RESEARCH ARTICLE

Discrete time Output Feedback Sliding Mode Control design for Uncertain Systems using Linear Matrix Inequalities

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An output feedback based sliding mode control design methodology for discrete time systems is considered in this paper. In previous work, it has been shown that by identifying a minimal set of current and past outputs, an augmented system can be obtained which permits the design of a sliding surface based upon output information only, if the invariant zeros of this augmented system are stable. In this work, a procedure for realizing discrete time controllers via a particular set of extended outputs is presented for non square systems with uncertainties. This method is applicable when unstable invariant zeros are present in the original system. The conditions for existence of a sliding manifold guaranteeing a stable sliding motion are given. A procedure to obtain a Lyapunov matrix, which simultaneously satisfies both a Riccati inequality and a structural constraint, is used to formulate the corresponding control to solve the reachability problem. A numerical method using linear matrix inequalities is suggested to obtain the Lyapunov matrix. Finally, the design approach given in the paper is applied to an aircraft problem and the use of the method as a reconfigurable control strategy in the presence of sensor failure is demonstrated.

Keywords: sliding modes; output feedback; discrete time implementation; reconfigurable control

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1 Introduction

Early contributions in the area of sliding mode control were developed in a framework in which all the system states are assumed available. This approach may not be realistic for practical engineering problems and has motivated the need to develop output feedback controllers. A number of algorithms have been developed for robust stabilization of uncertain systems which are based on sliding surfaces and output feedback control schemes (see, for example, Zak and Hui 1993, Edwards and Spurgeon 1995). A geometric condition has been developed to guarantee the existence of the sliding surface and the stability of the reduced order sliding motion (Zak and Hui 1993). An algorithm which is convenient for practical use has been derived in Edwards and Spurgeon (1995, 1998). Based on the work given in Zak and Hui (1993), some dynamic feedback sliding mode controllers have been proposed (Kwan

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2001, Shyu et al. 2001). In all the above output feedback sliding mode control schemes, it is an *a priori* requirement that the system under consideration is minimum phase and relative degree one and that a particular sub-system must be output feedback stabilisable (Edwards and Spurgeon 1998).

Compared with continuous time sliding mode control strategies, the design problem in discrete time is much less mature. Other than the early work in Sira-Ramirez (1991), much of the literature assumes all states are available (Furuta and Pan 2000, Hui and Zak 1999, Tang and Misawa 2002). Discrete sliding mode control schemes which have restricted themselves to output measurements alone have often been observer based schemes, with or without disturbance estimation (Lee and Lee 1999, Tang and Misawa 2000). Recent exceptions have been the work in Monsees (2002) which considers both static and dynamic output feedback problems, and the discrete time versions of certain higher-order sliding-mode control schemes (Bartolini et al. 2001, 2000).

For continuous time systems, it was shown in Floquet et al. (2007) that the relative degree condition associated with the solution of the existence problem can be weakened if a classical sliding mode observer is combined with sliding mode exact differentiators to generate additional independent output signals from the available measurements. In the discrete case, it was shown that by using the output signal at the current time instant together with a limited amount of information from previous sample instants, the class of discrete time systems for which an output feedback based sliding mode controller can be developed is significantly broadened (Govindaswamy et al. 2008, 2009). It has been shown that with this approach, discrete time output feedback based sliding mode control for systems with unstable invariant zeros is possible. It has also been shown that, by extending the outputs, the relative degree condition associated with the solution of the existence problem can be relaxed for output feedback based sliding mode control. This notion is further developed here, by designing discrete time sliding mode controllers for uncertain systems using the extended outputs.

The conditions for realizing the discrete state feedback min-max control law via the output have been treated in (Magana and Zak 1988, Sharav-Schapiro et al. 1998). In particular, it has been stated that a Riccati min-max control law can be realized via the outputs if the discrete algebraic Riccati equation satisfies a particular constraint (Sharav-Schapiro et al. 1998). It has been shown that the realization of the output feedback min-max control law is equivalent to the discrete positive realness of an auxiliary system generated from the original plant triple (Sharav-Schapiro et al. 1998). In Sharav-Schapiro et al. (1998) a simple existence condition for the output feedback min-max control has been given for square systems. For non square systems, a procedure for synthesizing a min-max controller using the measured outputs has been given in Lai et al. (2004). An explicit procedure to obtain a Lyapunov matrix that satisfies the discrete Riccati equation and simultaneously satisfies the constraint has also been given.

In this paper, by solving an algebraic Riccati equation that satisfies a particular constraint, it is shown that discrete time sliding mode control using output feedback can be performed for non square uncertain systems. Using the extended outputs, it is first shown that the sliding dynamics is a function of the disturbance and that the ideal sliding mode as defined for a nominal system is not possible. In this case, the reduced order dynamics is shown to be bounded about a region around the sliding surface. An additional criteria for choosing the extended outputs is given which will minimize the effect of the disturbance on the sliding surface. The required control law can be realized via the extended outputs provided a Lyapunov matrix exists that solves a constrained Riccati equation. The conditions for the existence of the control law are given.

The paper is structured as follows. Section 2 gives the problem motivation. The existence problem is given in Section 3 and the closed loop stability analysis is dealt with in Section 4. The reachability problem is dealt with in Section 5. A motivational problem using an aircraft system is presented to illustrate the methodology. The motivational problem will consider sensor failures. An augmented output will be created by extending the available outputs, such that the resulting plant triple has stable or no invariant zeros. In addition to the above criteria, the extended output will be chosen such that the effect of the disturbance on the sliding surface will be minimized. A reconfigurable control design will then be performed in the presence of sensor failure.

2   Motivation

Consider the uncertain discrete, linear, time invariant state space system representation with uncertainties as given below:

$$x_{k+1} = Ax_k + Bu_k + B_d d_k \tag{1}$$

$$y_k = [(y_1)_k...(y_p)_k]^T = Cx_k, (y_i)_k = C_i x_k \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the output vector, $u_k \in \mathbb{R}^m$ is the control input and $d_k \in \mathbb{R}^q$ is the disturbance vector. It is assumed that $m \leq p$, the pair (A,B) is controllable and without loss of generality, that rank$(C) = p$ and that column rank$(B) = m$. Assume that the uncertainties are lumped in d_k and are such that for some real $\rho_0 > 0$ and $\rho_1 > 0$, $\|d_k\| < \rho_0 \|x_k\| + \rho_1$. It is also assumed that the usual matching condition $rank(B|B_d) = rank(B)$ is not satisfied. The nominal part of the plant (1)-(2) is defined as the representation when the disturbance $d_k = 0$. If the sliding surface s is defined as:

$$s = \{x \in \mathbb{R}^n : FCx_k = 0\} \tag{3}$$

for some selected matrix $F \in \mathbb{R}^{m \times p}$ then it is well known that for a unique equivalent control to exist, the matrix $FCB \in \mathbb{R}^{m \times m}$ must have full rank. As

$$\text{rank}(FCB) \leq min\{\text{rank}(F), \text{rank}(CB)\} \tag{4}$$

it follows that both F and CB must have full rank. As F is a design parameter, it can be chosen to be full rank. A necessary condition for FCB to be full rank, and thus for solvability of the output feedback sliding mode design problem, becomes that CB must have rank m. If this rank condition holds and any invariant zeros of the triple (A,B,C) lie in the unit disk, then the existence of a matrix F defining the surface (3), which provides a stable sliding motion with a unique equivalent control is determined from the stabilisability by output feedback of a specific, well-defined subsystem of the plant (Edwards and Spurgeon 1998). The aim here is to extend the existing results such that a sliding mode control based on output measurements can be designed for the system (1)-(2) when $rank(CB) < m$ and/or unstable invariant zeros are present in the triple (A,B,C). It will be assumed that the matrix A is invertible. Such a property often occurs in discrete-time linear system representations and thus the assumption is not limiting. For instance, consider the continuous, linear, time invariant system

$$\dot{z}(t) = Nz(t) + Gu(t) \tag{5}$$

If the nominal part of the system (1)-(2) is the discretized form of the continuous time system (5) under sampling i.e,

$$A = e^{Nt}, B = \int_0^T e^{Nt} d\tau G \tag{6}$$

then it is shown by Rugh (1996) (page 386) that the state space matrix A is invertible.

An extended output matrix $\tilde{C}$ is then constructed without any *a priori* assumptions on the system

(1)-(2) relating to the stability of the invariant zeros

$$
\tilde{C} = \begin{bmatrix} C_1 \\ \vdots \\ C_p \\ \vdots \\ C_1 A^{-\mu_1 + 1} \\ \vdots \\ C_p A^{-\mu_p + 1} \end{bmatrix} \tag{7}
$$

where this $\tilde{C}$ is full rank, $\text{rank}(\tilde{C}B) = \text{rank}(B)$, and any invariant zeros of the triple $(A, B, \tilde{C})$ lie inside the unit disk. Also, the μ_i are chosen such that $\tilde{p} = \sum_{i=1}^{p} \mu_i$ is minimal. Note that $\tilde{C} = C$ means that the original system is output feedback stabilisable using existing methods (see the work of Edwards and Spurgeon (1998)).

It is now important to show the relationship between the invariant zeros of the original system triple (A, B, C) and the invariant zeros of the augmented system triple $(A, B, \tilde{C})$. For the triple (A, B, C) in (1)-(2) define z as an invariant zero of (1)-(2) iff there exist vectors $0 \neq x_z \in \mathbb{C}^n$ and $u_z \in \mathbb{C}^m$, such that the triple z, x_z, u_z satisfies

$$
\begin{bmatrix} zI - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_z \\ u_z \end{bmatrix} = 0 \tag{8}
$$

where, I is the usual identity matrix of dimension n. Here, x_z is called the state zero direction and u_z is called the input zero direction. It has been shown by Govindaswamy et al. (2008), that any invariant zeros of the triple $(A, B, \tilde{C})$ are amongst the invariant zeros of the triple (A, B, C). It can be shown that if an appropriate choice of extended outputs is available, the invariant zeros from the original triple (A, B, C) may disappear from the augmented system. This is particularly useful if any of the invariant zeros of (A, B, C) are unstable. The choice of augmented outputs to remove transmission zeros is described in the lemma below.

Lemma 2.1: *Let z be the invariant zero of the original triple (A, B, C) with zero direction x_z. Assume the zero direction x_z^T is spanned by the rows of the augmented output matrix $\tilde{C}$. Then the invariant zero z is not present in the augmented triple $(A, B, \tilde{C})$.(Govindaswamy et al. 2009)*

Note that by augmenting the number of outputs to the dimension of the state, the transmission zeros may all be removed if the system is observable. However, the objective of this paper is to augment the outputs only sufficiently so that $\tilde{C}B$ is full rank and any transmission zeros are stable. In general it is not required, and indeed may not be desirable, to augment the outputs to the dimension of the state; the greater the number of states the greater the complexity of the resulting controller. The next section will develop and analyze the sliding surface and the resulting sliding mode dynamics.

3   The Existence Problem

Assume that it is possible to choose $\tilde{C}$ as stated in (7). The problem now becomes one of finding a suitable sliding variable s_k that is a function of the augmented outputs only. Choose a set of past outputs

along with the present outputs, and form a new extended output vector $\tilde{y}_k$ as shown below:

$$\tilde{y}_k = \begin{bmatrix} (y_1)_k \\ \vdots \\ (y_p)_k \\ \vdots \\ (y_1)_{k-\mu_1+1} \\ \vdots \\ (y_p)_{k-\mu_p+1} \end{bmatrix} \tag{9}$$

From the system (1)-(2), it can be computed that:

$$x_k = A^{-1}(x_{k+1} - Bu_k - B_d d_k) \tag{10}$$

$$y_k = Cx_k \tag{11}$$

From the above equations the following relation between $\tilde{y}$, x and u can be derived. One has:

$$y_k = Cx_k$$

$$y_{k-1} = CA^{-1}(x_k - Bu_{k-1} - B_d d_{k-1})$$

$$\vdots$$

$$y_{k-j} = CA^{-j}x_k - CA^{-j}Bu_{k-1} - \cdots - CA^{-1}Bu_{k-j} - CA^{-j}B_d d_{k-1} - \cdots - CA^{-1}B_d d_{k-j}$$

$$= CA^{-j}x_k - \sum_{l=1}^{j} CA^{-l}Bu_{k-j+l-1} - \sum_{l=1}^{j} CA^{-l}B_d d_{k-j+l-1}$$

Thus

$$\tilde{y}_k = \tilde{C}x_k - \begin{bmatrix} 0_{p \times l_1} \\ \mathbf{M}_{(\tilde{p}-p) \times l_1} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_p \end{bmatrix} - \begin{bmatrix} 0_{p \times q_1} \\ \mathbf{M_d}_{(\tilde{p}-p) \times q_1} \end{bmatrix} \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \vdots \\ \bar{d}_p \end{bmatrix}$$

with $l_1 = m(\tilde{p} - p)$ and $q_1 = q(\tilde{p} - p)$

$$U_k = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_p \end{bmatrix} \in \mathbb{R}^{l_1}, \quad D_k = \begin{bmatrix} \bar{d}_1 \\ \bar{d}_2 \\ \vdots \\ \bar{d}_p \end{bmatrix} \in \mathbb{R}^{q_1} \tag{12}$$

and, for $i = 1, ..., p$:

$$\bar{u}_i = \begin{bmatrix} u_{k-1} \\ u_{k-2} \\ \vdots \\ u_{k-\mu_i+1} \end{bmatrix} \text{ and } \bar{d}_i = \begin{bmatrix} d_{k-1} \\ d_{k-2} \\ \vdots \\ d_{k-\mu_i+1} \end{bmatrix} \tag{13}$$

$$\mathbf{M} = diag\{\mathbf{M}_1, ..., \mathbf{M}_p\}$$

$$\mathbf{M}_i = \begin{pmatrix} C_iA^{-1}B & 0 & \cdots & \cdots & 0 \\ C_iA^{-2}B & C_iA^{-1}B & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & C_iA^{-1}B & 0 \\ C_iA^{-\mu_i+1}B & \cdots & \cdots & C_iA^{-2}B & C_iA^{-1}B \end{pmatrix} \in \mathbb{R}^{(\mu_i-1) \times (m(\mu_i-1))}$$

and the matrix $\mathbf{M_d}$ is

$$\mathbf{M_d} = diag\left\{\mathbf{M_{d1}},...,\mathbf{M_{d}}_p\right\}$$

$$\mathbf{M_{d}}_i = \begin{pmatrix} C_iA^{-1}B_d & 0 & \dots & \dots & 0 \\ C_iA^{-2}B_d & C_iA^{-1}B_d & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & C_iA^{-1}B_d & 0 \\ C_iA^{-\mu_i+1}B_d & \dots & \dots & C_iA^{-2}B_d & C_iA^{-1}B_d \end{pmatrix} \in \mathbb{R}^{(\mu_i-1)\times(q(\mu_i-1))}$$

Defining the sliding manifold in terms of known variables gives:

$$s_k = F\tilde{y}_k + F\begin{bmatrix} 0_{p\times l_1} \\ \mathbf{M}_{(\tilde{p}-p)\times l_1} \end{bmatrix}U_k \tag{14}$$

$$= F\tilde{C}x_k - F\begin{bmatrix} 0_{p\times q_1} \\ \mathbf{M_d}_{(\tilde{p}-p)\times q_1} \end{bmatrix}D_k \tag{15}$$

Now to analyze the stability of the sliding motion, it is convenient to introduce a coordinate transformation $x \rightarrow \hat{T}x = \hat{x}$ to the usual regular form, making the final $\tilde{p}$ states of the system directly dependent on the extended outputs (Edwards and Spurgeon 1998). The system (1)-(2) in this case will be given as:

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \hat{B}_d = \begin{bmatrix} \hat{B}_{d_1} \\ \hat{B}_{d_2} \end{bmatrix}, \hat{C} = \begin{bmatrix} 0 & T \end{bmatrix} \tag{16}$$

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

where $T \in \mathbb{R}^{\tilde{p}\times\tilde{p}}$ is an orthogonal matrix, $\hat{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the remaining sub-blocks in the system matrix are partitioned accordingly.

The corresponding switching surface matrix is given by

$$\overset{\tilde{p}-m}{\leftrightarrow}\ \overset{m}{\leftrightarrow}\\ \begin{bmatrix} F_1 & F_2 \end{bmatrix} = FT \tag{17}$$

where T is the matrix from equation (16). As a result

$$F\hat{C} = \begin{bmatrix} F_1C_f & F_2 \end{bmatrix} \tag{18}$$

where

$$C_f = \begin{bmatrix} 0_{(\tilde{p}-m)\times(n-\tilde{p})} & I_{(\tilde{p}-m)} \end{bmatrix} \tag{19}$$

Therefore $F\hat{C}\hat{B} = F_2B_2$ and the square matrix F_2 is nonsingular. Let the disturbance distribution matrix be defined as:

$$\Gamma = F\begin{bmatrix} 0_{p\times q_1} \\ \mathbf{M_d}_{(\tilde{p}-p)\times q_1} \end{bmatrix}$$

In the new coordinates Γ will be of the form

$$\overset{q_1-m}{\leftrightarrow}\ \overset{m}{\leftrightarrow}\\ \begin{bmatrix} \hat{\Gamma}_1 & \hat{\Gamma}_2 \end{bmatrix} = \hat{\Gamma} \tag{20}$$

Consider the nominal plant for the system (1)-(2), when $d_k = 0$, in the new coordinate system. The canonical form in (16) can then be viewed as a special case of the regular form normally used in sliding

mode controller design, and it can be shown that the reduced order sliding motion in this is governed by a free motion with system matrix

$$\hat{A}_{11}^s = \hat{A}_{11} - \hat{A}_{12} K C_f \tag{21}$$

which must therefore be stable and the gain $K \in \mathbb{R}^{m \times (\tilde{p}-m)}$ is defined as $K = F_2^{-1} F_1$. It should be emphasized that (21) describes the ideal sliding mode dynamics which are defined by the nominal system representation. However, for the system (1)-(2) in the presence of uncertainty, the sliding surface (15) is perturbed by the disturbance term and hence it is clear that the ideal sliding motion as given in (21) is not possible. Hence, it is necessary to show that the reduced order motion in the presence of the disturbance will be confined within an acceptable region around the sliding surface $\hat{s}$. To show this, let:

$$\hat{s}_k = 0 \tag{22}$$

so that

$$(\hat{x}_2)_k = F_2^{-1}(-F_1 C_f(\hat{x}_1)_k + \hat{\Gamma}\hat{D}_k) \tag{23}$$

Substituting for $(\hat{x}_2)_k$ in $(\hat{x}_1)_{k+1}$, one can obtain

$$(\hat{x}_1)_{k+1} = (\hat{A}_{11} - \hat{A}_{12} F_2^{-1} F_1 C_f)(\hat{x}_1)_k + \hat{A}_{12} F_2^{-1} \hat{\Gamma}\hat{D}_k + \hat{B}_{d_1}\hat{d}_k$$

$$(\hat{x}_1)_{k+1} = \hat{A}_{11}^s(\hat{x}_1)_k + \hat{A}_{12} F_2^{-1} \hat{\Gamma}\hat{D}_k + \hat{B}_{d_1}\hat{d}_k$$

The solution for $(\hat{x}_1)_{k+1}$ is:

$$(\hat{x}_1)_k = (\hat{A}_{11}^s)^k(\hat{x}_1)_o + \sum_{j=0}^{k-1}(\hat{A}_{11}^s)^{k-j-1}(\hat{A}_{12} F_2^{-1}\hat{\Gamma}\hat{D}_j + \hat{B}_{d_1}\hat{d}_j) \tag{24}$$

and thus

$$\|(\hat{x}_1)_k\| \le \|(\hat{A}_{11}^s)^k\|\|(\hat{x}_1)_o\| + \|\sum_{j=0}^{k-1}(\hat{A}_{11}^s)^{k-j-1}\|(\|\hat{A}_{12}\|\|F_2^{-1}\|\|\hat{\Gamma}\|\|\hat{D}_j\| + \|\hat{B}_{d_1}\|\|\hat{d}_j\|) \tag{25}$$

If $\hat{A}_{11}^s$ is designed by the choice of F_1 such that it has eigenvalues within the unit circle, then it can be shown that

$$\|(\hat{A}_{11}^s)^k\| \le \gamma\lambda^k$$

where $\gamma > 0$ and $0 < \lambda < 1$. It can further be shown that:

$$\sum_{j=0}^{k-j-1}\|(\hat{A}_{11}^s)^j\| \le \gamma \sum_{j=0}^{k-j-1}\lambda^j$$

$$\le \gamma \sum_{j=0}^{\infty}\lambda^j$$

$$\le \frac{\gamma}{1-\lambda}$$

Using the standard definition of a *supremum*, define $\rho_2 = \sup_{k>0}\|\hat{D}_k\|$ and $\rho_3 = \sup_{k>0}\|\hat{d}_k\|$, where ρ_2 and ρ_3 are the smallest constants such that $\|\hat{D}_k\| < \rho_2$ and $\|\hat{d}_k\| < \rho_3 \ \forall k > 0$. Hence bounds on $\|(\hat{x}_1)_k\|$ can be written as:

$$\|(\hat{x}_1)_k\| \le \gamma\lambda^k\|(\hat{x}_1)_o\| + \frac{\gamma}{1-\lambda}(\|\hat{A}_{12}\|\|F_2^{-1}\|\|\hat{\Gamma}_k\|\rho_2 + \|\hat{B}_{d_1}\|\rho_3) \tag{26}$$

and the reduced order sliding motion will then be uniformly ultimately bounded by $\frac{\gamma}{1-\lambda}(\|\hat{A}_{12}\|\|F_2^{-1}\|\|\hat{\Gamma}\|\rho_2 + \|\hat{B}_{d_1}\|\rho_3)$. Note that the band can be further minimised if the extended outputs are chosen so that the ideal sliding mode dynamics are minimally affected by the disturbance as described below.

Consider the matrix $\mathbf{M_d}$ in the new coordinate system. The i_{th} block of $\hat{\mathbf{M}}_\mathbf{d}$ can be written as

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$$\hat{\mathbf{M}}_{\mathbf{d_i}} = \begin{pmatrix} \hat{C}_i\hat{A}^{-1} & 0 & \dots & \dots & 0 \\ \hat{C}_i\hat{A}^{-2} & \hat{C}_i\hat{A}^{-1} & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & \hat{C}_i\hat{A}^{-1} & 0 \\ \hat{C}_i\hat{A}^{-\mu_i+1} & \dots & \dots & \hat{C}_i\hat{A}^{-2} & \hat{C}_i\hat{A}^{-1} \end{pmatrix} \begin{pmatrix} \hat{B}_d \\ \hat{B}_d \\ \vdots \\ \vdots \\ \hat{B}_d \end{pmatrix}$$

The effect of the disturbance $\hat{d}_k$ on the sliding surface $\hat{s}_k$ and hence the sliding mode dynamics can be nullified in certain cases by choosing the extended outputs such that the columns of the matrix $[\hat{B}_d^T\ \hat{B}_d^T\ \dots\ \dots\ \hat{B}_d^T]^T$ span the null space of the lower triangular matrix

$$\begin{pmatrix} \hat{C}_i\hat{A}^{-1} & 0 & \dots & \dots & 0 \\ \hat{C}_i\hat{A}^{-2} & \hat{C}_i\hat{A}^{-1} & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ \vdots & & & \hat{C}_i\hat{A}^{-1} & 0 \\ \hat{C}_i\hat{A}^{-\mu_i+1} & \dots & \dots & \hat{C}_i\hat{A}^{-2} & \hat{C}_i\hat{A}^{-1} \end{pmatrix}$$

The above condition is restrictive and it is possible that not all disturbance channels span the null space of the above lower triangular matrix. In this case it is necessary to choose the extended outputs so that the direct effect of the disturbance on the extended output, and hence the sliding mode dynamics, is minimized. In this work, the main criteria for choosing the extended outputs are that

   (i) The resulting triple $(\hat{A}, \hat{B}, \hat{C})$ has stable or no invariant zeros
  (ii) The effect of the disturbance on the sliding surface $\hat{s}_k$ is minimized

If the extended outputs are chosen with respect to the above criteria, the sliding surface $\hat{s}_k$ will be

$$\hat{s}_k = F\hat{\hat{y}}_k + F \begin{bmatrix} 0_{p \times l_1} \\ \hat{\mathbf{M}}_{(\tilde{p}-p) \times l_1} \end{bmatrix} \hat{U}_k \tag{27}$$

$$= F\hat{C}\hat{x}_k \tag{28}$$

For the above sliding surface, from (26), the reduced order sliding motion can be computed to be uniformly ultimately bounded as $\frac{\gamma}{1-\lambda}(\|\hat{B}_{d_1}\|\rho_3)$. It is important to note that, given the class of uncertainty considered this represents the behavior of the system in the ideal sliding mode i.e when $\hat{s}_k$ is identically zero.

Note that, for the existence of a stable bounded sliding motion, $\hat{A}_{11}^s = \hat{A}_{11} - \hat{A}_{12}KC_f$ must be stable, which implies that the pair $(\hat{A}_{11}, \hat{A}_{12})$ is controllable and $(\hat{A}_{11}, C_f)$ is observable. The former is ensured as $(\hat{A}, \hat{B})$ is controllable. The observability of $(\hat{A}_{11}, C_f)$, is not so straightforward, but can be investigated by considering the canonical form below.

Lemma 3.1: *Let $(\hat{A}, \hat{B}, \hat{C})$ be a linear system with $\tilde{p} > m$ and rank $(\hat{C}\hat{B}) = m$. Then a change of coordinates $\hat{x} \to \acute{T}\hat{x} = \acute{x}$ exists so that the system triple with respect to the new coordinates has the following structure:*

- *The system matrix can be written as*

$$\acute{A} = \begin{bmatrix} \acute{A}_{11} & \acute{A}_{12} \\ \acute{A}_{21} & \acute{A}_{22} \end{bmatrix}$$

where $\acute{A}_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$ and the sub-block $\acute{A}_{11}$ when partitioned has the structure

$$\acute{A}_{11} = \left[\begin{array}{cc|c} \acute{A}_{11}^o & \acute{A}_{12}^o & \acute{A}_{12}^m \\ 0 & \acute{A}_{22}^o & \\ \hline 0 & \acute{A}_{21}^o & \acute{A}_{22}^m \end{array}\right]$$

where $\acute{A}_{11}^o \in \mathbb{R}^{r\times r}$, $\acute{A}_{22}^o \in \mathbb{R}^{(n-\tilde{p}-r)\times(n-\tilde{p}-r)}$ and $\acute{A}_{21}^o \in \mathbb{R}^{(\tilde{p}-m)\times(n-\tilde{p}-r)}$ *for some* $r \geq 0$ *and the pair* $(\acute{A}_{22}^o, \acute{A}_{21}^o)$ *is completely observable.*

- *The input distribution matrix $\acute{B}$ and the output distribution matrix $\acute{C}$ have the structure in (16).*

For a proof and a constructive algorithm to obtain this canonical form see the work of Edwards and Spurgeon (1995). In the case where $r > 0$, the intention is to construct a new system $(\tilde{A}_{11}, \acute{A}_{122}, \tilde{C}_f)$ which is both controllable and observable with the property that

$$\lambda(\acute{A}_{11}^s) = \lambda(\acute{A}_{11}^o) \cup \lambda(\tilde{A}_{11} - \acute{A}_{122}K\tilde{C}_f).$$

To this end, as in (Edwards and Spurgeon 1995), partition the matrices $\acute{A}_{12}$ and $\acute{A}_{12}^m$ as

$$\acute{A}_{12} = \begin{bmatrix} \acute{A}_{121} \\ \acute{A}_{122} \end{bmatrix} \quad \text{and} \quad \acute{A}_{12}^m = \begin{bmatrix} \acute{A}_{121}^m \\ \acute{A}_{122}^m \end{bmatrix}$$

where $\acute{A}_{122} \in \mathbb{R}^{(n-m-r)\times m}$ and $\acute{A}_{122}^m \in \mathbb{R}^{(n-\tilde{p}-r)\times(\tilde{p}-m)}$ and form a new sub-system represented by the triple $(\tilde{A}_{11}, \acute{A}_{122}, \tilde{C}_f)$ where

$$\tilde{A}_{11} = \begin{bmatrix} \acute{A}_{22}^o & \acute{A}_{122}^m \\ \acute{A}_{21}^o & \acute{A}_{22}^m \end{bmatrix}$$

$$\tilde{C}_f = \begin{bmatrix} 0_{(\tilde{p}-m)\times(n-\tilde{p}-r)} & I_{(\tilde{p}-m)} \end{bmatrix} \tag{29}$$

It follows that the spectrum of $\acute{A}_{11}^s$ decomposes as

$$\lambda(\acute{A}_{11} - \acute{A}_{12}K C_f) = \lambda(\acute{A}_{11}^o) \cup \lambda(\tilde{A}_{11} - \acute{A}_{122}K\tilde{C}_f)$$

Lemma 3.2: *The spectrum of $\acute{A}_{11}^o$ represents the invariant zeros of $(A, B, \tilde{C})$.(Edwards and Spurgeon 1995)*

It follows directly that for a stable sliding motion, the invariant zeros of the system $(\acute{A}, \acute{B}, \acute{C})$ must lie inside the unit disk and the triple $(\tilde{A}_{11}, \acute{A}_{122}, \tilde{C}_f)$ must be stabilisable with respect to output feedback. The matrix $\acute{A}_{122}$ is not necessarily full rank. Suppose rank$(\acute{A}_{122}) = m'$ then, as in Edwards and Spurgeon (1995), it is possible to construct a matrix of elementary column operations $T_{m'} \in \mathbb{R}^{m\times m}$ such that

$$\acute{A}_{122}T_{m'} = \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} \tag{30}$$

where $\tilde{B}_1 \in \mathbb{R}^{(n-m-r)\times m'}$ and is of full rank. If $K_{m'} = T_{m'}^{-1}K$ and $K_{m'}$ is partitioned compatibly as

$$K_{m'} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \begin{array}{l} \updownarrow m' \\ \updownarrow m-m' \end{array}$$

then

$$\tilde{A}_{11} - \acute{A}_{122}K\tilde{C}_f = \tilde{A}_{11} - \begin{bmatrix} \tilde{B}_1 & 0 \end{bmatrix} K_{m'}\tilde{C}_f$$
$$= \tilde{A}_{11} - \tilde{B}_1 K_1 \tilde{C}_f$$

and $(\tilde{A}_{11}, \acute{A}_{122}, \tilde{C}_f)$ is stabilisable by output feedback if and only if $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_f)$ is stabilisable by output feedback. From the original controllability assumption, Lemma 3.1 and the construction (29), the arguments from Edwards and Spurgeon (1995) can be used to prove:

Lemma 3.3: *The pair $(\tilde{A}_{11}, \tilde{B}_1)$ is completely controllable and $(\tilde{A}_{11}, \tilde{C}_f)$ is completely observable.*

As can be seen, the triple $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_f)$ is both completely controllable and observable as required to realise an output feedback based controller. To design the sliding surface consider the method for the design of a static output feedback gain as given in Mehdi et al. (2004). It is important to note that the work of Mehdi et al. (2004) assumes observability of the system matrices. The technique in Mehdi et al. (2004) introduces slack variables to decouple the Lyapunov matrix and the static output feedback gain. With the additional slack variables and a chosen state space variable, an LMI problem is solved to obtain the static output feedback controller. The following theorem is required to formulate the design of the sliding surface using this method.

Theorem 3.4: *Let the matrix A_o be defined as $A_o = \tilde{A}_{11} + \acute{A}_{122}K_o$. A static output feedback gain K is stabilising for the triple $(\tilde{A}_{11}, \acute{A}_{122}, \tilde{C}_f)$ if and only if there exist a positive definite matrix $\Xi = \Xi^T > 0$ in $\mathbb{R}^{n \times n}$, non singular matrices $G_1 \in \mathbb{R}^{m \times m}$ and $E_4 \in \mathbb{R}^{n \times n}$, non-null matrices $E_1 \in \mathbb{R}^{n \times n}$ and $L \in \mathbb{R}^{m \times p}$ and arbitrary matrices $E_2 \in \mathbb{R}^{n \times n}$, $E_3 \in \mathbb{R}^{m \times n}$ such that the following LMI:*

$$\begin{bmatrix} E_1 A_o + A_o^T E_1^T - \Xi & * & * & * \\ E_2 A_o & -\Xi & * & * \\ E_3 A_o + \acute{A}_{122}^T E_1^T + (L\tilde{C}_f - G_1 K_o) & \acute{A}_{122}^T E_2^T & E_3 \acute{A}_{122} + \acute{A}_{122}^T E_3^T - (G_1 + G_1^T) & * \\ E_4 A_o - E_1^T & \Xi - E_2^T & E_4 \acute{A}_{122} - E_3^T & -E_4 - E_4^T \end{bmatrix} < 0 \qquad (31)$$

is feasible for a given state feedback gain K_o that stabilises the pair $(\tilde{A}_{11}, \acute{A}_{122})$ with the static output feedback gain given by

$$K = G_1^{-1} L \qquad (32)$$

For a conclusive proof of the above theorem, refer to (Mehdi et al. 2004). The section below will now consider selection of a suitable control law to stabilise the closed loop dynamics of the plant.

4   Closed loop stability analysis

Introduce a nonsingular state transformation of the form $\acute{x} \rightarrow \bar{T}\acute{x}$ where $\bar{T}$ is given as :

$$\bar{T} = \begin{bmatrix} I_{n-m} & 0 \\ -KC_f & I_m \end{bmatrix} \qquad (33)$$

where the matrices I_{n-m} and I_m are identity matrices of dimension $n-m$ and m respectively and the matrix $KC_f \in \mathbb{R}^{m \times (n-m)}$. With the above transformation, the system (1)-(2) is of the form:

$$\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \bar{B}_d = \begin{bmatrix} \bar{B}_{d_1} \\ \bar{B}_{d_2} \end{bmatrix} F\bar{C} = [0 \ F_2] \qquad (34)$$

Here, $\bar{A}_{11} = \bar{A}_{11}^s = \acute{A}_{11} - \acute{A}_{12}KC_f$ with $\bar{A}_{11} \in \mathbb{R}^{(n-m) \times (n-m)}$ and is stable by construction. The rest of the sub-matrices in the matrix $\bar{A}$ are conformably partitioned. The control law u_k is chosen here as a function of the known signals $\tilde{y}_k$ and U_k and has the form :

$$u_k = -(B^T P B)^{-1} s_k \qquad (35)$$

where the matrix P is symmetric positive definite and with $P \in \mathbb{R}^{n \times n}$. In the new coordinate system given in (34), the corresponding control $\bar{u}_k$ is:

$$\bar{u}_k = -(\bar{B}^T P \bar{B})^{-1} \bar{s}_k \qquad (36)$$

In the case of a discrete-time, uncertain system it is necessary to consider closed loop analysis of the full-order system. In continuous time, where infinite switching is assumed, it is possible to enforce the sliding mode condition so that $\bar{s}_k$ is identically zero after some finite time. This behavior will not be replicated in discrete time where motion close to $\bar{s}_k = 0$ is the best that can be achieved in the uncertain case. In this paper it will be demonstrated that $\bar{s}_k$ can be constrained to within some acceptably small region of $\bar{s}_k = 0$ after a finite number of steps.

To show that the control $\bar{u}_k$ stabilizes the system (34), define the Lyapunov candidate :

$$\bar{V}_k = \bar{x}_k^T \bar{P} \bar{x}_k \tag{37}$$

with $P \rightarrow \bar{T}^{-T} \acute{T}^{-T} \hat{T}^{-T} P \hat{T}^{-1} \acute{T}^{-1} \bar{T}^{-1} =: \bar{P}$. The Lyapunov matrix $\bar{P}$ is chosen such that

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^T & \bar{P}_3 \end{bmatrix} \tag{38}$$

with $\bar{P}_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and the rest of the sub matrices of $\bar{P}$ conformably partitioned. The forward difference of the Lyapunov function (37), $\Delta \bar{V}_k$ in the new coordinates is given as:

$$\Delta \bar{V}_k = \bar{x}_{k+1}^T \bar{P} \bar{x}_{k+1} - \bar{x}_k^T \bar{P} \bar{x}_k \tag{39}$$

The stability analysis of the system given in (34) together with the control (36) will now be considered. Before deriving the stability condition, the result from Yan et al. (2009) will be stated here without the proof, for use in the main result.

Lemma 4.1: *Let there be a matrix $N_1 \in \mathbb{R}^{m \times n}$ and let there be vectors $w \in \mathbb{R}^m$ and $r \in \mathbb{R}^n$. Then the inequality*

$$w^T N_1 r \leq \frac{1}{2\varepsilon} w^T N_1 N_2^{-1} N_1^T w + \frac{\varepsilon}{2} r^T N_2 r$$

holds for any symmetric positive definite matrix $N_2 \in \mathbb{R}^{n \times n}$ and any positive constant ε

Now the main result can be stated in the theorem given below.

Theorem 4.2 : *If there exists a matrix $F \in \mathbb{R}^{m \times \tilde{p}}$, a symmetric positive definite matrix $\bar{P} \in \mathbb{R}^n$, a symmetric positive definite matrix $\bar{Q} \in \mathbb{R}^{n \times n}$ that simultaneously solves the discrete Riccati equation*

$$\bar{A}^T \bar{P} \bar{A} - \bar{P} - \bar{A}^T \bar{P} \bar{B} (\bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \bar{A} = -\bar{Q} \tag{40}$$

and the linear constraint:

$$F \bar{C} = \bar{B}^T \bar{P} \bar{A} \tag{41}$$

then the control law (36) stabilizes the system (34) iff

$$\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} > \rho_0 \Big(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2 \Big)^{\frac{1}{2}} \tag{42}$$

where $\underline{\lambda}(\bar{Q}_1)$ is the minimum eigenvalue of $\bar{Q}_1$, and the matrix $\bar{Q}_1$ is defined as

$$\bar{A}^T \bar{P} \bar{A} - (1-\varepsilon)\bar{P} - \bar{A}^T \bar{P} \bar{B} (\bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \bar{A} = -\bar{Q}_1 \tag{43}$$

where ε is some scalar with $\varepsilon > 0$ and the matrix $\bar{E} \in \mathbb{R}^n$. Also the system states will be uniformly ultimately bounded with respect to the ellipsoid $\xi(r_1)$ with parameter

$$r_1 = \varepsilon_1 + \Big(\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} - \rho_0 (\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2)^{\frac{1}{2}} \Big)^{-2} \Big(\Big(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2 \Big) \rho_1^2 \Big) \underline{\lambda}(\bar{P}) \tag{44}$$

with $\varepsilon_1 > 0$ and arbitrarily small.

Proof: Using the above control law given in (36), the forward difference $\Delta \bar{V}_k$ can be shown to be:

$$\Delta \bar{V}_k = (\bar{x}_k^T \bar{A}^T + \bar{u}_k^T \bar{B}^T + \bar{d}_k^T \bar{B}_d^T) \bar{P} (\bar{A}\bar{x} + \bar{B}\bar{u}_k + \bar{B}_d \bar{d}_k) - \bar{x}_k^T \bar{P} \bar{x}_k \tag{45}$$

$$= \bar{x}_k^T (\bar{A}^T \bar{P} \bar{A} - \bar{P} - \bar{A}^T \bar{P} \bar{B} (\bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \bar{A}) \bar{x}_k - 2\bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B} (\bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \bar{A} \bar{x}_k \tag{46}$$

$$+ 2\bar{d}_k^T \bar{B}_d^T \bar{P} \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k$$

$$= -\bar{x}_k^T \bar{Q} \bar{x}_k + 2\bar{d}_k^T \bar{B}_d^T \bar{P} \bar{A} \bar{x}_k - 2\bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B} (\bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P} \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k$$

where the matrix $\bar{Q}$ is as defined in (42). The above equation can be further reduced to

$$= -\bar{x}_k^T \bar{Q} \bar{x}_k + 2\bar{d}_k^T \bar{B}_d^T (\bar{P} - \bar{P}\bar{B}(\bar{B}^T \bar{P} \bar{B})^{-1} \bar{B}^T \bar{P}) \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{47}$$

$$= -\bar{x}_k^T \bar{Q} \bar{x}_k + 2\bar{d}_k^T \bar{B}_d^T (\bar{P} - \bar{P}\bar{B}(B_2^T \bar{P}_3 B_2)^{-1} \bar{B}^T \bar{P}) \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{48}$$

$$= -\bar{x}_k^T \bar{Q} \bar{x}_k + 2\bar{d}_k^T \bar{B}_d^T (\bar{P} - \bar{P}\bar{B}(B_2^T \bar{P}_3 B_2)^{-1} \bar{B}^T \bar{P}) \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{49}$$

$$= -\bar{x}_k^T \bar{Q} \bar{x}_k + 2\bar{d}_k^T \bar{B}_d^T \bar{E} \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{50}$$

where the matrix $\bar{E}$ can be computed from $\bar{P} - \bar{P}\bar{B}(B_2^T \bar{P}_3 B_2)^{-1} \bar{B}^T \bar{P}$ as

$$\bar{E} = \begin{bmatrix} \bar{P}_1 - \bar{P}_2 \bar{P}_3^{-1} \bar{P}_2^T & 0 \\ 0 & 0 \end{bmatrix}$$

using lemma (4.1) with $N_1 = \bar{B}_d^T \bar{E} \bar{A}$, $w = \bar{d}_k$ and $r = \bar{x}_k$. Equation (50) can now be written as

$$\Delta \bar{V}_k = -\bar{x}_k^T \bar{Q} \bar{x}_k + 2\bar{d}_k^T \bar{B}_d^T \bar{E} \bar{A} \bar{x}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{51}$$

$$\leq -\bar{x}_k^T \bar{Q} \bar{x}_k + \varepsilon \bar{x}_k \bar{P} \bar{x}_k + \frac{1}{\varepsilon} \bar{d}_k^T \bar{B}_d^T \bar{E} \bar{A} \bar{P}^{-1} \bar{A}^T \bar{E} \bar{B}_d \bar{d}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{52}$$

$$\leq -\bar{x}_k^T \bar{Q}_1 \bar{x}_k + \frac{1}{\varepsilon} \bar{d}_k^T \bar{B}_d^T \bar{E} \bar{A} \bar{P}^{-1} \bar{A}^T \bar{E} \bar{B}_d \bar{d}_k + \bar{d}_k^T \bar{B}_d^T \bar{P} \bar{B}_d \bar{d}_k \tag{53}$$

where $\bar{Q}_1$ is defined as in (43). Using the assumption that $\|\bar{d}_k\| < \rho_0 \|\bar{x}_k\| + \rho_1$ in (53) gives

$$\Delta \bar{V}_k \leq -\underline{\lambda}(\bar{Q}_1) \|\bar{x}\|^2 + \left(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2\right) \|\bar{d}_k\|^2 \tag{54}$$

$$\leq -\underline{\lambda}(\bar{Q}_1) \|\bar{x}_k\|^2 + \left(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2\right) (\rho_0 \|\bar{x}_k\| + \rho_1)^2 \tag{55}$$

As stated before, $\underline{\lambda}(\bar{Q}_1)$ is the minimum eigenvalue of $\bar{Q}_1$. Hence the condition for stability can now be written as:

$$0 > -\underline{\lambda}(\bar{Q}_1) \|\bar{x}_k\|^2 + (\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2)(\rho_0 \|\bar{x}_k\| + \rho_1)^2 \tag{56}$$

$$\left(\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} - \rho_0(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2)^{\frac{1}{2}}\right) \|\bar{x}_k\| > \left(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2\right)^{\frac{1}{2}} \rho_1 \tag{57}$$

Hence

$$\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} > \rho_0(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2)^{\frac{1}{2}} \tag{58}$$

Also,

$$\|\bar{x}_k\| > \left(\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} - \rho_0(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2)^{\frac{1}{2}}\right)^{-1} \left(\frac{1}{\varepsilon} \|\bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d\|^2 + \|\bar{P}^{\frac{1}{2}} \bar{B}_d\|^2\right)^{\frac{1}{2}} \rho_1 \tag{59}$$

Hence the states $\bar{x}$ will ultimately enter an ellipsoid after a finite time k depending on the initial condition $\bar{x}_0$ and will thereafter remain within the ellipsoid which is given by

$$\xi(r_1) = \{\xi \in \mathbb{R}^n : \bar{V}(\xi) < r_1\} \tag{60}$$

where r_1

$$r_1 = \varepsilon_1 + \left(\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} - \rho_0 \left(\frac{1}{\varepsilon} \| \bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d \|^2 + \| \bar{P}^{\frac{1}{2}} \bar{B}_d \|^2 \right)^{\frac{1}{2}} \right)^{-2} \left(\left(\frac{1}{\varepsilon} \| \bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d \|^2 + \| \bar{P}^{\frac{1}{2}} \bar{B}_d \|^2 \right) \rho_1^2 \right) \underline{\lambda}(\bar{P}) \quad (61)$$

with $\varepsilon_1 > 0$ and is arbitrarily small.
Now consider the stability condition given in (58)

$$\underline{\lambda}(\bar{Q}_1)^{\frac{1}{2}} > \rho_0 \left(\frac{1}{\varepsilon} \| \bar{P}^{-\frac{1}{2}} \bar{A}^T \bar{E} \bar{B}_d \|^2 + \| \bar{P}^{\frac{1}{2}} \bar{B}_d \|^2 \right)^{\frac{1}{2}} \quad (62)$$

$$\Rightarrow 0 > (-\underline{\lambda}(\bar{Q}_1) I_q + \rho_0^2 \frac{1}{\varepsilon} \bar{B}_d^T \bar{E} \bar{A} \bar{P}^{-1} \bar{A}^T \bar{E} \bar{B}_d + \rho_0^2 \bar{B}_d^T \bar{P} \bar{B}_d) \quad (63)$$

$$\Longleftrightarrow$$

$$0 > \begin{bmatrix} -\underline{\lambda}(\bar{Q}_1) I_q & \rho_0 \sqrt{\frac{1}{\varepsilon}} \bar{B}_d^T \bar{E} \bar{A} & \rho_0 \bar{B}_d^T \bar{P} \\ \rho_0 \sqrt{\frac{1}{\varepsilon}} \bar{A}^T \bar{E} \bar{B}_d^T & -\bar{P} & 0 \\ \rho_0 \bar{P} \bar{B}_d & 0 & -\bar{P} \end{bmatrix} \quad (64)$$

where the second inequality follows from taking the schur complements, with the matrix $-\bar{P} < 0$. The variable ε in the above equation can be considered as a design variable and can be fixed for control design.

For the above matrix inequality (64) to be satisfied, it is pertinent that a matrix $\bar{P}$ exist that satisfies the constrained Riccati equation (40). Consider the linear constraint $F\bar{C} = \bar{B}^T \bar{P} \bar{A}$ together with the canonical structure given in (34). It can be seen that

$$[0 \ F_2] = \bar{B}^T \bar{P} \bar{A} \quad (65)$$

$$[0 \ F_2] = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}^T \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^T & \bar{P}_3 \end{bmatrix} \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \quad (66)$$

$$= \left[B_2^T \bar{P}_2^T \bar{A}_{11} + B_2^T \bar{P}_3 \bar{A}_{21} \ \ B_2^T \bar{P}_2^T \bar{A}_{12} + B_2^T \bar{P}_3 \bar{A}_{22} \right] \quad (67)$$

For the relation (65) to be satisfied, it can be seen that:

$$\bar{P}_2^T \bar{A}_{11} + \bar{P}_3 \bar{A}_{21} = 0 \implies \bar{P}_2^T = -\bar{P}_3 \bar{A}_{21} \bar{A}_{11}^{-1} \quad (68)$$

and

$$F_2 = B_2^T \bar{P}_2^T \bar{A}_{12} + B_2^T \bar{P}_3 \bar{A}_{22} \quad (69)$$

Using the results given above the matrix $\bar{Q}$ can be computed as:

$$\bar{Q} = \begin{bmatrix} \bar{P}_1 - \bar{A}_{11}^T \bar{P}_1 \bar{A}_{11} + \bar{A}_{21}^T \bar{P}_3 \bar{A}_{21} & (-\bar{P}_3 \bar{A}_{21} \bar{A}_{11}^{-1} - \bar{A}_{12}^T \bar{P}_1 \bar{A}_{11} + \bar{A}_{12}^T \bar{A}_{11}^{-T} \bar{A}_{21}^T \bar{P}_3 \bar{A}_{21})^T \\ * & \bar{P}_3 + \bar{A}_{12}^T \bar{A}_{11}^{-T} \bar{A}_{12}^T \bar{P}_3 \bar{A}_{21} \bar{A}_{11}^{-1} \bar{A}_{12} - \bar{A}_{12}^T \bar{P}_1 \bar{A}_{12} \end{bmatrix} \quad (70)$$

Also the closed loop matrix for the system (34) with the feedback control law (36) can be computed as:

$$\bar{A}_{cl} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \left([0 \ F_2] \bar{A}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right)^{-1} [0 \ F_2] \quad (71)$$

where the inverse of the matrix $\bar{A}$ can be written as :

$$\bar{A}^{-1} = \begin{bmatrix} \bar{A}_{11} + \bar{A}_{11}^{-1} \bar{A}_{12} (\bar{A}_{22} - \bar{A}_{21} \bar{A}_{11}^{-1} \bar{A}_{12})^{-1} \bar{A}_{21} \bar{A}_{11}^{-1} & -\bar{A}_{11}^{-1} \bar{A}_{12} (\bar{A}_{22} - \bar{A}_{21} \bar{A}_{11}^{-1} \bar{A}_{12})^{-1} \\ -(\bar{A}_{22} - \bar{A}_{21} \bar{A}_{11}^{-1} \bar{A}_{12})^{-1} \bar{A}_{21} \bar{A}_{11}^{-1} & (\bar{A}_{22} - \bar{A}_{21} \bar{A}_{11}^{-1} \bar{A}_{12})^{-1} \end{bmatrix} \quad (72)$$

Hence

$$\bar{A}_{cl} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ B_2 \end{bmatrix} (F_2(\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12})^{-1}B_2)^{-1} [0 \; F_2] \tag{73}$$

$$\bar{A}_{cl} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12} \end{bmatrix} \tag{74}$$

The relation between the matrix $\bar{Q}$ and the closed loop matrix $\bar{A}_{cl}$ is shown to be $\bar{Q} = \bar{P} - \bar{A}_{cl}^T \bar{P}\bar{A}_{cl}$. Hence, if the matrix $\bar{A}_{cl}$ is stable, it can be noticed that a $\bar{P}$ that satisfies the above closed loop Lyapunov equation, will also satisfy the open loop Riccati equation (40). The existence of a positive definite matrix $\bar{P}$ is guaranteed if the closed loop matrix $\bar{A}_{cl}$ is stable. Hence the necessary condition for the existence of a matrix $\bar{P}$ is to find a matrix K that would stabilize $\bar{A}_{cl}$. As in the previous section, the gain K is computed from the algorithm given in (Mehdi et al. 2004). For obtaining an optimum P, where $P = \hat{T}^T\acute{T}^T\bar{T}^T\bar{P}\bar{T}\acute{T}\hat{T}$, the lower bound for the matrix $\bar{Q}$ is fixed as I_n and the following set of linear matrix inequalities is formulated:

$$\bar{Q} > I \tag{75}$$

$$\begin{bmatrix} -\underline{\lambda}(\bar{Q}_1)I_q & \rho_0\sqrt{\frac{1}{\varepsilon}}\bar{B}_d^T\bar{E}\bar{A} & \rho_0\bar{B}_d^T\bar{P} \\ \rho_0\sqrt{\frac{1}{\varepsilon}}\bar{A}^T\bar{E}\bar{B}_d^T & -\bar{P} & 0 \\ \rho_0\bar{P}\bar{B}_d & 0 & -\bar{P} \end{bmatrix} < 0 \tag{76}$$

$$\bar{P} > 0 \tag{77}$$

with $\bar{P}_1$ and $\bar{P}_3$, as the decision variables. The unknown variable F_2 can be computed from (69). It can be shown that the control law (36) is dependent only on the parameter K and not on F_2. By simple algebraic manipulation it can be shown that:

$$\bar{u}_k = -(\bar{B}^T\bar{P}\bar{B})^{-1}\bar{s}_k \tag{78}$$

From (41) it can be shown that $F\bar{C}\bar{A}^{-1}\bar{B} = \bar{B}^T\bar{P}\bar{B}$ and using this in the above equation gives:

$$\bar{u}_k = -(F\bar{C}\bar{A}^{-1}\bar{B})^{-1}(F\bar{\bar{y}}_k + F\bar{M}\bar{U}_k) \tag{79}$$

$$= -([K \; I_m]\bar{C}\bar{A}^{-1}\bar{B})^{-1}([K \; I_m]\bar{\bar{y}}_k + [K \; I_m]\bar{M}\bar{U}_k) \tag{80}$$

The above control law can be used for feedback control, but it will be necessary to show the existence of a positive definite $\bar{P}$ and a negative definite $\bar{Q}$ that will satisfy the conditions of theorem (4.2) to prove stability.

To show that the nominal closed loop dynamics where the system contains no uncertainty, are a reduced order dynamics introduce a new change of coordinates of the form $\bar{x} \rightarrow T_c\bar{x} = x^c$, where T_c has the form

$$T_c = \begin{bmatrix} I_{n-m} & -\bar{A}_{11}^{-1}\bar{A}_{12} \\ 0 & I_m \end{bmatrix} \tag{81}$$

The nominal closed loop matrix A_{cl}^c in the new coordinates can be written as

$$A_{cl}^c = \begin{bmatrix} \bar{A}_{11} + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21} & 0 \\ \bar{A}_{21} & 0 \end{bmatrix}$$

The nominal closed loop eigenvalues in this case are clearly seen to be the eigenvalues of the matrix $\bar{A}_{11} + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21}$. Hence stability of the closed loop implies the stability of the matrix $\bar{A}_{11} + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21}$. It is now necessary to show that control (36) forces the state trajectories towards the sliding surface s_k from any initial condition and that the sliding dynamics are bounded within a region of the sliding surface in finite time.

5   Existence of a sliding mode

To show that the control law (36) forces the states of the uncertain system towards the sliding surface s_k, consider $\bar{s}_{k+1}$.

$$\bar{s}_{k+1} = F\bar{C}\bar{A}_{cl}\bar{x}_k + F\bar{C}\bar{B}_d\bar{d}_k \tag{82}$$

Let $\phi = F_2\bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12}F_2^{-1}$. The term $F\bar{C}\bar{A}_{cl}\bar{x}_k$ can be written as

$$F\bar{C}\bar{A}_{cl}\bar{x}_k = F_2\bar{A}_{21}(\bar{x}_1)_k + \phi\bar{s}_k$$

Hence $\bar{s}_{k+1}$ can be written as

$$\bar{s}_{k+1} = F_2\bar{A}_{21}(\bar{x}_1)_k + \phi\bar{s}_k + F\bar{C}\bar{B}_d\bar{d}_k \tag{83}$$

Consider the sliding surface at the instant $\bar{s}_{k-1}$. Then

$$\bar{s}_{k-1} = F\bar{C}\bar{x}_{k-1} \tag{84}$$

$$\bar{s}_{k-1} = F\bar{C}\bar{A}^{-1}\bar{x}_k - F\bar{C}\bar{A}^{-1}\bar{B}\bar{u}_{k-1} - F\bar{C}\bar{A}^{-1}\bar{B}_d\bar{d}_{k-1} \tag{85}$$

$$\tag{86}$$

Substituting for $\bar{u}_{k-1}$ in the above equation gives

$$\bar{s}_{k-1} = F\bar{C}\bar{A}^{-1}\bar{x}_k + F\bar{C}\bar{A}^{-1}\bar{B}(F\bar{C}\bar{A}^{-1}\bar{B})^{-1}\bar{s}_{k-1} - F\bar{C}\bar{A}^{-1}\bar{B}_d\bar{d}_{k-1} \tag{87}$$

$$\bar{s}_{k-1} = F\bar{C}\bar{A}^{-1}\bar{x}_k + \bar{s}_{k-1} - F\bar{C}\bar{A}^{-1}\bar{B}_d\bar{d}_{k-1} \tag{88}$$

$$F\bar{C}\bar{A}^{-1}\bar{x}_k = F\bar{C}\bar{A}^{-1}\bar{B}_d\bar{d}_{k-1} \tag{89}$$

Expanding (89) gives

$$-F_2(\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12})^{-1}\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{x}_1)_k + F_2(\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1}\bar{A}_{12})^{-1}(\bar{x}_2)_k = F_2(\bar{A}_{22} - \bar{A}_{21}\bar{A}_{11}^{-1})^{-1}$$
$$\cdot [-\bar{A}_{21}\bar{A}_{11}^{-1}\ I_m]\bar{B}_d\bar{d}_{k-1} \tag{90}$$

$$-\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{x}_1)_k + (\bar{x}_2)_k = [-\bar{A}_{21}\bar{A}_{11}^{-1}\ I_m]\bar{B}_d\bar{d}_{k-1} \tag{91}$$

$$\Rightarrow F_2^{-1}\bar{s}_k = \bar{A}_{21}\bar{A}_{11}^{-1}(\bar{x}_1)_k + [-\bar{A}_{21}\bar{A}_{11}^{-1}\ I_m]\bar{B}_d\bar{d}_{k-1} \tag{92}$$

$$\bar{s}_k = F_2\bar{A}_{21}\bar{A}_{11}^{-1}(\bar{x}_1)_k - F_2[\bar{A}_{21}\bar{A}_{11}^{-1}\ -I_m]\bar{B}_d\bar{d}_{k-1} \tag{93}$$

Substituting the above value for $\bar{s}_k$ in (83) and taking norms gives

$$\|\bar{s}_{k+1}\| \le (\|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|)\|(\bar{x}_1)_k\| - \|\phi F_2[\bar{A}_{21}\bar{A}_{11}^{-1}\ -I_m]\bar{B}_d\|\|\bar{d}_{k-1}\| + \|F\bar{C}\bar{B}_d\|\|\bar{d}_k\| \tag{94}$$

$$\le (\|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|)\|\bar{x}_k\| - \|\phi F_2[\bar{A}_{21}\bar{A}_{11}^{-1}\ -I_m]\bar{B}_d\|\|\bar{d}_{k-1}\| + \|F\bar{C}\bar{B}_d\|\|\bar{d}_k\| \tag{95}$$

The stability and uniform ultimate boundedness of $\bar{s}_{k+1}$ follows from the stability and uniform ultimate boundedness of $\bar{x}_k$ and the boundedness of the disturbance term $\bar{d}_k$. Note that equation (59) claims that the states $\bar{x}_k$ enter an ellipsoid $\xi(r_1)$ in a finite time k. Let $\hat{k}$ be the time instant at which the states enter the ellipsoid $\xi(r_1)$. Let $\|x_{\hat{k}}\| = \varepsilon$ at $\delta(\xi(r)_1)$. Then equation (94), at the instant $\hat{k}+1$ can be written as

$$\|\bar{s}_{\hat{k}+1}\| \le (\|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|)\|\varepsilon\| - \|\phi F_2[\bar{A}_{21}\bar{A}_{11}^{-1}\ -I_m]\bar{B}_d\|\|\bar{d}_{k-1}\| + \|F\bar{C}\bar{B}_d\|(\rho_0\varepsilon + \rho_1) \tag{96}$$

As can be seen the right hand side of the above equation will be bounded from above by a constant $\forall k > \hat{k}+2$. Hence

$$\|\bar{s}_k\| \le \rho_4 \ \forall k > \hat{k}+2$$

where

$$\rho_4 = (\|F_2\bar{A}_{21} + \phi F_2\bar{A}_{21}\bar{A}_{11}^{-1}\|)\|\varepsilon\| - (\|\phi F_2[\bar{A}_{21}\bar{A}_{11}^{-1} \quad -I_m]\bar{B}_d\| - \|F\bar{C}\bar{B}_d\|)(\rho_0\varepsilon + \rho_1)$$

It is clear from (96) that the properties of F_2 determine how close the system dynamics are to the ideal sliding mode dynamics. From (69) the matrix F_2 is seen to be a function of the matrix $\bar{P}_3$. Hence to make $\|s_k\|$ small, it is necessary for the matrix $\bar{P}_3$ to be as small as possible. This can be done by introducing a constraint on the upper bound of F_2 such that $\|F_2\| < \gamma_1$ for some scalar $\gamma_1 > 0$. Hence, considering $\|F_2\| < \gamma_1$ and taking the schur complements gives

$$F_2^T F_2 < \gamma_1 I_m \tag{97}$$

$$\begin{bmatrix} \gamma_1 I_m & F_2^T \\ F_2 & I_m \end{bmatrix} > 0 \tag{98}$$

The above linear matrix inequality with F_2 from (69), along with linear matrix inequalities (75)-(77) can then be solved for obtaining an optimum $\bar{P}$ with $\bar{P}_1$, $\bar{P}_3$, γ_1 as decision variables.

The above control design paradigm will now be applied to an aircraft problem in the below section and it will be shown that it is a useful for reconfigurable control. The example will be used to show the band on the finite time $\hat{k}$ required to attain an acceptable boundary of the sliding dynamics can be supported in practice.

6   Motivational Example

Consider the aircraft system from (Edwards and Spurgeon 1998) given below. The plant model represents the fifth-order lateral dynamics of an L-1011 fixed wing aircraft, with the actuator dynamics removed. The nominal plant system triple is given by:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 \\ 0 & 0.249 & -1.000 & -5.20 & 0 \\ 0.0386 & -0.996 & -0.0003 & -0.117 & 0 \\ 0 & 0.500 & 0 & 0 & -0.5 \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ -0.744 & -0.032 \\ 0.337 & -1.1200 \\ 0.02 & 0 \\ 0 & 0 \end{bmatrix} C = \begin{bmatrix} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with the disturbance matrix B_d taken as $[0\ 0\ 0\ 0\ 0.5]$. The plant states are

$$x = \begin{bmatrix} \phi & \text{bank angle (rad)} \\ r & \text{yaw rate (rad/sec)} \\ p & \text{roll rate (rad/sec)} \\ \beta & \text{side slip angle (rad)} \\ x_5 & \text{washed-out filter state} \end{bmatrix}$$

with inputs

$$u = \begin{bmatrix} \delta_r & \text{rudder deflection (rad)} \\ \delta_a & \text{aileron deflection (rad)} \end{bmatrix}$$

and outputs

$$y = \begin{bmatrix} r_{wo} & \text{washed out yaw rate (rad)} \\ p & \text{roll rate (rad)} \\ \beta & \text{sideslip angle (rad)} \\ \phi & \text{bank angle (rad)} \end{bmatrix}$$

The sampling rate chosen for discretizing the plant was $t = 0.1s$. Assume that the aircraft has lost the side slip angle and roll rate measurements. The invariant zeros for the resulting plant triple (A, B, C) are at $[1.00\ 0.9929\ -0.9671]$ and thus the resulting sliding mode dynamics is unstable with only the

remaining two outputs. Notice here that the invariant zero at 1.00 has the state zero direction

$$x_z^T = [-0.0000 \; 0.0154 \; 0.0000 \; -0.2135 \; 0.0154]$$

It is desirable that the columns of the disturbance distribution matrix B_d should span the null space of $\tilde{C}$. With this assumption it can be shown that the disturbance distribution matrix B_d spans the null space of $C_4 A^{-1}$ and $C_4 A^{-2}$ where C_4 is the fourth row in the output distribution matrix and corresponds to the bank angle. Hence for this particular example the augmented matrix Γ is chosen by augmenting the bank angle twice. The augmented matrix $\tilde{C}$ in this case is given as

$$\tilde{C} = \begin{bmatrix} 0.0000 & 1.0000 & 0.0000 & 0.0000 & -1.0000 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 1.0000 & 0.000402 & -0.1052 & -0.02702 & 0.0000 \\ 1.0003 & -0.00198 & -0.2214 & -0.11213 & 0.0000 \end{bmatrix}$$

Also as stated in Lemma 1 the chosen augmented outputs are such that x_z spans $\tilde{C}$ and it can be checked that $\tilde{C} x_z \neq 0$ and hence it is possible to remove the unstable invariant zero from the triple $(A, B, \tilde{C})$. Now introduce a transformation such that the triple $(A, B, \tilde{C})$ is in the form (16). Here the orthogonal transformation matrix T is

$$T = \begin{bmatrix} -0.0012 & -0.000379 & -0.08412 & -0.9964 \\ 0.9166 & 0.2919 & -0.2722 & 0.02179 \\ -0.21021 & 0.93304 & 0.29090 & -0.024685 \\ 0.33999 & -0.210211 & 0.91335 & -0.07750 \end{bmatrix}$$

and the subsystem $(\tilde{A}_{11}, A_{122}, \tilde{C}_f)$ obtained for the sliding surface design is

$$\tilde{A}_{11} = \begin{bmatrix} 1.0032 & 0.09729 & -0.07168 \\ 0.00322 & 2.9840 & -1.5281 \\ 0.00102 & 1.9340 & -0.49194 \end{bmatrix}$$

$$A_{122} = \begin{bmatrix} -0.037824 & -0.072574 \\ -0.560571 & 0.040835 \\ -0.54337 & 0.04266 \end{bmatrix} \quad \tilde{C}_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The initial state feedback gain K_o is obtained by pole placement and the poles are placed at $[\; 0.98 \; 0.90 \; 0.83 \;]$. Using the algorithm given by Mehdi et al. (2004), the gain K for the reduced order subsystem $(\tilde{A}_{11}, A_{122}, \tilde{C}_f)$ is calculated as by

$$K = \begin{bmatrix} 5.35302 & -2.28322 \\ 15.68429 & 3.72875 \end{bmatrix}$$

with $\lambda(\tilde{A}_{11} - A_{122} K \tilde{C}_f)$ at $[0.55255 \; 0.99107 + 0.00958i \; 0.99107 + 0.00958i]$. The plant was then transformed to the canonical form given in (34). The design constant ε was chosen as .05 and the matrices $\bar{P}_1$ and $\bar{P}_3$ were computed by applying the LMI solver to the feasibility problem (75)-(77) and were obtained as

$$\bar{P}_1 = \begin{bmatrix} .39968 & -9.03852 & 8.96266 \\ -9.03852 & 517.34801 & -510.53449 \\ 8.96266 & -510.53449 & 557.29478 \end{bmatrix}, \bar{P}_3 = \begin{bmatrix} 10.62381 & -1.35066 \\ -1.35066 & .353600 \end{bmatrix}$$

$$\bar{P} = \begin{bmatrix} .39968 & -9.03852 & 8.96266 & .031528 & .010857 \\ -9.03852 & 517.34801 & -510.53449 & 33.94485 & -7.211998 \\ 8.96266 & -510.53449 & 557.29478 & -35.45336 & 4.92901 \\ .031528 & 33.94485 & -35.45336 & 10.6238 & -1.35066 \\ .010857 & -7.211998 & 4.92901 & -1.35066 & .353600 \end{bmatrix}$$
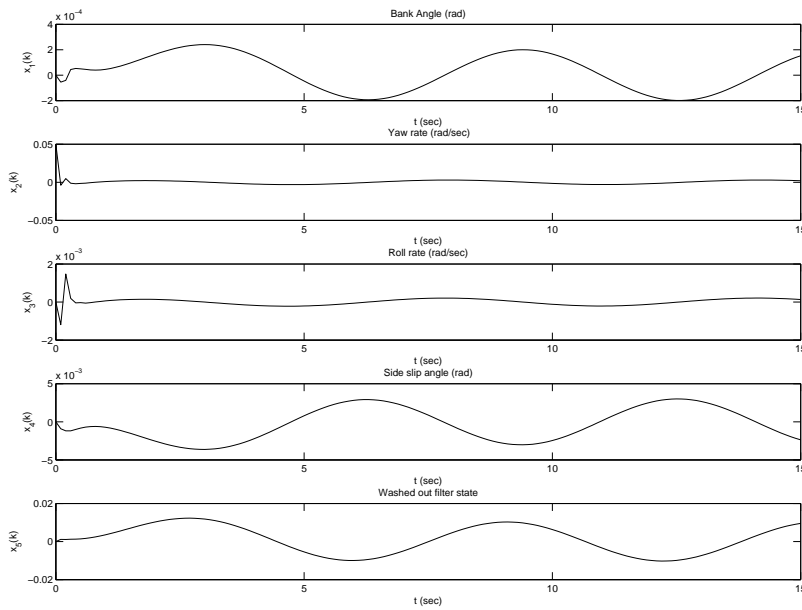
Figure 1. Response of the states x_k in the presence of uncertainty

and the matrix $\bar{Q}$ is obtained as

$$\bar{Q} = \begin{bmatrix} -.016055 & .4995 & -.33542 & .046864 & -.01913 \\ .4995 & -139.29522 & 100.21783 & -20.66627 & 5.47405 \\ -.33542 & 100.21783 & -101.7237 & 15.15744 & -2.67452 \\ .046864 & -20.66627 & 15.15744 & -4.69744 & .82527 \\ -.01913 & 5.47405 & -2.67452 & .82527 & -.29562 \end{bmatrix}$$

The value of γ_1 is 0.46055 and the matrix F_2 was computed from (69) as

$$F_2 = \begin{bmatrix} .163961 & .00417 \\ .17486 & -.02176 \end{bmatrix}$$

The output feedback gain F is then obtained as

$$F_2 \begin{bmatrix} K & I \end{bmatrix} T^T = \begin{bmatrix} -.01697 & -.80418 & .58063 & -.24663 \\ .00748 & -.45302 & .62464 & -.14181 \end{bmatrix}$$

and the matrix ϕ is given as

$$\phi = \begin{bmatrix} -.34098 & -.79723 \\ .05901 & -.09818 \end{bmatrix}$$

The simulations were performed with the control law given in (36) and with the initial conditions $[0 \ 0.05 \ 0 \ 0 \ 0]^T$. The disturbance is a slowly varying disturbance of the form $(sin(\omega t))$. Here, ω was chosen as 1 *rad/sec*, while the peak to peak amplitude for the sine wave was chosen as 0.04 *rad*. The values of the disturbance bounds ρ_0 and ρ_1 is taken as .02 and .02 respectively. Here $\|\bar{x}\|$ from (59) is computed as .034533. Using this value in (96) gives a value of $\rho_4 = .027225$. The actual bounds on $\bar{x}_k$ and $\bar{s}_k$ obtained from the simulations are .017974 and .000104. Hence, the values of the computed theoretical bounds on $\bar{x}_k$ and $\bar{s}_k$ are conservative, especially so for the sliding surface. The simulation results are shown in Figures 1 and 2. It is seen that despite the presence of the unstable invariant zero following sensor failure, the reconfigurable control strategy developed here is able to stabilise the plant, whilst minimising the ultimate bound on the switching function.
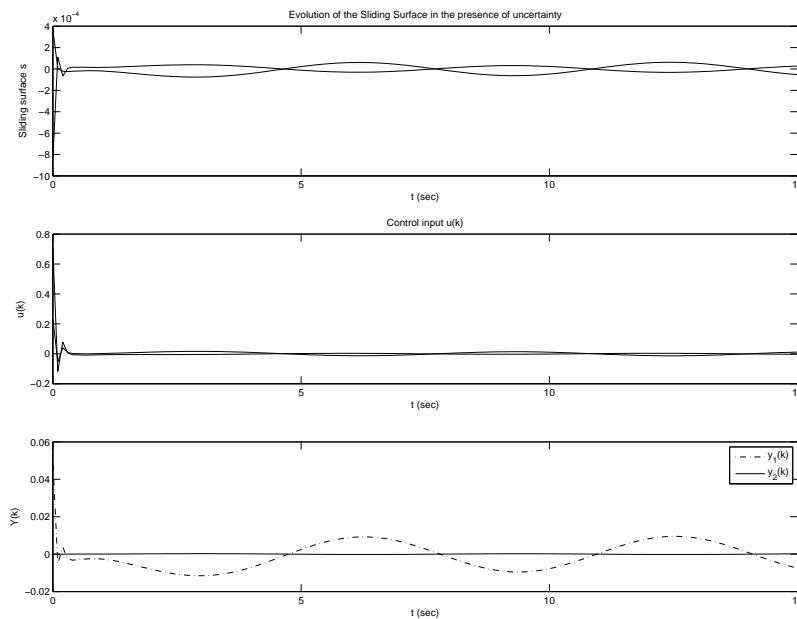
Figure 2.   The control input u_k, sliding surface s_k and the plant output y_k

7   Conclusion

An output feedback based sliding mode control design for discrete time systems has been developed in this paper. It has been shown that discrete time controllers can be realized via the extended outputs for non square systems with uncertainties. The conditions for the existence of such controllers have been given and a procedure to obtain a Lyapunov matrix, which solves a constrained Riccati inequality has been given. A numerical solution to obtain the Lyapunov matrix using linear matrix inequalities has been suggested and the design approach given in the paper has been applied to an aircraft problem. It is shown that in the presence of sensor failure the methodology yields an effective reconfigurable control approach.

Acknowledgement

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