Computer Science at Kent

Constructive Potential Theory: Foundations and Applications.

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Abstract

Stochastic analysis is now an important common part of computing and mathematics. Its applications are impressive, ranging from stochastic concurrent and hybrid systems to finances and biomedicine. In this work we investigate the logical and algebraic foundations of stochastic analysis and possible applications to computing. We focus more concretely on functional analysis theoretic core of stochastic analysis called potential theory. Classical potential theory originates in Gauss and Poincare's work on partial differential equations. Modern potential theory now study stochastic processes with their adjacent theory, higher order differential operators and their combination like stochastic differential equations. In this work we consider only the axiomatic branches of modern potential theory, like Dirichlet forms and harmonic spaces.

Due to the inherently constructive character of axiomatic potential theory, classical logic has no enough ability to offer a proper logical foundation. In this paper we propose the weak commutative linear logics as a logical framework for reasoning about the processes described by potential theory. The logical approach is complemented by an algebraic one. We construct an algebraic theory with models in stochastic analysis, and based on this, and a process algebra in the sense of computer science.

Applications of these in area of hybrid systems, concurrency theory and biomedicine are investigated.

Parts of this paper have been presented, in shorter form, at diverse conferences and workshops. This work represents a common 'umbrella' for all these presentations and offers an extended version for the (some time) very short published materials.

Keywords: *Constructive Stochastic Analysis, Process Algebra, Logic, Continuous Computation*

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Chapter 1 **Introduction**

In recent years a lot of interest was manifested in computation processes having a continuous nature. Examples abound around: hybrid systems, biological inspired computing paradigms such as data mining and neural networks, quantum computation, embedded systems. Continuous aspects can be identified in domain traditionally considered as branches of discrete mathematics: in real time systems the time domain value is often a dense (i.e. continuous in order) set, and Zeno phenomena (i.e. the familiar continuo convergence) should be considered.

Another important computation fields, where continuity is considered, are probabilistic and stochastic computation. Stochastic Petri nets, probabilistic process algebra like PEPA, stochastic hybrid systems use frequently continuous probabilistic distributions. Particular attention was given to this case in model checking.

Behaviour of many systems (like sensors monitoring heart beats, the management of the risks involved in using stocks to fund pensions) can be profoundly affected by stochastic fluctuations and randomness. Significant benefits can be achieved if one can be quantitative, and where possible deeply understand, these systems. For example, the ability to price risk, has radically changed the financial markets, and is at the present time causing a complete rearrangement of the conventional insurance industry. The mathematical study of such systems will almost certainly involve tools with names like martingales, paths spaces, Malliavin calculus, stochastic integrals, measure valued branching processes, stochastic partial differential equations etc. All are tools central to stochastic analysis. The mathematical analysis of stochastic systems, while perhaps a little inaccessible to outsiders, is undergoing quite rapid scientific development. One could say with some justification that Stochastic Analysis has emerged as a core area of late 20th century of mathematics. The Dirichlet Spaces have both deterministic and probabilistic models, and constitutes the most important way of making effective the functional analytic tools in stochastics. Other current researches uses the Dirichlet Spaces in the Markov processes theory, stochastic differential geometry, stochastic differential equations, mechanics and physics. Detailed informations about all of these can be found in Ma and Rockner's book [MR 90].

All these approaches have no constructive logical foundations and no adequate computational model. We could interpret the recent work in computer algebra systems and formalisation of mathematics in higher orders logic (we call now these calculational approaches from now on) as an indirect computational (and thus constructive) approach to mathematics. We can make a few observations here: *(a)* stochastic analysis has not being approached yet by any of these methods; *(b)* the price paid of being very effective is that they are very poor from a foundational perspective; *(c)* the computational model (which is, in most cases, a variant of rewriting systems) is not used explicitly and is not adequate to from a foundational point of view (it says how mathematical -symbolic-calculation and -limited forms of-reasoning can be executed on a machine rather than the actual computational content of a continuous mathematical theory). We look here at a more balanced approach:

- More *constructive* than the classical logic
- Less *effective* but more *foundational* than calculational approaches
- Making explicit a *computational model* for continuous (and especially stochastic) mathematical theories. This model should have the same role the Hilbert machines have for discrete computation: to abstract the operational features enough for a formal logic analysis.

In this work we aim to show that the non-commutative linear logic and the Hilbert machines are suitable abstract tools to a logical approach of these topics.

The Hilbert machines are an extension of classical models of computation with symbols in order to deal with concepts like infiniteness and similarity. In [Wik 98] H. Wiklicky introduced Hilbert machine as a new quantitative computational model. A Hilbert machine operates with data taken from a Hilbert space. Prominent examples of such model of computation include Girard's Geometry of Interaction, Neural Networks and Quantum Computation.

From a logical viewpoint, a fully constructive and foundational powerful logic proved to be linear logic. Linear logic was discovered motivated by profound proof theoretic arguments, and its applications in computing proved it as logic of resources and concurrency. The possibility of using LL for founding continuous mathematics came from its further developments, namely Geometry of Interaction and non-commutative LL. In the original operator formulation of Geometry of Interaction, projections play a very important role. The general C^* algebras may contain no projections other than O and I. In such cases, the Geometry of Interaction becomes a trivial theory. Therefore we are interested in sub-classes of C^* algebras rich in projections. Functional analytic arguments relate the existence of projections to the w-closability property. Such C^* algebras, closed in w-convergence are called von Neumann algebras. Historically, the von Neumann algebras were the first class of operators algebras introduced. The passage from the C^* algebras to the von Neumann algebras correspond in the commutative case to the passage from the algebra of continuous functions to the algebra of measurable functions. In the non-commutative case, this analogy leads to the interpretation of von Neumann algebras as "non-commutative measure theory".

The Dirichlet spaces were introduced in 1959 by A. Beurling and J. Deny [BD 95] as an axiomatic extension of classical Dirichlet integrals.

We associate to each Hilbert machine a Dirichlet space, providing in this way a logical and computational model to each class of applications of Dirichlet spaces.

Another contribution is a unifying approach to nondeterministic and deterministic continuous processes. Nondeterminism is considered by working directly with stochastic processes. This is useful in computing over the real numbers, since continuous distributions can often be more realistic, for instance in modelling economic systems or population growth. A decent formalisation must cover both discrete and continuous distributions.

In the next chapter we present background material used in the rest of the paper. The chapter is divided into three very different sections: first section introduces order causal relations, which are order relations enriched with concepts coming from concurrency theory; the second section presents domain computability for algebras; the third section introduces some basic abstract structures from continuous mathematics, namely von Neuman algebras and Dirichlet forms.

The second chapter presents a computational model for continuous processes, the Hilbert machines, and a logical foundation of these structures in linear logic.

In the third chapter we develop an abstract domain for the processes introduced before. The dynamic of these processes is modelled using the causal order relations introduced in the first chapter. The processes are abstracted via an algebraic axiomatization, suitable to apply the results presented in the first chapter. We show that most of the continuous processes modelled by differential equations and stochastic processes are domain computable.

The last chapter sketches some partial conclusions of this mathematical experiment.

Chapter 2 Background

2.1 Order Relations

Let $\prec$ be an order relation on the set B. We shall use the notations

$$\begin{aligned} \preccurlyeq \quad &= \prec \cup id_{|B}; \\ \succ \quad &= \prec^{-1}; \\ \lessdot \quad &= \prec - \prec^2; \end{aligned}$$

Definition 1 *Define the following algebraic relations on B:*

- the **concurrency** relation: $co \subset B \times B$ is a symmetric relation with $co \cap id_B = \emptyset$.
- the **causal** relation: $li \subset B \times B$ is a binary relation with $li \cap id_B = \emptyset$ such that the following interrelating properties holds

$$\begin{aligned} co \cap li &= \emptyset \\ co \cup li &= B \times B - id_B \end{aligned}$$

Proposition 1 *The following properties holds*

$$\begin{aligned} li &= li^{-1} \\ li &= B \times B - co \\ co &= B \times B - li \end{aligned}$$

Definition 2 *Proximity π and the immediate neighborhood lo relations are defined by*

$$\begin{aligned} \pi &= \{(a, b); \underline{li}_B[a] \subsetneq \underline{li}_B[b]\} \\ lo &= \pi \cup \pi^{-1} \end{aligned}$$

Remark 1 *lo express a notion of locality (even does not contain any topological or metric information) based on concurrency and causality alone.*

Definition 3 *A relation $\varpi \subseteq B \times B$ is a consistent orientation if the following conditions holds:*

$$\begin{array}{ll} \varpi \cup \varpi^{-1} = lo & (\textit{consistency of changes}) \\ \varpi \circ \varpi \subseteq li & (\textit{li between pre- and postset}) \\ \varpi \circ \varpi^{-1} \subseteq \underline{co}_B & (\underline{co}_B \textit{ within preset}) \\ \varpi^{-1} \circ \varpi \subseteq \underline{co}_B & (\underline{co}_B \textit{ within postset}) \end{array}$$

Remark 2 *We shall note*

$$\mathcal{O}(B) = \{\varpi \subseteq B \times B; \varpi \textit{ is a consistent orientation}\}$$

Axiom 1 *$\mathcal{O}(B) \neq \emptyset$ (consistent orientability)*

Axiom 2 *The Basic Concurrency Axioms*

$$\begin{array}{ll} \textit{.li-irreducibility.} & (\forall a, b \in B) : \underline{li}_B[a] = \underline{li}_B[b] \Rightarrow a = b \\ \textit{.co-irreducibility.} & (\forall a, b \in B) : \underline{co}_B[a] = \underline{co}_B[b] \Rightarrow a = b \\ \textit{.li-coherence.} & li^*_B = B \times B \\ \textit{.co-coherence.} & co^*_B = B \times B \\ \textit{.no changes of changes.} & \pi^2 = \emptyset \\ \textit{.re-coherence.} & lo^*_B = B \times B \\ \textit{.local co-transitivity.} & (\forall a \in B) : (co|_{lo[a]})^2 \subseteq \underline{co}_B|_{lo[a]} \\ \textit{.local orientability.} & (\forall a \in B) : (li|_{lo[a]})^2 \subseteq \underline{co}_B|_{lo[a]} \\ \textit{.local extensibility.} & (\forall a \in B) : id|_{lo[a]} \subseteq (\underline{li}_B|_{lo[a]})^2 \end{array}$$

Proposition 2 *We have*

$$\begin{array}{rcl} \pi \cap \pi^{-1} & = & \emptyset \\ \pi \cap id_B & = & \emptyset \\ \pi & \subseteq & li \\ lo & \subseteq & li \\ lo & = & lo^{-1} \\ \pi \circ co & \subseteq & co \\ \underline{li}_B \circ \pi & \subseteq & \underline{li}_B \\ \pi & \subseteq & co \circ li \\ \pi \cap li \circ co & = & \emptyset \\ \pi & = & li - li \circ co \\ (\forall a & \in & B) : lo[a] \neq \emptyset \\ (\forall a & \in & B) : |lo[a]| \geq 2 \end{array}$$

Definition 4 *A partial order $\prec \subseteq B \times B$ is called a causal order iff*

$$\prec \cup \succ = li.$$

Remark 3 *We shall note $\mathcal{O}(B)$ the class of causal orders on B.*

Definition 5 *Let $\prec$ be a causal order. Define*

•(reconstitution of causal relation)

$$li =: \prec \cup \succ \cup id_{|B}$$

•(reconstitution of concurrency relation)

$$co =: \overline{li} \cup id_{|B}$$

• for any $a \in B$:

$$\begin{array}{rcl} {}^{\blacklozenge}a & = & : \{b \in B; b \prec a\}, \\ a^{\blacklozenge} & = & : \{b \in B; b \succ a\} \end{array}$$

• $l \subseteq B$ is a *li-set* iff

$$(\forall a, b \in l) : (a, b) \in li$$

• $l \subseteq B$ is a *line* iff l is maximal w.r.t. li:

$$(\forall a \in B - l), (\exists b \in l) : (a, b) \in (B \times B) - li.$$

Let $L = L(B)$ be the set of lines of B.

• $c \subseteq B$ is a *co-set* iff

$$(\forall a, b \in co) : (a, b) \in co$$

• $c \subseteq B$ is a *cut* iff c is maximal w.r.t. co :

$$(\forall a \in B - c), (\exists b \in l) : (a, b) \in (B \times B) - co.$$

Let $C = C(B)$ be the set of cuts of B

Remark 4 *We have*

- $(a\ li\ b)$ or $(a\ co\ b)$;
- $(a\ li\ b) \& (a\ co\ b) \Leftrightarrow a = b$;
- A is a line iff

$$\begin{array}{ll} (i) & (\forall a, b \in A) : (a \prec b) \ or \ (b \prec a) \ or \ (a = b); \\ (ii) & (\forall b \in B - A), (\exists a \in A) : not(a \prec b \ or \ b \prec a); \end{array}$$

- A is a cut iff

$$\begin{array}{ll} (i) & (\forall a, b \in A) : not(a \prec b \; or \; b \prec a); \\ (ii) & (\forall b \in B - A), (\exists a \in A) : (a \prec b) \; or \; (b \prec a). \end{array}$$

Definition 6 *A Dedekind cut (D-cut for short) is a partition $(A, \overline{A})$ for which*

$$(\forall a \in A \forall b \in \overline{A}) : not(a \prec b).$$

Remark 5 *$(A, \overline{A})$ is a D-cut iff*

$$A = \downarrow A \; and \; \overline{A} = \uparrow \overline{A} \; (A \in D(B)).$$

For $A \subset B$, define $M(A) =: max(A) \cup \min(\overline{A})$.

Definition 7 *If $(A, \overline{A})$ is a D-cut then define*

$$\begin{aligned} Obmax(A) &= : \{a \in Max(A); \forall A' \in D(B) \forall l \in L : a \in Max(A' \cap l) \Rightarrow a \in Max(A')\}; \\ Obmin(A) &= : \{a \in Min(A); \forall A' \in D(B) \forall l \in L : a \in Min(A' \cap l) \Rightarrow a \in Min(A')\}; \\ c(A) &= : Obmax(A) \cup Ob\min(A). \end{aligned}$$

Proposition 3 *Let $A \in D(B)$ and $a \in Max(A)$, $a \in Min(\overline{A})$. We have*

$$\begin{aligned} a &\notin Ob\max(A) \Leftrightarrow \exists b \in B \exists l \in L : (a \prec b) \; and \; (l \cap [a, b] = \{a\}); \\ a &\notin Ob\min(A) \Leftrightarrow \exists b \in B \exists l \in L : (b \prec a) \; and \; (l \cap [b, a] = \{a\}). \end{aligned}$$

Definition 8 *A complete lattice is a partially ordered set in which every subset has a least upper bound and a greatest lower bound. A conditionally complete lattice is a lattice which have the property that every non-void bounded subset has a least upper bound and a greatest lower bound.*

2.2 Type 2 Computability

2.2.1 Types of Constructive Analysis

This report deals with computable analysis. The subject, as its name suggests, represents a marriage between analysis and physics on the one hand, and computability on the other. Computability, of course, brings to mind computers, which are playing an ever larger role in analysis and physical theory. Thus it becomes useful to know, at least theoretically, which computations in analysis and physics are possible and which are not.

The appearance of constructive mathematics as a serious contender for the attention of practising mathematicians may traced to that of L.E.J. Brouwer's doctoral dissertation, "Over de Grondslagen der Wiskunde", in 1907. Although it is true to say that a few individuals (for example Kronecker) had earlier expressed disapproval of the idealistic methods of some of their nineteenth century contemporaries, it is in Brouwer's polemic writings, beginning with the above and continuing throughout the next forty seven years, that the foundations of a precise and practical approach to constructive mathematics were laid.

Unfortunately-and perhaps inevitably, in the face of opposition from men of such stature as Hilbert - Brouwer's intuitionist school became more and more involved in quasi-mystical speculation about the nature of constructive thought, to the detriment of the practice of constructive mathematics itself. Thus it remains for Erret Bishop, in his seminal book "Foundations of Constructive Analysis", to resurrect constructive mathematics in practice and produce some outstanding constructive proofs of important theorems already known in their classical form: in particular many of the fundamental results in theories of Banach spaces, measure and locally compact groups.

Computable analysis is traditionally approached from two different directions. On the one hand, we have the machine-oriented work, where computations are performed on a certain kind of abstract machine. One the other hand, we have the analysis-oriented approach. Here the concepts from classical analysis are effectively presented and used to develop a computability theory for real numbers. This approach to computable analysis comes from the work of Grzegorczyk [Gre 57]. The work of Pour-El and Richards [PR 89] is based on this definition and is now well-established and frequently cited in various communities including by physicists like Penrose [Pen 89].

Recent researches use domain theory as an approach to computable analysis. Weihrauch [Wei 87] has called this approach type 2 (as opposite to recursive analysis, which is called type 1).) Various attempts have been made to use algebraic domains to represent classical spaces in mathematics. Weihrauch [Wei

87] and Schreiber constructed embeddings of Polish spaces into algebraic domains. Stoltenberg-Hansen and Tucker have shown how to represent complete local rings and topological algebras [ST 95] by algebraic domains. They also prove the equivalence between their definition of computable real function with the notion of Pour-El and Richards. Blank has shown how to embed complete metric spaces into algebraic domains.

This work is done in the framework of type 2 constructive analysis. Domain computability provides us a common semantic domain in which we can interrelate the machine-oriented model (Hilbert machines) and the logical model (non-commutative LL).

2.2.2 Computability of Topological Algebras

2.2.2.1 Domain Computability

Definition 9 *A cpo P is an algebraic cpo if for each $x \in P$, the set $approx(x) = \{a \in P_c : a \leq x\}$ is directed and $x = \bigvee approx(x)$.*

Definition 10 *A cpo P is a Scott-Ershov domain , or simply domain, if P is an algebraic cpo such that if the set $\{a, b\} \subseteq P_c$ is consistent in P (i.e. has an upper bound in P) then $a \vee b$ exists in P.*

Definition 11 *A partial order $P = (P; \leq, \perp)$ with least element $\perp$ is a conditional upper semilattice with least element (abbreviated cusl) if whenever $\{a, b\} \subseteq P$ is consistent in P (i.e. has an upper bound in P) then $a \vee b$ exists in P.*

Remark 6 *The compact elements in any Scott-Ershov domain form a cusl.*

Definition 12 *P is called consistently complete if every consistent set has a supremum.*

Remark 7 *Every Scott-Ershov domain is consistently complete.*

Definition 13 *Let $P = (P; \sqsubseteq, \perp)$ be a cusl. Then $I \subseteq P$ is an ideal if*

1. $\perp \in I$
2. if $a \in I$ and $b \leq a$ then $b \in I$
3. if $a, b \in I$ and $a \vee b$ exists then $a \vee b \in I$

Definition 14 *The principal ideal generated by $a \in P$ is defined by $[a] = \{b \in P : b \leq a\}$ and is the smallest ideal containing a.*

Definition 15 *The ideal completion of P is the structure $\bar{P} = (\bar{P}; \subseteq, [\perp])$ where $\bar{P} = \{I \subseteq P : I$ is an ideal$\}$.*

Let P be a cusl. Then the ideal completion $\bar{P} = (\bar{P}; \subseteq, [\perp])$ is a domain. Furthermore, $\bar{P}_c = \{[a] : a \in P\}$ and the map $\iota : P \to \bar{P}_c$ defined by $\iota(a) = [a]$ is an order-preserving bijection.

Theorem 1 *Let $P = (P; \leq, \perp)$ be a domain. Then $\bar{P}_c \cong P$, where $\bar{P}_c$ is the ideal completion of the cusl P_c and where the isomorphism is witnessed by an order-preserving bijection.*

Definition 16 *Let $P = (P; \leq, \perp)$ be a domain. The Scott topology on P is given by : $U \subseteq P$ is open if*

the Alexandrov condition	$(x \in U)\&(x \leq y) \Rightarrow (y \in U)$
the Scott condition	$(x \in U) \Rightarrow (\exists a \in approx(x))(a \in U)$

Remark 8 *A topological base for the Scott topology is given by*

$$\mathcal{B} = \{B_a : a \in P_c\}, \; B_a = \{x \in P : a \leq x\}$$

Proposition 4 *Every open cover of B_a , $a \in P_c$, has a subcover consisting of one open set.*

Corollary 1 *Every element $B_a, a \in P_c$ of $\mathcal{B}$ is compact.*

Remark 9 *This provides us with a technical reason for saying that the basic open sets are 'concrete' or 'finite' and hence that the compact elements of a domain are finite.*

Definition 17 *A function $f : P \to Q$ between domains is called order continuous if f is monotone and preserves suprema of directed sets, that is*

1. $x \leq y \Rightarrow f(x) \leq f(y)$
2. $f(\bigvee A) = \bigvee f(A)$ for each directed set $A \subseteq P$.

Proposition 5 *A function $f : P \to Q$ between domains is continuous in the topological sense with respect to the Scott topology iff one of the following conditions holds:*

1. it is order continuous.
2. $f(x) = \bigvee \{f(a) : a \in approx(x)\}$ for each $x \in P$.

Proposition 6 *For each monotone function $f : P_c \to Q$ has a unique continuous extension $\bar{f} : P \to Q$.*

2.2.2.2 $\Sigma-$Algebras and Computable Algebras

Definition 18 *A single sorted signature consists Σ consists of a sort name s, and a family $\langle \Sigma_k : k \in \mathbb{N} \rangle$ of sets, where each element c of Σ_0 is called a constant symbol (of sort s) and each element σ of Σ_k is called a k-ary function (of type $s^k \to s$).*

A signature is said to be *non-void*, instantiated or sensible if $\Sigma_0 \neq \emptyset$. Typically, a finite signature is written as a list of symbols: $< s; c_1, ...c_p, \sigma_1, ..., \sigma_q >$ where $c_i \in \Sigma_0$ and $\sigma_j \in \Sigma_{k(j)}$ for $1 \leq i \leq p$ and $1 \leq j \leq q$.

Let A be an algebra of signature Σ.

Definition 19 *A numbering of A consists of a set Ω_α of natural numbers and a surjection $\alpha : \Omega_\alpha \to A$ such that for each k-ary operation symbol $\sigma \in \Sigma_k (k \geq 0)$ with corresponding k-ary operation σ_A of A, there exists a total tracking function $f : \Omega_\alpha^k \to \Omega_\alpha$ such that for all $x_1, ..., x_k \in \Omega_\alpha$, $\sigma_A(\alpha(x_1), ..., \alpha(x_k)) = \alpha(f(x_1), ..., f(x_k))$.*

Remark 10 *We obtain a Σ-algebra R_α of natural numbers such that each numbering $\alpha : R_\alpha \to A$ is a Σ-epimorphism. Consider the kernel relation $\equiv_\alpha$ on the number algebra R_α defined by*

$$x \equiv_\alpha y \ \textit{iff} \ \alpha(x) = \alpha(y) \ , \ (\forall x, y \in \Omega_\alpha)$$

Remark 11 *The relation $\equiv_\alpha$ is a Σ-congruence on R and $A \cong R/ \equiv_\alpha$.*

Let $I = (I, \leq)$ be a directed set and $\{A_i : i \in I\}$ be an indexed family of $\Sigma-$algebras and $\phi_i^j : A_j \to A_i$ be a $\Sigma-$homomorphism, for each $i \leq j$.

Definition 20 *An inverse system of $\Sigma-$algebras is a couple $\{A_i : i \in I\}$, $\{\phi_i^j : i \leq j \in I\}$ such that: $\phi_i^i = id_{A_i}$, $\phi_i^j \circ \phi_j^k = \phi_i^k$. An inverse system is said to be surjective if each ϕ_i^j is surjective.*

Definition 21 *The projective or inverse limit of the inverse system $\{A_i : i \in I\}$, $\{\phi_i^j : i \leq j \in I\}$, if it exists, is a $\Sigma-$algebra $\hat{A}$ together with a family of $\Sigma-$homomorphisms $\hat{\phi}_i : \hat{A} \to A_i$, such that for each $i \leq j \in I$, $\hat{\phi}_i = \phi_i^j \circ \hat{\phi}_j$ which is a solution of the following universal problem. If B is a $\Sigma-$algebra and $\psi_i : B \to A_i$ is a family of $\Sigma-$homomorphisms for $i \in I$ such that for each $i \leq j \in I$, $\psi_i = \psi_i^j \circ \psi_j$, then there is a unique $\Sigma-$homomorphism θ making the diagrams below commute.*

$$\begin{tikzcd}
\hat{A} \arrow[rr, dotted, no head, "\theta"] \arrow[dr, "\hat{\phi}_j"] \arrow[ddr, "\hat{\phi}_i"'] & & B \arrow[dl, "\psi_j"'] \arrow[ddl, "\psi_i"] \\
& A_j \arrow[d, "\phi_i^j"] & \\
& A_i &
\end{tikzcd}$$

Definition 22 *An effective numbering α of A is a numbering that consists of a recursive set Ω_α of natural numbers and for each k-ary operation σ_A of A and a recursive tracking function $f : \Omega_\alpha^k \to \Omega_\alpha$.*

Definition 23 *Let $\alpha : \Omega_\alpha \to A$ be an effective numbering of A. The numbering α is called*

1. *a computable numbering* iff the relation $\equiv_\alpha$ is recursive on the recursive set Ω_α; in this case the algebra A is said to be computable under α.
2. *a semicomputable numbering* iff the relation $\equiv_\alpha$ is recursively enumerable on the recursive set Ω_α; in this case the algebra A is said to be *semicomputable under* α.
3. *a cosemicomputable numbering* iff the relation $\equiv_\alpha$ is co-recursively enumerable on the recursive set Ω_α; in this case the algebra A is said to be *cosemicomputable under* α.

Definition 24 *An algebra is computable, semicomputable, or cosemicomputable if there exists a computable, semicomputable, or cosemicomputable numbering for the algebra, respectively.*

2.2.2.3 Locally Compact Hausdorff Algebras

By a topological algebra we mean an algebra whose carrier set is a topological space and whose operations are continuous. The class of locally compact Hausdorff algebras is a large and natural class of algebras and includes many metric algebras (for example the ring of real numbers $\mathbb{R}$).

Proposition 7 *Let X be a locally compact Hausdorff space. Then X is regular. Further the family of compact neighborhoods of each point is a base for its neighborhood system.*

Let $P^{'}$ be a family of non-empty compact subsets of X and let $P = P^{'} \cup \{X\}$. We order P by reverse inclusion, that is

$$F \sqsubseteq F^{'} \Longleftrightarrow F \supseteq F^{'} \;\; (\forall F, F^{'} \in P)$$

Then $P = (P, \sqsubseteq, X)$ is a partial order with least element X.

Definition 25 *The structure $P = (P, \sqsubseteq, X)$ is a cusl of compact neighborhood systems if the following conditions hold:*

i) if $F, F^{'} \in P$ and $F \cap F^{'} \neq \emptyset$ then $F \cap F^{'} \in P$ and
ii) if $U \in \tau$ and $x \in U$ then $(\exists F \in P)(x \in F^{\circ} \& F \subseteq U)$.

The previous proposition tells us that every locally compact Hausdorff space X has a cusl of compact neighborhood systems, namely $\mathcal{H}(X) \cup \{X\}$, where $\mathcal{H}(X)$ is the set of all non-empty compact subsets of X.

Example 1 *For $\mathbb{R}$ we let $P = \{[p, q]; p \leq q, p, q \in \mathbb{Q}\} \cup \{\mathbb{R}\}$.*

The next two theorems are the key results for obtaining domain representability of fundamental structures of stochastic analysis. These results will be applied in Chapter 4.

Theorem 2 *[ST 95] Every topological $\Sigma-$algebra which is locally compact and Hausdorff is domain representable.*

Theorem 3 *[ST 95] Every metric $\Sigma-$algebra is domain representable.*

2.2.2.4 Topologies as Approximation Structures

Consider a set A.

Definition 26 *An approximation for A is a set P with a relation $\ll$ from P to A such that*

$$\begin{array}{rcl} (\forall a & \in & A)(\exists p \in P)(p \ll a) \\ (\forall a, b & \in & A)(a = b \Leftrightarrow \{p \in P; p \ll a\} = \{p \in P; p \ll b\}. \end{array}$$

Definition 27 *(The refinement order) We define the preorder relation $\sqsubseteq$ by*

$$p \sqsubseteq q \Leftrightarrow (\forall a \in A)(q \ll a \Rightarrow p \ll a)$$

Example 2 *Let $A = \mathbb{R}$ and*

$$P = \{[p,q]; p \leq q, p, q \in Q\} \cup \{(-\infty, \infty)\}$$

and the refinement order $\sqsubseteq$ given by the inverse inclusion. The approximation relation $\ll$ from P to $\mathbb{R}$ is given by

$$[p,q] \ll a \Longleftrightarrow a \in [p,q].$$

Remark 12 *$P = (P, \sqsubseteq)$ is a computable structure.*

Example 3 *Let $X = (X, \tau)$ be a T_0 topological space. The relation from $P = \tau$ to $A = X$ defined by*

$$U \ll x \Longleftrightarrow x \in \tau\ ,\ (\forall U \in \tau)(\forall x \in X)$$

is an approximation relation .

Remark 13 *It suffices to consider a topological base $\mathcal{B}$ for τ to obtain an approximation structure $(\mathcal{B}, \supseteq)$.*

Proposition 8 *Let $X = (X, \tau)$ be a T_0 topological space. Then $(\tau, \ll)$ is an approximation for X iff (X, τ) is a T_0-space.*

Let $(P, \ll)$ be an approximation for A.

Definition 28 *(The specialization order). The relation on A defined by*

$$a \lesssim b \Longleftrightarrow (\forall p \in P)(p \ll a \Rightarrow p \ll b)$$

is an approximation relation from A to A. Further, $\lesssim$ is a partial order and $(A, \lesssim)$ is an approximation for A relative to $\lesssim$.

Proposition 9 *Let $X = (X, \tau)$ be a T_0 topological space and let $\lesssim$ be the specialization order on X. Then $\lesssim$ is the discrete order (i.e. $\lesssim$ is $=$) iff X is a T_1-space.*

Proposition 10 *Let $P = (P; \leq, \perp)$ be a domain. Then the specialization order on P coincides with the domain order $\leq$.*

Corollary 2 *$(P_c; \leq)$ is an approximation for P relative to $\leq$.*

2.2.2.5 Domain Computable Algebras

Definition 29 *A structured domain or $\Sigma-$domain for a signature Σ is structure*

$$P = (P; \leq, \perp; x_1, ..., x_p, \Psi_1, ..., \Psi_q)$$

such that

1. $(P; \leq, \perp)$ is a domain;
2. each $x_i \in P$, where P is given by Σ;
3. each Ψ_j is a continuous n_j-ary operation on P, that is $\Psi_j : P^{n_j} \to P$ is continuous, where P^{n_j} is given the product topology, and q and the arities n_j are given by Σ.

Definition 30 *A topological $\Sigma-$algebra $A = (A; a_1, ..., a_p, \sigma_1, ..., \sigma_q)$ is called domain representable by the $\Sigma-$domain $P = (P; \leq, \perp; \hat{a}_1, ..., \hat{a}_p, \hat{\sigma}_1, ..., \hat{\sigma}_q)$ if there is a $\Sigma-$substructure $P_A = (P_A; \hat{a}_1, ..., \hat{a}_p, \hat{\sigma}_1, ..., \hat{\sigma}_q)$of P and a $\Sigma-$epimorphism $v_A : P_A \to A$ which is continuous with respect to the subspace topology of P_A.*

Definition 31 *A domain representation of the representable topological $\Sigma-$algebra A is a triple (P, P_A, v_A) as defined previous.*

Proposition 11 *Suppose that the topological $\Sigma-$algebras A and B are domain representable by the $\Sigma-$domains P and Q respectively. Then the topological $\Sigma-$algebra $A \times B$ is domain representable by the $\Sigma-$domain $P \times Q$.*

Proposition 12 *Suppose that the topological $\Sigma-$algebra A is domain representable by the $\Sigma-$domain P. Then*

1. each topological subalgebra B of A is domain representable by P.
2. each continuous homomorphic image B of A is domain representable by P.

Let us consider a $\Sigma-$algebra A together with a family $\{\equiv_n\}_{n\in\mathbb{N}}$ of separating congruences on A. Let (r_n) be a sequence of strictly decreasing positive real numbers such that $r_n \to 0$. Then define an ultrametric d on A by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ r_n & \text{if } x \neq y, \text{ where } n \text{ is least s.t. } x \not\equiv_n y \end{cases}$$

Similarly we define an ultrametric $\hat{d}$ on $\hat{A} = \underleftarrow{Lim}\ A \,/\, \equiv_n$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ r_n & \text{if } x \neq y, \text{ where } n \text{ is least s.t. } \hat{\phi}_n(x) \neq \hat{\phi}_n(y) \end{cases}$$

Proposition 13 *$\hat{A}$ is the ultrametric completion of A i.e. the following conditions holds:*

1. $\hat{A}$ is a complete ultrametric space.
2. The unique $\Sigma-$embedding $\theta : A \to \hat{A}$ is an isometry with respect to d and $\hat{d}$.
3. $\theta[A]$ is dense in $\hat{A}$.

Theorem 4 *Suppose A is an ultrametric $\Sigma-$algebra with non-expansive operations. Then there is a complete ultrametric $\Sigma-$algebra $\hat{A}$ with non-expansive operations, and a continuous $\Sigma-$embedding $\theta : A \to \hat{A}$ such that each $x \in \hat{A}$ is the limit of some sequence $(\theta(a_n))$ where (a_n) is a Cauchy sequence in A. In particular, $\theta[A]$ is dense in $\hat{A}$.*

Proof. We may without loss of topological generality assume that the ultrametric d on A is bounded. Define a family of separating congruences $\{\equiv_n\}_{n\in\mathbb{N}}$ by

$$x \equiv_n y \Leftrightarrow d(x,y) \leq r_n$$

where (r_n) is a strictly decreasing sequence of positive real numbers such that $r_n \to 0$. Then we obtain $\hat{A} = \underleftarrow{Lim}\ A/ \equiv_n$ and the $\Sigma-$homomorphism $\theta : A \to \hat{A}$ by the construction of the previous theorem. To see that θ is continuous, just observe that the identity on A is a homeomorphism between (A, d) and $(A, d^{'})$ where $d^{'}$ is the ultrametric on A obtained from $\{\equiv_n\}_{n\in\mathbb{N}}$. This observation also suffices for proving that $\theta[A]$ is dense in $\hat{A}$.

Corollary 3 *Each ultrametric $\Sigma-$algebra with non-expansive operations is domain representable.*

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2.3 Fundamental Structures in Functional Analysis

2.3.1 von Neumann Algebras

Let us fix H a complex Hilbert space and let $\mathcal{B}(H)$ be set of bounded linear operators on H.

Definition 32 *A selfdual (or semipolar) cone $\mathbb{S}$ in the complex Hilbert space H is a subset satisfying the property*

$$\{a \in H, \forall\sigma \in \mathbb{S} :< a, \sigma >\geq 0\} = \mathbb{S}.$$

$\mathbb{S}$ is then a closed,convex cone and H is the complexification of the real subspace

$$H^{\Lambda} =: \{a \in H :< a, \sigma >\in \mathbb{R}, \forall\sigma \in \mathbb{S}\}$$

whose elements are called Λ-real : $H = H^{\Lambda} \oplus iH^{\Lambda}$. Such $\mathbb{S}$ gives to a structure of ordered Hilbert space on H^{Λ} (denoted by $\leq$) and to an antiunitary involution Λ on H, which preserves $\mathbb{S}$ and H^{Λ} : $\Lambda(a + ib) =: a - ib$ for all $a, b \in H^{\Lambda}$.

Definition 33 *An abstract $C^*-algebra$, denoted by $\mathcal{B}$, is a complex Banach algebra with $1 \in \mathcal{A}$ together with an involution * with the followings properties:*

$$1^* = 1,\ \alpha^{**} = \alpha,\ (\alpha + \beta)^* = \alpha^* + \beta^*,$$
$$(\beta\alpha)^* = \alpha^*\beta^*,\ ||\alpha|| = \| \alpha^* \|,\ ||\alpha\alpha^*|| = \| \alpha \|^2$$

Definition 34 *An concrete $C^*-algebra$ is an abstract algebra $C^*-algebra$ $\mathcal{B} \subseteq \mathcal{B}(H)$ on a fixed Hilbert space H.*

An abstract $C^*-algebra$ is thus an abstract structure given by abstract axioms, ignoring the underlying Hilbert space.

Definition 35 *The commutant of a subset $\mathcal{M}$ of an abstract $C^*-algebra$ $\mathcal{B}$, denoted by $\mathcal{M}'$, is the set*

$$\mathcal{M}' = \{\alpha \in \mathcal{B}; \alpha\mu = \mu\alpha \text{ , for any } \mu \in \mathcal{M}\}.$$

We can define also $\mathcal{M}''$ the bicommutant of $\mathcal{M}$ by $\mathcal{M}'' = (\mathcal{M}')'$.

Remark 14 *It can easily check that:*
(i) $\mathcal{M} \subseteq \mathcal{M}''$
(ii)The commutant $\mathcal{M}'$ is an operator algebra which contains the identity operator $1 \in \mathcal{B}(H)$; moreover $\mathcal{M}'$ is weakly-closed.

Definition 36 *An abstract von Neumann algebra $\mathcal{A}$ is an abstract sub-C^*-algebra of an abstract C^*-algebra with $\mathcal{A} = \mathcal{A}''$.*

Definition 37 *A concrete von Neumann algebra $\mathcal{A}$ is a C^*-algebra of operators $\mathcal{A} \subset \mathcal{B}(H)$ with $\mathcal{A} = \mathcal{A}''$.*

Again the difference between an abstract von Neumann algebra and a concrete one consists only in ignoring the underlying Hilbert space and presenting it in an abstract, axiomatic manner.

Definition 38 *(Special operators of an abstract von Neumann algebra) An operator α of a von Neumann algebra $\mathcal{M}$ is called*

- *unitary if $\alpha\alpha^* = \alpha^*\alpha = 1$.*
- *hermitian $\alpha = \alpha^*$*
- *projector if α is hermitian and $\alpha^2 = \alpha$. We denote by $\Pi_{\mathcal{M}}$ the set of all projections of $\mathcal{M}$*
- *partial isometry if $\alpha\alpha^*$ is a projector*
- *symmetry if α is hermitian and unitary*
- *partial symmetry if α is hermitian and partial isometry*

Proposition 14 *(Properties for the special operators [KR 83])*

- if α and β are projectors that commutes then $\alpha\beta$ is a projector too.
- if α is a projector then $\beta = 2\alpha - 1$ is a symmetry
- if α is a symmetry then $\beta = 1/2.(\alpha + 1)$ is a projector
- if α is a projector then $\alpha\alpha^*$ is a projector called the *initial projector* and $\alpha^*\alpha$ is a projector too, called the *final projector*.

A crucial property of von Neumann algebras is that they are rich in projections (and thus in partial isometries and partial symmetries).
Let us denote by $\Pi_{\mathcal{M}}$ the set of all projections of the von Neumann algebra $\mathcal{M}$.

Proposition 15 *([KR 83])$\Pi_{\mathcal{M}}$ is a complete lattice in $\mathcal{M}$.*

Definition 39 *A standard form $(\mathcal{A}, H, \mathbb{S}, \Lambda)$ of the von Neumann algebra $\mathcal{A}$ (acting faithfully on the Hilbert space H) consists of a selfdual,closed, convex cone $\mathbb{S}$ in H, fulfilling the properties:*
(i) $\Lambda\mathcal{A}\Lambda = \mathcal{A}'$
(ii) $\Lambda\alpha\Lambda = \alpha^*, \forall\alpha \in C(\mathcal{A}) =: \mathcal{A} \cap \mathcal{A}'$ (the center of $\mathcal{A}$);
(iii) $\Lambda\sigma = \sigma, \forall\sigma \in \mathbb{S}$;
(iv) $\alpha\Lambda\alpha\Lambda(\mathbb{S}) \subseteq \mathbb{S}, \forall\alpha \in \mathcal{A}$.

We will show in the Section 5 that a (non-symmetric) Dirichlet space can be associated to every standard form of a von Neumann algebra.

2.3.2 Dirichlet Forms

Dirichlet forms have the origin in the energy method used by Dirichlet to solve the Dirichlet problem from the classical electrostatic. In the seminal papers, in 1959, Beurling and Deny [BD 59] set the study of the Dirichlet forms, identifying a very important contraction property, called the Markov property. The interest of studying Dirichlet forms has increased when the connection with the stochastic analysis has been set up. In seventies, Fukushima and Silverstein set, in their papers (see [Fuk 80]), an explicit connection with stochastic analysis. These papers offer a fruitful correspondence between the Dirichlet forms and the symmetric Markov processes. In '80s, the request for Markov processes study tools led to extensions of the Fukushima result from the local spaces to infinite dimensional spaces. The most general expression of relationships between Dirichlet spaces and stochastic analysis was done by Ma and Rŏckner [MR 92]. In conclusion, the Dirichlet forms play a prominent role in different mathematical fields as: stochastic, differential geometry, PDEs and they establish an effective connection among the different applications. This fact is possible because the Dirichlet forms permit a netrivial development of the stochastic analysis, under hypothesis of minimal regularity, for instance in very irregular spaces without differential structure like fractals or path spaces.

Let X be a separable locally compact space and μ a Radon measure, strictly positive on X (i.e., μ is a measure defined on the Borel $\sigma-$algebra $\mathcal{B}(X)$ of X, finite for the compact sets, strictly positive for the nonempty open sets and $supp(\mu) = X$). We denote by $L^2(X, \mu)$ the Hilbert space of all squared integrable functions defined on X with real values, provided with the inner product: $[u, v] := \int_X uv \; d\mu$.

Let H be a real Hilbert space with the inner product $(,)$.

Definition 40 *$\mathcal{E}$ is called* symmetric form *on H if the following conditions are satisfied:*

(a)$\mathcal{E}$:$D[\mathcal{E}]\times D[\mathcal{E}] \to \mathbb{R}$, where $D[\mathcal{E}]$ is a dens linear subspace of H,
(b)$\mathcal{E}(u, v) = \mathcal{E}(v, u), \mathcal{E}(u + v, w) = \mathcal{E}(u, w) + \mathcal{E}(v, w), a\mathcal{E}(u, v) = \mathcal{E}(au, v)$,
(c)$\mathcal{E}(u, u) \geq 0$;
for all $u, v, w \in D[\mathcal{E}], a \in \mathbb{R}$. $D[\mathcal{E}]$ is called the domain of $\mathcal{E}$.

If $\mathcal{E}$ is a symmetric form on H, for any $\alpha > 0$, one can define a new symmetric form on H: $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)$, $u, v \in D[\mathcal{E}]$, $D[\mathcal{E}_\alpha] = D[\mathcal{E}]$. $D[\mathcal{E}]$ is a pre-Hilbert space with the inner product $\mathcal{E}_\alpha$. $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ determine equivalent metrics on $D[\mathcal{E}]$ for different $\alpha, \beta > 0$.

Definition 41 *If $D[\mathcal{E}]$ is complete w.r.t. $\mathcal{E}_\alpha$ ($\alpha > 0$), then $\mathcal{E}$ is* closed, *i.e., we have:*

$u_n \in D[\mathcal{E}], \mathcal{E}_1(u_n - u_m, u_n - u_m) \to 0, n, m \to \infty \Longrightarrow$
$\Longrightarrow \exists u \in D[\mathcal{E}], \mathcal{E}_1(u_n - u, u_n - u) \to 0, n \to \infty$
If $\mathcal{E}^{(1)}$ and $\mathcal{E}^{(2)}$ are two symmetric forms, $\mathcal{E}^{(2)}$ is an *extension* of $\mathcal{E}^{(1)}$ if $D[\mathcal{E}^{(1)}] \subset D[\mathcal{E}^{(2)}]$ and $\mathcal{E}^{(1)} = \mathcal{E}^{(2)}$ on $D[\mathcal{E}^{(1)}]\times D[\mathcal{E}^{(1)}]$.
In the following, the Hilbert space, H, will be $L^2(X, \mu)$.

Definition 42 *A symmetric form $\mathcal{E}$ on $L^2(X, \mu)$ is called* Markovian *it satisfies the following condition:*

(**PM**)For all $\varepsilon > 0$, there exists a real function φ_ε such that
(i)$\varphi_\varepsilon(t) = t, \forall t \in [0, 1]$, $-\varepsilon \leq \varphi_\varepsilon(t) \leq 1 + \varepsilon, \forall t \in \mathbb{R}$, and $0 \leq \varphi_\varepsilon(t') - \varphi_\varepsilon(t) \leq t' - t$ if $t < t'$,
(ii)$u \in D[\mathcal{E}] \Rightarrow \varphi_\varepsilon(u) \in D[\mathcal{E}], \mathcal{E}(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \leq \mathcal{E}(u, u)$.

Definition 43 *If $\mathcal{E}$ is a symmetric form on $L^2(X, \mu)$, then the* unit contraction (resp. the normal contraction) operates *on $\mathcal{E}$ if the condition (UC) (resp. (NC)) holds:*

(**UC**) $u \in D[\mathcal{E}]$,$v = (0 \vee u) \wedge 1 \Rightarrow v \in D[\mathcal{E}]$, $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$;
(**NC**) $u \in D[\mathcal{E}]$, v is a normal contraction of $u \Rightarrow v \in D[\mathcal{E}]$, $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$.
A function v is called *normal contraction* of a function u if

$$|v(x)| \leq |u(x)| \, \forall x \in X; |v(x) - v(y)| \leq |u(x) - u(y)| \, \forall x, y \in X.$$

Remark 15 $(NC) \Rightarrow (UC) \Rightarrow (PM)$.

These conditions are equivalent if the symmetric form $\mathcal{E}$ is closed [Fuk 80].

Sometimes it is convenient to denote $D[\mathcal{E}]$ by $\mathcal{F}$ and the symmetric form by $(\mathcal{E}, \mathcal{F})$.

A nonnegative closed symmetric form $(\mathcal{E}, \mathcal{F})$ on $L^2(X, \mu)$ is called *Dirichlet form (* or *Dirichlet space* relative to $L^2(X, \mu)$) if it is Markovian.

Definition 44 *A family $(P_t)_{t>0}$ of linear operators on B with $D(P_i) = B$ for all $t > 0$ is called a strongly continuous contraction semigroup (on B) if*

1. $P_t P_s = P_{t+s}$ *for all* $t, s > 0$ *(semigroup property)*

2. P_t *is a contraction on* B *for all* $t > 0$

3. $\lim_{t\to} P_t u = u$ *for all* $u \in B$ *(strong continuity)*
   ($\Leftrightarrow t \longmapsto P_t u$ *is continuous on* $[0, \infty)$ *for all* $u \in B$*, with* $P_0 = Id_B$*).*

Given a strongly continuous contraction semigroup on B $(P_t)_{t>0}$, the linear operator $(L, D(L))$ on B defined by

$$D(L) = \{u \in B / \lim_{t\downarrow 0} \frac{1}{t}(P_t u - u) exists\},$$

$$Lu = \lim_{t\downarrow 0} \frac{1}{t}(P_t u - u), u \in D(L)$$

is called the *infinitesimal-generator* of $(P_t)_{t>0}$.

The *resolvent* $\mathcal{V} = (G_\alpha)_{\alpha>0}$ associated to the semigroup $(P_t)_{t>0}$ is

$$G_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt, \forall f \in L^2(X, \mu), x \in X$$

Definition 45 *A bounded linear operator S on $L^2(X;\mu)$ is called Markovian if $0 \le Su \le 1 \quad \mu - a.e.$ whenever $u \in L^2(X, \mu)$, $0 \le u \le 1$ μ-a.e.*

Remark 16 *Using some functional analysis results we can take $-L$, being the unique selfadjoint operator, positively defined which corresponds to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. The Hille-Yosida-Philips theorem implies the existence of the semigroup $(P_t)_{t>0} = (e^{tL})_{t>0}$ and the resolvent $(G_\alpha)_{\alpha>0} = (\alpha I - L)^{-1}_{\alpha>0}$ associated to $L \le 0$*

The Markov property of a Dirichlet form can be characterized in terms of semigroup and resolvent (so in terms of a Markov process) by the following result:

Theorem 5 (Fuk 80) *Let $\mathcal{E}$ be a closed symmetric form on $L^2(X;\mu)$. Let $\{T_t, t > 0\}$ and $\{G_\alpha, \alpha > 0\}$ be the strongly continuous resolvent on $L^2(X;\mu)$ which are associated with $\mathcal{E}$ .Then the next five conditions are equivalent to each other:*

(a) T_t is Markovian for each $t > 0$.
(b) αG_α is Markovian for each $\alpha > 0$.
(c) $\mathcal{E}$ is Markovian
(d) The unit contraction operates on $\mathcal{E}$.
(e) Every normal contractions operates on $\mathcal{E}$.

A semigroup (resp.a resolvent) on $L^2(X;\mu)$ satisfying conditions (a),(resp.(b)) is called a Markovian semigroup (resp. a Markovian resolvent).

Chapter 3 Logical and Computational Models

3.1 Hilbert Machines

The Hilbert machines are an attempt of reasoning with quantitative statements. Intuitively, they are mathematical machines whose symbols are interpreted as vectors in some vector space. As a consequence, their transition functions must be compatible with the algebraic structure of data space, i.e. they are linear operators. Moreover, in order to treat the issue of similarity, the data space is enriched with a compatible topology, yielding continuous transition functions. A well-known result from analysis characterizes the continuity of linear operators in terms of boundness of their norm. Thus it is natural to consider a topology on the data space generated by a Hilbertian structure. In this way we get bonus a kind of "expansion" factor associated to each transition function: its norm.

Definition 46 *A linear machine is a structure* (I, O, Q, T) *where*

- I is the input vector space
- O is the output vector space
- Q is the state vector space
- T is linear transition map defined by

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix} : Q \times I \to Q \times O$$

  with $T_1 : Q \to Q$, $T_2 : Q \to O$, $T_3 : I \to Q$, $T_4 : I \to O$.

Definition 47 *The execution of a linear machine at the 'moment' k on the input vector i_k at the vector state q_k consists in the state vector q_{k+1} and the output vector o_{k+1} defined by the linear equations* $q_{k+1} = T_1 q_k + T_3 i_k \quad , \quad o_{k+1} = T_2 q_k + T_4 i_k$

Remark 17 *If we consider a 'continuous' time in the previous definition, e.g. the interval* $(0,1)$, *we obtain the very important concept of semigroup of operators (although in the particular case when all the semigroup components are the same) as defining a continuous sequential evolution (a line in the language of Petri nets processes) of a linear machine.*

Definition 48 *A concrete Hilbert machine is a linear machine defined on a concrete Hilbert space and a concrete continuous operator on this space.*

Example 4 *We highly recommend the papers [Wik 98] and [Wik 96] for detailed examples of Hilbert machines, which include familiar computer science concepts like the classical finite automata, neural networks, quantum computation and linear logic.*

Definition 49 *An abstract Hilbert machine is given by an element of a von Neumann algebra* $\mathcal{A}$.

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3.2 The Weak Commutative Linear Logic ($\mathcal{WCLL}$)

We use the weak commutative linear logic as formulated by M.C. Abrusci in [Abr 9].

The sequent calculus

$$\frac{\vdash \Gamma, A}{\vdash A, \Gamma}\ (Cyclic\ Exchange)$$

$$Basic\ Rules \begin{cases} Identity & \dfrac{}{A \Rightarrow A}(id) \\ \\ Cut & \dfrac{\Gamma \Rightarrow \Delta_1, A, \Delta_2 \quad , \ \Gamma_1, A, \Gamma_2 \Rightarrow \Delta}{\Gamma\ \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta, \Delta_2}(cut) \end{cases} \begin{cases} \Delta_1{=}\Gamma_2{=}\emptyset\ or \\ \Delta_2{=}\Gamma_1{=}\emptyset\ or \\ \Gamma_1{=}\Gamma_2{=}\emptyset\ or \\ \Delta_1{=}\Delta_2{=}\emptyset \end{cases}$$

$$\begin{array}{lll}
(_)^{\perp}\ Rules & \dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A^{\perp}, \Delta}(^{\perp}(_), R), & \dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma, A^{\perp} \Rightarrow \Delta}(^{\perp}(_), L). \\
^{\perp}(_)\ Rules & \dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, ^{\perp} A}(^{\perp}(_), R), & \dfrac{\Gamma \Rightarrow A, \Delta}{^{\perp}A, \Gamma \Rightarrow \Delta}(^{\perp}(_), L). \\
1 - Rules & \dfrac{}{\Rightarrow 1}(1, R), & \dfrac{\Gamma_1, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, 1, \Gamma_2 \Rightarrow \Delta}(1, L). \\
T - Rule & \dfrac{}{\Gamma \Rightarrow \Delta_1, T, \Delta_2}(T) &
\end{array}$$

$$\otimes - Rules \left\{ \begin{array}{l}
\dfrac{\Gamma_1 \Rightarrow \Delta_1, A, \Delta_2 \quad , \ \Gamma_2 \Rightarrow \Delta_3, B, \Delta_4}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_3, \Delta_1, A \otimes B, \Delta_4, \Delta_2}(\otimes, R) \quad \left\{ \begin{array}{l} \Delta_3{=}\Gamma_2{=}\emptyset\ or \\ \Delta_2{=}\Gamma_1{=}\emptyset\ or \\ \Delta_3{=}\Delta_2{=}\emptyset \end{array} \right. \\
\dfrac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, A \otimes B, \Gamma_2 \Rightarrow \Delta}(\otimes, L)
\end{array} \right.$$

$$|| - Rules \left\{ \begin{array}{l}
\dfrac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, A||B, \Delta_2}(||, R) \\
\dfrac{\Gamma_1, A, \Gamma_2 \Rightarrow \Delta_1 \quad , \ \Gamma_3, B, \Gamma_4 \Rightarrow \Delta_2}{\Gamma_3, \Gamma_1, A||B, \Gamma_4, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}(cut) \quad \left\{ \begin{array}{l} \Delta_1{=}\Gamma_2{=}\emptyset\ or \\ \Delta_2{=}\Gamma_3{=}\emptyset\ or \\ \Gamma_3{=}\Gamma_2{=}\emptyset\ or \end{array} \right.
\end{array} \right.$$

$$\& - Rules \left\{ \begin{array}{l}
\dfrac{\Gamma \Rightarrow \Delta_1, A, \Delta_2 \quad \Gamma \Rightarrow \Delta_1, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, A\&B, \Delta_2}(\&, R) \\
\dfrac{\Gamma_1, A, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, A\&B, \Gamma_2 \Rightarrow \Delta}(\&, L1) \quad \dfrac{\Gamma_1, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, A\&B, \Gamma_2 \Rightarrow \Delta}(\&, L2)
\end{array} \right.$$

$$\oplus - Rules \left\{ \begin{array}{l}
\dfrac{\Gamma \Rightarrow \Delta_1, A, \Delta_2}{\Gamma \Rightarrow \Delta_1, A \oplus B, \Delta_2}(\oplus, R1) \quad \dfrac{\Gamma \Rightarrow \Delta_1, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, A \oplus B, \Delta_2}(\oplus, R2) \\
\dfrac{\Gamma_1, A, \Gamma_2 \Rightarrow \Delta \quad , \Gamma_1, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, A \oplus B, \Gamma_2 \Rightarrow \Delta}(\oplus, L)
\end{array} \right.$$

Definition 50 *A weak commutative phase space*

$$< P, \circ, 1, \perp >$$

is a noncommutative monoid $< P, \circ, 1 >$ *(whose elements are called phases) provided with a cyclic set* $\perp \subset P$ *i.e.*

$$a \circ b \in \perp \quad \textit{iff}\ b \circ a \in \perp$$

of elements called antiphases.

Definition 51 *We define the binary operations on the sets of phases* $\mathcal{P}(P)$ *(the power set of* P*)*

$$\begin{array}{llll}
\textit{Composition} & A \circ B & =: & \{a \circ b \mid a \in A \textit{ and } b \in B\}, \\
\textit{Linear Postimplication} & A \multimap B & =: & \{b \mid \forall a \in A\ b \circ a \in B\}, \\
\textit{Linear Retroimplication} & B \twoheadleftarrow A & =: & \{b \mid \forall a \in A\ a \circ b \in B\}, \\
\textit{Intersection} & A \cap B & =: & \{a \mid a \in A \textit{ and } a \in B\}, \\
\textit{Union} & A \cup B & =: & \{a \mid a \in A \textit{ or } a \in B\},
\end{array}$$

Proposition 16 $< \mathcal{P}(P), \circ, 1 >$ *is also an weak commutative phase space.*

We are particularly interested in non-commutative linear logic as a logical framework for studying the quantitative and continuous computation. Following the Wiclicky's previously cited papers, we quickly recall how we can encode the formulas of a computational logic as vectors in a Hilbert space. We start, as usual, with a fixed set of predicates (the 'qualities') $\mathbb{S}_u$ and consider the formulas F, in the disjunctive normal form, constructed over the qualities: $F = F_1 \vee F_2 \vee ... \vee F_a$ with $F_j = P_{u_1} \wedge P_{u_2} \wedge ... \wedge P_{u_b}$. This formula will be encoded as a set of vectors sets $V = \{v_1, ..., v_a\}$, each vector set $v_j = (v_{u_1}, v_{u_2}, ..., v_{u_b})$ representing the formula F_j, where each v_u is 1 whenever the predicate P_u is true and 0 otherwise. In the case of linear logic, each 'resource' P_u comes with a multiplicity (i.e. the counting of available copies of a some resource) which can be considered a real number (with the negative sign meaning a debt of a resource). Based on this encoding of formulas, a linear proof can be represented as continuous operator (or a pair of continuous operators in Girard's Geometry of Interaction [Gir 88]-[22]) on a Hilbert space, i.e. as a Hilbert machine.

3.3 Local Dirichlet Spaces

Definition 52 *A local Dirichlet space (Beurling, Deny [BD 9]) is a Hilbert Space $(H, <>)$ lattice ordered by a linear order $\leq$ in a compatible manner, i.e. the order duality implies the Hilbertian duality $s \wedge t = 0 \Rightarrow \langle s, t \rangle = 0$. We denote by H^+ the set of non-negative elements of H.*

Important Examples: l^2 and L^2

Define the temporal moments set T as a total ordered set, . Let $\mathcal{S}$ be the set of real values indexed by time

$$s = \{s_{(t)} \ , t \in T\} \Leftrightarrow s : T \to \mathbb{R}.$$

such that $Es = \int\limits_{t \in T} s^2_{(t)} dm(t) < \infty$, where T is a locally compact separable Hausdorff space (complicate definition for a set of times; in practice T will be a very simple value set) m is Radon measure on T (which will be degenerated in the most examples). $(\mathcal{S} \ , <>)$ is a Hilbert space under the definition

$$< s, s^{'} > =: \int\limits_{t \in T} s_{(t)} s^{'}_{(t)} dt$$

The space $L^2[a, b]$ is defined as the space $\mathcal{S}$ with $T = [a, b]$ for $a, b \in \mathbb{R}$. The space l^2 is defined as the space $\mathcal{S}$ with $T = \mathbb{N}$. We term *signals* the elements of this space. Note that l^2 and L^2 are Hilbertian the same, but they are very different as ordered structures (Birkhoff [Bir 9]). We can easily see that l^2 is a discrete lattice, whilst L^2 has a continuous order.

Define the temporal moments set T as a total ordered set with t_i the minimal element and t_f the maximal element.

Definition 53 *A signal is a family of real values indexed by time $s = \{s_{(t)} \ , t \in T\} \Leftrightarrow s : T \to \mathbb{R}$ with finite energy $Es = \int\limits_{t \in T} s^2_{(t)} dt < \infty$.*

We shall note by $\mathcal{S}$ the set of all the signals. $(\mathcal{S} \ , <>)$ is a Hilbert space under the definition

$$< s, s^{'} > =: \int\limits_{t \in T} s_{(t)} s^{'}_{(t)} dt$$

and a local Dirichlet space under pointwise order $(s \leq s' \Leftrightarrow s_t \leq s'_t, \forall t \in T)$.

Definition 54 *A discrete signal is a signal $s = \{s_{(n)} \ , n \in \mathbb{N}\} \Leftrightarrow s : \mathbb{N} \to \mathbb{R}$ for which $T = \mathbb{N}$. In this case $E_s = \sum\limits_{n=1}^{\infty} s^2_{(n)} < \infty$. We denote by $\mathcal{DS}$ the set of all the signals. $(\mathcal{DS} \ , <>)$ is a local Dirichlet space with the inner product having the expression $< s, s^{'} > =: \sum\limits_{n=1}^{\infty} s_{(n)} s^{'}_{(n)} dt$.*

Definition 55 *A continuous signal is a signal for which $T = [0, 1] : s : [0, 1] \to \mathbb{R}$. In this case $Es = \int_0^1 s^2(t) dt$.*

Definition 56 *A couple of interaction generators is a pair (∂, δ) of partial isometries such that*

$$\partial^* \partial = \delta \delta^* = 1$$

$$\partial \partial^* + \delta \delta^* = 1$$

Remark 18 *Each of $\partial\partial^*, \delta\delta^*$ is a projector. In the case of C^*-algebras only, as considered by J.Y. Girard and H. Wiklicky, there are possible situations when 1 and 0 are the only projections, so there are no interaction generators. This is an important reason to consider more well-behaved C^*-algebras like the von Neumann algebras, which are rich in projections, so in interaction generators too.*

THE DISCRETE CASE

Being more intuitive, we detail the $\mathcal{DS}$ space.

Example 5 *Let $e_n(m) = \begin{cases} 1 & , m = n \\ 0 & , m \neq n \end{cases}$ and $s =: \sum\limits_{n=1}^{\infty} s_{(n)} e_{(n)} = (s_1, s_2, ...)$,*

$t =: \sum_{n=1}^{\infty} t_{(n)} e_{(n)} = (t_1, t_2, ...)$ be discrete signals. Define operators ∂ and δ by

$$\partial\ (s) =: \sum_{n=1}^{\infty} s_{(n)} e_{(2n)} \quad , \quad \delta\ (s) =: \sum_{n=1}^{\infty} s_{(n)} e_{(2n+1)}$$

Then

$$\partial^*\ (s) =: \sum_{n=1}^{\infty} s_{(2n)} e_{(n)} \qquad \delta^*\ (s) =: \sum_{n=1}^{\infty} s_{(2n+1)} e_{(n)}$$

Thus $\partial\partial^*(s) = (0, s_2, 0, s_4, 0, ...)$, $\delta\delta^*(t) = (t_1, 0, t_3, 0, ...)$, $\partial\partial^* s + \delta\delta^* t = (t_1, s_2, t_3, s_4, ...)$

In order to obtain a connection with the weak commutative linear logic, we define a model of a weak commutative phase space in a local Dirichlet space. For this, we define the set of phases P as the set of all partial isometries on a fixed Hilbert space H and $\perp_P$ as the set of all nilpotent operators.

Definition 57 *We define the dualisation relation*

$$\alpha \perp_P \beta \Leftrightarrow \alpha\beta \text{ is nilpotent} \Leftrightarrow \alpha = \beta^{\perp} \Leftrightarrow \alpha^{\perp} = \beta$$

Definition 58 *We define*

$$\begin{array}{lll} A \otimes B & =: & \{\partial\alpha\partial^* + \delta\beta\delta^* \mid \alpha \in A \text{ and } \beta \in B\}^{\perp\perp} \\ !A & =: & \{1 \otimes \alpha \mid \alpha \in A\}^{\perp\perp} \\ A \multimap B & =: & (A \otimes B^{\perp})^{\perp} \\ A || B & =: & (A^{\perp} \otimes B^{\perp})^{\perp} \\ A \& B & =: & \{\alpha \mid \alpha \in A \text{ and } \alpha \in B\}^{\perp\perp} = A \cap B \\ A \oplus B & =: & \{a \mid \alpha \in A \text{ or } \alpha \in B\}^{\perp\perp} = (A \cup B)^{\perp\perp} \\ A \Rightarrow B & =: & (!A) \multimap B \end{array}$$

We can formulate the main result of this section.

Theorem 6 *For every abstract von Neumann algebra we can associate a local Dirichlet space.*

Remark 19 *The relevance of this result is that we can use the Hilbert machines as a computational model and the weak commutative linear logic as a logical framework for all the models of local Dirichlet spaces.*

Proof. The proof is a particular combination of the general results exposed in [BB 01], [Ito 9], [7] and [14]. But the construction requires some pieces of an algebraic theory, exposed briefly in the next section.
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3.4 Linear Logic Theory of Systems

Let ∂ and δ be two operators which satisfy $(P1)$ and $(P2)$.

Definition 59 *An abstract continuous system (ACS for short) is a pair $\Sigma = (\sigma, \omega)$, where σ and ω are operators in some matrix algebra $\mathcal{M}_{2m+n}(B(H))$ and*

1. *σ is a partial isometry, the coefficients $\sigma_{i,j}$ being weakly nilpotent (i.e. the sequence $(\sigma^n_{i,j})$ converges weakly to 0)*
2. *ω is the partial symmetry exchanging indices 1 and 2, 3 and 4,...,$2m-1$ and $2m$.*

$$\omega = \begin{cases} \omega_{2i,2i-1} = \omega_{2i-1,2i} = 1 & , i \in [m] \\ 0 & , otherwise \end{cases} \tag{3.1}$$

(ω is hermitian and satisfies $\omega^3 = \omega$).

Definition 60 *The abstract continuous process (acp for short) $ACP(\Sigma)$ associated to the ACS $\Sigma = (\sigma, \omega)$ is defined by*

$$ACP(\Sigma) = (1-\omega^2).\sigma.(1-\omega\sigma)^{-1}.(1-\omega^2) \tag{3.2}$$

Remark 20 *Let $\theta = 1-\omega^2$. Then θ is a projector.*

Definition 61 *For any acp $ACP(\Sigma)$, $\Sigma = (\sigma, \omega)$, we define its execution*

$$RUN(\Sigma) = \sigma.(1-\omega\sigma)^{-1} \tag{3.3}$$

Remark 21 *We have $ACP(\Sigma) = \theta.RUN(\Sigma).\theta$. Thus $RUN(\Sigma)$ memorizes the computation steps. For instance, if $\sigma^{'} = RUN(\Sigma)$, then*

$$\sigma^{'}.(1-\omega\sigma) = \sigma \Leftrightarrow \sigma^{'} = \sigma + \sigma^{'}\omega\sigma$$

and in the same way, $\sigma^{'} = \sigma + \sigma\omega\sigma^{'}$ (so $\sigma^{'}\omega\sigma = \sigma\omega\sigma^{'}$). From this we get $\sigma^{'} = \sigma.(1+\omega\sigma^{'})$, hence

$$\sigma = \sigma^{'}.(1+\omega\sigma^{'})^{-1} = (1+\omega\sigma^{'})^{-1}.\sigma^{'}$$

This shows that $RUN(\Sigma)$ keeps the memory of σ, etc. The expression $\theta.(...).\theta$ extracts the result and therefore it is not the computation, but something like displaying the result somewhere. Hence the running of a acp is represented by the operator $RUN(\Sigma)$.

Proposition 17 *The formula for acp always makes sense.*

Proof. The partial isometries $(1-\omega^2)\sigma(\omega\sigma)^n(1-\omega^2)$, $(1-\omega^2)\sigma(\omega\sigma)^m(1-\omega^2)$ have disjoint domains and codomains for $n \neq m$ and therefore the sum of all the $(1-\omega^2)\sigma(\omega\sigma)^n(1-\omega^2)$ makes sense in terms of weak or strong convergence.□

Definition 62 *An acp $ACP(\Sigma)$, $\Sigma = (\sigma, \omega)$, is called deadlock-free if $\omega\sigma$ is weakly nilpotent.*

The interpretation of the sequent $\vdash [\Delta], \Gamma$ will be an ACS $\Sigma = (\sigma, \omega)$.

Proposition 18 *If the ACS (σ, ω) is the interpretation of the sequent $\vdash [\Delta], \Gamma$ then $\omega\sigma$ is nilpotent and it acp is a partial symmetry.*

Proposition 19 *The ACS $\Sigma^{'} = (\sigma, \omega+\rho)$ is deadlock-free iff the ACSs $\Sigma = (\sigma, \omega)$ and $\Sigma^{''} = (ACP(\Sigma), \rho)$ are deadlock-free. In that case*

$$ACP(\Sigma^{'}) = ACP(ACP(\Sigma), \rho) = ACP(\Sigma^{''}) \tag{3.4}$$

Proof. : We shall consider the basic cases of cuts. Consider the following context:

$$\frac{\begin{array}{cc}\sigma' & \sigma''\\ \vdash A,\Gamma & \vdash A^{\perp},\Delta\end{array}}{\vdash [A]\,,\Gamma,\Delta}$$

a cut between two sequents, each of them being proved in a cut-free way; we shall assume that the last rules (R') and $\left(R''\right)$ applied to $\sigma^{'}$ and $\sigma^{'}$ are (up to exchange) logical rules for A to $A^{\perp}$. In that case, we have a way to replace the cut by other ones, and this process is the basic part of Gentzen's proof. If we denote by τ the proof obtained by this acp, our goal is to relate $ACP\left(\underline{\sigma},\omega\right)$ with $ACP\left(\underline{\tau},\varrho\right)$ where ϱ the partial symmetry expressing the new cuts of τ. We consider 6 cases:

I. $A=B\otimes C$, so that $A^{\perp}=B^{\perp}\|C^{\perp}$. Hence (up to exchanges, that we once for all ignore), σ'comes from proofs $\sigma_1$and σ_2 of sequents $\vdash B^{\perp},\Gamma_1$and $\vdash C,\Gamma_2$, (with $\Gamma=\Gamma_1,\Gamma_2$) by means of a $^{\otimes}$-rule, whereas σ'' comes from a proof σ_3 of $\vdash B^{\perp},C^{\perp},\Delta$, by a $\|$-rule. τ is defined by making a cut between $\sigma_1$and σ_3 which yields σ_0, proof of $\vdash B^{\perp},C^{\perp},\Gamma_1,\Delta$, and a second cut between σ_2 and σ_0 yields a proof τ of $\vdash [B,C]\,,\Gamma,\Delta$.

Example 6 *To see what happens, we shall assume that Γ_1,Γ_2 and Δ all consist of one formula , so that we can write a matrix, which is much more visual than indices : the matrices$\underline{\sigma_1},\underline{\sigma_2}$ and $\underline{\sigma_3}$* $(2\times 2,2\times 2,3\times 3)$ *are given :*

$$\begin{pmatrix}\alpha & \beta & \beta_1 & \gamma_1 & \gamma_2 & \alpha_3 & \beta_3\\ \gamma & \alpha_1 & \alpha_2 & \beta_2 & \gamma_3 & \alpha_4 & \beta_4\\ 0 & 0 & 0 & 0 & \alpha_5 & \beta_5 & \gamma_5\end{pmatrix}$$

Now $\underline{\sigma}$ is 5×5 :

$$\begin{pmatrix}\partial\alpha\partial^{*}+\delta\beta_1\delta^{*} & 0 & \partial\beta & \delta\gamma_1 & 0\\ 0 & \partial\gamma_2\partial^{*}+\partial\alpha_3\delta^{*}+\delta\gamma_3\partial^{*}+\delta\alpha_4\delta^{*} & 0 & 0 & \partial\beta_3+\delta\beta_4\\ \gamma\partial^{*} & 0 & \alpha_1 & 0 & 0\\ \delta\partial^{*} & 0 & 0 & \beta_2 & 0\\ 0 & \alpha_5\partial^{*}+\beta_5\delta^{*} & 0 & 0 & \gamma_5\end{pmatrix}$$

whereas Ξ^{*} is 7×7 :

$$\begin{pmatrix}\alpha & 0 & 0 & 0 & \beta & 0 & 0\\ 0 & \gamma_2 & 0 & \alpha_3 & 0 & 0 & \beta_3\\ 0 & 0 & \beta_1 & 0 & 0 & \gamma_1 & 0\\ 0 & \gamma_3 & 0 & \alpha_4 & 0 & 0 & \beta_4\\ \gamma & 0 & 0 & 0 & \alpha_1 & 0 & 0\\ 0 & 0 & \alpha_2 & 0 & 0 & \beta_2 & 0\\ 0 & \alpha_5 & 0 & \beta_5 & 0 & 0 & \gamma_5\end{pmatrix}$$

Moreover, whereas ω exchanges 1 with 2, ϱ exchanges 1 with 2, and 3 with 4. Define an isomorphism ψ from $M_7\left(B\left(H\right)\right)$ into $M_5\left(B\left(H\right)\right)$ by contracting indices $1,3$ into 1 and indices $2,4$ into 2 by means of ∂, δ in both cases, the indices $5,6,7$ being renamed $3,4,5$; then,
$\psi\left(\underline{\tau}\right)=\underline{\sigma}$, $\psi\left(r\right)=\omega\left(\partial\partial^{*}+\delta\delta^{*}\right)$.

So, $\omega\underline{\sigma}=\omega\left(\partial\partial^{\bullet}+\delta\delta^{\bullet}\right)\sigma$ *is nilpotent* $\Leftrightarrow$ $\varrho\underline{\tau}$ *is nilpotent* and $ACP\left(\underline{\sigma},\omega\right)=\psi\left(ACP\left(\underline{\tau},\varrho\right)\right)$. If we restrict our attention to the last 3×3 squares of both matrices, since ψ is identical on this square. We can say that $ACP\left(\underline{\tau},\varrho\right)=ACP\left(\underline{\tau},\omega\right)$.

II. $A=\forall\alpha B$ such that $A^{\perp}=\exists\alpha B^{\perp}$. This means that σ' is obtained from a proof σ_1 of $\vdash B,\Gamma$ by means of $\forall$-rule (so α is not free in Γ), whereas σ'' is obtained from a proof σ_2 of $\vdash B^{\perp}\left[\frac{C}{\alpha}\right],\Gamma$ then a cut σ_2 yields a proof τ of $\vdash B^{\perp}\left[\frac{C}{\alpha}\right],\Delta$ means $\exists$-rule ; in that case, $\sigma'^{\bullet}=\underline{\sigma_1},\sigma''^{\bullet}=\underline{\sigma_2}$. τ is defined as follows : we first form σ_3, proof of $\vdash B\left[\frac{C}{\alpha}\right],\Gamma$, then a cut with σ_2 yields a proof τ of $\vdash\left[B\left[\frac{C}{\alpha}\right]\right],\Gamma,\Delta$. Now there is a change in the size of matrices involved and $\alpha=\varrho$; moreover , by induction we can show that $\underline{\sigma_3}=\underline{\sigma_1}$, hence $ACP\left(\underline{\sigma},\omega\right)$ exists if $ACP\left(\underline{\tau},\varrho\right)$ does, in that case that they are equal.

III. $A=!B$, such that $A^{\perp}=?B^{\perp}$ Then σ' comes from a proof $\sigma_1$of $\vdash A,\Gamma$ ($whith$ $\Gamma\ =?\Gamma_1$) by means of a !-rule. Assume moreover that (R'') is the contraction rule, so that σ'' comes from a proof σ_2 of $\vdash ?B^{\perp},?B^{\perp},\Delta\tau$ is obtained by first making a cut between σ_1 and σ_2, so to get rid of first occurrence of $?B^{\perp}$, yielding thus a proof σ_3 of $\vdash[!B]\,,?B^{\perp},\Gamma,\Delta$,then another cut between σ_1 and σ_3, yields a proof σ_0 of $\vdash[!B,!B]\,,\Gamma,\Gamma,\Delta$. Finally, a sequence of contraction yields a proof τ of $\vdash[!B,!B]\,,\Gamma,\Delta$. In fact the formula holds only when Γ is empty.

Example 7 *To see what happens, let us assume that* Γ *is empty and that* Δ *consists of exactly one formula, so that we can use a matriceal representation: by hypothesis* $\underline{\sigma_1}$ *as* 1×1 *matrix, and* $\underline{\sigma_2}$ *is* 3×3 :

$$\begin{pmatrix} \alpha & \beta & \gamma & \alpha_1 \\ 0 & \beta_1 & \gamma_1 & \alpha_2 \\ 0 & \beta_2 & \gamma_2 & \alpha_3 \end{pmatrix}$$

and $\underline{\sigma_1}$ is therefore

$$\begin{pmatrix} \gamma_3 \otimes \alpha & 0 & 0 \\ 0 & \partial'\beta\delta'^* + \partial'\gamma\delta'^* + \delta'\beta_1\partial'^* + \delta'\gamma_1\delta'^* & \partial'\alpha_1 + \delta'\alpha_2 \\ 0 & \beta_2\partial'^* + \gamma_2\delta'^* & \alpha_3 \end{pmatrix}$$

with $\partial' = \partial \otimes \gamma_3, \delta' = \delta \otimes \gamma_3$. On the other hand

$$\tau_o = \begin{pmatrix} \gamma_3 \otimes \alpha & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & \gamma & \alpha_1 \\ 0 & 0 & 0 & \gamma_3 \otimes \alpha & 0 \\ 0 & \beta_1 & 0 & \gamma_1 & \alpha_2 \\ 0 & \beta_2 & 0 & \gamma_2 & \alpha_3 \end{pmatrix}$$

Consider the isomorphism ψ from $\mathcal{M}_5(B(H))$ to $\mathcal{M}_3(B(H))$ described informally as follows: the index 5 is renamed 3,and the indices 1,3 and 2,4 are respectively contracted into 1 and 2, by means of ∂' and δ'. By a direct calculation we obtain

$$\psi(r) = \omega\left(\partial'\partial'^* + \delta'\delta'^*\right)$$

Now $\psi(\underline{\tau})$ is almost $\underline{\sigma}$; the only difference lies in its first diagonal coefficient, which is $\left(\partial'\partial'^\bullet + \delta'\delta'^\bullet\right) \otimes \alpha$. We have

$$\begin{aligned} \omega\underline{\sigma} \textit{ is nilpotent} \quad &\Leftrightarrow \quad (1 \otimes \alpha) \times \left(\partial'\beta\partial'^\star + \partial'\gamma\delta'^* + \delta'\beta_1\partial'^* + \delta'\gamma_1\delta'^*\right) \textit{ is nilpotent} \\ &\Leftrightarrow \quad (1 \otimes \alpha) \times \left(\partial'\beta\partial'^* + \partial'\gamma\delta'^* + \delta'\beta_1\partial'^* + \delta'\gamma_1\delta'^*\right) \textit{ is nilpotent} \end{aligned}$$

and therefore $\omega\left(\partial'\partial + \delta'\delta\right)\psi(\underline{\tau})$ is nilpotent. So, in case of nilpotency, it follows

$$ACP(\underline{\sigma}, \omega) = ACP(\psi(\underline{\tau}), \omega\left(\partial'\partial + \delta'\delta\right)),$$

so

$$ACP(\underline{\sigma}, \omega) = \psi(ACP(\underline{\tau}, \varrho)),$$

but if we restrict to the last 2×2 squares on which ψ is identical we obtain

$$ACP(\underline{\sigma}, \omega) = ACP(\underline{\tau}, \varrho)$$

IV. As in III, but assume that σ' comes by dereliction from a proof σ_2 of $\vdash B^\perp, \Delta$; in that case, σ is defined as the result of cutting σ_1 with σ_2 , so that to get a proof of $\vdash [B], \Gamma, \Delta$ .Here again we shall work with the extra hypothesis that Γ is empty ,and illustrate the proof in the particular case were Δ consists of a formula.

Example 8 *Assume that* $\underline{\sigma_1}$ *and* $\underline{\sigma_2}$ *are respectively*

$$\begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha_1 & \beta_1 \end{pmatrix}$$

Then $\underline{\sigma}$ is

$$\begin{pmatrix} \gamma_3 \otimes \alpha & 0 & 0 \\ 0 & \partial\beta\partial^* & \partial\gamma \\ 0 & \alpha_1\partial^* & \beta_1 \end{pmatrix}$$

whereas $\underline{\tau}$ is

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & \gamma \\ 0 & \alpha_1 & \beta_1 \end{pmatrix}$$

and

$$\omega = \varrho \ .$$

The nilpotency of $\underline{\varrho\tau}$ means that $\beta\alpha$ is nilpotent.. On the other hand, using the fact that $\partial^* (1 \otimes \alpha) = \alpha\partial^*$, the nilpotency of $\underline{\sigma}$ is the same as the nilpotency of $(\partial\beta\partial^*)(1 \otimes \alpha) = \partial\beta\alpha\partial^*$. Since $\partial(\beta\alpha)\partial^*$ is nilpotent, $\underline{\varrho\tau}$ and $\omega\underline{\sigma}$ are simultaneously nilpotent. If one of them is nilpotent, then the unique coefficient of $ACP(\underline{\sigma}, \varrho)$ is $\beta_1 + \alpha_1\alpha\gamma^* + \alpha_1\alpha\beta\alpha\gamma^* + \alpha_1\alpha\beta\alpha\beta\alpha\gamma^* + ...$, whereas the unique coefficient of $ACP(\underline{\sigma}, \omega)$ is $\beta_1 + \alpha_1\partial^*(1 \otimes \alpha)\partial\gamma^* + \alpha_1\partial^*(1 \otimes \alpha)\partial\beta\partial^*(1 \otimes \alpha)\partial\gamma^* + ...$ which is equal, using $\partial(1 \otimes \alpha)\partial = \alpha$, to $\beta_1 + \alpha_1\alpha\gamma^* + \alpha_1\alpha\beta\alpha\gamma^* + ...$, i.e., once more $ACP(\underline{\tau}, \varrho) = ACP(\underline{\sigma}, \omega)$.

V. As in IV, but σ'' is obtained from a proof σ_2 of $\vdash \Delta$ by means of a weakening. τ is defined as follows :since all formulas of Γ begin with ?, simply apply weakening to σ_2 , to get a cut -free proof $\vdash \Gamma, \Delta$. Here again we shall assume that Γ is empty and that Δ consists of one formula. Hence $\underline{\sigma_1}$ and $\underline{\tau}$ have both dimensions 1, and $\varrho = 0$.

Example 9 *Let α and β be their respective coefficients. Then $\underline{\tau}$ is 1×1 matrix consisting of β, whereas $\underline{\sigma}$ is 3×3 :*

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}$$

Then $\underline{\sigma}\omega\underline{\sigma}$ hence if we ignore the first two rows /columns , $ACP(\underline{\sigma}, \omega)$ is equal to β, hence to $\underline{\tau}$; but since $\varrho = 0$,

$$ACP(\underline{\tau}, \varrho) = \underline{\tau} \ ,$$

and the property holds in that case too.

VI. Another case: Consider a proof σ of $\vdash [!B], \Delta, !C$ ending with a cut between $\vdash !B$ (proved by σ',which comes from a proof $\sigma_1$of $\vdash B$ by a !-rule) and $\vdash ?B^\perp, ?\Delta, !C$ (proved by σ'', which comes from a proof σ_2 of $\vdash ?B^\perp, ?\Delta, C$by !-rule). Here τ is usual defined as the result of first cutting σ' with σ_2 so that to get a proof σ_3 of $\vdash [!B], ?\Delta, C$ to which !-rule is then applied so that to get a proof τ of $\vdash [!B], \Delta, C$ to which a !-rule is then applied so that to get a proof τ of $\vdash [!B], \Delta, !C$.

Example 10 *We shall assume that Δ consists of one formula, so that we start with the following matrices for $\underline{\sigma_1}$ and $\underline{\sigma_2}$:*

$$\begin{pmatrix} \alpha_3 & \beta & \gamma & \alpha \\ 0 & \beta_1 & \gamma_1 & \alpha_1 \\ 0 & \beta_2 & \gamma_2 & \alpha_2 \end{pmatrix}$$

so that $\underline{\sigma}$ is :

$$\begin{pmatrix} 1 \otimes \alpha_3 & 0 & 0 & 0 \\ 0 & 1 \otimes \beta & 1 \otimes \gamma & 1 \otimes \alpha \\ 0 & 1 \otimes \beta_1 & 1 \otimes \gamma_1 & 1 \otimes \alpha_1 \\ 0 & 1 \otimes \beta_2 & 1 \otimes \gamma_2 & \gamma_2 \otimes \alpha_2 \end{pmatrix}$$

and $\underline{\tau}$ is :

$$\begin{pmatrix} 1 \otimes 1 \otimes \alpha_3 & 0 & 0 & 0 \\ 0 & 1 \otimes \beta & 1 \otimes \gamma & 1 \otimes \alpha_1 \\ 0 & 1 \otimes \beta_1 & 1 \otimes \gamma_1 & 1 \otimes \alpha_2 \\ 0 & 1 \otimes \beta_2 & 1 \otimes \gamma_2 & 1 \otimes \alpha_3 \end{pmatrix}$$

moreover, $\omega = \varrho$ and $\underline{\sigma} = \underline{\tau}$.

Remark 22 *If $(\underline{\sigma}, \omega)$ is the interpretation of a proof σ a sequent $\vdash [\Delta], \Gamma$, then $\omega\underline{\sigma}$ is nilpotent.*

Remark 23 *If Γ does not use the symbol "?" or "$\exists$",and τ is cut-free proof of $\vdash \Gamma$ obtained from σ by using standard Gentzen reduction steps in any order, then $ACP(\underline{\sigma}, \omega) = \underline{\tau}$.*

Of course this makes sense with the abuse consisting in removing from $ACP\,(\underline{\sigma},\omega)$ the rows/columns corresponding to Δ, which are filled with null coefficients.

Proposition 20 *The following assertions are equivalent:*

i) σ is weakly nilpotent
ii)$1-\sigma$ has an inverse among $DDPU$ (Densely Defined Preclosed Unbounded) operators
iii)$1-\sigma$ has an left inverse among $DDPU$ operators
iv)$1-\sigma$ has an right inverse among $DDPU$ operators

Proof. .
$i) \Rightarrow ii) We$ constructs $\tau = 1+\sigma+\sigma^2+...+\sigma^{n-1}$, where n is the order of nilpotency of σ. It follows that $\tau(1-\sigma) = (1-\sigma)\tau = 1$.

$ii) \Rightarrow iii)$ and $ii) \Rightarrow iv)$ are easy to show.

$iv) \Rightarrow i)$ We have $\tau(1-\sigma) = 1$ with an operator τ defined every where. We assume the existence of an enumeration

$$\ddot{m}_1, ..., \ddot{m}_k, ... with\ \sigma^n(\ddot{m}_n) \neq 0.$$

By 20,vi) we can choose $\ddot{m}_1, ..., \ddot{m}_k, ...$ pairwise distinct, so that we can form the vector $x = \sum(1/n)\ddot{m}_n$; then

$$\tau(x) - \tau(\sigma^n(x)) = x + \sigma(x) + ... + \sigma^{n-1}(x)$$

The right hand side is a sum of atomic messages with coefficients of the form $1/k$. For $k < n, 1/k$ occurs at least k times, which shows that the square of this expression is at least $(1+\frac{1}{2}+...+\frac{1}{n-1})$, hence can be made as big as desired. But $||\tau(x)-\tau(\sigma^n(x))|| \leq \dfrac{2\pi||\tau||}{\sqrt{6}}$. Contradiction! This means that there is an integer k such that $\sigma^k(\ddot{m}) = 0$ for all atomic message $\ddot{m}$. But then $\sigma^k(x) = 0$ in the linear span of atomic messages, and by continuity everywhere, so $\sigma^k = 0$.

$iii) \Rightarrow i)$applying the previous implication to σ^* yields σ^* nilpotent, hence σ is nilpotent.□

Proposition 21 *If σ and ω are weak observable. The following assertions are equivalent:*

i) $\sigma\sigma^{'}$ is weakly nilpotent
ii)$\sigma^{'}\sigma$ is weakly nilpotent
iii)the operators equation $\sigma^{'} = \sigma + \sigma^{'}\omega\sigma$ has a $DDPU$ solution $\sigma^{'}$
iv)the operators equation $\sigma^{'} = \sigma + \sigma\omega\sigma^{'}$ has a $DDPU$ solution $\sigma^{'}$
v)the operators equation $\sigma^{'} = \sigma + \sigma^{'}\omega\sigma = \sigma + \sigma\omega\sigma^{'}$ has a $DDPU$ solution $\sigma^{'}$

Proof. $i) \Rightarrow v)$ the sequence $< \sigma^k(x), x >$ converges to 0 by hypothesis, but if $\sigma^n(x) = x$, we get a contradiction by considering the integers $k = n$.

$v) \Rightarrow ii)$ since $\ker(1-\sigma) = \{x, \sigma(x) - x = 0\} = \{0\}, 1-\sigma$ is injective. If $\mathcal{D}$ is the range of $1-\sigma$, one can define an unbounded operator τ on $\mathcal{D}$, provided $\tau(1-\sigma)$ is 1 ,whereas $(1-\sigma)\tau$ is the identity on $\mathcal{D}$. The conclusion is immediate if we prove the closure of $(1-\sigma)\tau$ will be1. That results if we prove that $\mathcal{D}$ is dense, and that τ is preclosed. Consider, when $x \in H$, the vectors $x^n =: \sigma^n(x)$ and $x_n =: \dfrac{(x^1+x^2+...+x^n)}{n}$, $(\forall n \in \mathbb{N}^*)$. If we set $y_n := \dfrac{nx^0+(n-1)x^1+...+x^{n-1}}{n}$, $(\forall n \in \mathbb{N}^*)$. Then $(1-\sigma)(y_n) = x - x_n$, hence the vectors $x - x_n$ belongs to $\mathcal{D}$. In the particular case $x = \ddot{m}$, then the $\ddot{m}^n$ are either atomic messages or 0, pairwise orthogonal. So the norm of $\ddot{m}_n$ is at most $\dfrac{1}{\sqrt{n}}$, and the points $\ddot{m} - \ddot{m}_n$ of $\mathcal{D}$ can be chosen as close as desired from $\ddot{m}$. So the closure of $\mathcal{D}$ contains all atomic messages, and so $\mathcal{D}$ is dense in H. It remains to show that τ is closed, which is true since its graph is obtained from the graph of the continuous $1-\sigma$ by means of usual operation.

$ii) \Rightarrow iii), ii) \Rightarrow iv)$ are left to the reader.

$iv) \Rightarrow vi)$ let τ be a left inverse for $1-\sigma$, $\tau(1-\sigma) = 1$ holds on a dense vector space. τ can be assumed to be closed , and the equation must hold on the hole of H Let x be any vector of H; then $\tau(x - \sigma(x)) = x$, hence $\tau(x) = x + \tau(\sigma(x))$.
This yields more generally

$$\tau(x) - \tau(\sigma^k(x)) = x + \sigma(x) + ... + \sigma^{k-1}(x). \qquad (*)$$

Now assume that $\sigma^n(\ddot{m}) = \ddot{m}$ with $n > 0$, and let $k = np$, $x = m$. The left hand of (*) is null , the right hand side has a norm equal to $p\sqrt{n}$, so cannot be null,contradiction.

iii) $\Rightarrow$ *vi)* apply the previous case to $1 - \sigma^*$, and conclude that $\sigma^{*n}(\ddot{m}) = \ddot{m}$ is impossible. But from $\sigma^n(\ddot{m}) = \ddot{m}$, it follows that $\ddot{m}$ is in the domain of σ^n, hence $\sigma^{*n}\sigma^n(\ddot{m}) = \ddot{m}$, and so $\sigma^{*n}(\ddot{m}) = \ddot{m}$, contradiction.

vi) $\Rightarrow$ *i)* let $x, y \in H$ (that we take of norm 1, to simplify matters). There exists a weak message M with the property

$$||M(x)|| < \varepsilon \ , \ ||M(y)|| < \varepsilon,$$

where ε is given in advance, and such that $1 - M$ is finite sum of atomic messages. Hypothesis *vi)* shows that for a sufficiently great n, we have

$$(1 - M)\sigma^n(1 - M) = 0$$

Let us compute $<\sigma^n(x), y> = <\sigma^n((1 - M)(x)), (1 - M)(y)> + <\sigma^n((1 - M)(x)), M(y)>$
$+ <\sigma^n(M(x)), (1 - M)(y)> + <\sigma^n(M(x), M(y)>$
But

$$<\sigma^n((1 - M)(x), (1 - M)(y)> = <(1 - M)\sigma^n(1 - M)(x), M(y)> = 0,$$

and, using the Cauchy-Schwarz inequality and the isometric character of σ^n, M,
$1 - M$, we obtain a majorisation $|<\sigma^n(x), y>| < \varepsilon + \varepsilon + \varepsilon^2$. This shows that $<\sigma^n(x), y>$ converges to 0.□

Theorem 7 *If σ is a weak observable, the following are equivalent:*

i) σ is nilpotent
ii)$1 - \sigma$ has an inverse in $\mathcal{B}(H)$
iii)$1 - \sigma$ has a right inverse in $\mathcal{B}(H)$
iv)$1 - \sigma$ has a left inverse in $\mathcal{B}(H)$

Proof.
i) $\Rightarrow$ *ii)* We have $<(\sigma\omega)^n(\sigma x), \omega^*(y)> = <\omega(\sigma\omega)^n(\sigma x), y> = <(\sigma\omega)^{n+1}(x), y>$, therefore $<(\sigma\omega)^{n+1}(x), y>$ converges to 0.

ii) $\Rightarrow$ *i)* results from the previous implication by interchanging ω and σ.

ii) $\Rightarrow$ *v)* from the proposition 20 results that the operator equation $\tau(1 - \omega\sigma) = (1 - \omega\sigma)\tau = 1$ has a solution τ which can be assumed closed and therefore the operator equation $\tau(1 - \omega\sigma) = x$ holds everywhere. Consider the substitution: $\sigma' =: \sigma\tau$. Hence $\sigma'(x - \omega\sigma(x)) = \sigma(x)$ and for x in the domain $\mathcal{D}$ of ρ $\sigma'(x) = \sigma(x) + \sigma'\omega\sigma(x)$. If $\omega\sigma(x)$ is defined then $x = (x + y) - \omega\sigma(x + y)$ and x belongs to $\mathcal{D}$: this proves the equality $\sigma' = \sigma + \sigma'\omega\sigma$. On the other hand,

$$\begin{aligned}(1 - \sigma\omega)(\sigma'(x)) &= \sigma\tau(x) - \sigma\omega\sigma\tau(x) \\ &= \sigma\tau(1 - \omega\sigma)(x) \\ &= \sigma(x) \ , \ (\forall x \in \mathcal{D})\end{aligned}$$

hence

$$\sigma'(x) = (\sigma + \sigma\omega\sigma')(x) \ , \ (\forall x \in \mathcal{D}) \ .$$

iv) $\Rightarrow$ *ii)* Since $\sigma' = \sigma + \sigma\omega\sigma'$ we may define

$$\tau =: 1 + \omega\sigma' .$$

It follows

$$\begin{aligned}(1 - \omega\sigma)(\tau(x)) &= \tau(x) - \omega\sigma(\tau(x)) \\ &= x + \omega(\sigma'(x)) - \omega(\sigma(x)) - \omega(\sigma\omega\sigma'(x)) \\ &= x \ , \ (\forall x \in \mathcal{D})\end{aligned}$$

therefore $1 - \omega\sigma$ is left invertible and $\omega\sigma$ is weakly nilpotent from the Prop. 20.

v) $\Rightarrow$ *ii)* and *v)* $\Rightarrow$ *iv)* are left to the reader.□□

Definition 63 *Define the set of phases P as the set of all partial symmetries on a fixed Hilbert space H and $\perp_P$as the set of all nilpotent operators.*

Definition 64 *We define the dualisation relation*

$$\alpha \perp_P \beta \Leftrightarrow \alpha\beta \textit{ is nilpotent} \tag{3.5}$$

Definition 65 *Define*

$$\begin{array}{lll} A \otimes B & =: & \{\partial\alpha\partial^* + \delta\beta\delta^* \mid \alpha \in A \textit{ and } \beta \in B\}^{\perp\perp} \\ !A & =: & \{1 \otimes \alpha \mid \alpha \in A\}^{\perp\perp} \\ A \multimap B & =: & (A \otimes B^{\perp})^{\perp} \\ A \Rightarrow B & =: & (!A) \multimap B \end{array}$$

.

Chapter 4 **Continuous Process Algebra**

In this chapter we develop a nice algebraic theory which abstracts the basic properties of a Dirichlet space. In particular, this allows us to associate a Dirichlet space to each von Neumann algebra.

The behavior of a concurrent system can be described in terms of the actions it can be perform. A simple behavior of this kind is the set of all possible sequences of actions. Such a semantics is called an interleaving semantics. The concurrent execution of actions is seen as equivalent to arbitrary interleaving, i.e. to executing these activities in an arbitrary order. Thus concurrency is simply reduced to some form of nondeterminism. Alternatively, one could try to represent concurrency explicitly, e.g. by describing a system run by a partial order of actions. Such a semantic would be 'truly concurrent'. This semantic domain is described in Sect. 2. where we introduce also the concept of extended process.

A physical phenomena is often described as a mesh of the world lines of interacting particles in the same way as a partially ordered set can be imagined to be a mesh of its lines. In physical modelling, the world line of an individual particle is described by a continuous curve with properties akin to those of the line of reals. The property of $D - continuity$ is based on the analogy between the lines of a poset (and their interpretation as sequential subprocesses).

If $\alpha \preccurlyeq \beta$ then either $\alpha = \beta$ or α occurs earlier than β in the process described by $< \mathbb{M}, \mathbb{S}, \ell >$ (β is caused by α). In this interpretation a $li - set$ comprise elements of that occur in sequential order and the lines may be viewed as the sequential subprocesses of the process described by $< \mathbb{M}, \mathbb{S}, \ell >$. This interpretation of lines was initiated by C. A. Petri ([Pet 82]). The difference from Petri's point of view is that *co* relation is interpreted as the temporal simultaneity of two basic occurrences that do not interact with each other (in Petri nets theory the co-relation is interpreted as the relation of concurrency between basic occurrences). In our setting, the interaction between two basic occurrences α and β is their superposition $\alpha \odot \beta$[6.].

The space of basic occurrences $\mathbb{B}$ is a conditionally complete lattice in the essential order and a a lower complete lattice in the specific order.

We study a special class of processes named dissipative processes (processes for which progress in time produces the increasing of all parameters values).

Let X be a $\mathcal{B} - harmonic$ space in the sense of Constantinescu-Cornea [CC 72] , $V \subset X$ an open set and let $\mathbb{B}$ be the set of positive, superharmonic functions on X and $\mathbb{B}^V$ be the set of positive, superharmonic functions on V. We may recapture the local structure by means of the global one as follows : $\mathbb{B}^V$ is determined whenever $\mathbb{B}$ is known (because $(\beta - B_\alpha^{X-V})_{|V}$ is contained in $\mathbb{B}^V$ for any $\beta \in \mathbb{B}$ and any $\beta \in \mathbb{B}^V$ is the supremum of superharmonic functions of this type). The localization operator is an abstract formulation of this fact.

The sweeping is an abstract method for construction of elementary processes which has model in the Poincaré's sweeping out process for solving the Dirichlet problem.We give here a short description of mathematical facts beyond the abstract construction of sweeping. Poincaré's method is applicable to the boundary values which are taken on a surface S by any polynomial p in the Cartesian coordinates x, y, z or to the boundary values taken by a uniformly convergent sequence of such polynomials. The region S is covered by an enumerable sequence of spheres (B_n) which are 'swept' in the order

$$B_1, B_2, B_1, B_2, B_3, B_1, B_2, B_3, B_4, ...$$

Poincaré begins by replacing the polynomial p in B_1 by the harmonic function α_1 which takes on the boundary of B_1 the same values as p. This process, which 'sweeps' the charge of density $q = -\frac{1}{4\pi}\Delta p$ from the interior of B_1 onto the surface of B_1, is specified by Poisson's integral. The continuous function, equal to α_1 in B_1 and to p elsewhere in S, is denoted by β_1, and is then replaced in B_2 by the harmonic function α_2 which takes on the boundary of B_2 the same values as β_1. The continual repetition of this process yields a sequence of functions (β_n), each of which satisfies the prescribed boundary condition on S and which converges to the required harmonic function γ.

We introduce the key concepts of energy and system. The mutual energy of two basic occurrences is defined in a manner similar to the concept of mutual energy of two potentials of H. Cartan (see [Kel 29]). It will be shown in Theorem 14 that the energy of an elementary process uniquely determines that process. A system is an abstraction for a physical system modelled by an elliptic operator. The basic intuition behind the system's definition is the following. Let

$$La = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(u_{ij}\frac{\partial}{\partial x_j})\ a$$

be an elliptic operator and C be the bilinear form which transform the matrix $U = (u_{ij})_{1 \leq i,j \leq n}$ into the identity matrix I_n. A system is then $\Gamma a = C(a, a)$. To any system we can attach an extended process (as in Definition 4.3.).

The intuition behind the W-like process $\overline{[\mathbb{B}]}$ is that it represents the all weak solutions of Dirichlet problem

$$(\mathbf{DP}) \begin{cases} (La)_{(x)} = f(x) & , x \in \Omega \\ a_{|\partial\Omega} = 0 \end{cases}$$

for an elliptic equation i.e. the all possible executions of a system.

We consider $f \in L^2(\Omega)$ which is the prototype for the basic space $\overline{[\mathbb{S}]}$. The domain of the system L is

$$\mathbb{S} = \{a; a \in C^2(\overline{\Omega}), a_{|\partial\Omega} = 0\}.$$

The energy associated to the W-like process $\overline{[\mathbb{B}]}$ is

$$\mathcal{E}_{\mathcal{C}}[\alpha, \beta] =: \frac{\mathcal{C}[\alpha,\beta]+\mathcal{C}[\beta,\alpha]}{2} = -\int_{\Omega} \alpha\beta dx$$

from which we can derive the $\Gamma - energy$

$$\mathcal{E}_{\Gamma}[a, b] = \mathcal{E}_{\mathcal{C}}[\Gamma a, b] = -\int_{\Omega} b\frac{\partial a}{\partial x_i}(u_{ij}\frac{\partial a}{\partial x_j})dx.$$

We obtain the following characterization for an extended process

$$\overline{[\mathbb{B}]} = \{\beta_f; \mathcal{E}_{\Gamma}[b, \beta_f] = -\int_{\Omega} bf dx, (\forall b \in \overline{[\mathbb{S}]})\}.$$

Basic knowledge from the theory of second order partial differential operators and harmonic analysis are assumed.

4.1 Extended Processes

Definition 66 *A real space is defined as being a structure* $< \mathbb{M}, \prec >$ *such that*

$(\mathbf{M}_1)$ $< \mathbb{M}, \prec >$ *is a lower complete semi-lattice. The order* $\prec$ *will be called the causal order. We shall note by* $\curlywedge$ (resp. $\curlyvee$) *the infimum (resp. supremum if exists) of this semi-lattice* and
$(\mathbf{M}_2)$ if $(\alpha_i)_{i \in I}$ is increasing and dominated in $\mathbb{M}$ by α, $\alpha \in \mathbb{M}$, then there exists $\curlyvee_{i \in I} \alpha_i$.

Definition 67 *Let* $\mathbb{D} \subseteq \mathbb{M}$. *We call* $\mathbb{D}$

- *dense in order from below* (in $\mathbb{M}$) if for any $\alpha \in \mathbb{M}$ we have $\alpha = \curlyvee\{\gamma \in \mathbb{D}; \gamma \preccurlyeq \alpha\}$;
- *increasingly dense* if the set $\{\gamma \in \mathbb{D}; \gamma \preccurlyeq \alpha \}$ is increasing to α for any $\alpha \in \mathbb{M}$;

Definition 68 *A basic space is defined as being a structure* $< \mathbb{S}, \leq, \bot, \top, \odot >$ *where:*

$(\mathbf{S}_1)$ $< \mathbb{S}, \leq, \bot, \top >$ *is a lattice for which:*
• $\bot$ *the minimal element and* $\top$ *the greatest element ;*
• *the lattice* $(\mathbb{S} \backslash \{\top\}, \leq_{|\mathbb{S}\backslash\{\top\}}, \bot)$ *is lower complete and upper conditionally complete ;*
• $\leq$ *will be called the essential order;*
we shall note by $\vee$ resp. $\wedge$ *the supremum resp. infimum of this lattice;*
•$\bot$ *will be called the nil action*; $\top$ *will be called deadlock*;
$(\mathbf{S}_2)$ $(\mathbb{S}, \odot, \bot)$ is a monoid;
$(\mathbf{S}_3)$ $s = \bot$ if $s \odot s = \bot$ $(\forall s \in \mathbb{S})$;
$(\mathbf{S}_4)$ $s \odot \top = \top$ $(\forall s \in \mathbb{S})$;
$(\mathbf{S}_5)$ $s \odot (a \vee b) = (s \odot a) \vee (s \odot b)$ $(\forall a, b, s \in \mathbb{S})$;
$(\mathbf{S}_6)$ $a \odot b = (a \wedge b) \odot (a \vee b)$ $(\forall a, b \in \mathbb{S})$;

Definition 69 *Two elements* $a, b \in \mathbb{S}$ *are called strongly dual if* $a \wedge b = \bot$.

We shall denote $a \in b^{\perp}$ if a and b are orthogonal and $a^{\perp} =: \{s \in \mathbb{S}; a \bot s\}$.

Definition 70 *Let $\mathbb{S}$ be an basic space. The specific order $\leq_{\odot}$ is defined by*

$$a \ \leq_{\odot} b \text{ iff } (\exists c \in \mathbb{S}) : \ b = a \odot c.$$

We shall note by $\bigvee_{\odot}$ resp. $\bigwedge_{\odot}$ the supremum resp. infimum in this order (if they exists).

Definition 71 *$a : b$ is called the residuu of a by b and it is the greatest element (if exists) which holds*

$$b \odot (a : b) \leq a.$$

Definition 72 *An basic space $\mathbb{S}$ has the decomposition property if for any $s, s_1, s_2 \in \mathbb{S}$ such that $s \leq s_1 \odot s_2$ there exists $t_1, t_2 \in \mathbb{S}$ such that*

$$t_1 \leq s_1 \ \ , \ \ t_2 \leq s_2 \ \ , \ \ s = t_1 \odot t_2 \ .$$

Proposition 22 *Every basic space has the decomposition property.*

Proof. Define $t_1 =: s \wedge s_1$ and $t_2 =: s : t_1$. It follows $t_1 \leq s_1$ and $t_2 = (s : s_1) \vee \perp$ so $t_2 \leq s_2$.□

Lemma 1 *Let $\mathbb{S}$ be an basic space and $s, a, b \in \mathbb{S}$ .Then*

i) $a \odot b \geq a \vee b$
ii) if $a \leq b$ then $s \odot a \leq s \odot b$
iii) $(a \bigwedge_{\odot} b) \odot (a \bigvee_{\odot} b) = a \odot b$
iv) if $a, b \leq s$ and $a \perp b$ then $a \odot b \leq s$
Proof.
i) $a \odot b = (a \wedge b) \odot (a \vee b) \geq a \vee b$
ii) $(\ s \odot a) \wedge (s \odot b) = s \odot (a \wedge b) = s \odot a$
iii) Suppose that every pair $a, b \in \mathbb{S}$ has a supremum $a \bigvee_{\odot} b$. Since $a \odot b \geq_{\odot} a$ and $a \odot b \geq_{\odot} b$ it follows that $a \odot b \geq_{\odot} a \bigvee_{\odot} b$. Define $c = (a \odot b) : (a \bigvee_{\odot} b)$. Then
$c \odot (a \bigvee_{\odot} b) = a \odot b \leq_{\odot} (a \bigvee_{\odot} b) \odot b$
so $c \leq_{\odot} b$ and similarly $c \leq_{\odot} a$. On the other hand, if $c^{'} \leq_{\odot} a$ and $c^{'} \leq_{\odot} b$, then
$c^{'} \odot (a \bigvee_{\odot} b) = (c^{'} \odot a) \bigvee_{\odot} (c^{'} \odot b) \leq_{\odot} a \odot b$
so $c^{'} \leq_{\odot} c$, and it follows that
$c = (a \bigwedge_{\odot} b) \Leftrightarrow (a \bigwedge_{\odot} b) \odot (a \bigvee_{\odot} b) = a \odot b$;
iv) $a \odot b = (a \wedge b) \odot (a \vee b) = a \vee b$ and
$(a \odot b) \wedge s = (a \vee b) \wedge s = (a \wedge s) \vee (b \wedge s) = a \vee b = a \odot b$.□

Proposition 23 *We have $\leq_{\odot} \subseteq \leq$.*

Proof. $a \ \leq_{\odot} b \Leftrightarrow \exists c \in \mathbb{S}$: $b = a \odot c \geq a \vee c \Rightarrow a \leq b$.□

Proposition 24 *Any basic space is a distributive lattice.*

Proof. By Bergmann theorem it is enough to show that $a \wedge s = b \wedge s$ and $a \vee s = b \vee s$ imply $a = b$. But
$a \odot s = (a \wedge s) \odot (a \vee s) = (b \wedge s) \odot (b \vee s) = b \odot s \Leftrightarrow a = b$. □

Definition 73 *A subset A of a basic space is called linearisable if*
(L) *if $s \odot a \leq s \odot b$ then $a \leq b$ $(\forall a, b \in A)$.*

Definition 74 *We define the order topology $\tau_{\leq}$ on $< \mathbb{S}, \leq >$ by putting $(a_i)_{i \in I} \underset{\tau_{\leq}}{\rightarrow} a$ iff ($(a_i)_{i \in I}$ is increasing and dominated and $\bigvee\limits_{i \in I} a_i = a$) or ($(a_i)_{i \in I}$ is decreasing and $\bigwedge\limits_{i \in I} a_i = a$).*

Remark 24 *Analogously can be defined the specific order topology $\tau_{\leq_{\odot}}$ on $< \mathbb{S}, \leq_{\odot} >$*

Proposition 25 *The superposition is continuous in the order topology.*

Proof. We shall prove that the followings relations holds in any basic space:

(ID$_1$) for any increasing and dominated net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigvee_{i\in I}(s \odot s_i) = s \odot (\bigvee_{i\in I} s_i)$;

(ID$_2$) for any net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigwedge_{i\in I}(s \odot s_i) = s \odot (\bigwedge_{i\in I} s_i)$.

We prove first **(ID$_2$).** We set $a =: \bigwedge_{i\in I} s_i$ and $b =: \bigwedge_{i\in I}(s \odot s_i)$. Observe that $s \odot a \leq b$. From $b \leq s \odot s_i$ we obtain $b : s \leq s_i$ $(\forall i \in I)$. Therefore

$b : s \leq a \Leftrightarrow b \leq s \odot a \Leftrightarrow \bigwedge_{i\in I}(s \odot s_i) = s \odot (\bigwedge_{i\in I} s_i)$.

We prove now **(ID$_1$).** We set $a =: \bigvee_{i\in I} s_i$ and $b =: \bigvee_{i\in I}(s \odot s_i)$. Observe that $s \odot a \geq b$. From $b \geq s \odot s_i$ we obtain $b : s \geq s_i$ $(\forall i \in I)$. Therefore $b : s \geq a \Leftrightarrow b \geq s \odot a \Leftrightarrow \bigvee_{i\in I}(s \odot s_i) = s \odot (\bigvee_{i\in I} s_i)$.□

Remark 25 *The lattice operations $\vee$ and $\wedge$ are continuous in the order topology.*

Remark 26 *The previous result, combined with theorem 2 from section 2.2.2 entails that any basic space is domain representable. Thus, we can study using the type 2 computability tools presented in section 2.2, the computability of all processes introduced in this report.*

Lemma 2 *The followings relations holds in any basic space:*

(GD$_1$) for any increasing and dominated net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigvee_{i\in I}(s \bigwedge s_i) = s \bigwedge (\bigvee_{i\in I} s_i)$;

(GD$_2$) *f*or any net $(s_i)_{i\in I} \subset \mathbb{S}$ and any $s \in \mathbb{S}$ we have $\bigwedge_{i\in I}(s \bigvee s_i) = s \bigvee (\bigwedge_{i\in I} s_i)$.

Proof.

We prove first **(GD$_2$).** We set $a =: \bigwedge_{i\in I} s_i$ and $a_i =: s_i : a$, $(\forall i \in I)$. Obviously $\bigwedge_{i\in I} a_i = \perp$. We have $s \wedge a \leq (s \odot a_i) \vee (a \odot a_i) = (s \vee a) \odot a_i$. Thus $\bigwedge_{i\in I}(s \vee s_i) \leq \bigwedge_{i\in I}((s \vee a) \odot a_i) = s \vee a$. The converse inequality is immediate.

We prove now **(GD$_1$).** We set $a =: \bigvee_{i\in I} s_i$ and $a_i =: a : s_i$, $(\forall i \in I)$. Obviously $\bigwedge_{i\in I} a_i = \perp$. We have $s \vee s_i \leq (s \odot a_i) \wedge (s_i \odot a_i) = (s \wedge s_i) \odot a_i \leq \bigvee_{i\in I}((s \wedge s_i) \odot a_i)$. Thus $s \wedge a \leq \bigvee_{i\in I}(s \wedge s_i)$. The converse inequality is immediate.□

Definition 75 *An extended process is a three-tuple $< \mathbb{M}, \mathbb{S}, \ell >$, where $< \mathbb{M}, \prec >$ is a real space, $< \mathbb{S}, \leq, \perp, \top, \odot >$ is a basic space and $\ell : \mathbb{M} \to \mathbb{S}$ is an injective isotone labelling function such that, if $\mathbb{B} = \ell(\mathbb{M})$ then:*

($\mathbf{P}_1$) $\ell(\alpha \curlyvee \beta) \geq_{\odot} \ell(\alpha) \vee \ell(\beta)$ if $\alpha \curlyvee \beta$ exists
($\mathbf{P}_2$) if $\ell(\alpha \curlyvee \beta) = \top$ and $\gamma \succ \alpha \curlyvee \beta$ then $\ell(\gamma) = \top$
($\mathbf{P}_3$) $\perp \in \mathbb{B}$
($\mathbf{P}_4$) $< \mathbb{B}, \leq_{|\mathbb{B}}, \wedge >$ is a lower complete semi-lattice of $< \mathbb{S}, \leq >$
($\mathbf{P}_5$) $\mathbb{B}$ is linearisable ;
($\mathbf{P}_6$) $(\mathbb{B}, \odot, \perp)$ is a monoid;
($\mathbf{P}_7$) The superposition is continuous in the order topology on $\mathbb{B}$;
($\mathbf{P}_8$) $\mathbb{B}$ has the *decomposition property.*

Remark 27 *The elements of an extended process will be called basic occurrences and will be denoted by Greek letters: α, β,etc. Their labels $\ell(\alpha), \ell(\beta)$ will be called elementary processes. In the next we shall identify these concepts.*

Definition 76 *An extended process is called*

•*dense* iff $\lessdot = \varnothing \Longleftrightarrow \forall \alpha, \beta \in \mathbb{B} : \alpha \prec \beta \Rightarrow \exists \gamma \in \mathbb{B} : \alpha \prec \gamma \prec \beta$;
•*combinatorial* iff $\preccurlyeq = (\lessdot)^+$;
•*K-dense* iff $(\forall l \in L)$ $(\forall c \in C)$ $l \cap c \neq \emptyset$;
•*N-dense* iff $(\forall \alpha, \beta, \gamma, \delta \in \mathbb{B}) : (\gamma\ co\ \beta\ \&\ \beta\ co\ \alpha\ \&\ \alpha\ co\ \delta\ \&\ \alpha\ li\ \gamma\ \&\ \gamma\ li\ \delta\ \&\ \delta\ li\ \beta) \Rightarrow$
$(\exists e \in \mathbb{B} : e\ co\ \alpha\ \&\ e\ co\ \beta\ \&\ e\ li\ \gamma\ \&\ e\ li\ \delta)$.;
•*of finite degree* iff $\forall \beta \in \mathbb{B} : |{}^{\blacklozenge}\beta| < \infty$ and $|\ \beta^{\blacklozenge}| < \infty$;
•with *finite intervals* iff $(\forall \alpha, \beta \in \mathbb{B}) : |[\alpha, \beta]| < \infty$;
•*boundedly discrete* iff $(\forall \alpha, \beta \in \mathbb{B})$ $(\exists n \in \omega)$ $(\forall l \in L) : |[\alpha, \beta] \cap l| < n$.

Definition 77 *A discrete observer is a function $dob : \mathbb{B} \to \omega$:*

$$\alpha \prec \beta \Rightarrow dob(\alpha) \prec dob(\beta)$$

Definition 78 *An extended process is called discrete observable if admits a discrete observer.*

Definition 79 *An extended process is injectively observable iff there exists an injective discrete observer.*

Definition 80 *A continuous observer is a function* $cob : \mathbb{B} \to \overline{\mathbb{R}}_+$ *with the following properties:*

($\mathbf{CO_1}$) $\alpha \prec \beta \Rightarrow cob(\alpha) \leq cob(\beta)$, $(\forall \alpha, \beta \in \mathbb{B})$;
($\mathbf{CO_2}$) $cob(\beta) = \sup_{i\in I}(cob(\beta_i))$ if $(\beta_i)_{i\in I} \uparrow \beta$;
($\mathbf{CO_3}$) $(\forall \beta \in \mathbb{B})$ $(\exists (\beta_i)_{i\in I} \uparrow \beta)$: $cob(\beta_i) < \infty$.

Definition 81 *A continuous observer cob is called nondeterministic iff* $cob(\alpha \odot \beta) = \max(cob(\alpha), cob(\beta))$

Definition 82 *An extended process is called continuous observable if admits a continuous observer*

We shall state without proof the followings connections between observability and discreteness:

Proposition 26 (BF 90) *If an extended process is discrete observable then it is boundedly discrete. If the extended process is countable then the converse also holds.*

Proposition 27 (BF 90) *An extended process is injectively observable iff the extended process has finite intervals and it is countable.*

Definition 83 *An extended process* $\mathbb{B}$ *is called continuous if for any Dedekind-cut* $(\mathbb{A}, \overline{\mathbb{A}})$ *of* $\mathbb{B}$ *and any line* $l : |M(\mathbb{A}) \cap l| = 1$.

Proposition 28 *If the extended process* $\mathbb{B}$ *is continuous then* $\mathbb{B}$ *is dense.*

Proof. Suppose that $\lessdot \neq \varnothing$, then $(\exists \alpha, \beta \in \mathbb{B}) : \alpha \lessdot \beta$. We define $\mathbb{A} = \downarrow \beta - \{\beta\}$ and $\overline{\mathbb{A}} = \mathbb{B} - \mathbb{A}$. The pair $(\mathbb{A}, \overline{\mathbb{A}})$ is a $D - cut$. Since $\alpha \lessdot \beta$ and $\mathbb{A} = \downarrow \beta - \{\beta\}$ it follows $\alpha \in Max(\mathbb{A})$. From the construction of A it follows $\beta \in Min(\overline{\mathbb{A}})$. Let l be a line such that $\alpha, \beta \in l$. Then $|M(\mathbb{A}) \cap l| = 2$, a contradiction with the fact that $\mathbb{B}$ is continuous.□

Definition 84 *An extended process* $\mathbb{B}$ *is called*
•*gap-free iff* $\forall \mathbb{A} \in D(\mathbb{B})$ $\forall l \in L$: $|c(\mathbb{A}) \cap l| \neq 0$;
•*jump-free iff* $\forall \mathbb{A} \in D(\mathbb{B})$ $\forall l \in L$: $|c(\mathbb{A}) \cap l| \neq 2$.

Definition 85 *An extended process is called D-continuous if for any Dedekind-cut* $(\mathbb{A}, \overline{\mathbb{A}})$ *of* $\mathbb{B}$ *and any line* $l : |c(\mathbb{A}) \cap l| = 1$

Remark 28 *If the extended process* $\mathbb{B}$ *is combinatorial then*
•$Obmax(\mathbb{A}) = \{\alpha \in Max(\mathbb{A}) / |\alpha^{\blacklozenge}| \leq 1\}$;
•$Obmin(\mathbb{A}) = \{\alpha \in Min(\mathbb{A}) / |^{\blacklozenge}\alpha| \leq 1\}$.

Proposition 29 *Let* $\mathbb{A} \subset \mathbb{B}$ *be specifically decreasing. Then we have* $\bigwedge_{\odot} \mathbb{A} = \bigwedge \mathbb{A}$.

Proof. Let $\alpha' = \wedge \mathbb{A}$. Then $\alpha' \geq \bigwedge_{\odot} \mathbb{A}$. Let $\beta \in \mathbb{A}$ be fixed. It follows
$\alpha' = \bigwedge \{\alpha \in \mathbb{A}, \alpha \leq_{\odot} \beta\}$. The family $\{\beta : \alpha; \alpha \in \mathbb{A}, \alpha \leq_{\odot} \beta\}$ is increasing and $\beta = \alpha \odot (\beta : \alpha)$ implies
$\beta = (\bigwedge_{\alpha \in \mathbb{A},\ \alpha \leq_{\odot} \beta} \alpha) \odot (\bigvee_{\alpha \in \mathbb{A},\ \alpha \leq_{\odot} \beta} (\beta : \alpha))$
for any $\alpha \in \mathbb{A}$, $\alpha \leq_{\odot} \beta$. Hence $\alpha' \leq_{\odot} \beta$. Thus $\alpha' \leq_{\odot} \bigwedge_{\odot} \mathbb{A}$.□

Proposition 30 *Let* $\mathbb{A} \subset \mathbb{B}$ *be specifically increasing and dominated. Then we have* $\bigvee_{\odot} \mathbb{A} = \bigvee \mathbb{A}$.

Proof. Let $\alpha' = \bigvee \mathbb{A}$. Then $\alpha' \leq \bigvee_{\odot} \mathbb{A}$. Let $\beta \in \mathbb{A}$ be fixed. It follows
$\alpha' = \bigvee \{\alpha \in \mathbb{A} \ / \ \alpha \geq_{\odot} \beta\}$. The family $\{\beta : \alpha; \ \alpha \in \mathbb{A}, \ \alpha \leq_{\odot} \beta\}$ is increasing and $\alpha = \beta \odot (\alpha : \beta)$ implies
$\alpha' = \beta \odot (\bigvee_{\alpha \in \mathbb{A},\ \alpha \geq_{\odot} \beta} (\alpha : \beta))$ for any $\alpha \in \mathbb{A}$, $\alpha \geq_{\odot} \beta$. Hence $\alpha' \geq_{\odot} \beta$. Thus $\alpha' \geq_{\odot} \bigvee_{\odot} \mathbb{A}$.□

Corollary 4 *The order topology* $\tau_{\leq}$ *is finer than the specific order topology* $\tau_{\leq_{\odot}}$.

Proof. Results directly from Prop. 29 and Prop. 30.

Definition 86 *The set $U \subset \mathbb{S}$ will be said to have u as a strong supremum (str sup) provided u is a common supremum of U relative to essential and specific order.*

Remark 29 *Let $a, b \in B$. We shall note*

- ${}_aB =: \{b \in B \ / \ b \leq a\}$
- $B_a =: \{b \in B \ / \ b \geq a\}$
- $B^{\odot}_{a,b} =: \{s \in B \ / \ a \leq_{\odot} s$ and $b \leq_{\odot} s\}$
- ${}_{a,b}B^{\odot} =: \{s \in B \ / \ s \leq_{\odot} a$ and $s \leq_{\odot} b\}$
- $B_{a,b} =: \{s \in B \ / \ s \leq a$ and $s \leq_{\odot} b\}$
- $\overline{B}_{a,b} =: \{s \in B \ / \ s \geq a$ and $s \geq_{\odot} b\}$

Definition 87 *The set $A \subset B$ will be said to have A as a strong supremum (str sup) provided a is a common supremum of A relative to essential and specific order.*

Let $a, b \in B$. We shall denote by

$$\begin{aligned} a \sqcap b &=: \begin{cases} \sup_{\leq}(B_{a,b}) \text{ if } \sup_{\leq}(B_{a,b}) \ \leq_{\odot} b \\ \perp \ \text{otherwise} \end{cases} \\ a \sqcup b &=: \begin{cases} \inf_{\leq}(\overline{B_{a,b}}) \text{ if } \inf_{\leq}(\overline{B_{a,b}}) \ \geq_{\odot} b \\ \uparrow \ \text{otherwise} \end{cases} \end{aligned}$$

Definition 88 *A mixed extended process is an extended process $(B, \leq, \perp, \top, \prec, \sqcap, \sqcup)$ with additional operations $\sqcap$ and $\sqcup$ for which*

$$\begin{aligned} a \sqcap b &\neq \perp \ , \ a \sqcup b \neq \uparrow \ , \ (\forall a, b \in B). \\ (a \sqcap b) \odot (a \sqcup b) &= a \odot b, (\forall a, b \in B). \end{aligned} \tag{C}$$

Remark 30 *Let B be a mixed extended process and $a, b \in B$ then $(a \sqcup b) \leq_{\odot} b$, $(a \sqcup b) \leq a$, $(a \sqcup b) \leq b$.*

Suppose that s, a, b are elements of B such that $s \leq a \odot b$. Then there exists $a', b' \in B$ such that $a' \leq a, b' \leq b$, and $s = a' + b'$. In fact, the elements $a' = s \sqcap a$ and $b' \odot (s \sqcap a) = s$ (this is the *domination decomposition property*). Plainly, the explicit choice of a', b' in the statement yields $s = a' + b'$ and $a' \leq a$. Since the inequalities $a \leq_{\odot} a \odot b$ and $s \leq a \odot b$ result in $a \sqcup s \leq a \odot b$, we have $b' \odot (s \sqcap a) = s$ and $b' \odot a = a \sqcup s$, so $b' \leq b$.

Proposition 31 *We have*

- $a \leq_{\odot} b \Rightarrow (s \sqcap a) \leq_{\odot} (s \sqcap b)$ *and* $(s \sqcup a) \leq_{\odot} (s \sqcup b)$.
- $s \odot (a \sqcap b) = (s \odot a) \sqcap (s \odot b)$ *and* $s \odot (a \sqcup b) = (s \odot a) \sqcup (s \odot b)$
- $s \sqcap (a \odot b) \leq (s \sqcap a) \odot (s \sqcap b)$ *and* $s \sqcup (a \odot b) \leq (s \sqcup a) \odot (s \sqcup b)$
- $s \leq_{\odot} (a \odot b) \Rightarrow s \leq_{\odot} (a \sqcap s) \odot (b \sqcup s)$

Proof. First, we prove that the condition (C) is equivalent to the condition that, for all $a, b \in B$, there exists $a \sqcup b$ which satisfies the inequality $a \sqcup b \leq_{\odot} a \odot b$. For any $c \in B$ satisfying $c \leq_{\odot} a$ and $c \leq_{\odot} b$, there is a corresponding $c' \in B$ such that $c \odot c' = a \odot b$. That there exists a largest such c is evident from the fact that some c admitted in the previous equality yields $c' = a \sqcup b$, and this is the smallest possible c'. The first two properties are immediate. For the third one, we apply the domination decomposition property to $s \sqcap (a \odot b) \leq a \odot b$ to get the representation $s \sqcap (a \odot b) = a' \odot b'$ with $a' \leq a, b' \leq b$. Since we have $a' \leq_{\odot} a$ and $b' \leq_{\odot} b$, the asserted inequality follows.

Let T be a regular topological space and Ω the set of closed sets of T.

Definition 89 *An abstract carrier on (T, B) is a map $C : B \to \Omega$ with the following properties*

i) $a = \perp \Leftrightarrow C(a) = \emptyset$
ii)$a \leq b \Rightarrow C(a) \subset C(b)$
iii)$(\forall a \in B), (\forall U, V \in \Gamma) : U \cup V = T, (\exists a_U, a_V \in B)$:

$$a = a_U \odot a_V \ and \ C(a_U) \subset U \ , \ C(a_V) \subset V$$

Let $V \in \Omega$ and $a \in B$.
We shall define $B_V = \{b \in B | C(b) \subset V\}$ and $a_V = \bigvee_{b \in B_{V, b \leq a}} b$.

Lemma 3 *For any abstract carrier C we have:*

i) $C(a \wedge b) \subset C(a) \cap C(b)$
ii) $C(a) \cap C(b) = \emptyset \Rightarrow a \in b^{\perp}$.
iii) $C(a) \cup C(b) = C(a \vee b) = C(a \odot b)$

Proof.
i)$a \wedge b \leq a \Rightarrow C(\ a \wedge b) \subset C(a)$ and $a \wedge b \leq b \Rightarrow C(\ a \wedge b) \subset C(b)$ so $C(a \wedge b) \subset C(a) \cap C(b)$;
ii)$C(a) \cap C(b) = \emptyset \Rightarrow C(a \wedge b) = \emptyset \Rightarrow a \wedge b = \perp$;
iii)$a \vee b \geq a \Rightarrow C(\ a \vee b) \supset C(a)$ and
$a \vee b \geq b \Rightarrow C(\ a \vee b) \supset C(b)$ so $C(a \vee b) \supset C(a) \cup C(b)$. On the other hand $a \vee b \leq a \odot b$ so $C(a \vee b) \subset C(a \odot b)$. Let U be an open neighbourhood of $C(a) \cup C(b)$. By property 3 of an abstract carrier, there exists $a^{'}, b^{'} \in B$ such that
$a \odot b = a^{'} \odot b^{'}$, $C(a^{'}) \subset \overline{U}$, $C(b^{'}) \subset T - U$.
Hence we have
$a \wedge b^{'} = b \wedge b^{'} = \perp$,
$b^{'} = (a \odot b) \wedge b^{'} \leq (a \wedge b^{'}) \odot (b \wedge b^{'}) = \perp$,
$C(a \odot b) = C(b^{'}) \subset \overline{U}$.
Since U is arbitrary and T regular, we have $C(a \odot b) \subset C(a) \cup C(b)$.□

Proposition 32 *B_V is a pseudoband.*

Proof. Let $A \subset B_V$ and m its least upper bound in B. We shall prove that $m \in B_V$. Let U be an open neighborhood of V. There exists $m_1, m_2 \in B$ such that $m = m_1 \odot m_2$, $C(m_1) \subset \overline{U}$, $C(m_2) \subset T - U$. Then for any $b \in B$ we have $C(b \wedge m_2) \subset C(b) \cap C(m_2) = \emptyset$. Hence
$b \wedge m_2 = \perp$, $b = b \wedge m \leq (b \wedge m_1) \odot (b \wedge m_2) \leq m_1$.
Since b is arbitrary $m \leq m_1$, $C(m) \subset C(m_1) \subset \overline{U}$. Since U is arbitrary, it follows that $m \in B_V$.□

Definition 90 *Let $\mathbb{A} \subset \mathbb{S}$ be a set such that $< \mathbb{A}, \leq_{\mathbb{A}} >$ satisfies the axioms $(P_3) \div (P_7)$. Define $[\mathbb{A}]$ as followings : we introduce on $\mathbb{A} \times \mathbb{A}$ the following equivalence relation*

$$(a, b) \approx (a^{'}, b^{'}) \Leftrightarrow a \odot b^{'} = a^{'} \odot b.$$

We shall denote by $[\mathbb{A}]$ the quotient space of $\mathbb{A} \times \mathbb{A}$. For any $a, b \in \mathbb{A}$ we denote by $\widehat{(a,b)}$ the element of $[\mathbb{A}]$ generated by (a, b). Define on $[\mathbb{A}]$ the following relations and operations
•$\perp^{'} =: \widehat{(a,a)}$;
•$\widehat{(a,b)} \odot^{'} \widehat{(a^{'},b^{'})} =: \widehat{(a \odot a^{'}, b \odot b^{'})}$; $\widehat{(a,b)} :^{'} \widehat{(a^{'},b^{'})} =: \widehat{(a,b)} \odot^{'} \widehat{(b^{'},a^{'})}$;
•$\widehat{(a,b)} \leq^{'} \widehat{(a^{'},b^{'})}$ if $a \odot b^{'} \leq a^{'} \odot b$;
•$(\widehat{(a,b)})^{*} =: \widehat{(b,a)}$;
•$\widehat{(a,b)} \prec^{'} \widehat{(a^{'},t^{'})}$ iff $\widehat{((a,b))}^{*} \prec ((a^{'}, b^{'}))^{*}$;

Proposition 33 *The map $a \rightarrow \widehat{a} = \overline{(a,0)}$ is a one-to-one and ordered-preserving map of $\mathbb{A}$ into $[\mathbb{A}]_{\uparrow} =:$ $\{\widehat{a} \in [\mathbb{A}]; \widehat{a} \geq \perp\}$.*

Proof. Easy verification.□

Remark 31 *In the sequel we shall identify $< \mathbb{A}, \prec, \leq, \perp, \top >$ with its image in $< [\mathbb{A}], \prec^{'}, \leq^{'}, \perp^{'}, \top^{'} >$, so we can use consistently the same notation. Further $[\mathbb{A}] = \mathbb{A} : \mathbb{A}$, because $\widehat{(s,t)} = \widehat{(s,\perp)} \odot \widehat{(\perp,t)}$ $= \widehat{(s,\perp)} : \widehat{(t,\perp)} = s : t$. The causal order will be extended to $[\mathbb{A}]$ by putting $a \prec b$ iff $b^{*} \prec a^{*}$.*

The next three examples were inspired from [MR 92], where they have been presented in the context of non-commutative Dirichlet Forms.

Example 11 *Let τ be Lebesgue measure on $\mathbb{R}^n$, let X be an open set of $\mathbb{R}^n$ and let $a_{i,j}$ $(i,j=1..n)$ be τ-measurable functions satisfying the following conditions*

- $a_{i,j}=a_{j,i}$
- *there exists a real number $m\geq 1$ such that*

$$\frac{1}{m}\sum_{i=1}^{n}\xi_i^2\leq\sum_{i,j=1}^{n}a_{i,j}(x)\xi_i\xi_j\leq m\sum_{i=1}^{n}\xi_i^2\,,\ (\forall x\in X,\forall(\xi_i)_{1\leq i\leq n}\in\mathbb{R}^n) \tag{4.1}$$

For any open set $V\subset X$ we shall denote by $\mathcal{H}(V)$ the set of real, continuous functions h on V such that the derivatives of first order of h in the sense of distribution theory belong to $L^2_{loc}(\tau)$ and such that for any real function f of class $C^\infty(V)$ with compact carrier, we have $\int_U\sum_{i,j=1}^{n}a_{i,j}\frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_j}d\tau=0$. An extended process consists on all positive functions p for which $p_{|V}\in\mathcal{H}(V)$ for all open sets $V\subseteq X$.

Example 12 *Let $D\subset\mathbb{R}^n$ be an open set and define the basic space $\mathcal{U}$ as a harmonic sheaf on X such that :*

- *any $\mathcal{U}$-functions is of class C^2;*
- *$\mathcal{U}$ is non-degenerate at every point of D;*
- *the set of $\mathcal{U}$-regular sets is a base of D;*

Then there exists a system of real functions $u_{i,j},v_i,w$ $(i,j=1..n)$ on $\mathbb{D}$ such that:
i) $u_{i,j}=u_{j,i}$;
ii) $(u_{i,j})$ is a non-zero positive definite matrix at any point of D;
iii) for any $\mathcal{H}$-function h, we have

$$\sum_{i,j=1}^{n}u_{i,j}\frac{\partial^2 h}{\partial x_i\partial x_j}+\sum_{i=1}^{n}v_i\frac{\partial h}{\partial x_i}+wh=0;$$

iv) there exists an open, dense set $V\subset X$, such that $u_{i,j},v_i,c$ are continuous on V, and such that any solution of the above equation on any open subset of V, is an $\mathcal{U}$-function.;
An extended process consists on all positive functions p for which $p_{|V}\in\mathcal{H}(V)$ for all open sets $V\subseteq X$.

Example 13 *Let $D\subseteq\mathbb{R}^n$ be an open set, let the basic space $\mathbb{S}$ be the positive continuous real function on D , $a\in[\mathbb{S}]$ and let $(A_i)_{1\leq i\leq j}$, B be first order differential operators of class $\mathcal{C}^\infty(D)$. For any open set $V\subseteq D$, let $\mathcal{H}(V)$ be the set of real functions $h\in\mathcal{C}^\infty(V)$ and satisfying*

$$\sum_{i=1}^{j}(A_i)^2h+Bh+ah=0.$$

Suppose that the Lie algebra generated by the operators A_i and B for the operation

$$(\sum_{i=1}^{n}a_i\frac{\partial}{\partial x_i})\cdot(\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i})=\sum_{i=1}^{n}(\sum_{j=1}^{n}a_j\frac{\partial b_i}{\partial x_j}-b_j\frac{\partial a_i}{\partial x_j})\frac{\partial}{\partial x_i}$$

is of rank n at any point of D. An extended process consists on all positive functions p for which $p_{|U}\in\mathcal{H}(U)$ for all open sets $U\subseteq X$.

Remark 32 *In Examples 11. and 13. D endowed with $\mathcal{H}$ is a Brelot space [6].*

Remark 33 *Let $\mathbb{A}\subset\mathbb{B}$*

i)We shall use the notation $\bigvee_{[\mathbb{B}]}\mathbb{A}$ for the least upper bound of $\mathbb{A}$ in $[\mathbb{B}]$ if it exists. If $\mathbb{A}$ is increasing and dominated we have

$$\bigvee_{[\mathbb{B}]}\mathbb{A}=\bigvee\mathbb{A};$$

ii)We shall use the notation $\bigwedge_{[\mathbb{B}]}\mathbb{A}$ for the greatest lower bound of $\mathbb{A}$ in $[\mathbb{B}]$ if it exists.In this case

$$\bigwedge_{[\mathbb{B}]}\mathbb{A}=\bigwedge\mathbb{A}.$$

Definition 91 *For any $a \in [\mathbb{S}]$ we shall call the regular form of a the element $\overline{a} \in \mathbb{B}$ defined by*

$$\overline{a} = \bigwedge\{\beta \in \mathbb{B}; a \leq^{'} (\beta, \perp)\}$$

Lemma 4 *For any $a, b \in \mathbb{S}$*

i)$\overline{a} \leq \overline{b}$ if $a \leq b$
ii) $\overline{a \odot b} \leq \overline{a} \odot \overline{b}$
iii) $\overline{(\overline{a})} = \overline{a}$.
iv) $(\overline{s_i})_{i \in I} \uparrow (\overline{s})$ if $(s_i)_{i \in I} \uparrow s$.
v) $(\overline{s_i})_{i \in I} \downarrow (\overline{s})$ if $(s_i)_{i \in I} \downarrow s$.

Proof
i)Let $\mathbb{A} = \{\beta \in \mathbb{B} \ / \ a \leq^{'} (\beta, \perp)\}$ and $\mathbb{D} = \{\beta \in \mathbb{B} \ / \ b \leq^{'} (\beta, \perp)\}$. Then
$a \leq b \Rightarrow \mathbb{A} \supset \mathbb{D} \Leftrightarrow \bigwedge \mathbb{A} \leq \bigwedge \mathbb{D} \Leftrightarrow \overline{a} \leq \overline{b}$.
ii)Let $\mathbb{A} \odot \mathbb{D} = \{\beta \in \mathbb{B} \ / \ a \odot b \leq^{'} (\beta, \perp)\}$. Then
$\wedge(\mathbb{A} \odot \mathbb{D}) \leq (\wedge \mathbb{A}) \odot (\wedge \mathbb{D}) \Leftrightarrow \overline{a \odot b} \leq \overline{a} \odot \overline{b}$.
iii)-v) Results by direct verification.□

Lemma 5 *Let $\alpha, \beta, \gamma \in \mathbb{B}$ with $\beta = \alpha \odot \gamma$. Then*

$$\overline{\alpha} \leq_{\odot} \beta \text{ and } \beta : \overline{\alpha} = \bigvee\{\sigma \in ; \sigma \leq \gamma, \sigma \leq_{\odot} \beta\}.$$

Proof. Of course $\beta \leq \gamma \odot \ \overline{\alpha}$. According to the decomposition property
$(\exists \delta, \delta^{'} \in \mathbb{B}) : \beta = \delta \odot \delta^{'}$ and $\delta \leq \gamma$, $\delta^{'} \leq \overline{\alpha}$.
From $\delta^{'} = \beta : \delta \geq \beta : \gamma$ it follows $\delta^{'} \geq \alpha$ and therefore $\delta^{'} \geq \overline{\alpha}$, hence $\delta^{'} = \overline{\alpha}$. If $\sigma \in \mathbb{B}$ is such that $\sigma \leq \gamma$ and $\sigma \leq_{\odot} \beta$ it follows $\beta : \sigma \in \mathbb{B}$, $\beta : \sigma \geq \alpha$ and therefore $\beta : \sigma \geq \overline{\alpha}$. From $\overline{\alpha} \odot \delta = \beta$ we deduce $\delta \geq \sigma$.□

Proposition 34 *For any $\alpha, \beta \in \mathbb{B}$ we have $(\alpha \vee \beta) \leq_{\odot} (\alpha \odot \beta)$.*

Proof. We have $\alpha \vee \beta = \overline{\alpha \vee_{[\mathbb{B}]} \beta} = \overline{(\alpha \odot \beta) : (\alpha \wedge \beta)}$ hence $(\alpha \vee \beta) \leq_{\odot} (\alpha \odot \beta)$.□

Lemma 6 *Let $\mathbb{A} \subset \mathbb{B}$ and let $\beta \in \mathbb{B}$ be such that $\alpha \leq_{\odot} \beta$ $(\forall \alpha \in \mathbb{A})$. Then $\bigvee \mathbb{A} \leq_{\odot} \beta$.*

Proof. We first assume that $\mathbb{A}$ is increasing. It follows $\beta = (\bigvee \mathbb{A}) \odot (\bigwedge (\beta : \alpha))$ so $\bigvee\limits_{\alpha \in \mathbb{A}} \mathbb{A} \leq_{\odot} \beta$. We shall prove that $\mathbb{A}$ may be assumed increasing. Let $(\alpha_i)_{1 \leq i \leq n}$ be a finite subset of $\mathbb{A}$ and let $(\gamma_i)_{1 \leq i \leq n} \subset \mathbb{B}$ such that $\beta = \alpha_i \odot \gamma_i$, $(\forall i : 1 \leq i \leq n$). It follows that $\bigvee\limits_{1 \leq i \leq n} \alpha_i = \overline{\beta : (\bigwedge \gamma_i)} \leq_{\odot} \beta$.□

Theorem 8 *The space of basic occurrences $([\mathbb{B}], \leq^{'})$ is a conditionally complete lattice in the essential order.*

Proof. Let $\mathbb{A} \subset [\mathbb{B}]$ specifically dominated and let $\mathbb{D} =: \{\beta \in [\mathbb{B}]; \ \alpha \leq_{\odot} \beta \ , \forall \alpha \in \mathbb{A}\}$.

It is no loss of generality to assume $\perp \in A$. Then we have $\mathbb{D} \subset \mathbb{B}$. Let now $\beta^{'} = \wedge \mathbb{D}$. Since for any $\alpha \in A$ and $\beta \in \mathbb{D}$: $\beta : \alpha \in \mathbb{B}$ we get $\overline{\beta^{'} : \alpha} = \bigwedge\{\beta : \alpha; \ \beta \in \mathbb{D}\}$, $\beta^{'} : \alpha \in \mathbb{B}$ and therefore $\beta^{'} \in \mathbb{D}$. Let now $\underline{\beta \in \mathbb{D} \text{ be fixed}}$ and denote $\gamma = \overline{\beta : \beta^{'}}$. Then from Prop. 5 $\exists d \in \mathbb{S} : d = \beta : \gamma$ and $d \leq \beta^{'}$. From $\gamma = \overline{(\beta : \alpha) : (\beta^{'} : \alpha)}$ $(\forall \alpha \in \mathbb{A})$ it follows $\gamma \leq_{\odot} \beta : \alpha$. Hence there exists $e \in \mathbb{B}$ such that $\beta = \gamma \odot \alpha \odot e$, $d \odot \gamma = \gamma \odot \alpha \odot e$, $d = \alpha \odot e$. This implies $\gamma \geq_{\odot} \alpha$ and $\gamma \geq \beta^{'}$. Thus $\gamma = \beta^{'}$, $\beta \geq \beta^{'}$ and therefore $\beta^{'} = \bigvee_{\odot} \mathbb{A}$.□

Lemma 7 *Let $s \in [\mathbb{S}]$, $s = a : b$. Then $\overline{s} \leq_{\odot} a$.*

Proof. We have $a = s \odot b \leq \widehat{s} \odot b$. Let $a = a_1 \odot a_2$ with $a_1 \leq \overline{s}$ and $a_2 \leq b$. It follows $a = a_1 \odot a_2 \leq a_1 \odot b$ so $a_1 \geq a : b \geq s$. From definition of $\overline{s}$ results $a_1 \geq \overline{s}$. Then $\overline{s} = a_1 \leq_{\odot} a$.□

Theorem 9 *The space of basic occurrences $\mathbb{B}$ is a lower complete lattice in the specific order.*

Proof. Let $\mathbb{B}^{\odot}_{\alpha,\beta} =: \{\gamma \in \mathbb{B}; \gamma \leq_{\odot} \alpha, \gamma \leq_{\odot} \beta\}$ and $\alpha, \beta \in \mathbb{B}$, $\gamma \in \mathbb{B}^{\odot}_{\alpha,\beta}$. It follows that there exist $\gamma_1, \gamma_2 \in \mathbb{B}$ such that $\gamma = \alpha \odot \gamma_1 = \beta \odot \gamma_2$. Let $\delta = \bigwedge\{\gamma; \gamma \in \mathbb{B}^{\odot}_{\alpha,\beta}\}$, $\delta_1 = \bigwedge\{\gamma_1; \gamma \in \mathbb{B}^{\odot}_{\alpha,\beta}\}$, $\delta_2 = \bigwedge\{\gamma_2; \gamma \in \mathbb{B}^{\odot}_{\alpha,\beta}\}$. From $\delta = \alpha \odot \delta_1 = \beta \odot \delta_2$ it follows $\delta \in \mathbb{B}^{\odot}_{\alpha,\beta}$. Let $\theta = \overline{\gamma : \delta}$. Then $\theta = \overline{\gamma_1 : \delta_1} = \overline{\gamma_2 : \delta_2}$. From the decomposition property there exists $\sigma, \sigma_1, \sigma_2 \in \mathbb{B}$ with $\sigma \leq \delta$ such that $\gamma = \sigma \odot \theta$, $\gamma_1 = \sigma_1 \odot \theta$, $\gamma_2 = \sigma_2 \odot \theta$. It follows that $\sigma \in \mathbb{B}^{\odot}_{\alpha,\beta}$, $\sigma = \delta$, $\delta \leq_{\odot} \gamma$. Because $\gamma \in \mathbb{B}^{\odot}_{\alpha,\beta}$ is arbitrary, it follows $\delta = \alpha \bigwedge_{\odot} \beta$.□

Definition 92 *Let $U \subset A \subseteq \mathbb{S}$. We call U*
- *solid (in A) if $(u \in U,\ s \in A,\ s_{\updownarrow} \leq u_{\updownarrow}) \Rightarrow (s \in U)$;*
- *specifically solid (in A) if $(u \in U\ ,\ s \in A,\ s \leq_{\odot} u) \Rightarrow (s \in U)$.*

Proposition 35 *If $\mathbb{A} \subset \mathbb{B}$ is an substructure which is dense in order from below and specifically solid then $\mathbb{A}$ is increasingly dense.*

Proof. Let $a, \beta \in \mathbb{A}$. From Prop. 34. results $a \vee \beta \in \mathbb{A}$. Thus $\mathbb{A}$ satisfies the axioms of a basic space which is increasingly dense in $\mathbb{S}$.□

Proposition 36 *If the subsets $\mathbb{A}, \mathbb{A}^{'} \subset \mathbb{B}$ are solid and increasingly dense then $\mathbb{A} \cap \mathbb{A}^{'}$ is solid and increasingly dense.*

Proof. Let $\alpha \in \mathbb{B}$ and denote $Y = \{x \in \mathbb{A} \cap \mathbb{A}^{'} / x \leq \alpha\}$ and for $y \in \mathbb{A}$ with $y \leq \alpha$ denote $Y_y = \{x^{'} \in \mathbb{A}^{'} / x^{'} \leq y\}$. Then $Y_y \subset Y$ and $\alpha \geq \bigvee Y \geq \bigvee\limits_{y \in X,\ y \leq a} \bigvee Y_y = \alpha$. For any $y_1, y_2 \in Y$ there exists $x \in X$ such that $y_1 \leq x \leq a$ and $y_2 \leq x$. Since $\mathbb{A}^{'}$ is increasingly dense there exists $x^{'} \in X^{'}$ with $y_1 \leq x^{'} \leq x$ and $y_2 \leq x^{'}$. Obviously $x^{'} \in \mathbb{A}$ and therefore $x^{'} \in Y$.□

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4.2 Dissipative Processes

Definition 93 *An extended process is called dissipative if $\leq_{|\mathbb{B}} = \prec$.*

In this section every process is supposed to be dissipative and all continuous observers to be additives.

Definition 94 *The process image is $Im\mathbb{B} = \{cob : \mathbb{B} \to \overline{\mathbb{R}}_{+}; cob$ is an additive continuous observer$\}$*

Remark 34 *$Im\mathbb{B}$ can be ordered with the usual pointwise order $cob_1 \leq cob_2 \Leftrightarrow cob_1(\beta) \leq cob_2(\beta)$ $(\forall \beta \in \mathbb{B})$. In this order $Im\mathbb{B}$ is a lattice and*

$$\begin{aligned} (cob_1 \vee cob_2)_{(\beta)} &= \sup_{\beta_1 \odot \beta_2 \leq \beta} \{cob_1(\beta_1) + cob_2(\beta_2)\} \\ (cob_1 \bigwedge cob_2)_{(\beta)} &= \inf_{\beta_1 \odot \beta_2 = \beta} \{cob_1(\beta_1) + cob_2(\beta_2)\} \end{aligned}$$

Definition 95 *A couple of observers is a map $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \overline{\mathbb{R}}_{+}$ such that for the maps defined by*

$$[.]_{\alpha} : \mathbb{B} \to \overline{\mathbb{R}}_{+},\ [\beta]_{\alpha} = \mathcal{C}[\alpha, \beta] \text{ and } [.]_{\beta} : \mathbb{B} \to \overline{\mathbb{R}}_{+},\ [\alpha]_{\beta} = \mathcal{C}[\alpha, \beta]\ ,\ (\forall \alpha, \beta \in \mathbb{B})$$

we have $[.]_{\alpha}, [.]_{\beta} \in Im\mathbb{B}$.

Remark 35 *A couple of observers is not necessary symmetric.*

Definition 96 *A couple of observers $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \mathbb{R}_{+}$ will be called positive definite if $\mathcal{C}[\beta, \beta] \geq 0$, $(\forall \beta \in \mathbb{B})$.*

Lemma 8 *The couple of observers $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \mathbb{R}_{+}$ is positive definite iff*

$$\mathcal{C}[\alpha, \beta] + \mathcal{C}[\beta, \alpha] \leq \mathcal{C}[\alpha, \alpha] + \mathcal{C}[\beta, \beta]\ ,\ (\forall \alpha, \beta \in \mathbb{B}).$$

Definition 97 *A couple of observers $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \mathbb{R}_{+}$ will be called regular if $\mathcal{C}[\alpha, \overline{\beta}] = \mathcal{C}[\overline{\alpha}, \beta]$, $(\forall \alpha, \beta \in \mathbb{B})$.*

We set, without proofs, the followings immediate results:

Lemma 9 *If the couple of observers $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \mathbb{R}_{+}$ is regular then $\mathcal{C}[\overline{\beta}, \overline{\beta}] \leq \mathcal{C}[\beta, \beta]$, $(\forall \beta \in \mathbb{B})$.*

Lemma 10 *If the couple of observers $\mathcal{C} : \mathbb{B} \times \mathbb{B} \to \mathbb{R}_{+}$ is regular then $\mathcal{C}[\alpha, \overline{\alpha}] = \mathcal{C}[\overline{\alpha}, \alpha] = \mathcal{C}[\overline{\alpha}, \overline{\alpha}]$, $(\forall \alpha \in [\mathbb{A}])$.*

Definition 98 *A map $T : \mathbb{B} \to \mathbb{B}$ is called continuous in order from below iff $T\beta = \bigvee\limits_{\alpha \in \mathbb{A}} T\alpha$, for any family $\mathbb{A} \subset \mathbb{B}$ increasing to $\beta \in \mathbb{B}$.*

Definition 99 *A localisation is a map $L : \mathbb{B} \to \mathbb{B}$ with the following properties:*

($\mathbf{L_1}$) $L\beta \leq \beta$, $(\forall \beta \in \mathbb{B})$; **($\mathbf{L_2}$)** $L^2(\beta)$=$L\beta$, $(\forall \beta \in \mathbb{B})$; **($\mathbf{L_3}$)** $L(\beta \odot \beta^{'}) = L(\beta) \odot L(\beta^{'})$, $(\forall \beta, \beta^{'} \in \mathbb{B})$.

Remark 36 *For any localisation L we shall note $\mathbb{B}_L =: L(\mathbb{B})$.*

Definition 100 *The application $S_L =: id_{\mathbb{B}} : L$,where L is a localisation, will be called a sweeping if it is increasing.*

Remark 37 *For any sweeping S we shall note*

$$\begin{aligned} \mathbb{B}_S &= : S(\mathbb{B}) \\ Ker_S &= : \{\beta \in \mathbb{B}; S\beta = \perp\} \end{aligned}$$

Remark 38 *If S is a sweeping then $L_S =: id_{\mathbb{B}} : S$ is a localisation.*

Remark 39 *The class of all sweepings on $\mathbb{B}$ can be ordered by putting*

$$S \leq T \text{ iff } S\beta \leq T\beta \text{ , } (\forall \beta \in \mathbb{B})$$

for any sweepings S, T. This ordered set is a complete distributive lattice.

Definition 101 *For any sweeping S we define the sweeping $S^{'}$ as the smallest sweeping having the property*

$$S \vee S^{'} = id_{\mathbb{B}}.$$

Lemma 11 *Let $\alpha \in \mathbb{B}$ and $\beta = \alpha : (\alpha \bigwedge_{\odot} S_L\alpha)$. Then*

$$\begin{aligned} L\alpha &= L\beta \text{ and} \\ \beta \bigwedge_{\odot} S_L\beta &= \perp . \end{aligned}$$

Proof. Let $\gamma = \alpha \bigwedge_{\odot} S_L\alpha$. From $\gamma \leq_{\odot} S_L\alpha$ and $S_L = S_L^2$ it follows $S_L\gamma = \gamma$. We have also $\beta = \alpha : \gamma$ and $S_L\beta = S_L\alpha : \gamma$.$\square$

Lemma 12 *If $\alpha, \beta \in \mathbb{B}$ with $\alpha \odot S_L\beta \leq \beta \odot S_L\alpha$ then*

$$(L\alpha \odot S_L\beta) \in \mathbb{B}.$$

Proof. The decomposition property provides $\gamma, \delta \in \mathbb{B}$ with
$\gamma \leq S_L\alpha$, $\delta \leq \beta$ and $\alpha \odot S_L\beta = S_L\gamma \odot S_L\delta$.
Since $S_L\delta \leq S_L\beta$ and $S_L\gamma \leq S_L\alpha$ it follows $S_L\delta = S_L\beta$ and $S_L\gamma = S_L\alpha$ and therefore $(L\alpha \odot S_L\beta) = \delta \in \mathbb{B}$.$\square$

Corollary 5 *In the previous lemma, if $\alpha \bigwedge_{\odot} S\alpha = \perp$ then*

$$\begin{aligned} \alpha &\leq \beta \text{ and} \\ S\alpha &\leq_{\odot} S\beta. \end{aligned}$$

Proof. We have $L_S\alpha = S\beta$ and $\alpha \leq_{\odot} S\alpha \odot \delta$ and $\alpha \leq_{\odot} (\alpha \bigwedge_{\odot} \delta) \odot (\alpha \bigwedge_{\odot} S\alpha) \leq_{\odot} \delta \leq \beta$. Analogously $S\alpha \leq_{\odot} S\beta$.$\square$

Proposition 37 *The decomposition property holds for $\mathbb{B}_L$, for any localisation L.*

Proof. Let $\alpha, \beta, \gamma \in \mathbb{B}$ such that $L\alpha \leq L\beta \odot L\gamma$. From the decomposition property for the basic space there exists $\alpha^{'}, \beta^{'}, \gamma^{'} \in \mathbb{B}$ such that
$\alpha^{'} \leq S_L\alpha$, $\beta^{'} \leq \beta$, $\gamma^{'} \leq \gamma$ and $\alpha \odot S_L\beta \odot S_L\gamma = \alpha^{'} \odot \beta^{'} \odot \gamma^{'}$.
So $S_L\alpha^{'} = \alpha^{'} = S_L\alpha$, $S_L\beta = S_L\gamma$, $S_L\beta^{'} = S_L\gamma^{'}$, hence
$L\alpha = L\alpha^{'} \odot L\beta^{'}$ and $L\alpha^{'} \leq L\beta$, $L\beta^{'} \leq L\gamma$.$\square$

Proposition 38 *Let L be a localisation and $(\alpha_i)_{i \in I} \subset \mathbb{B}$ be such that $\alpha_i \bigwedge_{\odot} S_L\alpha_i = \perp$, $(\forall i \in I)$ then:*

i) if $(L\alpha_i)_{i\in I}$ is increasing and dominated in $[\mathbb{B}]$ by $L\alpha$, $\alpha \in B$, then $(\alpha_i)_{i\in I}$ is increasing and dominated in $\mathbb{B}$ by α and we have

$$\bigvee_{\substack{[\mathbb{B}]\\ i\in I}} L\alpha_i = L(\bigvee_{i\in I} \alpha_i) \ \ and \ \ \bigvee_{i\in I} S_L\alpha_i = S_L(\bigvee_{i\in I} \alpha_i).$$

ii) if $(L\alpha_i)_{i\in I}$ is decreasing then

$$\bigwedge_{i\in I} L\alpha_i = L(\bigwedge_{i\in I} \alpha_i) \ and \ \bigwedge_{\substack{\odot\\ i\in I}} S_L\alpha_i = S_L(\bigwedge_{i\in I} \alpha_i).$$

Proof.
i)From Lemma 12. and Cor. 5 we have $(\alpha_i)_{i\in I}$ increasing and $\alpha_i \leq \alpha$, $(\forall i \in I)$. Let $\beta = \bigvee_{i\in I} \alpha_i$ and $\gamma = \bigvee^{\odot}_{i\in I} S_L\alpha_i$. From Prop. 30. we have $\gamma \leq \beta$. Since $\gamma \leq_{\odot} S_L\alpha$ it follows $\gamma = S_L\gamma \leq S_Lb$. From $L(\alpha_i) \leq L\alpha$, $(\forall i \in I)$ we get $\beta \odot S_L\alpha \leq \alpha \odot \gamma \Rightarrow S_Lb \odot S_L\alpha \leq S_L\alpha \odot S_L\gamma \Rightarrow S_Lb \leq S_L\gamma$, hence $S_Lb = \gamma$. For $i \leq j$ we have $L(\alpha_i) \odot S_L(\alpha_j) \leq \alpha_j$ and therefore $L(\alpha_i) \odot \gamma \leq \beta$, $L(\alpha_i) \leq L\beta$.
Let $\alpha^{'} : \beta^{'} \in [\mathbb{B}]$ be such that $L(\alpha_i) \leq \alpha^{'} : \beta^{'}$, $(\forall i \in I)$. It follows $\alpha_i \odot \beta^{'} \leq \alpha^{'} \odot S_L(\alpha_i)$, $\beta \odot \beta^{'} \leq \alpha^{'} \odot \gamma$, $L\beta \leq \alpha^{'} : \beta^{'}$ and therefore $L\beta = \bigvee_{\substack{[B]\\ i\in I}} L\alpha_i$.
ii)Let $\beta = \bigwedge_{i\in I} \alpha_i$ and $\gamma = \bigwedge^{\odot}_{i\in I} S_L\alpha_i$. From $\gamma \leq_{\odot} S_L\alpha_i$, $(\forall i \in I)$ we get $S_L\gamma = \gamma \leq \beta$ and therefore $\gamma \leq S_Lb$. From Lemma 12. and Cor. 5. $(S_L\alpha_i)_{i\in I}$ is specifically decreasing. From $S_L\beta \leq S\alpha_i$ and from Prop. 29. it follows $S_L\beta \leq \bigwedge_{i\in I} S_L\alpha_i = \bigwedge^{\odot}_{i\in I} S_L\alpha_i = \gamma$, hence $\gamma = S_L\beta$. Let $\alpha^{'}, \beta^{'} \in \mathbb{B}$ be such that $L(\alpha_i) \geq \alpha^{'} : \beta^{'}$, $(\forall i \in I)$. Then $L(\alpha_i) \geq L\beta \geq \alpha^{'} : \beta^{'}$, $(\forall i \in I)$. Hence $L\beta = \bigvee_{\substack{[B]\\ i\in I}} L\alpha_i$.□

Proposition 39 *$\mathbb{B}_L$ is a inferior semi-lattice and a lattice ideal, for any localisation L.*

Proof. We prove first that if $\alpha, \beta \in \mathbb{B}_L$ then $\alpha \wedge \beta \in \mathbb{B}_L$. Let
$\alpha = L\alpha^{'}$, $\beta = L\beta^{'}$, $\alpha^{'}, \beta^{'} \in \mathbb{B}$, and $\gamma = (\alpha^{'} \odot S_L\beta^{'}) \wedge (\beta^{'} \odot S_L\alpha^{'})$.
Since $S_L\gamma = S_L(\alpha^{'} \odot \beta^{'})$ it follows $\alpha \wedge \beta = L\gamma \in \mathbb{B}_L$.
We prove now that if $\alpha \in \mathbb{B}_L$ and $\delta \in \mathbb{B}$ then $\alpha \wedge \delta \in \mathbb{B}_L$. Let
$\alpha = L\alpha^{'}$, $\alpha^{'} \in \mathbb{B}$, and $\delta^{'} = \alpha^{'} \wedge (\delta \odot S_L\alpha^{'})$.
Since $S_L\delta^{'} = S_L\alpha^{'}$ it follows $\alpha \wedge \delta = L\delta^{'} \in \mathbb{B}_L$.□

Lemma 13 *If S and T are two sweepings then $S \leq T$ is equivalent with each one of the followings conditions:*

$$S \circ T = T \circ S = S; \ Ker_T \subset Ker_S$$

Proposition 40 *If S and T are two sweepings such that $S \vee T = id_{\mathbb{B}}$ and $S \circ T = T \circ S$. Then*

$$id_{\mathbb{B}} \odot (S \circ T) = S \odot T \ \ and \ S \circ T = S \wedge T.$$

Proof. Let $\beta \in [\mathbb{B}]$. We have
$L_S\beta \in Ker_S$, $(S \odot T)(L_S\beta) = (T \odot S)(L_S\beta) =\perp$, $(L_S\beta) : T(L_S\beta) \in Ker_S \cap Ker_T$.
Since $S \vee T = id_{\mathbb{B}}$ we have $Ker_S \cap Ker_T = \{\perp\}$ and therefore $(L_S\beta) : T(L_S\beta) =\perp$ which is equivalent with $\beta \odot (S \circ T)\beta = S\beta \odot T\beta$.

The last assertion follows immediately since $S \circ T$ is a sweeping.□

Remark 40 *Let $\alpha^{(n)}$ means $\alpha \odot \alpha ... \odot \alpha$ by n times.*

Lemma 14 *If $\alpha, \beta, \gamma \in [\mathbb{B}]_{\uparrow}$ and $\alpha^{(n)} \leq \beta$, $(\forall n \in \mathbb{N}) \Rightarrow \alpha =\perp$ then the following assertions are equivalent:*

i) $\beta \wedge \alpha^{(n)} \leq \gamma$, $(\forall n \in \mathbb{N})$; ii) $(\beta : \gamma) \wedge \alpha \leq\perp$.
Proof.
i) ⇒ ii) Let $\sigma, \delta \in \mathbb{B}$ such that $\sigma = (\beta : \gamma)_{\odot}$, and $\delta^{(p)} = (\alpha^{(p)} : \gamma)_{\odot}$ where $p \in \mathbb{N}$ is fixed. Then we have
$(\beta : \gamma) \wedge (\alpha^{(n)} : \gamma) \leq\perp$, $(\forall n \in \mathbb{N})$

thus $\sigma \wedge \delta^{(p)} =\perp$ and therefore $\sigma \wedge \delta =\perp$. Then we get
$((\beta : \gamma) \wedge \alpha)^{(p)} = (\beta : \gamma)^{(p)} \wedge \alpha^{(p)} = (\beta : \gamma)^{(p)} \wedge (\alpha^{(p)} : \gamma \odot \gamma)$

$\leq \sigma^{(p)} \wedge (\delta^{(p)} \odot \gamma) \leq (\sigma \wedge \delta)^{(p)} \odot (\sigma^{(p)} \wedge \gamma) \leq \gamma.$

ii) $\Rightarrow$ *i)* We have $\sigma \wedge \alpha =\perp$ and therefore $\sigma \wedge \alpha^{(n)} \leq \sigma^{(n)} \wedge \alpha^{(n)} = (\sigma \wedge \alpha)^{(n)} =\perp$ hence $\beta \wedge \alpha^{(n)} = (\beta : \gamma \odot \gamma) \wedge \alpha^{(n)} \leq (\sigma \odot \gamma) \wedge \alpha^{(n)} \leq (\sigma \wedge \alpha^{(n)}) \odot (\gamma \wedge \alpha^{(n)}) \leq \gamma.\square$

Theorem 10 *Let $\beta \in [\mathbb{B}]_{\uparrow}$. The map $S_\beta : \mathbb{B} \to \mathbb{B}$ defined by*

$$S_\beta(\alpha) = \bigvee_{n\in\mathbb{N}} \overline{(\alpha \wedge \beta^{(n)})}$$

is a sweeping.

Proof. Let $\alpha, \alpha^{'} \in [\mathbb{B}]_{\odot}$ .From definition of S_β it follows
$S_\beta(\alpha) \leq \alpha$ and $S_\beta(\alpha) \leq S_\beta(\alpha^{'})$ if $\alpha \leq \alpha^{'}$. From $(\alpha \odot \alpha^{'}) \wedge \beta^{(n)} \leq (\alpha \wedge \beta^{(n)}) \odot (\alpha^{'} \wedge \beta^{(n)})$, $(\forall n \in \mathbb{N})$ we get $S_\beta(\alpha \odot \alpha^{'}) \leq S_\beta(\alpha) \odot S_\beta(\alpha)$.
Further let $\sigma, \sigma^{'} \in \mathbb{B}$ be such that $\sigma \leq S_\beta(\alpha) \leq \alpha$, $\sigma^{'} \leq S_\beta(\alpha^{'}) \leq \alpha^{'}$, $S_\beta(\alpha \odot \alpha^{'}) = \sigma \odot \sigma^{'}$. Since $(\alpha \odot \alpha^{'}) \wedge \beta^{(n)} \leq S_\beta(\alpha \odot \alpha^{'}) = \sigma \odot \sigma^{'}$ we get, using the Prop. 14, $\beta \wedge ((\alpha \odot \alpha^{'}) : (\sigma \odot \sigma^{'})) =\perp$ and therefore $\beta \wedge (\alpha : \sigma) =\perp \Leftrightarrow \alpha \wedge \beta^{(n)} \leq \sigma \Rightarrow \sigma \geq S_\beta(\alpha)$.
Analogously we deduce $\sigma^{'} \geq S_\beta(\alpha^{'})$, hence $S_\beta(\alpha \odot \alpha^{'}) = S_\beta(\alpha) \odot S_\beta(\alpha^{'})$.
From $\alpha \wedge \beta^{(n)} \leq S_\beta(\alpha)$ we get $\alpha \wedge \beta^{(n)} \leq S_\beta(\alpha) \wedge \beta^{(n)}$ and therefore $S_\beta(S_\beta(\alpha)) = S_\beta(\alpha)$. $\square$

Lemma 15 *S_β is continuous in order from below.*

Proof. Let $\mathbb{A} \subset \mathbb{B}$ increasing to $\gamma \in \mathbb{B}$. From $\gamma \wedge \beta^{(n)} = \bigvee_{\alpha\in\mathbb{A}} (\alpha \wedge \beta^{(n)}) \leq \bigvee_{\alpha\in\mathbb{A}} S_\beta(\alpha)$ we deduce that S_β is continuous in order from below.$\square$

Definition 102 *Let $\mathbb{A}$ be a specifically solid ideal of $\mathbb{B}$ and denote*

$$P_{\mathbb{A}}\beta = \bigvee\{\alpha \in \mathbb{A}; \alpha \leq \beta\}.$$

Theorem 11 *If for any $\mathbb{A} \subset \mathbb{B}$ increasing to $\gamma \in \mathbb{A}$ we have $\bigwedge_{\alpha\in A} \overline{(\gamma : \alpha)} =\perp$ then $P_{\mathbb{A}}$ is a sweeping and continuous in order from below.*

Proof. We prove first that $id_{\mathbb{B}} : P_{\mathbb{A}}$ satisfy $(\mathbf{L}_4)$.
Let $\alpha, \beta \in \mathbb{S}$ and $\gamma \in \mathbb{A}$ such that $\gamma \leq \alpha \odot \beta$. Then there exists $\alpha^{'}, \beta^{'} \in \mathbb{B}$ such that

$$\gamma = \alpha^{'} \odot \beta^{'}, \alpha^{'} \leq \alpha \ , \ \beta^{'} \leq \beta.$$

Therefore $\alpha^{'}, \beta^{'} \in \mathbb{A}$ and $\gamma \leq (P_{\mathbb{A}}\alpha) \odot (P_{\mathbb{A}}\beta)$ and, equivalently $P_{\mathbb{A}}(\alpha \odot \beta) \leq (P_{\mathbb{A}}\alpha) \odot (P_{\mathbb{A}}\beta)$.
On the other hand $P_{\mathbb{A}}(\alpha \odot \beta) \geq \alpha^{'} \odot \beta^{'}$ and therefore $P_{\mathbb{A}}(\alpha \odot \beta) \geq (P_{\mathbb{A}}\alpha) \odot (P_{\mathbb{A}}\beta)$.

We prove now that $P_{\mathbb{A}}$ is continuous in order from below.
Let $s \in \mathbb{S}$, $(s_i)_{i\in I} \uparrow s$ and $\gamma \in \mathbb{A}$ with $\gamma \leq s$. For any s_i we denote $(\gamma, s_i) =: \gamma : \overline{(\gamma : (\gamma \wedge s_i))}$. From the hypothesis and Prop. 5. we obtain: $(\gamma, s_i) \in \mathbb{A}$, $(\gamma, s_i) \leq \gamma$, $\bigvee_{i\in I} (\gamma, s_i) = \gamma$. Hence $\gamma \leq \bigvee_{i\in I} P_{\mathbb{A}}(s_i)$. Because $\gamma \leq s$ is arbitrary we get $P_{\mathbb{A}}(s) \leq \bigvee_{i\in I} P_{\mathbb{A}}(s_i)$. On the other hand $P_{\mathbb{A}}(s_i) \leq P_{\mathbb{A}}(s)$, $(\forall i \in I)$ so $\bigvee_{i\in I} P_{\mathbb{A}}(s_i) \leq P_{\mathbb{A}}(s)$. The rest of the axioms results by easy verification.$\square$

Proposition 41 *$\forall cob \in \operatorname{Im}\mathbb{B}$, $\forall \alpha, \beta \in \mathbb{B}$, $\exists$ $cob_1, cob_2 \in \operatorname{Im}\mathbb{B}$:*

$$\begin{aligned} cob &= cob_1 + cob_2 \ \textit{and} \\ cob(\alpha \wedge \beta) &= cob_1(\alpha) + cob_2(\beta) \end{aligned}$$

Proof. Let $\alpha, \beta \in B$, $\beta^{'} = \alpha \wedge \beta$, $\gamma = \beta : \beta^{'}$ and $cob \in \mathrm{Im}\,\mathbb{B}$. We shall define $cob_1 : \mathbb{B} \to \mathbb{R}$ by

$$cob_1(\sigma) = \sup_{n\in\omega} (\sigma \wedge \gamma^{(n)}).$$

Then $cob_1(\sigma \odot \sigma^{'}) = \sup_{n\in\omega} ((\sigma \odot \sigma^{'}) \wedge \gamma^{(n)}) = \sup_{n\in\omega} ((\sigma \wedge \gamma^{(n)}) \odot (\sigma^{'} \wedge \gamma^{(n)})) = cob_1(\sigma) + cob_1(\sigma^{'})$.
If $\sigma \leq \sigma^{'}$ then $cob_1(\sigma) \leq cob_2(\sigma^{'})$. Let $U \subset \mathbb{B}$ be increasing to u. We shall prove that cob_1 is continuous in order from below. We have
$cob_1(u) \geq \sup_{v\in U} cob_1(v) \geq \sup_{v\in U}\sup_{n\in\omega} cob_1(v \wedge c^{(n)}) \geq \sup_{n\in\omega} cob_1(u \wedge \gamma^{(n)}) = cob_1(u)$.
From $\sigma \odot (\sigma^{'} \wedge \gamma^{(n)}) \leq \sigma^{'} \odot (\sigma \wedge \gamma^{(n)})$ we get
$cob_1(\sigma) + cob_1(\sigma^{'} \wedge \gamma^{(n)}) \leq cob_1(\sigma^{'}) + cob_1(\sigma \wedge \gamma^{(n)})$
and $cob(\sigma) + cob_1(\sigma^{'}) = cob_1(\sigma) + cob(\sigma^{'})$.
Therefore $cob_2 =: cob - cob_1$ satisfies axioms of a *continuous observer*.□

Definition 103 *For any additive continuous observer cob we shall define a regular form* $\bar{\cdot}$ *by*

$$(\overline{cob})(\beta) = cob(\overline{\beta}) \ , \ (\forall \beta \in \mathbb{B}).$$

Proposition 42 *We have:* $\overline{(cob_1 - cob_2)}_{(\beta)} = \sup_{\alpha\leq\beta, cob_2(\beta)<\infty} \{cob_1(\alpha) - cob_2(\alpha)\}$.

Proof. The map $cob(\beta) = \sup_{\alpha\leq\beta, cob_2(\beta)<\infty} \{cob_1(\alpha) - cob_2(\alpha)\}$ is an additive continuous observer and satisfy the properties $cob_1 - cob_2 \leq cob \leq ob$ for any $ob \in \mathrm{Im}\,\mathbb{B}$, $ob \geq cob_1 - cob_2$.□

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4.3 Energetic Spaces

Definition 104 *The mutual energy* $\mathcal{E}[a, b]$ *of two elements* a, b *is a map* $\mathcal{E} : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ *with the following properties:*

($\mathbf{EN_1}$) $\mathcal{E}[a \odot b, s] = \mathcal{E}[a, s] + \mathcal{E}[b, s]$ (the *superposition principle*)
($\mathbf{EN_2}$) $\mathcal{E}[a, b] = \mathcal{E}[b, a]$ (the *symmetry condition*)
($\mathbf{EN_3}$) $\mathcal{E}[s] = \mathcal{E}[s, s]$ (the *energy* of the element s)
($\mathbf{EN_4}$) $\mathcal{E}[s] > 0$ if $s \neq \perp$ ($\mathcal{E}$ is *positive definite*)
($\mathbf{EN_5}$) $|\mathcal{E}[a, b]|^2 \leq \mathcal{E}[a, b] \cdot \mathcal{E}[a, b]$ (the *weak sector condition*)

Remark 41 *We can extend the energy to* $[\mathbb{S}] \times [\mathbb{S}]$ *by* $\mathcal{E}[a : b, c : d] = \mathcal{E}[a, c] + \mathcal{E}[b, d] - \mathcal{E}[a, d] - \mathcal{E}[b, c]$.

Definition 105 *Two elements* $a, b \in \mathbb{S}$ *are called dual in energy (noted* $a \in b^{\perp}_{\mathcal{E}}$*) if* $\mathcal{E}[a, b] = 0$.

Lemma 16 *For any* $a, b \in [\mathbb{S}]$

i) $\mathcal{E}[\perp] = 0$; ii) $\mathcal{E}[a, \perp] = 0$; iii) $\mathcal{E}[a] > 0$ if $a \neq \perp$;
iv) $\mathcal{E}[a^*] = \mathcal{E}[a]$; v) $\mathcal{E}^{\frac{1}{2}}[a \odot b] \leq \mathcal{E}^{\frac{1}{2}}[a] + \mathcal{E}^{\frac{1}{2}}[b]$;
vi) $\mathcal{E}[a \odot b] + \mathcal{E}[a : b] = 2(\mathcal{E}[a] + \mathcal{E}[b])$;

Proof.
i) In ($\mathbf{EN_1}$) take $a = b = s = \perp$;
ii) $\mathcal{E}[a, \perp] = \mathcal{E}[a, \perp \odot \perp] = \mathcal{E}[a, \perp] + \mathcal{E}[a, \perp]$;
iii) Let $a = u : v$. Then

$$\begin{aligned} \mathcal{E}[a] &= \mathcal{E}[u : v, u : v] \\ &= \mathcal{E}[u] + \mathcal{E}[v] - 2\mathcal{E}[u, v] \\ &\geq \mathcal{E}[u] + \mathcal{E}[v] - 2\mathcal{E}^{\frac{1}{2}}[u]\mathcal{E}^{\frac{1}{2}}[v] \\ &= (\mathcal{E}^{\frac{1}{2}}[u] - \mathcal{E}^{\frac{1}{2}}[v])^2; \end{aligned}$$

iv) Let $a = u : v$. Then $a^* = v : u$ and

$$\begin{array}{rcl}\mathcal{E}[a^*] & = & \mathcal{E}[v : u, v : u] \\ & = & \mathcal{E}[v] + \mathcal{E}[u] - 2\mathcal{E}[v, u] \\ & = & \mathcal{E}[u : v, u : v] \\ & = & \mathcal{E}[a];\end{array}$$

v) We have

$$\begin{array}{rcl}\mathcal{E}[a \odot b] & = & \mathcal{E}[a \odot b, a \odot b] \\ & = & \mathcal{E}[a] + \mathcal{E}[b] + 2\mathcal{E}[a, b] \\ & \leq & \mathcal{E}[a] + \mathcal{E}[b] + 2\mathcal{E}^{\frac{1}{2}}[a]\mathcal{E}^{\frac{1}{2}}[b] \\ & = & (\mathcal{E}^{\frac{1}{2}}[a] + \mathcal{E}^{\frac{1}{2}}[b])^2;\end{array}$$

vi) We have

$$\begin{array}{rcl}\mathcal{E}[a \odot b] + \mathcal{E}[a & : & b] = \mathcal{E}[a \odot b, a \odot b] + \mathcal{E}[a : b, a : b] \\ & = & \mathcal{E}[a] + \mathcal{E}[a, b] + \mathcal{E}[b, a] + \mathcal{E}[b] + \mathcal{E}[a] - \mathcal{E}[a, b] - \mathcal{E}[b, a] + \mathcal{E}[b] \\ & = & 2(\mathcal{E}[a] + \mathcal{E}[b]).\square\end{array}$$

Definition 106 *We define the energy metric* $d : [\mathbb{S}] \times [\mathbb{S}] \to \mathbb{R}_+$ *by*

$$d(a, b) = \begin{cases} \mathcal{E}^{\frac{1}{2}}[a : b] & \text{if } a, b \in \mathbb{S} \\ \mathcal{E}^{\frac{1}{2}}[(u \odot v^{'}) : (v \odot u^{'})] & \text{if } a, b \in [\mathbb{S}], a = (u, v), b = (u^{'}, v^{'}) \end{cases}$$

Remark 42 *We can define the energy topology* τ_d *on* $[\mathbb{S}]$ *by*

$$(a_n)_{n \in N} \underset{\tau_d}{\to} a \text{ iff } (d(a_n, b))_{n \in N} \underset{\mathbb{R}}{\to} 0.$$

Corollary 6 *The energy topology is a Hausdorff topology.*

Definition 107 *We shall note by* $\overline{[\mathbb{S}]}$ *the completion of* $[\mathbb{S}]$ *in the energy topology.*

Remark 43 *The theorem 3 from section 2.2.2 shows that every* $\overline{[\mathbb{S}]}$ *is domain representable, so we can study the computability of energetic spaces.*

Remark 44 *The energy* $\mathcal{E}$ *can be extended to* $\overline{[\mathbb{S}]}$ *by* $\mathcal{E}[a, b] = \lim\limits_{n \to \infty} \mathcal{E}[a_n, b_n]$, $(a, b \in \overline{[\mathbb{S}]})$, *where* $(a_n) \to a$, $(b_n) \to b$, $(a_n) \subset [\mathbb{S}]$, $(b_n) \subset [\mathbb{S}]$.

Definition 108 *An energetic space is a structure* $< [\mathbb{S}], \mathcal{E} >$ *such that* $[\mathbb{S}]$ *is an extended space,* $\mathcal{E} : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ *is an energy and*

- **(ES$_1$)** $[\mathbb{S}] = \overline{[\mathbb{S}]}$;
- **(ES$_2$)** $a \in b^{\perp} \Rightarrow a \in b^{\perp}_{\mathcal{E}}$, $(\forall a, b \in [\mathbb{S}])$.

Theorem 12 *The structure* $< [\mathbb{S}], \mathcal{E} >$ *is an energetic space iff* $[\mathbb{S}]$ *is closed in the energy topology and the energy* $\mathcal{E}$ *is a lattice valuation.*

Proof. We prove first that $\mathcal{E}$ is a lattice valuation iff

$$\mathcal{E}[a \wedge b, a \vee b] = \mathcal{E}[a, b] \ , \ (\forall a, b \in \overline{[\mathbb{S}]}).$$

Using axiom $(\mathbf{S_6})$ we obtain
$\mathcal{E}[(a \wedge b) \odot (a \vee b)] = \mathcal{E}[a \odot b, a \odot b] \Leftrightarrow$
$\mathcal{E}[a \wedge b] + \mathcal{E}[a \vee b] + 2 \cdot \mathcal{E}[a \wedge b, a \vee b] = \mathcal{E}[a] + \mathcal{E}[b] + 2 \cdot \mathcal{E}[a, b]$.
We prove now that in any energetic space the energy $\mathcal{E}$ is a lattice valuation. Let $c = a \wedge b$. We have $a = c \odot a^{'}$, $b = c \odot b^{'}$, $a^{'} \in (b^{'})^{\perp}_{\mathcal{E}}$, $a \vee b = c \odot (a^{'} \vee b^{'}) = c \odot (a^{'} \odot b^{'})$.

Therefore

$$\begin{aligned}\mathcal{E}[a\wedge b, a\vee b] &= \mathcal{E}[a^{'}\odot b^{'}\odot c, c]\\ &= \mathcal{E}[a^{'}\odot c, c]+\mathcal{E}[b^{'}, c]\\ &= \mathcal{E}[a^{'}\odot c, b^{'}\odot c]-\mathcal{E}[a^{'}\odot c, b^{'}]+\mathcal{E}[b^{'}, c]\\ &= \mathcal{E}[a,b]-\mathcal{E}[a^{'}, b^{'}]\\ &= \mathcal{E}[a,b].\end{aligned}$$

Conversely, if $\mathcal{E}$ is a lattice valuation then
$\mathcal{E}[a,b]=\mathcal{E}[a\wedge b, a\vee b]=\mathcal{E}[\perp, a\vee b]=0\Leftrightarrow a\in b^{\perp}$.□

Example 14 *Let $\overline{[\mathbb{S}]}$ be the class of all the spaces of excessive functions $\xi_{\mathcal{V}}$ [BBC 81] of all sub-Markovian resolvents $\mathcal{V}$ which are in duality (with respect to a finite measure μ) and for which the initial kernels are proper. For any $\xi_{\mathcal{V}}$, $\xi_{\mathcal{W}}\subseteq\overline{[\mathbb{S}]}$ and $a\in\xi_{\mathcal{V}}$, $b\in\xi_{\mathcal{W}}$ define the mutual energy $\mathcal{E}[a,b]$ of a and b by*

$$\mathcal{E}[a,b]=:\sup\{\int f\,Wg\,d\mu\,;\,a,b\in\mathfrak{F},\,Vf\leq s,\,Wg\leq t\}$$

where V is the initial kernel for $\mathcal{V}$, W is the initial kernel for $\mathcal{W}$ and $\mathfrak{F}$ denotes the set of all $\mathfrak{B}$-measurable positive numerical functions on X, $(X,\mathfrak{B},\mu)$ being the measurable space.

Example 15 *Let $D\subset R^n$ be Greenean set (with the Green function G) and let $\overline{[\mathbb{S}^{\prime}]}$ be the class of all Borel measures on D. The mutual energy $\mathcal{E}[a,b]$ of two measures $a^{'}=\mu, b^{'}=\nu$, $a^{'}, b^{'}\in\overline{[\mathbb{S}^{\prime}]}$ is defined by $\mathcal{E}^{'}[a^{'}, b^{'}]=:\int\int G(x,y)\,d\mu(x)\,d\nu(y)$.*

Remark 45 *If we denote by $a(x)=:\int G(x,y)\,d\mu(x)$, $b(x)=:\int G(x,y)\,d\nu(y)$, there exist resolvents $\mathcal{V}$, $\mathcal{W}$ which are in duality (with respect to a finite measure μ), such that $a\in\xi_{\mathcal{V}}$, $b\in\xi_{\mathcal{W}}$ and $\mathcal{E}[a,b]=\mathcal{E}^{'}[a^{'}, b^{'}]$.*

Example 16 *Let $\overline{[\mathbb{S}]}$ be the class of all absolute continuous functions f on (x,y) with $f^{'}\in L^2(x,y)$ and $f(x)=f(y)=0$. One can define the mutual energy $\mathcal{E}[a,b]$ of a and b by $\mathcal{E}[a,b]=:\int_x^y a^{'}b^{'}dt$.*

Definition 109 *For $\widehat{s}\in\overline{[\mathbb{S}]}$ let $\widehat{s}_{\uparrow}=\widehat{s}\bigvee_{\overline{[\mathbb{S}]}}0$, $\widehat{s}_{\downarrow}=(\perp:\widehat{s})\bigvee_{\overline{[\mathbb{S}]}}0$, $\widehat{s}_{\updownarrow}=\widehat{s}_{\uparrow}\odot\widehat{s}_{\downarrow}$.*

Proposition 43 *The energy is continuous in the energy topology.*

Proof. Let $(a_n)\subset[\mathbb{S}]$, $(b_n)\subset[\mathbb{S}]$ with $(a_n)\to a$, $(b_n)\to b$. We must show that $\lim_{n\to\infty}\mathcal{E}[a_n,b_n]=\mathcal{E}[a,b]$. But

$$\begin{aligned}|\mathcal{E}[a,b]-\mathcal{E}[a_n,b_n]| &\leq |\mathcal{E}[a,b:b_n]|+|\mathcal{E}[a:a_n,b]|\\ &\leq \mathcal{E}^{\frac{1}{2}}[a]\mathcal{E}^{\frac{1}{2}}[b:b_n]+\mathcal{E}^{\frac{1}{2}}[b]\mathcal{E}^{\frac{1}{2}}[a:a_n]\\ &\leq \mathcal{E}[a]\cdot d(b,b_n)+\mathcal{E}[b]\cdot d(a,a_n)\end{aligned}$$

which converges to zero.□

Lemma 17 *The energy metric is translation invariant.*

Proof. $d^2(a\odot s, b\odot s)=\mathcal{E}[(a\odot s):(b\odot s)]=\mathcal{E}[(a:b)\odot(s:s)]=\mathcal{E}[a:b]=d^2(a,b)$.□

Proposition 44 *The superposition is continuous in the energy topology.*

Proof. Let $(a_n)_n\subset[\mathbb{S}]$, $(b_n)_n\subset[\mathbb{S}]$ with $(a_n)\to a$, $(b_n)\to b$. We must show that $\lim_{n\to\infty}(a_n\odot b_n)=a\odot b$. But
$d(a_n\odot b_n, a\odot b)\leq d(a_n\odot b_n, a\odot b_n)+d(a\odot b, a\odot b_n)\leq d(a_n,a)+d(b_n,b)$ which converges to zero.□

We want now to characterize the extended process for which the basic space are energetic spaces.

Definition 110 *An extended process $\mathbb{B}$ is called W-like process if there exists a map $\urcorner:\mathbb{B}\to\operatorname{Im}\mathbb{B}$ such that :*

(W$_1$) $\urcorner[\alpha\odot\beta]=\urcorner[\alpha]+\urcorner[\beta]$, and

$\alpha\leq\beta\Leftrightarrow\urcorner[\alpha]\leq\urcorner[\beta]$, $(\forall\alpha,\beta\in\mathbb{B})$;

(W$_2$) $\urcorner[\mathbb{B}]$ is solid and increasingly dense in $Im\mathbb{B}$;

(W$_3$) $\urcorner[R(\alpha)]=\tilde{R}(\urcorner[\alpha])$, $(\forall\alpha\in\mathbb{B})$;

(W$_4$) for any two sweepings S and T on $\mathbb{B}$ such that $S\vee T=id_{\mathbb{B}}$ we have $S\circ T=T\circ S$.

Let $\mathcal{C}:\mathbb{B}\times\mathbb{B}\to\overline{\mathbb{R}}_+$ defined by $\mathcal{C}[\alpha,\beta]=\urcorner[\beta](\alpha)$, $(\forall\alpha,\beta\in\mathbb{B})$. For any W-like process $\mathbb{B}$ we define $\mathbb{B}^f=:\{\beta\in\mathbb{B};\mathcal{C}[\beta,\beta]<\infty\}$.

Lemma 18 *$\mathcal{C}$ is a couple of observers.*

Lemma 19 *The couple of observers $\mathcal{C}$ has the followings properties:*

*)For any $\beta\in\mathbb{B}$ the maps $\alpha\leq\beta\Rightarrow\mathcal{C}[\sigma,\alpha]\leq\mathcal{C}[\sigma,\beta]$.

**) If $(\alpha_i)_{i\in I}\uparrow\alpha$ then $\bigvee\limits_{i\in I}\mathcal{C}[\sigma,\alpha_i]=\mathcal{C}[\sigma,\alpha]$.

***) For any $cob\in\operatorname{Im}\mathbb{B}$ there exists $\beta\in\mathbb{B}$ such that $cob(\alpha)=\mathcal{C}[\beta,\alpha]$, $(\forall\alpha\in\mathbb{B})$.

Proof.
*) Results from $\mathcal{C}[\sigma,\alpha]=\urcorner[\alpha](\sigma)$, $\mathcal{C}[s,\beta]=\urcorner[\beta](\sigma)$ and $\alpha\leq\beta\Leftrightarrow\urcorner[\alpha]\leq\urcorner[\beta]$.
**) Results from $\bigvee\limits_{i\in I}\mathcal{C}[\sigma,\alpha_i]=\urcorner[\bigvee\limits_{i\in I}\alpha_i](\sigma),\mathcal{C}[\bigvee\limits_{i\in I}\alpha_i,\sigma]=\bigvee\limits_{i\in I}\urcorner[\alpha_i](\sigma)$, and if $(cob_i)_{i\in I}\subset\operatorname{Im}\mathbb{B}$, $(cob_i)_{i\in I}\uparrow cob\in\operatorname{Im}\mathbb{B}$ then $cob[\sigma]=\bigvee\limits_{i\in I}cob_i[\sigma]$.
***) Follows from the fact that $\urcorner$ is a bijection.□

Corollary 7 *The axioms **W$_1$**) **W$_2$**) are logical equivalent with the properties *), **), ***). The axiom **W$_3$**) is logical equivalent with the following property for any sweeping S on $\mathbb{B}$*

$$\mathcal{C}[S\alpha,\beta]=\mathcal{C}[\alpha,S\beta]\ ,\ (\forall\alpha,\beta\in\mathbb{B}).$$

For any $\beta\in\mathbb{B}$ we define $\mathbb{B}_\beta=:\{\alpha\in\mathbb{B}^f;\exists m,n\in\mathbb{N},\alpha^{(m)}\leq\beta^{(n)}\}$.

Remark 46 $\mathbb{B}^f=\bigcup\limits_{\beta\in\mathbb{B}^f}\mathbb{B}_\beta$.

Proposition 45 *$\mathbb{B}^f$ is solid and increasingly dense in $\mathbb{B}$.*

Proof. If $\alpha\leq\beta^f$ and $\beta^f\in\mathbb{B}^f$ we have $\mathcal{C}[\alpha,\alpha]\leq\mathcal{C}[\beta^f,\beta^f]<\infty$ so $\alpha\in\mathbb{B}^f$. Let now $\beta\in\mathbb{B}$. There exists a net $(\alpha_i)_{i\in I}\uparrow\alpha$ such that $\mathcal{C}[\alpha_i,\alpha_i]\leq\mathcal{C}[\alpha_i,\beta]<\infty$ so $(\alpha_i)_{i\in I}\subset\mathbb{B}^f$.□

Lemma 20 *$\mathbb{B}^f$ is a basic space if $\mathcal{C}[\beta^f,\beta^f]\geq 0$ for any $\beta^f\in[\mathbb{B}^f_\alpha]$ and $\alpha\in\mathbb{B}^f$.*

Proof. Let $\alpha,\beta\in\mathbb{B}$ and $(\gamma_i^f)_{i\in I}\subset\mathbb{B}^f$ increasing to $\alpha\odot\beta$. Define $\alpha_i=(\overline{\gamma_i^f:\alpha})$, $\beta_i=(\overline{\gamma_i^f:\beta})$. Then we have $\alpha_i\odot\beta_i\leq\gamma_i^f$, $(\forall i\in I)$ and the net $(\alpha_i)_{i\in I}$ increases to α and the net $(\beta_i)_{i\in I}$ increases to β .□

Corollary 8 *For any $\alpha,\beta\in\mathbb{B}$*

$$\mathcal{C}[\alpha,\beta]+\mathcal{C}[\beta,\alpha]\leq\mathcal{C}[\alpha,\alpha]+\mathcal{C}[\beta,\beta]\qquad(\mathbf{PO})$$

and

$$\mathcal{C}[\alpha,\alpha]=0\Rightarrow\alpha=\perp.$$

Proof. Because $\alpha_i,\beta_i\in\mathbb{B}^f_{\gamma_i^f}$, $(\forall i\in I)$ we have $\mathcal{C}[(\alpha_i:\alpha_i),(\beta_i:\beta_i)]\geq 0$ which is equivalent to $\mathcal{C}[\alpha_i,\beta_i]+\mathcal{C}[\beta_i,\alpha_i]\leq\mathcal{C}[\alpha_i,\alpha_i]+\mathcal{C}[\beta_i,\beta_i]$, $(\forall i\in I)$. Passing to the limit we obtain (PO). If $\mathcal{C}[\alpha,\alpha]=0$ then for any $\beta^f\in\mathbb{B}^f$ we have $\frac{1}{2}\mathcal{C}[\alpha,\beta]\leq(\mathcal{C}[\alpha,\alpha])^{\frac{1}{2}}\cdot(\mathcal{C}[\beta,\beta])^{\frac{1}{2}}$, so $\mathcal{C}[\alpha,\beta]=0$ and therefore $\alpha=\perp$.□

Lemma 21 *Let $\beta\in[\mathbb{B}']$, $\mathbb{B}'\subseteq\mathbb{B}$ be solid in $\mathbb{B}$ with respect to the specific order and such that $\mathcal{C}[\beta]<\infty$, $\beta=\alpha:\alpha'$, $\alpha:\alpha'\in\mathbb{B}$ and $(\beta_n)_{n\in N}$ be the sequence defined by $\beta_1=\beta$, $\beta_{n+1}=\overline{\beta}_n:\beta_n$. Then*

$$\mathcal{C}[\beta]=\sum_{n=1}^{\infty}\mathcal{C}[\overline{\beta}_n].$$

Proof. Since $\mathbb{B}^{'}$ is solid in $\mathbb{B}$ with respect to the specific order $\beta_n \in [\mathbb{B}^{'}]$. The formula $\mathcal{C}[\beta] = \sum_{j=1}^{n} \mathcal{C}[\overline{\beta}_j] + \mathcal{C}[\beta_{n+1}]$ can be proved by induction. Using relations

$$\mathcal{C}[\beta_{n+1}] = \mathcal{C}[\overline{\beta}_{n+1}, \beta_{n+1}] = \mathcal{C}[\beta_{n+1}, \overline{\beta}_{n+1}]$$

we have

$$\begin{aligned} \mathcal{C}[\beta_{n+1}] &= \mathcal{C}[\overline{\beta}_{n+1}, \beta_{n+1}] + \mathcal{C}[\beta_{n+1} : \overline{\beta}_{n+1}, \beta_{n+1}] \\ &= \mathcal{C}[\overline{b}_{n+1}] + \mathcal{C}[\beta_{n+1} : \overline{\beta}_{n+1}, \beta_{n+1} : \overline{\beta}_{n+1}] + \mathcal{C}[\beta_{n+1} : \overline{\beta}_{n+1}, \overline{\beta}_{n+1}] \\ &= \mathcal{C}[\overline{\beta}_{n+1}] + \mathcal{C}[\beta_{n+2}], \end{aligned}$$

and therefore $\mathcal{C}[\beta] = \sum_{j=1}^{n+1} \mathcal{C}[\overline{\beta}_j] + \mathcal{C}[\beta_{n+2}]$.
We now construct inductively the sequences $(\alpha_n)_{n\in N}, (\alpha_n^{'})_{n\in N}$ in $\mathbb{B}^{'}$ such that

$$\alpha_1 = \alpha \ , \ \alpha_1^{'} = \alpha^{'} \ , \ \alpha_{n+1} = \alpha_n : \overline{(\alpha_n : \alpha_n^{'})} \ , \ \alpha_{n+1}^{'} = \alpha_n^{'} : \overline{(\alpha_n^{'} : \alpha_n)}.$$

The sequences $(\alpha_n)_{n\in N}, (\alpha_n^{'})_{n\in N}$ are decreasing with respect to the specific order and $\alpha_{n+1}^{'} \leq \alpha_{n+1}$, $\alpha_{n+1} \leq \alpha_n^{'}$. Hence $\bigwedge_{\odot} \alpha_n = \bigwedge \alpha_n = \bigwedge \alpha_n^{'} = \bigwedge_{\odot} \alpha_n^{'}$.
But $\alpha_n : \alpha_n^{'} = \beta_{2n-1}$ (formula that can be proved by induction) and therefore

$$\mathcal{C}[\beta_{2n-1}] = \mathcal{C}[\alpha_n : \alpha_n^{'}] \ , \ \lim_{n\to\infty} \mathcal{C}[\beta_{2n-1}] = 0 \, , \ \mathcal{C}[\beta] = \sum_{n=1}^{\infty} \mathcal{C}[\overline{\beta}_n]. \square$$

Lemma 22 *Let $\mathbb{A} \subset \mathbb{B}$ a inferior semilattice, solid with respect the specific order and $\mathcal{C}[\alpha] < +\infty$, $(\forall \alpha \in \mathbb{A})$. If the couple of observers $\mathcal{C}$ is regular, then*

$$\mathcal{C}[S_\alpha \sigma, \sigma^{'}] = \mathcal{C}[\sigma, S_\alpha \sigma^{'}] \ , \ (\forall \alpha \in [\mathbb{A}]_{\uparrow} \ , \ \forall \sigma, \sigma^{'} \in \mathbb{A}).$$

Proof. We prove first

$$\mathcal{C}[S_\alpha \sigma, \sigma] = \mathcal{C}[\sigma, S_\alpha \sigma] = \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma] \ , \ (\forall \alpha \in [\mathbb{A}]_{\uparrow}, \forall \sigma \in \mathbb{A}) \qquad \textbf{(AD)}$$

Let $\alpha \in [\mathbb{A}]_{\uparrow}$, $\sigma \in \mathbb{A}$. For any $n \in N$ we define $\alpha_n =: \sigma \wedge \alpha^{(n)}$. Then $(\alpha_n)_{n\in N}$ is increasing in $\mathbb{A}$. Further $(\overline{\alpha}_n)_{n\in N}$ is an increasing sequence in $\mathbb{A}$ and we have $S_\alpha \sigma = \bigvee_{n\in N} \overline{\alpha}_n$. Since

$$S_\alpha \sigma = \overline{(S_\alpha \sigma)^{(2)} : \sigma} = \bigvee_{n\in N} \overline{((\overline{\alpha_n})^{(2)} : \sigma)}$$

we obtain

$$\begin{aligned} \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma] &= \bigvee_{n\in N} \mathcal{C}[\overline{(\overline{\alpha_n})^{(2)} : \sigma}, \overline{(\overline{\alpha_n})^{(2)} : \sigma}] \\ &= \bigvee_{n\in N} \mathcal{C}[\overline{(\overline{\alpha_n})^{(2)} : \sigma}, (\overline{\alpha_n})^{(2)} : \sigma] \\ &= \bigvee_{n\in N} \mathcal{C}[\overline{(\overline{\alpha_n})^{(2)} : \sigma}, (\overline{\alpha_n})^{(2)}] - \bigvee_{n\in N} \mathcal{C}[\overline{(\overline{\alpha_n})^{(2)} : \sigma}, \sigma] \\ &= 2 \cdot \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma] - \mathcal{C}[S_\alpha \sigma, \sigma] \end{aligned}$$

so $\mathcal{C}[\sigma, S_\alpha \sigma] = \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma]$. Analogously we obtain $\mathcal{C}[S_\alpha \sigma, \sigma] = \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma]$.
Using relation **(AD)** we prove now the conclusion. We have
$\mathcal{C}[S_\alpha(\sigma \odot \sigma^{'}), \sigma \odot \sigma^{'}] = \mathcal{C}[S_\alpha(\sigma \odot \sigma^{'}), S_\alpha(\sigma \odot \sigma^{'})]$,
$\mathcal{C}[\sigma^{'}, S_\alpha \sigma^{'}] = \mathcal{C}[S_\alpha \sigma^{'}, S_\alpha \sigma^{'}]$, $\mathcal{C}[\sigma, S_\alpha \sigma] = \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma]$
Thus $\mathcal{C}[\sigma, S_\alpha \sigma^{'}] + \mathcal{C}[\sigma^{'}, S_\alpha \sigma] = \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma^{'}] + \mathcal{C}[S_\alpha \sigma^{'}, S_\alpha \sigma] = 2 \cdot \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma^{'}]$.
Since $\mathcal{C}[\sigma, S_\alpha \sigma^{'}] \geq \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma^{'}]$ and $\mathcal{C}[\sigma, S_\alpha \sigma^{'}] \geq \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma^{'}]$ we get
$\mathcal{C}[\sigma^{'}, S_\alpha \sigma] = \mathcal{C}[S_\alpha \sigma^{'}, S_\alpha \sigma]$. Analogously we obtain the relation $\mathcal{C}[\sigma, S_\alpha \sigma^{'}] = \mathcal{C}[S_\alpha \sigma, S_\alpha \sigma^{'}]$.$\square$

Proposition 46 *Let $\mathbb{B}$ be a W-like process. Then $< [\mathbb{B}_\alpha^f], \mathcal{E}_{\mathcal{C}} >$ is a energetic space, $(\forall \alpha \in [\mathbb{B}])$.*

Proof.

We prove now condition (**ES**). We prove first

(**SW**) $S_\beta(\beta) = \beta$, for any $\beta \in [\mathbb{B}^f_\alpha]$.

Let S and T be two sweepings on $\mathbb{B}$ such that $S \vee T = id_{\mathbb{B}}$. Take $T = S^{'}$.We have then $S \circ S^{'} = S^{'} \circ S$ and therefore $S_\beta \circ S^{'}_\beta = S^{'}_\beta \circ S_\beta$, $(\forall \beta \in [\mathbb{B}^f_\alpha])$.From Proposition we have $S_\beta(id_{\mathbb{B}} : S^{'}_\beta) = id_{\mathbb{B}} : S^{'}_\beta$.
Since $\beta \in Ker_{S^{'}_\beta}$ it follows $S_\beta(\beta) = \beta$, for any $\beta \in [\mathbb{B}^f_\alpha]$.

Let now $\alpha, \beta \in [\mathbb{B}^f_\alpha]$ be such that $\alpha \in \beta^\perp$. We have

$\mathcal{E}_{\mathcal{C}}[\alpha, \beta] = \mathcal{E}_{\mathcal{C}}[S_\alpha\alpha, \beta] = \mathcal{E}_{\mathcal{C}}[S_\alpha\alpha, S_\alpha\beta] = \mathcal{E}_{\mathcal{C}}[S_\alpha\alpha, \perp] = 0$

so $\alpha \in \beta^\perp_{\mathcal{E}_{\mathcal{C}}}$.□

Definition 111 *Let $\mathbb{B}$ be a W-like process. The map $\mathcal{E}_{\mathcal{C}} : [\mathbb{S}] \times [\mathbb{S}] \to R$ defined by*

$$\mathcal{E}_{\mathcal{C}}[\alpha, \beta] =: \frac{\mathcal{C}[\alpha, \beta]+\mathcal{C}[\beta, \alpha]}{2}$$

will be called the energy associated to the W-like process $\mathbb{B}$.

Definition 112 *A system is a map $\Gamma : \overline{[\mathbb{S}]} \to \overline{[\mathbb{S}]}$ such that*

($\mathbf{S_1}$) $\Gamma[a \odot b] = \Gamma[a] \odot \Gamma[b]$;
($\mathbf{S_2}$) Γ is continuous in τ_d;
($\mathbf{S_3}$) there exists $m = m_\Gamma \in \mathbb{R}_+$ such that $\frac{1}{m} \cdot \mathcal{E}[a] \leq \mathcal{E}[\Gamma a] \leq m \cdot \mathcal{E}[a]$, $(\forall a \in \overline{[\mathbb{S}]})$;
($\mathbf{S}_4$) $\Gamma[[\mathbb{B}]]$ is dense in $\overline{[\mathbb{S}]}$;
($\mathbf{S}_5$) $\mathcal{E}[a, b] = \dfrac{\mathcal{E}[\Gamma a, b] + \mathcal{E}[a, \Gamma b]}{2}$.

Definition 113 *For any system Γ we can associate its $\Gamma-energy$ $\mathcal{E}_\Gamma$ defined by*

$$\mathcal{E}_\Gamma[a, b] = \mathcal{E}[\Gamma a, b].$$

For any system Γ we define the space $[\mathbb{B}_\Gamma] =: \{\alpha \in \overline{[\mathbb{S}]}; \mathcal{E}_\Gamma[\alpha, s] \geq 0, \forall s \in \overline{[\mathbb{S}]}_\uparrow\}$ called *the extended process associated to system* Γ (or the $\Gamma-extended\ process$).

Theorem 13 *The lattice operations $\vee$ and $\wedge$ are continuous in the $\Gamma-energy$ topology.*

Proof. If $(a_n)_{n\in N} \underset{\tau_d}{\rightarrow} a$ and $b_n =: a : a_n$ then we have
$(b_n)_{n\in N} \underset{\tau_d}{\rightarrow} \perp$, $(b_n)_\uparrow \underset{\tau_d}{\rightarrow} \perp$, $(b_n)_\downarrow \underset{\tau_d}{\rightarrow} \perp$, $(b_n)_\updownarrow \underset{\tau_d}{\rightarrow} \perp$,
and $(a_n)_\uparrow = (a_\uparrow \odot (b_n)_\downarrow) : (a_\uparrow \wedge (b_n)_\uparrow) \odot (a_\downarrow \wedge (b_n)_\downarrow)$.
From this and from ($\mathbf{S}_3$) it is sufficient to show that if $s \in \overline{[\mathbb{S}]}_\uparrow$ and $(s_n)_{n\in N} \underset{\tau_d}{\rightarrow} \perp$, $(s_n)_{n\in N} \subset \overline{[\mathbb{S}]}_\uparrow$ then $\mathcal{E}_\Gamma[(s \wedge s_n)]_{n\in N} \to 0$ (therefore $(s \wedge s_n)_{n\in N} \to \perp$). We have $\perp \leq s \wedge s_n \leq s_n$, $(\forall n \in \mathbb{N})$. Hence for any $\beta \in [\mathbb{B}_\Gamma]_\uparrow$ we get $0 \leq \mathcal{E}_\Gamma[\beta, s \wedge s_n] \leq \mathcal{E}_\Gamma[\beta, s_n]$ so $\mathcal{E}_\Gamma[\beta, s \wedge s_n]_{n\in N} \to 0$. From ($\mathbf{S}_4$) it follows that $\mathcal{E}_\Gamma[s, s \wedge s_n]_{n\in N} \to 0$ $(\forall s \in \overline{[\mathbb{S}]}_\uparrow)$.

From $\mathcal{E}_\Gamma[s, s_n] + \mathcal{E}_\Gamma[s \wedge s_n] - \mathcal{E}_\Gamma[s, s \wedge s_n] - \mathcal{E}_\Gamma[s \wedge s_n, s_n] = \mathcal{E}_\Gamma[s : (s \wedge s_n), s_n : (s \wedge s_n)] \leq 0$ it follows $\mathcal{E}_\Gamma[(s \wedge s_n)]_{n\in N} \to 0$.□

Definition 114 *For any $s \in \overline{[\mathbb{S}]}$ define the energy-reduite $\underline{s} \in [\mathbb{B}_\Gamma]$ as the unique element which satisfies $\mathcal{E}_\Gamma[s : \underline{s}, \underline{s}] = 0$.*

Proposition 47 *We have: $\mathcal{E}_\Gamma[\underline{s}] \leq \mathcal{E}_\Gamma[s \odot t]$, $(\forall t \in \overline{[\mathbb{S}]}_\uparrow)$.*

Proof. $\mathcal{E}^2_\Gamma[\underline{s}] = \mathcal{E}^2_\Gamma[s, \underline{s}] \leq \mathcal{E}^2_\Gamma[s \odot t, \underline{s}] \leq \mathcal{E}_\Gamma[\underline{s}] \cdot \mathcal{E}_\Gamma[s \odot t]$.□

Corollary 9 *For any $s \in \overline{[\mathbb{S}]}$ we have: $\underline{s} = \overline{s}$.*

Proof. From definition results $\underline{s} \in [\mathbb{B}_\Gamma]$ and $\underline{s} \geq s$. Let $\alpha \in [\mathbb{B}_\Gamma]$ with $\alpha \geq \underline{s}$. From $\alpha \wedge \underline{s} \in [\mathbb{B}_\Gamma]$ and $(\alpha \wedge \underline{s}) \geq s$ we obtain
$(\alpha \wedge \underline{s}) : s \in \overline{[\mathbb{S}]}_\uparrow$, $\mathcal{E}_\Gamma[\alpha \wedge \underline{s}] \leq \mathcal{E}_\Gamma[\underline{s}] \leq \mathcal{E}_\Gamma[s \odot t]$, $(\forall t \in \overline{[\mathbb{S}]}_\uparrow)$;
so $\alpha \wedge \underline{s} = \underline{s}$, $\alpha \geq \underline{s}$ and therefore $\underline{s} = \overline{s}$.□

Lemma 23 *Any increasing and dominated net is τ_d convergent.*

Proof. Let $(\alpha_i)_{i\in I} \subset [\mathbb{B}_\Gamma]$ be a net increasing in $[\mathbb{B}_\Gamma]$ and dominated in $\overline{[\mathbb{S}]}$. Let also $s \in \overline{[\mathbb{S}]}$ be such that $\alpha_i \leq s$, $(\forall i \in I)$. Results then $\alpha_i \leq \overline{s}$, $(\forall i \in I)$ so the net $(\alpha_i)_{i\in I}$ is increasing in $[\mathbb{B}_\Gamma]$ and dominated by $\overline{s}$ in $[\mathbb{B}_\Gamma]$.We have for $i, j \in I$, $i > j$,

$$\begin{aligned} d^2(\alpha_i &: \alpha_j) = \mathcal{E}_\Gamma[\alpha_i : \alpha_j, \alpha_i : \alpha_j] \\ &= \mathcal{E}_\Gamma[\alpha_i] - 2\mathcal{E}_\Gamma[\alpha_i, \alpha_j] + \mathcal{E}_\Gamma[\alpha_j] \\ &\leq \mathcal{E}_\Gamma[\alpha_i] - \mathcal{E}_\Gamma[\alpha_j]. \end{aligned}$$

From the fact that the family $(\mathcal{E}_\Gamma[\alpha_i])_{i\in I}$ is increasing and dominated in $\mathbb{R}$ results the convergence of $(\alpha_i)_{i\in I}$. $\square$

Lemma 24 *Any decreasing net is τ_d convergent.*

Proof. Let $(\alpha_i)_{i\in I} \subset [\mathbb{B}_\Gamma]$ decreasing. We have for $i, j \in I$, $i > j$,

$$\begin{aligned} d^2(\alpha_i &: \alpha_j) = \mathcal{E}_\Gamma[\alpha_i : \alpha_j, \alpha_i : \alpha_j] \\ &= \mathcal{E}_\Gamma[\alpha_i] - 2\mathcal{E}_\Gamma[\alpha_i, \alpha_j] + \mathcal{E}_\Gamma[\alpha_j] \\ &\leq \mathcal{E}_\Gamma[\alpha_i] - \mathcal{E}_\Gamma[\alpha_j]; \end{aligned}$$

From the fact that the family $(\mathcal{E}_\Gamma[\alpha_i])_{i\in I}$ of positive elements is decreasing in $\mathbb{R}$ results the convergence of $(\alpha_i)_{i\in I}$.$\square$

Corollary 10 *For any $\mathbb{A} \subset [\mathbb{B}_\Gamma]$ we have $\bigwedge_{\overline{[\mathbb{S}]}} \mathbb{A} \in [\mathbb{B}_\Gamma]$.*

Definition 115 *For any set $A \subset \overline{[\mathbb{S}]}$ we define its polar A° by $A^\circ =: \{s \in A^\circ; \mathcal{E}_\Gamma[a, s] \leq 0, \forall a \in A\}$.*

Proposition 48 *The energy $\mathcal{E}_\Gamma$ is isotone on $[\mathbb{B}_\Gamma]$.*

Proof. Let $\alpha, \beta \in [\mathbb{B}_\Gamma]$ with $\alpha \leq \beta$. We have
$\mathcal{E}_\Gamma[\alpha] = \mathcal{E}_\Gamma[\alpha, \alpha] \leq \mathcal{E}_\Gamma[\alpha, \beta] \leq \mathcal{E}_\Gamma[\beta, \beta] = \mathcal{E}_\Gamma[\beta]$.$\square$

Theorem 14 *Any $\Gamma -$ elementary process is uniquely determined by its energy.*

Proof. Let $\alpha, \beta \in [\mathbb{B}_\Gamma]$ with $\mathcal{E}_\Gamma[\alpha] = \mathcal{E}_\Gamma[\beta]$. We have

$$\begin{aligned} \mathcal{E}_\Gamma[\alpha &: \beta] = \mathcal{E}_\Gamma[\alpha] - 2\mathcal{E}_\Gamma[\alpha, \beta] + \mathcal{E}_\Gamma[\beta] \\ &\leq \mathcal{E}_\Gamma[\alpha] - 2\mathcal{E}_\Gamma[\beta] + \mathcal{E}_\Gamma[\beta] \\ &= 0 \end{aligned}$$

so $\alpha = \beta$.$\square$

Proposition 49 *We have $\overline{[\mathbb{B}_\Gamma]} = \overline{[\mathbb{S}]}$.*

Proof. Let $s \in \overline{[\mathbb{S}]}$ such that $\mathcal{E}_\Gamma[\alpha, s] = 0$, $(\forall \alpha \in [\mathbb{B}_\Gamma])$. It follows $s_\updownarrow \in ([\mathbb{B}_\Gamma])^\circ$. From definition of $[\mathbb{B}_\Gamma]$ and from the bipolar theorem results
$([\mathbb{B}_\Gamma])^\circ = \overline{[\mathbb{S}]}_\uparrow$ so $s_\updownarrow \in \overline{[\mathbb{S}]}_\uparrow \Leftrightarrow s = \perp$.$\square$

Proposition 50 *For any system Γ the space $\mathbb{B}_\Gamma =: [\mathbb{B}_\Gamma]_\uparrow$ is an extended process.*

Proof.
($\mathbf{P_3}$) Of course $\mathcal{E}_\Gamma[\perp, s] = 0$, $(\forall s \in \overline{[\mathbb{S}]}_\uparrow)$ so $\perp \in \mathbb{B}_\Gamma$;
($\mathbf{P_4}$) Let $\alpha, \beta \in \mathbb{B}_\Gamma$. It is sufficient to show that $\alpha \wedge \beta = nf(\alpha \wedge \beta)$. Indeed, if we denote $\gamma = nf(\alpha \wedge \beta)$ we prove that $\gamma = \gamma \wedge \alpha$ and $\gamma = \gamma \wedge \beta$. From the definition of γ we have $\mathcal{E}_\Gamma[\gamma] = \mathcal{E}_\Gamma[\gamma, \gamma \wedge \alpha]$ and therefore

$$\begin{aligned} \mathcal{E}_\Gamma[\gamma, \gamma &: (\gamma \wedge \alpha)] = \mathcal{E}_\Gamma[\gamma, \gamma : (\alpha \wedge \beta)] \\ &= \mathcal{E}_\Gamma[\gamma, (\alpha \wedge \beta) : (\gamma \wedge \alpha)] \\ &= 0 \end{aligned}$$

Also, since $((\alpha:(\gamma\wedge\alpha))\in(\gamma:(\alpha\wedge\gamma))^{\perp}$ we get $\mathcal{E}_{\Gamma}[\alpha:(\gamma\wedge\alpha),\gamma:(\alpha\wedge\gamma)]=0$. Hence we deduce that

$$\mathcal{E}_{\Gamma}[\gamma:(\alpha\wedge\gamma)]=\mathcal{E}_{\Gamma}[\gamma,\gamma:(\alpha\wedge\gamma)]+\mathcal{E}_{\Gamma}[\alpha:(\gamma\wedge\alpha),\gamma:(\alpha\wedge\gamma)]-\mathcal{E}_{\Gamma}[\alpha,\gamma:(\alpha\wedge\gamma)]$$

and therefore $\mathcal{E}_{\Gamma}[\gamma:(\alpha\wedge\gamma)]=0$ so $\gamma=\alpha\wedge\gamma$. Analogously we get $\gamma=\beta\wedge\gamma$;

($\mathbf{P_5}$) and ($\mathbf{P_6}$) are direct consequences of ($\mathbf{EN}_1$);

($\mathbf{P_7}$) Results from ($\mathbf{ID_1}$) and ($\mathbf{ID_2}$) and from Lemma 23 and from Lemma 24.

($\mathbf{P_8}$) Let $\alpha,\beta,\gamma\in\mathbb{B}_{\Gamma}$ such that $\gamma\leq\alpha\odot\beta$ and let $\alpha^{'}=nf(\gamma:\beta)$, $\beta^{'}=\gamma:\alpha^{'}$. Then $\alpha^{'},\beta^{'}\in\mathbb{B}_{\Gamma}$, $\alpha^{'}\leq\alpha$, $\beta^{'}\leq\beta$, $\gamma=\alpha^{'}\odot\beta^{'}$.□

Proposition 51 *We have:* $\alpha:\overline{(\alpha:\beta)}\in\mathbb{B}_{\Gamma}$, $(\forall\alpha,\beta\in\mathbb{B}_{\Gamma})$.

Proof. Let $\alpha,\beta\in\mathbb{B}_{\Gamma}$. Then
$\overline{\alpha:\overline{(\alpha:\beta)}}\in\mathbb{B}_{\Gamma}\Leftrightarrow\overline{(\alpha:\overline{(\alpha:\beta)})}=\alpha:\overline{(\alpha:\beta)}$.
Note $\gamma=\alpha:\overline{\alpha:\beta}$. We have
$\mathcal{E}_{\Gamma}[\overline{\gamma}:\gamma]=\mathcal{E}_{\Gamma}[\overline{\gamma},\overline{\gamma}:\gamma]-\mathcal{E}_{\Gamma}[\gamma,\overline{\gamma}:\gamma]=\mathcal{E}_{\Gamma}[\gamma,\gamma:\overline{\gamma}]$
From $\gamma\leq\beta$ we deduce $\overline{\gamma}\leq\beta$ and

$$\begin{aligned}\mathcal{E}_{\Gamma}[\gamma,\gamma &: \overline{\gamma}]=\mathcal{E}_{\Gamma}[\alpha:\overline{(\alpha:\beta)},\gamma:\overline{\gamma}]\\ &= \mathcal{E}_{\Gamma}[\alpha,\gamma:\overline{\gamma}]-\mathcal{E}_{\Gamma}[\overline{(\alpha:\beta)},\gamma:\overline{\gamma}]\\ &\leq \mathcal{E}_{\Gamma}[\alpha,\gamma:\overline{\gamma}]-\mathcal{E}_{\Gamma}[\overline{(\alpha:\beta)},\beta:\gamma]\\ &\leq \mathcal{E}_{\Gamma}[\overline{\alpha:\beta},(\beta:\alpha)\odot\overline{(\alpha:\beta)}]\\ &= \mathcal{E}_{\Gamma}[\overline{\alpha:\beta},\overline{(\alpha:\beta)}:(\alpha:\beta)]\\ &= 0.\end{aligned}$$

Hence $\mathcal{E}_{\Gamma}[\overline{\gamma}:\gamma]=0\Leftrightarrow\overline{\gamma}=\gamma$.□

We define

$$\begin{aligned}[\mathbb{S}]^{S} &= :KerS,\\ \mathbb{S}^{S} &= :KerS\cap\mathbb{S},\\ \Gamma_{S} &= :\Gamma_{[\mathbb{S}]^{S}},\end{aligned}$$

The structure $<[\mathbb{S}]^{S},\mathcal{E}_{\Gamma}>$ is the energetic space associated to the system Γ_S .

Proposition 52 *We have*

i) $\mathbb{B}^{S}$ is solid in the Γ_S-*extended process* $\mathbb{B}_{\Gamma_S}$;

ii) for any $\beta\in\mathbb{B}_{\Gamma_S}$ there exists a sequence $(\beta_n)_{n\in N}\subset\mathbb{B}^{S}$ such that $\beta=\bigodot\limits_{n=1}^{\infty}\beta_n$;

iii) for any $\alpha\in[\mathbb{S}]^{S}$ such that $\beta\in\mathbb{B}^{S}\Rightarrow\alpha\wedge\beta\in\mathbb{B}^{S}$ we have $\alpha\in\mathbb{B}_{\Gamma_S}$;

iv) for any $\beta\in\mathbb{B}$ and any $\alpha\in\mathbb{B}_{\Gamma_S}$ we have $\alpha\wedge\beta\in\mathbb{B}_{\Gamma_S}$.

Proof.

Let $\alpha\in[\mathbb{S}]^{S}$. We show that $\alpha\in\mathbb{B}_{\Gamma_S}$ iff there exists a sequence $(\beta_n)_{n\in N}\subset\mathbb{B}_{\Gamma}$ such that $L_S(\beta_n)\underset{\tau_d}{\rightarrow}\alpha$.
Indeed we have
$\alpha\in\mathbb{B}_{\Gamma_S}\Leftrightarrow S\alpha\in([\mathbb{S}]^{S}\cap\mathbb{S})^{\circ}\Leftrightarrow S\alpha\in\overline{([\mathbb{S}]^{S})^{\circ}-(\mathbb{S})^{\circ}}\Leftrightarrow\alpha\in\overline{\mathbb{B}_{\Gamma}\odot\Gamma^{-1}(([\mathbb{S}]^{S})^{\circ})}$.
Since $\Gamma^{-1}(([\mathbb{S}]^{S})^{\circ})=[\mathbb{S}]^{S}$ we have $\alpha\in\mathbb{B}_{\Gamma_S}\Leftrightarrow\alpha=\lim\limits_{\tau_d}(\beta_n\odot S\gamma_n)$ where $(\beta_n)_{n\in N}\subset[\mathbb{S}]$. Since $\alpha\in\mathbb{B}_{\Gamma_S}$ we have
$S\alpha=\perp$, $\lim\limits_{\tau_d}(S\beta_n\odot S\gamma_n)=\perp$, $\alpha\in\mathbb{B}_{\Gamma_S}\Leftrightarrow\alpha=\lim\limits_{\tau_d}L_S(\beta_n)$.

i) Let $\alpha\in\mathbb{B}_{\Gamma_S}$, $\beta\in\mathbb{B}^{S}$ and $(\beta_n)_{n\in N}\subset\mathbb{B}_{\Gamma}$ be a sequence such that $\alpha=\lim\limits_{\tau_d}L_S(\beta_n)$, $\alpha\leq L_S(\beta)$. Since $\mathbb{B}_{\Gamma}$ is a inferior semi-lattice we may assume $L_S(\beta_n)\leq L_S(\beta)$ We have $L_S(\beta_n)\odot S\beta\in\mathbb{B}^{S}$ and $\alpha\odot S\beta\in\mathbb{B}^{S}$ and therefore $\alpha=(\alpha\odot S\beta):S(\alpha\odot S\beta)\in\mathbb{B}^{S}$;

ii) Let $\alpha\in\mathbb{B}_{\Gamma_S}$. It is sufficient to show that, for any $\varepsilon>0$, there exists $\alpha_\varepsilon\in\mathbb{B}_{\Gamma_S}$ and $\beta_\varepsilon\in\mathbb{B}^{S}$ such that $\alpha=\alpha_\varepsilon\odot\beta_\varepsilon$ and $\mathcal{E}_{\Gamma}(\beta_\varepsilon)\leq\varepsilon$. For this purpose let $(\beta_n)_{n\in N}\subset\mathbb{B}_{\Gamma}$ such that $\beta_n\underset{\tau_d}{\rightarrow}\alpha$ and $\alpha_n=:nf(\alpha:\beta_n)-in\ [\mathbb{S}]^{S}$. We have $\mathcal{E}_{\Gamma}(\beta_n)\rightarrow0$ and $\alpha:\alpha_n\in\mathbb{B}_{\Gamma_S}$, $\alpha:\alpha_n\leq\beta_n$. We may choose $\alpha_\varepsilon=\alpha_n$ for a sufficiently large n and $\beta_\varepsilon=\alpha:\alpha_\varepsilon$.

iii) Let $\alpha^{'}=:\overline{\alpha}(-in\ [\mathbb{S}]^{S})$ and let $(\beta_n)_{n\in N}\subset\mathbb{B}_{\Gamma}$ such that $\beta_n\underset{\tau_d}{\rightarrow}\alpha^{'}$. We have $\alpha=\alpha\wedge\alpha^{'}=\lim\limits_{\tau_d}(\alpha\wedge\beta_n)\in\mathbb{B}_{\Gamma_S}$.

iv) Follows from iii) remarking that $\alpha\in\mathbb{B}_{\Gamma}$, $\beta\in\mathbb{B}^{S}\Rightarrow\alpha\wedge\beta\in\mathbb{B}^{S}$.□

Corollary 11 *Let S be a sweeping on $\mathbb{B}_\Gamma$. If we denote $\mathbb{B}^S =: (\mathbb{B}_\Gamma)_{L_S}$ then $\mathbb{B}^S$ is an extended process and*

$$(\alpha \wedge L_S\beta) \in \mathbb{B}^S \ , \ (\forall \alpha, \beta \in \mathbb{B}_\Gamma);$$
$$(S\alpha) \odot (L_S\beta) \in \mathbb{B}_\Gamma \ if \ (L_S\alpha) \geq (L_S\beta).$$

Proof. Results from the fact that $\mathbb{B}^S = Ker_S$.□

Example 17 *Let $V \subset \mathbb{R}^n, n \geq 1$, V open, $m = dx$ be the Lebesgue measure on V and $C_0^\infty(V)$ denotes the set of all infinitely differentiable functions on V with compact support. Let $u_{ij} : V \to \mathbb{R}, 1 \leq i, j \leq n$, such that*

i) $u_{ij} = u_{ji}$ for all $1 \leq i, j \leq n$
ii) $\sum_{i,j=1}^n u_{ij}(x)\xi_i, \xi_j \geq 0$ for all $\xi_i, ..., \xi_n \in \mathbb{R}$, $dx - a.e.x \in U$.
iii) $u_{ij} \in L^2_{loc}(U, dx)$, $\frac{\partial}{\partial x_i} u_{ij} \in L^2_{loc}(U, dx), 1 \leq i, j \leq n$, where the derivatives are taken in the sense of Schwartz distributions.

Define $[\mathbb{B}] =: C_0^\infty(V)$, $[\mathbb{S}] =: L^2(V; dx)$ and the system by the linear operator Γ on $[\mathbb{S}]$

$$\Gamma\alpha = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(u_{ij}\frac{\partial}{\partial x_j})\alpha \ , \ (\alpha \in [\mathbb{B}]).$$

It is necessary to have $\Gamma\alpha \in [\mathbb{S}]$ for every $\alpha \in [\mathbb{B}]$. Define the energy by

$$\mathcal{E}_\Gamma[\alpha, \beta] =: \mathcal{E}[\Gamma\alpha, \beta] = \sum_{i,j=1}^n \int \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_j} u_{ij} dx \ , \ (\alpha, \beta \in [\mathbb{B}]).$$

Then $< \mathcal{E}, [\mathbb{B}] >$ is closable on $[\mathbb{S}]^6$. Since $[\mathbb{B}]$ is dense in $[\mathbb{S}]$, its closure is a symmetric closed form on $L^2(V; dx)$.

Example 18 *Consider the previous example with $u_{ij} := \frac{1}{2}\delta_{ij}$, i.e. $\Gamma = \frac{1}{2}\Delta$ with domain $[\mathbb{B}]$. We denote the corresponding energy by $\mathcal{E}$, and its domain by $[\mathbb{S}] = H_0^{1,2}(V)$ (since the completion of $C_0^\infty(U)$ w.r.t. Γ is by definition the (1,2)- Sobolev space on V with Dirichlet boundary conditions).*

Example 19 *The Laplacian Δ is defined on all of $L^2(V; dx)$ in the sense of Schwartz distributions.Then $\Gamma =: \frac{1}{2}\Delta$ with domain $\{u \in H_0^{1,2}(V) \mid \Delta u \in L^2(V; dx)\}$ is the system corresponding to $< \mathcal{E}, [\mathbb{B}] = H_0^{1,2}(V) >$ on $[\mathbb{S}] = L^2(V; dx)$.*

Example 20 *Define $H^{1,2}(V) := \{a \in L^2(V; dx) \mid \frac{\partial a}{\partial x_i} \in L^2(V; dx), 1 \leq i \leq n\}$ with derivatives in the Schwartz distributions sense (i.e., $H^{1,2}(V)$ is the(1,2)-Sobolev space on V with Neumann-boundary conditions). Define the energy $\mathcal{E}^\#$*

$$\mathcal{E}^\#(a, b) := \frac{1}{2}\sum_{i=1}^n \int \frac{\partial a}{\partial x_i}\frac{\partial b}{\partial x_i} dx \ ; \ (u, v \in H^{1,2}(V)).$$

Then $< \mathcal{E}^\#, [\mathbb{B}^\#] =: H^{1,2}(V) >$ is a energy on $[\mathbb{S}] =: L^2(V; dx)$ which extends $< \mathcal{E}, [\mathbb{B}] = H_0^{1,2}(V) >$. Note that in general $H^{1,2}(V) \neq H_0^{1,2}(V)$, e.g. if V is a ball then $1 \in H^{1,2}(V)$, but $1 \notin H_0^{1,2}(V)$. This is different if $V = \mathbb{R}^n$.

Remark 47 $H_0^{1,2}(\mathbb{R}^n) = H^{1,2}(\mathbb{R}^n)$.

Example 21 *Let $m = dx$ and let "$\hat{\cdot}$ "resp."$\check{\cdot}$ " denote Fourier transform, i.e. $f(x) = (2\pi)^{-n/2} \int \exp[i < x, y >_{L^2}] f(y) dy$, resp. its inverse. Define for $0 < \alpha \leq 1$ $(-\Delta)^\alpha a := (|x|^{2\alpha} \hat{u})^{\vee}$ $(\in L^2(\mathbb{R}^n; dx)); a \in C_0^\infty(\mathbb{R}^n)$. Then $(-\Delta)^\alpha$ is a system on $[\mathbb{S}] =: L^2(\mathbb{R}^n; dx)$ with dense basic space $[\mathbb{B}] =: C_0^\infty(\mathbb{R}^n)$. Define the energy $\mathcal{E}^{(\alpha)}_{(-\Delta)^\alpha}$*

$$\mathcal{E}^{(\alpha)}_{(-\Delta)^\alpha}(a, b) =: \frac{1}{2}\int \hat{u}\bar{\hat{v}} \, |x|^{2\alpha} \, dx \ ; \ (a, b \in C_0^\infty(\mathbb{R}^n))$$

where "$\bar{\cdot}$ " means complex conjugation. Its closure $< \mathcal{E}^{(\alpha)}_{(-\Delta)^\alpha}, [\mathbb{B}] =: H^{\alpha,2}(\mathbb{R}^n) >$ is hence a symmetric closed form on $[\mathbb{S}] =: L^2(\mathbb{R}^n; dx)$.

Example 22 *If* $0 < \alpha < 1$ *then* $a \in [\mathbb{B}]$ *if and only if* $\iint \frac{|a(x)-a(y)|^2}{|x-y|^{2\alpha+n}} dxdy < \infty$ *and for* $a, b \in H^{\alpha,2}(\mathbb{R}^n)$

$$\mathcal{E}[a,b] = K_\alpha \iint \frac{(a(x)-a(y))(b(x)-b(y))}{|x-y|^{2\alpha+n}} dxdy$$

for some constant K_α *(independent of* a, b*).*

Example 23 *Let* $\mathbb{B} := V \subset \mathbb{R}^n$ *,* V *open,* $m := \sigma \cdot dx$ *for some* $\sigma \in L^1_{loc}(V; dx)$ *, such that* $\int_v \sigma\ dx > 0$ *for all* $V \subset V$ *,* V *open. Let* $\underline{p} := (p_1, ..., p_n)$ *with* $p_i \in L^1_{loc}(V; dx)$*,* $p_i \geq 0$ dx *- a.e. and define for* $a, b \in C_0^\infty(V)$*.*

$$\mathcal{E}_{\underline{p}}(u,v) := \sum_{i=1}^n \int \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} p_i dx.$$

Then $< \mathcal{E}_{\underline{p}}, [\mathbb{B}] =: C_0^\infty(V))$ *is a densely defined symmetric positive definite bilinear form on* $L^2(V; \sigma \cdot dx)$ *. We want to give conditions on* p_i, σ *so that* $< \mathcal{E}_{\underline{p}}, [\mathbb{B}] >$ *is closable on* $[\mathbb{S}] =: L^2(V; \sigma \cdot dx)$ *. Define for* $p \in B^+(V)$

$$R(p) := \{x \in V \mid \int_{\{y \in V \mid |x-y| \leq \varepsilon\}} p^{-1}(y)dy < \infty \ for\ some\ \varepsilon > 0\}.$$

Here we use the convention that $\frac{a}{0} =: (sign\ a) \cdot \infty$ *. Then we ask* $R(p)$ *is open and* $p > 0$ dx *-a.e. on* $R(p)$ *and* $R(p)$ *is the largest open set* $U \subset V$ *such that* $p^{-1} \in L^1_{loc}(U; dx)$ *.*

Because any local Dirichlet space is an energetic space, the previous theorem shows us how one can associate a local Dirichlet space to a von Neumann algebra.

Definition 116 *Let* $\mathbb{B}$ *be a W-like process. The map* $\mathcal{E}_\mathcal{C} : [\mathbb{S}] \times [\mathbb{S}] \to R$ *defined by*

$$\mathcal{E}_\mathcal{C}[\alpha, \beta] =: \frac{\mathcal{C}[\alpha,\beta] + \mathcal{C}[\beta,\alpha]}{2}$$

is an energy which will be called the energy associated to the W-like process $\mathbb{B}$*.*

It is an easy exercise to verify that any symmetric Dirichlet form is an energy, and its associated potentials form a W-like process.

Chapter 5 Unified Theory of Deterministic and Stochastic Processes

5.1 Types of Continuous Processes

Definition 117 *An extended process $\mathbb{B} = (B, \leq, 0, \infty, \prec)$ is called C-like (ces for short) if for any $A \subset B$ with a specific order majorant we have:*

(C_1) $\bigvee_{\odot} A$ is the specific order supremum of a countable subset of A ;

(C_2) if $A^{'}$ is the class of specific order majorants of A, then

$$\bigvee A \leq_{\odot} A^{'};$$

Definition 118 *We shall introduce the unary internal operation $\lfloor . \rfloor$ on B such that*

$$\begin{aligned} \lfloor a \odot b \rfloor &= \lfloor a \rfloor \odot \lfloor b \rfloor \\ \lfloor \lfloor a \rfloor \rfloor &= \lfloor a \rfloor \end{aligned}$$

Remark 48 *The previous definitions can be extended to arbitrary families of elements Δ.*

Example 24 *Let a be a positive superharmonic function on a set X, which has a subharmonic minorant. Then $\lfloor a \rfloor$ is the greatest harmonic minorant*

Example 25 *Let $\Delta = \{x_t(.), \mathcal{F}(.), t \in T\}$ be a class of stochastic processes with a common linearly ordered parameter set T, on a common probability space, and adapted to a common filtration $\mathcal{F}(.)$. Then $\lfloor a \rfloor$ is the greatest martingale minorant.*

Example 26 *We assume a specified filtered probability space with an arbitrary linearly ordered parameter set T. Denote by $(\mathcal{C}, \leq)$ the lattice, in the essential order, of those stochastic process equivalence classes under standard modification which contain positive supermartingales. The space $([\mathcal{C}], \leq)$ consists then of the stochastic process equivalence classes which contain supermartingales having positive supermartingale essential order majorants.$(\mathcal{C}, \leq, 0, \infty, \prec)$ is a model for the $\mathcal{C}$-like process.*

Remark 49 *The lattice ideas was first introduced in martingale theory by Kricksberg ([Kri 56]).*

Definition 119 *A continuous extended process is an process ${}^c\mathcal{C} \subset \mathcal{C}$ with the following properties:*

C_1)the regularized act as an application $\ddot{\cdot} : [\mathcal{C}] \to [{}^c\mathcal{C}]$ such that

- if $a \leq_{\odot} b$ in $[C]$ then $\ddot{a} \leq_{\odot} \ddot{b}$ in $[{}^cC]$, $(\forall a, b \in \mathcal{C})$
- if $a \in \mathcal{C}$ then $\ddot{a} \in^c \mathcal{C}$, $(\forall a \in \mathcal{C})$

C_2)$a \leq_{\odot} b$ in $[{}^c\mathcal{C}]$ iff $a \leq_{\odot} b$ in $[\mathcal{C}]$, $(\forall a, b \in \mathcal{C})$

Example 27 *In the continuous parameter context the lattice $({}^c\mathcal{C}, \leq)$ consists of the equivalence classes containing positive right continuous supermartingales . If $T = \mathbb{R}^+$ an equivalence class in $[{}^c\mathcal{C}]$ contains a right continuous supermartingale if the class contains almost surely right continuous supermartingale and two almost surely right continuous supermartingales in the same equivalence class are indistinguishable. If $a(\cdot) \in [{}^c\mathcal{C}]$ then the regularized has the form*

$$\ddot{a}(x) = \liminf_{r \downarrow t, r \in Q} a(r)$$

Remark 50 *(Rao [MR 69]) An adapted almost surely right continuous L^1 bounded process is a member of $[{}^c\mathcal{C}]$ iff it is a quasimartingale (as defined in Fisk 1965).*

We can also exploit the correspondence from (Follmer [Fol 73]) between member of $[{}^c\mathcal{C}]$ and measures on the σ algebra of predictable sets determined by the filtration of quasimartingales.

Example 28 *Let D be a Greenian subset of $\mathbb{R}^n$, $n \geq 2$. Denote by ${}^c\mathcal{C}$ the class of positive superharmonic functions on D, in pointwise order and with addition. It follows that $[{}^c\mathcal{C}]$ is the class of superharmonic functions on D with positive superharmonic majorants, ordered by pointwise inequality and with addition.*

Proposition 53 $({}^c\mathcal{C}, \prec, \leq, , \perp, \top)$ *with an causal order such that $\odot$ is the usual superposition (+),* 0 *as* $\perp$, ∞ *as deadlock is an continuous $\mathcal{C}$-like process.*

Proof. Axioms B_1), B_3), and B_4) are satisfied because $({}^c\mathcal{C}, \leq)$ is a vectorial lattice.

Axiom B_5) (with $\odot$ as the usual-linear-superposition): trivial for finite-valued functions, is true in general case because $a = b$ on the set of finiteness of a; so $a = b$ quasi-everywhere and therefore everywhere.

Axiom B_2) : if $A \subset^c \mathcal{C}$ has an h-superharmonic minorant then according to the Fundamental Convergence Theorem, the lower semicontinuous smoothing of the pointwise infimum of A is superharmonic, and this function is $\bigwedge A$. If A has a superharmonic majorant then $\vee A$ exists and is the $({}^c\mathcal{C}, \leq)$ infimum of the class of superharmonic majorants of A. Thus $({}^c\mathcal{C}, \leq)$ is a conditionally complete lattice.

Axiom B_6): if A has an superharmonic majorant, some countable subset of A has the same $({}^c\mathcal{C}, \leq)$ supremum as A. If A is directed upward $\bigvee A$ is the pointwise supremum; if A is not directed upward, we shall apply the result in the directed case to the set of $({}^c\mathcal{C}, \leq)$ suprema of finite subsets of A.□

Definition 120 *The $\mathcal{H}$-like process is a specific pseudoband $\mathcal{H}$ with the following properties:*

(H_1) $\leq_\odot$ *and* $\leq$ *coincides on* $\mathcal{H}$

$$\leq_{\odot|\mathcal{H}} = \leq_{|\mathcal{H}}$$

(H_2) *If* $A \subset \mathcal{H}$ *then* $\bigwedge_\odot A = \lfloor A \rfloor$

(H_3) $(\mathcal{H}, \leq_\mathcal{H})$ *is a complete lattice*

Definition 121 *An $\mathcal{H}$-like continuous process is an $\mathcal{H}$-like space ${}^c\mathcal{H}$ which is a conditionally complete sublattice of* $({}^c\mathcal{C}, \leq_\odot)$

Definition 122 *Denote by $\mathcal{H}(D)$ the set of all positive harmonic functions on D .*

Proposition 54 *$\mathcal{H}(D)$ is a model for $\mathcal{H}$-like continuous extended processes.*

Proof. A superharmonic function specific order majorized by an harmonic function is itself harmonic. Let H be a set of positive harmonic functions on D with $\bigvee_\odot H = h$, then h is a specific order majorant of each member of H; so h is positive and superharmonic and its greatest harmonic minorant on D $\lceil h \rceil$ is a specific order majorant of H. Hence $h = \lceil h \rceil$ and h is harmonic. The essential and specific orders coincides on$\mathcal{H}(D)$.□

Proposition 55 *The set of the finite signed measures on ∂D of the ball D is a model for $\mathcal{H}$-like continuous extended processes.*

Proof. The Riesz representation theorem establish a lattice isomorphism between the set of the finite signed measures on ∂D and $\mathcal{H}(D)$.□

Proposition 56 *The set of stochastic process equivalence classes under standard modification which contain positive martingales is a model for $\mathcal{H}$-like continuous extended processes.*

Definition 123 *The B-like process is a specific closed specific pseudoband B such that*

$$\mathcal{B} = \mathcal{H}^\perp \text{ and } \mathcal{C} = \mathcal{H} \odot \mathcal{B}$$

Definition 124 *The B-like continuous process is B-like space ${}^cB$ which is a conditionally complete sublattice of $({}^c\mathcal{C}, \leqslant)$ such that*

$${}^c\mathcal{B} = ({}^c\mathcal{H})^\perp \text{ and } {}^c\mathcal{C} = ({}^c\mathcal{H}) \odot ({}^c\mathcal{B}).$$

Proposition 57 *The set of stochastic process equivalence classes under standard modification which contains positive supermartingale potentials (i.e. positive supermartingale a for which $\lfloor a \rfloor = 0$) is a model for $\mathcal{H}$-like extended processes.*

Remark 51 *A martingale is in an $\mathcal{H}$ equivalence class iff the martingale is L^1 bounded.*

Proposition 58 *The set of stochastic process equivalence classes under standard modification which contain right continuous positive supermartingale potentials is a model for $\mathcal{H}$-like continuous extended processes.*

Remark 52 *If a is L^1 bounded almost surely right continuous martingale, then $a \in^c \mathcal{H}$.*

Proposition 59 *The set of all positive superharmonic potentials on D (noted $B(D)$) is a model for B-like continuous extended processes.*

Proof. An element of B(D) specific order majorized by a superharmonic potential $G_D\mu$ of a measure is itself such a potential. If A is a set of such potentials with $\bigvee_{\odot} A = a$, then s is a specific order majorant of each member of A ; so s is positive and superharmonic and the potential $a : \overline{a}$ is also a specific order majorant of A and therefore must be a; that is, a is a potential.□

Proposition 60 *The set M_a^+ of measures on D whose potentials are superharmonic is a model for B-like continuous extended processes.*

Proof. The map $\mu \mapsto G\mu$ is a one-to-one order preserving map from M_a^+ onto B(D).□

Definition 125 *An Qb-like process is a set like*

$$\mathrm{Q} = \{a \in \mathcal{C} \ ; \ a = \bigvee_{\odot} A, \ A \subset \mathcal{B} \textit{is a set of bounded elements } \}$$

Proposition 61 *The set of stochastic process equivalence classes under standard modification which contain quasi-bounded positive supermartingales is a model for Qb-like basic spaces.*

Proposition 62 *An Qb-like continuous process is an Qb-like process which is a conditionally complete sublattice of $({}^c\mathcal{C}, \leq_{\odot})$.*

Proposition 63 *The set of stochastic process equivalence classes under standard modification which contain quasi-bounded positive supermartingales and all bounded positive supermartingales involved in definition are supposed to be almost surely right continuous is a model for Qb-like continuous process.*

Proposition 64 *If D is provided with a boundary ∂D by a metric compactification, then the Perron-Wiener-Brelot (PWB) method ([Bre 60, Doo 80) solutions for this boundary are model for the Qb-like process.*

Remark 53 *We have also a converse result for Prop. 64. If the boundary is internally resolutive then every element of Qb-like process is a PWB solution.*

Definition 126 *Define*

$$\mathcal{HQ} = \mathcal{H} \cap \mathcal{Q} \ \ \textit{and } \mathcal{B}\mathrm{Q} = \mathcal{B} \cap Q.$$

Remark 54 *$\mathcal{H}Q$ is the band in the set of stochastic process equivalence classes under standard modification which contain quasi-bounded positive supermartingales generated by the equivalence class of process all of whose random variables are identically* 1.

Proposition 65 *We have*

$$\mathcal{B}Q = (\mathcal{H}Q)^{\perp} \ \textit{and } Q = \mathcal{B}Q \odot \mathcal{H}Q$$

Definition 127 *Define*

$${}^c\mathcal{H}Q = ({}^c\mathcal{H}) \cap ({}^cQ) \ \textit{and } {}^c\mathcal{B}Q = ({}^c\mathcal{B}) \cap ({}^cQ).$$

Proposition 66 *We have*

$${}^c\mathcal{B}Q = ({}^c\mathcal{H}Q)^{\perp} \ \textit{and } {}^cQ = ({}^c\mathcal{B}Q) \odot ({}^c\mathcal{H}Q)$$

Definition 128 *Let*

$$\begin{aligned} \mathcal{R} &= Q^{\perp}, \\ {}^c\mathcal{R} &= ({}^cQ)^{\perp}. \end{aligned}$$

Proposition 67 *The class of stochastic process equivalence classes under standard modification for which every bounded $\mathcal{C}$-specific order minorant is a standard modification of the identically zero process is a model for $\mathcal{R}$.*

The class of functions for which every bounded ${}^c\mathcal{C}$-specific order minorant is the identically zero function is a model for ${}^c\mathcal{R}$.

Definition 129 *Let*

$$\begin{aligned} \mathcal{HR} &= \mathcal{H} \cap \mathcal{R},\ \mathcal{BR} = \mathcal{B} \cap \mathcal{R} \\ & \quad \textit{and} \\ {}^c\mathcal{HR} &= ({}^c\mathcal{H}) \cap ({}^c\mathcal{R}),\ {}^c\mathcal{BR} = ({}^c\mathcal{B}) \cap ({}^c\mathcal{R}) \end{aligned}$$

Proposition 68 *A function a is member of the ${}^c\mathcal{HR}$ model iff the Martin representation of a has a representing measure which is singular.*

Proposition 69 *A stochastic process is a member of $B\mathcal{R}$ iff it is local martingale (as defined by Ito and Watanabe in [IW 65]).*

Proposition 70 *A necessary condition for $h \in^c \mathcal{H}$ to belongs to ${}^c\mathcal{HR}$ is $h \wedge k \in^c \mathcal{B}$ for every strictly positive constant k and it is a sufficient condition if the above condition is satisfied for some strictly positive constant k*

Definition 130 *Let*

$$\mathcal{BR} = \mathcal{B} \cap \mathcal{R} \textit{ and } {}^c\mathcal{BR} = ({}^c\mathcal{B}) \cap ({}^c\mathcal{R}).$$

Proposition 71 *We have*

$$\begin{aligned} \mathcal{C} = \mathcal{H}Q \odot \mathcal{HR} \odot \mathcal{B}Q \odot \mathcal{BR} & \textit{ and} \\ {}^c\mathcal{C} \quad = \quad ({}^c\mathcal{H}Q) \odot ({}^c\mathcal{HR}) \odot ({}^c\mathcal{B}Q) \odot ({}^c\mathcal{BR}). \end{aligned}$$

Definition 131 *A discrete process is an process ${}^d\mathcal{C} \subset \mathcal{C}$ with the following properties:*

D_1) the regularized act as an application $\ddot{\cdot} : [\mathcal{C}] \to [{}^d\mathcal{C}]$ such that

- if $a \leq_\odot b$ in $[C]$ then $\ddot{a} \leq_\odot \ddot{b}$ in $[{}^dC]$, $(\forall a, b \in \mathcal{C})$
- if $a \in \mathcal{C}$ then $\ddot{a} \in^d \mathcal{C}$, $(\forall a \in \mathcal{C})$

D_2) $a \leq_\odot b$ in $[{}^d\mathcal{C}]$ iff $a \leq_\odot b$ in $[\mathcal{C}]$, $(\forall a, b \in$C)
In the discrete parameter context $T = \mathbb{Z}^+$.

5.2 Semantic Developments

Definition 132 *By an $\mathcal{E}$-model we mean a fixed connected Greenian set $D \subset \mathbb{R}^n$ such that*

$\mathcal{H}$	*means*	*the class of all positive harmonic functions on D*
$\mathcal{B}$	*means*	*the class of all positive superharmonic potentials on D*
${}^c\mathcal{C}$	*means*	*the class of all positive superharmonic functions on D*
Q	*means*	*the class of all positive quasi-bounded functions on D*

$\mathcal{D}$	*means*	*the class of all real valued Borel measurable functions u on D for which if $\xi$is in D and if B_n is an increasing sequence of open relatively compact subsets of D with union D then the sequence $[u_{\vert\partial B_n}, \mu_{B_n}(\xi,\cdot)]_{n\in\mathbb{Z}^+}$ of coupled functions and measures is uniformly integrable.*
$\mathcal{L}^m$	*means*	*the class of all real valued Borel measurable functions u on D for which if $\xi$is in D and if B_n is an increasing sequence of open relatively compact subsets of D with union D then $sup_{n\in\mathbb{Z}^+}\mu_{B_n}(\xi, \vert u\vert^m) < \infty$*

Definition 133 *By an $\mathcal{M}$-model we mean stochastic processes with parameter set R^+, on a complete probability measure space $(\Omega, \mathfrak{F}, Br)$ provided with a right continuous filtration $\mathfrak{F}$ such that $\mathfrak{F}(0)$ contains the null sets*

$\mathcal{H}$	*means*	*The class of stochastic process equivalence classes under standard modification which contain positive martingales,*
$\mathcal{B}$	*means*	*The class of stochastic process equivalence classes under standard modification which contain positive supermartingale potentials,*
${}^c\mathcal{C}$	*means*	*The class of stochastic process equivalence classes under standard modification which contain positive right continuous supermartingales,*
Q	*means*	*The class of stochastic process equivalence classes under standard modification which contain quasi-bounded positive supermartingales.*

$\mathcal{D}$	*means*	*the class of all stochastic processes* $x(\cdot)$ *for which the family* $\{x(T) : T$ *optional, countable valued with values in* $I\}$ *of random variables is uniformly integrable.*
$\mathcal{L}^m$	*means*	*the class of all stochastic processes* $x(\cdot)$ *for which* $sup\{M(\|x(T)\|^m) : T$ *optional, countable valued with values in* $I\} < \infty$

Remark 55 *For* $m > 1$ *we have*

$$
\begin{array}{ll}
{}^c\mathcal{C} \subset \mathcal{L}^1 & \textit{in both } \mathcal{M}-model \textit{ and } \mathcal{E}-model \\
{}^c\mathcal{C} \cap \mathcal{L}^m \subset \mathcal{D} & \textit{in both } \mathcal{M}-model \textit{ and } \mathcal{E}-model \\
\mathcal{L}^m \subset \mathcal{D} \quad \textit{in } \mathcal{M}-model &
\end{array}
$$

Proposition 72 *Let* $b \in Q$ *and* $a \in^c \mathcal{C}$ *. Define the class of sets:* $X_r =: \{b > r\}$ *and the sweeping operator:* $\sigma_{X_r}[b] =: \bigvee_{x \in X} (b \wedge x)$*. We have*

- in the $\mathcal{M}-model$: $\underset{n\to\infty}{lim} (\sigma_{X_n}[b(\cdot)]) = 0$ except a set of null capacity.
- in the $\mathcal{E}-model$: $\underset{r\to\infty}{lim} (\sigma_{X_r}[b]) = 0$ except a set of null capacity.

Proposition 73 *Let* $h \in \mathcal{H}$*. We have*

- in the $\mathcal{M}-model$ a necessary condition for the stochastic process $h(\cdot)$ to belongs to $Q^{\perp}$ is
$$h(\cdot) \wedge k \in \mathcal{B}$$
for every strictly positive constant k and it is a sufficient condition if the above condition is satisfied for some strictly positive constant k .
- in the $\mathcal{E}-model$ a necessary condition for $h \in Q^{\perp}$ is
$$h \wedge k \in \mathcal{B}$$
for every strictly positive constant k and it is a sufficient condition if the above condition is satisfied for some strictly positive constant k or equivalently
$$\underset{r\to\infty}{lim} \; b(r) = 0 \text{ a.s.}$$

Proposition 74 *Let* $a \in^c \mathcal{C}$ *with a component* $a_Q \in Q^{\perp}$*. We have*

- in the $\mathcal{M}-model$: $\underset{n\to\infty}{lim} (\sigma_{X_n}[a(\cdot)]) = a_Q$ except a set of null capacity.
- in the $\mathcal{E}-model$: $\underset{r\to\infty}{lim} (\sigma_{X_r}[a]) = a_Q$ except a set of null capacity.

Proposition 75 *Let* $a \in^c \mathcal{C}$*. We have*

- in the $\mathcal{M}-model$ a necessary and sufficient condition for the stochastic process $a(\cdot)$ to belongs to $Q^{\perp}$ is $\sigma_{X_n}[a(\cdot)] = a$ except a set of null capacity, for every strictly positive constant k .
- in the $\mathcal{E}-model$ a necessary and sufficient condition for $a \in Q^{\perp}$ is $\sigma_{X_n}[a] = a$ for every strictly positive constant k .

Proposition 76 *Let* $a \in B$*. The following assertions are equivalent*

- in the $\mathcal{M}-model$

  i) the process $a(\cdot)$ belongs to BQ;

  ii) the process $a(\cdot)$ belongs to D;

  iii) $\underset{n\to\infty}{lim} (\sigma_{X_n}[a(\cdot)]) = 0$ except a set of null capacity.
- in the $\mathcal{E}-model$, $a = G_D\mu$

  i) $a \in BQ$; ii) $a \in D$;

  iii) $\underset{k\to\infty}{lim} (\sigma_{X_k}[a]) = 0$ except a set of null capacity;

  iv) μ vanishes on sets of null capacity.

5.3 Models from Biomedicine

This chapter is rather technical and intends to provide strong mathematical arguments that biopotentials appearing in human muscles and heart activity are models of our process algebra. We show these by a two step method: we show that the mathematical models of biopotentials are solutions of Laplace and Poisson equations, and thus potentials in the sense of classical potential theory. We gave in the previous chapters examples showing that potentials in axiomatic potential theory are models of continuous process algebra. The way potentials in classical potential theory are axiomatized in the modern approaches can be found in the potential theory literature.

5.3.1 $\mathcal{H}$B-like processes in cardiac electrogenesis

We want to express the fact that the basic occurrences of processes associated with concurrent biosystems such as the heart are modelled by potentials defined in $\mathcal{H}$B-like processes. More concerns on parallel evolution and interaction between the basic occurrences will be considered in a forthcoming paper.

If we consider the body as an insulated volume conductor bounded by an irregular surface, containing intracardiac current sources, then the total current $\overrightarrow{J}$ can be expressed everywhere in the volume conductor as the sum of a passive term $\sigma\overrightarrow{E}$ and a source term $\overrightarrow{J}_i$, where $\overrightarrow{E}$ is the electric field, σ is the conductivity for isotropic media and $\overrightarrow{J}_i$. Thus $\overrightarrow{J} = \sigma\overrightarrow{E} + \overrightarrow{J}_i$.

Since the electric field $\overrightarrow{E}$ is conservative, it can be expressed as the gradient of a scalar **potential** function

$$\overrightarrow{E} = -\nabla p$$

The source term $\overrightarrow{J_i}$, where the index "i" stands for impressed current, represents the contribution of a non-conservative field which accounts for the locally generated current density due to conversion of chemical to electrical energy. This conversion occurs in the cell membrane.

The charge conservation law states that the rate of change of the quantity of charge within a volume v bounded by a fixed surface S, is always equal to the flux of change per unit time through this surface. The law may thus be written as $\frac{dq}{dt} = -\int_S \overrightarrow{J} \cdot d\overrightarrow{S}$. But $\frac{dq}{dt} = \frac{d}{dt}\int_v \rho dv = \int_v \frac{\partial\rho}{\partial t} dv$ and by means of the divergence theorem for the vector $\overrightarrow{J}$ we obtain $\nabla\overrightarrow{J} + \frac{\partial\rho}{\partial t} = 0$ everywhere in v. The conductivity σ is a real quantity due to the resistive nature of biological tissues at the low frequencies contained in bioelectric signals. This is true everywhere in v except in the membrane,which has both resistive and capacitive properties. The bioelectric current field may then be treated as stationary since it does vary with time, but the time dependence adds no distinct source term. Then, under stationary conditions $\nabla\overrightarrow{J} = 0$. If the volume conductor is isotropic and if the membrane is an ideal surface with zero thickness, then $\nabla\overrightarrow{J} = \nabla(-\sigma\nabla p + \overrightarrow{J_i}) = 0$,

$$\nabla^2 a = \frac{\nabla\overrightarrow{J_i}}{\sigma} \tag{5.1}$$

Thus a satisfies a **Poisson's equation** which reduces to the following **Laplace's equation**

$$\nabla^2 a = \Delta p = 0 \tag{5.2}$$

in regions of the volume conductor where there are no sources. The source function $-\nabla\overrightarrow{J_i}$ corresponds to the charge density as defined in electrostatics and $-\nabla\overrightarrow{J_i}dv$,where dv is a volume element,can be considered as contributing to the **potential** field in the same way as a point charge contributes to the electrostatic **potential** field. Since the current density $\overrightarrow{J_i}$ refers to the active membrane and has a given volume distribution,then

$$a(\overrightarrow{x}) = \frac{1}{4\pi\sigma}\int_v \frac{-\nabla\overrightarrow{J_i}}{d} dv \tag{5.3}$$

where $d = |\overrightarrow{x} - \overrightarrow{y}|$ and x, y refer to the field point and to the source element position respectively. The assumption is here made that the medium is infinite and homogeneous.

An equivalent expression for the **potential** can be obtained by using the vector identity $\nabla(\overrightarrow{J_i} \cdot \frac{1}{d}) = \frac{1}{d}\nabla\overrightarrow{J_i} + \overrightarrow{J_i}\nabla(\frac{1}{d})$ and by integrating both sides within a volume that completely contains all sources $\overrightarrow{J_i}$. By applying the divergence theorem the first volume integral reduces to a surface integral and since $\overrightarrow{J_i}$ is necessarily zero over this bounding surface its contribution is zero: $0 = \int_v \frac{\nabla \overrightarrow{J_i}}{d} dv + \int_v \overrightarrow{J_i} \cdot \nabla(\frac{1}{d}) dv$. Then the **potential** $a(\overrightarrow{x})$ is given by

$$a(\overrightarrow{x}) = \frac{1}{4\pi\sigma} \int_v \overrightarrow{J_i} \cdot \nabla(\frac{1}{d}) dv \tag{5.4}$$

In electrostatics the **potential** at a point $\overrightarrow{x}$ due to a dipole at a point $\overrightarrow{y}$is obtained by the **superposition** (i.e. $\odot$) of the **potential** due to a positive point charge b at a point $\overrightarrow{y}$,i.e. b/d with $d = |\overrightarrow{x} - \overrightarrow{y}|$, and the **potential** due to an equal charge of opposite sign $-b$ at a very small distance from the first charge. Let this small distance be taken along one coordinate axis, e.g. Δy_1. Since the **potential** function of a point charge has a first order singularity at the position of the charge, i.e.$x = y$, $d = 0$, then the new system of two charges has a higher order singularity at $d = 0$. This procedure is equivalent to differentiating the **potential** function q/r with respect to the source variable y_1. We note that this **potential** is also a solution of **Laplace's equation** since it is the derivative of one of its solutions. Then $a(\overrightarrow{x}) = \frac{1}{4\pi\sigma} \frac{\partial(b/d)}{\partial y_1} \Delta y_1 = \frac{q\Delta y_1}{4\pi\sigma} \cdot \frac{x_1 - y_1}{d^3}$. By defining $\lim_{b \to \infty, \Delta y_1 \to 0} b \cdot \Delta y_1 = \overrightarrow{a}$ as the dipole moment,then $a(\overrightarrow{x}) = \frac{1}{4\pi\sigma} \overrightarrow{a} \cdot \nabla(\frac{1}{d})$ where ∇ is the gradient vector operator acting on source points $\overrightarrow{y}$.

We want to obtain an interpretation of the volume density distribution of the dipole moment. The sources are located in the cellular membrane. If we consider the membrane as a thin layer we can treat it as a mathematical surface separating two **potential** distributions just inside and outside of the cell membrane. In such a system the following conditions are satisfied:

- continuity of the normal component of the current density across the membrane $\sigma_i \frac{\partial p}{\partial n} |_i = \sigma_o \frac{\partial p}{\partial n} |_o$where $\overrightarrow{n}$ is the normal unit vector to the membrane surface which is approached from inside (i) and outside (o) respectively and σ_i and σ_o are the conductivities on both sides of the surface;

- the **potential** is discontinuous across the membrane $a_m = a_i - a_o$ where a_m is the transmembrane **potential**. For a single fiber immersed in an infinite volume conductor the scalar function σp, with $\sigma = \sigma_i$ inside the fiber, and $\sigma = \sigma_o$ outside the fiber, satisfies the following **Laplace's equation**

$$\nabla^2(\sigma p) = 0 \tag{5.5}$$

In addition, the function σp is discontinuous across the fiber membrane and its normal derivative is continuous. These properties of the function σp characterize the presence of a double layer on the surface element dS. The moment of the double layer is normal to dS, has strength $(\sigma_o a_o - \sigma_i a_i) \cdot dS$ and is oriented according to the outward normal Then the function σp is given by $\sigma p(\overrightarrow{x}) = \frac{1}{4\pi} \int_B (\sigma_o a_o - \sigma_i a_i) \cdot \nabla(\frac{1}{d}) \cdot d\overrightarrow{S}$ and the **potential function** is

$$a(\overrightarrow{x}) = \frac{1}{4\pi\sigma} \int_S (\sigma_o a_o - \sigma_i a_i) \nabla(\frac{1}{d}) \cdot d\overrightarrow{S} \tag{5.6}$$

where $\sigma = \sigma_o$ if the field point is external to the fiber and $\sigma = \sigma_i$ if the field point is internal to the fiber. We note that $\nabla(1/d) \cdot d\overrightarrow{S} = d\Omega$ is the solid angle subtended by the source element dS at the field point $\overrightarrow{x}$.

$$a(\overrightarrow{x}) = \frac{1}{4\pi\sigma} \int_S (\sigma_o a_o - \sigma_i a_i) d\Omega$$

In the extracellular volume the **potential** due to an active fiber is given by a dipole layer source covering the membrane. The dipole moment magnitude per unit area is given by the discontinuity of the function σp across the membrane surface.

Whenever there is an electrical source distribution on a given surface S (single layer for charge distribution, double layer for dipole distribution) then the source magnitude is determined by the associated scalar or vector function which is discontinuous across the surface S. The value of the discontinuity of the function is proportional to the magnitude of the surface distribution. The function which is discontinuous across the surface is the one which contains the product of the source magnitude times a term which is the solid angle subtended at a point by the surface element. For the single layer the charge distribution is proportional to the discontinuity of the normal derivative of the **potential** (normal component of the electric field) at the surface S.

If we consider an idealized uniform cylindrical fiber in an infinite, homogeneous conducting medium, then the equation may be transformed into

$$a(\overrightarrow{x}) = \frac{1}{4\pi\sigma_o} \int_v \frac{I_m}{d} dv \tag{5.7}$$

where I_m is the transmembrane current density and the volume of integration is that of the cylinder. This result is obtained by assuming the core conductor model to be valid for the cylindrical fiber. Under this assumption the transmembrane current density I_m is given by the spatial change of the intracellular longitudinal current I. From linear cable theory, in a cylinder oriented along coordinate axis x_1 it is $I_m = -\frac{\partial I_{x_1}}{\partial x_1}$. The intracellular current density I_{x_1} is generated by a spatial gradient of the intracellular **potential** $a_i : I_{x_1} = -\sigma x_1 \frac{\partial p_i}{\partial x_1}$ where σ_{x_1} is the intracellular conductivity along the cylinder axis. Thus $I_m = \sigma_{x_1} \frac{\partial^2 a_i}{\partial x_1^2}$ is proportional to the second derivative of the intracellular **potential** and is constant over any cross section of the cylinder.

Remark 56 *Inverse and Forward Models in Electrocardiology*

It is the purpose of clinical electrocardiology to obtain some information on the intracardiac electrical events from body surface measurements. Over a century ago Helmholtz showed that an endless variety of electrical generators (electro-motive forces) located within a volume conductor can produce one and the same **potential** distribution at its surface. This statement can be proved by the following reasoning. Let an arbitrary surface S, internal to a volume conductor, surround all current sources so that no current source exists outside or on the surface S. On the surface S it is always possible to spread a double layer distribution and leave the sources inside so that no current will flow outside the volume bounded by S. When this is done, surface S behaves as the boundary between a conductor and an insulator. Let a be the **potential** function of the real sources in the absence of the double layer on S, and b the **potential** function of the double layer on S which would prevent the currents generated by the internal sources from flowing through S. The **superposition** $a \odot b$ of a and b must satisfy the following conditions within the entire volume conductor

$$\nabla^2 a = \frac{\nabla \overrightarrow{J}_i}{\sigma} \textit{ at points occupied by sources} \tag{5.8}$$

$$\nabla^2 b = -\frac{\mu}{\sigma} \textit{ at the surface S of the double layer} \tag{5.9}$$

$$\nabla^2 (a \odot b) = 0 \textit{ elsewhere} \tag{5.10}$$

$$\frac{\partial (a \odot b)}{\partial n} = 0 \textit{ on } S \tag{5.11}$$

where $\nabla \overrightarrow{J_i}$ is the internal source function and μ is the magnitude of the double layer density distribution normal to S and oriented along the outward normal. Both b and a are **harmonic functions** outside S and due to condition (5.10), according to a general theorem (Kellog 1926), it must be

$$a \odot b = const \tag{5.12}$$

outside S, or $a = -b$ but for an additive constant which can be assumed to be zero, i.e. $\frac{1}{4\pi\sigma} \int_v \frac{-\nabla \overrightarrow{J_i}}{d} dv = -\frac{1}{4\pi\sigma} \int_S \mu \nabla(\frac{1}{d}) \cdot d\overrightarrow{S}$.

This equations provide a pure biological model for our extended processes.

At a point x of the surface S the **potential** function a is continuous while the **potential** function b has a discontinuity which equals the strength of the double layer, i.e. $\frac{\mu(S)}{\sigma} = \Phi_S(S)$, where Φ_S is the **potential** measured at the surface S. It follows from Eq that the **potential** outside S can be given either by **potential** function

$$a = \frac{1}{4\pi\sigma} \int_v \frac{-\nabla \overrightarrow{J_i}}{d} dv \tag{5.13}$$

relating to internal sources or by the **potential** function

$$-b = \frac{1}{4\pi\sigma} \int_S \Phi_S \nabla(\frac{1}{d}) d\overrightarrow{S} \tag{5.14}$$

relating to the non-homogeneous double layer normal to S, and oriented as the outward normal. Since the surface S has been chosen in an arbitrary manner with the only restraint that all sources be internal to it, then an infinite number of double layer distributions, each on a different surface, will generate the same **potential** field in the volume conductor external to all of them.

It follows from the Helmoltz theorem that the distribution of electrical generators within the cardiac muscle cannot be uniquely reconstructed from electrical signals recorded at the surface of the human body. However, this is only true if we have no a priori knowledge of the generator. Practically, physiological constraints may greatly reduce the theoretical indeterminacy. The problem of gaining some insight into the distribution of intracardiac events from external measurements is called the "inverse problem" of electrocardiology.

The counterpart of the inverse problem can be stated as follows: given the distribution of intracardiac generators, find the **potential** distribution within the body and on its surface. The problem is referred to as the "forward problem" of electrocardiology. There is a fundamental problem between these two problems. In the forward problem, a complete knowledge of the cardiac electrical generators and the properties of the surrounding medium enables us to determine an unique surface **potential** distribution (within a constant). To achieve this purpose one must have a quantitative knowledge of torso geometry, inhomogeneities and anisotropy of the body tissues. The forward approach plays an important role in electrocardiology in that it makes possible to evaluate the performance of different models of cardiac generators. In addition, the forward approach is a necessary intermediate step in the solution of the inverse problem.

5.3.2 Anisotropy of Cardiac Muscle and Conduction of Excitation

The theory which assumes a uniform double (dipole) layer as an equivalent current generator during cardiac excitation has been recently challenged by experimental evidence. In 1977, Corbin and Scher proposed an "axial" double layer model for the depolarization wavefront where the dipole moment density is oriented along the axis of the fibers. The **potential** distribution generated by such an axial model was in qualitative agreement with three dimensional **potential** patterns measured in the thickness of the ventricular walls after epicardial or midwall stimulation, in open chest dogs. Current was flowing toward resting tissue from those portions of the front that moved along fiber direction. Spach developed a three dimensional model where the cardiac sources are represented by transmembrane currents of non uniform distribution due to anisotropic intracellular conductivity. The **potential** a_0 in the intracellular homogeneous volume of conductivity σ_0 is given by eq. For a three dimensional network of myocardial fibers the volume density of the transmembrane current is given by
$I_m = -(\frac{\partial I_{y_1}}{\partial y_1} + \frac{\partial I_{y_2}}{\partial y_2} + \frac{\partial I_{y_3}}{\partial y_3}) = \frac{\partial}{\partial y_1}(\sigma_{i_{y_1}}\frac{\partial p_i}{\partial y_1}) + \frac{\partial}{\partial y_2}(\sigma_{i_{y_2}}\frac{\partial p_i}{\partial y_2}) + \frac{\partial}{\partial y_3}(\sigma_{i_{y_3}}\frac{\partial p_i}{\partial y_3})$
where $\sigma_{i_{y_1}}, \sigma_{i_{y_2}}, \sigma_{i_{y_3}}$ are the intracellular conductivities along three orthogonal axes, and $I_{y_1}, I_{y_2}, I_{y_3}$ are the intracellular curent densities along the same directions. When the arrangement of myocardial fibers has axial symmetry $\sigma_{1=\sigma_{i_{y_1}}}$ and $\sigma_t = \sigma_{i_{y_1}} = \sigma_{i_{y_3}}$ where σ_1 and σ_t represent the longitudinal and transverse fiber conductivities respectively. The extracellular **potential**s simulated by Spach showed good agreement with the **potential**s measured on a stimulated two dimensional sheet of ventricular muscle in a tissue bath. The intracellular gradients were measured by means of intracellular needles and the conductivities were deduced from measured conduction velocities by applying Hodgkin equation.

The Forward Problem: The Integral Equation Approach

As stated above, the forward problem deals with the computation of the **potential** distribution within the body and at its surface, when the distribution of intracardiac current sources is known. Let us, at first, consider a homogeneous volume conductor v with internal current sources, bounded by a surface S at the conductor-air interface. According to Green's second identity it is $\int_v (p\nabla^2\psi - \psi\nabla^2 a)dv = \int_S (p\nabla\psi - \psi\nabla p)\cdot d\overrightarrow{S}$ where a and ψ are harmonic functions in the domain considered. If a is the **potential** function and $\psi = 1/d$ then $\int_v p\nabla^2(\frac{1}{d})dv - \int_v \frac{\nabla^2 a}{d}dv = \int_S p\nabla(\frac{1}{d})\cdot d\overrightarrow{S} - \int_S \frac{\nabla p}{d}\cdot d\overrightarrow{S}$. It can be shown that the volume integral $\int_v p\nabla^2(\frac{1}{d})dv$ equals $-4\pi p$ within volume v, $-2ap$ on the surface S and zero outside volume v. In eq the last surface integral is zero because the surface is insulated. For the field points belonging to the surface S, eq is then transformed into

$$-2\pi p(\overrightarrow{x}) - \int_v \frac{\nabla^2 a}{d}dv = \int_S a(\overrightarrow{y})\nabla(\frac{1}{d})\cdot d\overrightarrow{S},$$

$$a(\overrightarrow{x}) = \frac{1}{2\pi}\int_v \frac{-\nabla^2 a}{d}dv - \frac{1}{2\pi}\int_S a(\overrightarrow{y})\nabla(\frac{1}{d})\cdot d\overrightarrow{S} \tag{5.15}$$

where $\overrightarrow{x}$ is the point vector defining the point on surface S where we are computing the **potential** and $\overrightarrow{y}$ is the point vector defining all the elements of area on surface S. In eq the volume integral represents the source term, i.e. the **potential** we would measure at the surface S if the medium were infinite and homogeneous $a_{im}(\overrightarrow{x}) = \frac{1}{4\pi}\int_v \frac{-\nabla^2 a}{d}dv$, where $a_{im}(\overrightarrow{x})$ stands for "infinite medium" **potential** on surface

S. Then eq may be written as

$$a(\overrightarrow{x}) = 2p_{im}(\overrightarrow{x}) + \int_S a(\overrightarrow{y})d\Omega \tag{5.16}$$

where $d\Omega = \nabla(1/d)\cdot d\overrightarrow{S}$ is the solid angle subtended by $\overrightarrow{dS}$ at point $\overrightarrow{x}$. This is an Fredholm equation. Let the surface $S = S_1$ contain n internal regions of non-homogeneity, each region homogeneous and bounded by a close surface $S_j(j = 2, ..., n)$. It can be shown that for $\overrightarrow{x}$ belonging to surface $S_d = (d = 1, 2, ..., n)$,
$a(\overrightarrow{x}) = 2p_{im}(\overrightarrow{x})+ \sum_{S=1}^{n} (-\frac{1}{2\pi})\frac{\sigma_S^i - \sigma_S^0}{\sigma_d^i + \sigma_d^o} \int_{S_S} a(\overrightarrow{y})\frac{\partial}{\partial n}(\frac{1}{d})dS,$
with $\overrightarrow{y}$ point of S_S and σ_S^i,σ_S^0 internal and external conductivities of surface S_S respectively. If all surfaces are discretized into triangular elements the integral equation is transformed into a system of linear equations which can be solved numerically with particular procedures due to the fact that this system is singular. Once the integral equation is solved and the **potential** is known on all surfaces $S_d(d = 1, 2, ..., n)$ then the **potential** at points $\overrightarrow{x}$ of the volume conductor bounded by S_d is computed as

$$a(\overrightarrow{x}) = a_{im}(\overrightarrow{x})+ \sum_{S=1}^{n} (-\frac{1}{4\pi})\frac{\sigma_S^i - \sigma_S^0}{\sigma_d^i + \sigma_d^o} \int_{S_S} a(\overrightarrow{y})\frac{\partial}{\partial n}(\frac{1}{d})dS, \tag{5.17}$$

where $a(\overrightarrow{x})$ is given by known quantities. The forward problem for the **potential** function is then completely defined: given the internal sources, it is always possible to compute the **potential** anywhere in the volume conductor and at its surface.

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Chapter 6 **Conclusions**

In this paper we have proposed a computer science perspective on processes associated traditionally with continuous mathematics is really beneficial. As result, we have established the computability of very general structures and founded them in a more constructive logic. We have presented the Hilbert machine as computational model and the weak commutative linear logic as a logical framework for the stochastic analysis, in its axiomatic form of Dirichlet spaces. A general process algebra, named continuous process algebra or continuous information processing systems has been developed. An important application of this process algebra is that we can associate a Dirichlet space and a weak commutative phase space (a linear logic) to each von Neumann algebra, and therefore to any Hilbert machine of it.

Other important applications include hybrid systems [BB 01 c], [BB 01 d], models in biomedicine [Buj 01 e] and the mathematical knowledge representation [BB 02 a], [Buj 01 e]. Future developments will include quantum mechanics [Buj 01 d] and financial mathematics applications [BB 01 b].

An important conclusion of this mathematical experiment is that partial orders constitute a very powerful modelling tool, for studying structures arising from both stochastic analysis and computer science. They provide us a rich semantic domain and we believe it is possible to extend the approach towards verification.

This work is still in its early stages. It is pioneering in trying to interrelate very different in nature mathematical disciplines. We have tried not being superfluous by avoiding too much mathematical text and not risk transforming our ideas into cheap speculations. The price is this document is very technical and less intuitive. Our choice was motivated by the need of providing a document, which can serve as a basis for further more concise developments and presentations.

This work was initiated in the first author's MSc thesis [Buj 98 a] and further developed in [BB 01 a], [BB 01 d], [BB 02 a] and other presentations cited in the bibliography. We want to point out here some important related approaches. A similar program was initiated by Herbert Wiklicky in [Wik 96] and further developed in [Wik 98]. His work and Jean Yves Girard's Geometry of Interaction program were great sources of inspiration for us. Michele Abrusci's work [Abr 91] in non-commutative LL was essential in letting us continuing the mentioned programs and to interrelate them with our previous work in axiomatic stochastic analysis.

The axiomatic trend in stochastic analysis was explosively developing in the last 50 years. It generated new and very rich mathematical theories like harmonic spaces (Brelot, Bucur, Constantinescu, Cornea see the monographs [CC 72] and [Bre 70] for a full exposition), Dirichlet spaces (Beurling, Deny [BD 59], Ma, Rockner [MR 92]) and H-Cones (Boboc, Bucur, Cornea [BBC 81]). We consider that the most general approach is that based on H-cones. H-cones can be associated to both harmonic spaces and Dirichlet forms. Their theory combines lattice theory ([Bir 68]) with functional analysis, and has applications to both differential equations and Markov processes. It allows us to abstract the continuous mathematics machinery, hardly inaccessible to logicians and computer scientists. The direction was even further developed by Arsove and Leutwiller in [AL 74], [AL 80]. For a foundation approach to stochastic analysis, this work is priceless. The theory of H-cones is the most straightforward way to give models to our algebraic theory. Anyway, this work helps only partially our needs. For an axiomatic representative of stochastic analysis, we have chosen the theory of Dirichlet spaces. We have taken as an abstraction of Dirichlet spaces the work from [BBC 81] and [AL 80] as a starting point, but our developments followed a different direction.

The idea of modelling dynamics of concurrent systems with partial orders comes from the theory of Petri nets (see the monograph [BF 90] for many detailed explanations). The decomposition of a causal order into sequential and concurrency relations was used for the first time in [Ste 96].

A following accompanying paper will present some important omissions, like applications to hybrid systems and control theory. Hybrid systems mix continuous and discrete behaviours, and their semantics should provide a uniform treatment to all components. We have provided above, references to our work in using this theory as a semantic domain for (stochastic) hybrid systems. Also, the interpretation of this work as a way of representing the (continuous) mathematical knowledge in linear logic has not been exposed here. Possible applications in data mining also deserves more explanations.

Important applications can also be done in the fields of software engineering. The presentation [BB 02 a] sketches some applications of LL in software testing of stochastic and/or hybrid systems. We believe the theory can be fruitfully applied to the control of stochastic hybrid systems, as well to their formal specification.

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