems. Even though the above chapters have developed warranty policy optimisation frameworks collectively and comprehensively, this research can also be improved in many aspects. As such, in Chapter 6, the sale volume modelling, renewing warranty policy optimisation and copula selection are discussed. Chapter 7 wraps up the research and discusses future research.

A list of papers that published during my PhD study

Journal papers

Luo, M., & Wu, S. (2018). A mean-variance optimisation approach to collectively pricing warranty policies. *International Journal of Production Economics,* 196, 101-112.

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Conference papers

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Abbreviations

1-D — one-dimensional

2-D — two-dimensional

BMS — Bonus-Malus System

CVaR — Conditional Value-at-Risk

FBNR — Failed But Not Reported

GLM — Generalised Linear Model

HPP — Homogeneous Poisson Process

IBNR — Incurred But Not Report

KKT — Karush-Kuhn-Tucker

MCMC — Markov Chain Monte Carlo

MEMS — Microelectro-mechanical systems

NFBR — Non-Failed But Reported

NFRW — Non-renewing Free Replacement Warranty

NHPP — Non-homogeneous Poisson Process

OEM — Original Equipment Manufacturer

RF — Radio frequency

SRGM — Software Reliability Growth Model

TDC — Tail Dependence Coefficient

VaR — Value-at-Risk

WEF — Warranty Execution Function

WSN — Wireless Sensor Network

Table 1: Notation table

$X_{k,i}$	Cost of the ith warranty claim of product k
$N_k(t)$	Number of warranty claims of product k within time interval $(0,t)$
$S_k(t)$	Total cost of warranty claims of product k within time interval $(0,t)$
P_k	Price of product k
T_k	Warranty length of product k
$\boldsymbol{P}$	Vector of prices of products
$\boldsymbol{T}$	Vector of lengths of warranties
λ_k	Parameter of the claim arrival process of product k
μ_k	Expected cost per claim of product k
σ_k	Standard deviation of $X_{k,i}$
c_k	Fixed manufacturing cost of product k
$M_k(P_k, T_k)$	Sales volume of product k when the warranty price is P_k and the warranty length is T_k
$\mathbf{M}$	Vector of all sales volume, $\mathbf{M} = [M_1, ..., M_n]'$, where $[.]'$ denotes the transpose of a matrix/vector,
$\omega_k(P_k, T_k)$	Profit of product k when the warranty price is P_k and the warranty length is T_k
$\Omega(\mathbf{P}, \mathbf{T})$	Total profit of the manufacturer when the warranty prices is $\mathbf{P} = [P_1, ..., P_n]'$ and the warranty length is $\mathbf{T} = [T_1, ..., T_n]'$.
$\lambda_1(t)$	intensity function of a NHPP of hardware failures
$\lambda_2(t)$	intensity function of a NHPP of software failures
M	total sales volume of the product
$F_3(t)$	cumulative distribution function of NFBR claims
$p_1(t)$	probability of a claim being made at time t, given that a hardware failure has occurred
p_2	probability that a software failure results in a warranty claim
c_1	cost on a claim due to a hardware failure during the warranty period
c_{s1}	cost on a claim due to a software failure during the useful lifetime
c_{s2}	cost on producing and releasing a software update/patch
$\Delta\tau$	constant update release interval under time-based policy
m	number of updates released within product lifetime under time-based policy
N	threshold of number of reported failures for releasing updates under number-based policy
k	expected number of updates released within product lifetime under number-based policy
c_{31}	administration cost per NFBR claim
c_{32}	expected cost on fixing the cause of an NFBR claim
T_r	software release time

Chapter 1

Introduction

Warranty plays an important role in consumer and commercial transactions. It is essentially offered with most durable products to promote product sales. Warranty is a contractual obligation incurred by a manufacturer (vendor or seller) in connection with the sale of a product (Blischke, 1993). The manufacturer should compensate customers through offering repair or replacement service in the event of premature failures of the warranted items or items' inability to perform its intended function (Karim & Suzuki, 2005; Wu, 2014b). A typical warranty transaction is: a consumer pays the warranty price when purchasing an item (items); then the manufacturer provides free repair or replacement service for the failures of the item(s) that occur during the warranty period. Warranty is a tool to assure consumers' satisfaction with product performance over the warranty period (Liao, 2016). It can be seen as a guarantee or promise about the reliability of a product and may be legally mandated or market driven: the European Union (EU) passed legislation requiring a two-year warranty for all products sold in Europe (Wu, 2014b). There are also several warranty acts enacted in the U.S. over the last 100 years (UCC, Magnusson Moss Warranty Act, Tread Act, for example) (Wu, 2014b; Murthy & Djamaludin, 2002).

Warranty expense is an important part of the manufacturer's operating expense. In the manufacturing industries, warranty incurs huge amount of costs, for example, the total cost of worldwide warranty claims of U.S.-based firms was $26.4 billion in 2015; and the balance of their total warranty reserves was $43.35 billion at the end of 2015 (WarrantyWeek, 2016b). This is especially the case for some industries, for example, the automotive industry is the most warranty-intensive of all: warranty claims paid by

the total automotive OEM (original equipment manufacturer) in the entire U.S.-based manufacturers was $10,097 million in 2015 (WarrantyWeek, 2016b).

Warranty expense belongs to the operational expenses of the warranty provider who needs to cover labour and parts costs for repairs within the warranty period. Hence, unanticipated failures of sold items within warranty coverage may cause losses, which may include economic losses and reputation damage, to the warranty provider. Therefore, accurately forecasting the expected liability due to warranty claims is important in warranty related modelling, which involves modelling of the claim frequency and severity (Wu, 2012).

Warranty is a type of insurance. Insurance is a tool through which the risk of a loss can be transferred from an encountered entity to another in exchange for payment. The principle of insurance can be explained as an insurer raising funds from a group of similar policyholders to pay the policyholders who suffer losses within the group. The money raised from the policyholders is called premium, and the money paid for the claims is called claim amount/size/severity. An insurance company should hold the balance between the total premium and the total claim amount. The expected total claim amount is always treated as the expected liability of insurance companies. Insurance companies normally wish to forecast the expected total claim amount more accurately (Czado et al., 2012). Warranty and insurance share commonalities in modelling. For example, the compound Poisson process plays an important role in both warranty modelling and insurance modelling.

The discipline that applies mathematical and statistical methods to assess risk and model related issues in insurance, financial and other industries is named as actuarial science. In this research, warranty modelling means *applying mathematical and statistical methods to model the mechanism of processes and issues in the warranty area to support warranty data analysis, warranty policy optimisation and related activities*. It has been noticed that actuarial science has attracted more researchers and has been better established than warranty modelling. Learning from actuarial science is a good approach to improving warranty modelling.

1.1 Research objectives

According to the literature in actuarial science and warranty management, one can find that there is a bulk of research questions that has been studied in actuarial science but not well researched in warranty management. For example, variable dependence in insurance data analysis, such as dependence between claim types, claim amount, and claim frequency, has been well studied in insurance data analysis and insurance policy optimisation (Yau et al., 2003; De Jong et al., 2008; Frees & Valdez, 2008; Mikosch, 2009; Zhao & Zhou, 2012; Shi & Valdez, 2014a), but it has less investigated in the literature of warranty management.

As such, the research mainly aims to fulfil the following research objectives:

- To compare the advances of the actuarial science and warranty management in order to find knowledge gaps: what can warranty researchers learn from the actuarial science?
- To develop a collective warranty modelling method to optimise the warranty policies when the dependence among the warranty claims of the products produced by the same manufacturer is considered;
- To improve the risk measure method applied in the collective warranty policies optimisation framework when the downside risk is considered;
- To develop a comprehensive warranty modelling method to optimise the warranty policies when the interplay among the subsystems, such hardware and software subsystems, of a product are considered.

The needs for investigating the above objectives include the following reasons and are not exhaustedly listed.

1. *Avoidance of information loss.* In warranty data analysis, Jung & Bai (2007); Wu (2014a) has modelled the dependence between age and usage. However, little has been done on modelling the dependence among the number of warranty claims,

warranty claim cost and even warranty claims of different products, and even on modelling the dependence among the warranty claims of the products produced by the same manufacturer. While in the current competitive market, it has always been vital to improve model performance. Hence there is a need to include the dependence in warranty modelling to avoid information loss.

2. *Availability of modelling tools.* Copulas make it possible to incorporate both continuous and discrete random variables in a joint probability distribution. It is also well known that it is easy to develop models with copula as it separates the modelling procedures into two steps: (1) modelling marginal distributions, and (2) integrating the marginal distributions with copula. These are copula's advantages. In this research, the number of failure modes is limited and the occurrence of failure modes in warranty claims therefore are modelled with a discrete probability distribution, whereas the other random variables such as age, usage (typically mileage for automobile), and claim amount should be modelled with continuous probability distributions. The advantages of copula enable us to do such research.

1.2 Thesis structure

This thesis focuses on collectively and comprehensively optimising warranty policies with consideration of dependence among the products produced by the same manufacturer and the interplay among the software and hardware subsystems and the user of a product.

The rest of this thesis consists of 6 chapters: literature review, a mean-variance approach to jointly optimising warranty pricing for a portfolio of products, a value-at-risk approach to optimisation of warranty policy, warranty cost analysis considering the interplay of product subsystems, discussion and conclusion and future work.

In Chapter 2, due to the commonalities shared by warranty modelling and actuarial science, this chapter compares existing modelling approaches to insurance and warranty with regard to modelling of claim frequency, modelling of claim size, policy pricing and policy optimisation. The potential aspects to improve warranty policy optimisation are

summarised.

In Chapter 3, a real world issue is considered that a manufacturer commonly produces more than one product, or a portfolio of products, and provides warranty servicing for them. However, many researchers have attempted to optimise warranty policy to maximise the profit or minimise the cost of each individual product. Warranty claims of the products produced by the same manufacturer, however, may be due to common causes, since the products may be designed by the same engineer team or using the same type of components. This implies that the numbers of warranty claims of different products may be related, and optimisation of warranty policies for each individual product may therefore cause biased decisions. To overcome this disadvantage, this chapter aims to collectively optimise a manufacturer's total profit for a portfolio of different products by using a mean-variance optimisation approach. A tool from the probability theory, the *copula*, is used to depict the dependence among the warranty claims of different products. Numerical examples are provided to illustrate the application of the proposed methods.

Chapter 4 takes a further development on the basis of Chapter 3. The dependence among the warranty claims of different products produced by the same manufacturer is also considered and a real world issue is used to illustrate the situation. Furthermore, this chapter also collectively optimises warranty policy for a set of different products whose failures are statistically dependent. However, according to the literature in the financial literature, even though the variance of a random variable is applied as a risk measure by the overwhelming influential models of portfolio selection, the symmetry of this measure cannot reflect the risk comprehensively. In this chapter, the downside risk measures, borrowed from financial mathematics, such as Value-at-Risk and Conditional Value-at-Risk, are employed. Then, the existences of the optimal solutions for different scenarios are proved . Numerical examples are used to validate the applicability of the proposed methods.

In Chapter 5, the products consisting of software and hardware are considered. The causes of warranty claims of such products may be attributed to software specific failures, hardware specific failures, software-hardware interaction failures and human errors.

This chapter investigates warranty costs incurred due to those three subsystems with a main focus on their interactions. It estimates the costs due to different causes, develops integrated warranty cost models and optimises warranty policies considering the above possible combinations. Numerical examples are given to illustrate the proposed models.

Chapter 6 discusses the potential improvements of this research in three aspects including the forms of sales volume functions, the optimisation of renewing warranty policy, and the selection of copulas.

Finally, Chapter 7 concludes the findings of this thesis and proposes the future research that can be proceeded from both warranty data analysis and warranty policy optimisation perspectives.

Chapter 2

Literature review

Warranty shares commonalities with insurance in many aspects. Research on actuarial science has attracted much more attention than on warranty modelling. This chapter compares existing modelling approaches to insurance and warranty with regard to claim frequency, claim size, policy pricing and policy optimisation. The suggestions for improving warranty optimisation are summarised.

2.1 Comparison

2.1.1 Comparison of general features of warranty and insurance

Although warranty can be treated as a type of insurance from a mechanism aspect, warranty and insurance still have some differences in their general fractures including coverages, policies and data.

Coverages

Warranty can be classified as two types, base warranty and extended warranty. Base warranty is provided when a product is sold, whereas extended warranty is purchased separately and voluntarily by the buyer as a service contract. That is, base warranty covers the early period of a product, starting from the first day when an item is put in use, whereas extended warranty covers a certain period after the base warranty has expired. Warranty can be one-dimensional or two-dimensional. A one-dimensional warranty is characterised by an interval which is defined by time or usage, and a two-dimensional warranty is characterised by a region in a two-dimensional plane which consists of the time axis and the usage axis (Blischke et al., 2011). Warranty covers the period set by the warranty provider and regulated by the market authority.

Insurance can be classified as life insurance and non-life insurance by the perils they insured. The peril insured by life insurance is death, and non-life insurance typically protects the insured from loss or damage caused by specific risks. Non-life insurance is a concept used in continental Europe which covers all insurance products except life insurance, but the same concepts used in the UK and the US are general insurance and property and casualty insurance respectively (Wüthrich & Merz, 2008). Insurance covers the period that both the insurer and the insured agree on.

The coverage of warranty is similar to the coverage of non-life insurance. To improve warranty modelling, the investigation of actuarial science methods and techniques should be focused on the regime of non-life insurance.

Policies

A warranty policy is characterised by both duration and scope (full/limited, labor/parts, which parts, replacement/repair, etc). There are four commonly used types of warranty policies for consumer goods: renewing free-replacement warranty, renewing pro-rata warranty, non-renewing free-replacement warranty, and non-renewing pro-rata warranty. Under a renewing warranty policy, an item that fails within its warranty period is repaired and the warranty is also renewed. Under a non-renewing warranty, the original warranty is not altered by a failed item and the warranty provider only guarantees satisfactory service on the item within the original warranty period. Under a free replacement policy, the warranty provider agrees to provide free repair services within the warranty period. However, if an item that fails during its warranty period is repaired at a certain cost to the consumer, the policy is a pro-rata warranty policy.

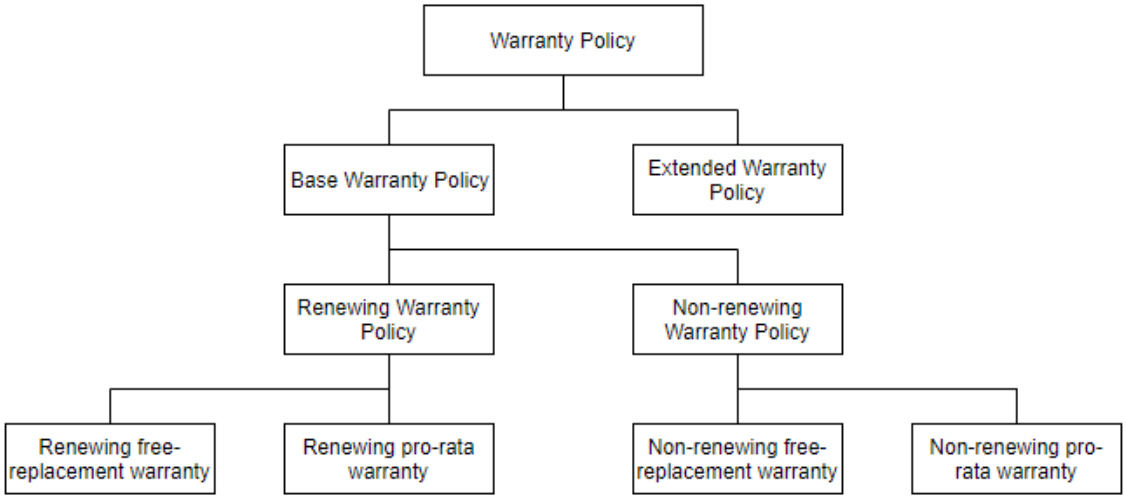

Figure 2.1: Warranty policy classification

An insurance policy is a legal document which confers the policyholder the right to make claims and obliges the policy issuer to accept claims when covered events occur (Czado et al., 2012). In life insurance, during the coverage period of one policy, the number of claims may not be greater than one (i.e., one can only die once). However, in warranty policy or non-life insurance, it is common that the number of claims is more than once during the coverage period. Figure 2.1 and Figure 2.2 present the classifications

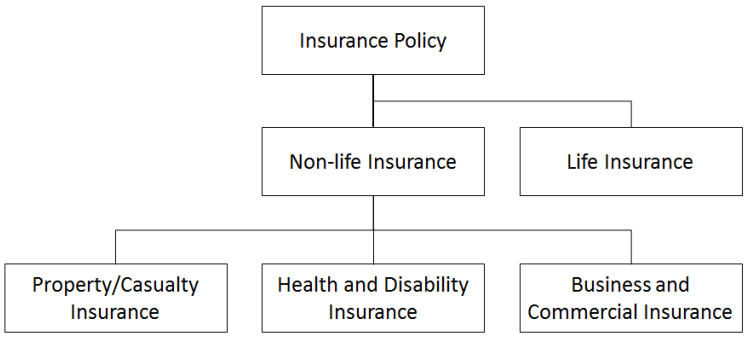

Figure 2.2: Insurance policy classification

of insurance policy and warranty policy. The range of the insured subjects of insurance policy is much wider than that of warranty policy. The not involving product development warranty policy is similar to some non-life insurance policies, for instance, car warranty is similar to auto-mobile insurance which belongs to property/casualty insurance. Hence, in this research, comparison is conducted between warranty and general insurance (non-life insurance).

Data

Warranty data consist of claims data and supplementary data. Warranty claims data are lifetime data and collected during the servicing of warranty claims (Wu, 2013), which commonly include manufacture date, manufacture volume, sales date, sales volume, claims date and claims cost. Warranty claims data are the data collected when warranty claims happen. The supplementary data are additional data, which are not always collected. Such data may be from those unfailed items whose warranty has not been claimed, even though such data may be analysed (Wu, 2012).

General insurance data are normally recorded by the insurer as a two-way array includes cases and variables. In general insurance, a claim is a demand for payment of damages covered under an insurance policy (Weke & Ratemo, 2013). The claim size is the severity of the effect associated with claims and may be referred to in the literature of actuarial science as severity, loss, size or amount of damage and cost of a claim. The policies, claims, policyholders, accidents, etc. can be denoted as cases, while the level of injury, gender, claim amount, etc. can be recorded as variables (De Jong et al., 2008).

In insurance, variables in claim reports can be quantitative or qualitative. For example, in the datasets used by (De Jong et al., 2008), the variables recorded in personal injury insurance include claim amount, injury type, accident date, reporting date and finalisation date. Variables recorded in one-year vehicle insurance may include policyholder's age and gender, area of residence, vehicle value, vehicle age, vehicle type, claim occurrence and claim amount. Sometimes, the original records of insurance policies contain more detailed information. For instance, in the study of (Frees & Valdez, 2008), the data collected from an insurance company in Singapore include vehicle and driver characteristics, insurance coverage and annual claims experience. Insurers normally record detailed information about policies and claims for accounting and premium rating purposes. The original data are tedious, the distribution of claim size and other data can be estimated through data pre-processing. Moreover, the claims data are often limited as data may be missing or corrupted (Cizek et al., 2005).

Compared with warranty data, insurance data may contain more information such as information of the policyholders and they are normally panel data (Zhao & Zhou, 2012; Shi & Valdez, 2014a), whereas warranty data are normally not presented as panel data format. Insurance claims occur due to one of the specific perils agreed in the policy whereas warranty claims occur due to product failures. Insurance data provide longer term information than warranty data, for example, car owners may need to purchase car insurance annually, even after the car's warranty has expired.

As a summary, Table 2.1 presents the comparison between insurance and warranty claims data.

Table 2.1: Comparison between insurance and warranty claims data

	Insurance	**Warranty**
Number of variables	many (including the) characteristics of policyholders)	few (only including product and repair data)
Occurrence	usually once	maybe recurrently
Claim amount	maybe large	normally small
Causes	perils (accidents)	failures
Covered interval	may cover the whole life of a product	the early life of a product

Warranty claims data and insurance claims data may contain basic information of

claims such as claim size and occurred date. However, insurance claims data may have more variables/covariates than warranty claims data, for example, insurance data may include policy holders' information such as the driver's gender, age and other characteristics whereas warranty data do not contain such detailed information. It should be noted that the quality of warranty data, which are normally collected from field, may not always be perfect, as they may be aggregated, delayed and censored.

2.1.2 Comparison of modelling techniques in actuarial science and warranty

Warranty is a good instrument for the manufacturer to enhance sales and a good protection for the customer from poor quality of products; but warranty also leads additional costs to the manufacturer, these costs depend on the warranty parameters, such as warranty periods, repairing method and costs, etc., which are stated by warranty policies. In this research, the warranty policy optimisation is conducted from a manufacturer's perspective, and aims to find the optimal warranty policy settings to maximise the manufacturer's expected total profit.

Fig. 2.3 states the mechanism of the warranty policy optimisation in this research. To maximise the total profit, the warranty cost and sales volume/demand of product, influenced by warranty policy, should be investigated; and such investigation is carried out based on proper modelling techniques, which are raised from warranty data analysis in the literature of warranty management involving modelling of number of warranty claims, warranty claim cost, etc.

The aim of warranty data analysis is to extract useful information and help in decision making by analysing warranty data with either statistical or computer algorithms (Wu, 2012). Warranty data analysis services in five areas presented in Figure 2.4. The first two areas, early detection of reliability problems and suggestion on design modification all aim to detect abnormalities of products from warranty data. However, the perils covered in insurance policy are not "produced" by the insurance company, hence we do not considering these two areas in this project. The purposes of warranty data analysis in the last

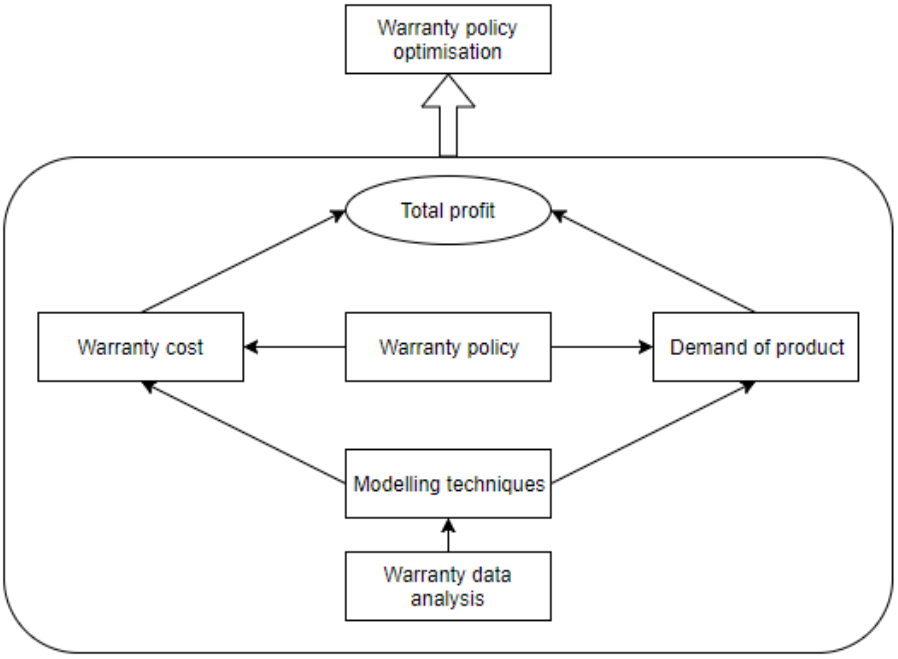

Figure 2.3: Warranty policy optimisation

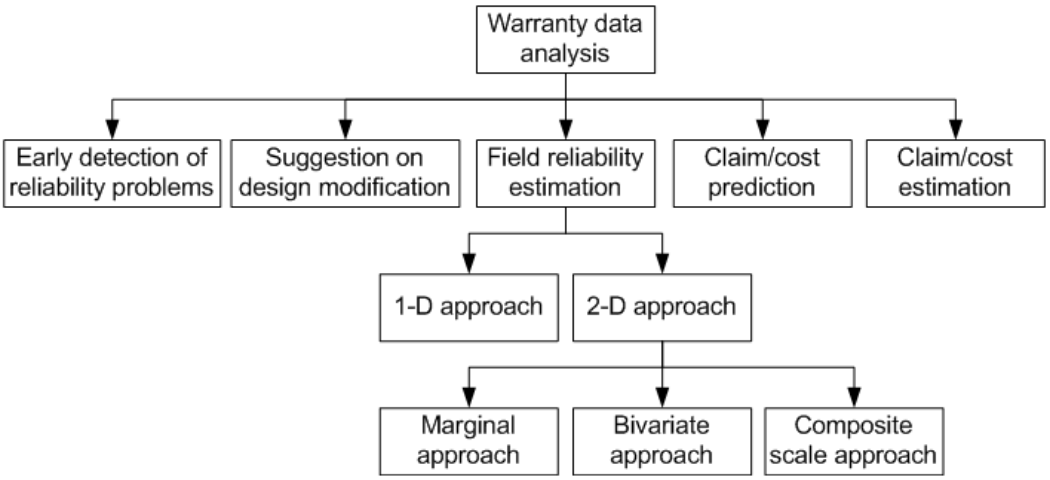

Figure 2.4: Warranty data analysis (Wu, 2012)

three areas are: estimating product reliability for deciding on warranty policy and appropriate maintenance policy, predicting future claims and warranty cost, and estimating and explaining the costs of warranty claims. It should be noted that warranty claim estimation is for a hypothetical infinite population of items, of which those sold are considered a random sample, whereas in warranty claim prediction, the population of items that is eventually sold is finite (Wu, 2012). The modelling techniques raised from warranty data analysis are also used to support warranty policy optimisation.

Considering the similarities between warranty and insurance, warranty modelling technologies can be improved referring to insurance modelling technologies, which are also originated from insurance data analysis. Insurance data analysis primarily service two purposes, ratemaking and total reserves prediction and estimation. Ratemaking aims to price insurance policy based on individual's characteristics and claim history, in which the policyholder's related risk level and expected claims will be estimated, and the price and coverage of the insurance policy will be determined based on the estimation. Total reserves prediction or estimation is predicting or estimating the total cost caused by related insurance claims. Figure 4 presents the components of insurance data analysis and the relationship between them.

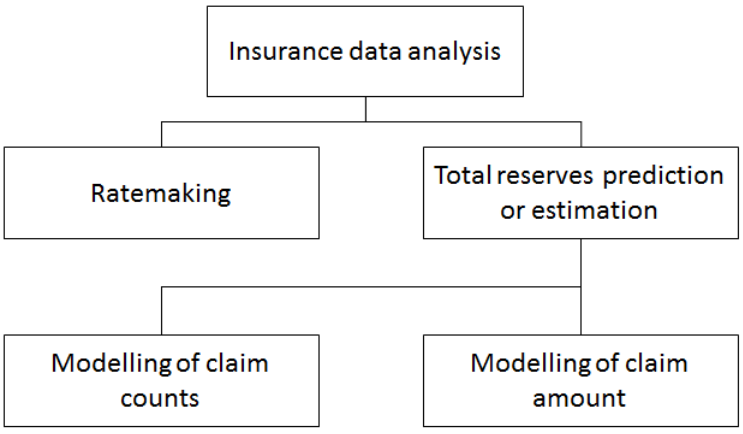

Figure 2.5: Insurance data analysis

Field reliability estimation and ratemaking all service in policy level, while claim/cost prediction or estimation and total reserves prediction or estimation involve processing claims data collected from a specific period. All of them involve modelling of claim

counts, and modelling of claim size. Table 2 presents the similarities between insurance and warranty data analysis from the aspect of application.

Table 2.2: Similarities between insurance and warranty data analysis application

Warranty	Field reliability estimation	Claim/cost prediction or estimation
Insurance	Ratemaking	Total reserves prediction
Purpose	service the pricing of policies	predicting the total cost of claims

The reserving modelling and premium rating are the two major areas of insurance modelling. Reserving modelling aims to forecast claims reserve, which is an insurer's future obligation and equals to the present value of an insurance policy's future cash flows. The insurer's total liability is the sum of the claims reserves of all individual policies issued. Premium rating is a process within which the amount that policyholder should pay the insurer for an insurance policy is determined. In this process, various properties of the insured object and the policyholder are taken into consideration.

Reserving modelling in both warranty and general insurance data analyses involves three components, modelling of number of warranty claims, modelling of claim size or severity and modelling of total claim amount. The claim frequency is the number of claims during a given period; the claim size or severity is the monetary amount of each claim and the total claim amount is the sum of the monetary losses of all claims.

Modelling of number of warranty claims

In general insurance modelling, the Poisson distribution and the Poisson process are benchmark models in modelling claim counts data, and the variants of those models are widely used (Mikosch, 2009). For example, when the interarrival times between claims are independent and identically distributed according to the exponential distribution, with parameter λ, then the probability that there are k claims is given by

$$P\{N(t) = k\} = \frac{(\lambda T)^k e^{-\lambda T}}{k!} \tag{2.1}$$

where $N(T) =$ cumulative number of failures from time 0 to time T.

In order to improve the performance of the modelling, one may also use other modelling techniques such as those reviewed below.

In the research of Yau et al. (2003), a generalised linear mixed model (GLMM) is applied to analyse repeated insurance claim frequency data. A conditionally fixed effect vector is incorporated into the linear predictor to model the inherent correlation. The study is based on a motor insurance dataset. The authors argue that the Poisson regression model is not proper for modelling the dependence between the observations from the same policyholder and they then extend the Poisson regression to a random effects model, in which a random effect vector is used for modelling the shared effects of repeated observations for the same policyholders.

Boucher et al. (2006) discuss fixed and random effects models used on longitudinal data in insurance claim modelling, with a case study based on three consecutive year data of a motor's third party liability insurance portfolio in Belgium. The data include the annual number of claims and the characteristics of the insured, such as sex and age of drivers, power of the vehicle and the size of city. The number of claims per year is assumed to follow the Poisson distribution with specific parameter for each policyholder. The parameter of the Poisson distribution is treated as a linear combination of the observable characteristics multiplying a static factor. In the random effects model, the static factor is expressed as a random variable with unit mean, and in the fixed effects model, the static factor is expressed as an estimated parameter of each individual. The parameters are estimated through the maximum likelihood. The static factors (i.e. heterogeneity parameters) are modelled by the Poisson-Gamma model, the Poisson-log normal model and the Poisson-inverse Gaussian model, respectively. They conclude that in a short period or with a few observations, the random effects model is better than the fixed effects model. However, in the data of the motor's third party liability insurance portfolio, the heterogeneity is not identically distributed across the insured, which can lead to inconsistency in the random effects model. Then they relax the assumption that the heterogeneity term is i.i.d., and use a regression on the individual heterogeneity terms to link the fixed effects model to the random effects model.

In the research of Shi & Valdez (2011), the Frank copula is used to jointly model the type of coverage and the number of accidents, in which the dependence parameter of the copula depicts the relationship between the frequency of accidents and the choice of coverage. Based on the copula model calibrated by one-year cross-sectional claims data collected from a major Singaporean automobile insurer, they find a significant positive coverage-risk relationship. This method is able to help derive the pure premium through demonstrating the effect of coverage choice on the incidence of accidents. Similarly, Zhao & Zhou (2014) use copulas to model the dependence between the class occupied by the insured and the claim frequency, and take the zero-excess phenomenon into account.

Zhao & Zhou (2012) investigate longitudinal data models of claim counts with excess-zeros and consider the dependence between claims from two successive periods. Copula is used to model the dependence. They discuss two models: The first one is a discrete margin copula model, which is applied to fit the time series data of longitudinal observations. The second one uses a copula to model the time-dependent unobservable random effects. In the first model, they use the Poisson distribution to model the number of claims with a mean claim frequency that incorporates the covariate information. Considering that the insured may not claim on the two successive periods, they use the zero-inflated Poisson model. In the second model, the residual heterogeneity in tariff cells is considered and modelled by a random effect, which is presented as a random effect variable in the mean of the Poisson distribution with excess zeros.

Shi & Valdez (2014a) provide a copula-based method to model the number of claims within a longitudinal context. The authors aim to predict the number of claims in a subsequent period based on the data in the previous periods. In order to reduce the computational difficulties, the claim amount is converted to a continuous random variable by a technique called *jittering* and then the joint distribution of the number of claims in successive periods is modelled through copulas. Jittering is a method in which an independent continuous random variable is subtracted from each component of the multivariate discrete data, aiming to convert discrete variables to continuous ones.

Matsui (2014) introduces a non-homogeneous Poisson cluster model to cope with the

reducing property of payment processes. The Poisson cluster process is used to model the total number or amount of payments for the claims arriving in a year and being paid in the interval $[0, t]$ ($t \geq 1$), with different clusters. The paper aims to predict the claims occurring in a future interval $(t, t+s], t \geq 1, s > 0$. The author considers a model with additive Levy processes, non-homogeneous Poisson clusters and non-homogeneous negative binomial clusters.

Modelling of claim size

In general insurance modelling, the exponential-inverse Gaussian distribution, which has a shorter tail than the Pareto distribution and allows for incorporating covariates, is introduced in Frangos & Karlis (2004) to model claim size. In their model the claim size is assumed to follow the exponential distribution, whose parameter is $\theta_i t_i$, where θ_i follows the inverse Gaussian distribution and the logarithm of t_i has a linear relationship with covariates. This model can be treated as an exponential regression model with random effects, in which the random effects are related to the inverse Gaussian distribution.

Albrecher & Teugels (2006) consider the possible dependence between the claim size and the waiting time for a claim in an insurance portfolio, which differs from the conventional assumption in which time between claims and claim sizes are independent. The authors assumed inter-claim time and its subsequent claim size to be dependent according to an arbitrary copula structure.

Frees & Valdez (2008) propose a Bayesian hierarchical approach to estimating joint multivariate distributions that model claim amounts of various claim types. The hierarchical model is decomposed to three components related to the frequency, type, and severity of claims, respectively. In this research, initially, a negative binomial regression model is used to assess claim frequency. The main contribution of this research lies in their introduction of a multivariate claim distribution to handle long-tailed, correlated claims with covariates.

Resti et al. (2010) also investigate the independent assumption between claim types, and then apply the Gaussian, Frank and Clayton copulas to model the dependence between

claim types. This research is based on a motor insurance claim dataset from Malaysia.

In order to estimate the loss reserves for incurred but not reported (IBNR) claims through individual claim loss models instead of aggregated claim loss models, Zhao & Zhou (2010) use the semi-survival copula and the semi-competing risk copula to model the dependence between the event times with delays in the individual claim loss model. The performances of their proposed methods are evaluated through simulation. In this research, the assumption, that relationship between the event time and the delay is independent, is relaxed; and the dependence structure between the event time and the delay is characterised by a copula with the margins of the event times and the delays.

Czado et al. (2012) discard a crucial assumption that the number of claims and the claim sizes are independent. A mixed copula approach is provided to model the dependence between the number of claims and the average claim size through the Gaussian copula. The number of claims in a group of characterised policyholders is modelled by Poisson regression, and the average claim size of the group is modelled through Gamma regression. They then construct the joint distribution of the two marginal distributions through the Gaussian copula. They find that this model performs well overall, but the extreme values are not presented very well. This may be caused by copula selection which is suggested for future study.

To estimate the size of claim reserving, Peters et al. (2014) develop a type of hierarchical Bayesian-based claim models that considers claims payments and incurred losses information. Their method extends the claims reserving models of Hertig (1985) and Gogol (1993). They also use a data-augmented mixture Copula paidincurred claims model.

2.2 Policy pricing and optimisation

The policy pricing, basically, means to find an optimal price of a specific insurance policy based on the policy content. The policy optimisation is more likely to policy designing, which aims to find an optimal combination of policy elements including policy pricing, policy period, compensation method, etc.

Premium rating or ratemaking in insurance, is the core activity in general insurance

policy pricing. The key principle in ratemaking is cost-based pricing of individual risks. The price charged to the policyholders for the coverage in an insurance policy is the estimated present value of the future costs incurred by the covered peril. In the pure premium approach, the price of an insurance policy is defined as the ratio of the predicted costs of all future claims against the insurance policy coverage. The price is in effect to the risk exposure and expenses (Denuit et al., 2007). Additionally, the property ratemaking is based on the distributions of claim frequency and loss, which is similar to estimating the total reserves (Mikosch, 2009). That is, ratemaking aims to price insurance policy based on individual's characteristics and claim history, in which the policyholder's related risk level and expected claims will be estimated, and the price and coverage of the insurance policy will be determined based on the estimation.

Denuit et al. (2007) state that actuaries have to design a tariff structure to fairly allocate the burden of claims among policyholders in a competitive market. The policies are categorised into classes. The policyholders in the same class are charged the same premium. In practice, property and vehicle insurers use risk classification plans to create classes. The classification variables are priori variables as those values are determined before the policyholder starts to use their property. Premiums for motor liability coverage are often set based on territory the vehicle garaged in, the use of the vehicle, and driver's individual characteristics such as age, gender, occupation, etc. A priori classification can be achieved with generalised regression models.

Currently, in motor insurance, a popular approach is using experience rating to link premium amounts to individual's past claims experience. In this approach, the insured drivers who are responsible for accidents are penalised by premium surcharges (or maluses), and the claim-free policyholders are rewarded by discounts (or bonuses). The systems applying this approach are called Bonus-Malus systems, experience rating, no-claim discounts, or merit rating (Denuit et al., 2007). Shi & Valdez (2011) also state that in general insurance ratemaking is a classical actuarial problem. In principle, the distribution of insurance claims is the basis for determining pure premiums.

All in all, the ratemaking approaches in insurance can be divided into two categories;

the first one determines the risk classification of the policyholders based on the policyholders' supplementary information (covariates) and historical data of the similar policy, and the premium will be computed according to the risk classification. The second one is the experience rating which determines the premium based on the policyholder's individual past claims experience. These two approaches can also be applied together when the first one determines the basement of the premium and the second one makes correction on the basement.

N. Frangos & Vrontos (2001) introduce an optimal Bonus-Malus System (BMS) in automobile insurance which considers the frequency and severity of a policyholder's historical accidents simultaneously instead of the major BMS designed based on the frequency of accidents and disregarding the incurred severity. In their optimal BMS the frequency of accidents and the severity of accidents are assumed independent. They model the frequency component of optimal BMS with the negative binomial distribution, which is a gamma-Poisson mixture distribution, and model the severity component with the Pareto distribution, which is a conjugated distribution of exponential and inverse-gamma distributions.

i Morata (2009) applies the bivariate Poisson regression model in priori ratemaking. Priori ratemaking is conducted based on the priori variables, which are used to segment the insurance policies portfolio in homogeneous classes. i Morata (2009) relaxes the assumption that the number of claims of different types are independent and uses bivariate Poisson regression to model the number of claims for third-party liability and the number of claims for other guarantees. The result shows the independence assumption between claim types should be rejected.

According to Shi & Valdez (2011), insurance claim data have the unique semi-continuous feature in the sense that a positive continuous component is associated with a significant fraction of zeros. In the actuarial practice, two methods are widely used as standards: the Tweedie generalised linear model (GLM) and the frequency-severity model. The former incorporates a percentage of zeros into a continuous distribution through a compound Poisson process, where the resulting Tweedie distribution is featured with a mass prob-

ability at zero and thus is suitable for modelling the semi-continuous claim data. The latter decomposes the total claims amount into frequency and severity parts. An insurance policy usually provides multiple types of coverage. The unique contract is designed for each type such as deductibles or coverage limits, insurers must deal with the payment separately. Plus, the insurers can obtain additional insights by decomposing aggregated claims into different categories and investigating their joint behaviour (Shi, 2016).

Bermúdez & Karlis (2011) find the independent assumption of different types of claims is not realistic in automobile insurance. They point out that in the classical priori tariff system not all risk factors are identified: the tariff classes can be heterogeneous and the unobserved heterogeneity and serial dependence may lead to over dispersion. Hence they apply a multivariate Poisson regression model such as the zero-inflated model. As their model has computational difficulties, they introduce Bayesian inference based on Markov chain Monte Carlo (MCMC) method to solve it.

Zhao & Zhou (2014) discuss the Bonus-Malus system in posteriori ratemaking. As an innovation, they use a bivariate copula function to model the dependence between claim frequency and policyholder's risk class. The marginal distribution of the number of claims is modelled with Poisson regression. The distribution of the classes is represented through a matrix and the joint distribution is modelled by the Clayton copula.

Antonio & Valdez (2012) review statistical tools used in risk classification for ratemaking. They state that, in general insurance, a priori ratemaking is used to build the basis when a policyholder is new and insufficient information is available, and a posteriori ratemaking is used to correct and adjust the priori premium when the historical information about policyholder becomes available.

However, an insurance company, normally, sells more than one policy and even more than one type of policies, i.e. the company has to deal with a bunch/portfolio of policies. Alyakina & Khisamova (2014) indicate a methodology for rating of insurance portfolio allowing the insurance company to ensure the financial stability and solvency. The methods to assess the insurance portfolio are divided into mathematical methods, economic analysis and subjective characteristics analysis of the portfolio. Boyko (2011) takes the

reinsurance operations in to the economic and mathematic model to optimize the policy portfolio structure of an insurance company. Oliynyk (2015) investigate the optimal structure of insurance portfolio on the basis of nonlinear programming and allows the insurance company to reach maximal profit with minimal risk by choosing the specific optimal structure of insurance portfolio. Buerle & Blatter (2011) optimize proportional reinsurance and investment policies in a multidimensional Lvy-driven insurance model to maximize exponential utility, with the consideration of the dependencies of the risk reserves in different business lines. The dependence is modelled through Archimedean Lvy copula.

2.3 Current and Future Developments: suggestions for warranty modelling and policy optimisation

To improve warranty modelling and support efficient warranty policy optimisation, some techniques applied in insurance modelling, such as dependence modelling, random effects and other policy pricing and optimisation approaches, should be considered.

2.3.1 Dependence modelling

In the development process of insurance claim modelling, a conventional assumption is the independence of claims Yau et al. (2003) on the ground of the fact that the insurance claims in a group of policies are caused by independent accidents. However, many researchers find that, in practice, the independence assumption is too restrictive and may not hold. As such, dependence between claims in different periods claimed by a policyholder is considered, based on which many modelling approaches are developed, see De Jong et al. (2008); Zhao & Zhou (2012); Shi & Valdez (2014a); Mikosch (2009); Yau et al. (2003); Rolski et al. (2009), for example.

According to Wu (2012), most literatures on warranty claim data analysis for products with one-dimensional warranty policies do not consider the dependences within claims data. When analysing one-dimensional warranty data, existing approaches include estimating mixed distributions (Majeske, 2003), fitting the Weibull distribution (Ion et al.,

2007), considering sales delay (Wilson et al., 2009), etc. Those approaches do not consider any dependence within basic claims data, such as the dependence between claims and failure modes, which may cause the loss of information and result in biased decision making.

When analysing two-dimensional warranty data, some types of dependence are considered, for example, Lawless et al. (1995), Davis (1999) and Baik & Murthy (2008) discuss modelling the dependence of failures on age and mileage, which directly estimates a joint bivariate distribution depicting the dependence between age and usage (Singpurwalla & Wilson, 1993; Moskowitz & Chun, 1994; Jung & Bai, 2007; Wu, 2014a), the dependence between failures and age/usage (Lawless et al., 2009), and the dependence between recurrent events and usage (J. Lawless & Crowder, 2010).

Having reviewed the existing papers in warranty data analysis and insurance data analysis, we find that copula-based approaches are widely used in insurance data analysis, but barely applied in warranty data analysis.

Copula is a tool that models the dependence structures between random variables. It was first introduced by Sklar (1959) and has attracted considerable attention in theoretical and application aspects in recent years. In insurance data analysis, copula-based approaches are used to model the dependence between different claim types (Frees & Valdez, 2008; Shi & Valdez, 2014b), between accident date and reported date (Zhao & Zhou, 2010), between policy coverage and number of claims (Shi & Valdez, 2011; Zhao & Zhou, 2014; Bolancé et al., 2014), between claim counts in successive periods (Zhao & Zhou, 2012; Shi & Valdez, 2014a), between claims in different business lines (Diers et al., 2012; Zhang & Dukic, 2013), between number of claims and average claim size (Czado et al., 2012), and between time-to-claim and claim size (Weke & Ratemo, 2013).

In warranty data analysis, however, there is only one article using copulas to model dependence (Wu, 2014a), in which a new method of constructing asymmetric copulas is introduced, and the asymmetric copulas are used to capture the tail dependence between the pair of age and usage in two-dimensional reliability data.

In general, the existing methods of warranty data analysis lack a systematic research

on the dependence between warranty claims of different products and within warranty claims of the same products. Learnt from insurance data analysis, warranty data analysis can be improved by applying copula-based modelling approaches.

2.3.2 Policy pricing and optimisation

Warranty policies are extensively studied in the literature of warranty management, but most of them are based on assumptions such as "warranted items being repaired minimally" or "being repaired minimally", which are very much concepts widely used in reliability and maintenance engineering, see Bouguerra et al. (2012); Settanni et al. (2014), for example. As mentioned above, different from base warranty, extended warranty is purchased separately and voluntarily by the buyer as a service contract. This suggests that the price of the extended warranty may be optimised on the basis of operating condition, usage intensity, consumers' information, etc, which may be collected from a variety of sources in the era of big data. Based on such information, warranty providers will be able to sale extended warranties with different prices to different consumers. For this reason, modelling techniques of risk classification in policy pricing in insurance may be adapted to warranty data analysis.

In the literature, as an insurance company may issue more than one insurance policy, the pricing and optimisation of insurance policies portfolio are investigated by researchers. In warranty area, the similar situation, that one manufacturer may produce more than one product and provide warranty service with them, also should not be ignored in warranty modelling and warranty policy optimisation.

2.4 Summary

This chapter investigate the techniques, approaches and theories applied in insurance modelling, and compares those to warranty modelling from different aspects.

- dependence modelling
- modern portfolio theory with

  - the mean-variance approach, and
  - the value-at-risk (VaR) and conditional value-at-risk (CVaR) approaches,
- copulas

It finds that

- little research on the dependence between warranty claims of different products and their interactions has been conducted;
- different warranty policies provided by one manufacturer are investigated on the basis of individual products, little research on warranty policy optimisation for a portfolio of products has been conducted; and
- copulas are becoming more widely used in insurance modelling than in warranty modelling; but they are seldom used in warranty policy optimisation.

To fulfil knowledge gaps, in the rest of this thesis, the warranty policies of the products produced by an individual manufacturer are optimised integrally through the Modern Portfolio Theory; the dependence among the policies are modelled through copula in the optimisation; and, furthermore, the effects of subsystems' interplay on product warranty are also investigated to make the optimisation framework comprehensive.

Chapter 3

A mean-variance approach to jointly optimising warranty pricing for a portfolio of products

Warranty policy can influence the profit and cost of a product. In practice, a manufacturer commonly produces more than one product, or a portfolio of products, and provides warranty servicing for them. Many authors have attempted to optimise warranty policy to maximise the profit or minimise the cost of each individual product. Warranty claims of the products produced by the same manufacturer, however, may be due to common causes, since the products may be designed by the same engineer team or using the same type of components. This implies that the numbers of warranty claims of different products may be related, and optimisation of warranty policies for each individual product may therefore cause biased decisions. To overcome this disadvantage, this chapter aims to collectively optimise a manufacturer's total profit for a portfolio of different products by using a mean-variance optimisation approach. A tool from the probability theory, *copula*, is used to depict the dependence among the warranty claims of different products. Numerical examples are provided to illustrate the application of the proposed methods.

3.1 Introduction

3.1.1 Background

A typical warranty transaction is: a consumer pays the warranty price when purchasing the product and the manufacturer (or dealer) provides repair or replacement service for product failures occurring during the warranty period. A good warranty policy will raise a firm's brand image and reputation among consumers. The assurance of warranty can reduce the costs associated with failure of the product purchased from a consumer's perspective. Warranties highlight product reliability and quality: a longer warranty can send out a strong signal about the products and service quality (Liao, 2016). However, warranty providers should be aware of two major uncertainties in warranty management:

Uncertainty of warranty cost. The demands for repairs or replacements always come out unexpectedly, and the number of warranty claims received during the warranty period normally appears only as probability-based forecasts (Yenipazarli, 2014).

Uncertainties of sales volume. The sales volume of most goods may increase with the decrease of warranty price. It is true that high sales price of a product may increase the profit per item; but it may become less attractive to its consumers and therefore reduce the product total demand. Meanwhile, from a consumer's perspective, longer warranty length, which implies higher warranty price for the manufacturer, indicates better product quality and reliability. It is reported that longer warranties bring more unnecessary cost to manufacturers Aggrawal et al. (2014) than short ones. A manufacturer should consider its total cost and profit comprehensively. The total profit is determined by the total revenue and cost, where the total revenue is product price multiplying sales volume, and the total cost is the sum of production cost, warranty cost and other operation costs multiplying sales volume.

Both product price and warranty length are normally set by the manufacturer to reflect the current market. The goal of most manufacturers is to maximise profit, which may be achieved by optimally pricing the warranty policy of their products. Meanwhile, warranty

cost can be treated as a random negative cash flow which depends on the sales volume and the length of warranty period. By *random*, it means uncertainty, presenting the manager the expected warranty cost, which is difficult to forecast. As such, accurately forecasting warranty cost is vitally important.

In existing publications relating optimisation of warranty policies, researchers have introduced many methods to maximise profit through optimising product price and warranty length in different scenarios. For example, Yazdian et al. (2016) jointly optimises the acquisition price, remanufacturing degree, selling price and warranty length of remanufacturing products, assuming there are linear and non-linear demand functions. Wei et al. (2015) investigates the optimal strategies on product price and warranty length of two complementary products from two manufacturers in a two-stage game theoretic perspective. Yeh & Fang (2015) introduces a model to optimise product price and warranty length considering the manufacturer's production capacity and preventive maintenance program. Aggrawal et al. (2014) present a method to optimise price and warranty length for a product based on a two dimensional innovation diffusion model. Wu et al. (2009) develop a decision model to determine the optimal price, warranty length and the production rate of a product to maximise profit based on the pre-determined life cycle in a static demand market. Similar to these articles, all of the other existing research, including Lin et al. (2009); Matis et al. (2008); Ladany & Shore (2007); Huang et al. (2007), etc., only maximises the profit of each individual product for a manufacturer.

3.1.2 Motivation and novelty

The numbers of warranty claims of products produced by the same manufacturer may not be statistically independent because they may be designed by the same team of engineers, manufactured on the same production lines and share same types of components. As a result, they may have common causes. For example, Ford's turbocharged and direct injection gasoline engines, belonging to the EcoBoost family, are applied on many different types of Ford cars, including Focus, Fiesta, Mondeo, etc. If any design or quality problems happened on the EcoBoost, warranty claims from different products will crop up

during a short period.

As can be seen from the above literature review, however, existing literature has been concentrated on warranty reserve optimisation for each individual product separately. Little attention has been paid to collectively optimising warranty policies and reserves for a portfolio of different products. To maximise the total profit for a manufacturer that sells a number of products, the prices of all products should be optimised collectively, which is the aim of this chapter.

This chapter is the first attempt that collectively optimises the prices of a portfolio of products for a manufacturer, which creates novelty. It proposes to use an approach borrowed from the modern portfolio theory, which is widely applied in the financial sector. It develops a novel method that collectively optimises warranty reserves for a set of different products, considering the dependence of the numbers of warranty claims. The warranty prices of the set of products are then optimised under the mean-variance optimisation framework.

This chapter uses a tool from the probability theory, *copula*, to depict the dependence among warranty claims of different products, which can reduce the bias that may be caused in modelling a complicated dependence with a simple method such as covariance estimation. This is because the covariance matrix can only reflect the linear correlation whereas copula can model more complicated non-linear dependence

3.1.3 Overview

The other sections of this chapter are structured as follows. Section 3.2 formulates the problem. Section 3.3 investigates the existence of the optimal values for different scenarios and uses copulas to depict the dependence among warranty claims of different products. Section 3.4 offers numerical examples to illustrate the proposed methods. Section 3.5 concludes this chapter and proposes future work.

3.2 Formulation of the problem

Assume a manufacturer offers non-renewing free replacement warranty (NFRW) policies. Under a NFRW policy, the manufacturer provides its customers with repair or replacement at no cost within the warranty period, the original warranty is not altered upon a failed item, and the manufacturer only guarantees satisfactory service on the item within the original warranty period. We also assume the repair time is negligible. The items are new at $t = 0$ when they are sold. The number of claims follows the homogeneous Poisson process (HPP). The numbers of warranty claims and the claim cost are statistically independent.

In practice, warranty policies can be categorised into one- and two-dimensional (1-D and 2-D). A 1-D policy is characterised by an interval, such as age or usage, as warranty limit and a 2-D policy is characterised by a region in the 2-D plane (Ye & Murthy, 2016), such as age and usage. In the literature, many methods are used to deal with the 2-D situations, for example, some authors use a so-called composite scale approach that integrates the two scales (age and usage) to create a single composite scale and model the claim arrival process (Wu, 2012). In this chapter, only one scale, the length of warranty, noted by T_k, is considered in modelling. Of course, one mat regard T_k as the usage in 1-D warranty or the composite scale in 2-D warranty.

3.2.1 The expected number of warranty claims and cost

Suppose a manufacturer produce n products. The aggregated warranty cost of product k follows a stochastic process $\{S_k(t)\}_{t\geq 0}$ over the time interval $(0, t)$, which is expressed by the following equation,

$$S_k(t) = \sum_{i=1}^{N_k(t)} X_{k,i}. \tag{3.1}$$

In Eq. (3.1), the cost of claim i of product k is described by the random variable $X_{k,i}$ and the counting process $N_k(t)$, which is the number of claims during $(0, t)$ and is assumed to take a form of a homogeneous Poisson process (HPP) with intensity $\lambda_k > 0$, $P(N_k(t) = i) = \frac{(\lambda_k t)^i e^{-\lambda_k t}}{i!}$. For a given k, $X_{k,i}$ are independent and identically distributed random

variables which have finite values on the positive half-line $\mathbb{R}_{>0}$ with the probabilities $P(X_{k,i})$. The frequency $N_k(t)$ and severity $X_{k,i}$ are assumed to be independent.

Then, the expected value of $S_k(t)$ is given by

$$\mathrm{E}[S_k(t)] = \mathrm{E}[N_k(t)]\mathrm{E}[X_k] = \lambda_k \mu_k t, \tag{3.2}$$

and the variance of $S_k(t)$ is given by

$$\begin{aligned} \mathrm{Var}\left[S_k(t)\right] &= \mathrm{E}[N_k(t)]\mathrm{Var}[X_k] + \mathrm{Var}[N_k(t)]\mathrm{E}[X_k]^2 \\ &= \lambda_k t(\mathrm{Var}[X_k] + \mathrm{E}[X_k]^2) \\ &= \lambda_k t(\sigma_k^2 + \mu_k^2), \end{aligned} \tag{3.3}$$

where μ_k and σ_k are the mean and variance of X_k, respectively.

3.2.2 The sales volume and profit

In the literature, the sales volume of a product is mainly assumed to be non-random and static. The models may be expressed in a linear (Yazdian et al., 2016; Lin et al., 2009) or non-linear form (Ladany & Shore, 2007; Xie et al., 2014; Huang et al., 2007), depending on different scenarios. For simplicity, this chapter uses the linear form proposed by Yazdian et al. (2016). However, in the real world, the sales volume of a product is changing with time randomly. This fact should be considered by future research.

In this research, the linear relationship among the length of warranty coverage, warranty price and sales amount of a product is assumed as following:

$$M_k = A_k - \beta_k P_k + \eta_k T_k, \tag{3.4}$$

where P_k is the warranty price, T_k is the warranty period of product k, M_k is the sales volume of product k, respectively. A_k,β_k and $\eta_k > 0$ are positive real numbers. The profit of one item of product k is

$$r_k = P_k - S_k(T_k) - c_k, \tag{3.5}$$

where c_k is the fixed cost of one item of product k, including manufacturing cost, management expenditures, etc.

Then, the profit of product k is

$$\omega_k(P_k, T_k) = M_k[P_k - S_k(T_k) - c_k], \tag{3.6}$$

the expected value of $\omega_k(P_k, T_k)$ is

$$\mathrm{E}[\omega_k(P_k, T_k)] = (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k\lambda_k\mu_k - c_k), \tag{3.7}$$

and the variance of $\omega_k(P_k, T_k)$ is

$$\mathrm{Var}[\omega_k(P_k, T_k)] = M_k^2 \mathrm{Var}[S_k(t)] = M_k^2 T_k \lambda_k(\sigma_k^2 + \mu_k^2).$$

For the n products, the total profit of the manufacturer is

$$\Omega(\boldsymbol{P}, \boldsymbol{T}) = \sum_{k=1}^{n} \omega_k(P_k, T_k). \tag{3.8}$$

$\Omega(\boldsymbol{P}, \boldsymbol{T})$ can be re-written as

$$\begin{aligned} \Omega(\boldsymbol{P}, \boldsymbol{T}) &= \sum_{k=1}^{n} \omega_k(P_k, T_k) \\ &= \sum_{k=1}^{n} (A_k - \beta_k P_k + \eta_k T_k)\left[P_k - S_k(T_k)\right]. \end{aligned} \tag{3.9}$$

The mean and variance of $\Omega(\boldsymbol{P}, \boldsymbol{T})$ are

$$\begin{aligned} \mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] &= \sum_{k=1}^{n} \mathrm{E}[\omega_k(P_k, T_k)] \\ &= \sum_{k=1}^{n} (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k\lambda_k\mu_k - c_k), \end{aligned} \tag{3.10}$$

and

$$\mathrm{Var}(\Omega(\boldsymbol{P}, \boldsymbol{T})) = \boldsymbol{M'VM}, \tag{3.11}$$

respectively, where

$$\boldsymbol{M'} = \left[M_1, M_2, \ldots, M_n\right], \tag{3.12}$$

and

$$\boldsymbol{V} = \begin{bmatrix} \mathrm{Var}(S_1(T_1)) & \mathrm{Cov}(S_1(T_1), S_2(T_2)) & \ldots & \mathrm{Cov}(S_1(T_1), S_n(T_n)) \\ \mathrm{Cov}(S_2(T_2), S_1(T_1)) & \mathrm{Var}(S_2(T_2)) & \ldots & \mathrm{Cov}(S_2(T_2), S_n(T_n)) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Cov}(S_n(T_n), S_1(T_1)) & \mathrm{Cov}(S_n(T_n), S_2(T_2)) & \ldots & \mathrm{Var}(S_n(T_n)) \end{bmatrix}. \tag{3.13}$$

A manufacturer may wish to optimise $\boldsymbol{P}$ and $\boldsymbol{T}$ in the quantity $\Omega(\boldsymbol{P}, \boldsymbol{T})$ in Eq. (3.8) to achieve the maximum profit.

Each T_k in $\boldsymbol{T}$ may be either a commonly agreed quantity or an endogenous variable: If it is an endogenous variable, T_k may be determined by the warranty provider; if it is a commonly agreed quantity, T_k may be served as an exogenous variable. Since in reality, the length of warranty is pre-specified and it can be from a small number of discrete positive integers, 12 months, 24 months, 30 months, etc, for example. In this chapter, each T_k in $\boldsymbol{T}$ is regarded as a commonly agreed quantity. In other words, we mainly focus on optimally pricing warranty policies for a given $\boldsymbol{T}$, that is, to seek $\boldsymbol{P}$ so that the expected profit $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T}]$ can be maximised.

3.3 Optimisation of the profit

For a manufacturer, $\Omega(\boldsymbol{P}, \boldsymbol{T})$ in Eq. (3.8)can be maximised through seeking the optimal values of P. Another idea is to use the mean-variance approach: since the variance of $\Omega(\boldsymbol{P}, \boldsymbol{T})$ can be regarded as a risk measure, one may integrate this risk when optimising $\Omega(\boldsymbol{P}, \boldsymbol{T})$. These two ideas are discussed in the following two sub-sections, respectively.

3.3.1 Maximising the expected profit

For the convenience of further discussion, this subsection gives the optimisation objective function and does not consider the risk (i.e., the variance) of the portfolio of the warranty claim costs of the n products and merely aims to optimise $\Omega(\boldsymbol{P}, \boldsymbol{T})$, which is a commonly used setting.

The profit of product k can then be derived and expressed by

$$\mathrm{E}[\omega_k(P_k, T_k)] = (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k \lambda_k \mu_k - c_k), \tag{3.14}$$

which is a bivariate quadratic function with respect to T_k and P_k.

Hence, we have the following proposition.

Proposition 1. *If both $\boldsymbol{P}$ and $\boldsymbol{T}$ are vectors of decision variables and there are no other constraints, the optimum solution that maximises the expected profit, $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$ does*

not exist.

Proof. In this case, the expected profit of product k is

$$\begin{aligned}
\mathrm{E}[\omega_k(P_k, T_k)] &= (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k\lambda_k\mu_k - c_k) \\
&= -\beta_k P_k^2 - \eta_k\lambda_k\mu_k T_k^2 + (A_k + \beta_k c_k)P_k - (A_k\lambda_k\mu_k + \eta_k c_k)T_k \\
&\quad +(\beta_k\lambda_k\mu_k + \eta_k)P_k T_k - A_k c_k,
\end{aligned}$$

which is a bivariate quadratic function with respect to T_k and P_k, respectively and can be denoted as,

$$\mathrm{E}[\omega_k(P_k, T_k)] = AP_k^2 + BT_k^2 + CP_k + DT_k + EP_k T_k + F,$$

the discriminant for the existence of a bivariate quadratic function's minimum and maximum is $\Delta = 4AB - E^2$. Then, we have

$$\begin{aligned}
\Delta &= 4AB - E^2 \\
&= 4(-\beta_k)(-\eta_k\lambda_k\mu_k) - (\beta_k\lambda_k\mu_k + \eta_k)^2 \\
&= -(\beta_k\lambda_k\mu_k - \eta_k)^2 \leq 0.
\end{aligned}$$

When $\Delta < 0$, i.e. $\beta_k\lambda_k\mu_k \neq \eta_k$, $\mathrm{E}[\omega_k(P_k, T_k)]$ dose not have maximum nor minimum. When $\Delta = 0$, i.e. $\beta_k\lambda_k\mu_k = \eta_k$, because $4AB - E^2 = 0$, $DE - 2CB = 2AD - CE = 0$, and $A < 0$, $\mathrm{E}[\omega_k(P_k, T_k)]$ has a constant maximum with regardless of T_k.

□

Proposition 1 is raised because the sales volume is modelled by a linear function. Obviously, the existence of optimal solution is affected by the form of sales volume function. As higher price can lead to higher unit profit but lower sales volume, and longer warranty period can lead to higher sales volume but lower unit profit, optimization is to find a proper trade off between. The location of this trade off is determined by the form of sales volume function. To support practice and further research, varying sales volume functions referring to the literature are discussed in the Discussion chapter of this thesis.

In practice, the warranty length $\boldsymbol{T}$ is normally a discrete variable in a finite range, such as 24 months, 30 months, etc. If $\boldsymbol{T}$ is given, it is easy to prove that there is a maxima in $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$.

Proposition 2. *If $\boldsymbol{T}$ is known and $\boldsymbol{P}$ is a vector of decision variables, there exists an optimum solution that maximises the expected profit* $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$.

Proof. If $\boldsymbol{T}$ is known,

$$\mathrm{E}[\Omega(\boldsymbol{P})] = \sum_{k=1}^{n} \mathrm{E}[\omega_k(P_k)],$$

to optimise $\mathrm{E}[\Omega(\boldsymbol{P})]$ is equivalent to optimise $\mathrm{E}[\omega_k(P_k)]$.

By expanding $\mathrm{E}[\omega_k(P_k)]$, we have

$$\begin{aligned} \mathrm{E}[\omega_k(P_k)] &= (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k\lambda_k\mu_k - c_k) \\ &= -\beta_k P_k^2 + (A_k + \beta_k c_k + \beta_k\lambda_k\mu_k T_k + \eta_k T_k)P_k - \eta_k\lambda_k\mu_k T_k^2 \\ &\quad -(A_k\lambda_k\mu_k + \eta_k c_k)T_k - A_k c_k. \end{aligned}$$

Let

$$\frac{d\mathrm{E}[\omega_k(P_k)]}{dP_k} = -2\beta_k P_k + A_k + \beta_k c_k + \beta_k\lambda_k\mu_k T_k + \eta_k T_k = 0,$$

then,

$$P_k = \frac{A_k + \beta_k c_k + \beta_k\lambda_k\mu_k T_k + \eta_k T_k}{2\beta_k}.$$

Because $-2\beta_k < 0$, then, when $P_k = \frac{A_k+\beta_k c_k+\beta_k\lambda_k\mu_k T_k+\eta_k T_k}{2\beta_k}$, $\mathrm{E}[\Omega(\boldsymbol{P})]$ is maximised. □

In the following section, we assume $\boldsymbol{T}$ is known, and investigate the existence of the optimum solutions of $\boldsymbol{P}$ when considering the risk against profit.

3.3.2 Mean-variance approach to pricing warranty policies

In this section, we use the mean-variance optimisation approach to pricing warranty policies.

When both profits and risk are taken into consideration, manufacturers may have the following options:

Option 1. to maximise a combination of the profit and the risk of the estimated profit;

Option 2. to maximise the profit and meanwhile to limit the risk of the estimated profit; and

Option 3. to minimise the risk of the estimated profit subject to the constraint that the lower bound of the profit is greater than a pre-specified value.

The above three options are equivalent to the following three optimisation problems, respectively.

Option 1. Maximise $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]-\mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$, subject to $\boldsymbol{M}\geq\mathbf{1}$, where $\mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is the standard deviation of total profit and $\boldsymbol{M}\geq\mathbf{1}$ means the sale volume of each product should not be less than 1 unit. $\mathbf{1}$ is the unit matrix with the same dimensions as those of $\boldsymbol{M}$.

Option 2. Maximise $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$, subject to $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]\leq\varphi$, and $\boldsymbol{M}\geq\mathbf{1}$, which implies maximising the expected total profit at a given risk level.

Option 3. Minimise $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$, subject to $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]\geq\psi$, and $\boldsymbol{M}\geq\mathbf{1}$. It implies minimising the risk at a given expected total profit.

On Option 1, we have the following Propositions 3 and 4.

Proposition 3. *For product k, if the warranty length T_k is known and the price P_k is a decision variable, there exists an optimum solution that maximises* $\mathrm{E}[\omega_k(P_k,T_k)]-\mathrm{SD}[\omega_k(P_k,T_k)]$.

Proof. The optimisation problem is

$$\begin{aligned}\max\qquad & \mathrm{E}[\omega_k(P_k,T_k)]-\mathrm{SD}[\omega_k(P_k,T_k)],\\ s.t.\qquad & A_k-\beta_kP_k+\eta_kT_k\geq 1,\\ & P_k\geq 0,\end{aligned}$$

where T_k is known.

The second-order derivatives of $\mathrm{E}[\omega_k(P_k,T_k)]-\mathrm{SD}[\omega_k(P_k,T_k)]$ respect to price P_k is

$$\frac{d^2\{\mathrm{E}[\omega_k(P_k,T_k)]-\mathrm{SD}[\omega_k(P_k,T_k)]\}}{dP_k^2}=-2\beta_k<0,$$

means $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ is concave. Hence, the optimisation problem is equivalent to:

$$\begin{aligned} \min \qquad & -\mathrm{E}[\omega_k(P_k, T_k)] + \mathrm{SD}[\omega_k(P_k, T_k)], \\ s.t. \qquad & -A_k + \beta_k P_k - \eta_k T_k + 1 \leq 0, \\ & -P_k \leq 0. \end{aligned}$$

Let

$$\begin{aligned} L &= -\mathrm{E}[\omega_k(P_k, T_k)] + \mathrm{SD}[\omega_k(P_k, T_k)] - h(A_k - \beta_k P_k + \eta_k T_k - 1) - uP_k \\ &= (A_k - \beta_k P_k + \eta_k T_k) \left[-(P_k - T_k \lambda_k \mu_k - c_k) + \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} - h \right] + h - uP_k, \end{aligned}$$

where, $h \geq 0$ and $u \geq 0$ are Lagrange multipliers.

The KKT (Karush-Kuhn-Tucker) conditions of this problem are

$$\frac{\partial L}{\partial P_k} = 2\beta_k P_k - (\beta_k \lambda_k \mu_k + \eta_k) T_k - \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} - \beta_k c_k - A_k + \beta_k h - u = 0,$$

$$h(A_k - \beta_k P_k + \eta_k T_k - 1) = 0,$$

$$uP_k = 0, \quad h \geq 0, \quad u \geq 0,$$

$$-A_k + \beta_k P_k - \eta_k T_k + 1 \leq 0,$$

$$-P_k \leq 0.$$

Then, when

$$\frac{(\beta_k \lambda_k \mu_k + \eta_k) T_k + \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} + \beta_k c_k + A_k}{2\beta_k} \leq \frac{A_k + \eta_k T_k - 1}{\beta_k},$$

the solution is

$$P_k = \frac{(\beta_k \lambda_k \mu_k + \eta_k) T_k + \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} + \beta_k c_k + A_k}{2\beta_k};$$

when

$$\frac{(\beta_k \lambda_k \mu_k + \eta_k) T_k + \beta_k \sqrt{T_k \lambda_k (\sigma_k^2 + \mu_k^2)} + \beta_k c_k + A_k}{2\beta_k} > \frac{A_k + \eta_k T_k - 1}{\beta_k},$$

the solution is

$$P_k = \frac{A_k + \eta_k T_k - 1}{\beta_k}.$$

□

As mentioned above, the warranty claims of different products could be dependent, which implies the variance-covariance matrix of the portfolio includes non-zero covariance.

The covariance between the aggregated warranty costs of product k and l is:

$$\begin{aligned}
\mathrm{Cov}[S_k(T_k), S_l(T_l)] &= \mathrm{E}[S_k(T_k)S_l(T_l)] - \mathrm{E}[S_k(T_k)]\mathrm{E}[S_l(T_l)] \\
&= \mathrm{E}[\mathrm{E}[(\sum_{i=1}^{N_k(T_k)} X_{k,i})(\sum_{i=1}^{N_l(T_l)} X_{l,i})|N(T_k), N(T_l)]] - \lambda_k\mu_k T_k \lambda_k \mu_l T_l \\
&= \mathrm{E}[N_k(T_k)N_l(T_l)]\mathrm{E}[X_k X_l] - \lambda_k\mu_k T_k \lambda_k \mu_l T_l \\
&= \mathrm{Cov}[N_k(T_k), N_l(T_l)]\mu_k\mu_l \qquad (3.15)
\end{aligned}$$

This result is derived under the assumption that the costs of individual warranty claims of product k and l are independent, i.e. X_k and X_l are independent. The dependence among the warranty claims of products produced by one manufacturer relies on the dependent failure rates or time-to-failure caused by the common causes in manufacturing process. Regarding the costs of individual warranty claims, different products have varying requirements of repairing, these can lead to varying and independent costs. Hence, the covariance between the aggregated warranty costs of product k and l is derived under the assumption that the numbers of claims $N_k(T_k)$ and $N_l(T_l)$ are dependent and the costs of individual warranty claims X_k and X_l are independent. Then we have the following Propositions 4, 5 and 6.

Proposition 4. *If $\boldsymbol{P}$ is a vector of decision variables and $\boldsymbol{T}$ is known, there exists an optimum solution that maximises the combination of the profit and the risk of the estimated profit* $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] - \mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$.

Proof. The optimisation problem is

$$\max \qquad F(\boldsymbol{P}, \boldsymbol{T}) = \mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] - \mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})],$$

$$s.t. \boldsymbol{M} \geq 1,$$

$$\boldsymbol{P} \geq 0,$$

where $\boldsymbol{T}$ is known.

Because $\boldsymbol{T}$ is known, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ can be denoted as

$$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k),$$

and

$$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\le k\le l\le n}^{n-1} M_k M_l \sigma_{k,l}^2,$$

where $a_k = \eta_k T_k$, $b_k = T_k\lambda_k\mu_k + c_k$, $d_k = \lambda_k T_k(\sigma_k^2 + \mu_k^2)$ and $\sigma_{k,l}^2 = \mathrm{Cov}[S_k(T_k), S_l(T_l)]$ all are constants.

Then,

$$\begin{aligned} F(\boldsymbol{P},\boldsymbol{T}) &= \mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \\ &= \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k) - \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k \\ &\quad -2\sum_{1\le k\le l\le n}^{n-1} M_k M_l \sigma_{k,l}^2. \end{aligned}$$

The partial derivative of $F(\boldsymbol{P},\boldsymbol{T})$ with respect to P_k is

$$\frac{\partial F}{\partial P_k} = -2\beta_k P_k + \beta_k b_k + A_k + a_k + 2\beta_k(A_k - \beta_k P_k + a_k)d_k + 2\beta_k \sum_{1\le k\le l\le n}^{n-1} M_l \sigma_{k,l}^2,$$

the second order derivatives are

$$\frac{\partial^2 F}{\partial P_k^2} = -2\beta_k(1 + \beta_k d_k),$$

and

$$\frac{\partial^2 F}{\partial P_k \partial P_l} = -2\beta_k\beta_l\sigma_{k,l}^2.$$

The Hessian matrix is

$$
\begin{aligned}
\boldsymbol{H} &= \begin{bmatrix} \frac{\partial^2 F}{\partial P_1^2} & \frac{\partial^2 F}{\partial P_1 \partial P_2} & \cdots & \frac{\partial^2 F}{\partial P_1 \partial P_n} \\ \frac{\partial^2 F}{\partial P_2 \partial P_1} & \frac{\partial^2 F}{\partial P_2^2} & \cdots & \frac{\partial^2 F}{\partial P_2 \partial P_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 F}{\partial P_n \partial P_1} & \frac{\partial^2 F}{\partial P_n \partial P_2} & \cdots & \frac{\partial^2 F}{\partial P_n^2} \end{bmatrix} \\
&= -2 \begin{bmatrix} \beta_1(1+\beta_1 d_1) & \beta_1\beta_2\sigma_{1,2}^2 & \cdots & \beta_1\beta_n\sigma_{1,n}^2 \\ \beta_2\beta_1\sigma_{2,1}^2 & \beta_2(1+\beta_2 d_2) & \cdots & \beta_2\beta_n\sigma_{2,n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \beta_n\beta_1\sigma_{n,1}^2 & \beta_2\beta_2\sigma_{n,2}^2 & \cdots & \beta_n(1+\beta_n d_n) \end{bmatrix} \\
&= -2 \left(\begin{bmatrix} \mathrm{Var}[S_1(T_1)] & \mathrm{Cov}[S_1(T_1), S_2(T_2)] & \cdots & \mathrm{Cov}[S_1(T_1), S_n(T_n)] \\ \mathrm{Cov}[S_2(T_2), S_1(T_1)] & \mathrm{Var}[S_2(T_2)] & \cdots & \mathrm{Cov}[S_2(T_2), S_n(T_n)] \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Cov}[S_n(T_n), S_1(T_1)] & \mathrm{Cov}[S_n(T_n), S_2(T_2)] & \cdots & \mathrm{Var}[S_n(T_n)] \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} \frac{1}{\beta_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\beta_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{\beta_n} \end{bmatrix} \right) \\
&= -2(\boldsymbol{V} + \boldsymbol{B}),
\end{aligned}
$$

where $\boldsymbol{V}$ is a covariance matrix, and $\boldsymbol{B}$ is a positive definite matrix; hence $\boldsymbol{H}$ is negative definite and $F(\boldsymbol{P}, \boldsymbol{T})$ is concave.

Let $\frac{\partial F}{\partial P_k} = 0$, then, we have:

$$-2\beta_k P_k + \beta_k b_k + A_k + a_k + 2\beta_k(A_k - \beta_k P_k + a_k)d_k + 2\beta_k \sum_{1 \le k \le l \le n}^{n-1} M_l \sigma_{k,l}^2 = 0,$$

$$P_k = \frac{\beta_k b_k + A_k + a_k + 2\beta_k(A_k + a_k)d_k + 2\beta_k \sum_{1 \le k \le l \le n}^{n-1} M_l \sigma_{k,l}^2}{2\beta_k(1+\beta_k)}$$

It means the solution, which maximises the objective function $F(\boldsymbol{P}, \boldsymbol{T})$ exists.

□

Proposition 5. *If $\boldsymbol{P}$ is a vector of decision variables and $\boldsymbol{T}$ is known, there exists an optimum solution that maximises* $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$*, subject to* $\mathrm{Var}(\Omega(\boldsymbol{P}, \boldsymbol{T})) \le \varphi$.

Proof. The optimisation problem is

$$\max \qquad \mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})],$$

$$s.t. \qquad \mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \le \varphi,$$

where $\boldsymbol{T}$ is known.

According to the proof of Proposition 2, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ is a concave function on $\boldsymbol{P}$. Then, this optimisation problem is equivalent to

$$\begin{aligned} \min \qquad & -\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})], \\ s.t. \qquad & \mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq \varphi, \end{aligned}$$

where $\boldsymbol{T}$ is known.

Similar to the proof of Proposition 4, $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ can be rewrote as

$$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)(P_k - b_k),$$

and

$$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\leq k\leq l\leq n}^{n-1} M_k M_l \sigma_{k,l}^2,$$

where $a_k = \eta_k T_k$, $b_k = T_k\lambda_k\mu_k + c_k$, $d_k = \lambda_k T_k(\sigma_k^2 + \mu_k^2)$ and $\sigma_{k,l}^2 = \mathrm{Cov}[S_k(T_k), S_l(T_l)]$ all are constants.

However,

$$\frac{\partial \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_k} = -2\beta_k(A_k - \beta_k P_k + a_k)d_k - 2\beta_k \sum_{1\leq k\leq l\leq n}^{n-1} M_l \sigma_{k,l}^2,$$

Then,

$$\begin{cases} \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_k^2} = 2\beta_k^2 d_k \\ \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_k \partial P_l} = 2\beta_k\beta_l\sigma_{k,l}^2. \end{cases}$$

The Hessian matrix,

$$\begin{aligned} \boldsymbol{H} &= \begin{bmatrix} \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_1^2} & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_1 \partial P_2} & \cdots & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_1 \partial P_N} \\ \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_2 \partial P_1} & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_2^2} & \cdots & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_2 \partial P_N} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_N \partial P_1} & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_N \partial P_2} & \cdots & \frac{\partial^2 \mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))}{\partial P_N^2} \end{bmatrix} \\ &= 2\begin{bmatrix} \sigma_1^2 & \sigma_{1,2}^2 & \sigma_{1,3}^2 & \cdots & \sigma_{1,n}^2 \\ \sigma_{2,1}^2 & \sigma_2^2 & \sigma_{2,3}^2 & \cdots & \sigma_{2,n}^2 \\ \sigma_{3,1}^2 & \sigma_{3,2}^2 & \sigma_3^2 & \cdots & \sigma_{3,n}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n,1}^2 & \sigma_{n,2}^2 & \sigma_{n,3}^2 & \cdots & \sigma_n^2 \end{bmatrix}, \end{aligned}$$

is positive-definite, means $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ is a convex function, and the $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T})) \leq \varphi$ is a convex set.

The Lagrange function is

$$L = -\sum_{k=1}^{n}(A_k-\beta_k P_k+a_k)(P_k-b_k)+\rho[\sum_{k=1}^{n}(A_k-\beta_k P_k+a_k)^2 d_k+2\sum_{1\le k\le l\le n}^{n-1} M_k M_l \sigma_{k,l}^2-\varphi],$$

where, $\rho > 0$ is Lagrange multiplier φ is a positive constant.

Take the partial derivative respect to P_k and ρ, and let them equal 0, we have

$$\frac{\partial L}{\partial P_k} = 2\beta_k(1+\rho d_k)P_k - 2\rho\beta_k(A_k d_k + a_k d_k + \sum_{1\le k\le l\le n}^{n-1} M_l\sigma_{k,l}^2) - \beta_k b_k - A_k - a_k = 0,$$

where $k = 1, 2, 3 \ldots n$, and

$$\frac{\partial L}{\partial \rho} = \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\le k\le l\le n}^{n-1} M_k M_l \sigma_{k,l}^2 - \varphi = 0.$$

Hence, the optimal solution is the solution of the following equation set,

$$\begin{cases} 2\beta_1(1+\rho d_1)P_1 - 2\rho\beta_1(A_1 d_1 + a_1 d_1 + \sum_{l\neq 1, l\le n}^{n-1} M_l\sigma_{1,l}^2) - \beta_1 b_1 - A_1 - a_1 = 0 \\ 2\beta_2(1+\rho d_2)P_2 - 2\rho\beta_2(A_2 d_2 + a_2 d_2 + \sum_{l\neq 2, l\le n}^{n-1} M_l\sigma_{2,l}^2) - \beta_2 b_2 - A_2 - a_2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 2\beta_n(1+\rho d_n)P_n - 2\rho\beta_n(A_n d_n + a_n d_n + \sum_{l<n}^{n-1} M_l\sigma_{n,l}^2) - \beta_2 b_2 - A_2 - a_2 = 0 \\ \sum_{k=1}^{n}(A_k - \beta_k P_k + a_k)^2 d_k + 2\sum_{1\le k\le l\le n}^{n-1} M_k M_l \sigma_{k,l}^2 - \varphi = 0 \end{cases}.$$

This set of equations consists of $n+1$ equations with $n+1$ variables, it is complex to provide the proof of the existence of the set's solution in this research. In practice, the number of equations and number of variables are limited, this set of equations can be resolved through numeric simulation. In the numeric example section of this chapter, the optimal set of prices for a portfolio of three products is solved through simulation.

□

Proposition 6. *If $\boldsymbol{P}$ is a vector of decision variables and $\boldsymbol{T}$ is known, there exists an optimum solution that minimises* $\mathrm{Var}(\Omega(\boldsymbol{P}, \boldsymbol{T}))$*, subject to* $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \ge \psi$.

Proof. According to the proof of Proposition 5, when $\boldsymbol{P}$ is the decision variables, and $\boldsymbol{T}$ is known, the $\mathrm{Var}(\Omega(\boldsymbol{P}, \boldsymbol{T}))$ is a convex function, and $-\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$ is a convex function. Hence, -$\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \le -\psi$ is a convex set; and there exists an optimum solution that minimises $\mathrm{Var}(\Omega(\boldsymbol{P}, \boldsymbol{T}))$, subject to $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \ge \psi$. The optimum solution can be obtained through the method of Lagrange multipliers.

□

3.3.3 Methods of estimating $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$

For a given set of warranty claim data, one must estimate $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ for the three options discussed in Section 3.3.2 before any optimisation problems may be discussed. One may estimate them with two methods:

Method 1. The non-parametric method, for example, the method of moments estimation, with which there is no need to assume a probability distribution.

Method 2. The parametric method, for example, the maximum likelihood estimation method. With this method, one needs to estimate a joint distribution of the numbers of warranty claims and then derive $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ and $\mathrm{Var}(\Omega(\boldsymbol{P},\boldsymbol{T}))$ from the distribution.

Similar to the fact that the Pearson correlation coefficient can only describe the linear relationship between two random variables, the covariance can merely measure a linear correlation. As such, one should note, Method 1 may not be able to capture the nonlinear relationship among the numbers of warranty claims. In such cases, Method 2 may be advocated. Given the difficulty of estimating a joint multivariate probability distribution, a tool, *copula*, which is a widely studied topic in recent years and can be used to depict nonlinear relationships between random variables, is borrowed to model the joint distribution of the numbers of warranty claims.

Copulas are widely used in constructing multivariate distributions and formalising the dependence structures between random variables, whatever discrete or continuous. Abe Sklar first introduced the notion of copula in 1959, in recent years copula has attracted considerable attention in both theoretical and application aspects. The Theorem of Sklar states that any cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula that describes the dependence structure between the variables (Wu, 2014a).

Assume $(X_1, ..., X_d)$ is a given vector of random variables, its cumulative distribution function is $H(x_1, ..., x_d) = P(X_1 \leq x_1, ..., X_d \leq x_d))$, and its marginals are $F_k(x_k) = P(X_k \leq x_k)$, where $k = 1, ...d$. Sklar proved that $H(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$,

where $C(.)$ is defined as a copula. Copulas are useful in statistical applications because they allow one to model the marginals and the copula separately when modelling and estimating the distribution of a random vector (Wu, 2014a).

Similar to its work in modelling the dependence among continuous random variables, copula can also be used in constructing the joint probability distribution of discrete variables (Nikoloulopoulos & Karlis, 2010). In this area, Joe & Hu (1996) introduce multivariate parametric families of copulas which are mixtures of max-infinitely divisible (max-id) bivariate copulas. Nikoloulopoulos & Karlis (2010) show this class of copulas has superiority to others, because it allows flexible dependence among the random variables and has a closed form cdf (cumulative distribution function) and thus computations are rather easy.

In our case, the claim arrival process of product k follows a homogeneous Poisson process, i.e. $N_k \sim \text{Pois}(\lambda_k t_k)$. Assume that M_k items of product k have been sold to the market and the claim processes of a given product's items are mutual independent. Denote $u_k(n_k) = P(N_k \leq n_k)$, the joint distribution of each products' number of claim is

$$H(n_1, n_2, \ldots, n_k) = C(u_1(n_1), u_2(n_2), \ldots, u_k(n_k)), \tag{3.16}$$

where C is a copula. Then, the joint probability mass function is,

$$h(n_1, n_2, \ldots, n_k) = \sum_{\substack{y_j \in \{n_j, n_j - 1\} \\ 1 \leq j \leq k}} \left(H(y_1, y_2, \ldots, y_k) \prod_{i=1}^{k} \text{sgn}(y_i) \right), \tag{3.17}$$

where y_j is equal to n_j or $n_j - 1$, and

$$\text{sgn}(y_j) = \begin{cases} 1, y_j = n_j \\ -1, y_j = n_j - 1. \end{cases}$$

Apparently, $\sum_{\substack{y_j\in\{n_j,n_j-1\}\\1\le j\le k}}$ has 2^k elements in total. For example,

$$\begin{aligned}
h(n_1,n_2,n_3) &= \sum_{\substack{y_j\in\{n_j,n_j-1\}\\1\le j\le 3}} [H(y_1,y_2,y_3)\prod_{i=1}^{3}\mathrm{sgn}(y_i)] \\
&= H(n_1,n_2,n_3)-H(n_1-1,n_2,n_3)-H(n_1,n_2-1,n_3)-H(n_1,n_2,n_3-1) \\
&\quad +H(n_1-1,n_2-1,n_3)+H(n_1-1,n_2,n_3-1)+H(n_1,n_2-1,n_3-1) \\
&\quad -H(n_1-1,n_2-1,n_3-1).
\end{aligned}$$

The expected total number of warranty claims of the manufacturer is

$$\mathrm{E}[N]=\sum_{j=1}^{m}[h(n_{1,j},n_{2,j},\ldots,n_{k,j})(n_{1,j}+n_{2,j}+\cdots+n_{k,j})],$$

where m is an integer determined by the sampling frequency in calculation. The expected total profit is

$$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]=\sum_{j=1}^{m}\{h(n_{1,j},n_{2,j},\ldots,n_{k,j})[M_1(P_1-n_{1,j}\mu_1-c_i)+\cdots+M_k(P_k-n_{k,j}\mu_k-c_k)]\}. \tag{3.18}$$

The variance of total expected profit is

$$\begin{aligned}
\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] &= \sum_{k=1}^{n}(A_k-\beta_kP_k+\eta_kT_k)^2\lambda_kT_k(\sigma_k^2+\mu_k^2) \\
&\quad +2\sum_{1\le k<l\le n} M_kM_l\mu_k\mu_l\mathrm{Cov}[N_k(T_k),N_l(T_l)], \qquad (3.19)
\end{aligned}$$

where $\mathrm{Cov}[N_k(T_k),N_l(T_l)]$ can be calculated according to the joint probability mass function $h(n_1,n_2,\ldots,n_k)$.

Furthermore, the joint distribution of the total profit can also be estimated based on the joint probability mass function $h(n_1,n_2,\ldots,n_k)$,

$$\begin{aligned}
&\mathrm{P}\left\{(A_1-\beta_1P_1+\eta_1T_1)\left[P_1-S_1(T_1)\right]=z_1,...,(A_n-\beta_nP_n+\eta_nT_n)\left[P_n-S_n(T_n)\right]=z_n\right\} \\
= \; & h(n_1,n_2,\ldots,n_n)\prod_{i=1}^{n_1}P(X_{1,i}=\frac{z_1}{n_1})\prod_{i=1}^{n_2}P(X_{2,i}=\frac{z_2}{n_2})\cdots\prod_{i=1}^{n_n}P(X_{n,i}=\frac{z_n}{n_n}). \qquad (3.20)
\end{aligned}$$

In this section, the estimation of the joint distribution of the numbers of warranty claims of the products produced by a manufacturer is demonstrated based on a copula approach. The covariance among the products' profits and the variance of the total profit of the manufacturer are calculated based on the joint distribution.

3.4 Numeric example

Assume a manufacturer produces 3 products. The numbers of warranty claims follow homogeneous Poisson processes with intensity functions λ_1, λ_2, and λ_3, respectively. The warranty claim costs of the three products follow normal distributions with means μ_1, μ_2, μ_3, respectively; and standard deviations σ_1, σ_2, and σ_3, respectively.

The related parameters of these products are presented in Table 3.1. $\boldsymbol{A}$, $\boldsymbol{\beta}$, and $\boldsymbol{\eta}$ are the parameter vectors of sales volume function; $\boldsymbol{\lambda}$ is the intensity vector of the HPPs; $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are the mean and standard deviation of claim cost; and $\boldsymbol{c}$ is the fixed cost vector.

Table 3.1: Parameters

	$\boldsymbol{A}$	$\boldsymbol{\beta}$	$\boldsymbol{\eta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	$\boldsymbol{\sigma}$	$\boldsymbol{c}$
Product 1	$A_1 = 8000$	$\beta_1 = 80$	$\eta_1 = 5$	$\lambda_1 = 0.08$	$\mu_1 = 6$	$\sigma_1 = 1$	$c_1 = 40$
Product 2	$A_2 = 9000$	$\beta_2 = 85$	$\eta_2 = 6$	$\lambda_2 = 0.09$	$\mu_2 = 5$	$\sigma_2 = 0.8$	$c_2 = 50$
Product 3	$A_3 = 6000$	$\beta_3 = 60$	$\eta_3 = 10$	$\lambda_3 = 0.1$	$\mu_3 = 7$	$\sigma_3 = 1.5$	$c_3 = 30$

In this section, the unit of warranty length is assumed to be month and the unit of cost is assumed to be GBP (Great Britain Pound).

3.4.1 Maximising the expected profit

The maximum expected total profit is the sum of all products' maximum expected profit. However, if P_k and T_k both are decision variables, the function, $\mathrm{E}[\omega_k(P_k, T_k)] = (A_k - \beta_k P_k + \eta_k T_k)(P_k - T_k \lambda_k \mu_k - c_k)$, does not have a maximum value.

Based on Proposition 1, Fig. 3.1 shows the relationship between the maximised expected profit $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ and the warranty length T_k of product k. For each given T_k, $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ exists.

Based on Proposition 2, Fig. 3.2 presents the case that when $T_k = 24$ ($k = 1, 2, 3$), the expected-profit against price curves are parabolic. There exists an optimal value P_k to maximise $\mathrm{E}[\omega_k(P_k, T_k)]$ for each product. If the warranty length of the products is 2 years, i.e. 24 months, Table 3.2 presents the optimal prices and the maximised expected profits, i.e. the values at peak points of the curves in Fig. 3.2.

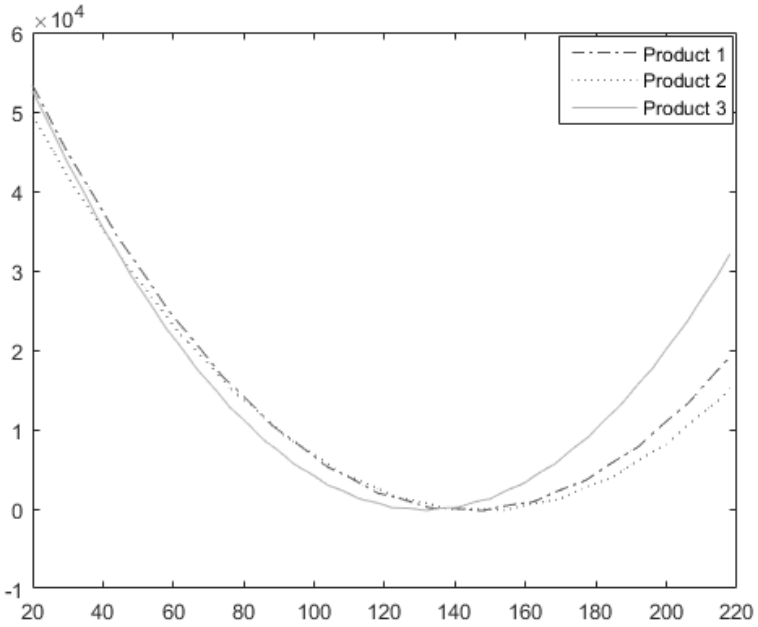

Figure 3.1: $\mathrm{Max}\{\mathrm{E}[\omega_k(P_k, T_k)]\}$ (on the Y-axis) against T_k (on the X-axis).

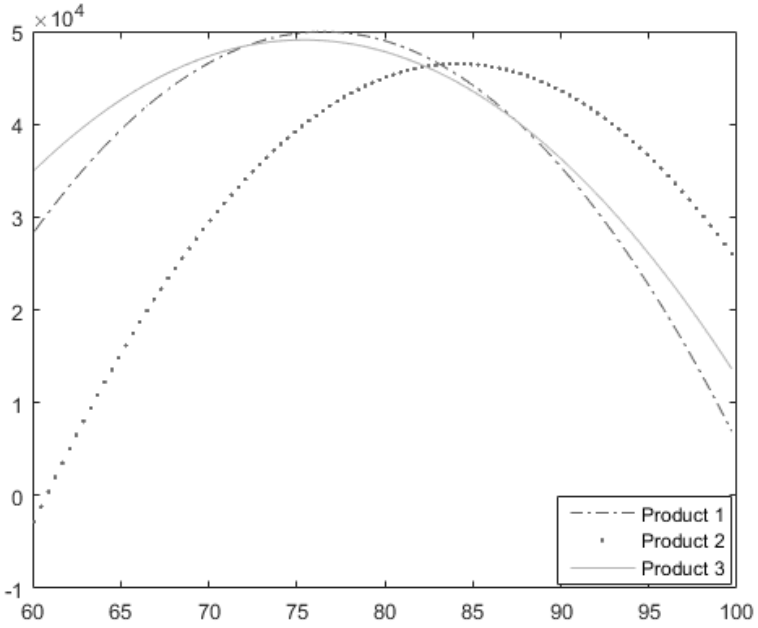

Figure 3.2: $\mathrm{E}[\omega_k(P_k, T_k)]$ (on the Y-axis) against P_k (on the X-axis).

Table 3.2: Maximum profits of products with given warranty length, 2 years.

Warranty length	T_1=24	T_2=24	T_3=24
Optimal price	P_1=76.51	P_2=84.19	P_3=75.40
Maximised expected profit	49,960.01	46,495.81	49,077.60

3.4.2 Mean-variance optimisation approach

Based on Proposition 4, the manufacturer's profit can be optimised when the risk is taken into consideration. For product k, Fig. 3.3 presents the results of Option 1, as discussed in Section 3.2, in which the manufacturer tries to maximise $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ when $T_k = 24$ ($k = 1, 2, 3$). This figure shows there exists an optimal P_k which maximise $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ for each product.

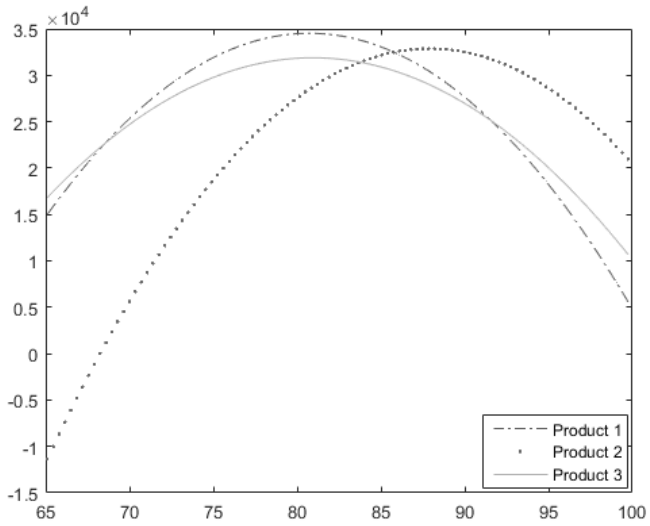

Figure 3.3: $\mathrm{E}[\omega_k(P_k, T_k)] - \mathrm{SD}[\omega_k(P_k, T_k)]$ (on the Y-axis) against P_k (on the X-axis).

For Option 2 mentioned in Section 3.2, the covariance $\boldsymbol{V}$ shown in Eq. (3.13) should be estimated. In this case, instead of estimating $\boldsymbol{V}$, we simply assume its correlation matrix, as shown in the following:

$$\boldsymbol{\rho} = \begin{Bmatrix} 1 & 0.3 & 0.4 \\ 0.3 & 1 & 0.5 \\ 0.4 & 0.5 & 1 \end{Bmatrix}. \tag{3.21}$$

Table 3.3 shows the result of Option 2, which agrees with Proposition 5 that there exists an optimal solution. Under the assumption that the manufacturer's goal is to maximise profit when the variance of the profit should be less than 100,000. Table 3.3 presents the

optimal prices when $\boldsymbol{T}$ are varying within 24-month and 36-month. If the dependence

Table 3.3: Maximum profits with different $\boldsymbol{T}$ at given variance, where $\mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \leq 100,000$.

Warranty length	T_1=24, T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=98.37 P_2=100.63 P_3=104.00	P_1=102.25 P_2=99.66 P_3=102.46	P_1=100.62 P_2=106.00 P_3=100.73	P_1=98.71 P_2=105.30 P_3=106.00
Maximised expected profit	35,235.27	31,303.88	24,449.68	22,106.10

among the products are ignored, i.e. use identity matrix $\boldsymbol{I}$ instead of the correlation matrix $\boldsymbol{\rho}$ in computing, the optimal results under the same assumption are presented in Table 3.4. The differences between the contents of Tables 3.3 and 3.4 indicate that if the depen-

Table 3.4: Maximum profits with different $\boldsymbol{T}$ at given variance ignoring dependence, where $\mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \leq 100,000$.

Warranty length	T_1=24, T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=92.92 P_2=96.98 P_3=103.81	P_1=100.95 P_2=95.20 P_3=99.85	P_1=99.80 P_2=102.93 P_3=98.53	P_1=96.65 P_2=101.02 P_3=106.00
Maximised expected profit	61,632.49	53,951.25	42,455.64	39,544.61

dence among products are ignored, the manufacturer may misprice individual products and overvalue the expected total profit.

Similarly, when the same $\boldsymbol{\rho}$ in Eq. (3.21) is used, Table 3.5 shows the result of Option 3, which agrees with Proposition 6 that there exists an optimal solution. Under the assumption that the manufacturer's goal is to minimise the variance of the profit when the expected profit should not be less than 50,000, Table 3.5 presents the optimal prices when $\boldsymbol{T}$ are varying within 24-month and 36-month. When the dependences among products are ignored, the optimal results under the same assumption are presented in Table 3.6. The results in Table 3.5 and 3.6 indicate if the manufacturer does not recognise the dependences among the products, the total risk level will be undervalued.

Table 3.5: Minimum variance with different $\boldsymbol{T}$ at given expected profit level, where $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq 50,000$.

Warranty length	T_1=24,T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=94.33 P_2=100.36 P_3=103.65	P_1=100.22 P_2=98.14 P_3=99.89	P_1=97.85 P_2=103.52 P_3=97.36	P_1=94.70 P_2=100.86 P_3=104.26
Minimised variance	147,995.66	176,226.69	218,509.72	258,176.05

Table 3.6: Minimum variance with different $\boldsymbol{T}$ at given expected profit level ignoring dependence, where $\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq 50,000$.

Warranty length	T_1=24,T_2=24 T_3=24	T_1=36, T_2=24 T_3=24	T_1=36, T_2=36 T_3=24	T_1=36, T_2=36 T_3=36
Optimal price	P_1=95.29 P_2=98.83 P_3=104.00	P_1=101.45 P_2=95.46 P_3=100.43	P_1=98.89 P_2=102.22 P_3=97.48	P_1=94.43 P_2=98.75 P_3=106.00
Minimised variance	76,432.03	90,178.34	120,554.13	134,952.03

3.4.3 Measuring dependence with copulas

In case more complicated dependence is assumed, as discussed in Section 3.3, the copulas can be applied. Here, we assume that the multivariate Gumbel copula is applied. The multivariate Gumbel copula can capture strong upper tail dependence and can be used in the case, for example, there is a design problem on a type engine used on different types of car, many similar failures, which causes warranty claims, may occur in the early life period, or the premature period. The multivariate Gumbel copula is,

$$C(u_1,u_2,u_3;\theta_1,\theta_2) = \exp\{-([(-\ln u_1)^{\theta_2} + (-\ln u_2)^{\theta_2}]^{\frac{\theta_1}{\theta_2}} + (-\ln u_3)^{\theta_1})^{\frac{1}{\theta_1}}\}. \quad (3.22)$$

Assume $\theta_1 = \theta_2 = 2$ in this case. In practice, these parameters may be estimated by

Table 3.7: Optimal solution based on trivariate Gumbel copula, where $\boldsymbol{T'} = \{24,24,24\}$.

	Option 1	Option 2	Option 3
Constraint	$\boldsymbol{M} \geq 1$	$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})] \leq 100,000$ $\boldsymbol{M} \geq 1$	$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] \geq 1,000$ $\boldsymbol{M} \geq 1$
Optimal price	P_1=83.76 P_2= 113.53 P_3= 85.81	P_1=83.75 P_2= 107.56 P_3= 85.80	P_1=101.49 P_2= 107.57 P_3= 103.98
Optimised objective value	$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})] - \mathrm{SD}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ =301,582.62	$\mathrm{E}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ =299,770	$\mathrm{Var}[\Omega(\boldsymbol{P},\boldsymbol{T})]$ =12,031

empirical data. It is easy to obtain Eq. (3.20) by substituting $\theta_1 = \theta_2 = 2$ into Eq. (3.22). Then, the three options mentioned in Section 3.2 can be solved. The results are shown in Table 3.7 and interpreted below.

- In the 2nd column, the objective function $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] - \mathrm{SD}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$ is maximised with the constraints that $\boldsymbol{T'} = \{24, 24, 24\}$ and $\boldsymbol{M} = \boldsymbol{A} - \boldsymbol{\beta P} + \boldsymbol{\eta T} \geq 0$.
- In the 3rd column, the objective function $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$ is maximised with the constraints that $\boldsymbol{T'} = \{24, 24, 24\}$, $\mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \leq 100,000$ and $\boldsymbol{M} \geq 1$.
- In the 4th column, the objective function $\mathrm{Var}[\Omega(\boldsymbol{P}, \boldsymbol{T})]$ is minimised with the constraints that $\boldsymbol{T'} = \{24, 24, 24\}$, $\mathrm{E}[\Omega(\boldsymbol{P}, \boldsymbol{T})] \geq 1,000$ and $\boldsymbol{M} \geq 1$.

3.4.4 Remarks

In this section, the optimal prices of the three products with various warranty lengths are illustrated with numerical examples. Section 3.4.1 demonstrates the traditional warranty analysis method, which does not consider the risk caused by the uncertainty of warranty claim. Sections 3.4.2 and 3.4.3 demonstrate the optimisation problems with a consideration of risks represented by variance. Compared with the traditional methods, the proposed new methods have two advantages: firstly both the expected profit and corresponding risk are considered in warranty policy optimisation, which can help the manufacturer to manage its operational risk in an efficient manner; secondly, the warranty risk of a manufacturer are estimated collectively, which means the warranty policies are optimised under more meaningful and accurate constraints than those of the traditional methods.

3.5 Summary

This chapter optimally prices warranty policies when the dependence among warranty claims of different products is taken into consideration. Such optimisation is performed using the mean-variance optimisation approach, which considers the profit of a portfolio of different products with correlated numbers of warranty claims. The variance of the

total profit of the portfolio of products is calculated based on a copula-based discrete joint distribution of the number of warranty claims of the products produced by a manufacturer.

This chapter provides a collective warranty policy optimisation method when the expected profit and corresponding risk are taken into consideration. From a practical and applicable perspective, this method emphasizes the risk and potential dependence in warranty management and provides decision makers with a new approach to optimising the trade-off between the profit and risk in operation.

In the discussion in this chapter, the process of warranty claims is assumed to be the homogeneous Poisson process (HPP). In addition to HPP, other processes, including the non-homogeneous Poisson process, the doubly Poisson process, and the like, may be used to model the numbers of warranty claims. In real application, one may analyse warranty claim data and then decide which a stochastic process should be used.

Our future research will aim to answer the following questions:

(1) the mean-variance optimisation approach to the situation that the compound non-homogeneous Poisson process is applied;

(2) the selection of a proper copula to construct the joint distribution;

(3) the modelling of the varying dependence in case that the dependences among the products are varying.

Chapter 4

A value-at-risk approach to optimisation of warranty policy

This chapter is developed from Chapter 3. The dependence among the warranty claims of different products produced by the same manufacturer is also considered a real world issue is considered; and this chapter also collectively optimises warranty policy for a set of different products whose failures are statistically dependent. However, according the literature in financial area, even though the variance of a random variable is applied as a risk measure by the overwhelming influential models of portfolio selection, the symmetry of this measure cannot reflect the risk comprehensively. In this chapter, the downside risk measures, borrowed from financial mathematics, such as Value-at-Risk and Conditional Value-ar-Risk are employed. The distributions of the warranty costs of a single product and the portfolio of products are provided. Then, the existences of the optimal solutions for different scenarios are proved on copula-based models. Numerical examples are used to validate the applicability of the proposed methods.

4.1 Introduction

4.1.1 Background and literature review

To improve a manufacturer's operating expense management, an efficient warranty policy management supported by warranty data analysis and modelling is necessary. It is known that the number of warranty claims and the associated cost are uncertain, and products with longer warranty periods may attract more buyers than those with shorter warranty periods. From a manufacturer's perspective, however, providing a longer warranty period implies more resources that are needed to handle warranty claims. Hence, there is a need to develop approaches to optimising the warranty price and the warranty length.

In the literature, many methods aiming to optimise the warranty price and the warranty length of an individual product have been proposed. Fig. 4.1 illustrates the evolution of the research in warranty policy optimisation, which shows that the research evolves from simple and unrealistic assumptions to more complex and realistic ones.

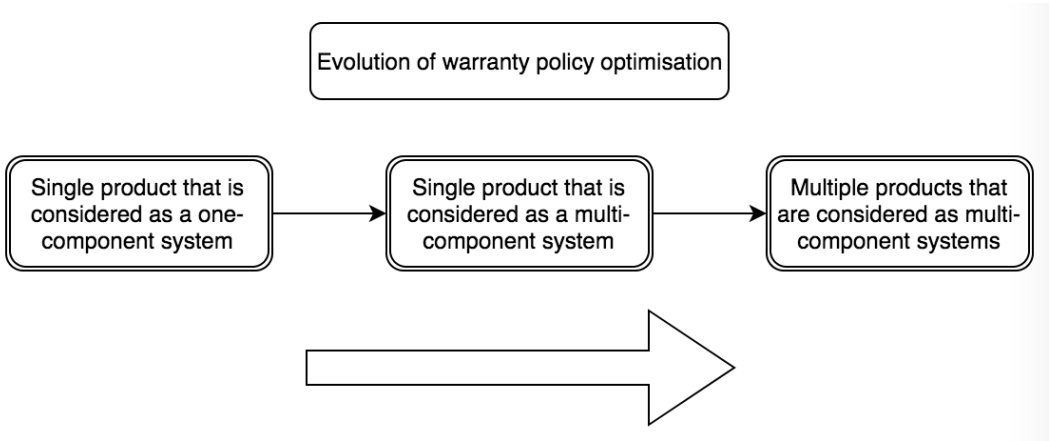

Figure 4.1: Evolution of warranty policy optimisation

At the early stage, many researchers attempt to find the optimal price and warranty length, assuming that the product is composed of only one component. At the same time, some other factors, such as production rate, market competition and demand, etc., are also considered. Ladany & Shore (2007) address a method to determine the optimal warranty period with considering the products lifetime and market demand. Lin et al. (2009) optimise the price, warranty length and production rate of a one component system dy-

namically. Wu et al. (2009) develop a decision model to determine the optimal price, the length of warranty and the production rate to maximise profit based on the pre-determined life cycle in a static demand market. Aggrawal et al. (2014) present a method to optimise warranty price and the length of warranty for a product based on a two-dimensional innovation diffusion model, and estimate the overall maximum profit for the manufacturer. Wei et al. (2015) investigates the optimal strategies on product price and the length of warranty of two products produced by two manufacturers and sold by one dealer. Yazdian et al. (2016) jointly optimises the acquisition price, re-manufacturing degree, selling price and the length of warranty of re-manufacturing products under linear and non-linear demand functions.

The one-component assumption of a single product may be too unrealistic. Researchers then consider the assumption that a product is composed of multiple components. Huang et al. (2007) develop a model to determine the optimal product reliability, price and warranty strategy to achieve the maximum total integrated profit for a general repairable multi-component product sold under a free replacement-repair warranty strategy. Matis et al. (2008) explore the optimal price and pro rate warranty length for a multi-component product with considering the different repair options on the components. Bai & Pham (2006) investigate optimisation of warranty policies for single products composed of multiple components. Liu et al. (2015) also investigate the warranty cost for a single product consisting of multiple components; meanwhile, as an improvement, the failure interactions between the components are considered. Ahmadi (2016) addresses an optimal replacement problem for complex multi-component systems by determining an optimal operating time which balances income and cost to maximizes the expected profit over a cycle. Adkins & Paxson (2017) construct a general replacement model for a multi-component product with considering the salvage value and depreciation in operating. Chen et al. (2017) seek to maximize the total profit per item of a multi-component product through optimally determine the production run length and the warranty period.

All of the literature mentioned above solely maximises the profit of individual products produced by a manufacturer. Little research, however, has been devoted to optimising

warranty policy of a set of products collectively. However, in real world, a manufacturer may assemble different types of product by sharing some key components; then, the manufacturer has to deal with the warranty policies for multiple products considered as multi-component systems, which is the third stage of warranty policy optimisation research (see the third rectangle in Fig. 4.1 and investigated in this chapter.

4.1.2 Motivation and our approach

The warranty claim arrival processes of products may not be statistically independent because the claims may have common causes such as similar design, same production lines and same types of components installed in the products. For example, Apple, iPhone 6, iPhone 6 Plus and iPad Mini 4, have the same type of CPU. If any design or quality problems happen on one of the products, warranty claims of the other products will crop up during a short period.

As can be seen from the above literature review, however, little attention has been paid to collectively optimise warranty policies for a portfolio of different products. This motivates us to develop novel approaches to filling in the knowledge gap.

We propose to optimise warranty policy through maximising product profit. The method collectively optimises the warranty price and the warranty length of a set of different products whose warranty claims are statistically dependent, considering the uncertainty of the product profit. The value-at-risk theory is borrowed to manage the uncertainty. The dependence is modelled by *copula*, a tool from the probability theory. The use of copula provides a more flexible tool to model more complicated dependence than a simple method such as covariance estimation.

The novelty of this chapter lies in the fact that it is the first attempt to collectively optimise the warranty policies of a portfolio of products from a manufacturer's perspective.

4.1.3 Overview

The rest sections of the chapter are structured as follows. Section 4.2 lists the assumptions of the optimisation problems. It also formulates the profits and warranty costs of

individual products and portfolio of products, respectively. Section 4.3 investigates the existence of the optimal solutions for the different optimisation problems and uses copula to model the dependence among warranty claims of different products. Section 4.4 offers numerical examples to illustrate the proposed methods and to validate the applicability of the proposed methods. Section 4.5 concludes this chapter and proposes our future work.

4.2 Formulation of the problem

Assume a manufacturer offers non-renewing free replacement warranty (NFRW) policies. Under an NFRW policy, the manufacturer provides its customers with repair or replacement at no cost within the warranty period; the original warranty is not altered upon a failed item; and the manufacturer only guarantees satisfactory service on the item within the original warranty period. Assume that repair time is negligible and the repair is minimal repair. Products are new at $t = 0$ when they are sold. The number of claims follows the non-homogeneous Poisson process (NHPP). The numbers of warranty claims and the claim cost are statistically independent.

4.2.1 The sales volume and profit

For a product, there are two critical marketing variables: the selling price and the warranty length (Chen et al., 2017). The sales volume of a product is negatively related to its selling price and positively related to its warranty length. Denote P_k and T_k as the selling price P_k and the warranty length T_k of product k, respectively. Both P_k and T_k can influence the sales volume, M_k, and profit, ω_k. In what follows, the profit of product k is the revenue deducting the warranty cost, i.e. $\omega_k = M_k P_k - S_k(T_k)$, where $S_k(T_k)$ is the aggregated warranty cost of product k within T_k.

In the literature, the sales volume of product k, M_k, is expressed by a function of product price P_k and length of warranty T_k in different forms, including linear Lin et al. (2009); Yazdian et al. (2016) and non-linear ones (Huang et al., 2007; Ladany & Shore, 2007; Xie et al., 2014). For simplicity, a linearity form, introduced by Yazdian et al.

(2016), is used in this chapter. The sales volume is defined by

$$M_k = A_k - \beta_k P_k + \eta_k T_k, \tag{4.1}$$

where $A_k(> 0)$ is a constant relating to the market size of product k, and $\beta_k(> 0)$ and $\eta_k(> 0)$ are the price and length of warranty elasticities, respectively.

4.2.2 The distribution of the aggregated warranty cost

Suppose a manufacturer produces n products. The aggregated warranty cost of product k follows a stochastic process $\{S_k(T_k)\}_{T_k \geq 0}$ over the warranty period $(0, T_k)$, which is expressed by the following equation,

$$S_k(T_k) = \sum_{j=1}^{N_k(T_k)} X_{k,j}, \tag{4.2}$$

where $X_{k,j}$ is the cost of the j-th claim of product k and $N_k(T_k)$ is the number of claims during $(0, T_k)$. $\{X_{k,1}, X_{k,2}, \ldots, X_{k,j}\}$ are independent and identically distributed random variables which have finite values on the positive half-line $\mathbb{R}_{>0}$ with the probabilities $P(X_{k,j})$. The cost of claims is assumed to follow the log-normal distribution. The counting process $N_k(T_k)$ is assumed to take a form of the NHPP with cumulative intensity $M_k\Lambda_k(T_k)$, and $P(N_k(T_k) = n) = \frac{(M_k\Lambda_k(T_k))^n}{n!} e^{-M_k\Lambda_k(T_k)}$. $N_k(t)$ and $X_{k,i}$, are assumed to be statistically independent.

The expected value of $S_k(T_k)$ is given by

$$\mathrm{E}[S_k(T_k)] = \mathrm{E}[N_k(T_k)]\mathrm{E}[X_k] = M_k\Lambda_k(T_k)\mu_k, \tag{4.3}$$

and the variance of $S_k(T_k)$ is given by

$$\begin{aligned} \mathrm{Var}\left[S_k(T_k)\right] &= \mathrm{E}[N_k(T_k)]\mathrm{Var}[X_k] + \mathrm{Var}[N_k(T_k)]\mathrm{E}[X_k]^2 \\ &= M_k\Lambda_k(T_k)(\mathrm{Var}[X_k] + \mathrm{E}[X_k]^2) \\ &= M_k\Lambda_k(T_k)(\sigma_k^2 + \mu_k^2), \end{aligned} \tag{4.4}$$

where μ_k and σ_k are the mean and the variance of X_k, respectively.

Denote F_{S_k} and f_{S_k} as the probability density function (pdf) and the cumulative distribution function (cdf) of $S_k(T_k)$, respectively. The characteristic function of X_k

is $\varphi_{X_k}(t) = \int_{-\infty}^{\infty} f_{X_k}(x)e^{itx}dx$, where i is a unit imaginary number. Denote $\varphi_{S_k}(t)$ as the characteristic function of S_k. The probability generation function of $N_k(T_k)$ is $\psi_k(s) = \sum_{n=0}^{\infty} s^n p_{k,n}$, where $p_{k,n} = \Pr\{N(T_k) = n\}$. According to the Levy-Khintchine formula, one can obtain

$$\varphi_{S_k}(t) = \sum_{n=0}^{\infty} (\varphi_{X_k}(t))^n p_{k,n} = e^{M_k \Lambda_k(T_k)(\varphi_{X_k}(t)-1)}. \tag{4.5}$$

Then, the density of $S_k(T_k)$ can be calculated through the inverse Fourier transform, that is

$$f_{S_k}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{S_k}(t)e^{-itx}dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{M_k \Lambda_k(T_k)(\varphi_{X_k}(t)-1)-itx}dt.$$

Apparently, $S_k(T_k)$ is non-negative. According to Luo & Shevchenko (2009), the probability density and cumulative distribution function of $S_k(T_k)$ are given by

$$f_{S_k}(z) = \frac{2}{\pi} \int_0^{\infty} Re[\varphi_{S_k}(t)]\cos(tz)dt, \tag{4.6}$$

and

$$F_{S_k}(z) = \frac{2}{\pi} \int_0^{\infty} Re[\varphi_{S_k}(t)]\frac{\sin(tz)}{t}dt, \tag{4.7}$$

respectively, where $z \geq 0$.

In practice, computing the density and cumulative distribution through the Fourier transform requires high computing power to deal with the underflow and overflow problems. Another popular approach is to use an approximating distribution to avoid direct calculation of Eq. (4.7) and Eq. (4.6)). According to Bee (2016), if the distribution of X_k is subexponential, the compound $S_k(T_k)$ inherits the subexponentiality property from X_k. In this case, X_k is assumed to follow the log-normal distribution, which is subexponential. Hence, $S_k(T_k)$ is also subexponential, and X_k and $S_k(T_k)$ are tail equivalent. For details about the definition of *tail equivalent*, the reader is referred to (Bee, 2016).

To make this chapter concise, we use the log-normal distribution to approximate $S_k(T_k)$.

4.2.3 Product profit

In this subsection, we derive the profit of products for different scenarios.

The profit of one product

The profit of product k is given by

$$\omega_k = M_k P_k - S_k(T_k). \tag{4.8}$$

Denote F_{ω_k} and f_{ω_k} as the cdf and pdf of ω_k, respectively. Then

$$\begin{aligned} F_{\omega_k}(z) &= P[\omega_k \leq z] \\ &= P[M_k P_k - S_k(T_k) \leq z] \\ &= P[S_k(T_k) \geq M_k P_k - z] \\ &= 1 - F_{S_k}(M_k P_k - z), \end{aligned} \tag{4.9}$$

which can be calculated on the basis of Eq. (4.7)).

The expected value of ω_k is

$$\mathrm{E}[\omega_k] = M_k[P_k - \Lambda_k(T_k)\mu_k], \tag{4.10}$$

and the variance of ω_k is

$$\mathrm{Var}[\omega_k] = \mathrm{Var}[S_k(t)] = M_k \Lambda_k(T_k)(\sigma_k^2 + \mu_k^2). \tag{4.11}$$

The profit of a portfolio of products

The total profit of a portfolio consisting of N products is given by

$$\Omega = \sum_{k=1}^{N} \omega_k = \sum_{k=1}^{N} [M_k P_k - S_k(T_k)]. \tag{4.12}$$

The distribution of the profit of product portfolio can be expressed by

$$F_\Omega(z) = P\{\Omega \leq z\} = F^{(N)}(z), \tag{4.13}$$

where $F^{(N)}(z)$ is the N-fold convolution of the distribution of ω_k. The expected total profit is

$$\mathrm{E}[\Omega] = \sum_{k=1}^{N} \mathrm{E}[\omega_k] = \sum_{k=1}^{N} M_k[P_k - \Lambda_k(T_k)\mu_k]. \tag{4.14}$$

As we mentioned in Section 1.2, the warranty claim arrival processes of the products produced by the same manufacturer may be correlated. Hence the variance of the portfolio

profit is related to the correlation among the products. If the claim arrival processes of different products are linearly correlated, the variance of Ω can be calculated based on the covariance matrix of the portfolio, which is

$$\mathrm{Var}[\Omega] = \boldsymbol{I}^T \boldsymbol{V} \boldsymbol{I}, \tag{4.15}$$

where $\boldsymbol{I}^T = [1, 1, \ldots, 1]$ and

$$\boldsymbol{V} = \begin{bmatrix} \mathrm{Var}(S_1(T_1)) & \mathrm{Cov}(S_1(T_1), S_2(T_2)) & \ldots & \mathrm{Cov}(S_1(T_1), S_n(T_n)) \\ \mathrm{Cov}(S_2(T_2), S_1(T_1)) & \mathrm{Var}(S_2(T_2)) & \ldots & \mathrm{Cov}(S_2(T_2), S_n(T_n)) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Cov}(S_n(T_n), S_1(T_1)) & \mathrm{Cov}(S_n(T_n), S_2(T_2)) & \ldots & \mathrm{Var}(S_n(T_n)) \end{bmatrix}, \tag{4.16}$$

where $\mathrm{Cov}(S_k(T_k), S_i(T_i)) = \rho_{k,i}\sqrt{\mathrm{Var}(S_k(T_k))\mathrm{Var}(S_i(T_i))}$, and $\rho_{k,i}$ is the Pearson correlation coefficient. The warranty costs of different products may have different types of dependence such as a rank correlation or a tail-dependence. In such cases, the metrics that can only measure a linear correlation may be inappropriate in the case where the relationship of the variables is not linear. As such, we employ a powerful tool, *copula*, to model the dependence among the products. According to Boubaker & Sghaier (2013), Kendall's tau and copula parameters, especially the parameters of the Archimedean copulas, can be used as substitutes (and more comprehensive metrics) for measuring both the linear and the nonlinear relationships in the covariance matrix in portfolio optimisation.

4.3 Mean-risk optimisation

In this chapter, the mean-risk optimisation is used to maximise the expected profit under an acceptable risk level. The objective functions are illustrated in Eq. (4.10) and Eq. (4.14) for one product and multiple products scenarios, respectively. The constraints of optimisation problems are given for different risk measures.

In the one product scenario, the manufacturer aims to maximise the following function:

$$\mathrm{E}[\omega_k] = M_k[P_k - \Lambda_k(T_k)\mu_k].$$

Suppose in this case, the NHPP following the cumulative intensity $\Lambda_k(T_k) = a_k T_k^{b_k}$

(where $b_k > 1$) are used. Then, the objective function is defined by

$$\mathrm{E}[\omega_k] = A_k P_k - A_k \mu_k a_k T_k^{b_k} - \beta_k P_k^2 + \beta_k \mu_k P_k a_k T_k^{b_k} + \eta_k P_k T_k - \eta_k \mu_k a_k T_k^{b_k+1}. \quad (4.17)$$

In case both of P_k and T_k are decision variables, based on the properties of this function, we have Proposition 7.

Proposition 7. *Depending on whether P_k and T_k are known, one can prove the following results.*

- *If both of P_k and T_k are decision variables, the global maxima of $\mathrm{E}[\omega_k]$ does not exist.*
- *If T_k (or P_k) is the decision variable and P_k (or T_k) is known, the global maxima of the function $\mathrm{E}[\omega_k](T_k)$ (or $\mathrm{E}[\omega_k](P_k)$) exists.*

Proof. The Hessian matrix of $\mathrm{E}[\omega_k]$ is

$$\boldsymbol{H}_{f(P_k,T_k)} = \begin{bmatrix} -2\beta_k & a_k b_k \beta_k \mu_k T_k^{b_k-1} + \eta_k \\ a_k b_k \beta_k \mu_k T_k^{b_k-1} + \eta_k & -a_k b_k \mu_k T_k^{b_k-2}[(b_k-1)(A_k - \beta_k P_k + \eta_k T_k) + 2\eta_k T_k] \end{bmatrix},$$

and the eigenvalues of $\boldsymbol{H}_{\mathrm{E}[\omega_k]}$ are

$$x_1 = -\{2\beta_k + a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}$$
$$+ \sqrt{\{2\beta_k - a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}^2 + 4(a_k b_k \beta_k \mu_k T^{b-1} + \eta_k)^2},$$

and

$$x_2 = -\{2\beta_k + a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}$$
$$- \sqrt{\{2\beta_k - a_k b_k \mu_k T^{b-2}[(b_k-1)A_k - (b_k-1)\beta_k P_k + (b_k+1)\eta_k T_k]\}^2 + 4(a_k b_k \beta_k \mu_k T^{b-1} + \eta_k)^2},$$

respectively. It can be seen that $x_1 > 0$ and $x_2 < 0$. As such, the Hessian matrix is indefinite. As a result, the global minima of $\mathrm{E}[\omega_k]$ does not exist.

Similarly, the Hessian matrix of $\mathrm{Var}[\omega_k]$ is

$$\boldsymbol{H}_{\mathrm{Var}[\omega_k]} = \begin{bmatrix} 0 & -\beta_k(\sigma_k^2 + \mu_k^2) a_k b_k T_k^{b_k-1} \\ -\beta_k(\sigma_k^2 + \mu_k^2) a_k b_k T_k^{b_k-1} & a_k b_k (\sigma_k^2 + \mu_k^2) \\ & \times[2\eta_k T_k^{b_k-1} + (A_k - \beta_k P_k + \eta_k T_k)(b_k-1) T_k^{b_k-2}] \end{bmatrix},$$

and the eigenvalues of $\boldsymbol{H}_{\mathrm{Var}[\omega_k]}$ are $x_1 = a_k b_k T_k^{b_k-1}(\sigma_k^2 + \mu_k^2)\eta_k + a_k b_k T_k^{b_k-1}(\sigma_k^2 + \mu_k^2)\sqrt{\eta_k^2 + \beta_k^2} > 0$ and $x_2 = a_k b_k T_k^{b_k-1}(\sigma_k^2+\mu_k^2)\eta_k - a_k b_k T_k^{b_k-1}(\sigma_k^2+\mu_k^2)\sqrt{\eta_k^2 + \beta_k^2} < 0$, which implies that the Hessian matrix is indefinite and the feasible region of (P_k, T_k) defined by the constraint is therefore infinite.

Consequently, the global maxima of the objective function does not exist.

□

In a multiple product scenario, the objective function to be maximised is

$$\mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k(P_k - \Lambda_k(T_k)\mu_k).$$

Since the prices and sales volumes are assumed to be mutually independent in this chapter, Proposition 7 is also valid in the multiple products scenario. The dependence among the warranty claims of the products is reflected in the constraints of optimisations.

4.3.1 The risk measure

In the financial discipline, Artzner et al. (1999) define risk as the variability of the future value of a position due to uncertain events. Babaei et al. (2015) point out that risk is used to characterise the situation in which a portfolio is exposed to vulnerabilities and enforces losses to the institutions. Risk measures are introduced for the requirement of quantifying the losses that may be incurred. The variance of a random variable is considered as a risk measure by the overwhelming influential models of portfolio selection. However, since the variance is a symmetric risk measure, researchers turn to using downside risk measures, such as Value-at-Risk (VaR) and Conditional VaR (Expected Shortfall), which can reflect a better notion of risk (Babaei et al., 2015). Furthermore, the variance as a risk measure is normally applied under the assumption that the correlation is linear. Such an assumption is not imposed in the VaR and CVaR theories. In this chapter, we focus on maximising the profit under the mean-risk framework, in which the risk may be measured by the variance, VaR or CVaR.

4.3.2 One product scenario

In the one product scenario, the manufacturer aims to maximise the expected profit of one product at an acceptable risk level. The optimisation problems with different risk measures are discussed in the following subsections.

Mean-variance framework

A mean-variance framework aims to maximise the expected profit of product k under a given/acceptable value of variance. The optimisation problem is defined by

$$\begin{aligned} \max \quad & \mathrm{E}[\omega_k] = M_k[P_k - \Lambda_k(T_k)\mu_k], \\ s.t. \quad & \mathrm{Var}[\omega_k] = M_k\Lambda_k(T_k)(\sigma_k^2 + \mu_k^2) \leq \psi, \\ & M_k = A_k - \beta_k P_k + \eta_k T_k \geq 0, \\ & P_k \geq 0, \quad T_k \geq L_T, \end{aligned}$$

where ψ is the acceptable risk level of the manufacturer, and L_T is the legal minimum limit of the length of warranty, e.g. the manufacturers should provide at least 2-year warranty in Europe. According to Proposition 1, only one variable, either P_k or T_k, is treated as the decision variable in optimisation. Regarding this mean-variance optimisation, for example, if P_k is the decision variable, the constraint is $P_k \geq \frac{1}{\beta_k}\left(A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2+\mu_k^2)}\right)$, and the objective function can achieve the global maxima at $P_k = \frac{A_k+\beta_k\mu_k a_k T_k^{b_k}+\eta_k T_k}{2\beta_k}$. If $\frac{1}{\beta_k}\left(A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2+\mu_k^2)}\right) < \frac{A_k+\beta_k\mu_k a_k T_k^{b_k}+\eta_k T_k}{2\beta_k}$, the objective function is maximised at $P_k = \frac{A_k+\beta_k\mu_k a_k T_k^{b_k}+\eta_k T_k}{2\beta_k}$, otherwise, the objective function is maximised at $P_k = \frac{1}{\beta_k}\left(A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2+\mu_k^2)}\right)$. Then, we have Proposition 8.

Proposition 8. *If T_k (or P_k) is the decision variable, P_k (or T_k) is known and the power law parameter $b_k > 1$, then the optimal solution, which maximises the expected profit of product k under a given variance level, exists.*

Proof. If P_k is known, the first order derivative of the objective function $\mathrm{E}[\omega_k](T_k)$ is given by

$$\frac{d\mathrm{E}[\omega_k](T_k)}{dT_k} = -\eta_k\mu_k a_k(b_k+1)T_k^{b_k} - (A_k - \beta_k P_k)\mu_k a_k b_k T_k^{b_k-1} + \eta_k P_k,$$

and the second order derivative of $\mathrm{E}[\omega_k](T_k)$ is given by

$$\frac{d^2\mathrm{E}[\omega_k](T_k)}{dT_k^2} = -a_k b_k \mu_k T_k^{b_k-2}[(b_k-1)(A_k - \beta_k P_k + \eta_k T_k) + 2\eta_k T_k] < 0,$$

as a result, the function $\mathrm{E}[\omega_k](T_k)$ is concave for $T_k \geq 0$.

If P_k is known, the first order derivative of $\mathrm{Var}[\omega_k](T_k)$ is

$$\frac{d\mathrm{Var}[\omega_k](T_k)}{dT_k} = a_k(\sigma_k^2 + \mu_k^2)[\eta_k T_k^{b_k} + (A_k - \beta_k P_k + \eta_k T_k) b_k T_k^{b_k-1}] > 0,$$

and the second order derivative of $\mathrm{Var}[\omega_k](T_k)$ is

$$\frac{d^2\mathrm{Var}[\omega_k](T_k)}{dT_k^2} = a_k b_k(\sigma_k^2 + \mu_k^2)[2\eta_k T_k^{b_k-1} + (A_k - \beta_k P_k + \eta_k T_k)(b_k-1) T_k^{b_k-2}] > 0,$$

which implies that the function $\mathrm{Var}[\omega_k](T_k)$ is convex and monotonously increases for $T_k \geq 0$ and that the feasible range of T_k defined by $\mathrm{Var}[\omega_k](T_k) \leq \psi$ is finite. Hence, when P_k is known. As a result, the solution of the optimisation problem exists.

If T_k is known, the first order derivative of the objective function $\mathrm{E}[\omega_k](P_k)$ is given by

$$\frac{d\mathrm{E}[\omega_k](P_k)}{dP_k} = A_k - 2\beta_k P_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k;$$

and the second order derivative of the objective function $\mathrm{E}[\omega_k](P_k)$ is given by

$$\frac{d^2\mathrm{E}[\omega_k](P_k)}{dP_k^2} = -2\beta_k < 0.$$

The fact that the second order derivative is negative implies: the objective function $\mathrm{E}[\omega_k](P_k)$ achieves the global maxima at $P_k = \frac{A_k + \beta_k \mu_k a_k T_k^{b_k} + \eta_k T_k}{2\beta_k}$. If T_k is known, the constraint is $P_k \geq \frac{A_k + \eta_k T_k - \frac{\psi}{a_k T_k^{b_k}(\sigma_k^2 + \mu_k^2)}}{\beta_k}$. Hence, the optimal solution exists if T_k is known.

□

Mean-VaR framework

In general, value-at-risk is the α-quantile of a distribution, where α is a given confidence level. In this case, ω_k is the profit of product k, we defined the $\mathrm{VaR}_\alpha(\omega_k)$ as the minimum profit at the $(1-\alpha)$ level, i.e. there is a $(1-\alpha)$ probability that the profit of product k will be greater than $\mathrm{VaR}_\alpha(\omega_k)$. The $\mathrm{VaR}_\alpha(\omega_k)$ can be expressed as

$$\mathrm{VaR}_\alpha(\omega_k) = F_{\omega_k}^{-1}(\alpha). \tag{4.18}$$

According to Eq. (4.9)),

$$\mathrm{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1-\alpha). \tag{4.19}$$

Then, the optimisation problem can be defined by

$$\begin{aligned} \max \quad & \mathrm{E}[\omega_k] = M_k(P_k - \Lambda_k(T_k)\mu_k), \\ s.t. \quad & \mathrm{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1-\alpha) \geq \phi, \\ & M_k \geq 0, \quad P_k \geq 0, \quad T_k \geq L_T, \end{aligned}$$

where α is a given confidence level, and ϕ is the minimum profit at the $(1-\alpha)$ confidence level set by the manufacturer. If $\phi < 0$, the maximum loss should be less than $-\phi$ at the $(1-\alpha)$ confidence level. If P_k is the decision variable and T_k is known, then $M_k P_k$ in $\mathrm{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1-\alpha) \geq \phi$ is a parabola with a negative coefficient on the quadratic term, and $F_{S_k}^{-1}(1-\alpha)$ can be derived from the distribution of warranty cost S_k. If T_k is known, the mean and the variance of S_k are $(A_k - \beta_k P_k + \eta_k T_k)\Lambda_k(T_k)\mu_k$ and $(A_k - \beta_k P_k + \eta_k T_k)\Lambda_k(T_k)(\sigma_k^2 + \mu_k^2)$, respectively. Both of the quantities monotonously decrease with respect of P_k. $F_{S_k}^{-1}(1-\alpha)$ decreases monotonously with P_k as well. As such, the feasible range of P_k is $(\max(0, P_l), P_u)$, where P_l and P_u are defined by $\mathrm{VaR}_\alpha(\omega_k) = M_k P_k - F_{S_k}^{-1}(1-\alpha) \geq \phi$, respectively.

Mean-CVaR framework

Denote $\mathrm{CVaR}_\alpha(\omega_k)$ as the conditional value-at-risk, where α is a given probability. if the profit exceeds the $\mathrm{VaR}_\alpha(\omega_k)$ on the left tail, $\mathrm{CVaR}_\alpha(\omega_k)$ is the expected profit of the product k. In other word, $\mathrm{CVaR}_\alpha(\omega_k)$ is the expected profit of the product k in the worst $\alpha\%$ of cases. In this chapter, $\mathrm{CVaR}_\alpha(\omega_k)$ is expressed as

$$\mathrm{CVaR}_\alpha(\omega_k) = \frac{1}{\alpha}\int_{-\infty}^{\mathrm{VaR}_\alpha(\omega_k)} z f_{\omega_k}(z) dz, \tag{4.20}$$

where $f_{\omega_k}(z)$ is the pdf of ω_k. The distribution of ω_k is determined by the distribution of S_k. Hence $\mathrm{CVaR}_\alpha(\omega_k)$ can be calculated by

$$\mathrm{CVaR}_\alpha(\omega_k) = M_k P_k - \frac{1}{\alpha}\int_{F_{S_k}^{-1}(1-\alpha)}^{+\infty} z f_{S_k}(z) dz, \tag{4.21}$$

where $f_{S_k}(z)$ and $F^{-1}_{S_k}(1-\alpha)$ are defined by Eq. (4.6) and Eq. (4.7)), respectively. They can be approximated by the log-normal distribution, as shown in Section 4. The optimisation problem can therefore be expressed by

$$\begin{aligned} \max \quad & \mathrm{E}[\omega_k] = M_k(P_k - \Lambda_k(T_k)\mu_k), \\ s.t. \quad & \mathrm{CVaR}_\alpha(\omega_k) = M_k P_k - \frac{1}{\alpha}\int_{F^{-1}_{S_k}(1-\alpha)}^{+\infty} z f_{S_k}(z)dz \geq \delta, \\ & M_k \geq 0, P_k \geq 0, T_k \geq LT, \end{aligned}$$

where α is a given confidence level and δ is the acceptable minimum level of $\mathrm{CVaR}_\alpha(\omega_k)$ set by the manufacturer.

Referring to Propositions 7 and 8, and the above discussions in Section 3.2.2 and 3.2.3, one can obtain the following proposition.

Proposition 9. *For T_k and P_k, if one of them is the decision variable and the other is known, then the optimal solution, which maximises the expected profit of product k under a given value-at-risk level or conditional value-at-risk level, exists.*

4.3.3 Multiple product scenario

The preceding section investigates the scenarios of the optimisation problems of individual products under a risk-informed consideration.

Assuming that a manufacturer produces N products, one can easily estimate the expected total profit through estimating and then summing the expected profit of each individual product. Considering the uncertainty in estimation of the total profit of the N products, one may consider the statistical dependence among the warranty claim arrival processes. In this section, the portfolio optimisation is investigated under the mean-variance, mean-VaR and mean-CVaR frameworks.

Mean-Variance framework

Under the mean-variance framework for the N products, the optimisation problem is defined by

$$\max \quad \mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k(P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \mathrm{Var}[\Omega] = \sum_{k=1}^{N} M_k T_k \lambda_k (\sigma_k^2 + \mu_k^2) + 2 \sum_{i \neq j}^{N} \rho_{i,j} \sqrt{M_i M_j T_i T_j \lambda_i \lambda_j (\sigma_i^2 + \mu_i^2)(\sigma_j^2 + \mu_j^2)} \leq \psi.$$

In this optimisation problem, the dependence among the warranty claims of the different products is measured with the Pearson correlation coefficient $\rho_{i,j}$, and the objective function is the sum of the expected profits of the N products. According to Proposition 8, one can obtain the following proposition.

Proposition 10. *For $\boldsymbol{T}$ and $\boldsymbol{P}$, if one of them is the decision variable vector and the other is known, then the optimal solution, which maximises the expected total profit $\mathrm{E}[\Omega]$ of the product portfolio under a given variance level, exists.*

Mean-VaR framework

Under the mean-VaR framework for the N products, the optimisation problem is defined by

$$\max \quad \mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k (P_k - \Lambda_k(T_k)\mu_k),$$
$$s.t. \quad \mathrm{VaR}_\alpha[\Omega] \geq \phi.$$

In practice, one may need to obtain the optimal solution. According to Babaei et al. (2015), even though the definition of the VaR is intuitive and easy to interpret, calculating the VaR of a portfolio is not easy. In financial mathematics, to calculate the portfolio VaR, there are three commonly used methods, which are the variance-covariance, stochastic simulation, and historical simulation methods. The variance-covariance method assumes that the risk factors are jointly normally distributed. The normality assumption may not hold in our case and therefore the variance-covariance method is not applicable. The VaR of the total profit of the product portfolio will be calculated using the stochastic simulation method in this research. The historical simulation method will be investigated in our future work. To conduct an exact and efficient simulation, copula, an important tool in the probability theory is borrowed.

Copulas are widely used in constructing multivariate distributions and formalising the dependence structures between random variables, whatever discrete or continuous. Abe Sklar first introduced the notion of copula in 1959 (Sklar, 1959). In recent years, copula has attracted considerable attention in both theoretical and application aspects. Sklar's theorem states that any cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula that describes the dependence structure between the variables (Sklar, 1959). Assume $(X_1, ..., X_d)$ is a given vector of random variables, its cumulative distribution function is $H(x_1, ..., x_d) = P(X_1 \leq x_1, ..., X_d \leq x_d))$, and its marginals are $F_k(x_k) = P(X_k \leq x_k)$, where $k = 1, ...d$. Sklar proved that $H(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$, where $C(.)$ is a copula. Copulas are useful in statistical applications because they allow one to estimate the marginals and the copula separately when modelling and estimating the distribution of a random vector. It has recently been used in modelling warranty claims (Wu, 2014a).

Denote the joint distribution of the products' number of claims by

$$H(z_1, z_2, \ldots, z_k) = C(F_{N_1}(z_1), F_{N_2}(z_2), \ldots, F_{N_k}(z_k)), \tag{4.22}$$

where $C(.)$ is a copula, and F_{N_k} is the CDF of the number of claims of product k. The density of the joint distribution is given by

$$h(z_1, z_2, \ldots, z_N) = c(F_{N_1}(z_1), F_{N_2}(z_2), \ldots, F_{N_k}(z_k)) \prod_{k=1}^{n} f_{N_k}(z_k), \tag{4.23}$$

where $c(.)$ is the density of copula $C(.)$.

Then, the joint distribution of the products' profits can be simulated based on Eq. (4.6)), Eq. (4.7)), Eq. (4.22) and Eq. (4.23)).

Let $F_\Omega(z)$ be the distribution of the total profit of the products, where $\Omega = \sum_{k=1}^{N} \omega_k$. In a copula-based model, the VaR of the total profit can be calculated through simulation. It is clear that calculating $F_\Omega(z)$ is mainly a numerical issue (Bernard & Vanduffel, 2015).

Mean-CVaR framework

Under the mean-CVaR framework, the optimisation problem is defined by

$$\max \quad \mathrm{E}[\Omega] = \sum_{k=1}^{N} M_k(P_k - \Lambda_k(T_k)\mu_k),$$

$$s.t. \quad \mathrm{CVaR}_\alpha[\Omega] \geq \delta.$$

This optimisation problem implies that the manufacturer aims to maximise the total profit under the constraint that the expected extreme profit at confidence level α is not less than δ. This constraint can also be expressed by the copula and marginal distributions.

Based on the property of the above objective function and Proposition 9, one can derive the following proposition

Proposition 11. *For $\boldsymbol{T}$ and $\boldsymbol{P}$, if one of them is the decision variable vector and the other is known, then the optimal solution, which maximises the expected total profit $\mathrm{E}[\Omega]$ of the product portfolio under a given value-at-risk level or conditional value-at-risk level, exists.*

4.4 Numerical examples

4.4.1 One product scenario

Assume that the cost of each warranty claim of one product, X_1, follows a log-normal distribution with mean $\mu_1 = 200$ and standard deviation $\sigma = 40$; assume the warranty claim arrival process is a Non-homogeneous Poisson process with cumulative intensity $\Lambda_1(T_1) = 0.004T_1^{1.04}$; and the sales volume of this product is defined by $M_1 = 1,000 - 0.2P_1 + 0.13T_1$. In Table 4.1, L_{μ_1} and L_{σ_1} are logarithmised mean and standard deviation

Table 4.1: Parameters for one product

Log-normal			
$\mu_1 = 200$	$\sigma_1 = 40$	$L_{\mu_1} = 5.2787$	$L_{\sigma_1} = 0.1980$
NHPP			
$a = 0.004$		$b = 1.04$	
Sales volume			
$A_1 = 1,000$	$\beta_1 = 0.2$	$\eta_1 = 0.13$	

respectively, and they are calculated by

$$L_{\mu_1} = \ln\left(\frac{\mu_1}{\sqrt{1+\frac{\sigma_1^2}{\mu_1^2}}}\right),$$

and

$$L_{\sigma_1} = \sqrt{\ln\left(1+\frac{\sigma_1^2}{\mu_1^2}\right)}.$$

Mean-Variance

Under the mean-variance framework, the optimisation problem is

$$\max \quad \mathrm{E}[\omega_1] = (1,000 - 0.2P_1 + 0.13T_1)(P_1 - 0.004T_1^{1.04} \times 200)$$

$$s.t. \quad \mathrm{Var}[\omega_1] = (1,000 - 0.2P_1 + 0.13T_1) \times 0.004T_1^{1.04} \times (200^2 + 40^2) \leq \psi.$$

Suppose $T_1 = 720$. The above objective function is parabola and has the global maximum $\mathrm{E}[\omega_1] = 1,113,257.59$ at $P_1 = 3,108.70$ with the corresponding sales volume $M_1 = 472$. Meanwhile, the feasible range of P_1 is $[5,468 - 0.000032\psi, 5,468]$, which means: $P_1 = 3,108.70$ is the optimal solution if the variance limitation $\psi \geq 147,103,140.52$.

If P_1 is known, for example, let $P_1 = 3,000$. The global maxima is $\mathrm{E}[\omega_1] = 1,200,600$ when $T_1 \approx 32$. The sales volume and the variance of total profit at $T_1 \approx 32$ are $M_1 = 404$ and $\mathrm{Var}[\omega_1] = 2,472,080.61$, respectively. Furthermore, the function of $\mathrm{Var}[\omega_1]$ monotonously increases with T_1. Hence, if the variance of the total profit is less than ψ with $\psi \geq 2,472,080.61$, $T_1 = 32$ is the optimal solution. However, in practice, the length of warranty is regulated by the authorities, such as at least 2 years warranty is required in Europe, this regulation may also be considered in optimisation.

Mean-VaR

Under the mean-VaR framework, given that the confidence level $\alpha = 0.05$, the optimisation problem is

$$\max \quad \mathrm{E}[\omega_1] = (1,000 - 0.2P_1 + 0.13T_1)(P_1 - 200 \times 0.004T_1^{1.04})$$

$$s.t. \quad \mathrm{VaR}_{0.05}(\omega_1) = (1,000 - 0.2P_1 + 0.13T_1)P_1 - F_{S_1}^{-1}(0.95) \geq \phi.$$

In this scenario, the company may aim to optimise the price and length of warranty under the constraint that the total profit is not less than ϕ at 95% probability, i.e. $\mathrm{VaR}_{0.05} \geq \phi$. In order to resolve this optimisation problem, the effects of P_1 and T_1 on $\mathrm{VaR}_{0.05}$ should be investigated. However, the closed form of the cdf of S_1 is difficult to obtain, and we cannot obtain the close form of $F_{S_1}^{-1}(0.95)$ either. Then a log-normal distribution

with mean, $\mathrm{E}[S_1] = 0.8T_1^{1.04} \times (1,000 - 0.2P_1 + 0.13T_1)$, and variance, $\mathrm{Var}[S_1] = 166.4T_1^{1.04} \times (1,000 - 0.2P_1 + 0.13T_1)$, is used to approximate the distribution $F_{S_1}(x)$.

Initially, we focus on the situation that T_1 is known. According to Section 4.1.1, if $T_1 = 720$, the objective function has a global maximum $\mathrm{E}[\omega_1] = 1,113,257.59$ at $P_1 = 3,108.70$. then we have

$$\mathrm{VaR}_{0.05}(\omega_1) = (1,093.6 - 0.2P_1)P_1 - F_{S_1}^{-1}(0.95). \tag{4.24}$$

If the constraint on $\mathrm{VaR}_{0.05}(\omega_1)$ is not considered, the feasible range of P_1 is $[0, 5, 468]$. Within this range, the relationship between P_1 and the first term in Eq. (4.24)), i.e. the revenue $(1,093.6 - 0.2P_1)P_1$, is illustrated in Fig. 4.2. The relationship between P_1 and the second term $F_{S_1}^{-1}(0.95)$ in Eq. (4.24)), i.e. the right tail 5% VaR of warranty cost, is shown in Fig. 4.3. The relationship between P_1 and $\mathrm{VaR}_{0.05}(\omega_1)$ is illustrated in Fig. 4.4.

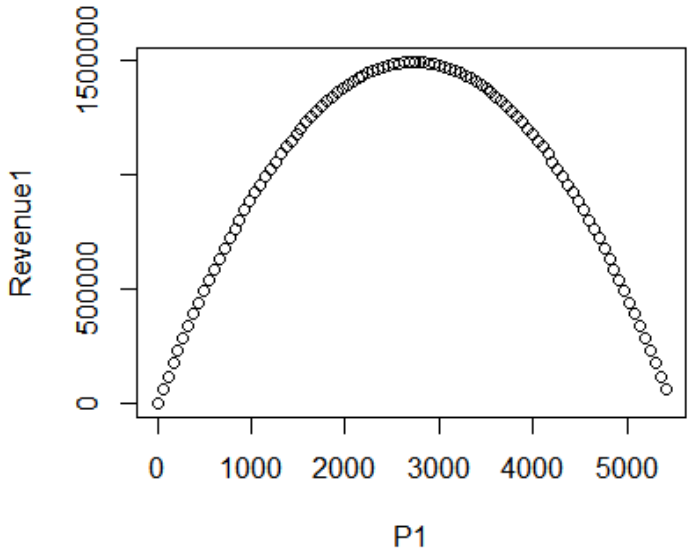

Figure 4.2: Total revenue (on the Y-axis) against price P_1 (on the X-axis).

Fig. 4.3 reveals that the right tail 5% VaR of warranty cost monotonically decreases with P_1 within the feasible range of P_1. Then, the relationship between $\mathrm{VaR}_{0.05}(\omega_1)$ and P_1 is determined by the total revenue which is a quadratic function of P_1. Hence, with the

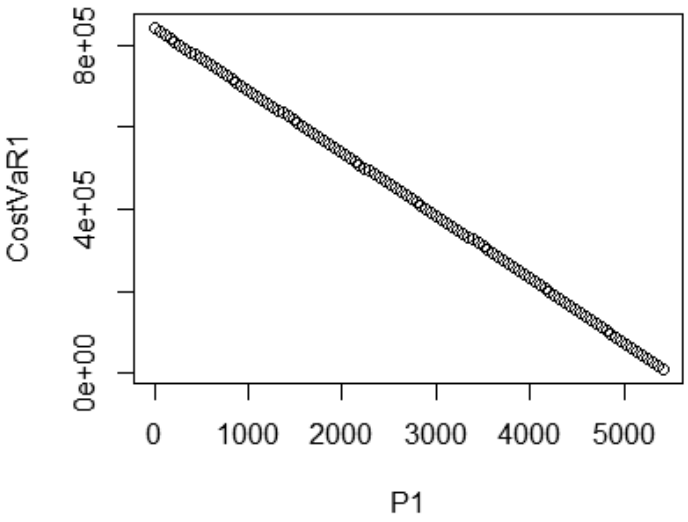

Figure 4.3: Right tail 5% VaR of warranty cost (on the Y-axis) against P_1 (on the X-axis).

optimal constraint, one can derive a feasible range of P_1, (p_{l1}, p_{u1}), where p_{l1} and p_{u1} are determined by ϕ.

In accordance with the market behaviour, one may set market price P_1, said $P_1 = 3,000$. The optimal length of warranty T_1 can then be determined and one can obtain

$$\mathrm{VaR}_{0.05}(\omega_1) = 3,000 \times (400 + 0.13T_1) - F_{S_1}^{-1}(0.95). \tag{4.25}$$

Fig. 4.5 and Fig. 4.6 reveal the effects of T_1 on the right tail 5% VaR of the warranty cost and $\mathrm{VaR}_{0.05}(\omega_1)$. The values of T_1 in these figures are discrete values from 2 years (720 days) to 10 years (3,600 days) with half of a year step (180 days). Then the optimal constraint gives a feasible range of T_1, $[720, t_u]$, where t_u is determined by ϕ.

Mean-CVaR

Under the mean-CVaR framework, the objective function is the same as that under the mean-VaR framework, but the constraint is given by

$$\mathrm{CVaR}_{0.05}(\omega_1) = (1,000 - 0.2P_1 + 0.13T_1)P_1 - \frac{1}{0.05}\int_{F_{S_1}^{-1}(0.95)}^{+\infty} z f_{S_1}(z)dz \geq \delta. \tag{4.26}$$

Eq. (4.26) implies: the company requires that the expected profit is not less than δ even if the worst event at 5% probability occurs. The first term of Eq. (4.26) also is the total

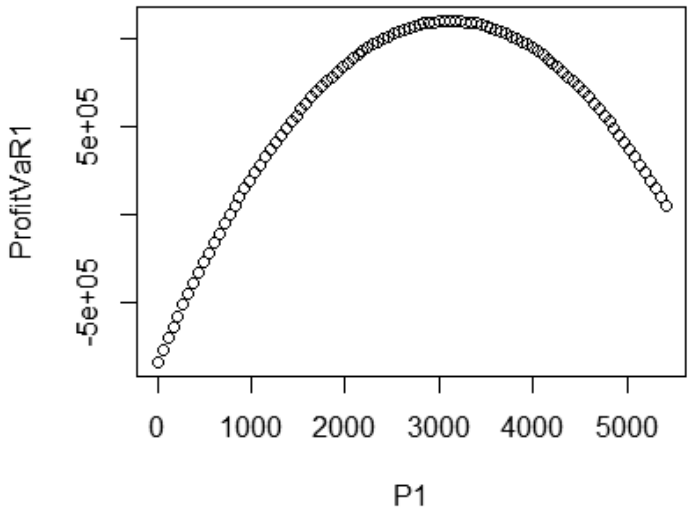

Figure 4.4: $\mathrm{VaR}_{0.05}(\omega_1)$ (on the Y-axis) against P_1 (on the X-axis).

revenue of Product 1, and the second term is the right tail 5% CVaR of the total warranty cost of Product 1. f_{S_1} can also be approximated by the log-normal distribution.

If $T_1 = 720$ is known, the relationship between P_1 and $\mathrm{CVaR}_{0.05}(\omega_1)$ is illustrated in Fig. 4.7. The feasible range of P_1 under the constraint is $[cp_l, cp_u]$, where both cp_l and cp_u are determined by δ.

If $P_1 = 3,000$ is known and $T_1 = 720 + 180v$ (for $v = 1, 2, ..., 16$), the relationship between T_1 and $\mathrm{CVaR}_{0.05}(\omega_1)$ is illustrated in Fig. 4.8. Then the feasible range of T_1 under the constraint is $[720, ct_u]$ where ct_u is determined by δ.

4.4.2 Three product scenario

Assume 7 types of components are installed in 3 products, each of which is composed of 4 or 5 components, as illustrated in Table 4.2. The components in each product are structured in series, which implies: if a component in an product item fails, the item fails.

Assume that the claim arrival processes of the components are NHPPs with a cumulative failure intensity, $\Lambda(t) = at^b$. The parameters in the claim arrival process models

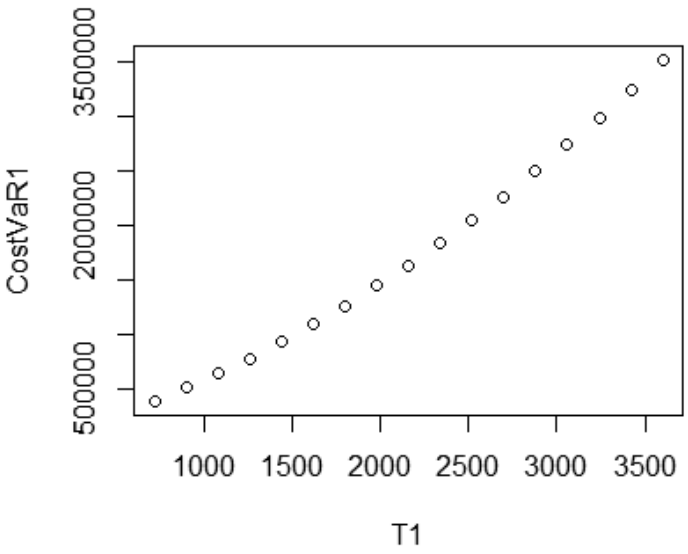

Figure 4.5: Right tail 5% VaR of warranty cost (on the Y-axis) against T_1 (on the X-axis).

Table 4.2: Components of products

	Components						
Product	C1	C2	C3	C4	C5	C6	C7
Product1	✓	✓		✓	✓	✓	
Product2	✓		✓		✓	✓	
Product3	✓			✓		✓	✓

and the warranty claim costs of the components are presented in Table 4.3. Then, the claim arrival processes of the products are NHPPs. As such their intensities are given by $\Lambda_1(t) = a_1 t^{b_1} + a_2 t^{b_2} + a_4 t^{b_4} + a_5 t^{b_5} + a_6 t^{b_6}$, $\Lambda_2(t) = a_1 t^{b_1} + a_3 t^{b_3} + a_5 t^{b_5} + a_6 t^{b_6}$ and $\Lambda_3(t) = a_1 t^{b_1} + a_4 t^{b_4} + a_6 t^{b_6} + a_7 t^{b_7}$, respectively.

According to the above setting, one can generate three data sets, each of which contains claim times and cost of claims of M_k items of product k. These three data sets are correlated, based on which we can estimate the correlation parameters of our model.

Initially, we investigate the model if T_1, T_2 and T_3 are known. Assume all products have 2-year (720 days) warranty, i.e. $T_1 = T_2 = T_3 = 720$. The products' sales volume and warranty cost parameter are then given in Table 4.4.

The expected warranty cost of the three products are, $\mathrm{E}[S_1] = 879.73 \times (1,093.6 - 0.2P_1)$, $\mathrm{E}[S_2] = 601.65 \times (2,079.2 - 0.5P_2)$ and $\mathrm{E}[\omega_3] = 572.91 \times (2,672 - 0.6P_3)$,

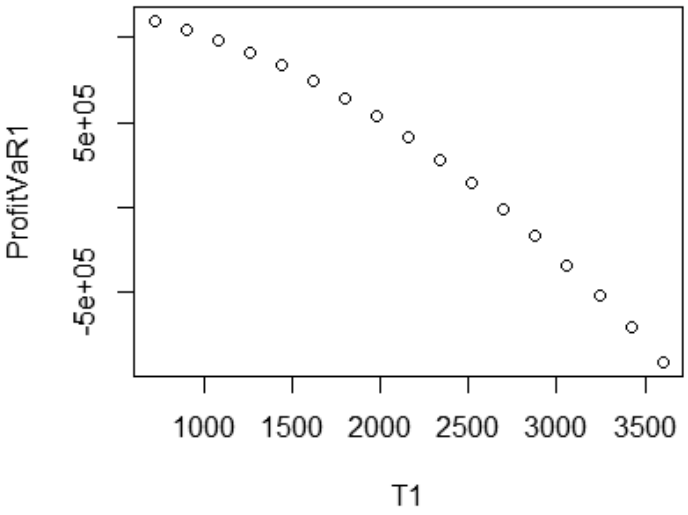

Figure 4.6: $\mathrm{VaR}_{0.05}(\omega_1)$ (on the Y-axis) against T_1 (on the X-axis).

Table 4.3: Parameters of Components

Components	a	b	Cost of each claim
C1	0.0037	1.02	100
C2	0.0037	1.01	120
C3	0.0028	1.02	80
C4	0.0028	1.02	90
C5	0.0024	1.03	65
C6	0.0019	1.03	60
C7	0.0014	1.04	50

respectively. The variances of the warranty costs of the three products are $\mathrm{Var}[\omega_1] = 83,376.02 \times (1,093.6 - 0.2P_1)$, $\mathrm{Var}[\omega_2] = 49,020.67 \times (2,079.2 - 0.5P_2)$ and $\mathrm{Var}[\omega_3] = 48,238.49 \times (2,672 - 0.6P_3)$, respectively. Based on these expected values and variances, the warranty cost distributions, $F_{S_1}(z_1)$, $F_{S_2}(z_2)$ and $F_{S_3}(z_3)$, can be approximated.

To model the dependence among the products, one may construct a trivariate copula. In practice, there are many different copula families existing, a suitable copula can be constructed or selected in two steps. In the first step, referring to the physical situation of the products or the features of empirical operating, a proper copula family can be selected. For example, if the dependence is linear, a copula from the elliptical family can be selected; and if a rank correlation is found in the data, a copula from the

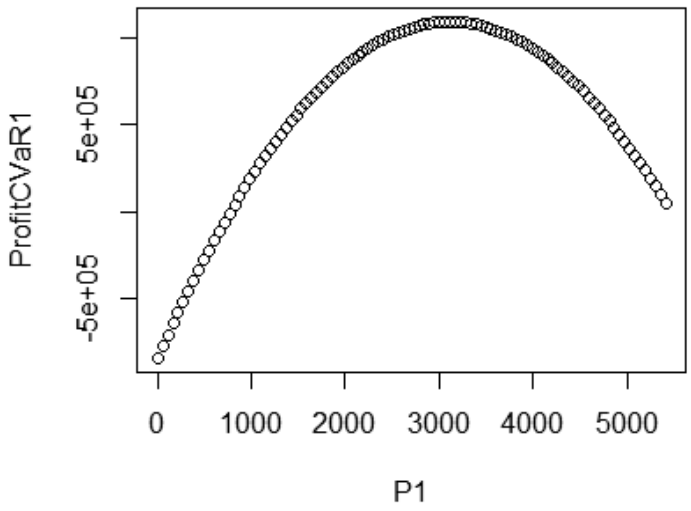

Figure 4.7: $\mathrm{CVaR1}$ (on the Y-axis) against P_1 (on the X-axis).

Table 4.4: Parameters of 3 products

	Parameters of sales volume			Parameters of warranty cost	
	A_k	β_k	η_k	μ_k	σ_k
Product 1	1,000	0.2	0.13	89.36	21.99
Product 2	2,000	0.5	0.11	78.22	15.95
Product 3	2,600	0.6	0.10	79.04	20.20

Archimedean family can be selected. Additionally, the form of marginal distribution, the tail-dependence, etc. all can influence copula selection. In the second step, the goodness-of-fit of the initially selected copula can be compared by mean squared errors, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The details of copula selection in warranty data analysis will be investigated in future research.

In this case, the numerical examples are demonstrated by simulation, considering the non-elliptical marginal distributions and the potential upper tail-dependence, a trivariate Gumbel copula is simply constructed as an example. Then the joint distribution of the number of claims is

$$\begin{aligned} H(z_1, z_2, z_3) &= C(u_1, u_2, u_3; \theta_1, \theta_2) \\ &= \exp\{-[(-\ln u_1)^{\theta_1} + [(-\ln u_2)^{\theta_2} + (-\ln u_3)^{\theta_2}]^{\frac{\theta_1}{\theta_2}}]^{\frac{1}{\theta_1}}\}, \end{aligned} \quad (4.27)$$

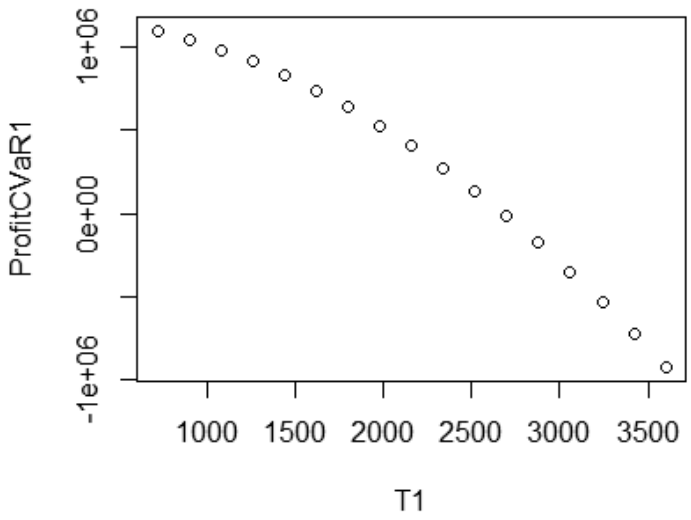

Figure 4.8: $\mathrm{CVaR}_{0.05}(\omega_1)$ (on the Y-axis) against P_1 (on the X-axis).

where $u_1 = F_{N_1}(z_1)$, $u_2 = F_{N_2}(z_2)$ and $u_3 = F_{N_3}(z_3)$. With simulation, $\theta_1 = \frac{1}{1-0.59} \approx 2.44$ and $\theta_2 = \frac{1}{1-0.71} = 3.45$.

The density of the joint distribution is given by

$$h(z_1, z_2, z_3) = c(u_1, u_2, u_3) f_{N_1}(z_1) f_{N_2}(z_2) f_{N_3}(z_3). \tag{4.28}$$

The cumulative distribution function of the total number of claims of the three products, $\boldsymbol{N} = N_1 + N_2 + N_3$, of the manufacturer is given by,

$$F_{\boldsymbol{N}}(z) = \sum_{z_1=0, z_2 \leq z-z_1}^{z} h(z_1, z_2, z - z_1 - z_2) \tag{4.29}$$

Let $P_1 = 3,173.86$, $P_2 = 2,380.02$ and $P_3 = 2,513.12$, then the maximum expected profits can be obtained, $\mathrm{E}[\omega_1] = 5,263,057$, $\mathrm{E}[\omega_2] = 3,162,622$ and $\mathrm{E}[\omega_3] = 3,764,428$. At the same time, the VaR and the CVaR of the total profit Ω can be determined based on Eqs. (4.27), (4.28), and (4.29). In Table 4.5, the $\mathrm{VaR}_{0.05}$ and the $\mathrm{CVaR}_{0.05}$ of the total profit Ω with different values of θ_2 and θ_1 are presented. θ_2 represents the correlation between the profits of Products 2 and 3; θ_1 represents the correlation between the profit of Products 1 and the profit of Products 2 and 3. Larger θ_1 and θ_2 indicates a stronger correlation. $\theta_1 = \theta_2 = 1$ implies that the profits of the three products are not correlated.

According to Table 4.5, the $\mathrm{VaR}_{0.05}$ and the $\mathrm{CVaR}_{0.05}$ of the total profit Ω are increasing when the correlations among the products becomes stronger, which implies: if a manufacturer ignores the dependence among warranty claims of products, it will underestimate the upcoming total warranty cost. Such ignorance may cause bias in decision making.

Table 4.5: The VaR and CVaR of Ω with different dependences

$P_1 = 3,173.86$, $P_2 = 2,380.02$, $P_3 = 2,513.12$				
Copula parameter	$\theta_1 = 1, \theta_2 = 1$	$\theta_1 = 2.44, \theta_2 = 3.25$	$\theta_1 = 2.44, \theta_2 = 4$	$\theta_1 = 3, \theta_2 = 4$
$\mathrm{VaR}_{0.05}(\Omega)$	4,884,922	4,859,833	4,859,018	4,858,978
$\mathrm{CVaR}_{0.05}(\Omega)$	4,873,141	4,855,315	4,854,950	4,853,788

4.5 Summary

In the real world, a manufacturer normally produces many different products that have common components installed. Consequently, the frequencies of warranty claims of different products are statistically dependent, which conflicts the fact that the existing methods in the literature of warranty management solely focus on individual product and ignore the claim dependence.

This chapter proposes a method to collectively optimise warranty policy for a portfolio of different products. Using the value-at-rick theory, it attempts to maximise the total profit of a set of products through optimising the warranty price and the warranty length. The numerical example shows that the dependence problem can be properly addressed with the proposed method.

The chapter only investigates the optimisation of warranty policy for a portfolio of products covered by a one-dimensional warranty policy. That is, the warranty only covers one dimension, which can be either the usage or the age dimension, but not both. For some products (see Ye & Murthy (2016), for example), however, a warranty policy may cover both age and usage (e.g., the warranty of a car may cover both age and mileage), which is called a two-dimensional warranty policy. Our future work aims to investigate the optimisation problem for products with a two-dimensional warranty coverage.

Chapter 5

A comprehensive analysis of warranty claims and optimal polices

Nowadays many products, such as 3C products (Computer, Communication and Consumer Electronics) and cars, consist of software and hardware. The causes of warranty claims of such products may be attributed to software specific failures, hardware specific failures, software-hardware interaction failures and human errors. Apparently, those causes may be dependent. For example, one may claim warranty due to the malfunction of the embedded software in a product item and then the entire item may be replaced. Nevertheless, the existing research on warranty management studies mainly concentrates on warranty analysis of hardware subsystems, assuming that the warranty claims are statistically independent of those caused by the failures of software subsystems or human factors, that is, the interactions between those causes are neglected.

This chapter investigates warranty costs incurred due to those three subsystems with a main focus on their interactions. It estimates the costs due to different cause, develops integrated warranty cost models and optimises warranty policies considering the above possible combinations. Numerical examples are given to illustrate the proposed models.

5.1 Introduction

A warranty is a contractual obligation incurred by a manufacturer in connection with the sale of a product. In broad terms, the purpose of warranty is to establish liability in the event of a premature failure of an item or the inability of the item to perform its intended function (Blischke & Murthy, 1992). For products sold with warranty, manufacturers bear additional cost incurred due to warranty servicing. Such cost, often referred to as warranty servicing cost, is generally substantial. For example, according to WarrantyWeek (2016a), Apple paid $1.25 billion or more for warranty claims during a single quarter in the second half of 2015; and during the same time period, HP Inc. paid around $300 million per quarter. Therefore, accurately estimating warranty cost is indispensable to the manufacturers.

Nowadays, many products, such as 3C products (Computer, Communication and Consumer Electronics) and cars consist of two subsystems, hardware and embedded software. The designed functions of the products are performed based on the reliable collaboration of their hardware and software subsystems. That is, a hardware failure or/and a software failure may cause a warranty claim. For example, the failure of the control software, which is embedded in the control panel, may cause the entire control panel to be replaced.

It should also noted that warranty claims are not always triggered by the failures of product items, some users' behaviours (human factors) may also contribute warranty claims (Wu, 2011).

5.1.1 Related work

Warranty policy optimisation

In the literature, many methods aiming to optimise the warranty price and the warranty length of an individual product have been proposed. Fig. 5.1 illustrates the evolution of the research in warranty policy optimisation, which shows that the research evolves from simple and unrealistic assumptions to more complex and realistic ones.

At the early stage, many researchers attempt to find the optimal price and warranty

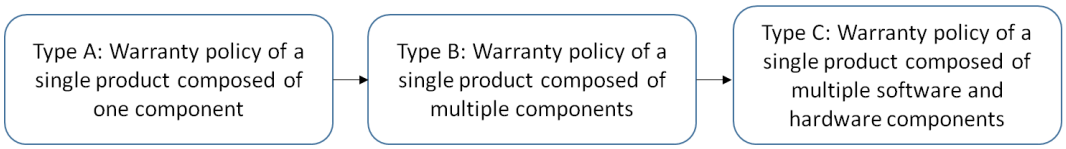

Figure 5.1: Evolution of warranty policy optimisation

length, assuming that the product is composed of only one component. Some other factors, such as production rate, market competition and demand, etc., may also be considered. Ladany & Shore (2007) address a method to determine the optimal warranty period with considering the products lifetime and market demand. Lin et al. (2009) optimise the price, warranty length and production rate of a one component system dynamically. Wu et al. (2009) develop a decision model to determine the optimal price, the length of warranty and the production rate to maximise profit based on the pre-determined life cycle in a static demand market. Dai et al. (2012) indicate warranty costs are incurred by both the supplier and the manufacturer, and provide the structural properties of the equilibrium strategies with considering warranty length in warranty management. Ding et al. (2014) investigate the relationship between the sales revenues and the repair costs under warranty coverage with considering the partial information about product reliability. Yazdian et al. (2016) jointly optimises the acquisition price, re-manufacturing degree, selling price and the length of warranty of re-manufacturing products under linear and non-linear demand functions.

The assumption that a product is composed of only one component is too simplistic and even unrealistic. Researchers then consider the assumption that a product is composed of multiple components. Huang et al. (2007) develop a model to determine the optimal product reliability, price and warranty strategy to achieve the maximum total integrated profit for a general repairable multi-component product sold under a free replacement-repair warranty strategy. Matis et al. (2008) explore the optimal price and pro rate warranty length for a multi-component product with considering the different repair options on the components. Ahmadi (2016) addresses an optimal replacement problem for complex multi-component systems by determining an optimal operating time which balances

income and cost to maximizes the expected profit over a cycle. Chen et al. (2017) seek to maximize the total profit per item of a multi-component product through optimally determine the production run length and the warranty period.

All of the work mentioned simply considers hardware subsystems. However, currently, as many product consists of hardware and software components, the difference and interplay between these two different types of components should be considered in warranty policy.

Hardware failures under warranty are usually rectified by the manufacturer, with no fee or partial fee to the consumer, based on the type of warranty policy used (Murthy & Djamaludin, 2002).

Software subsystems plays a vitally important role in many products. In this chapter, we chiefly discuss software subsystems embedded in hardware subsystems. In spite of great advancements in software reliability/quality assurance, potential faults may still be introduced into the software during its development process (Kimura et al., 1999; Williams, 2007). Software failures are usually caused by incorrect logic, incorrect statements, incorrect input data, and what not. In order to satisfy the reliability requirement and/or reduce the operating cost, software testing actions are normally performed to detect and remove software faults before the software is released. Software reliability can be improved by increasing the testing effort and by correcting detected faults. Therefore, in terms of software management, the determination of the optimal software release time, i.e. the optimal testing time, may be an important decision problem, which is called the optimal software release problem in the literature related to software operation. The reader is referred to Kimura et al. (1999) for more details on this problem.

In addition to hardware and software subsystems, product users, may be considered as another essential sub-system in many situations. Warranty claims are not always triggered by hardware or software failures, they may also be due to human factors. According to Wu (2011), there are at least two types of human factors in the context of warranty management: (1) consumers might not be bothered to claim warranty for failed items that are still under warranty, which is called failed-but-not-reported (FBNR) phenomenon; (2)

consumers may conduct a fraudulent warranty claim, or claim failure due to misuse or many other human factors, which is referred to as non-failed but reported (NFBR) claims. The first type of human factor relaxes a common assumption in warranty literature — all failures may cause warranty claims whereas the second type relaxes the assumption — all claims reported are due to product failures. Furthermore, after the updates of the software released by the manufacturer, whether and when to download and install the updates are decided by the customers. This implies the software update adoption rate, which can affect the software's reliability, may also be influenced by human factors.

Apparently, a product with lower price can enhance its sales volume; but reduce the unit profit of the product. Regarding the effect of warranty on manufacturer's profit, Wu et al. (2009) state that, in practice, consumers may predict the quality of a product based on its warranty, and a satisfactory warranty will certainly enhance consumers' purchase willingness, which is the well-known warranty's *signalling theory*. Products with longer warranty length may increase the total warranty cost to the manufacturer (Dai et al., 2012). Hence, it is important to trade-off the price and warranty length of a product in practice.

According to the literature, the failures of a product item consisting of hardware and software subsystems can be divided into three categories: hardware specific, software specific and hardware-software interaction failures (Fernandez & Stol, 2017; Roy et al., 2015; Teng et al., 2006). In order to estimate the warranty cost more accurately, this chapter proposes a new model that considers hardware and software failures of a product and even its user's behaviour integrally.

The sales volume

The sales volume of a product is affected by two critical marketing variables: the selling price and the warranty length (Chen et al., 2017). For example, these two variables, selling price P and warranty length T, can influence the sales volume, M, and profit, ω. The sales volume of a product is negatively related to its selling price and positively related to its warranty length. The profit of product k in this chapter is the revenue deducting the warranty cost, i.e. $\omega = MP - S(T)$, where $S(T)$ is the aggregated warranty cost of the

product within T.

In the literature, the sales volume M, is expressed by a function of product price P and length of warranty T in different forms. For simplicity, a linearity form, introduced by Yazdian et al. (2016), is used in this chapter. The sales volume is defined by

$$M = A - \beta P + \eta T, \tag{5.1}$$

where $A(> 0)$ is a constant relating to the market size of the product, and $\beta(> 0)$ and $\eta(> 0)$ are the price and length of warranty elasticities, respectively.

5.1.2 Novelty and contributions

The existing research on warranty management focuses on hardware subsystems (Xie & Ye, 2016; Ye & Murthy, 2016), software subsystems (Pham & Zhang, 1999) or human factors (Wu, 2011) separately. Little research, however, has been devoted to investigate the warranty claims due to the interplay of those three sub-subsystems.

This is the first research that takes a holistic consideration of warranty claims caused by different factors: hardware failure, software failure and user behaviours. The interplays between hardware and software failures are investigated for five different aspects.

This chapter has important managerial implications. In warranty management, optimising warranty policy and forecasting warranty claims are two of the most important activities. This requires analysts to understand the interplay of different warranty claim causes in order to make a precise forecasting and warranty optimisation. The methods proposed in this chapter, offers a better way than existing ones. The methods therefore advance the state-of-the-art in warranty claim forecasting and policy optimisation, and offer warranty managers theoretically established methods that can be used in their projects.

5.1.3 Summary

The rest part of this chapter is structured as follows. Section 5.2 gives assumptions and notations that will be used in this chapter. Section 5.3 categories the routes of warranty claims into different situations, derives warranty cost models assuming that warranty

claims due to hardware and software failures are statistically independent, and optimises the warranty policies through maximising the expected total profit from a manufacturer perspective. Section 5.4 derives cost models when the interplay of different subsystems is considered. Section 5.5 integrates the cost models derived from its preceding sections. Section 5.6 gives numerical examples illustrating the derived models. Section 5.7 concludes the chapter.

5.2 Assumptions

In this chapter, we analyse warranty cost from a manufacturer's perspective. The following assumptions are made:

(i) Products are new at $t = 0$ when they are sold.

(ii) Non-renewing free replacement warranty (NFRW) policy is offered to protect hardware failures. Under this policy, the manufacturer provides its customers with repair or replacement on hardware failures at no cost within the warranty period, the original warranty is not altered upon a failed item, and the manufacturer only guarantees satisfactory service on the item within the original warranty period.

(iii) Hardware failures require rectification to restore the products to an operating state.

(iv) Repair on hardware failures is assumed to be minimal repair, i.e. an item with a hardware failure is restored to the operating state that is exactly before it failed. The repair time is assumed so short that it is negligible.

(v) Software failures can be fixed through the removal of problems by debugging errors.

(vi) When a software failure occurs, the manufacturer can detect the fault which and remove it, failures due to this error may not occur again. No new fault is introduced in this process.

(vii) An individual consumer makes at most one non-failed but reported (NFBR) claim. Upon a NFBR claim, only administration cost is incurred to the manufacturer.

(viii) The number of claims follows the non-homogeneous Poisson process (NHPP).

(ix) The hardware and software of a product have the same warranty period.

5.3 Independent profit analysis

5.3.1 Possible warranty claim routes

A typical warranty claim process is shown in Figure 5.2. This process starts from the time when the item is thought to be failed and ends in five different routes.

Route 1 If a user reports a failure to the manufacturer (or the warranty servicing agent of her area), and the failure is diagnosed as a hardware failure covered by the warranty policy, the manufacturer may offer the user free repair or replacement of the item. Then, the process ends (End 1 in Figure 5.2). The cost of the manufacturer on this event consists of the hardware repair/replacement cost and the related management cost.

Route 2 If a user does not report a failure to the manufacturer, this process ends (End 2 in Figure 5.2). This phenomenon is called as the *failed-but-not-reported* (FBNR) event, which may be due to various reasons, for example, an item is not expensive so that the user is not bothered to claim warranty (Wu, 2011). This event does not incur any cost to the manufacturer.

Route 3 If a user reports a failure to the manufacturer, and the failure is not covered by the warranty policy or the item is not really failed, this process ends (End 3 in Figure 5.2). This phenomenon is named as *not-failed-but-reported* (NFBR) claim, which may be due to misuse, fraud, etc (Wu, 2011). The manufacturer may pay the related management cost, such as diagnosis fee, caused by this event.

Route 4 If a user reports a failure to the manufacturer, and the failure is diagnosed as a software failure covered by the warranty policy, the manufacturer should offer the user free repair of the software system. If the software is not connected to the internet, this process ends (End 4 in Figure 5.2). The cost of the manufacturer

on this event consists of the software debugging and repairing cost and the related management cost.

Route 5 If a user reports a failure to the manufacturer, and the failure is diagnosed as a software failure covered by the warranty policy, the manufacturer should offer the user free repair of the software system. If the software is connected to the internet, the manufacturer may develop and release the related update/patch on-line. Then, this process ends (End 5 in Figure 5.2). The cost of the manufacturer on this event consists of the software debugging and repairing cost, the update/patch developing and releasing cost and the related management cost.

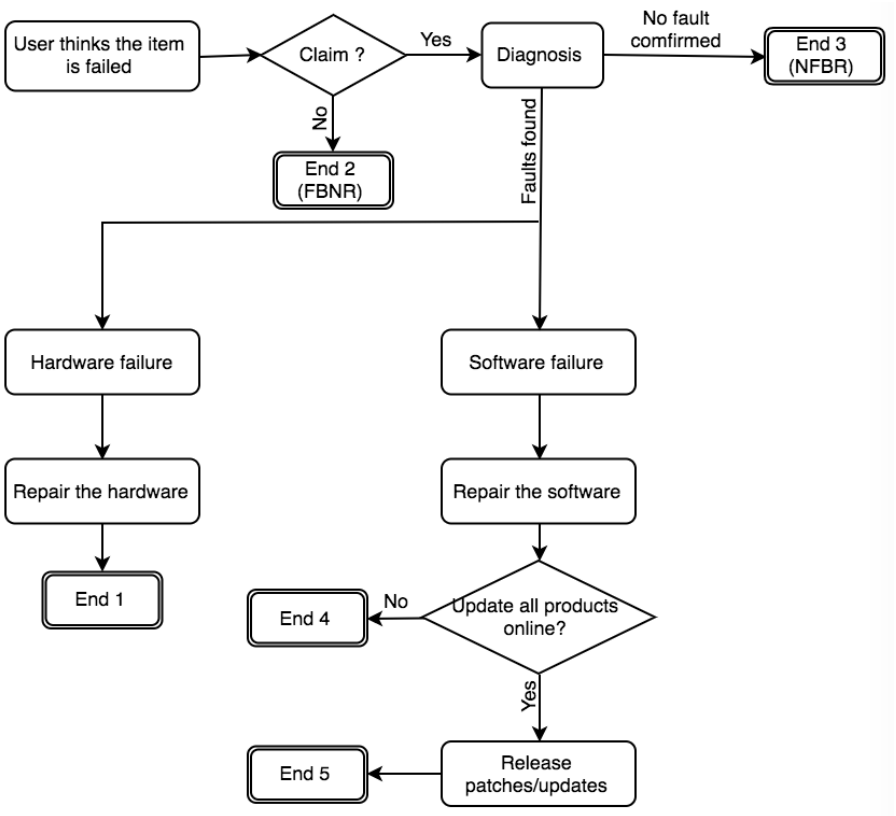

Figure 5.2: Warranty claim process

The above five routes and their associated costs should be considered by the manufacturer to support precise warranty management. Sometimes, Route 3 and Route 4 or Route 5 may occur concurrently, because some interactions may exist between hardware and software subsystems. This chapter aims to optimise warranty policies comprehensively

to maximise the manufacturer's profit with considering the issues related to the hardware, software and user of a products.

5.3.2 Warranty claims due to hardware failure

A hardware failure during the operating phase may be rectified through repair or replacement, and the warranty may be non-renewing or renewing. More specifically, hardware failures under warranty are usually rectified by the manufacturer, with no fee or partial fee to the consumer, based on the type of warranty policy used (Murthy & Djamaludin, 2002).

Suppose the manufacturer takes non-renewing warranty policy, minimal repair on hardware failures is performed and the time on repair is negligible. Thus, the arrival process of warranty claims due to hardware failures following a Non-Homogeneous Poisson Process (NHPP). The expected warranty cost on pure hardware failures is

$$\mathrm{E}[C_h(T)] = c_{h1}\Lambda_1(T) = c_{h1}\int_0^T \lambda_1(t)dt, \tag{5.2}$$

where T is the length of warranty, c_{h1} is the expected cost of each warranty claim due to hardware failure.

Assuming the NHPP follows a power law intensity in this chapter, the intensity and cumulative intensity functions of a product are $\lambda_1(t) = Ma_1b_1t^{b_1-1}$ and $\Lambda_1(t) = Ma_1t^{b_1}$, where M is the sales volume of the product, $a_1 > 0$ indicates the initial intensity and $b_1 > 1$ means that the hardware reliability is decreasing over time.

According to Eq. (5.2)), the expected profit of a hardware product is

$$\omega_h = MP - \mathrm{E}[C_h(T)] = (A - \beta P + \eta T)(P - c_{h1}a_1T^{b_1}). \tag{5.3}$$

Then we have the following result.

Proposition 12. *If P (or T) is decision variable and T (or P) is known, the optimal solution, which maximizes the expected profit of a hardware product ω_h, exists.*

Proof. The expected profit of a hardware product is

$$\omega_h = MP - \mathrm{E}[C_h(T)] = (A - \beta P + \eta T)(P - c_{h1}a_1T^{b_1}).$$

If T is known, the first order derivatives of ω_h is $\frac{d\omega_h}{dP} = A - 2\beta P + \eta T + \beta c_{h1} a_1 T^{b_1}$, and the second order derivatives of ω_h is $\frac{d^2\omega_h}{dP^2} = -2\beta < 0$. Then, ω_h is maximised at $P = \frac{A+\eta T+\beta c_{h1} a_1 T^{b_1}}{2\beta}$.

If P is known, the first order derivatives of ω_h is $\frac{d\omega_h}{dT} = -\eta c_{h1} a_1 (b_1 + 1) T^{b_1} + (\beta P - A) c_{h1} a_1 b_1 T^{b_1 - 1} + \eta P$, and the second order derivatives of ω_h is

$$\frac{d^2\omega_h}{dT^2} = -\eta c_{h1} a_1 (b_1 + 1) b_1 T^{b_1 - 1} + (\beta P - A) c_{h1} a_1 b_1 (b_1 - 1) T^{b_1 - 2}.$$

Because $M = A - \beta P + \eta T \geq 0$, $T \geq 0$ and $b_1 > 1$, then, $\frac{d^2\omega_h}{dT^2} < 0$ and the optimal solution of T, maximising ω_h, exists.

□

5.3.3 Warranty claims due to software failure

Potential faults or causes of failures are introduced into the software during its development process. Once a fault is diagnosed and removed, some of the software's errors may be debugged and the total number of potential faults may be reduced, which results in a growing reliability of software. In recent times, there is a trend that software patches are provided during early software release and updating. To satisfy customers concern of reliable software, manufacturers may provide warranty on the embedded software. Within the warranty period, the manufacturer provides assurance to the customers that the software may work properly and if any defect is found, the manufacturer may either repair or replace the software without charging the customer (Kansal et al., 2016; Singh et al., 2015).

The software warranty policies can be divided into two types:

Type I The consumer is entitled to return the software, and the manufacturer should provide support to bring the software back up to its operating mode. Every product is repaired independently under this type of warranty policy.

Type II The manufacturers may collect error reports via the internet, and then debug the errors and release updates or patches online according to the reports. All consumers who buy the products can download and install the updates free of charge.

Denote a counting process representing the cumulative number of software failures detected at time t by $\{N(t), t \geq 0\}$. The software reliability growth model (SRGM) based on the NHPP can be formulated as: $P(N(t) = n) = \frac{\Lambda_2(t)^n}{n!} \exp[-\Lambda_2(t)]$, where $\Lambda_2(t) = \int_0^t \lambda_2(t)dt$ is the cumulative intensity.

If the manufacturer takes Type I software warranty policy, the expected warranty cost on software failures is

$$\mathrm{E}[C_{s1}(T)] = c_{s1}\Lambda_2(T) = c_{s1}\int_0^T \lambda_2(t)dt, \tag{5.4}$$

where c_{s1} is the expected warranty cost of each claim on software failure under Type I software warranty policy, and the intensity, $\lambda_2(t) = Ma_2b_2e^{-b_2t}$, and cumulative intensity, $\Lambda_2(t) = Ma_2(1 - e^{-b_2t})$, where $a_2, b_2 > 0$, are constructed according to the most well-known NHPP-based SRGM, Goel-Okumoto (G-O) model (Wang et al., 2015).

If the manufacturer takes Type II software warranty policy, to model the warranty cost we have the following assumptions:

(i) if a software system failure occurs before the corresponding update is executed, the software is brought back to the operating mode as the same version, i.e. the error isn't removed; and

(ii) the manufacturer releases the updates/ patches based on a pre-specified time schedule, for example, releasing updates online in every 6 months.

Under this type of warranty policy, the expected warranty cost on software failures is:

$$\mathrm{E}[C_{s2}(T)] = nc_{s2} + c_{s1}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + c_{s1}\int_{\tau_n}^{T} \lambda_{2,n}(t)dt, \tag{5.5}$$

where $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is the number of releasing patches; M is the number of product sold at $t = 0$; τ_i is the time of ith patches/updates release under time-based policy, $\tau_i - \tau_{i-1} = \Delta\tau$ and $\tau_0 = 0$; $\lambda_{2,i}$ is the failures intensity after ith patches/updates released, $\lambda_{2,i} = \theta\lambda_{2,i-1}$ and $\lambda_{2,0}(t) = \lambda_2(t)$; and c_{s2} is the expected total cost of releasing an update. In this model, $\Delta\tau$ is a pre-specified time length, such as 1 month, 6 months, etc.; and θ is the expected percentage of software faults removed within time length $\Delta\tau$.

Regarding the expected profit of software product, under Type I software warranty, according to Eq. (5.4)), the expected profit of a software product is

$$\omega_{s1} = MP - \mathrm{E}[C_{s1}(T)] = (A - \beta P + \eta T)(P - c_{s1}a_2(1 - e^{-b_2 T})). \tag{5.6}$$

If T is known as a constant, the right hand side of Eq. (5.6) becomes to a parabolic function. On the other hand, if P is known as a constant, the first order derivative of ω_{s1} on T is $\frac{d\omega_{s1}}{dT} = c_{s1}a_2e^{-b_2T}(Ab_2 + \beta Ab_2P + \eta - \eta T b_2) + \eta P - \eta c_{s1}a_2$. Thus, we have the following result.

Proposition 13. *(1) If P is decision variable and T is known, the optimal solution, which maximizes the expected profit of a software product ω_{s1} under Type I software warranty, exists. (2) If T is decision variable and P is known, the optimal solution, which maximizes the expected profit of a software product ω_{s1} under Type I software warranty, does not exist.*

Proof. The expected profit of a software product is

$$\omega_{s1} = MP - \mathrm{E}[C_{s1}(T)] = (A - \beta P + \eta T)(P - c_{s1}a_2(1 - e^{-b_2 T})),$$

If T is known, the above function becomes to a parabolic function, it means the optimal P, which maximises ω_{s1}, exists.

If P is known, the first order derivative of ω_{s1} on T is $\frac{d\omega_{s1}}{dT} = c_{s1}a_2e^{-b_2T}(Ab_2 + \beta Ab_2P + \eta - \eta T b_2) + \eta P - \eta c_{s1}a_2$, then the second order derivative of ω_{s1} on T is $\frac{d^2\omega_{s1}}{dT^2} = b_2c_{s1}a_2e^{-b_2T}\left[b_2(A - \beta P + \eta T) - 2\eta\right] > 0$, hence, the optimal T, which maximises ω_{s1}, does not exist.

□

When the manufacturer takes Type II software warranty, according to Eq. (5.5)), the expected profit of a software product is

$$\begin{aligned} \omega_{s2} &= MP - \mathrm{E}[C_{s2}(T)] \\ &= MP - [nc_{s2} + c_{s1}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + c_{s1}\int_{\tau_n}^{T}\lambda_{2,n}(t)dt], \end{aligned} \tag{5.7}$$

There are three variables in this situation, which are: product price P, warranty length T and updating interval $\Delta\tau$. Then we have the following two propositions.

Proposition 14. *If $\Delta\tau$ is pre-specified, P is decision variable and T is known, the optimal solution, which maximizes the expected profit of a software product ω_{s2} under Type II software warranty, exists.*

Proof. If the manufacturer takes Type II software warranty, according to Eq. (5.5)), the expected profit of a software product is

$$\omega_{s2} = MP - [nc_{s2} + c_{s1}\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + c_{s1}\int_{\tau_n}^{T}\lambda_{2,n}(t)dt].$$

If $\Delta\tau = \tau_i - \tau_{i-1}$ and T are known, the number of release n is also known, then the expected profit is

$$\omega_{s2} = MP - \left[nc_{s2} + c_{s1}\sum_{i=1}^{n}\theta^{i-1}\left[\Lambda_2(\tau_i) - \Lambda_2(\tau_{i-1})\right] + c_{s1}\theta^n\left[\Lambda_2(T) - \Lambda_2(\tau_n)\right]\right];$$

it is equal to

$$\omega_{s2} = (A - \beta P + \eta T)(P - B) - nc_{s2},$$

where $B = c_{s1}a_2\sum_{i=1}^{n}\theta^{i-1}(e^{-b_2\tau_{i-1}} - e^{-b_2\tau_i}) + c_{s1}a_2\theta^n(e^{-b_2\tau_n} - e^{-b_2T})$ is a constant.

Obviously, the above function also is a parabolic function, then the optimal P, which maximises ω_{s2}, exists.

□

Proposition 15. *If both P and T are known and $\Delta\tau$ is a decision variable, the optimal solution, which maximizes the expected profit of a software product ω_{s2} under Type II software warranty, exists.*

Proof. If the manufacturer takes Type II software warranty, according to Eq. (5.5)), the expected profit of a software product is

$$\omega_{s2} = MP - \left[nc_{s2} + c_{s1}\sum_{i=1}^{n}\theta^{i-1}\left[\Lambda_2(\tau_i) - \Lambda_2(\tau_{i-1})\right] + c_{s1}\theta^n\left[\Lambda_2(T) - \Lambda_2(\tau_n)\right]\right].$$

If P and T are known, this function is equal to

$$\omega_{s2} = (A - \beta P + \eta T)(P - c_{s1}a_2 D) - nc_{s2},$$

where $D = \sum_{i=1}^{n}\theta^{i-1}(e^{-b_2\tau_{i-1}} - e^{-b_2\tau_i}) + \theta^n(e^{-b_2\tau_n} - e^{-b_2T})$ and $-nc_{s2}$ are the non-constant terms.

As $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is an integer greater than 0, and ω_{s2} decreases with n, then, when $n = 1$ t, ω_{s2} can be maximised. When $n = 1$, $\frac{T}{2} \leq \Delta\tau \leq T$. Then $D = (1-e^{-b_2\tau_1})+\theta(e^{-b_2\tau_1} - e^{-b_2T})$, a local optimal $\Delta\tau$, which maximises ω_{s2}, exists.

□

5.3.4 Warranty claims due to users

The NFBR and FBNR events should be considered in warranty cost analysis, however there is another human factor, the *adoption rate*, which should not be ignored either. The adoption rate is the percentage of the users who have downloaded and installed the updates/patches of the software embedded in the product. Sometimes users may not download and install software patches/updates immediately after they are released. The adoption rate may be a function of time, for example, the adoption rate of the software embedded iPhone may increase over time.

According to Wu (2011), it is reasonable to assume that a consumer makes at most one NFBR claim. Wu (2011) proposes three models to estimate the expected warranty cost when both NFBR claims and FBNR phenomenon are considered. Manufacturers responses to NFBR claims may be different: (1) some manufacturers may cease the warranty contract for consumers with NFBR claims, and (2) some may not cease the warranty contract, as it is not easy to tell if a NFBR claim is intentionally or unintentionally committed. However, both responses incur costs to the manufacturers, and therefore should be considered in estimating warranty cost. Following Wu (2011), we assume that time to a NFBR claims is a random variable Z with distribution function $H_1(t) = 1 - e^{-(t/\alpha_1)^{\alpha_2}}$, $\alpha_1, \alpha_2 > 0$. Then, we consider the following two scenarios.

(i) A NFBR claim may not cause warranty to be ceased, the manufacturer may check and return the item. Then the expected warranty cost is given by

$$\mathrm{E}[C_{hu1}(T)] = Mc_{hu1}H_1(T), \tag{5.8}$$

where c_{hu1} is the administration cost per NFBR claim.

(ii) A NFBR claim may cause warranty to be ceased, the manufacturer may fix and return the item. Once the warranty ceases, there are no further costs to the manufacturer. Then the expected warranty cost is given by

$$\mathrm{E}[C_{hu2}(T)] = Mc_{hu2}H_1(T), \tag{5.9}$$

where c_{hu2} is the expected cost on fixing the cause of an NFBR claim.

Regarding the FBNR phenomenon, the consumers' willingness to claim warranty may diminish with time, then the probability of consumers being inclined to claim warranty is assumed to be

$$q_1(t) = e^{-\gamma_1 - \gamma_2 t}, \tag{5.10}$$

which is called a warranty execution function (WEF) (Wu, 2011), where $\gamma_1, \gamma_2 > 0$.

Regarding the effects of delayed updating behaviour on warranty cost, the proportion of users installing the update, i.e. adoption rate, $q_2(\Delta t)$, increases with time after releasing. $\Delta t = t - \tau_i$, where t is the current time and τ_i is the time when the ith update is released. The delayed updating behaviour does not incur new cost directly, but it can affect the cost $\mathrm{E}[C_{s2}(T)]$.

5.4 Comprehensive profit analysis and optimisation

In the literature, most of the researchers use the Markov process to model the interaction between hardware and software based on the physic structures of products (Roy et al., 2015; Teng et al., 2006), in this chapter, briefly, the hardware-software interaction failures are modelled under two different situations.

(i) the interaction between hardware and software failures can be categorised into two categories, hardware-failure-caused software failure and software-failure-caused hardware failure; and

(ii) the causes of hardware-software interaction failures can not be determined.

Below we discuss the above two situations.

5.4.1 The interplay between software and hardware

In this section, the cost of the ith warranty claim due to hardware failure is denoted by $c_{h,i}$, with $\mathrm{E}(c_{h,i}) = c_{h1}$, and the cost of jth warranty claim due to software failure is denoted by $c_{s,j}$, with $\mathrm{E}(c_{s,j}) = c_{s1}$, where $c_{h,i}$ and $c_{s,j}$ follow non-negative continuous probability distributions. In practice, a product may be composed of many different hardware subsystems, which may be controlled by one or more software subsystems. To investigate the interplay between software and hardware subsystems, we consider the following 5 scenarios.

Scenario 0 The occurrences of hardware and software failures are statistically independent. If the software system cannot be updated online, the total warranty cost of m sold items of a product during the warranty period is

$$C_{01}(T) = \sum_{i=1}^{N_h(T)} c_{h,i} + \sum_{j=1}^{N_{s1}(T)} c_{s,j}, \tag{5.11}$$

where $N_h(T)$ is the total number of warranty claims of M items due to hardware failures during the warranty period T, and $N_{s1}(T)$ is the total number of warranty claims of M items due to software failures during the warranty period T. $N_h(T)$ and $N_{s1}(T)$ follow NHPPs with cumulative intensities $\Lambda_1(T)$ and $\Lambda_2(T)$, respectively.

The expected total cost in this situation is

$$\begin{aligned} \mathrm{E}[C_{01}(T)] &= c_{h1}\Lambda_1(T) + c_{s1}\Lambda_2(T) \\ &= M c_{h1} a_1 T^{b_1} + M c_{s1} a_2 (1 - e^{-b_2 T}), \end{aligned} \tag{5.12}$$

and the expected total profit in this scenario is

$$\begin{aligned} \omega_{01} &= MP - \mathrm{E}[C_{01}(T)] \\ &= M\left[P - c_{h1} a_1 T^{b_1} - c_{s1} a_2 (1 - e^{-b_2 T})\right]. \end{aligned} \tag{5.13}$$

Assume that the software can be updated online. Once a fault is reported, confirmed and repaired, the software that is embedded all of the sold items of this type of

product will be repaired. Then, the total warranty cost of the of a product during the warranty period is

$$C_{02}(T) = \sum_{i=1}^{N_h(T)} c_{h,i} + \sum_{j=1}^{N_{s2}(T)} c_{s,j} + \sum_{k=1}^{n} c_{u,k}, \tag{5.14}$$

where n is the number of software updates during the warranty period, and $c_{u,k}$ is the cost of the kth debugging and updating, $\mathrm{E}(c_{u,k}) = c_{s2}$. $N_{s2}(T)$ is more complicated than $N_{s1}(T)$ because the intensity function of $N_{s2}(T)$ is influenced by software updating activities.

$$\mathrm{E}[N_{s2}(T)] = \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T} \lambda_{2,n}(t)dt, \tag{5.15}$$

The parameters in Eq. (5.15) are defined the same as those in Eq. (5.5)). Then the expected total cost is

$$\begin{aligned} \mathrm{E}[C_{02}(T)] &= Mc_{h1}a_1T^{b_1} + nc_{s2} \\ &\quad + c_{s1}\left[\sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T} \lambda_{2,n}(t)dt\right], \end{aligned} \tag{5.16}$$

and the expected total profit in this scenario is

$$\omega_{02} = MP - \mathrm{E}[C_{02}(T)]. \tag{5.17}$$

Scenario 1 The occurrence of hardware and software failures are statistical dependent; but no failure is due to the physical interaction of the hardware-software subsystems. A software failure can lead to hardware failures of the M sold items with probability $p(t)$ at time t. If the software cannot be updated online, the total warranty cost of the M sold items of a product during the warranty period is given by

$$C_{11}(T) = \sum_{i=1}^{N_{hs1}(T)} c_{h,i} + \sum_{j=1}^{N_{s1}(T)} c_{s,j}, \tag{5.18}$$

where $N_{hs1}(T) = N_h(T) + N_{s1}(T)\int_0^T p(t)dt$. Then, the expected total warranty cost is

$$\begin{aligned} \mathrm{E}[C_{11}(T)] &= Mc_{h1}\left[a_1T^{b_1} + a_2(1 - e^{-b_2T})\int_0^T p(t)dt\right] \\ &\quad + Mc_{s1}a_2(1 - e^{-b_2T}). \end{aligned} \tag{5.19}$$

Then the expected total profit is

$$\begin{aligned}\omega_{11} &= MP - \mathrm{E}[C_{11}(T)] \\ &= MP - Mc_{h1}\left[a_1 T^{b_1} + a_2(1-e^{-b_2 T})\int_0^T p(t)dt\right] \\ &\quad -Mc_{s1}a_2(1-e^{-b_2 T}). \end{aligned} \tag{5.20}$$

If the software can be updated online, once a fault is reported, confirmed and repaired, all of the sold items' embedded software will be repaired. Then, the total warranty cost of the of a product during the warranty period is

$$C_{12}(T) = \sum_{i=1}^{N_{hs2}(T)} c_{h,i} + \sum_{j=1}^{N_{s2}(T)} c_{s,j} + \sum_{k=1}^{n} c_{u,k}, \tag{5.21}$$

where $N_{hs2}(T) = N_h(T) + N_{s2}(T)\int_0^T p(t)dt$. Then, the expected total warranty cost is

$$\begin{aligned}\mathrm{E}[C_{12}(T)] &= Mc_{h1}a_1 T^{b_1} + c_{h1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]\int_0^T p(t)dt \\ &\quad +c_{s1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right] + nc_{s2}, \end{aligned} \tag{5.22}$$

and the expected total profit in this scenario is

$$\omega_{12} = MP - \mathrm{E}[C_{12}(T)]. \tag{5.23}$$

The updating time interval, $\Delta\tau$, is pre-specified and decided by the manufacturer.

Scenario 2 The occurrence of hardware and software failures are statistical dependent; and the failures of the product are not only caused by pure hardware/software factors but by some design problems related to hardware-software interaction. In this scenario, repairing the software is impossible or the software repairing cost is huge, the manufacturer may decide to replace the hardware subsystems of all items to reduce the failure rate of the product. Then, the total warranty cost of the M sold items of a product during the warranty period is

$$C_{21}(T) = \sum_{i=1}^{N_h(T)} c_{h,i} + \sum_{j=1}^{N_{hr}(T)} Mc_{hr,j}, \tag{5.24}$$

where $c_{hr,j}$ is the cost of replace hardware components for the jth software-caused hardware failure, $\mathrm{E}(c_{hr,j}) = c_{h2}$, and $N_{hr}(T) = N_s(T)\int_0^T p(t)dt = \Lambda_2(T)\int_0^T p(t)dt$. Then, the expected total warranty cost is

$$\mathrm{E}[C_{21}(T) = M\left\{c_{h1}a_1T^{b_1} + Mc_{h2}a_2(1-e^{-b_2T})\right\}. \tag{5.25}$$

The expected total profit is

$$\omega_{21} = M\left\{P - c_{h1}a_1T^{b_1} - Mc_{h2}a_2(1-e^{-b_2T})\right\}. \tag{5.26}$$

Scenario 3 The hardware and software are interplaying; and the failures of the product are not only caused by pure hardware/software factors but by some problems related to hardware-software interplay (eg. bad design). Different from Scenario 2, in this scenario, it is impossible to replace a hardware system. The manufacturer may decide to develop/improve the software to reduce the failure rate of the product. Then, if the product cannot be updated online, the total warranty cost of m sold items of a product during the warranty period is

$$C_{31}(T) = \sum_{i=1}^{N_{hs3}(T)} c_{s,i}, \tag{5.27}$$

where $N_{hs3}(T) = N_h(T) + N_{s1}(T)$ and $\mathrm{E}(c_{s,i}) = c_{s1}$.

Then the expected cost is

$$\mathrm{E}[C_{31}] = c_{s3}M\left[a_1T^{b_1} + a_2(1-e^{-b_2T})\right], \tag{5.28}$$

and the expected total profit is

$$\omega_{31} = M\left\{P - c_{s3}\left[a_1T^{b_1} + a_2(1-e^{-b_2T})\right]\right\}. \tag{5.29}$$

If the product can be updated online, then the total warranty cost is

$$C_{32}(T) = \sum_{i=1}^{N_h(T)} c_{u1,i} + \sum_{j=1}^{N_{s2}(T)} c_{s,j} + \sum_{k=1}^{n} c_{u,k}, \tag{5.30}$$

where $c_{u1,i}$ is the cost of develop software to resolve the ith hardware failure, $N_{hs3}(T) = N_h(T) + N_{s1}(T)$ and $\mathrm{E}(c_{u1,i}) = c_{s3}$.

Then, the expected total warranty cost is

$$\mathrm{E}[C_{32}(T)] = Mc_{s3}a_1T^{b_1} + c_{s1}\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right] + nc_{s2}, \tag{5.31}$$

and the expected total profit in this scenario is

$$\omega_{32} = MP - \mathrm{E}[C_{32}(T)]. \tag{5.32}$$

The updating time interval, $\Delta\tau$, is pre-specified and decided by the manufacturer.

Scenarios 4 This scenarios describes a potential situation in the near future. The Artificial Intelligence (AI) technology allows software subsystems themselves to program automatically to deal with the dynamic state of the product, which implies that faults may be introduced into the software not only during the developing process before released but also during the operating phase. Then, the total warranty cost in this scenarios is

$$C_{41}(T) = \sum_{i=1}^{N_{hs1}(T)} c_{h,i} + \sum_{j=1}^{N_{hs4}(T)} c_{s,j}, \tag{5.33}$$

where $N_{hs1}(T) = N_h(T) + N_{s1}(T)\int_0^T p(t)dt$ and $N_{hs4}(T) = N_h(T)\int_0^T q(t,M)dt + N_{s1}(T)$. $\int_0^T q(t,M)dt$ is the probability that a hardware failure leads to software failure at time T. Then the expected cost is

$$\begin{aligned}\mathrm{E}[C_{41}(T)] &= Mc_h\left[a_1T^{b_1} + a_2(1-e^{-b_2T})\int_0^T p(t)dt\right] \\ &\quad + Mc_{s1}\left[a_1T^{b_1}\int_0^T q(t)dt + a_2(1-e^{-b_2T})\right],\end{aligned} \tag{5.34}$$

and the expected total profit in this scenario is

$$\omega_{41} = MP - \mathrm{E}[C_{41}(T)]. \tag{5.35}$$

According to Proposition 12 and Proposition 13, in all of the above 5 scenarios, we have the following two propositions.

Proposition 16. *If the software cannot be updated online, P is decision variable and T is known, the optimal solution, which maximizes the expected profit of the product, exists.*

Proposition 17. *If the software cannot be updated online, P is decision variable and T and $\Delta\tau$ are known, the optimal solution, which maximizes the expected profit of the product, exists. If $\Delta\tau$ is decision variable and T and P are known, the optimal solution, which maximizes the expected profit of the product, exists.*

5.5 Integrated warranty models

The integrated warranty models are built in three situations. In the first situation, the warranty claims of a hardware product and the human factors are integrated. In the second situation, the warranty claims of a software product and the human factors are integrated. In the third situation, the warranty claims of a product consisting of hardware and software and the human factors are integrated.

5.5.1 Hardware warranty with considering human factors

If a product is pure hardware, the NFBR and FBNR phenomena can influence the total warranty cost of it. Routes 1, 2 and 3 of warranty claim process discussed in Section 3.1 can occur.

If the NFBR may not cause warranty to be ceased, according to Eq. (5.2)), (5.8 and (5.10), the expected total cost is

$$\begin{aligned} \mathrm{E}[C_{int11}(T)] &= \mathrm{E}[C_h(T)]q_1(T) + \mathrm{E}[C_{hu1}(T)] \\ &= Mc_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} + Mc_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{5.36}$$

Then, the expected profit of the product is

$$\begin{aligned} \omega_{int11} &= MP - \mathrm{E}[C_{int11}(T)] \\ &= (A-\beta P+\eta T)(P - c_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} - c_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{5.37}$$

If the NFBR may cause warranty to be ceased, according to Eq. (5.2)), (5.9 and (5.10),

the expected total cost is

$$\begin{aligned}
\mathrm{E}[C_{int12}(T)] &= [\mathrm{E}[C_h(T)]q_1(T) + Mc_{hu2}]\, H_1(T) + \mathrm{E}[C_h(T)]q_1(T)(1 - H_1(T)) \\
&= M\left[c_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1 - e^{-(t/\alpha_1)^{\alpha_2}}) \\
&\quad + Mc_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}. \qquad (5.38)
\end{aligned}$$

Then, the expected profit of the product is

$$\begin{aligned}
\omega_{int12} &= MP - \mathrm{E}[C_{int12}(T)] \\
&= MP - M\left[c_{h1}a_1T^{b1}e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1 - e^{-(t/\alpha_1)^{\alpha_2}}) \\
&\quad + c_{h1}a_1MT^{b1}e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}). \qquad (5.39)
\end{aligned}$$

5.5.2 Software warranty with considering human factors

If only the software warranty is considered and the software cannot be updated online, the NFBR and FBNR phenomena may influence the total warranty cost of it, but the delayed updating is not applied on this situation. Routes 1, 2 and 4 of warranty claim process discussed in Section 3.1 can occur.

If the NFBR may not cause warranty to be ceased, according to Eq. (5.4)), (5.8 and (5.10), the expected total cost is

$$\begin{aligned}
\mathrm{E}[C_{int21}(T)] &= \mathrm{E}[C_{s1}(T)]q_1(T) + \mathrm{E}[C_{hu1}(T)] \\
&= Mc_{s1}a_2(1 - e^{-b_2T})e^{-\gamma_1-\gamma_2T} + Mc_{hu1}(1 - e^{-(t/\alpha_1)^{\alpha_2}}). \qquad (5.40)
\end{aligned}$$

Then, the expected profit of the product is

$$\begin{aligned}
\omega_{int21} &= MP - \mathrm{E}[C_{int21}(T)] \\
&= M(P - c_{s1}a_2(1 - e^{-b_2T})e^{-\gamma_1-\gamma_2T} - c_{hu1}(1 - e^{-(t/\alpha_1)^{\alpha_2}})). \qquad (5.41)
\end{aligned}$$

If the NFBR may cause warranty to be ceased, according to Eq. (5.4)), (5.9 and (5.10), the expected total cost is

$$\begin{aligned}
\mathrm{E}[C_{int22}(T)] &= [\mathrm{E}[C_{s1}(T)]q_1(T) + Mc_{hu2}]\, H_1(T) + \mathrm{E}[C_{s1}(T)]q_1(T)(1 - H_1(T)) \\
&= M\left[c_{s1}a_2(1 - e^{-b_2T})e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1 - e^{-(t/\alpha_1)^{\alpha_2}}) \\
&\quad + Mc_{s1}a_2(1 - e^{-b_2T})e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}. \qquad (5.42)
\end{aligned}$$

Then, the expected profit of the product is

$$\begin{aligned}\omega_{int22} &= MP - \mathrm{E}[C_{int22}(T)] \\ &= MP - M\left[c_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T} + c_{hu2}\right](1-e^{-(t/\alpha_1)^{\alpha_2}}) \\ &\quad -Mc_{s1}a_2(1-e^{-b_2T})e^{-\gamma_1-\gamma_2T}e^{-(t/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{5.43}$$

If the software can be updated online, the NFBR, FBNR and delayed updating phenomena all can influence the total warranty cost of it. Routes 1, 2 and 5 of warranty claim process discussed in Section 3.1 can occur.

Assume the proportion of users installed the update, i.e. adoption rate, is $q_2(\Delta t) = 1 - e^{-d\Delta\tau}$ where $d > 0$, and the FBNR is considered then Eq. (5.5) can be modified to

$$\mathrm{E}[C'_{s2}(T)] = nc_{s2} + q_2(\Delta\tau)c_{s1}\sum_{i=1}^{n} q_1(\tau_i)\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + q_2(T-\tau_n)q_1(T)c_{s1}\int_{\tau_n}^{T}\lambda_{2,n}(t)dt. \tag{5.44}$$

Take the NFBR and FBNR into account, if the NFBR may not cause warranty to be ceased, according to Eq. (5.8)), (5.10 and (5.44), the expected total cost is

$$\mathrm{E}[C_{int23}(T)] = \mathrm{E}[C'_{s2}(T)] + \mathrm{E}[C_{hu1}(T)], \tag{5.45}$$

then, the expected profit of the product is

$$\begin{aligned}\omega_{int23} &= MP - \mathrm{E}[C_{int23}(T)] \\ &= MP - nc_{s2} - q_1(\Delta\tau)q_2(\Delta\tau)c_{s1}\sum_{i=1}^{n}\theta^{i-1}\left[\Lambda_2(\tau_i) - \Lambda_2(\tau_{i-1})\right] \\ &\quad -q_1(T-\tau_n)q_2(T-\tau_n)c_{s1}\theta^n\left[\Lambda_2(T) - \Lambda_2(\tau_n)\right] \\ &\quad -Mc_{hu1}(1-e^{-(t/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{5.46}$$

If the NFBR may cause warranty to be ceased, according to Eq. (5.9)), (5.10 and (5.44), the expected total cost is

$$\mathrm{E}[C_{int24}(T)] = \left(\mathrm{E}[C'_{s2}(T)] + Mc_{hu2}\right)H_1(T) + \mathrm{E}[C'_{s2}(T)](1-H_1(T)), \tag{5.47}$$

then, the expected profit of the product is

$$\omega_{int24} = MP - \mathrm{E}[C_{int24}(T)]. \tag{5.48}$$

5.5.3 Hybrid warranty with considering human factors

If a product is composed of hardware and software subsystems, the warranty cost of such product is influenced by the warranty claims on both hardware and software failures, the three human factors and the interplay between hardware and software subsystems. In Section 4.1, 5 different scenarios of hardware and software subsystems interplay are discussed, in this section, the hybrid model is constructed based on the interplay described in Scenario 1, i.e. a software failure can lead to hardware failure with probability $p(t)$ at time t within the M sold items of a product.

If the software cannot be updated online, the NFBR and FBNR phenomenon may affect the total warranty cost, but the delayed updating is not applied in this situation. If the NFBR may not cause warranty to be ceased, according to Eq. (5.8)), (5.10 and (5.19), the expected total warranty cost is

$$\begin{aligned}\mathrm{E}[C_{int31}(T)] &= \mathrm{E}[C_{11}(T)]q_1(T) + \mathrm{E}[C_{hu1}(T)] \\ &= Mq_1(T)\left\{c_{h1}\left[a_1T^{b_1} + a_2(1-e^{-b_2T})\int_0^T p(t)dt\right] + c_{s1}a_2(1-e^{-b_2T})\right\} \\ &\quad + Mc_{hu1}(1-e^{-(T/\alpha_1)^{\alpha_2}}). \end{aligned} \tag{5.49}$$

Then the expected total profit is

$$\omega_{int31} = MP - \mathrm{E}[C_{int31}(T)]. \tag{5.50}$$

If the NFBR may cause warranty to be ceased, according to Eq. (5.9)), (5.10 and (5.19), the expected total warranty cost is

$$\mathrm{E}[C_{int32}(T)] = \left[\mathrm{E}[C_{11}(T)]q_1(T) + Mc_{hu2}\right]H_1(T) + \mathrm{E}[C_{11}(T)]q_1(T)(1-H_1(T)). \tag{5.51}$$

Then the expected total profit is

$$\omega_{int32} = MP - \mathrm{E}[C_{int32}(T)]. \tag{5.52}$$

If the software can be updated online, the delayed updating effect is applied. Then,

the expected warranty cost considering delayed updating and FBNR is

$$\begin{aligned}
\mathrm{E}[C'_{12}(T)] &= Mq_1(T)c_{h1}a_1T^{b_1} + nc_{s2} \\
&\quad + c_{h1}\left[q_1(\Delta\tau)q_2(\Delta\tau)\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt\right. \\
&\quad \left. + q_1(T-\tau_n)q_2(T-\tau_n)\int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]\int_0^T p(t)dt \\
&\quad + c_{s1}\left[q_1(\Delta\tau)q_2(\Delta\tau)\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt\right. \\
&\quad \left. + q_1(T-\tau_n)q_2(T-\tau_n)\int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]. \qquad (5.53)
\end{aligned}$$

If the NFBR may not cause warranty to be ceased, according to Eq. (5.8)), (5.10 and (5.53), the expected total warranty cost is

$$\mathrm{E}[C_{int33}(T)] = \mathrm{E}[C'_{12}(T)] + \mathrm{E}[C_{hu1}(T)], \qquad (5.54)$$

and the expected total profit in this scenario is

$$\omega_{int33} = MP - \mathrm{E}[C_{int33}(T)]. \qquad (5.55)$$

If the NFBR may cause warranty to be ceased, according to Eq. (5.9)), (5.10 and (5.53), the expected total warranty cost is

$$\mathrm{E}[C_{int34}(T)] = \left[\mathrm{E}[C'_{12}(T)] + Mc_{hu2}\right]H_1(T) + \mathrm{E}[C'_{12}(T)](1 - H_1(T)), \qquad (5.56)$$

and the expected total profit is

$$\omega_{int34} = MP - \mathrm{E}[C_{int34}(T)]. \qquad (5.57)$$

The effect of the variables P and T on then expected profits ω_{int31}, ω_{int32}, ω_{int33} and ω_{int34} are explored by the corresponding numerical examples.

5.6 Numeric examples

In this section, the models of product profit considering warranty costs due to hardware failures and software failure, and the integrated models are illustrated through numeric examples. The sales volume parameters are set: market size parameter, $A = 1000$, coefficient of product price, $\beta = 0.1$, and coefficient of warranty length, $\eta = 21$.

5.6.1 The expected total profit considering hardware failures and software failures independently

If the hardware is under a non-renewing warranty, the values of the parameters of Eq. (5.3) are set: Expected cost per hardware claim $c_{h1} = 100$, Power law parameter, $a_1 = 0.1$, Power law parameter, $b_1 = 1.04$.

Then, the expected total product profit only considering warranty cost due to hardware failures is

$$\omega_h(P,T) = (1000 - 0.1P + 21T)(P - 100 \times 0.1 \times T^{1.04}). \tag{5.58}$$

According to Proposition 12, if the warranty length is known, for example, set $T = 24$ months; then the expected total profit can be maximised at $P = 7656.27$, and the maximum expected total profit is $\omega_h = 5,451,953$. The relationship between ω_h and P is reflected by the Figure 5.3. When the product is known, for example, set $P = 2000$; then the expected total profit can be maximised at $T = 64$, and the maximum expected total profit is $\omega_h = 2,667,529$. The relationship between ω_h and T is showed by the Figure 5.4.

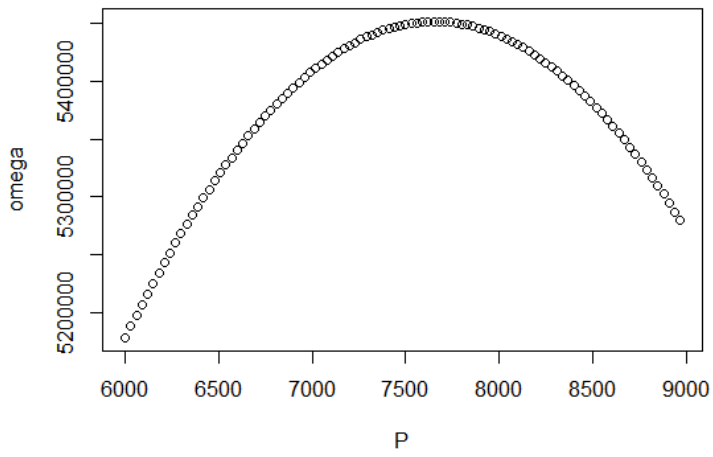

Figure 5.3: The expected total profit ω_h against product price P, $T = 24$.

If the manufacturer takes Type I software warranty policy, the parameters of software warranty cost and the software reliability growth model (SRGM) are set as: c_{s1} = 100, a_2

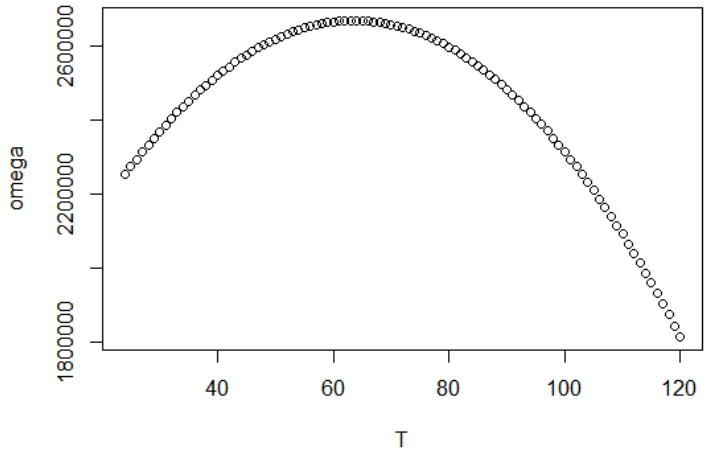

Figure 5.4: The expected total profit ω_h against warranty length T, $P = 2,000$.

= 50, and b_2 = 0.05.

Then, the cumulative intensity function is

$$\Lambda_2(T) = 50(1 - e^{-0.05T}),$$

and the expected total profit of a software product is

$$\omega_{s1}(P, T) = (1000 - 0.1P + 21T)(P - 100 \times 50 \times (1 - e^{-0.05T})). \tag{5.59}$$

When the warranty length is known, for example, set $T = 24$ month, then, the expected total profit of a software product can be maximised at $P = 9267.44$, and the maximum expected total profit is $3,332,737$. The following Figure 5.5 presents the expected total profit against the product price P.

However, if P is known and $T \geq 0$, the first order derivative of Eq. (5.6) cannot be zero, i.e. the Eq. (5.6) does not have any maxima or minima. The surface in Figure 5.6 presents the expected total profit of a software product against P and T.

If the manufacturer takes Type II software warranty policy, the related parameters are set in the following Table 5.1. Then, the expected total profit is

$$\omega_{s2}(P, T, \Delta\tau) = MP - [500n + 100 \sum_{i=1}^{n} \int_{\tau_{i-1}}^{\tau_i} \lambda_{2,i-1}(t)dt + 100 \int_{\tau_n}^{T} \lambda_{2,n}(t)dt], \tag{5.60}$$

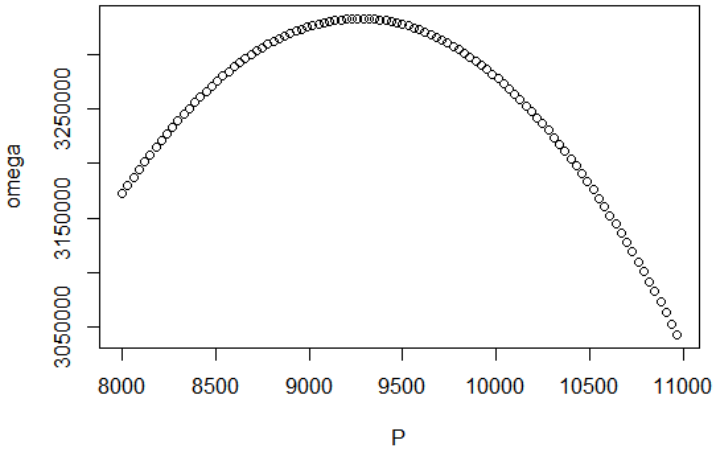

Figure 5.5: The expected total profit of a software product when $T = 24$.

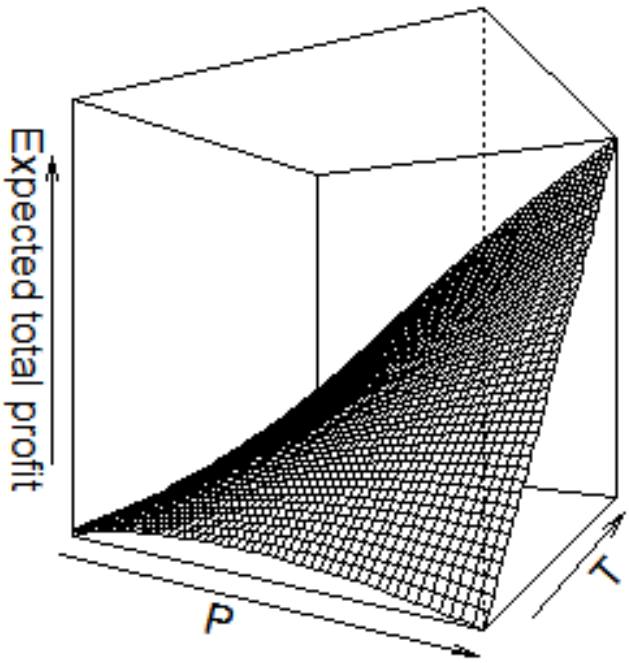

Figure 5.6: The expected total profit of a software product against P and T.

where $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is the number of releasing patches; M is the number of product sold at $t = 0$; τ_i is the time of ith patches/updates release under time-based policy, $\tau_i = \tau_{i-1} + \Delta\tau$ and $\tau_0 = 0$; $\lambda_{2,i}$ is the failures intensity after ith patches/updates release, $\lambda_{2,i} = 90\% \times \lambda_{2,i-1}$ and $\lambda_{2,0}(t) = \lambda_2(t)$.

Then, if the warranty length is 2-year, i.e. $T = 24$, and the software's update is

Table 5.1: Parameters for software warranty under Type II policy.

Software claim cost	SRGM parameter		Update cost	The changes after update
c_{s1}	a_2	b_2	c_{s2}	θ
100	50	0.05	500	90%

released quarterly, i.e. $\Delta\tau = 3$; the maximum expected total profit of this product $\omega_{s2} = 1,958,190$ can be achieved at $P = 10,614.86$. The Figure 5.7 presents the relationship between the expected total profit and product price under Type II software warranty policy if T and $\Delta\tau$ are known.

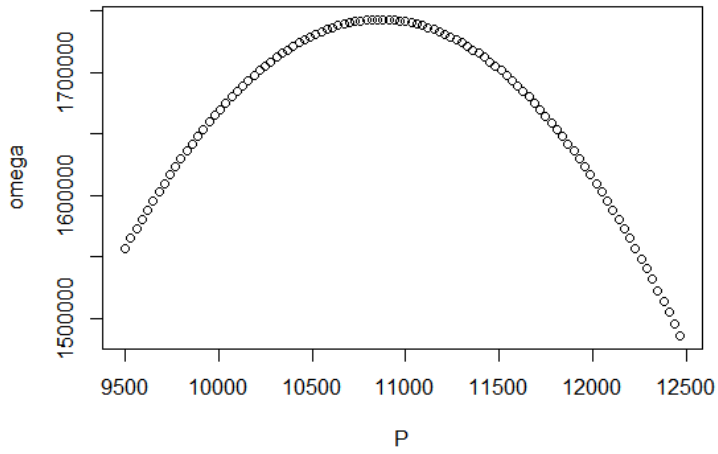

Figure 5.7: The expected total profit ω_{s2} against P, when $T = 24$ and $\Delta\tau = 3$.

If the warranty length is still 2-year, and the market price of this product is $P = 6500$; then the optimal update releasing interval is $\Delta\tau = 13.6$ month, and the maximised expected total profit is $2,227,765$. The Figure 5.8 presents the relationship between the expected total profit and the update releasing interval under Type II software warranty policy if P and T are known. The curve in Figure 5.8 is not continuous because the times of update releasing $n = \lfloor \frac{T}{\Delta\tau} \rfloor$ is an integer.

5.6.2 Expected total profit considering hardware-software interactions

There are 5 scenarios of hardware-software interactions discussed, in this section, the numeric examples for Scenario 0 and Scenario 1 are provided. For the product consists

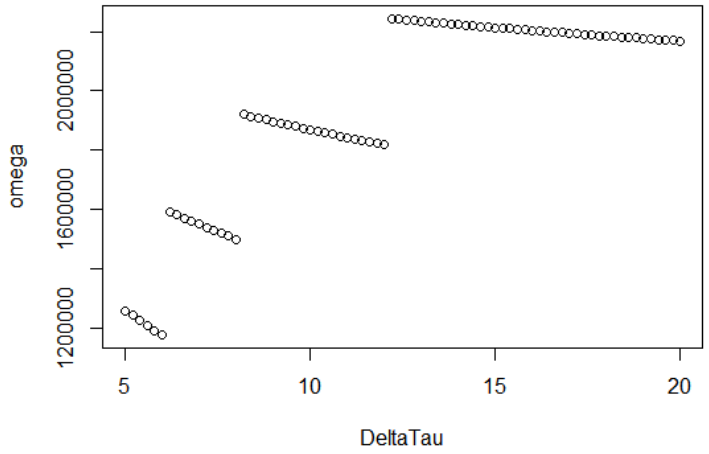

Figure 5.8: The expected total profit ω_{s2} against $\Delta\tau$, when $P = 6500$ and $T = 24$.

of hardware and software subsystems, the parameters are set in the Table 5.2.

Table 5.2: Parameters for Scenario 0 and 1 of interaction.

c_{h1}	a_1	b_1	c_{s1}	a_2	b_2	c_{s2}	θ
100	0.1	1.04	100	50	0.05	500	90%

Scenario 0

In Scenario 0, the product consists of hardware and software subsystems, but the failures of these two subsystems are assumed independently. The expected total profit under the first condition (software cannot be updated online) of Scenario 0 is

$$\omega_{01} = (1000 - 0.1P + 21T)\left[P - 10T^{1.04} - 5000(1 - e^{-0.05T})\right]. \tag{5.61}$$

When $T = 24$ is known, then the expected total profit is maximised at $P = 9403.55$, and the maximum expected total profit is $3,177,261$. The curve of expected total profit ω_{01} against product price P is displayed in Figure 5.9.

If the software can be updated online and the pre-specified updating interval is $\Delta\tau =$

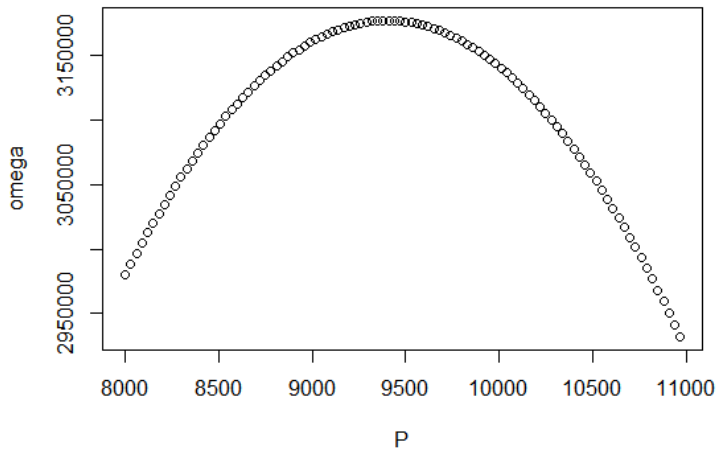

Figure 5.9: The expected total profit ω_{01} against P, when $T = 24$.

6months the expected total profit in Scenario 0 is

$$\begin{aligned}\omega_{02} &= (1000 - 0.1P + 21T)P - 10 \times (1000 - 0.1P + 21T)T^{1.04} \\ &\quad +100 \times \left[\sum_{i=1}^{n} 0.9^{i-1} \left[\Lambda_2(\tau_i) - \Lambda_2(\tau_{i-1})\right] + 0.9^n \left[\Lambda_2(T) - \Lambda_2(\tau_n)\right]\right] \\ &\quad +500n. \end{aligned} \tag{5.62}$$

If the warranty length is $T = 24$ months, then $n = 4$ the expected total profit is maximised at $P = 8,743.16$ and the maximum expected profit is $7,930,037$. The curve of expected total profit ω_{02} against product price P is displayed in Figure 5.10.

If the warranty length is $T = 24$ months and the product price $P = 8000$, the maximum expected total profit is $8,345,104$ achieved at $\Delta\tau = 12.25$. The curve of expected total profit ω_{02} against the updating interval $\Delta\tau$ is displayed in Figure 5.11.

Scenario 1

In Scenario 1, the occurrence of hardware and software failures are statistical dependent: a software failure can lead to hardware failures with probability $p(t)$ at time t. Assume $p(t) = 0.1e^{-0.1t}$ in this case. If the software cannot be updated online, the expected total

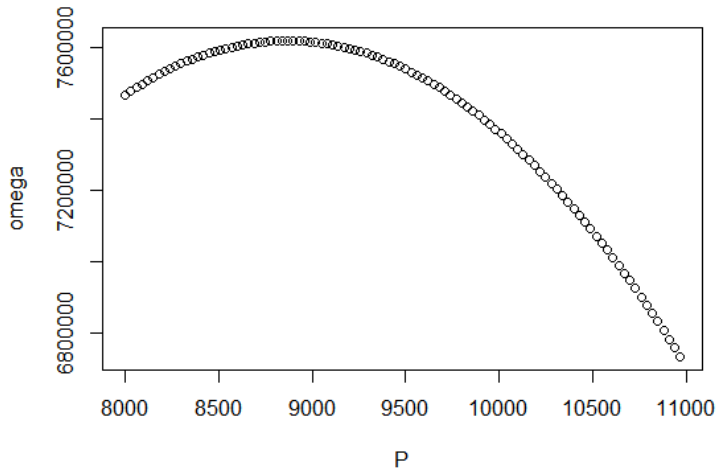

Figure 5.10: The expected total profit ω_h against warranty length T, $P = 2,000$.

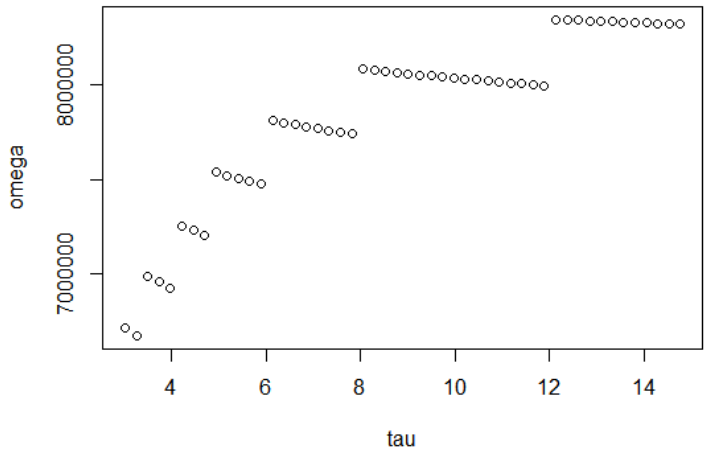

Figure 5.11: The expected total profit ω_{02} against $\Delta\tau$, when $T = 24$ and $P = 8000$.

profit is Then the expected total profit is

$$\begin{aligned}\omega_{11} &= (1000 - 0.1P + 21T)P \\ &\quad -(1000 - 0.1P + 21T) \times 100 \left[0.1T^{1.04} + 50(1 - e^{-0.05T}) \int_0^T 0.1e^{-0.1t}dt\right] \\ &\quad -(1000 - 0.1P + 21T) \times 100 \times 50(1 - e^{-0.05T}). \end{aligned} \tag{5.63}$$

When $T = 24$ is known, then the maximum expected total profit $\omega_{11} = 1,638,784$ can be reached at $P = 10,991.81$. The curve of expected total profit ω_{11} against product price P is displayed in Figure 5.12.

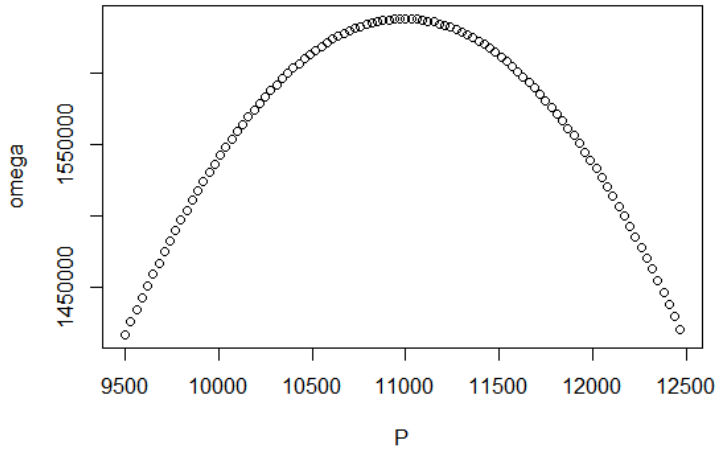

Figure 5.12: The expected total profit ω_{11} against P, when $T = 24$.

If the software can be updated online, then the expected total profit is

$$\begin{aligned}
\omega_{12} &= (1000 - 0.1P + 21T)P - \mathrm{E}[C_{12}(T)] \\
&= (1000 - 0.1P + 21T)P - (1000 - 0.1P + 21T) \times 10T^{1.04} \\
&\quad -100\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right]\int_0^T 0.1e^{-0.1t}dt \\
&\quad -100\left[\sum_{i=1}^{n}\int_{\tau_{i-1}}^{\tau_i}\lambda_{2,i-1}(t)dt + \int_{\tau_n}^{T}\lambda_{2,n}(t)dt\right] - \lfloor\frac{T}{\Delta\tau}\rfloor \times 500. \qquad (5.64)
\end{aligned}$$

If the software is updated online in every 6 month, i.e. $\Delta\tau = 6$, the warranty length is $T = 24$ and the times of updating is $n = \lfloor\frac{T}{\Delta\tau}\rfloor = 4$; then, the maximum expected total profit $\omega_{12} = 1,336,010$ can be achieved at $P = 11,384.86$. The curve of expected total profit ω_{12} against product price P is displayed in Figure 5.13.

If the warranty length is $T = 24$ months and the product price $P = 12,000$, the maximum expected total profit is $1,448,997$ achieved at $\Delta\tau = 13.03$. The curve of expected total profit ω_{12} against the updating interval $\Delta\tau$ is displayed in Figure 5.14.

5.6.3 Expected total profit considering hardware-software interactions and human factors

If a product is composed of hardware and software subsystems, the warranty cost of such product is influenced by the warranty claims on both hardware and software failures, the

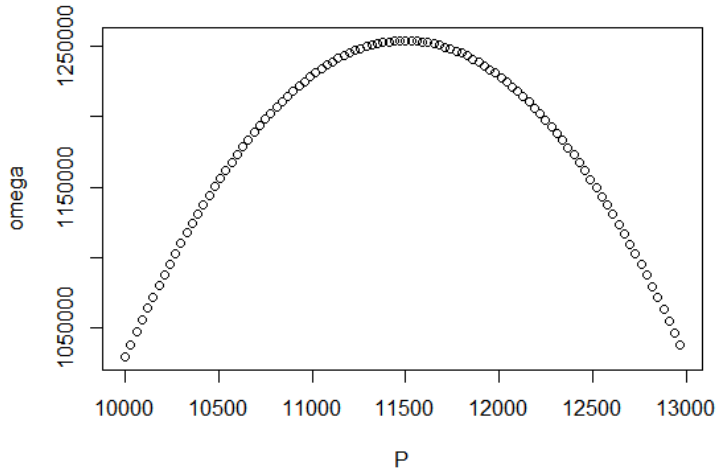

Figure 5.13: The expected total profit ω_{12} against P, when $T = 24$ and $\Delta\tau = 6$.

three human factors and the interplay between hardware and software subsystems. In this case, the hardware and software warranty parameters are set as same as above examples and the human parameters are set in Table 5.3.

Table 5.3: Parameters for human factors.

NFBR parameters				FBNR parameters		Delayed updating parameter
α_1	α_2	c_{hu1}	c_{hu2}	γ_1	γ_2	d
60	2	50	80	0.01	0.018	0.9

Off-line situation

If Scenario 1 of hardware-software interplay occurs, the software cannot be updated on-line, the NFBR and FBNR phenomena affect the total warranty cost, and the NFBR does not cause warranty to be ceased, the expected total profit is

$$\begin{aligned}
& \omega_{int31} \\
= \; & MP - \mathrm{E}[C_{int31}(T)] = MP - \mathrm{E}[C_{11}(T)]q_1(T) - \mathrm{E}[C_{hu1}(T)] \\
= \; & MP - \\
& Me^{-0.01-0.018T}\left\{100\left[0.1T^{1.04} + 50(1-e^{-0.05T})\int_0^T 0.1e^{-0.1t}dt\right] + 5000(1-e^{-0.05T})\right\} \\
& -50M(1-e^{-(T/60)^2}).
\end{aligned} \tag{5.65}$$

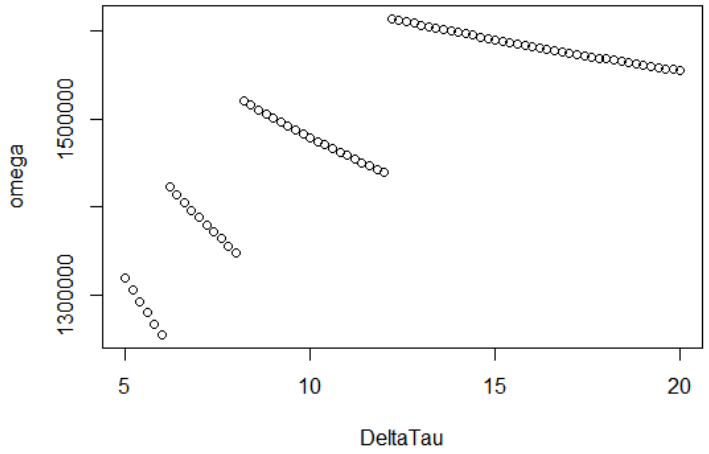

Figure 5.14: The expected total profit ω_{12} against $\Delta\tau$, when $T = 24$ and $P = 12,000$.

If the warranty length is $T = 24$ months, the maximum expected total profit $\omega_{int31} = 2,786,918$ can be achieved at $P = 10,000$. The curve of expected total profit ω_{int31} against the price P is displayed in Figure 5.15 by the line curve; in this figure, the point-curve, which is under the line curve, represents the total profit ω_{int31} against the price P without considering the human factors. Figure 5.15 indicates that if the human factors are

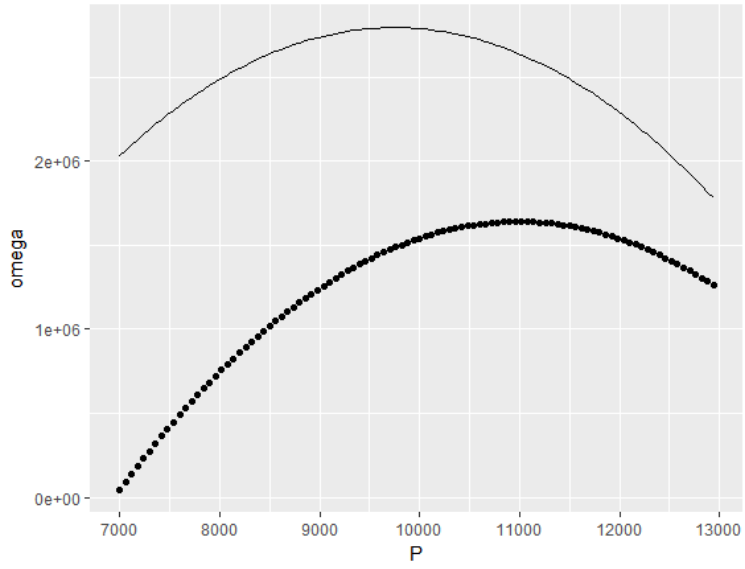

Figure 5.15: The expected total profit ω_{int31} against P with and without considering human factors, when $T = 24$.

not taken into account, the expected total profit is undervalued in this case.

If the NFBR may cause warranty to be ceased, the expected total profit is

$$\begin{aligned}\omega_{int32} &= MP - \mathrm{E}[C_{int32}(T)] \\ &= MP - [\mathrm{E}[C_{11}(T)]q_1(T) + Mc_{hu2}]\, H_1(T) - \mathrm{E}[C_{11}(T)]q_1(T)(1 - H_1(T))\end{aligned} \tag{5.56}$$

If the warranty length is $T = 24$ months, the maximum expected total profit $\omega_{int32} = 2,784,683$ can be achieved at $P = 9,999$. This result indicate that whether the NFBR event causes warranty to be ceased or not, the optimal value of price P is not influenced significantly.

Online situation

If the software can be updated online, the delayed updating effect is applied. Meanwhile, if the NFBR may not cause warranty to be ceased, the expected total profit is $\omega_{int33} = MP - \mathrm{E}[C_{int33}(T)]$, where $\mathrm{E}[C_{int33}(T)]$ consists of the cost of the NFBR event cost and the warranty cost modified with considering FBNR and delayed updating phenomena. According to Eq. (5.51)), (5.52 and (5.53), the optimal P and $\Delta\tau$ can be found by simulation. The simulation result shows that if the warranty length is $T = 24$ months and the updating interval is $\Delta\tau = 6$, the maximum expected total profit $\omega_{int33} = 1,521,743$ can be achieved at $P = 11,139.05$. The curve of expected total profit ω_{int33} against the price P is displayed in Figure 5.16 by the line curve; in this figure, the point-curve, which is under the line curve, represents the total profit ω_{int33} against the price P without considering the human factors.

If the warranty length $T = 24$ and the price $P = 12000$ are known, the optimal updating interval may also be determined by simulation. The result shows the maximum expected total profit $\omega_{int33} = 1,584,178$ is achieved at $\Delta\tau = 13.03$. The plot of expected total profit ω_{int33} against the updating interval $\Delta\tau$ is displayed in Figure 5.17 by the points; in this figure, the stars, which is under the points, represent the total profit ω_{int33} against $\Delta\tau$ without considering the human factors.

Figure 5.16 and 5.17 indicate that if the human factors are not taken into account, the expected total profit is undervalued under this online situation.

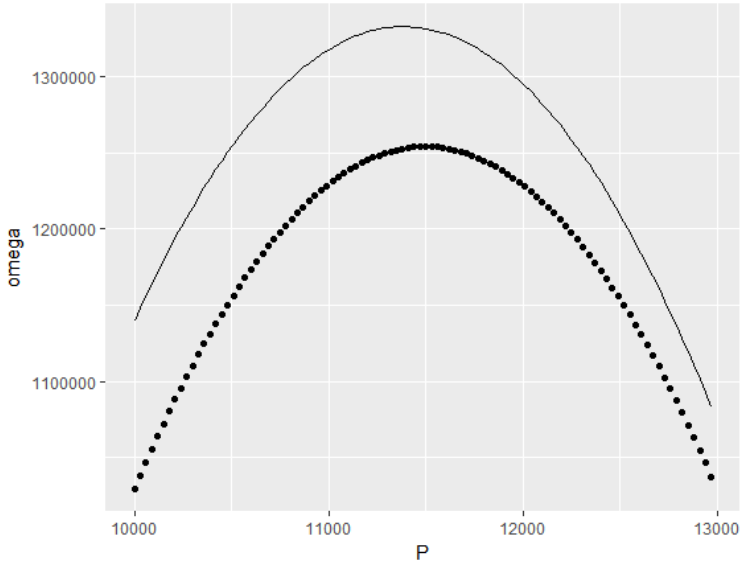

Figure 5.16: The expected total profit ω_{int33} against P with and without considering human factors, when $T = 24$ and $\Delta\tau = 6$.

5.7 Summary

The warranty cost of a product with embedded software system should be modelled in a manner that the failure of hardware, software and users. This chapter developed the models of warranty costs incurred by hardware specific, software specific and hardware-software interaction failures and provided integrated models.

Future work in this area includes the form of the cumulative intensity function of hardware-software interaction failure and the relationships among the three types of failures in the second scenario.

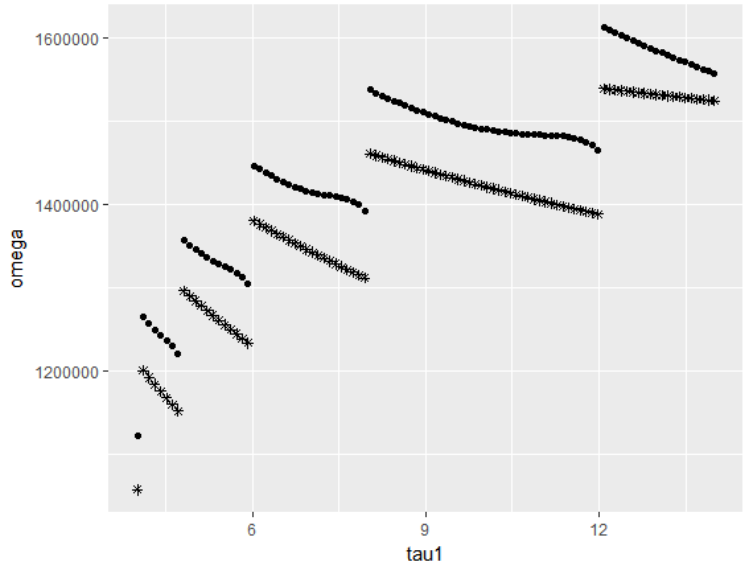

Figure 5.17: The expected total profit ω_{int33} against $\Delta\tau$ with and without considering human factors, when $T = 24$ and $P = 12000$.

Chapter 6

Discussion

The preceding chapters discussed optimisation of warranty policies for a portfoio of products and considered the effect of the interplay of hardware subsystems, software subsystems, and the end-users. In Chapter 3, the modern portfolio theory and copula are borrowed from the actuarial science and financial mathematics to build a mean-variance framework to collectively optimise the warranty policies of the products produced by a manufacturer. In Chapter 4, the mean-VaR framework is employed to deal with the extremal events and dependence among the warranty claims of different products. In Chapter 5, the warranty optimisation is conducted in a more comprehensive manner than that in Chapters 3 and 4 with a consideration of the interaction among hardware, software and users of a product in different scenarios. These three chapters provide frameworks to collectively and comprehensively optimise warranty policies , which have improved the existing research and made a contribution to the existing literature. It is known that the sales volume may be modelled in linear or non-linear forms and static or dynamic forms; and warranty policy can be renewing or non-renewing. Furthermore, copula selection is still a critical task in such research. These issues are discussed in this chapter.

6.1 Sales volume modelling

In the above chapters, the sales volume is modelled in a static linear form introduced by Yazdian et al. (2016), who assumes the following linear relationship among the length of warranty coverage, warranty price, and sales amount of a product, $M_k = A_k - \beta_k P_k +$

$\eta_k T_k$, where, $\beta_k > 0$ and $\eta_k > 0$. However, in the literature related to warranty management and marketing research, the product sales volume can be assumed in linear or non-linear and static or dynamic forms.

Ladany & Shore (2007) provide a non-linear expression of the sale volume as following

$$M_k = \alpha_k P_k^{-\beta_1} T_k^{\beta_2}, \tag{6.1}$$

where $\alpha_k > 0$, $\beta_1 > 0$ and $\beta_2 > 0$. This function means the sale volume decreases with product price increasing and increases with warranty length increasing, however, if warranty is not provided the demand is zero.

According to Eq. (3.7), if the sale volume is Eq. (6.1), the expected profit of a single product is

$$\mathrm{E}[\omega_k] = (\alpha_k P_k^{-\beta_1} T_k^{\beta_2})(P_k - \lambda_k \mu_k T_k). \tag{6.2}$$

The effects of P_k and T_k on $\mathrm{E}[\omega_k]$ can be revealed based on the derivatives. Taking the logarithm of both sides of Eq. (6.2), we can obtain

$$\begin{aligned}\ln(\mathrm{E}[\omega_k]) &= \ln(\alpha_k P_k^{-\beta_1} T_k^{\beta_2}) + \ln(P_k - \lambda_k \mu_k T_k) \\ &= \ln \alpha_k - \beta_1 \ln P_k + \beta_2 \ln T_k + \ln(P_k - \lambda_k \mu_k T_k).\end{aligned}$$

According to implicit differentiation, the first order derivative of $\mathrm{E}[\omega_k]$ on P_k is

$$\frac{\partial \mathrm{E}[\omega_k]}{\partial P_k} = (\alpha_k P_k^{-\beta_1} T_k^{\beta_2})(P_k - \lambda_k \mu_k T_k)(-\frac{\beta_1}{P_k} + \frac{1}{P_k - \lambda_k \mu_k T_k}),$$

as $(\alpha_k P_k^{-\beta_1} T_k^{\beta_2}) > 0$ and $(P_k - \lambda_k \mu_k T_k) > 0$, the sign of the first order derivative is determined by $(-\frac{\beta_1}{P_k} + \frac{1}{P_k - \lambda_k \mu_k T_k})$. Then, if $\beta_1 > 1$, when $0 < P_k \leq \frac{\beta_1 \lambda_k \mu_k T_k}{\beta_1 - 1}$, $\frac{\partial \mathrm{E}[\omega_k]}{\partial P_k} \geq 0$, i.e. when T_k is known, the expected profit of a single product $\mathrm{E}[\omega_k]$ can be maximised at $P_k = \frac{\beta_1 \lambda_k \mu_k T_k}{\beta_1 - 1}$; if $0 < \beta_1 < 1$, when $P_k \geq \frac{\beta_1 \lambda_k \mu_k T_k}{\beta_1 - 1}$, $\frac{\partial \mathrm{E}[\omega_k]}{\partial P_k} \geq 0$, it means the expected profit of a single product $\mathrm{E}[\omega_k]$ can be minimised at $P_k = \frac{\beta_1 \lambda_k \mu_k T_k}{\beta_1 - 1}$, i.e. when T_k is known, the optimal solution which maximised the expected profit does not exist.

Meanwhile, the first order derivative of $\mathrm{E}[\omega_k]$ on T_k is

$$\frac{\partial \mathrm{E}[\omega_k]}{\partial T_k} = (\alpha_k P_k^{-\beta_1} T_k^{\beta_2})(P_k - \lambda_k \mu_k T_k)(\frac{\beta_2}{T_k} - \frac{\lambda_k \mu_k}{P_k - \lambda_k \mu_k T_k}),$$

as $(\alpha_k P_k^{-\beta_1} T_k^{\beta_2}) > 0$ and $(P_k - \lambda_k \mu_k T_k) > 0$, the sign of the first order derivative is determined by $(\frac{\beta_2}{T_k} - \frac{\lambda_k \mu_k}{P_k - \lambda_k \mu_k T_k})$. As $\beta_2 > 0$, if $(\frac{\beta_2}{T_k} - \frac{\lambda_k \mu_k}{P_k - \lambda_k \mu_k T_k}) \geq 0$, $0 < T_k \leq \frac{\beta_2 P_k}{\lambda_k \mu_k (1+\beta_2)}$. It means that if P_k is known, the expected profit of a single product $\mathrm{E}[\omega_k]$ can be maximised at $T_k = \frac{\beta_2 P_k}{\lambda_k \mu_k (1+\beta_2)}$.

Xie et al. (2014) extends the model in Ladany & Shore (2007) to allow the demand existing when warranty is not provided,

$$M_k = \kappa_1 P_k^{-\beta_1} (T_k + \kappa_2)^{\beta_2}, \tag{6.3}$$

where $\kappa_1, \kappa_2 > 0$, $\beta_1 > 0$, and $0 < \beta_2 < 1$.

According to Eq. (3.7), if the sale volume is Eq. (6.3), the expected profit of a single product is

$$\mathrm{E}[\omega_k] = \kappa_1 P_k^{-\beta_1} (T_k + \kappa_2)^{\beta_2} (P_k - \lambda_k \mu_k T_k). \tag{6.4}$$

Taking the logarithm of both sides of Eq. (6.4), then, we have

$$\ln(\mathrm{E}[\omega_k]) = \ln \kappa_1 - \beta_1 \ln P_k + \beta_2 \ln(T_k + \kappa_2) + \ln(P_k - \lambda_k \mu_k T_k),$$

taking the first order derivative on P_k of both sides, we obtain

$$\frac{1}{\mathrm{E}[\omega_k]} \frac{\partial \mathrm{E}[\omega_k]}{\partial P_k} = -\frac{\beta_1}{P_k} + \frac{1}{P_k - \lambda_k \mu_k T_k},$$

then

$$\frac{\partial \mathrm{E}[\omega_k]}{\partial P_k} = \kappa_1 P_k^{-\beta_1} (T_k + \kappa_2)^{\beta_2} (P_k - \lambda_k \mu_k T_k) (-\frac{\beta_1}{P_k} + \frac{1}{P_k - \lambda_k \mu_k T_k}). \tag{6.5}$$

As $\kappa_1 P_k^{-\beta_1} (T_k + \kappa_2)^{\beta_2} (P_k - \lambda_k \mu_k T_k) > 0$, then the sign of Eq. (6.5) is determined by $(-\frac{\beta_1}{P_k} + \frac{1}{P_k - \lambda_k \mu_k T_k})$. Similar to the result of the sale volume form Eq. (6.1) discussed above, if $\beta_1 > 1$, when $0 < P_k \leq \frac{\beta_1 \lambda_k \mu_k T_k}{\beta_1 - 1}$, $\frac{\partial \mathrm{E}[\omega_k]}{\partial P_k} \geq 0$, i.e. when T_k is known, the expected profit of a single product $\mathrm{E}[\omega_k]$ can be maximised at $P_k = \frac{\beta_1 \lambda_k \mu_k T_k}{\beta_1 - 1}$.

If P_k is known, the first order derivative of $\mathrm{E}[\omega_k]$ on T_k is

$$\frac{\partial \mathrm{E}[\omega_k]}{\partial T_k} = \kappa_1 P_k^{-\beta_1} (T_k + \kappa_2)^{\beta_2} (P_k - \lambda_k \mu_k T_k) (\frac{\beta_2}{T_k + \kappa_2} - \frac{\lambda_k \mu_k}{P_k - \lambda_k \mu_k T_k}). \tag{6.6}$$

As $\kappa_1 P_k^{-\beta_1} (T_k + \kappa_2)^{\beta_2} (P_k - \lambda_k \mu_k T_k) > 0$, then the sign of Eq. (6.5) is determined by $(\frac{\beta_2}{T_k + \kappa_2} - \frac{\lambda_k \mu_k}{P_k - \lambda_k \mu_k T_k})$. Because $\kappa_1, \kappa_2 > 0$, $\beta_1 > 0$, and $0 < \beta_2 < 1$, if $0 < T_k <$

$\frac{\beta_2 P_k - \kappa_2 \lambda_k \mu_k}{(\beta_2+1)\lambda_k \mu_k}$, $\frac{\partial \mathrm{E}[\omega_k]}{\partial T_k} > 0$. It means that if P_k is known, the expected profit, expressed by Eq. (6.4), can be maximised at $T_k = \frac{\beta_2 P_k - \kappa_2 \lambda_k \mu_k}{(\beta_2+1)\lambda_k \mu_k}$.

The expressions discussed above show that the optimisation of warranty policies can also be conducted based on the non-linear sales volume functions. However, in the real market, especially in the consumer market, the items of one product cannot be sold out at the same time, i.e. $t = 0$ assumed above. The sales process is dynamic and the sales volume normally varies with time passing.

In the literature, the dynamic sales volume, i.e. dynamic demand, of a product is noticed by researchers. Huang et al. (2007) introduces a dynamic model, in which $T_k(t)$ is duration of warranty period for products sold at time t, $P_k(t)$ is price at time t, $M_k(t)$ is accumulated sales, $M_{M,k}$ is maximum sales potential, and $M_{0,k}$ is parameter characterizing past sales of production experience.

$$\begin{aligned} M_k(t) &= M_{0,k} + \int_0^t m_k(t)dt, \\ m_k(t) &= k_1(T_k(t)+k_2)^{\alpha} P_k(t)^{-\beta}[1 - \frac{M_k(t)}{M_M}] \times [\psi + \frac{M_k(t)}{M_M}], \\ M_k(0) &= M_{0,k} \end{aligned}$$

where k_1, k_2, α, $-\beta$ and ψ are positive.

As extension of Huang et al. (2007), Lin et al. (2009) gives three different dynamic demand functions to describe log-linear, limited growth and logistic growth forms. According to Lin et al. (2009), the dynamic sales volume functions of one product can be expressed as following:

- Log-linear sales volume:

$$m(P,T,t) = k_1 p(P,t)^{-a}(k_2+T)^{b}; \tag{6.7}$$

- Limited growth sales volume:

$$m(P,T,t) = k_1 p(P,t)^{-a}(k_2+T)^{b}(\bar{M} - M(P,T,t)); \tag{6.8}$$

and

- Logistic growth sales volume:

$$m(P,T,t) = k_1 p(P,t)^{-a}(k_2+T)^b M(P,T,t)(\bar{M} - M(P,T,t)). \tag{6.9}$$

In Eq. (6.7), (6.8) and (6.9), $m(P,T,t)$ is the sales rate at time t, $M(P,T,t) = \int_0^t m(u)du$ is the cumulative sales at time t, P is the initial price of the product at time $t = 0$ and T is the non-negative warranty length variable. $p(P,t)$ is a continuous derivable of P and t and represents the retail price of the product at time t, $p(P,0) = P$. The parameters in these three functions are $k_1 > 0$, $k_2 \geq 0$, $a > 1$, $0 < b < 1$ and $\bar{M} > 0$ is the market potential.

For instance, if the log-linear sales volume function is employed to model the sales volume, assume the warranty length T is known and the selling period of the product is also presupposed as T_s, then the total sales volume of this product is

$$M(P,T_s) = \int_0^{T_s} m(P,t)dt = \int_0^{T_s} k_1 p(P,t)^{-a}(k_2+T)^b dt, \tag{6.10}$$

the expected profit of this product is

$$\mathrm{E}[\omega] = \int_0^{T_s} f(P,t)dt = \int_0^{T_s} k_1 p(P,t)^{-a}(k_2+T)^b [p(P,t) - \lambda\mu T]dt. \tag{6.11}$$

Then, the partial derivative of $\mathrm{E}[\omega]$ on P is

$$\begin{aligned}
\frac{\partial \mathrm{E}[\omega]}{\partial P} &= \left\{ \int f(P,t)dt \middle| t = T_s \right\}' - \left\{ \int f(P,t)dt \middle| t = 0 \right\}' \\
&= \left\{ p(P,T_s)^{-a} \left[p(P,T_s) - \lambda\mu T \right] \left[(1-a)p(P,T_s)^{-a} + ap(P,T_s)^{-a-1}\lambda\mu T \right] p'(P,T_s) \right. \\
&\quad \left. - P^{-a}(P - \lambda\mu T) \left[(1-a)P^{-a} + aP^{-a-1}\lambda\mu T \right] \right\} k_1^2 (k_2+T)^{2b}.
\end{aligned} \tag{6.12}$$

In Eq. (6.12), $P^{-a}(P - \lambda\mu T)\left[(1-a)P^{-a} + aP^{-a-1}\lambda\mu T\right] > 0$, the sign of $\frac{\partial \mathrm{E}[\omega]}{\partial P}$ is determined by $p'(P,T_s)$. Then, if function $p(P,T_s)$ is monotonously decreasing, i.e. $p'(P,T_s) < 0$, the expected profit $\mathrm{E}[\omega]$ cannot be maximised; if function $p(P,T_s)$ is not monotone decreasing, the optimal solution which maximises the expected profit may exist.

If the limited growth sales volume function (6.8) is employed and the length of warranty T is known, then the expected profit of this product is

$$\mathrm{E}[\omega] = \int_0^{T_s} k_1 p(P,t)^{-a}(k_2+T)^b (\bar{M} - M(P,t))[p(P,t) - \lambda\mu T]dt. \tag{6.13}$$

However, as $M(P,t) = \int m(P,t)$, Eq. (6.8) is a differential equation. To investigate the properties of Eq. (6.5), the differential equation should be resolved initially, the solution is shown as following

$$\begin{aligned}
m(P,t) &= k_1 p(P,t)^{-a}(k_2+T)^b(\bar{M} - M(P,t)) \\
\frac{dM(P,t)}{dt} &= k_1 p(P,t)^{-a}(k_2+T)^b(\bar{M} - M(P,t)) \\
\frac{1}{\bar{M} - M(P,t)} dM(P,t) &= k_1 (k_2+T)^b p(P,t)^{-a} dt \\
-\ln(\bar{M} - M(P,t)) &= k_1 (k_2+T)^b \int p(P,t)^{-a} dt \\
\bar{M} - M(P,t) &= e^{-k_1(k_2+T)^b \int p(P,t)^{-a} dt} \\
M(P,t) &= \bar{M} - e^{-k_1(k_2+T)^b \int p(P,t)^{-a} dt}, \qquad (6.14)
\end{aligned}$$

then

$$m(P,t) = k_1 p(P,t)^{-a}(k_2+T)^b e^{-k_1(k_2+T)^b \int p(P,t)^{-a} dt}. \qquad (6.15)$$

According to Eq. (6.15), the expected profit of this product is

$$\mathrm{E}[\omega] = \int_0^{T_s} f(P,t)dt = \int_0^{T_s} m(P,t)[p(P,t) - \lambda\mu T]dt, \qquad (6.16)$$

calculate through integration by parts method,

$$\begin{aligned}
\mathrm{E}[\omega] &= \int_0^{T_s} m(P,t)[Pe^{-ct} - \lambda\mu T]dt \\
&= M(P,t)[Pe^{-ct} - \lambda\mu T] + \int M(P,t)cPe^{-ct}dt, \qquad (6.17)
\end{aligned}$$

then the effects of product price P on the expected profit $\mathrm{E}[\omega]$ can be investigated based on the first order derivative of $\mathrm{E}[\omega]$ on P, which is also determined by the form of function $p(P,t)$. Furthermore, the logistic growth sales volume function is similar to the limited growth sales volume function discussed above. The solution of the warranty policy optimisation problem based on the logistic growth sales volume is also affected by the form of $p(P,t)$.

In the real world, the price of a product may change over time, and the market price of the product at time t may be determined by the market participants and its initial price P. For instance, according to the consumer electronics data based of PriceSpy (priceSpy

(2016)), the prices of iPhone and its major competitor, Samsung, are all decreasing with time. Fig. 6.1 and Fig. 6.2 present the price history of the products in three years after

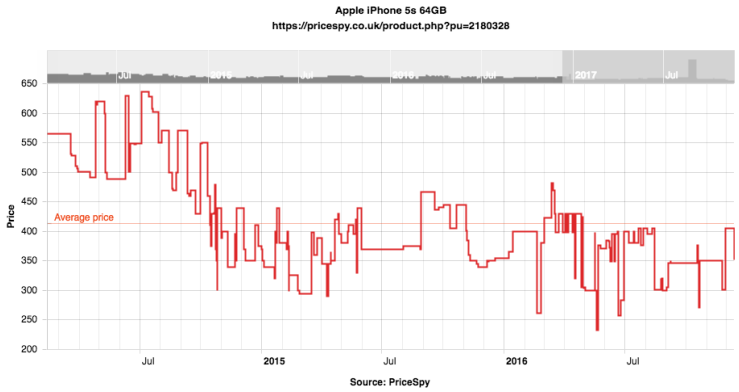

Figure 6.1: Apple iPhone 5s 64GB price history

being published. These figures reveal that the patterns of the price movement are complex and varying, even though the prices of both products have decreasing trend. It is necessary to investigate the proper time-varying price model to support warranty policy optimisation in future research.

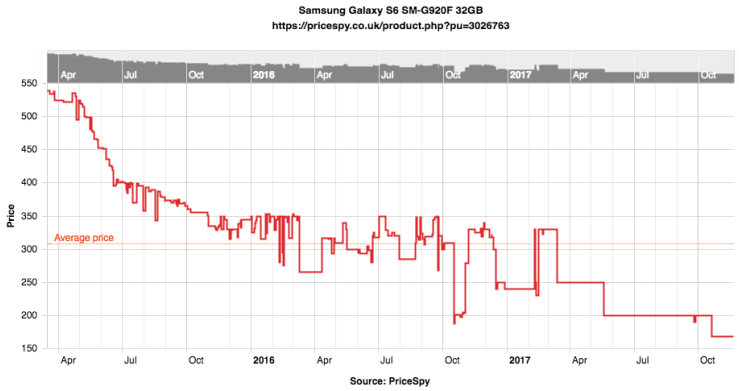

Figure 6.2: Samsung Galaxy S6 32GB price history

Additionally, in above discussion, different static and dynamic sales volume functions are investigated, however, in the real world, the most complex dynamic function may be still too simple to model the sales volume change. Also according to the database of PriceSpy, Fig. 6.3 and Fig. 6.4 present the popularity changes of these two products, which reflect the demand of these products in the market.

According to Fig. 6.3 and Fig. 6.4, the popularities of these two products have totally

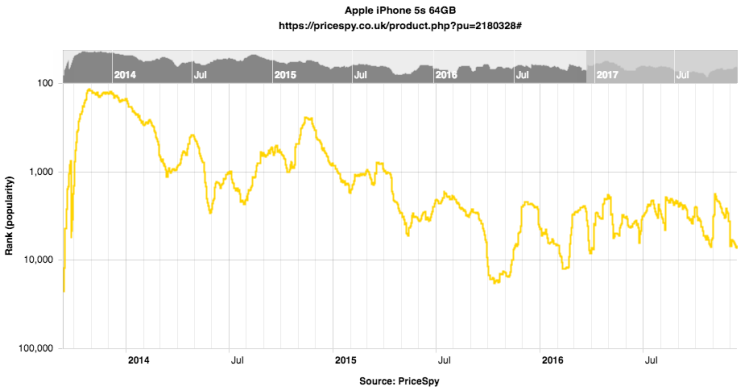

Figure 6.3: Apple iPhone 5s 64GB popularity history

different trends, even their prices all decrease with time, which means, in practice, to optimise the warranty policy of a particular product, the mode of its demand/sales volume should be investigated and considered comprehensively.

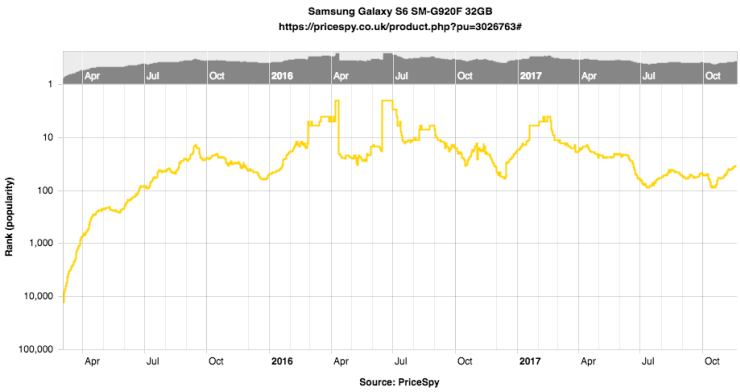

Figure 6.4: Samsung Galaxy S6 32GB popularity history

6.2 Renewing warranty

In Chapters 3, 4 and 5, the warranty policies are assumed to be non-renewing, however, in the real market, renewing warranty policies are provided to some unrepairable products and consumer electronic products. A renewing warranty policy means under this policy if an item fails within the warranty period, the item is replaced by a new one at the cost of the manufacturer or at the prorated cost to the user upon failure during the warranty period and the warranty is renewed (K. Jung et al., 2010). This process will continue until the manufacturer decides to stop supporting the product, i.e. the length of product useful

life has ended. According to Wu & Xie (2008), in this case, assuming that the product is sold with renewing warranty, after being sold the occurrence of the first failure of one item of the product has two possible cases:

Case A the product's first failure occurs within the warranty period $(0, T)$, and a new identical product replaces it; and

Case B the first failure of the product occurs after T, and it does not incur any replacement cost to the manufacturer.

To discuss warranty policy optimisation under renewing warranty, we consider the pure hardware product described in Chapter 5 for an example. The following assumptions are applied:

A1 if an item fails within the warranty period, the item is replaced by a new item at the cost of manufacturer, and

A2 the replacement cost of one item of a specified product, c_{rn}, is fixed. The total warranty cost under renewing warranty policy is $C_{rn} = nc_{rn}$, where n is the number of replacements.

If the item fails within its initial warranty period $(0, T)$, it is replaced by a new one with warranty under the same terms, i.e. the actual warranty period is extended. If the item survives beyond its warranty period, the actual warranty period ends at the warranty expiration time. Denote the distribution of the time between $(i-1)$th and ith failures t_i by $F_t(x)$ with *cdf* $f_t(x)$. Then the actual warranty period, denoted by W_T, is

$$W_T = \begin{cases} T, & if \quad t_1 > T \\ T + \sum_{i=1}^{n} t_i, & if \quad t_1 \leq T, \ldots, t_n \leq T, t_{n+1} > T \end{cases}, \tag{6.18}$$

and the distribution of the number of claims N_{rn} can be explained as

$$N_{rn} = \begin{cases} 0, & with \quad 1 - F_t(T) \\ 1, & with \quad F_t(T)(1 - F_t(T)) \\ 2, & with \quad F_t(T)^2(1 - F_t(T)) \\ \vdots & \\ n, & with \quad F_t(T)^n(1 - F_t(T)) \\ \vdots & \end{cases}. \tag{6.19}$$

Obviously, the distribution of the number of claims under renewing warranty policy is a geometric distribution with parameter $1 - F_t(T)$.

$$\Pr(N_{rn} = n) = F_t(T)^n(1 - F_t(T)) \tag{6.20}$$

Then, the expected number of claims is

$$\mathrm{E}[N_{rn}] = \frac{F_t(T)}{1 - F_t(T)}, \tag{6.21}$$

and the expected total warranty claim is

$$\mathrm{E}[C_{rn}] = \mathrm{E}[N_{rn}]c_{rn} = \frac{F_t(T)}{1 - F_t(T)}c_{rn}. \tag{6.22}$$

If t_i follows identical independent exponential distribution with parameter λ_{rn}, then the expected total warranty cost is

$$\begin{aligned} \mathrm{E}[C_{rn}] &= \frac{F_t(T)}{1 - F_t(T)}c_{rn} \\ &= \frac{1 - e^{-\lambda_{rn}T}}{e^{-\lambda_{rn}T}}c_{rn} \\ &= e^{\lambda_{rn}T}c_{rn} - c_{rn}. \end{aligned} \tag{6.23}$$

Then, the expected profit of one product under a renewing warranty policy with linear sales volume function is

$$\begin{aligned} \mathrm{E}[\omega_{rn}] &= M(P - \mathrm{E}[C_{rn}]) \\ &= (A - \beta P + \eta T)(P - e^{\lambda_{rn}T}c_{rn} - c_{rn}) \\ &= AP - Ae^{\lambda_{rn}T} - Ac_{rn} - \beta P^2 + \beta P e^{\lambda_{rn}T} + \beta P c_{rn} \\ &\quad +\eta TP - \eta T e^{\lambda_{rn}T} - \eta T c_{rn}. \end{aligned} \tag{6.24}$$

The solutions maximising the expected profit of one product under renewing warranty can be investigated according to the derivatives of (6.24) on P and T. The first order derivative of (6.24) on P is

$$\frac{\partial \mathrm{E}[\omega_{rn}]}{\partial P} = -2\beta P + A + \beta e^{\lambda_{rn}T} + \beta c_{rn}, \tag{6.25}$$

and the second order derivative of (6.24) on P is

$$\frac{\partial^2 \mathrm{E}[\omega_{rn}]}{\partial P^2} = -2\beta < 0. \tag{6.26}$$

It means if T is known, the expected profit of one product under renewing warranty can be maximised at $P = \frac{A+\beta e^{\lambda_{rn}T}+\beta c_{rn}}{2\beta}$. The first order derivative of (6.24) on T is

$$\frac{\partial \mathrm{E}[\omega_{rn}]}{\partial T} = -e^{\lambda_{rn}T}\left[\lambda_{rn}(A-\beta P+\eta T)+\eta\right]+\eta(P-c_{rn}), \tag{6.27}$$

and the second order derivative of (6.24) on T is

$$\frac{\partial^2 \mathrm{E}[\omega_{rn}]}{\partial T^2} = -e^{\lambda_{rn}T}\lambda_{rn}\left[\lambda_{rn}(A-\beta P+\eta T)+2\eta\right] < 0. \tag{6.28}$$

It means when P is known, if $\frac{\partial \mathrm{E}[\omega_{rn}]}{\partial T} = 0$ has non-negative solution, the expected profit of one product under renewing warranty can be maximised at the non-negative solution of $\frac{\partial \mathrm{E}[\omega_{rn}]}{\partial T} = 0$.

6.3 Copula selection

The copula theory states that any cumulative distribution function of a random vector can be written in terms of marginal distribution functions and a copula that describes the dependence structure between the variables, no matter whether the variables are continuous or discrete. The flourishing families of copula lead to the difficulty in copula selection in research and practice. In the literature, the overwhelming methods are the goodness-of-fit tests, such as the Kolmogorov-Smirnov test, the Anderson-Darling test, etc., based on various statistics, such as Kolmogorov-Smirnov distance, Anderson-Darling distance, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), etc. In practice, the goodness-of-fit tests can select out the proper copula, but a large amount of work needs to be accomplished because many copulas should be tested and compared.

Besides the goodness-of-fit tests, some properties of copulas can also help in selection, such as the dependence type, symmetry and tail dependence of copula. The researchers and practitioners can filter out the potentially proper copulas by these properties to generate efficient works.

6.3.1 Dependence type of copula

Elliptical and Archimedean families of copulas are the most popular copula families in the current literature. Elliptical copulas are parametric and include the Gaussian copula and the t copula. Gaussian and t-copulas contain the dependency structure from the multivariate normal and multivariate t-distributions respectively, and they are directly derived through the Sklars theorem (Pešta & Okhrin, 2014). Archimedean copulas are another associative class of copulas, different from the Gaussian copula, most common Archimedean copulas admit an explicit formula and allow modelling dependence in arbitrarily high dimensions with only one parameter.

The dependence between random variables can be linear or non-linear, and different copulas can cope with different types of dependence. For instance, the linear correlation coefficient, Pearson's ρ, based on the covariance of two variates is preserved by the Elliptical copulas but not the Archimedean copulas. The Gaussian copula, from the Elliptical family is constructed from a n-dimension normal distribution over a n-dimension matrix of Pearson's ρ through the probability integral transform. However, in the Archimedean family, two pairs of correlated variables with the same copula can have different linear correlation coefficient; while Kendall's τ and Spearman's ρ, the rank correlation coefficients are constants of the copulas, i.e. any correlated variable with the same copula will have the Kendall's τ and Spearman's ρ of that copula.

Let X and Y be continuous random variables with copula C, respectively. Then, Kendall's tau for X and Y is

$$\tau_C = 4 \int \int_{I^2} C(u,v) dC(u,v) - 1 = 4 \int \int_{I^2} C(u,v) c(u,v) du dv - 1 \tag{6.29}$$

Then Spearman's rho for X and Y is

$$\rho_C = 12 \int \int_{I^2} C(u,v) du dv - 3 = 12 \int \int_{I^2} [C(u,v) - uv] du dv \tag{6.30}$$

As a summary, the dependence type of the random variables and the value of corresponding correlation coefficient can guide the selection of copula in practice.

6.3.2 Tail dependence of copula

Generally, the concept of tail dependence indicates the degree of dependence in the lower-left-quadrant or upper-right-quadrant tails of a bivariate distribution. The degree of dependence is measured by the tail-dependence coefficients in common. This measure is widely used in the financial mathematics. Actually, in the financial market, the dependencies between financial asset returns are frequently observed in the last decades. However, common dependence measures such as the Pearsons correlation coefficient are not always suited for a proper understanding of dependencies in the market, particularly regarding the dependencies between extreme events such as extreme negative stock returns or large portfolio losses. As such, an alternative dependence measures to support beneficial asset-allocation strategies is needed (Frahm et al., 2005).

The following expression is the most common definition of tail dependence. Let (X, Y) be a pair of random variables with joint distribution F, and X and Y have marginal distributions G and H , respectively. The coefficient

$$\lambda_U = \lim_{z \to 1^-} \{G(X) > z | H(Y) > z\}, \tag{6.31}$$

is the upper tail dependence coefficient (upper TDC of the pair of random variables). If $\lambda_U > 0$, (X, Y) is upper tail dependent and if $\lambda_U = 0$, (X, Y) is upper tail independent, then, the coefficient

$$\lambda_L = \lim_{z \to 0^+} \{G(X) \leq z | H(Y) \leq z\}, \tag{6.32}$$

is the lower tail dependence coefficient (lower TDC) of the pair of random variables. If $\lambda_L > 0$, (X, Y) is lower tail dependent and if $\lambda_L = 0$, (X, Y) is lower tail independent, the above definition indicates that the TDC is a conditional probability that one margin exceeds a threshold when the other margin exceeds a threshold.

According to Sklar's theorem of copula, the TDC can also be expressed based on the notion of copula when the marginal distributions G and H are continuous. The joint distribution F can be represented as:

$$F(x, y) = C(G(x), H(y)),$$

where C is a copula. Then, the upper TDC of (X, Y) is

$$\lambda_U = \lim_{z \to 1^-} \frac{1 - 2z + C(z, z)}{1 - z}, \tag{6.33}$$

and the lower TDC is

$$\lambda_L = \lim_{z \to 0^+} \frac{C(z, z)}{z}. \tag{6.34}$$

The TDC can help in copula selection, because different copulas have different tail dependence properties. In the Elliptical family, the Gaussian copula does not have tail dependence, and t copula has tail dependence depended on its degrees of freedom. In the Archimedean family, Clayton copula is only lower tail dependent on lower TDC, $\lambda_L = 2^{-\frac{1}{\theta_C}}$, determined by its parameter θ_C; Gumbel copula is only upper tail dependent on upper TDC, $\lambda_U = 2 - 2^{\frac{1}{\theta_G}}$, determined by its parameter θ_G; and Frank copula is both upper tail independent and lower tail independent.

However, the above discussion only involves the tail dependences in the lower-left-quadrant and upper-right-quadrant, which means the upper and lower TDC can work well in copula selection only when the copula is symmetric/exchangeable. If the copula is asymmetric, the statuses of the joint distribution in the lower-right-quadrant and upper-left-quadrant should be considered. Wu (2014a) introduces two new tail dependence measures to cope with this issue. According to Wu (2014a), the TDC in the lower-right-quadrant, i.e. upper-lower TDC, is

$$\lambda_{u,l} = \lim_{z \to 0^+} \{G(X) > 1 - z | H(Y) \leq z\}, \tag{6.35}$$

and the upper-left-quadrant TDC, i.e. lower-upper TDC is

$$\lambda_{l,u} = \lim_{z \to 0^+} \{G(X) \leq z | H(Y) > 1 - z\}. \tag{6.36}$$

For a symmetric/exchangeable copula, $\lambda_{u,l} = \lambda_{l,u}$ The 4 TDCs described above can be considered comprehensively to select proper copulas before goodness-of-fit testing.

Chapter 7

Conclusions and future work

7.1 Conclusions

This research started with a multidisciplinary critical and comprehensive thinking across warranty modelling and actuarial science. At the first stage of this research, a large bulk of literature in these two disciplines are reviewed. In the progress of literature review, initially, the perils and events coverages, types and contents of policies and features of data in warranty and insurance are compared generally. Then, the techniques, approaches and theories applied in the actuarial science and warranty modelling are investigated and compared. According to such comparison, it finds that warranty modelling, particularly, warranty policy optimisation, may be improved referring to actuarial science with considerations of dependence modelling, using the modern portfolio theory (measuring risk through variance or VaR) and copulas. The literature review indicates that the dependence among some variables and the interactions among the related factors are merely considered in warranty modelling. In the real world, a manufacturer normally produces many different products that have common components installed. Consequently, the frequencies of warranty claims of different products are statistically dependent, which conflicts the fact that the existing methods in the literature in warranty modelling solely focus on individual products and ignore the claim dependence. In addition, in the literature, a product is normally considered as a pure hardware or software item, but the interplay between hardware and software is also ignored. As such, two major knowledge gaps are identified that the warranty policies of the products produced by one manufacturer are not investi-

gated integrally; and the effects of the interplay among a product's different subsystems, such as hardware and software, even and its user on warranty modelling are also merely considered in the literature of warranty management.

To fulfil the first knowledge gap, in Chapter 3, warranty policies of products produced by one manufacturer are modelled collectively through a model constructed based on the mean-variance methods, which is a method of the Modern Portfolio Theory. This chapter optimises warranty policies when the dependence among warranty claims of different products is taken into consideration. The dependence among the warranty claims of the products produced by one manufacturer is modelled through copulas in this new established collective warranty policy optimisation framework. Such optimisation is performed using the mean-variance optimisation approach, which considers the profit of a portfolio of different products with correlated numbers of warranty claims. The variance of the total profit of the portfolio of products is calculated based on a copula-based discrete joint distribution of the number of warranty claims of the products produced by a manufacturer. This chapter is the first attempt to provide a method of collectively optimising warranty policy when the expected profit and corresponding risks are taken into consideration. From a practical and applicable perspective, this method emphasizes the risk and potential dependence in warranty management and provides decision makers with a new approach to optimising the trade-off between the profit and risk in operation. This new framework also shows that if the dependence among the warranty claims of the products produced by one manufacturer is ignored, this manufacturer may undervalue the operational risk implied in the future warranty claims, and harm its operation.

However, in Chapter 3, the arrival process of warranty claims is assumed to be the homogeneous Poisson process (HPP), which may be too simple to cope with the facts in the real world. In addition to HPP, other processes, including the non-homogeneous Poisson process (NHPP), the doubly Poisson process, and the like, may be used to model the numbers of warranty claims. Furthermore, in Chapter 3, the risk, i.e. the uncertainty of total profit, is measured by variance, a symmetric risk measure; with progress of this research, the limits of such symmetric risk measure are uncovered in dealing with the

extreme events. This symmetric measure cannot reflect the tail of the distribution of total profit properly, hence some downside risk measures, such as Value-at-Risk, can be applied to improve the model.

Then, in Chapter 4, the non-homogeneous Poisson process and the downside risk measures, Value-at-Risk and Conditional Value-at-Risk, are introduced into the collective warranty policy optimisation framework. In this chapter, the number of warranty claims is modelled by NHPP with a power law intensity, and such process is compounded with independent and identically distributed log-normal distributions to model the total warranty cost, on which the total profit is derived based. This model is more complicated and realistic than that in Chapter 3, and the existence of optimal solutions, including warranty price and length, is also proven. To cope with the multi-product scenario, with support from multivariate copula, the joint distribution of the profits of products is constructed and the existence of optimal solutions in this scenario is proven. Then, another difficult task is raised, that is how to calculate or approximate the distribution of the sum of random variables, i.e. the distribution of the portfolio's total profit. To overcome this difficulty, a numerical method is introduced to calculate the aggregated distribution through Fast Fourier Transformation and an approximation method is applied to reveal the tail properties of the aggregated distribution. Lastly, numerical examples in single-product and multi-product scenarios are provided to contribute to the application and literature.

Chapter 3 and Chapter 4 are carried out based on an implied assumption that the products are composed only by hardware components. However, considering the technology development, in the real world more and more products consist of hardware and software subsystems, then the conventional warranty modelling and warranty policy optimisation approaches based on pure hardware products need improving.

In Chapter 5, the hardware, software and user subsystems of a product are taken into account together to build a more comprehensive warranty policy optimisation framework than those in Chapters 3 and 4. The warranty cost of a product with embedded software subsystem is modelled in a manner that the failure of hardware and software. This chapter developed the models of warranty costs incurred by hardware specific, software specific

and hardware-software interaction failures. The warranty policy optimisation methods based on the comprehensive framework are demonstrated in 5 different scenarios of the interplay between hardware and software subsystems. Then, the user's behaviours on warranty claiming such as FBNR and NFBR are also introduced to make the model more complete. This is the first research in the literature of warranty management which integrally considers the interactions among hardware subsystems, software subsystems and users of a product in warranty modelling, warranty policy optimisation and even reliability areas. This framework may contribute to the literature significantly.

The above three research chapters provide collective and comprehensive warranty policy optimisation frameworks, which have improved the research. However, some improvements can also be made on the models. For example, the sales volume can be modelled in linear or non-linear and static or dynamic forms; the warranty policy can be renewing or non-renewing; and copula selection is still a critical task in both theoretical and application perspectives. Then, these issues are discussed in Chapter 6.

All in all, this dissertation rethinks warranty policy optimisation from the operational perspective of a manufacturer with the consideration of risk and portfolio management. Its novelty lies in

- it is the first attempt to collectively optimise warranty policies of products produced by one manufacturer with a consideration of the risk of total profit,
- it is the first attempt to comprehensively optimise the warranty policy of a hardware product with embedded software subsystem with a consideration of the interplay between hardware and software subsystems and the effects of its user's behaviours.

As such, this research has significantly contributed to the literature in the warranty modelling.

7.2 Future work

It is noted that Chapter 5 considers the interplay of the subsystems including hardware, software and end-users, of individual products. Apparently, one may consider using the

mean-variance method, the VaR and the CVaR methods to optimise warranty policy for a portfolio of products considering the interplay of the subsystems of those products, which will be considered in the future.

Furthermoew, based on the new established collective and comprehensive frameworks, the future research can be undertaken in many aspects, such as considering random effects in warranty modelling, optimising two-dimensional warranty policies, optimising maintenance policy based on the collective and comprehensive framework, optimising group warranty policy (i.e. a warranty policy covering a group of items). Those future research focuses are briefed in the following.

7.2.1 Random effects

In the literature, there are two existing approaches to dealing with random effects, which are frequentist and Bayesian methods. For example, the former includes linear modelling and the latter includes hierarchical Bayesian modelling methods. Copulas are recently widely applied in the former category.

As discussed in the above review, in the actuarial science random effects has been studied by many authors, see Zhao & Zhou (2012); Shi & Valdez (2014a); Yau et al. (2003); Boucher et al. (2006) and Frangos & Karlis (2004), for example. Many researchers model insurance problems with a consideration of random effects with copulas (Yau et al., 2003; Frangos & Karlis, 2004; Boucher et al., 2006; Zhao & Zhou, 2012; Shi & Valdez, 2014a). For example, the dependence of the claims frequency of heterogeneous units in successive periods is depicted by the dependence between the random effect variables and modelled by copula (Zhao & Zhou, 2012).

In warranty modelling, random effects exist as well. For example, Akbarov & Wu (2012) argues that random effects exist in warranty data due to the fact that products are manufactured by different production lines and warehoused in different regions with different environments. other researchers have also studied random effects in warranty modelling area. For example, Fredette & Lawless (2007) use flexible non-homogeneous Poisson processes as forecasting methods for warranty claims, where the possible hetero-

geneity among the products is modelled through random effects; Lawless et al. (2009) introduce a mixed Poisson process to model repeated events with both age and usage scales, and the heterogeneity in both usage and event rates across product units are accommodated with independently and identically distributed random effects.

However, in the literature of warranty management, the random effects in warranty modelling are not considered with dependence, and a few works have been done in warranty policy optimisation with a consideration of random effects. Hence there is a need to consider the random effects within unobserved heterogeneity in warranty modelling and warranty policy optimisation to avoid information loss. For example, one of the future works is to develop copula-based warranty policy optimisation framework when random effects is considered.

Such future work can be conducted with two reasons, avoidance of information loss and availability of modelling tools. Avoidance of information loss means that, in the literature of warranty data analysis, the unobserved heterogeneity implied in warranty data collected from the same product or the same user is dependent; hence there is a need to consider the dependence within unobserved heterogeneity in warranty policy optimisation to avoid information loss. This work can be carried out referring to some research in actuarial science, in which the dependence of the claims frequency of heterogeneous units in successive periods is represented as the dependence between the random effect variables and modelled through copula (Zhao & Zhou, 2012). Availability of modelling tools means that, in the literature related actuarial science and warranty management, the random effects models are well studied and applied widely, such as the Cox Poisson process; but, it is hard to model the variables dependently via the dependence within random effects through conventional methods; however, the advantages of copula can overcome it. Copula makes it possible to incorporate both continuous and discrete random variables in a joint probability distribution. It is also well known that it is easy to develop models with copula as it separates the modelling procedures into two steps: (1) modelling marginal distributions, and (2) integrating the marginal distributions with copula. Such advantages of copula make it possible that one can collectively model and optimise warranty policies

with a consideration of random effects.

7.2.2 Two-dimensional and n-dimensional warranty policies

The three research chapters in this thesis only investigate the optimisation of warranty policy for a portfolio of products covered by one-dimensional warranty policies, and the dimension is the age of a product. In the literature, one-dimensional warranty policy is the mainstream; that is, the warranty only covers one dimension, which can be either the usage or the age dimension, but not both. For some products (see Ye & Murthy (2016), for example), however, a warranty policy may cover both age and usage (e.g., the warranty of a car may cover both age and mileage), which is called a two-dimensional warranty policy.

To optimise two-dimensional warranty policies, the relationship between the two dimensions should be modelled. Wu (2014a) proposes a new method of constructing asymmetric copulas and applies such method to model two-dimensional reliability data with a consideration of the dependence between the two dimensions, age and usage. Wang & Su (2016) propose a new two-dimensional preventive maintenance strategy and identify the optimal strategy to minimise the total expected warranty servicing cost from the manufacturer's perspective, in which the usage is treated as a random function of age. There are also many other studies carried out on the two-dimensional scenarios, the models are transferable for optimising warranty policies in the similar scenarios.

However, the policies discussed above are conventional, no matter if they are one-dimensional or two-dimensional. Currently, the academia and industrial sector start to enter the so called Big Data era with technology development. In this era, n-dimensional warranty policies ($n > 2$) become realistic with support from new techniques and huge field data.

For example, in recent years, the area of Wireless Sensor Networks (WSNs) attracts more and more researchers' interests. WSNs are supported by the advances in wireless technologies, such as the increasing miniaturisation of RF devices and microelectromechanical systems (MEMS). WSNs can help collect information about parameters of

the physical world. A WSN consists of a group of sensors integrated into a wireless network. Recent advances in wireless sensor technology allow to distribute the low-cost, low-power, multi-functional, small size sensor nodes diversely in demand for data collection in different applications scenarios. Such property has attracted a wide interest from academia and industry.

For instance, a WSN can be embedded into a car to collect and process data of operating variables, such as speed, temperature, location, status of key components and driver's behaviours, such as times and force of braking. Then a dynamic picture of the actual operating state of the car can be draw, thus, in principle, allowing for designing warranty policy efficient with a consideration of other dimensions or variables of car in operating except the age and mileage (usage). Furthermore, Bumblauskas et al. (2017) introduces a smart maintenance decision support systems based on corporate big data analytics motivated by the rapidly developments on big data analytics and predictive maintenance decision making. In which the warranty cost minimisation is considered. This research provides another aspect in which warranty policy optimisation can be improved in the Big Data era.

7.2.3 Maintenance policy optimisation

Maintenance policy optimisation is one of the most popular research areas in the reliability community. Optimisation of preventive and predictive maintenance policies is useful for real-world engineers and is also desired by them.

In the literature, there is a large bulk of maintenance policies modelling that have been developed, see Jardine et al. (2006); Wu & Zuo (2010); Si et al. (2011); Shafiee & Chukova (2013); Wang & Su (2016) and Wu et al. (2017). In the meantime, many authors have considered optimisation of maintenance policy for systems under warranty, see Bouguerra et al. (2012); Kurata & Nam (2013); Wang et al. (2015); Su & Wang (2016). Although Wu et al. (2017) has considered using the mean-variance approach to optimising maintenance policies for a set of systems collectively. However, they did not consider the scenario of products produced by one manufacturer under warranty or

measuring the risk using VaR and CVaR approaches in maintenance policy optimisation. Inspired by those developments conducted in this thesis, my future research may focus on developing or optimising warranty policies with considering various maintenance policies for a portfolio of different products produced by one manufacturer and also to develop maintenance policies under risk-related constraints established based on VaR and CVaR measures.

7.2.4 Optimisation of policies of cumulative warranty

According to Blischke & Murthy (1992), which has been reviewed in Chapter 2, warranty can be on individual items and groups of items. The warranty on a group of items is called as cumulative warranty. According to Yun et al. (2002), a cumulative free replacement warranty covers a group of n items for a total service time, i.e. warranty length of nT, rather than covering only one item for a period T. If any item in the group failed before $t = nT$, a free replacement item is supplied. This type of warranty policy is normally provided with industrial and commercial equipment which is bought in batch as spares and used one by one, for example, the mechanical or electronic modules used in military or commercial airline equipment.

However, in the literature related to warranty and maintenance management, optimisation of warranty and maintenance policies has been chiefly focusing on individual systems/products. Hence, in my future research, the approaches of the mean-variance, VaR and CVaR to optimising warranty policies and maintenance policies in the cumulative warranty scenario will be investigated. Furthermore, according to the properties of cumulative warranty, in the consuming electronics industry, a new warranty implementing method may be designed like that, the manufacturer provides cumulative warranty policies to its retailers which may provide individual warranty policies to the consumers.

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Appendix A

Appendix

A.1 Basic modelling techniques

A.1.1 Probability Distributions

Generally, the insurance claims modelling serves for estimating the loss reserve or claims reserve which is the provision for the future payment of incurred claims and denoted as obligations of an insurance company. As Tse (2009) states, the model of insurance claim losses, future payment of incurred claims, includes two components, claim frequency and claim severity. Claim frequency indicates number of claims in a block of insurance policies over a period of time; and claim severity is the cost of a claim, i.e. the claim size. These two components are normally modelled separately and then are combined to derive the aggregate loss.

In claims modelling, there are four discrete distributions, binomial, geometric, negative binomial, and Poisson distributions, applied to model claim frequency; and four continuous distributions, exponential, Gamma, Weibull, and Pareto distributions, applied to model claim size.

Claim Frequency

Naturally claim frequency is modelled as a non-negative discrete random variable. Basically, claim frequency can be modelled by four discrete distributions,

- Binomial distribution
- Geometric distribution

- Negative binomial distribution
- Poisson distribution

, which only take non-negative integer values. Furthermore, some new distributions can also be created following compound distributions and mixture distributions. For example, the compound Poisson distribution is a compound distribution in which the primary distribution is Poisson and the secondary distribution is arbitrary.

Binomial distribution is the discrete probability distribution of the number of successes in a sequence of n independent and identically distributed (i.i.d.) Bernoulli experiments, of which the success probability is p. n and p also are the parameters of the binomial distribution, $p \in [0, 1]$ and n is a positive integer. A random variable X follows the binomial distribution with parameters n and p, denoted as $X \sim B(n, p)$. The probability mass function (pmf) of X is

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \cdots, n \tag{A.1}$$

where $\binom{n}{x} = \frac{n!}{x!(n-x)!}$. The mean and variance of X are $E(X) = np$ and $Var(X) = np(1-p)$.

Geometric distribution is the probability distribution of the number of i.i.d. Bernoulli experiments occurred before getting one success in a experiments sequence. This distribution just has one parameter, p, which is the success probability of individual experiment, and $p \in [0, 1]$. A random variable X follows the binomial distribution with parameter p, denoted as $X \sim GM(p)$. The pmf of X is given by

$$f_X(x) = p(1-p)^x, \text{ for } x = 0, 1, \cdots \tag{A.2}$$

The mean and variance of X are $E(X) = \frac{1-p}{p}$ and $Var(X) = \frac{1-p}{p^2}$.

Negative binomial distribution is a discrete probability distribution of the number of fails in a sequence of i.i.d. Bernoulli experiments before a specified (non-random) number

of successes occurs. The parameters are p and r. A random variable X follows this distribution denoted by $X \sim NB(r, p)$, where r is a positive integer and indicates the number of success, and $p \in [0, 1]$ is the probability of success of individual experiment. The pmf of X is

$$f_X(x) = \binom{x+r-1}{r-1} p^r (1-p)^x, \text{ for } x = 0, 1, \cdots . \tag{A.3}$$

Its mean and variance are $E(X) = \frac{r(1-p)}{p}$ and $Var(X) = \frac{r(1-p)}{p^2}$. Comparing equation (2.2) and equation (2.3), it is easy to find that the geometric distribution is a special case of the negative binomial distribution with $r = 1$. The random variable which follows negative binomial distribution is just the sum of r independently variates which have geometric distribution.

Poisson distribution is one of the most widely used discrete distributions in empirical applications. It is commonly applied to model the number of arrivals of certain events within a period of time (such as the number of insurance claims in a year), the number of defective items in production, and as an approximation of the binomial distribution, among others. It is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time if these events occur with a known average rate and independently of the time since the last even. A non-negative discrete random variable X has a Poisson distribution with parameter λ,which is the known average occurrence rate, denoted by $P(\lambda)$. The pmf of it is

$$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \text{ for } x = 0, 1, \cdots , \tag{A.4}$$

where $\lambda > 0$ and is also equal to the expected value and variance of X, $E(X) = Var(X) = \lambda$.

New distributions can be created through two method, compound distributions and mixture distributions, based on these four basic discrete distribution, some new distributions.

Let $X_1, \cdots, X_n$ be i.i.d non-negative integer random variables. Denote the sum of their random variables by S,

$$S_n = \sum_{i=1}^{n} X_i. \tag{A.5}$$

It N is also a non-negative integer random variable and distributed independently of $X_1, \cdots, X_n$, then S is said to have a compound distribution. The distribution of N_t is called the primary distribution, and the distribution of X is called the secondary distribution. If N follows Poisson distribution and X follows geometric distribution, S has a Poissongeometric distribution. A compound Poisson distribution is a compound distribution where the primary distribution is Poisson, for any secondary distribution.

The mixing distributions is another method to create new distributions. Let $X_1, \cdots, X_N$ be random variables with corresponding pmf or probability density function,pdf, $f_{X_1}(x), \cdots, f_{X_n}(x)$. A new random variable X can be created with pmf or pdf $f_X(x)$ given by

$$f_X(x) = \sum_{i=1}^{n} p_i f_{X_i}(x), \tag{A.6}$$

where $p_i \geq 0$ for $i = 1, \cdots, n$, and $\sum_{i=1}^{n} p_i = 1$. Thus, p_i form a well-defined probability distribution. A new random variable Y by $P_r(Y = i) = f_Y(i) = p_i$. Hence, X may be regarded as a random variable which is equal to X_i with probability p_i, and Y may be interpreted as a random variable which takes value i if and only if $X = X_i$.

Claim Amount

Different from the claim frequency, the claim size is normally treated as a non-negative continuous random variable. Regarding to modelling claim size, there are some standard continuous distributions include

- Exponential distribution
- Gamma distribution
- Weibull distribution
- Pareto distribution.

Some new distributions can also be created through transformation, mixture distribution, and splicing methods. Additionally, the extreme values of claim size which always occur in the upper tail of the distribution also should be considered (Tse, 2009).

Exponential distribution is used to describe the inter-arrival time of an event, such as the breakdown of a machine. It is related to the Poisson distribution, such as if the inter-arrival time of an event is distributed as an exponential random variable with parameter λ, the number of occurrences of the event in a unit time interval is distributed as a Poisson with parameter λ. It is also the continuous analogue of the geometric distribution.

A random variable X follows an exponential distribution with parameter λ, denoted as $X \sim Exp(\lambda)$, its pdf is

$$f_X(x) = \lambda e^{-\lambda x}, \text{ for } x \geq 0, \tag{A.7}$$

where $\lambda > 0$. The cumulative distribution function (cdf) of X is

$$F_X(x) = 1 - e^{-\lambda x}. \tag{A.8}$$

The mean and variance of X are $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$

Gamma distribution is a two-parameter family of continuous probability distributions, and the two parameters are positive real numbers. In this realm of insurance and warranty claims modelling, we denote that a random variable X has a gamma distribution as $X \sim G(\alpha, \beta)$, and the pdf is

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, x \geq 0, \tag{A.9}$$

where, the function $\Gamma(\alpha)$ is the gamma function, when $\alpha > 1$, the gamma function satisfies the recursion that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$; If α is a positive integer, we have $\Gamma(\alpha) = (\alpha - 1)!$. The mean and variance of X are $E(X) = \alpha\beta,$ and $Var(X) = \alpha\beta^2$.

The common exponential distribution is a special case of the gamma distribution, because, from equation (2.15) we can find that the pdf of $\Gamma(1, \beta)$ is the same as that of $Exp(\frac{1}{\beta})$. Hence, the sum of i.i.d. exponential distributions follows a gamma distribution with a positive integer-valued α.

Weibull distribution is a continuous probability distribution, and widely used in reliability engineering and elsewhere, as it gives the distribution of lifetimes of objects. Weibull distribution also has two parameters, a random variable X has a Weibull distribution can be denoted as $X \sim W(\alpha, \lambda)$. The pdf is

$$f_X(x) = (\frac{\alpha}{\lambda})(\frac{x}{\lambda})^{\alpha-1} e^{-(\frac{x}{\lambda})^{\alpha}}, \text{ for } x \geq 0, \tag{A.10}$$

where, α and λ are positive, and are shape parameter and scale parameter respectively. The mean and variance of X are $E(X) = \mu = \lambda\Gamma(1+\frac{1}{\alpha}$ and $Var(X) = \lambda^2\Gamma(1+\frac{2}{\alpha}) - \mu^2$

Pareto distribution is a power law probability distribution and initially applied economics. A random variable X has a Pareto distribution can be denoted as $X \sim PT(\alpha, \gamma)$. The pdf is

$$f_X(x) = \frac{\alpha\gamma^{\alpha}}{(x+\gamma)^{\alpha+1}}, \text{ for } x \geq 0, \tag{A.11}$$

where, the parameters α and γ are positive. The mean and variance of X are $E(X) = \frac{\gamma}{\alpha-1}$ and $Var(X) = \frac{\alpha\gamma^2}{(\alpha-1)^2(\alpha-2)}$ for $\alpha > 2$.

Based on the claim frequency and size models, the aggregate loss, which is the sum of all losses incurred in a block of insurance policies, can be modelled through the individual risk and collective risk approaches. In the individual risk model, the aggregate loss is treated as the sum of the losses of all individual policies which are independently and identically distributed. However, in the collective risk model, the aggregate loss follows a compound distribution, in which the claim frequency represents the primary distribution and the claim size represents the secondary distribution.

A.1.2 Stochastic Processes

The process of claim arrival can be treated as a stochastic point process. The process can be simulated through the arrival times $\{T_i\}$, or the inter-arrival times $W_i = T_i - T_{i-1}$. The widely used stochastic process modelling techniques include

- Poisson process
- Non-homogeneous Poisson process

- Cox porcess
- Compound Poisson process
- Renewal process.

Poisson process is the most common and best known claim arrival point process. In this review, 'Poisson process' denotes the classical homogeneous Poisson process, which is different from the non-homogeneous Poisson process would be discussed later.

A continuous-time stochastic process$\{N_t : t \geq 0\}$ is a Poinsson process with intensity $\lambda > 0$, means this process starts at 0, $N(0) = 0$; the expected number of claims occur per unit time is λ; the number of claims in the time interval $(T_i, T_i + \tau)$, $(N(T_i + \tau) - N(T_i))$, the increments, are independent, stationary, and follow Poisson distribution with the parameter $\lambda\tau$, which only depends on the length of the interval; and inter-arrival times W_i are i.i.d. and follows an exponential distribution with parameter λ.

Non-homogeneous Poisson process is very closely related to the Poisson process, and can be treated as as a Poisson process with a variable intensity defined by the deterministic intensity (rate) function $\lambda(t)$. It means the increments do not have to be stationary, i.e. the intensity of the distribution of the number of claims in the time interval $(T_i, T_i + \tau)$ is dertermined by a function and depended on the time, $\lambda(t)$, instead of only depended on the length of the interval, $\lambda\tau$. The homogeneous Poisson process implies that the size of the insurance policies portfolio cannot increase or decrease is practice. For instance, it cannot describe situations, like in motor insurance, where claim occurrence epochs are likely to depend on the time of the year or of the week. For this situation the non-homogeneous Poisson process suits much better than the homogeneous one (Cizek et al., 2005).

Cox process is also known as a doubly stochastic Poisson process, it is also based on the Poisson process with some changes of the property of the intensity. This process provides more flexibility than the non-homogeneous Poisson process by allowing the intensity to be a stochastic process, instead of only depended on time. This process can be treated

as a two-step randomisation procedure: an intensity process $\{\Lambda(t)\}$ is used to generate another process $\{N_t\}$ by acting as its intensity, i.e. $\{N_t\}$ is a Poisson process conditional on $\{\Lambda(t)\}$. If $\{\Lambda(t)\}$ is deterministic, then $\{N_t\}$ is a non-homogeneous Poisson process. If $\Lambda(t) = \Lambda$ for some positive random variable Λ, then $\{N_t\}$ is a mixed Poisson process (Cizek et al., 2005).

Compound Poisson process is a continuous-time stochastic process with jumps. The arrival process of the jumps is a Poisson process with intensity λ, and the size of the jumps is random and follows a specified probability distribution G. This process can be denoted by

$$CP(t) = \sum_{i=1}^{N(t)} D_i \tag{A.12}$$

where, $\{N(t) : t \geq 0\}$ is a Poisson process with intensity λ, and D_i are i.i.d. random variables with distribution G.

Renewal process can be viewed as a generalised Poisson process. Regarding to the non-homogeneous Poisson process and Cox process, which are can also be viewed as generalised Poisson process, where the intensity λ is make non-constant. In the Renewal Process, the distribution of the inter-arrival times, waiting times, W_i, is made a variety of different distributions, instead of the exponential distribution in the Poisson process; even though the random variables W_i are still i.i.d. and positive. Additionally, the homogeneous Poisson process is also a special cause of renewal process.

A.1.3 Copulas

A copula is a multivariate distribution with simple marginal distribution for all the random variables (frequently uniform on the interval [0, 1]). As Sklars Theorem states that for any multivariate joint distribution, there exists a copula function such that the joint distribution can be represented as a function of its marginals through this copula (Sklar, 1959).In order to represent dependence, some researchers construct multivariate distributions through copula, because copula is often easier to model dependency between tractable random

variables and then to transform to the distributions of interest, rather than to start with an arbitrary multivariate distribution (Czado et al., 2012). According to recent research, the popular copulas applied in claims modelling are

- Elliptical copulas:

  Gaussian (Normal) copula, and

  t-copula

- Archimedean copulas:

  Clayton copula,

  Gumbel copula, and

  Frank copula.

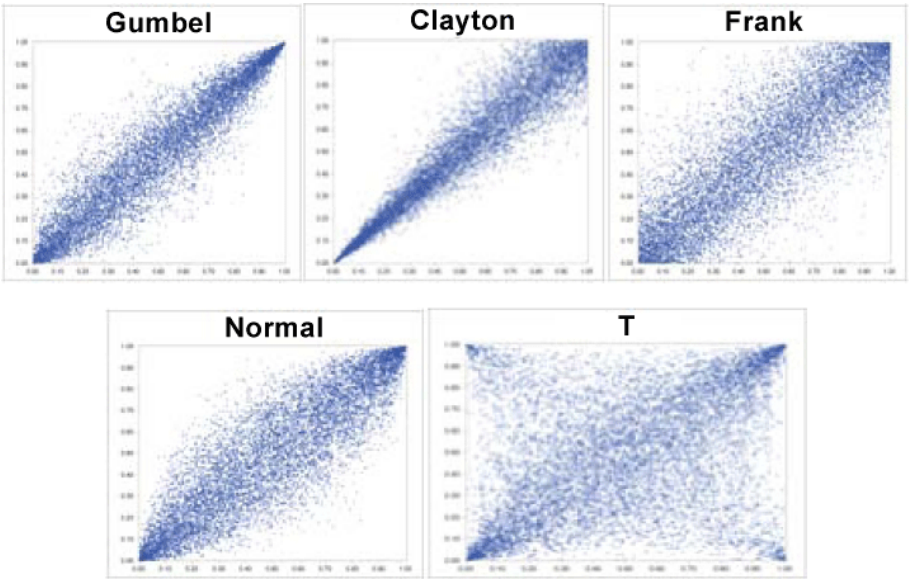

Figure A.1: Copulas (Naimy, 2012)

As Figure 6 shown, elliptical and Archimedean copulas, exhibit a certain degree of symmetry or are restricted to certain correlation structures (Diers et al., 2012). Regrading to the elliptical copulas and Frank copula, they exhibit a symmetric dependence, while the Gumbel and Clayton copula present the tail dependence which is asymmetric.

Elliptical copulas are parametric and include Gaussian copula and t copula. Gaussian and t-copulae contain the dependency structure from the multivariate Normal and multi-

variate t-distribution respectively, and they are directly derived through the Sklars theorem (Pešta & Okhrin, 2014).Gaussian copula and t copula all induce symmetric dependence structures (Diers et al., 2012).

Czado et al. (2012) employ Gaussian copula and provide a typical procedure of constructing a bivariate regression model for number of claims and claim sizes. Firstly, the Poisson regression and Gamma regression are used to represent the number of claims and corresponding average claim size respectively.

Archimedean copulas are another associative class of copulas, different from the Gaussian copulas, most common Archimedean copulas admit an explicit formula. As Archimedean copulas allow modeling dependence in arbitrarily high dimensions with only one parameter; hence they are popular in practice.

A.1.4 Credibility Theory and Prediction

The claim frequency, claim size and aggregate loss all can be denoted as the random loss variable X, and the random insurance loss in next period can be predicted. In order to update the prediction of the insurance loss based recent data of claims, the credibility theory is introduced. In the classical credibility approach, i.e. the limited-fluctuation credibility approach, the updated prediction of the loss is a weighted average of observed loss and predicted loss, which is based on prior information. As an improvement, Buhlmann's credibility updates the predicted loss based on a linear predictor using past observations through the least mean squared error approach with the assumption that the past observations are identically distributed. In order to extend and generalize the Buhlmann theory, this assumption is relaxed in Buhlmann-Straub credibility model, even though the observations are still assumed to be independent (Tse, 2009). In Buhlmann's credibility model and its generalisation, the random loss variable X, is distributed based on a parameter θ, which varies with different risk groups, and is assumed to be unknown and fixed. However, in Bayesian approach, the random variable Θ has a statistical distribution, and the parameter θ, which is the realisation of Θ can be estimated through likelihood function. Under Bayesian approach, the prediction of future random losses can be formulated, and

the Bayes prediction is the posterior mean of the future loss and has the minimum mean squared error.