

Optimal design for inspection and maintenance policy based on the CCC chart

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Abstract

In the paper, the concept of cumulative count of conforming chart (CCC chart) is applied in inspection and maintenance planning for systems where minor inspection, major inspection, minor maintenance and major maintenance are available. Several inspection and maintenance plans are defined and studied quantitatively. Analytic expressions of relevant statistics and their expectations are derived. These inspection and maintenance plans are optimized from an economic consideration.

Keywords. Inspection, maintenance, manufacturing, fraction nonconforming, geometric random variable, quality control, cumulative count of conforming chart, economic design.

1. Introduction.

This paper attempts to apply the concept of cumulative count of conforming chart (CCC chart) to monitor a system with three states.

Statistical process control is a widely used tool in the industry for process monitoring. Very commonly, control charts are merely used for the purpose of “fire fighting” for removal of assignable causes each time when the process parameter has shifted. However, occurrence of assignable causes can sometimes be the result of gradual deterioration of

the system such as aging of parts. Thus, signals on the control charts may be used as indications of whether inspection or maintenance of the system should be carried out, rather than just the state of fitness of the process. Tagaras and Lee (1988) and Tagaras(1998) separated the \bar{X} -chart into several zones and optimized the chart for monitoring a process whose deterioration can be classified into two states in which one state requires minor restoration and the other requires major restoration, and considered economic design of the control chart. Cassady *et al* (2000) considered economic design of control charts for optimization of preventive maintenance policies for systems. See also Page (1955), Weindling *et al* (1970), Gordon and Weindling (1975), Montgomery (1980), Chan *et al* (2000), Montgomery (2001), Chan *et al* (2002), Chan (2003), Liao *et al* (2006) for related work.

In what follows, we shall assume that as production continues, the production system will deteriorate from its normal state to a state that requires minor maintenance, and if minor maintenance is not carried out the system will eventually deteriorate to a state that requires major maintenance. We shall use the CCC chart defined based on the geometric distribution to develop inspection and maintenance policies for a production system in which each item produced can be classified as either conforming or nonconforming. The CCC chart, first introduced by Calvin (1983), further developed by Goh (1987), and extended by and Ranjan *et al* (2003), is an alternative to the classical np - and p -charts that have serious drawbacks when the process nonconforming rate is low (Calvin(1983), Lucas(1989), Chan (2000), and Chan *et al* (1997)). The CCC chart can be used effectively for monitoring high-quality processes in which the fraction of nonconforming is low, even as low as at the level of ppm (parts per million) or ppb (parts per billion).

2. The states and investigations of a system

We consider process monitoring and maintenance of a system in which the fraction of nonconforming items produced is $p \in (0, 1)$, and all items are inspected. The number of items inspected, n , until a nonconforming item is observed, is a geometric random variable with the following probability function $f(n)$ and cumulative distribution function $F(n)$:

$$f(n) = p(1 - p)^{n-1}, \quad F(n) = 1 - (1 - p)^n \quad (n = 1, 2, \dots). \quad (2.1)$$

In monitoring such a system using the CCC chart, control limits are obtained from $F(n)$ at $p = p_0$, and an observed value of n falling beyond the control limits is a signal for out of control which indicates possible shift of p from its target value p_0 .

Assume that the system has three states, the normal state S_0 with nonconforming fraction $p = p_0 > 0$, the slightly deteriorated state S_1 with $p = p_1 > p_0$, and the seriously deteriorated state S_2 with $p = p_2 > p_1$. (Throughout this paper we shall use S_0, S_1, S_2 with Roman type “S” to denote the states of a system, while S_1, S_2 with italic “S” will be reserved to represent mathematical quantities. Likewise, in what follows, $m_\gamma, M_\gamma, i_\gamma, I_\gamma$ and s_γ ($\gamma = 0, 1, 2$ or $1 + 2$) with Roman type “m”, “M”, “i”, “I” and “s” will be used to denote “maintenance”, “inspection” and “signal” as appropriate, while m, i, s in italic (with or without subscript) will be reserved to represent mathematical quantities.) Suppose that two grades of maintenance actions, minor maintenance m_1 and major maintenance m_2 are available, where m_1 is less thorough than m_2 but is less costly. For notational convenience, from time to time we shall use m_0 to mean that no maintenance action will be carried out. Based on practical consideration, we assume the following:

1. No maintenance is required for S_0 , and when the system is in S_0 neither m_1 nor m_2 will change its state.
2. When the system is in S_0 , it will deteriorate to S_1 with probability $\pi_{01} > 0$, imme-

diately before an item is produced.

3. Either m_1 or m_2 can bring the system from S_1 back to S_0 .
4. When the system is in S_1 , it will deteriorate to S_2 with probability $\pi_{12} > 0$, immediately before an item is produced.
5. m_1 has no effect on S_2 , but m_2 can bring the system from S_2 to S_0 .
6. The system will not deteriorate from S_0 to S_2 directly without passing through S_1 .
7. Complete failure (that is, malfunctioning) of the system does not occur, even though as the system deteriorates from S_0 to S_1 and then to S_2 , the nonconforming fraction p increases.

Suppose that $n_1 > n_2 > 0$. Divide the set of all positive integers Z^+ into the following three sets:

$$Z_0 = \{n \in Z^+ : n_1 + 1 \leq n < \infty\},$$

$$Z_1 = \{n \in Z^+ : n_2 + 1 \leq n \leq n_1\},$$

$$Z_2 = \{n \in Z^+ : 1 \leq n \leq n_2\}.$$

If a nonconforming item is observed at $n \in Z_\gamma$ ($\gamma = 0, 1, 2$), a type γ signal s_γ is said to have occurred. When s_0 has occurred, no inspection of the system will be carried out, and inspection will be carried out when s_1 or s_2 has occurred. When the system is in S_0 , we expect that s_0 will occur most of the time, and thus s_0 is regarded as an indication that the system is in its normal state S_0 . As the system deteriorates (p increases), the chance for s_1 to occur increases. When the system is in S_2 , we expect that s_2 will occur most of the time, and thus we regard s_2 as an indication that the system is in the most deteriorated state S_2 . However, any of s_0, s_1, s_2 may occur when the system is in S_0, S_1 or S_2 . Thus, when the system is in S_0 , signal s_1 or s_2 (which indicates system deterioration) may be regarded as “Type I error”, and when the system is in S_2 , signal s_0 or s_1 (which

indicates that the system is in state S_0 or S_1) may be regarded as “Type II error”. In Section 6, we shall take into account such errors in economic design of inspection and maintenance policies where expressions for the expected numbers of s_γ ($\gamma = 0, 1, 2$) when the system is S_0 , S_1 or S_2 will be required, and such expressions will be given in Section 5.

Suppose that two types of inspection, minor inspection i_1 and major inspection i_2 , are available, where i_1 is not as thorough as i_2 but is less costly. For notational convenience, from time to time we shall use i_0 to mean that no investigation will be carried out. When the system is in S_0 or S_1 , both i_1 and i_2 will correctly reveal the true state of the system, but when the system is in S_2 , i_1 will incorrectly indicate that the system is in S_1 and only i_2 will correctly reveal that the system is in S_2 .

From the economic of view, whether to carry out inspection depends on the inspection cost relative to the maintenance cost, and to the penalty of wrong indication of the state of the system. Consider the following three inspection arrangements, I_{1+2} , I_2 , I_0 , and three maintenance arrangements M_{1+2} , M_2 , M_0 .

I_{1+2} – Both i_1 and i_2 are employed. When s_γ occurs, i_γ will be carried out ($\gamma = 1, 2$).

I_2 – Only i_2 is employed. When either s_1 or s_2 occurs, i_2 will be carried out.

I_0 – Neither i_1 nor i_2 will be employed. When s_1 or s_2 occurs, maintenance work will be carried out without first inspecting the system.

M_{1+2} – Both m_1 and m_2 are employed.

M_2 – Only m_2 is employed.

M_0 – Neither m_1 nor m_2 is employed. That is, no maintenance work will be carried out.

The above concepts on states of the system and types of inspection and maintenance are illustrated by the following real-life example in maintenance of buses in a bus com-

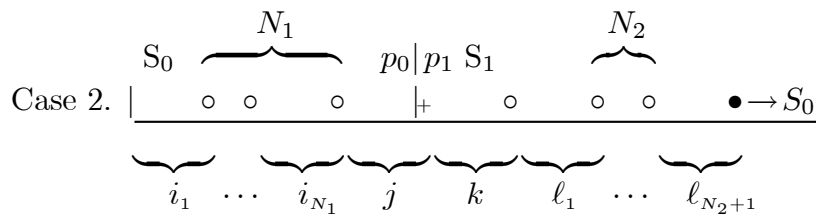
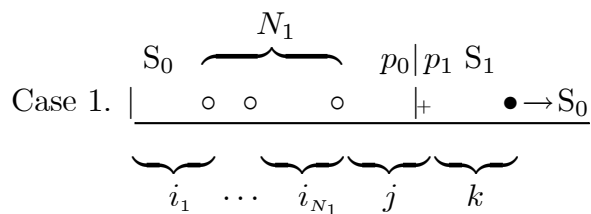
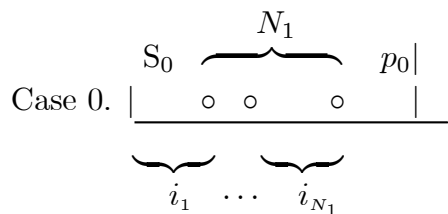
pany. A dynamo supplies the power to a cooling fan in the cooling system of the engine. By design, unfortunately this dynamo is installed near the engine with high temperature, and a cooling fan system called System C is designed for cooling the dynamo.

- (a) S_0 is the state where System C is not blocked by dust and operates efficiently, and the coolant will reach its boiling point with a small probability $p_0 > 0$ at the end of a day's operation.
- (b) Checking whether the coolant has reached its boiling point at the end of a day's operation is "inspection of an item".
- (c) S_1 is the state where System C is partially blocked by dust, so that the dynamo is over-heated and supplies insufficient electric current to drive the cooling fan, and as a result the probability for the coolant to reach its boiling point at the end of a day's operation will be $p_1 > p_0$.
- (d) After a day's operation in state S_0 , a bus will change to S_1 with probability $\pi_{01} > 0$.
- (e) Minor inspection i_1 opens the chassis cover and inspects whether System C requires dust cleaning. Minor maintenance m_1 removes the dust accumulated in System C. If m_1 is not performed, System C will be eventually blocked by dust resulting in over-heating and dynamo damage, which is state S_2 .
- (f) After a day's operation in state S_1 , a bus will change to S_2 with probability $\pi_{12} > 0$. In state S_2 , there is a probability $p_2 (> p_1)$ for the coolant to reach its boiling point which could possibly result in engine damage.
- (g) Major inspection i_2 disassembles System C and inspects whether the dynamo has been damaged due to over-heating, and major maintenance m_2 is to replace the dynamo.
- (h) The values of p_0 , p_1 , p_2 , π_{01} , and π_{12} depend on the bus route.
- (i) The costs for performing i_1 , i_2 , m_1 , and m_2 depend on the model of the bus.

3. Cycle of operation for the system.

A cycle of operation is said to be completed when the system starts at S_0 , deteriorates to S_1 or S_2 , and then is restored back to S_0 by m_1 or m_2 .

According to the framework of inspection and maintenance defined in Section 2, to study the system analytically we need to consider different scenarios for a cycle to complete as shown in Figure 1. The small circles, “o’s”, denote nonconforming items that either do not lead to any maintenance action or lead to a maintenance action that does not change the state of the system. The bullet, “•”, denotes the last nonconforming item in a cycle that leads to maintenance action that restores the system from states S_1 or S_2 back to S_0 . Hence a cycle ends only after a bullet • has occurred. Seven different cases (Case 0, ..., Case 6) are possible, and they are described below. A cycle may end under one of the six cases, Case 1, ..., Case 6.



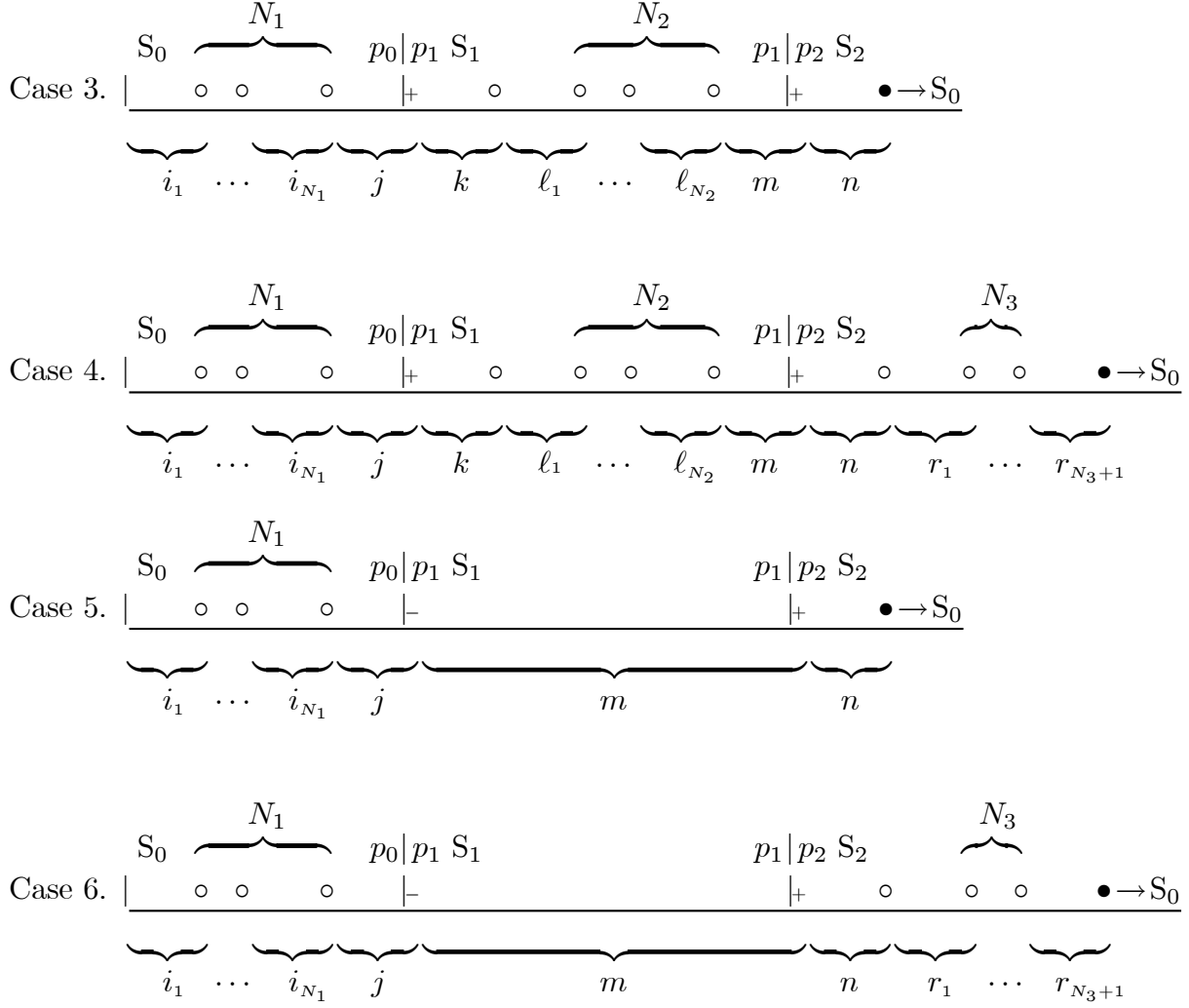


Figure 1. Occurrence of nonconformity items.

Case 0. While the system is in S_0 , $N_1(\geq 0)$ nonconforming items \circ 's have occurred. As the system is in S_0 and has not yet changed to S_1 , no maintenance action is necessary, there is no bullet \bullet in the figure, and a cycle has not yet been formed. The figure shows that before a nonconforming item \circ occurs, $i_u - 1$ ($u = 1, \dots, N_1$) conforming items are produced.

Case 1. While the system is in S_0 , $N_1(\geq 0)$ nonconforming items \circ 's have occurred. Next, another $j - 1$ conforming items have also occurred which is followed by changing

of the system from S_0 to S_1 , and then the j^{th} item occurs. Immediately after the system has changed from S_0 to S_1 , $k + 1(\geq 1)$ items are produced, where the first k of these $k + 1$ items are conforming items but the last one is a \bullet . The plus sign “+” represents the first item produced immediately after the system has changed to S_1 , where this item is a conforming item if $k > 0$, and is a nonconforming item \bullet if $k = 0$.

Case 2. This case represents the situation when the last nonconforming item in Case 1 is not a \bullet but a \circ . In this case, when the system is in S_1 , $N_2 + 1(\geq 1)$ \circ 's occur before the \bullet . The plus sign “+” represents a conforming item if $k > 0$ and represents a nonconforming item \circ if $k = 0$. The figure shows that when the system is in S_1 , $\ell_u - 1(\geq 0)$ ($u = 1, \dots, N_2 + 1$) items are produced before each of the last $N_2 + 1$ nonconforming items.

Case 3. This case represents the situation when the last nonconforming item in Case 2 is not a \bullet but a \circ . In this case, when the system is in S_1 , $N_2 + 1(\geq 1)$ \circ 's have occurred. Next, another $m - 1$ conforming items have occurred which is followed by changing of the system from S_1 to S_2 , and then the m^{th} item occurs. Immediately after the system has changed from S_1 to S_2 , $n + 1(\geq 1)$ items are produced, where the first n of these $n + 1$ items are conforming items but the last one is a \bullet . The plus sign “+” above the embrace of “ m ” represents the first item produced immediately after the system has changed to S_2 , where this item is a conforming item if $n > 0$ and is a nonconforming item \bullet if $n = 0$.

Case 4. This case represents the situation when the last nonconforming item in Case 3 is not a \bullet but a \circ . In this case, immediately after the system has changed from S_1 to S_2 , $N_3 + 1(\geq 1)$ \circ 's have occurred before the bullet \bullet . The plus sign “+” above the embrace of “ m ” represents a conforming item if $n > 0$ and represents a nonconforming item \circ if $n = 0$. The figure shows that when the system is in S_2 , $r_u - 1(\geq 0)$ ($u = 1, \dots, N_3 + 1$)

conforming items are produced before each of the last $N_3 + 1$ nonconforming items.

In Cases 5 and 6, no nonconforming item occurs when the system is in S_1 , and the bar “-” denotes a conforming item.

Case 5. While the system is in S_0 , $N_1(\geq 0)$ nonconforming items \circ 's have occurred. Next, another $j - 1$ conforming items have occurred which is followed by changing of the system from S_0 to S_1 , and then the j^{th} item occurs which is a conforming one. When the system is in S_1 , all the m items produced are conforming. Immediately after the system has changed from S_1 to S_2 , $n + 1(\geq 1)$ items are produced, where the first n of these $n + 1$ items are conforming but the last one is a \bullet . The plus sign “+” represents a conforming item if $n > 0$, and represents the \bullet if $n = 0$.

Case 6. This the case represents the situation when the last nonconforming item in Case 5 is not a \bullet but a \circ . In this case, immediately after the system has changed from S_1 to S_2 , $N_3 + 1(\geq 1)$ \circ 's have occurred before the bullet \bullet . The plus sign “+” above the embrace of “ m ” represents a conforming item if $n > 0$ and represents a nonconforming item \circ if $n = 0$. The figure shows that when the system is in S_2 , $r_u - 1(\geq 0)$ ($u = 1, \dots, N_3 + 1$) conforming items are produced before each of the last $N_3 + 1$ nonconforming items.

Because a nonconforming item can only show up as a circle \circ or a bullet \bullet but not both, from the different positions of occurrences of the bullets \bullet 's in Figure 1, it is not difficult to see that Case 1, ..., Case 6 are mutually exclusive. These cases are also exhaustive for the system to complete a cycle, because a bullet \bullet can occur only when the system is in either stage S_1 or S_2 , and Case 1, ..., Case 6 include all possible cases for the bullet \bullet to occur. At the end of Section 4 we shall prove that the total probability of occurrence of these six cases is 1, which confirms that these cases are exhaustive.

4. Quantitative Analysis of the Model ($\mathbf{I}_{1+2}, \mathbf{M}_{1+2}$)

In this section, we obtain the probabilities and expected items inspected for each of Case 0, ..., Case 6. Based on these results, expected numbers of inspections and maintenance actions can be derived, and using these results optimal inspection and maintenance strategies can be obtained in Section 7.

Consider the inspection and maintenance model ($\mathbf{I}_{1+2}, \mathbf{M}_{1+2}$), where all of i_1, i_2, m_1 and m_2 are employed. Under this model, when s_γ ($\gamma = 1, 2$) appears, i_γ will be carried out. If an inspection shows that the system is in S_γ , m_γ will be carried out ($\gamma = 1, 2$). When the system is in S_0 , the \circ 's in Figure 1 may indicate s_0, s_1 or s_2 . When the system is in S_1 , the \circ 's indicate s_0 , and the bullet \bullet indicates either s_1 or s_2 . When the system is in S_2 , the \circ 's may indicate either s_0 or s_1 , and the \bullet indicates s_2 .

In what follows, for convenience we let $\beta_0 = (1 - \pi_{01})(1 - p_0)$, $\beta_1 = (1 - \pi_{12})(1 - p_1)$.

In Case 0, while the system remains in S_0 , the probability for $(i-1)$ conforming items occurring followed by a nonconforming item \circ is $(1 - \pi_{01})^i (1 - p_0)^{i-1} p_0 = \beta_0^{i-1} (1 - \pi_{01}) p_0$.

The total probability over the entire range of i is

$$\begin{aligned} I. = I_{.;i} &= \sum_{i=1}^{\infty} \beta_0^{i-1} (1 - \pi_{01}) p_0 = \frac{(1 - \pi_{01}) p_0}{1 - \beta_0} = \left(\sum_{i \in Z_0} + \sum_{i \in Z_1} + \sum_{i \in Z_2} \right) \beta_0^{i-1} (1 - \pi_{01}) p_0 \\ &= I_{0;i} + I_{1;i} + I_{2;i} = I_0 + I_1 + I_2, \end{aligned}$$

where $I_{0;i}, I_{1;i}, I_{2;i}$ denote the last three summations, respectively, and for simplicity, we also denote them by I_0, I_1, I_2 , respectively. (In what follows, similar notations will be introduced to denote different summations.) The expected value of i in Case 0 is

$$\begin{aligned} I. [i] = I_{.;i}[i] &= \sum_{i=1}^{\infty} i \beta_0^{i-1} (1 - \pi_{01}) p_0 = \frac{(1 - \pi_{01}) p_0}{(1 - \beta_0)^2} = \frac{I_{(0)}}{1 - \beta_0} \\ &= \left(\sum_{i \in Z_0} + \sum_{i \in Z_1} + \sum_{i \in Z_2} \right) i \beta_0^{i-1} (1 - \pi_{01}) p_0 = I_{0;i}[i] + I_{1;i}[i] + I_{2;i}[i]. \end{aligned}$$

Refer to Case 1. Consider the following sequence of events: (A1) $j - 1 (\geq 0)$ consecutive conforming items are produced while the system remains in S_0 , (A2) the system changes from S_0 to S_1 , (A3) $k (\geq 0)$ consecutive conforming items are produced while the system remains in S_1 , and (A4) when the system is in S_1 , the $(k + 1)^{th}$ item produced is nonconforming. The probability for the events (A1) – (A4) to occur in sequence is $(1 - \pi_{01})^{j-1}(1 - p_0)^{j-1}\pi_{01} \times (1 - \pi_{12})^k(1 - p_1)^k p_1 = \beta_0^{j-1} \beta_1^k \pi_{01} p_1$. The total probability over the entire ranges of j and k is

$$\begin{aligned}
JK. &= JK_{:,j,k} = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \beta_0^{j-1} \beta_1^k \pi_{01} p_1 = \frac{\pi_{01}}{1 - \beta_0} \times \frac{p_1}{1 - \beta_1} \\
&= \left(\sum_{\substack{j+k \in Z_0 \\ j \geq 1; k \geq 0}} + \sum_{\substack{j+k \in Z_1 \\ j \geq 1; k \geq 0}} + \sum_{\substack{j+k \in Z_2 \\ j \geq 1; k \geq 0}} \right) \beta_0^{j-1} \beta_1^k \pi_{01} p_1 \\
&= JK_{0;j,k} + JK_{1;j,k} + JK_{2;j,k} = JK_0 + JK_1 + JK_2.
\end{aligned}$$

The expected value of $j + k$ for Case 1 is

$$\begin{aligned}
JK_{:,j,k}[j+k] &= JK_{j,k}[j+k] = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} (j+k) \beta_0^{j-1} \beta_1^k \pi_{01} p_1 \\
&= \left(\sum_{\substack{j+k \in Z_0 \\ j \geq 1; k \geq 0}} + \sum_{\substack{j+k \in Z_1 \\ j \geq 1; k \geq 0}} + \sum_{\substack{j+k \in Z_2 \\ j \geq 1; k \geq 0}} \right) (j+k) \beta_0^{j-1} \beta_1^k \pi_{01} p_1 \\
&= JK_{0;j,k}[j+k] + JK_{1;j,k}[j+k] + JK_{2;j,k}[j+k].
\end{aligned}$$

In Case 2, while the system is in S_1 , the probability for $(\ell - 1)$ consecutive conforming item to occur followed by a nonconforming item \circ is $(1 - \pi_{12})^{\ell}(1 - p_1)^{\ell-1} p_1 = \beta_1^{\ell-1}(1 - \pi_{12}) p_1$. The total probability over the entire range of ℓ is

$$\begin{aligned}
L. &= L_{:, \ell} = \sum_{\ell=1}^{\infty} \beta_1^{\ell-1} (1 - \pi_{12}) p_1 = \frac{(1 - \pi_{12}) p_1}{1 - \beta_1} = \left(\sum_{\ell \in Z_0} + \sum_{\ell \in Z_1} + \sum_{\ell \in Z_2} \right) \beta_1^{\ell-1} (1 - \pi_{12}) p_1 \\
&= L_{0;\ell} + L_{1;\ell} + L_{2;\ell} = L_0 + L_1 + L_2.
\end{aligned}$$

The expected value of ℓ in Case 2 is

$$\begin{aligned} L_{\cdot;\ell}[\ell] &= L_\ell[\ell] = \sum_{\ell=1}^{\infty} \ell \beta_1^{\ell-1} (1 - \pi_{12}) p_1 = \frac{(1 - \pi_{12}) p_1}{(1 - \beta_1)^2} \\ &= \left(\sum_{\ell \in Z_0} + \sum_{\ell \in Z_1} + \sum_{\ell \in Z_2} \right) \ell \beta_1^{\ell-1} (1 - \pi_{12}) p_1 = L_{0;\ell}[\ell] + L_{1;\ell}[\ell] + L_{2;\ell}[\ell]. \end{aligned}$$

Refer to Case 3. Consider the following sequence of events: (B1) $m - 1 (\geq 0)$ consecutive conforming items are produced while the system remains in S_1 , (B2) the system changes from S_1 to S_2 , (B3) $n (\geq 0)$ consecutive conforming items are produced while the system remains in S_2 , and (B4) when the system is in S_2 , the $(n + 1)^{th}$ item produced is nonconforming. The probability for the events (B1) – (B4) to occur in sequence is $(1 - \pi_{12})^{m-1} (1 - p_1)^{m-1} \pi_{12} \times (1 - p_2)^n p_2 = \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2$. The total probability over the entire ranges of m and n is

$$\begin{aligned} MN_{\cdot} &= MN_{\cdot;m,n} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2 = \frac{\pi_{12}}{1 - \beta_1} \\ &= \left(\sum_{\substack{m+n \in Z_0 \\ m \geq 1; n \geq 0}} + \sum_{\substack{m+n \in Z_1 \\ m \geq 1; n \geq 0}} + \sum_{\substack{m+n \in Z_2 \\ m \geq 1; n \geq 0}} \right) \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2 \\ &= MN_{0;m,n} + MN_{1;m,n} + MN_{2;m,n} = MN_0 + MN_1 + MN_2. \end{aligned}$$

The corresponding expected number of items inspected for Case 3 is

$$\begin{aligned} MN_{\cdot}[m+n] &= MN_{\cdot;m,n}[m+n] = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (m+n) \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2 \\ &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} m \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2 + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} n \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2 \\ &= \frac{\pi_{12}}{(1 - \beta_1)^2} \times \frac{\pi_{12}}{1 - \beta_1} \times \frac{1}{p_2} = \frac{\pi_{12}(p_2 + 1 - \beta_1)}{(1 - \beta_1)^2 p_2} \\ &= \left(\sum_{\substack{m+n \in Z_0 \\ m \geq 1; n \geq 0}} + \sum_{\substack{m+n \in Z_1 \\ m \geq 1; n \geq 0}} + \sum_{\substack{m+n \in Z_2 \\ m \geq 1; n \geq 0}} \right) (m+n) \beta_1^{m-1} \pi_{12} (1 - p_2)^n p_2 \\ &= MN_{0;m,n}[m+n] + MN_{1;m,n}[m+n] + MN_{2;m,n}[m+n]. \end{aligned}$$

In Case 4, when the system is in S_2 , the probability for $(r - 1)$ conforming to be produced followed by a nonconforming item is $(1 - p_2)^{r-1}p_2$. The total probability over the entire range of r is

$$\begin{aligned} R_{\cdot} = R_{\cdot;r} &= \sum_{r=1}^{\infty} (1 - p_2)^{r-1} p_2 = 1 = \left(\sum_{r \in Z_0} + \sum_{r \in Z_1} + \sum_{r \in Z_2} \right) (1 - p_2)^{r-1} p_2 \\ &= R_{0;r} + R_{1;r} + R_{2;r} = R_0 + R_1 + R_2. \end{aligned}$$

The expected value of r for Case 4 is

$$\begin{aligned} R_{\cdot}[r] = R_{\cdot;r}[r] &= \sum_{r=1}^{\infty} r(1 - p_2)^{r-1} p_2 = \frac{1}{p_2} = \left(\sum_{r \in Z_0} + \sum_{r \in Z_1} + \sum_{r \in Z_2} \right) r(1 - p_2)^{r-1} p_2 \\ &= R_{0;r}[r] + R_{1;r}[r] + R_{2;r}[r]. \end{aligned}$$

Refer to Case 5. Consider the events: (C1) immediately after the system has changes from S_0 to S_1 , $m(\geq 1)$ consecutive conforming items are produced, (C2) the system changes from S_1 to S_2 , (C3) immediately after the system has changes from S_1 to S_2 , $n(\geq 0)$ consecutive conforming items are produced while the systems remains in S_2 , (C4) when the systems is in S_2 , the $(n + 1)^{th}$ item produced is nonconforming. The probability for events (A1), (A2), (C1), (C2), (C3), (C4) to occur in sequence is $(1 - \pi_{01})^{j-1}(1 - p_0)^{j-1}\pi_{01}(1 - p_1) \times (1 - \pi_{12})^{m-1}(1 - p_1)^{m-1}\pi_{12} \times (1 - p_2)^n p_2] = \beta_0^{j-1}\beta_1^{m-1}\pi_{01}(1 - p_1)\pi_{12}(1 - p_2)^n p_2$. The total probability over the entire ranges of j, m, n , is

$$\begin{aligned} JMN_{\cdot} = JMN_{\cdot;j,m,n} &= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \beta_0^{j-1}\beta_1^{m-1}\pi_{01}(1 - p_1)\pi_{12}(1 - p_2)^n p_2 \\ &= \frac{\pi_{01}}{1 - \beta_0} \times \frac{\pi_{12}(1 - p_1)}{1 - \beta_1} \\ &= \left(\sum_{\substack{j+m+n \in Z_0 \\ j \geq 1; m \geq 1; n \geq 0}} + \sum_{\substack{j+m+n \in Z_1 \\ j \geq 1; m \geq 1; n \geq 0}} + \sum_{\substack{j+m+n \in Z_2 \\ j \geq 1; m \geq 1; n \geq 0}} \right) \beta_0^{j-1}\beta_1^{m-1}\pi_{01}(1 - p_1)\pi_{12}(1 - p_2)^n p_2 \\ &= JMN_{0;j,m,n} + JMN_{1;j,m,n} + JMN_{2;j,m,n} = JMN_0 + JMN_1 + JMN_2. \end{aligned}$$

The expected value of $j + m + n$ for Case 5 is

$$\begin{aligned}
JMN. &= JMN_{.j,m,n}[j + m + n] \\
&= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} (j + m + n) \beta_0^{j-1} \beta_1^{m-1} \pi_{01} (1 - p_1) \pi_{12} (1 - p_2)^n p_2 \\
&= \frac{\pi_{01}}{(1 - \beta_0)^2} \times \frac{\pi_{12}(1 - p_1)}{1 - \beta_1} + \frac{\pi_{01}}{1 - \beta_0} \times \frac{\pi_{12}(1 - p_1)}{(1 - \beta_1)^2} + \frac{\pi_{01}}{1 - \beta_0} \times \frac{\pi_{12}(1 - p_1)}{1 - \beta_1} \times \frac{1}{p_2} \\
&= \frac{\pi_{01} \pi_{12} (1 - p_1)}{(1 - \beta_0)^2 (1 - \beta_1)^2 p_2} \left(\frac{1}{1 - \beta_0} + \frac{1}{1 - \beta_1} + \frac{1}{p_2} \right) \\
&= \left(\sum_{\substack{j+m+n \in Z_0 \\ j \geq 1; m \geq 1; n \geq 0}} + \sum_{\substack{j+m+n \in Z_1 \\ j \geq 1; m \geq 1; n \geq 0}} + \sum_{\substack{j+m+n \in Z_2 \\ j \geq 1; m \geq 1; n \geq 0}} \right) (j + m + n) \beta_0^{j-1} \beta_1^{m-1} \pi_{01} (1 - p_1) \pi_{12} (1 - p_2)^n p_2 \\
&= JMN_{0;j,m,n}[j + m + n] + JMN_{1;j,m,n}[j + m + n] + JMN_{2;j,m,n}[j + m + n].
\end{aligned}$$

In Case 6, when the system is in S_2 , the probability for $(r - 1)$ conforming to be produced followed by a nonconforming item is $(1 - p_2)^{r-1} p_2$. The total probability over the entire range of r and the expected value of r are the same as those in Case 4, that is,

$$\begin{aligned}
R. &= R_{.,r} = \sum_{r=1}^{\infty} (1 - p_2)^{r-1} p_2 = 1 = \left(\sum_{r \in Z_0} + \sum_{r \in Z_1} + \sum_{r \in Z_2} \right) (1 - p_2)^{r-1} p_2 \\
&= R_{0;r} + R_{1;r} + R_{2;r} = R_0 + R_1 + R_2, \\
R. [r] &= R_{.,r}[r] = \sum_{r=1}^{\infty} r (1 - p_2)^{r-1} p_2 = \frac{1}{p_2} = \left(\sum_{r \in Z_0} + \sum_{r \in Z_1} + \sum_{r \in Z_2} \right) r (1 - p_2)^{r-1} p_2 \\
&= R_{0;r}[r] + R_{1;r}[r] + R_{2;r}[r].
\end{aligned}$$

Let P_1, \dots, P_6 be the probabilities for Cases 1, ..., 6 to occur, respectively. We have

$$P_1 = \sum_{N_1=0}^{\infty} I^{N_1} (JK_1 + JK_2),$$

$$P_2 = \sum_{N_1=0}^{\infty} I^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} (L_1 + L_2)$$

$$P_3 = \sum_{N_1=0}^{\infty} I^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} MN_2$$

$$P_4 = \sum_{N_1=0}^{\infty} I^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} (MN_0 + MN_1) \times \sum_{N_3=0}^{\infty} (R_0 + R_1)^{N_3} R_2$$

$$= \sum_{N_1=0}^{\infty} I^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} (MN_0 + MN_1),$$

$$P_5 = \sum_{N_1=0}^{\infty} I^{N_1} JMN_2,$$

$$P_6 = \sum_{N_1=0}^{\infty} I^{N_1} (JMN_0 + JMN_1) \times \sum_{N_3=0}^{\infty} (R_0 + R_1)^{N_3} R_2 = \sum_{N_1=0}^{\infty} I^{N_1} (JMN_0 + JMN_1).$$

Since Case 1, ..., Case 6 are mutually exclusive, the sum $(P_1 + P_2 + (P_3 + P_4)) + (P_5 + P_6) = 1$ ensures that these six cases include all the possible cases for the system to complete a cycle.

5. Expected values associated with models $(\mathbf{I}_{1+2}, \mathbf{M}_{1+2})$ and $(\mathbf{I}_0, \mathbf{M}_{1+2})$

We first consider the model $(\mathbf{I}_{1+2}, \mathbf{M}_{1+2})$. In what follows, for any function $g(\nu)$ the summation $\sum_{\nu=u}^w g(\nu)$ will be interpreted as 0 if $u > w$.

The closed form of the formulas displayed below are established using the formulas for the summations $S_1, S_{11}, S_2, S_{22}, S_3, S_{33}$ given in the Appendix.

From the definitions of $I, I_0, I_1, I_2, J_0, J_1, \dots, JMN_2$, we shall derive the expressions for the expected number of items inspected $E_\eta[\text{isp}]$, the expected number of type γ ($\gamma = 0, 1, 2$) signals $E_\eta[\text{s}_\gamma]$, the expected number of type γ investigation $E_\eta[\text{i}_\gamma]$, and the expected number type γ maintenance $E_\eta[\text{m}_\gamma]$ for Case 1, ..., Case 6.

For convenience in presentation, we shall use the notation

$$I. = I_{.;i} = \sum_{i=1}^{\infty} \beta_0^{i-1} (1 - \pi_{01}) p_0, \quad I_{.;isp} = I_{.;i}[i] = \sum_{i=1}^{\infty} \beta_0^{i-1} (1 - \pi_{01}) p_0 \times i, \dots,$$

$$JK_{1+2;isp} = JK_{1;isp} + JK_{2;isp} = JK_{1+2;j,k}[j+k] = \sum_{j+k \in Z_1 \cup Z_2} \beta_0^{j-1} \beta_1^k \pi_{01} p_1 \times (j+k), \dots,$$

and so on.

In all the cases in the Figure, when the system is in S_0 , each of the N_1 circles \circ 's represents a nonconforming item which gives either a type 0 signal s_0 , a type 1 signal s_1 , or a type 2 signal s_2 ; since these three types of signal occur in the ratio $I_0 : I_1 : I_2$ and $I. = I_0 + I_1 + I_2$, the expected numbers of type 0, type 1 and type 2 signals are $\frac{I_0}{I.} \times N_1$, $\frac{I_1}{I.} \times N_1$ and $\frac{I_2}{I.} \times N_1$, respectively. When the system is in S_1 , a circle \circ represents a nonconforming item which gives a signal s_0 . When the system is in S_2 , a circle \circ represents a nonconforming item which gives either a signal s_0 or s_1 .

When the system is in S_1 , a bullet \bullet represents a nonconforming item which gives either a signal s_1 or s_2 . When the system is in S_2 , a bullet \bullet represents a nonconforming item which gives a signal s_2 .

Since s_γ causes i_γ ($\gamma = 0, 1, 2$), for all $\eta = 1, \dots, 6$ we have

$$E_1[i_0] = E_1[s_0], \quad E_1[i_1] = E_1[s_1], \quad E_1[i_2] = E_1[s_2].$$

Derivations of the expected numbers of inspections and signals are based on the expressions of the probabilities P_1, \dots, P_6 given in Section 4. The calculations are straightforward but cumbersome, and here an illustration is given for Case 1. Corresponding formulas for Cases 2 – 6 can be obtained in a similar way, but the expressions are lengthy and they will not be displayed here.

Expected numbers associated with Case 1.

For Case 1, it is shown from Figure 1 that N_1 circles \circ 's have occurred, and that inspection is required if and only if $j + k \in Z_1$ or $j + k \in Z_2$. The expected number of items inspected is $E_1[\text{isp}] = E_{N_1} [E_{i_1} \cdots [E_{i_{N_1}} [E_{j,k} [i_1 + \cdots + i_{N_1} + j + k]]]]]$. Here when $N_1 = 0$, we interpret the sum $i_1 + \cdots + i_{N_1}$ as 0 and interpret $I_{:,i_1} \cdots I_{:,i_{N_1}}$ in the following derivation for $E_1[\text{isp}]$ as 1. (This convention will be used for the rest of this paper.)

We recall that the probability associated with Case 1 is

$$\begin{aligned} P_1 &= \sum_{N_1=0}^{\infty} \left\{ \sum_{i_1=1}^{\infty} \beta_0^{i_1-1} (1 - \pi_{01}) p_0 \cdots \sum_{i_{N_1}=1}^{\infty} \beta_0^{i_{N_1}-1} (1 - \pi_{01}) p_0 \right\} \times \sum_{\substack{j+k \in Z_1 \cup Z_2 \\ j \geq 1; k \geq 0}} \beta_0^{j-1} \pi_{01} p_1 \\ &= \sum_{N_1=0}^{\infty} \{ I_{:,i_1} \cdots I_{:,i_{N_1}} \} JK_{1+2;j,k} = \sum_{N_1=0}^{\infty} I_{:,i_{N_1}}^{N_1} (JK_1 + JK_2), \end{aligned}$$

It follows that

$$\begin{aligned} E_1[\text{isp}] &= E_{N_1} [E_{i_1} \cdots [E_{i_{N_1}} [E_{j,k} [i_1 + \cdots + i_{N_1} + j + k]]]]] \\ &= \sum_{N_1=0}^{\infty} \left\{ \sum_{i_1=1}^{\infty} \beta_0^{i_1-1} (1 - \pi_{01}) p_0 \cdots \sum_{i_{N_1}=1}^{\infty} \beta_0^{i_{N_1}-1} (1 - \pi_{01}) p_0 \right\} \times \\ &\quad \sum_{\substack{j+k \in Z_1 \cup Z_2 \\ j \geq 1; k \geq 0}} \beta_0^{j-1} \pi_{01} p_1 [i_1 + \cdots + i_{N_1} + j + k] \\ &= \sum_{N_1=0}^{\infty} \{ I_{:,i_1} \cdots I_{:,i_{N_1}} \} JK_{1+2;j,k} [i_1 + \cdots + i_{N_1} + j + k] \\ &= \sum_{N_1=0}^{\infty} \left(I_{:,i_1} [i_1] \times I_{:,i_2} \times \cdots \times I_{:,i_{N_1}} \times JK_{1+2;j,k} + \right. \\ &\quad \vdots \\ &\quad \left. I_{:,i_1} \times \cdots \times I_{:,i_{N_1-1}} \times I_{:,i_{N_1}} [i_{N_1}] \times JK_{1+2;j,k} + \right. \\ &\quad \left. I_{:,i_1} \times I_{:,i_{N_1}} \times \cdots \times I_{:,i_{N_1}} \times JK_{1,2;j,k} [j+k] \right) \\ &= \sum_{N_1=0}^{\infty} (N_1 I_{:,isp} I_{:,i_{N_1}}^{N_1-1} (JK_1 + JK_2) + I_{:,i_{N_1}}^{N_1} JK_{1+2;isp}) \\ &= \frac{I_{:,isp} (JK_1 + JK_2)}{(1 - I_{:,isp})^2} + \frac{JK_{1+2;isp}}{1 - I_{:,isp}}. \end{aligned}$$

For Case 1, as shown in Figure 1 the last of the $j + k$ items inspected indicated by the bullet • gives either a signal s_1 or s_2 , and the expected number of occurrence of these two types of signal are in the ratio $JK_1 : JK_2$. Hence from the expression for P_1 we have

$$\begin{aligned}
E_1[s_0] &= \sum_{N_1=0}^{\infty} I^{N_1}(JK_1 + JK_2) \left(\frac{I_0}{I} \times N_1 \right) = \frac{I_0(JK_1 + JK_2)}{(1 - I)^2}, \\
E_2[s_1] &= \sum_{N_1=0}^{\infty} I^{N_1}(JK_1 + JK_2) \left(\frac{I_1}{I} \times N_1 + \frac{JK_1}{JK_1 + JK_2} \times 1 \right) \\
&= \frac{I_1(JK_1 + JK_2)}{(1 - I)^2} + \frac{JK_1}{1 - I}, \\
E_2[s_2] &= \sum_{N_1=0}^{\infty} I^{N_1}(JK_1 + JK_2) \left(\frac{I_2}{I} \times N_1 + \frac{JK_2}{JK_1 + JK_2} \times 1 \right) \\
&= \frac{I_2(JK_1 + JK_2)}{(1 - I)^2} + \frac{JK_2}{1 - I}.
\end{aligned}$$

When the system is in S_0 , no maintenance will be carried out. When the system is in S_1 , either i_1 or i_2 will indicate the true state of the system, and m_1 will be carried out, while m_2 will not be necessary. Thus, looking at Case 1 in Figure 1 and using the expressions of $E_1[s_1]$ and $E_1[s_2]$, we obtain

$$\begin{aligned}
E_1[m_1] &= \sum_{N_1=0}^{\infty} I^{N_1}(JK_1 + JK_2) \left(\frac{JK_1 + JK_2}{JK_1 + JK_2} \times 1 \right) = P_1, \\
E_1[m_2] &= 0.
\end{aligned}$$

Model (I_0, M_{1+2}) .

As for this model investigations will not be carried out, and signal s_γ will immediately trigger maintenance action m_γ ($\gamma = 0, 1, 2$). This situation occurs when the costs of investigations i_1 and i_2 are nearly as high as the costs of maintenance m_1 and m_2 , so that it is economically not viable to carry out investigation. Under the model (I_0, M_{1+2}) , all the expressions for P_1, \dots, P_6 are identical to those of the model (I_{1+2}, M_{1+2}) , and so

are the expressions $E_1[\text{isp}], \dots, E_6[\text{isp}]$ for expected number of items inspected, and the expressions $E_1[s_i], \dots, E_6[s_i]$ for the expected numbers of type i signals ($i = 0, 1, 2$). It is clear that

$$E_\eta[i_\gamma] = 0, \quad E_\eta[m_\gamma] = E_\eta[s_\gamma] \quad (\eta = 1, \dots, 6; \gamma = 0, 1, 2).$$

6. Other models without minor investigation.

When the cost of minor investigation i_1 is nearly as high as i_2 , it will be economically viable to replace i_1 by i_2 . When the costs of i_1 and i_2 are nearly as high as m_1 and m_2 , it will be economically viable to replace i_1 by m_1 and replace i_2 by m_2 . When the cost of m_1 is nearly as high as m_2 , it will be economically viable to replace m_1 by m_2 .

Under the models (I_2, M_{1+2}) , (I_2, M_2) and (I_0, M_2) , when the system is in S_0 , the \circ 's in Figure 1 may indicate signals s_0, s_1 or s_2 . When the system is in S_1 , the \circ 's indicate signal s_0 , and the bullet \bullet indicate either s_1 or s_2 . When the system is in S_2 , the \circ 's indicate s_0 , and the \bullet indicate either s_1 or s_2 . Under these models, either s_1 or s_2 will cause i_2 or i_0 , and therefore n_2 is irrelevant and only n_1 will appear in the formulas for various probabilities and expected values. (For the definitions of n_1, n_2 , refer to the sets Z_0, Z_1, Z_2 defined in Section 1.) Under the model (I_0, M_0) , neither investigation nor maintenance will be carried out, and both n_1 and n_2 are irrelevant.

6.1. Model (I_2, M_{1+2}) .

Under this model, when either s_1 or s_2 appears, i_2 will be carried out. If i_2 shows that the system is in state S_γ , m_γ ($\gamma = 1, 2$) will be carried out which will restore the system back to S_0 .

The probabilities for Cases 1 and 2 in Figure 1 to occur are identical to those shown in Section 4 for models (I_{1+2}, M_{1+2}) and (I_0, M_{1+2}) ; and thus the expected values $E_\eta[\text{isp}]$,

$E_\eta[s_\gamma]$ and $E_\eta[m_\gamma]$ ($\eta = 1, 2$, $\gamma = 0, 1, 2$) are identical to those of the models (I_{1+2}, M_{1+2}) and (I_0, M_{1+2}) . We denote the probabilities for Cases 3, 4, 5 and 6 in Figure 1 by P_3^* , P_4^* , P_5^* and P_6^* , respectively. These probabilities are as follows.

$$\begin{aligned}
P_1 &= \sum_{N_1=0}^{\infty} I_1^{N_1} (JK_1 + JK_2), \\
P_2 &= \sum_{N_1=0}^{\infty} I_1^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} (L_1 + L_2) \\
P_3^* &= \sum_{N_1=0}^{\infty} I_1^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} (MN_1 + MN_2) \\
P_4^* &= \sum_{N_1=0}^{\infty} I_1^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} MN_0 \times \sum_{N_3=0}^{\infty} R_0^{N_3} (R_1 + R_2) \\
&= \sum_{N_1=0}^{\infty} I_1^{N_1} JK_0 \times \sum_{N_2=0}^{\infty} L_0^{N_2} MN_0, \\
P_5^* &= \sum_{N_1=0}^{\infty} I_1^{N_1} (JMN_1 + JMN_2), \\
P_6^* &= \sum_{N_1=0}^{\infty} I_1^{N_1} JMN_0 \times \sum_{N_3=0}^{\infty} R_0^{N_3} (R_1 + R_2) = \sum_{N_1=0}^{\infty} I_1^{N_1} JMN_0.
\end{aligned}$$

It is straightforward to show that $P_3^* + P_4^* = P_3 + P_4$, and $P_5^* + P_6^* = P_5 + P_6$.

Refer to Figure 1 for Cases 3, 4, 5, 6. Under the model (I_2, M_{1+2}) , when the system is in S_1 , the $1 + N_2$ circles \circ 's represent type 0 signals s_0 's. In Cases 6, when the system is in S_2 , the $1 + N_3$ circles \circ 's represent type 0 signals s_0 's, and the bullet \bullet represents either a type 1 signal s_1 or a type 2 signal s_2 , where these two types of signal occur in the ratio $R_1 : R_2$.

Replacing i_1 in the model (I_{1+2}, M_{1+2}) by i_2 forms the model (I_2, M_{1+2}) , and hence for model (I_2, M_{1+2}) we have

$$E_\eta[i_0] = E_\eta[s_0], \quad E_\eta[i_1] = 0, \quad E_\eta[i_2] = E_\eta[s_1] + E_\eta[s_2] \quad (\eta = 1, \dots, 6).$$

For Cases 1 and 2, all the expected values of the number of signals, inspections and maintenance actions are identical to those for the model (I_{1+2}, M_{1+2}) . For Cases 3 – 6,

corresponding results can be obtained based on the probabilities $P_3^*, P_4^*, P_5^*, P_6^*$, and the detailed are omitted here.

6.2 Model (I_2, M_2) .

Under this model, an s_1 or s_2 will be followed by i_2 which will correctly reveal the state of the system, and if the system is in S_1 or S_2 , m_2 will be carried out which will restore the system back to S_0 . The expressions P_3^*, \dots, P_6^* for the probabilities of Cases 3, ..., 6 in Figure 1 are as the same as those for the model (I_2, M_{1+2}) in Sub-section 6.1. The expressions P_1 and P_2 for the probabilities of Cases 1 and 2 in Figure 1 are identical to those of the model (I_{1+2}, M_{1+2}) in Section 4.

Since i_1 is not carried out, we have

$$E_\eta[i_0] = E_\eta[s_0], \quad E_\eta[i_1] = 0, \quad E_\eta[i_2] = E_\eta[s_1] + E_\eta[s_2] \quad (\eta = 1, \dots, 6).$$

For Cases 1 and 2, it can be seen that $E_1[m_0]$ and $E_2[m_0]$ are also identical to those of the model (I_{1+2}, M_{1+2}) , and we have $E_1[m_2] = P_1$, $E_2[m_2] = P_2$.

Refer to Cases 3, 4, 5, 6 in the Figure 1. Under model (I_2, M_2) , when the system is in S_1 , the $1 + N_2$ circles \circ 's represent type 0 signals s_0 's. In Case 4, when the system is in S_2 , the $1 + N_3$ circles \circ 's represent type 0 signals s_0 's, and the bullet \bullet represents either a type 0 signal s_0 or a type 1 signal s_1 , where these two types of signal occur in the ratio $R_1 : R_2$. Hence from the expression for the numbers of items inspected $E_4[isp]$, $E_4[s_\gamma]$, $E_4[isp]$, $E_4[s_\gamma]$ ($\gamma = 0, 1, 2$) are identical to those of the model (I_2, M_{1+2}) . In Cases 3, 4, 5 and 6, since m_1 is absent and the \bullet represent m_2 , the expressions for $E_\eta[m_\gamma]$ ($\eta = 3, 4, 5, 6$, $\gamma = 0, 1, 2$) are identical to those of model (I_2, M_{1+2}) .

6.3 Model (I_0, M_2) .

Under this model, no inspection will be carried out, and s_1 or s_2 will always lead to m_2 . If the system is in S_i ($i = 1, 2$), m_2 will always restore the system back to S_0 . The expressions of the probabilities for Cases 1 and 2 in Figure 1 are the same as the P_1 and P_2 for model (I_{1+2}, M_{1+2}) in Section 4. The expressions for the probabilities for Case 3, ..., 6 in Figure 1 and the expected number of items inspected are the same as the P_3^*, \dots, P_6^* and $E_3[\text{isp}], \dots, E_6[\text{isp}]$ for model (I_2, M_{1+2}) in Sub-Section 6.1.

For the model (I_0, M_2) , for all $\eta = 1, \dots, 6$ we have

$$E_\eta[i_1] = E_\eta[i_2] = E_\eta[i_3] = 0,$$

$$E_\eta[m_0] = E_\eta[s_0], \quad E_\eta[m_1] = 0, \quad E_\eta[m_2] = E_\eta[s_1] + E_\eta[s_2].$$

6.4 Model (I_0, M_0) .

Under this model, none of i_1 , i_2 , m_1 and m_1 will be employed. The system starts at S_0 , deteriorates to S_1 if $\pi_{01} > 0$, and then deteriorates to S_2 if $\pi_{12} > 0$. This model may be applied when the cost required to carry out investigation and maintenance are high compared to the penalty cost due to occurrence of nonconforming items.

7. Economic design of inspection and maintenance policy

In reality, various factors determine whether to perform investigation and maintenance on the system. These factors include resources available, operational convenience, loss due to system failure and nonconforming items, and others. Generally speaking, investigation and maintenance should be carried out if the loss due to occurrence of nonconforming items is considerable.

We consider the following three types of cost: (1) penalty cost C_{nc} due to occurrence of nonconforming items, (2) investigation cost C_{ivs} , (3) maintenance cost C_m . If N_{tol} is

the expected total number of items produced in a cycle, the average total cost per item per cycle is

$$\bar{c} = (C_{nc} + C_{ivs} + C_m)/N_{tol}.$$

In accordance with the investigation and maintenance policies defined in the previous sections, we consider the following fixed costs:

c_{nc} = the penalty cost incurred when a nonconforming item is produced,

$c_{ivs,1}$ = cost of carrying out i_1 each time,

$c_{ivs,2}$ = cost of carrying out i_2 each time,

$c_{m,1}$ = cost of carrying out m_1 each time,

$c_{m,2}$ = cost of carrying out m_2 each time.

In what follows, \sum_{η} will denote the summation over $\eta = 1, \dots, 6$ for Cases 1, ..., 6 in Figure 1. Since all items are inspected, for all the models (I_{1+2}, M_{1+2}) , (I_2, M_{1+2}) , (I_0, M_{1+2}) , (I_2, M_2) and (I_0, M_2) we have

$$N_{tol} = \sum_{\eta} E_{\eta}[\text{isp}],$$

where the expressions for $E_{\eta}[\text{isp}]$ for the different models are given in Sections 4, 5, 6 (and so are the expressions for $E_{\eta}[s_0]$, $E_{\eta}[s_1]$, $E_{\eta}[s_2]$, $E_{\eta}[m_1]$, $E_{\eta}[m_2]$ in the following C_{nc} , C_{ivs} , C_m).

Since each nonconforming item gives a signal (either s_0 , s_1 or s_2), the expected total number of nonconforming items is $\sum_{\eta} (E_{\eta}[s_0] + E_{\eta}[s_1] + E_{\eta}[s_2])$, giving

$$C_{nc} = c_{nc} \sum_{\eta} (E_{\eta}[s_0] + E_{\eta}[s_1] + E_{\eta}[s_2]).$$

As for C_{ivs} and C_m , we have

$$C_{ivs} = c_{ivs,1} \sum_{\eta} E_{\eta}[i_1] + c_{ivs,2} \sum_{\eta} E_{\eta}[i_2],$$

$$C_m = c_{m,1} \sum_{\eta} E_{\eta}[m_1] + c_{m,2} \sum_{\eta} E_{\eta}[m_2].$$

For the model (I_0, M_0) , the system will change to state S_2 eventually and remain in S_2 indefinitely. The average number of items produced when the system is in states S_0 and S_1 are $1/\pi_{01}$ and $1/\pi_{12}$, respectively. Therefore the average total cost is

$$\bar{c} = \lim_{N \rightarrow \infty} \frac{p_0 c_{nc}/\pi_{01} + p_1 c_{nc}/\pi_{12} + p_2 N c_{nc}}{1/\pi_{01} E_1 + 1/\pi_{12} + N} = p_2 c_{nc},$$

where N denotes the number of items produced after the process has shifted to state S_2 .

Referring to the sets Z_0, Z_1, Z_2 defined in Section 2, we see that the \bar{c} for each inspection and maintenance model can be minimized with respect to n_1 and n_2 in the region $\mathcal{N} = \{(n_1, n_2) : 2 \leq n_2 + 1 \leq n_1 \leq \infty\}$. Since \bar{c} is an algebraic function of n_1 and n_2 , minimization can be carried out easily using software such as *Mathematica*. The minimum of \bar{c} for the six maintenance models, say \bar{c}_{min} , can be compared, and the most cost-effective model can be selected. It is clear that \bar{c} depends on the parameters $p_0, p_1, p_2, \pi_{01}, \pi_{12}, c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}$, and the minimum cost \bar{c}_{min} may be attained under any of the five models $(I_{1+2}, M_{1+2}), (I_2, M_{1+2}), (I_0, M_{1+2}), (I_0, M_2), (I_0, M_0)$. Table 1 gives the numerical values of \bar{c} for different values of the parameters and for different models. In each of the Cases (a) – (h) in Table 1, the minimum \bar{c} 's among the different models are high-lighted with square brackets. In Cases (a), (b), (c), the minimum \bar{c} is attained under model (I_{1+2}, M_{1+2}) . In Cases (d), (e), (f), (g), the minimum \bar{c} is attained under the models $(I_0, M_{1+2}), (I_2, M_{1+2}), (I_0, M_2), (I_0, M_0)$, respectively.

Since $c_{m,1} < c_{m,2}$, given any values of $p_0, p_1, p_2, \pi_{01}, \pi_{12}, c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}$, it is easy to see from the expressions of C_m that the \bar{c} for model (I_2, M_2) cannot be smaller than that of model (I_2, M_{1+2}) . This is based on the assumption that $c_{m,2}$ is the same for both M_2 and M_{1+2} . However, in some cases an additional cost Δ per maintenance action is required to provide two types of maintenance services m_1 and m_2 , rather than just a single type m_2 . If Δ is included into the maintenance cost, the cost to carry

out m_2 each time under the model (I_2, M_{1+2}) will be $c_{m,2} + \Delta$ which is larger than the cost $c_{m,2}$ under the model (I_2, M_2) . In this case, it is possible that \bar{c}_{min} under the model (I_2, M_2) will be smaller than that under the model (I_2, M_{1+2}) , and this is illustrated by Case (h) in Table 1. This idea can be extended to the case when an additional cost is incurred in order to maintain two types of inspection (minor i_1 and major i_2), instead of just one type (major i_2).

For models (I_{1+2}, M_{1+2}) and (I_0, M_{1+2}) , the minimum of \bar{c} can be attained in the interior or on the boundary of \mathcal{N} . Cases (a) and (g) in Table 1 illustrate that for model (I_{1+2}, M_{1+2}) , the minimum of \bar{c} is attained in the interior of \mathcal{N} (where $2 < n_2 + 1 < n_1 < \infty$), and Cases (b), (c), (d), (e), (f), (h) illustrate that the minimum of \bar{c} is attained on the boundary of \mathcal{N} (where either $1 < n_2 < n_1 = \infty$, $n_2 = n_1 = \infty$ or $2 < n_2 + 1 = n_1 < \infty$). In Case (b), $n_1 = \infty$ means that i_0 does not exist, and occurrence of a nonconforming item is either i_1 or i_2 . In Case (c), $n_1 = n_2 = \infty$ indicates that both i_0 and i_1 do not exist, and any occurrence of nonconforming item is i_2 .

Table 1. Numerical examples.

Here $(p_0, p_1, p_2, \pi_{01}, \pi_{12}) = (0.015, 0.019, 0.05, 0.0004, 0.0035)$.

(a) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (2.3, 4.8, 10, 4.9, 260)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	[0.11024]	0.16171	0.11365	0.13279	0.17553	0.115
$N_{tol} =$	2671.82	2834.03	2705.25	3141.35	3141.35	∞
$n_2 =$	6	1	—	—	—	—
$n_1 =$	17	15	9	1	1	—

(b) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (120, 3, 18, 11, 22)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	[1.90466]	1.93549	1.92969	1.93181	1.93550	6
$N_{tol} =$	2559.19	2674.08	2666.93	2666.93	2679.38	∞
$n_2 =$	6	12	—	—	—	—
$n_1 =$	∞	13	14	14	12	—

(c) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (220, 0.1, 0.5, 10, 20)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	[3.33543]	3.47997	[3.33543]	3.33876	3.48091	11
$N_{tol} =$	2547.64	2633.66	2547.64	2547.64	2644.92	∞
$n_2 =$	∞	16	—	—	—	—
$n_1 =$	∞	24	∞	∞	19	—

(d) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (11, 2.7, 15, 2.8, 17)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	0.20333	[0.19967]	0.20272	0.20490	0.20036	0.55
$N_{tol} =$	2784.10	2754.99	2804.82	2804.82	2804.82	∞
$n_2 =$	4	3	1	1	1	—
$n_1 =$	5	9	4	4	4	—

(e) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (120, 5, 10, 411, 650)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	2.11196	2.73581	[2.08753]	2.14071	2.58369	6
$N_{tol} =$	2621.72	2824.78	2627.87	2648.57	2927.14	∞
$n_2 =$	27	3	—	—	—	—
$n_1 =$	28	4	25	18	2	—

(f) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (20, 9, 15, 10, 17)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	0.35242	0.34863	0.34984	0.35073	[0.34742]	1
$N_{tol} =$	2736.77	2756.84	2748.31	2748.31	2770.65	∞
$n_2 =$	6	5	—	—	—	—
$n_1 =$	7	6	6	6	5	—

(g) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (1, 4.8, 10, 4.9, 260)$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	0.08977	0.13875	0.09315	0.11371	0.20811	[0.05]
$N_{tol} =$	2871.82	2843.04	2705.25	2927.14	2927.14	∞
$n_2 =$	6	1	—	—	—	—
$n_1 =$	17	14	9	2	2	—

(h) $(c_{nc}, c_{inv,1}, c_{inv,2}, c_{m,1}, c_{m,2}) = (120, 5, 10, 11, 250)$ under M_2 and $= (120, 5, 10, 11 + \Delta, 250 + \Delta)$ under M_{1+2} , where $\Delta = 400$.

Investigation =	I_{1+2}	I_0	I_2	I_2	I_0	I_0
Maintenance =	M_{1+2}	M_{1+2}	M_{1+2}	M_2	M_2	M_0
$\bar{c}_{min} =$	2.11196	2.73581	2.08753	[1.98957]	2.25668	6
$N_{tol} =$	2621.72	2824.78	2627.87	2644.92	2846.52	∞
$n_2 =$	27	3	—	—	—	—
$n_1 =$	28	4	25	19	3	—

8. Discussion.

In this paper, the authors apply the concept of cumulative count of conforming chart (CCC chart) to develop inspection and maintenance plans for production systems where an item produced can be classified as either conforming or nonconforming. All products items produced are inspected, and the number of items inspected until a nonconforming item is observed is used as an indicator for system monitoring. When a nonconforming item is observed, either minor investigation, major investigation or no investigation on the system may be carried out, and subsequently, either minor maintenance, major maintenance or no maintenance on the system may be carried out. The author derived analytic expressions of the probabilities for different cases to occur, expected numbers of signals, expected numbers of investigations and maintenance actions. With these results, optimized investigation and maintenance strategies are obtained numerically from an economic approach.

This paper assumes that minor deterioration of the system can be detected by minor

investigation, and major deterioration of the system can only be detected only by major investigation, which is illustrated by an example on maintenance of a bus fleet. In reality, there is another situation where minor deterioration in the system is well-hidden and can be detected only by major inspection, whereas major deterioration is conspicuous and can be detected by minor inspection. Separate models can be established for this situation, and optimized investigation and maintenance strategies can be obtained.

This paper considers the case when system has three discrete states S_0, S_1, S_2 and changes from state S_γ to state $S_{\gamma+1}$ with probability $\pi_{\gamma,\gamma+1}$ ($\gamma = 0, 1$). This assumption has been used in the literature (for example, in Das and Jain (1997), Das *et al* (1996), Lorenzen and Vance (1986)) for the purpose of economic design of control charts. Separate studies can be carried out for the situation when deterioration of the system is continuous and the system has continuous states (see for example, Liao *et al* (2006)).

In practice, due to cost constraint or operational constraints, maintenance may not be carried out immediately even when investigations show that the system is in an early state of deterioration (Christer (1999), Liao *et al* (2008)). Signals on the control chart indicating minor deterioration of the system may be used as warnings, and actions will be taken only when there are signals showing more serious deterioration.

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Appendix.

To find the expected number of items inspected and the expected number of each of type of signal in different cases, we need to find closed-form expressions of I_0, I_1, \dots, JMN_2 . The following identities will be used, in which $i, j, k, u, w \geq 1$ are integers, $u < w$, $a \neq 1$, $b \neq 1$ and $c \neq 1$. These identities are derived using $1 + u + \dots + u^n = (1 - u^{n+1})/(1 - u)$ and its derivative, where $u \neq 1$. Corresponding results when $a = 1$, $b = 1$ or $c = 1$ can be more easily derived.

$$S_1 = \sum_{i=u}^w a^{i-1} = \frac{a^{u-1} - a^w}{1 - a}.$$

$$S_{11} = \sum_{i=u}^w ia^{i-1} = \frac{a^u - a^{w+1}}{(1-a)^2} + \frac{ua^{u-1} - (w+1)a^w}{(1-a)} = \frac{aS_1}{1-a} + \frac{ua^{u-1} - (w+1)a^w}{1-a}$$

$$= \frac{a^{u-1}(a+u-au) - a^w(a+w-aw)}{(1-a)^2}.$$

$$S_2 = \sum_{i+j=u}^w a^{i-1}b^{j-1}$$

$$= \begin{cases} \frac{a^{u-1} - a^w}{(1-a)(a-b)} - \frac{b^{u-1} - b^w}{(1-b)(a-b)}, & (a \neq b), \\ \frac{(u-1)a^{u-2} - (u-2)a^{u-1} - wa^{w-1} + (w-1)a^w}{(1-a)^2}, & (a = b). \end{cases}$$

$$S_{22} = \sum_{i+j=u}^w (i+j)a^{i-1}b^{j-1} = \left(\frac{1}{1-a} + \frac{1}{1-b} \right) \times S_2 + \frac{1}{(1-a)(1-b)} \times [Q_{22}], \text{ where}$$

$$[Q_{22}] = \begin{cases} \frac{(b^{u-1} - a^{u-1}) + (u-1)[(1-b)a^{u-1} - (1-a)b^{u-1}]}{a-b} \\ \quad - \frac{(b^w - a^w) + w[(1-b)a^w - (1-a)b^w]}{a-b}, & (a \neq b), \\ -(u-1)a^{u-2} + (u-1)a^{u-2}[(u-1) - (u-2)a] \\ \quad + wa^{w-1} - wa^{w-1}[w - (w-1)a], & (a = b). \end{cases}$$

$$\begin{aligned}
S_3 &= \sum_{i+j+k=u}^w a^{i-1} b^{j-1} c^{k-1} \\
&= \begin{cases} \frac{(a^{u-1} - a^w)}{(1-a)(a-b)(a-c)} - \frac{(b^{u-1} - b^w)}{(1-b)(a-b)(b-c)} + \frac{(c^{u-1} - c^w)}{(1-c)(a-c)(b-c)}, & (a \neq b \neq c), \\ \frac{(a^{u-1} - a^w)}{(1-a)(a-b)^2} - \frac{a((u-1)b^{u-2} - wb^{w-1})}{(1-b)^2(a-b)^2} \\ + \frac{(1+a)((u-2)b^{u-1} - (w-1)b^w)}{(1-b)^2(a-b)^2} - \frac{(u-3)b^u - (w-2)b^{w+1}}{(1-b)^2(a-b)^2}, & (a \neq b = c), \\ \frac{a^{u-1} - a^w}{(1-a)^3} + \frac{(u-1)a^{u-2} - wa^{w-1}}{(1-a)^2} + \frac{(u-1)(u-2)a^{u-3} - w(w-1)a^{w-2}}{2(1-a)}, & (a = b = c). \end{cases}
\end{aligned}$$

$$\begin{aligned}
S_{33} &= \sum_{i+j+k=u}^w (i+j+k)a^{i-1}b^{j-1}c^{k-1} \\
&= \left(\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} - 1 \right) \times S_3 + \frac{1}{(1-a)(1-b)(1-c)} \times [Q_{33}], \text{ where} \\
[Q_{33}] &= \begin{cases} \frac{(1-b)(1-c)[(u-1)a^{u-1} - wa^w] + (-1+bc)[a^{u-1} - a^w]}{(a-b)(a-c)} \\ - \frac{(1-a)(1-c)[(u-1)b^{u-1} - wb^w] + (-1+ac)[b^{u-1} - b^w]}{(a-b)(b-c)} \\ + \frac{(1-a)(1-b)[(u-1)c^{u-1} - wc^w] + (-1+ab)[c^{u-1} - c^w]}{(a-c)(b-c)}, & (a \neq b \neq c), \\ \frac{(1-b)^2((u-1)a^{u-1} - wa^w) + (-1+b^2)(a^{u-1} - a^w)}{(a-b)^2} \\ + \frac{-(1-a)((u-1)[(u-3)b^u - (u-2)(1+a)b^{u-1} + a(u-1)b^{u-2}])}{(a-b)^2} \\ + \frac{(1-a)(w[(w-2)b^{w+1} - (w-1)(1+a)b^w + awb^{w-1}])}{(a-b)^2} \\ + \frac{a((u-3)b^u - (w-2)b^{w+1})}{(a-b)^2} + \frac{-(1+a^2)((u-2)b^{u-1} - (w-1)b^w)}{(a-b)^2} \\ + \frac{a((u-3)b^u - (w-2)b^{w+1})}{(a-b)^2}, & (a \neq b = c), \\ = \frac{3(a^u - a^{w+1})}{(1-a)^4} + \frac{3(ua^{u-1} - (w+1)a^w)}{(1-a)^3} + \frac{3(u(u-1)a^{u-2} - (w+1)wa^{w-1})}{2(1-a)^2} \\ + \frac{u(u-1)(u-2)a^{u-3} - (w+1)w(w-1)a^{w-2}}{2(1-a)}, & (a = b = c). \end{cases}
\end{aligned}$$