Warranty claims data analysis considering sales delay

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Abstract

Sales delay is the time interval from the date of manufacture to the date of sale. In analysing warranty claims data, the existing research relating to the sales delay has mainly focused on estimating the probability distribution of the sales delay. Longer sales delay may lead to more warranty claims as it can have an impact on the post-sale reliability of products. However, research into this problem has received little attention.

This paper estimates the expected number of warranty claims under both renewing and non-renewing warranty policies taking into account the sales delay. We consider the case with three states, the sales delay state, the operating state and the failed state. We extend the three state case into an n state system case, where $n \geq 3$. We then give numerical examples to demonstrate the application of the derived equations. We also present a simulation and a case study were we estimate the reliability of products with three states.

Keywords: warranty claims, multistate components, sales delay, failure rate, non-homogeneous Poisson process.

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Notation

t_s	Time of sale.
$F_0(t)$	Lifetime distribution during operating state.
$F_1(t)$	Lifetime distribution of a product in sales delay state.
$F_2(t\|t_s)$	Lifetime distribution during operating state for a product sold at time t_s.
$F^{(k)}(t)$	k^{th} convolution of F.
$F^k(t)$	k^{th} power of F.
$\bar{F}_i(t) = 1 - F_i(t)$	Survival function of $F_i(t)$ for $k = 0, 1, 2$.
$r_i(t)$	Failure rate function of $F_i(t)$ for $k = 0, 1, 2$.
$G(x)$	Sales delay distribution.
w	Warranty period.
N	Total production amount.
s_t	Number of products sold in month t.
n_t	Number of warranty claims in month t.
μ_t	Expected number of warranty claims in month t.

1 Introduction

Warranty is a duty attached to a product and requires manufacturers to offer a pre-specified compensation to buyers when the product fails to perform its designated functions under normal usage within the warranty period. It is intended to assure buyers that faulty products will either be repaired or replaced by the manufacturer at no or partial cost. Such an obligation can be considered a burden imposed on the manufacturer. In the modern economy, however, warranty is increasingly seen as an opportunity and an effective marketing tool that can provide a competitive edge. This is due to the prevalent perception that the duration of the warranty is an indicator of the product's quality. Thus, selecting a suitable warranty policy is an optimisation process in which both costs and profits should be considered from the manufacturer's perspective.

Warranty requires the manufacturer to reserve a certain amount of its resources to cover its warranty obligation. Achieving the optimal tradeoff between warranty related reserves and the actual warranty claims is necessary in order to maintain higher levels of customer satisfaction and profits. As such, estimation of the expected number of warranty claims is crucial for formulating an optimal warranty policy and requires warranty claims data analysis. The warranty claims data is, however, often comes in an incomplete or aggregated form, where some of the data necessary for statistical analysis is missing.

Sales delay is the time from the date of manufacture to the date of sale. The sales delay has been considered in the existing literature, where the main focus has been on estimating its probability distribution. Suzuki et al.[1,2] estimate the sales delay distribution from a sample of warranty data, where the dates of sale are known for reported warranty claims. Wang et al.[3] estimate monthly sales amounts using warranty data where the claims data represent the number of claims for products sold in a given month. They assume that the total amount of sales for the period of time under study can be obtained from other additional data sources. Lim[4] uses a non-homogeneous Poisson process (NHPP) model to estimate the sales delay, where it is referred to as holding times. Karim[5] also estimates the sales delay distribution using an NHPP model. Karim[6] models the sales delay distribution using a lognormal distribution, where this distribution is estimated from a sample of sales delay data. The literature on modelling the sales delay makes extensive use of the ideas and concepts discussed in relation to the modelling of the so-called reporting delays (Kalbfleisch and Lawless[7], Kalbfleisch et al.[8], and Lawless[9]). The reporting delays themselves can often be assumed to be negligible.

Unlike reporting delays that can have impact only on warranty claims data analysis without bearing any relevance to product consumers, the sales delay can have an impact on both manufacturers and product consumers. The impact of the sales delay on product reliability has been noted before, for example, Robinson and McDonald[10] note that longer sales delays result in larger number of warranty claims. Nevertheless, the existing literature has been mainly focused on estimation of the sales delay distribution. So far, little research has been done on estimating the impact of the sales delay on product reliability and/or warranty policy optimisation.

There are two main types of warranty policies, renewing and non-renewing. Under renewing warranty, if a product fails during the warranty period, the warranty of the repaired or newly replaced product is renewed from anew at no or a pre-specified cost. Under non-renewing warranty, the warranty of the original product is carried over to the repaired or newly replaced product. In this paper we consider both of these warranty policies.

In its lifecycle, a product can be in a number of states. For example, a product can be in the following three states, sales delay state, operating state and failed state. This is an example of a three-state system. Warranty cost analysis for such systems has been studied by Wu and Li[11] and Wu and Xie[12], where similar systems with dormant, operating, and failed states have been considered. These studies discuss three-state building services systems.

In this study we derive the number of expected warranty claims for products under renewing and non-renewing warranty policies. We also consider estimation of the reliability of products during the sales delay and operating state from a set of real life data. We show that models based on considering the reliability of a product during the sales delay period and during the operating period yield better results than those with a single reliability function. The novelty of this paper can be summarised in the following points.

- Sales delay has been studied in warranty claims analysis but has not been considered from customers perspective. This paper is the first to address this issue.

- The existing literature focuses on estimating the distribution of the sales delay, but does not consider the impact of the sales delay on products reliability. This paper presents the first attempt to explore this issue.

The paper is structured as follows. Section 2 develops models for estimating the expected number of warranty claims for three-state systems under renewing warranty and non-renewing warranty policies. Section 3 presents a numerical example and demonstrates how the models discussed in the previous section can be put into practical use. Section 4 presents a simulation and a case study and discusses the results of implementing models with different failure rates to field data from the electronics industry. The last section discusses future work plans and summarises the main conclusions of this paper.

2 Model development

In this paper we make the following assumptions.

A1 Products under consideration are non-repairable. A failed product is replaced with an identical product, where replacement times are negligible.

A2 Products can be in the following three states: sales delay state, operating state and failed state. Products in the sales delay state have a different lifetime distribution from those in the operating state.

A3 Products that fail during the sales delay state are replaced with identical ones only at the time of sale.

A4 The sales delay times are assumed to be independent and identically distributed and so are failure times.

The reliability of products during the sales can deteriorate due to different reasons depending on the type of products. For example, electronic equipment can be effected by damp storage conditions.

In a three state system, the first failure of a product can occur in the following three cases.

Case A. A product fails to operate at the time of sale, t_s, where it is replaced with a new identical product with failure rate $r_0(t)$, where t starts from 0.

Case B. A product enters an operating state at time t_s, where its failure rate changes from $r_1(t)$ to $r_2(t|t_s)$. The first failure of the product occurs within (t_s, t_s+w), where it is replaced with a new identical product with failure rate $r_0(t)$.

Case C. The first failure of a product occurs after the warranty term expires. This case does not incur any replacement cost to the manufacturer, so it is not discussed any further.

2.1 Non-renewing warranty policy

From renewal theory (Ross[13]), the number of replacements of a new product, $N(t)$, within time interval $(0, t)$ is given by a renewal process with the time between adjacent renewals distributed according to $F_0(t)$. The probability of k renewals in $[0, t)$ is given by $\Pr\{N(t) = k\} = F_0^{(k)}(t) - F_0^{(k+1)}(t)$, where $F^{(k)}(t)$ is the k^{th} convolution of F. The expected number of renewals/replacements, $M(t)$, is given by $M(t) = \sum_{k=1}^{\infty} F_0^{(k)}(t)$, where $M(t)$ satisfies the renewal integral equation $M(t) = F_0(t) + \int_0^t M(t-x)dF_0(x)$.

For Case A, the probability of an event that a product fails to operate at time t_s is $F_1(t_s)$. So, the expected number of replacements for products that fail during the sales state, $M_{1A}(w)$, is given by:

$$M_{1A}(w) = N(1 + M(w)) \int_0^{\infty} F_1(x)dG(x), \tag{1}$$

where N is the total number of products.

For Case B, if the first failure occurs within $(t_s, t_s + y]$ with $y < w$, the expected number of replacements within time interval $(t_s + y, w)$ is $M(w - y)$. From time t_s to the occurrence of the first failure, the failure rate of a product is $r_2(t|t_s)$ and the lifetime distribution is $F_2(t|t_s)$. Thus, the expected number of replacements within $(t_s, t_s + w]$ for products that fail after being sold, Case B, is given by:

$$M_{1B}(w) = N \int_0^{\infty} \left[\bar{F}_1(x) \int_0^{w} (1 + M(w-y))dF_2(y|x) \right] dG(x). \tag{2}$$

As the expected number of failures for both cases is $M_{1A}(w) + M_{1B}(w)$, we can obtain the following result:

If products are sold under non-renewing warranty, the expected total number of warranty claims for N products within warranty period w is given by

$$\begin{aligned} S_1(w) &= N \Big\{ (1 + M(w)) \int_0^{\infty} F_1(x)dG(x) \\ &\quad + \int_0^{\infty} \left[\bar{F}_1(x) \int_0^{w} (1 + M(w-y))dF_2(y|x) \right] dG(x) \Big\} \end{aligned} \tag{3}$$

2.2 Renewing warranty policy

Under the renewing warranty policy, when a failed product is replaced, the warranty term is renewed. Therefore, for Case A, the expected number of replacements is given by

$$\begin{aligned} M_{2A}(w) &= (1 + \sum_{j=1}^{\infty} F_0^j(w)) N \int_0^{\infty} F_1(x)dG(x) \\ &= \frac{N}{\bar{F}_0(w)} \int_0^{\infty} F_1(x)dG(x), \end{aligned} \tag{4}$$

where F^j is the j^{th} power of F.

For Case B, if the first failure occurs within $(t_s, t_s + w]$, then the expected number of replacements is $F_2(w|t_s)$. After the first replacement, the expected number of replacements is $\left(\sum_{j=1}^{\infty} F_0^j(w)\right) N \int_0^\infty F_2(w|x)dG(x)$.

Therefore, the expected total number of replacements in Case B is given by

$$\begin{aligned} M_{2B}(w) &= \left(1 + \sum_{j=1}^{\infty} F_0^j(w)\right) N \int_0^\infty \bar{F}_1(x)F_2(w|x)dG(x) \\ &= \frac{N}{\bar{F}_0(w)} \int_0^\infty \bar{F}_1(x)F_2(w|x)dG(x). \end{aligned} \quad (5)$$

Since, the total for both cases is $M_{2A}(w) + M_{2B}(w)$, we have the following result:

If products are sold under renewing warranty, the expected total number of warranty claims for N products within warranty period w is given by

$$S_2(w) = \frac{N}{\bar{F}_0(w)} \int_0^\infty \{F_1(x)(1 - F_2(w|x)) + F_2(w|x)\} \, dG(x) \quad (6)$$

2.3 Extension to multistate systems

From a theoretical and practical point of view it is possible to extend the three state model described in the previous section into a multistate model. As discussed earlier a three state model has three states, namely, sales delay, operating and failed states. The sales delay state and operating state can be considered as two different operating modes. As the intensity of usage affects the operating intensity, different operating modes can have different failure rate functions. In order to distinguish *mode* from *state*, we define *state* as an operating *mode* in which the component has a different failure rate function from others. We refer to such components as multistate components.

[Insert Figure 1 here]

Suppose a component has m operating modes, and is operated with an operating mode pattern, $M_1 \rightarrow M_2 \rightarrow ... \rightarrow M_m \rightarrow M_1 \rightarrow M_2 \rightarrow ...$, repeatedly. As time passes, the component deteriorates, and the adjacent two identical operating modes can have different failure rates. Therefore, the component can have more than m states (see Figure 1), theoretically, it can even have infinite number of states. This type of multistate, which is called *type I multistate* hereafter, is different from the common multistate component defined as (El-Neweihi et al.[14]):

As time passes a component, starting in state M, deteriorates and enters state $M-1$, deteriorates further entering state $M-2$, and so on.

We call the above multistate component as *type II multistate* in what follows. There are two main differences between type I and type II multistate components:

- a transition from state i to state $i-1$ for a type I multistate component is due to an artificial interference (e.g., people force the component to change from one state to a another by triggering a certain functionality of the component), whereas a transition from state i to state $i-1$ for a type II multistate component is due to deterioration, and

- for a type I multistate component, state i can transit to state 0 without going through states $i-1, i-2, ..., 1$; whereas for a type II multistate component, state i can only transit to state $i-1$, then to $i-2$, and so on.

Suppose that a type I multistate component has n states, $S_1, ..., S_n$ from the sales date to the end of the warranty, where the usage time of S_i $(i = 1, ..., n)$ is t_i, with t_i satisfying $\sum_{i=1}^{n} t_i = w$. Assume that the lifetime distribution of a component in the full load is an exponential distribution with the scale parameter γ, that is $F(t) = 1 - e^{-\gamma t}$, and the expected failure rate function of S_i is $r_i = \alpha_i \gamma$, where $0 < \alpha_i \leq 1$. Then there are $\alpha_i \gamma t_i$ failures within t_i time units. As discussed previously, we also have the lifetime distribution during the sales delay given by $F_1(t) = 1 - e^{-\lambda\gamma t}$.

2.3.1 Multistate non-renewing warranty policy for exponential model

The expected total number of replacements, from the time of sale to the end of the warranty period is $\sum_{i=1}^{n} \alpha_i \gamma t_i$. Because of the memoryless property of the exponential distribution, $F_2(y|t_s) = F_0(y)$, where (3) can be simplified to:

$$\begin{aligned} S_1(w) &= N\left\{(1 + M(w)) \int_0^{\infty} F_1(x) dG(x) \right. \\ &\quad \left. + \int_0^{\infty} \left[\bar{F}_1(x) \int_0^{w} (1 + M(w-y)) dF_2(y|x)\right] dG(x)\right\} \\ &= N\left\{\int_0^{\infty} F_1(x) dG(x) + M(w)\right\} \end{aligned} \quad (7)$$

Thus, we have the following result:

If products are sold under non-renewing warranty, and have n states after the time of sale, the expected total number of warranty claims for N products within warranty period w is given by:

$$S_3(w) = N\left\{\int_0^{\infty} F_1(x) dG(x) + \sum_{i=1}^{n} \alpha_i \gamma t_i\right\} \quad (8)$$

2.3.2 Multistate renewing warranty policy for exponential model

The probability of a failure of type I multistate component within the warranty time period w, denoted by $\bar{H}_0(w)$, is given by

$$\begin{aligned} H_0(w) &= H_1(t_1) + \bar{H}_1(t_1)H_2(t_2) + \bar{H}_1(t_1)\bar{H}_2(t_2)H_3(t_3) + ... + \bar{H}_1(t_1)...\bar{H}_{n-1}(t_{n-1})H_n(t_n) \\ &= H_1(t_1) + \sum_{i=2}^{n}\prod_{k=1}^{i-1}\bar{H}_k(t_k)H_i(t_i) \end{aligned}$$

where $H_k(t_k) = 1 - e^{-\alpha_k \gamma t_k}$ for $k = 1, ..., n$.

Therefore, the expected total number of replacements after the time of sale, is given by:

$$\sum_{i=1}^{n} H_0^i(w) = \frac{H_0(w)}{1 - H_0(w)}$$

Thus, from (6), we have

If products are sold under renewing warranty, and have n states after the time of sale, the expected total number of warranty claims for N products within warranty period w is given by:

$$S_4(w) = \frac{N}{1 - H_0(w)} \int_0^{\infty} \{F_1(x)(1 - H_0(w)) + H_0(w)\}\, dG(x) \tag{9}$$

3 Numerical examples

In this section we present numerical examples on estimating the expected number of warranty claims and show how it is possible to establish a relationship between the failure rates of products in the sales delay and operating states. Following Wu and Xie [12], we assume that this relationship takes the following form.

The failure rate during the sales delay state is assumed to be related to the failure rate during the operating state through the following relationship:

$$r_1(t) = \lambda r_0(\nu t), \tag{10}$$

where $t \in (0, t_s]$, $0 < \nu < 1$ and $0 < \lambda < 1$. It follows that:

$$F_1(t) = 1 - (\bar{F}_0(\nu t))^{\frac{\lambda}{\nu}} \tag{11}$$

The residual lifetime distribution of products that survive after the time of sale, t_s, or the lifetime distribution within time interval $(t_s, +\infty)$, can be obtained based on the lifetime distribution during the sales delay state, $F_1(t)$. However, within the time interval $(0, t_s]$, the products are in the sales delay state, whereas within the time interval $(t_s, +\infty)$, they are in the operating state. Hence, the lifetime distribution within $(t_s, +\infty)$ is different from the residual lifetime distribution that is derived from the distribution within time interval $(0, t_s]$. The lifetime distribution of a product that survives after the sales delay state, t_s, is given by:

$$F_2(t|t_s) = \frac{F_0(\nu t_s + t) - F_0(\nu t_s)}{1 - F_0((\nu t_s))}, t \geq 0. \tag{12}$$

$F_0(\nu t_s)$ is the distribution of the scaled time t, with scaling parameter ν. The scaling of age in such a way is commonly used in reliability and maintenance analysis.

3.1 Example of a three-state model

Without loss of generality we assume that the sales delay distribution is given by the following lognormal distribution with mean μ and standard deviation σ:

$$G(x) = \int_0^x \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}} dt \tag{13}$$

For this example we assume that $F_0(t)$ is a CDF of Weibull distribution, which is one of the most common distributions used in reliability:

$$F_0(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^\alpha}. \tag{14}$$

This leads to the following $F_1(t)$ and $F_2(t)$:

$$F_1(t) = 1 - e^{-\frac{\lambda}{\nu}\left(\frac{\nu t}{\eta}\right)^\alpha}, \quad \text{and} \tag{15}$$

$$F_2(t|t_s) = 1 - e^{-\left(\frac{\nu t_s + t}{\eta}\right)^\alpha}. \tag{16}$$

For this example we approximate the renewal function for Weibull distribution, $M(t)$, using methods suggested by Jiang[15]. Setting the relevant parameters, as shown in Table 1, we can obtain the following results.

[Insert Table 1 here]

For three-state components, if the values of λ change over interval $(0, 1)$, the resulting values of $S_1(w)$ and $S_2(w)$ are shown in Figure 2. When λ becomes larger, the number of warranty claims for N products under non-renewing warranty and renewing warranty becomes larger. It is also clear that the difference between the two policies remains the same for different values of λ.

[Insert Figure 2 here]

Figure 3 depicts the relationship between the expected number of warranty claims and values of ν. It can be seen from the graph that the difference between the renewing and non-renewing policy becomes larger as the values of ν increase.

[Insert Figure 3 here]

Figure 4 depicts the relationship between the expected number of warranty claims and the mean of the sales delay distribution. It is clear from the figure that as the sales delay time increases the number of expected claims also increases. The difference between the renewing and non-renewing policies also increases as the sales delays become longer.

[Insert Figure 4 here]

Figure 5 demonstrates the difference between the failure rate during the sales delay and during the operating state. The magnitude of this difference is dependent on the values of λ and ν.

[Insert Figure 5 here]

3.2 Example of a multistate model

Using the results for multistate models from the previous section we can set the parameters as shown in Table 2. The resulting expected number of warranty claims under non-renewing and renewing warranty policies, S_3 and S_4, respectively, are given in Table 3. It can be seen from the table that as λ increases the expected number of warranty claims for both policies increase. It is also clear that there is little difference between the two policies for this example. The reason for this is that the parameters α_k and γ are small. This results in small $H_0(w)$.

[Insert Table 2 here]

[Insert Table 3 here]

4 Simulation and case study

This section presents the results of simulation and a case study. In both cases we consider data recorded on monthly basis matched to the months of manufacture. That is, the data represents failure times since the date of manufacture, Y, which is the sum of two time periods, namely, the sales delay, S, and the failure time T, $Y = S + T$. In this study, we assume that the distribution of S is known. The total number of months considered here is 60, 24 months for fitting the models and 12, 24 and 36 months for estimating the prediction accuracy. It is not uncommon in the electronics industry to use the first 24 months of the data to predict the expected number of failures for coming months.

For simplicity, we consider only failure times during the sales delay and failure times during the operating state for products that survived the sales delay period. That is, we focus only on times to first failure. This is justified as we are looking at how the sales delay can affect the reliability of the products.

In this paper we consider the following three models.

- **Model 1**. This model assumes that there are no failures during the sales delay time. For this model, the objective is to estimate the distribution of T.
- **Model 2**. This model assumes that a constant proportion, p, of sold products is found to be in the failed state at the point of sale. Thus, the objective is to estimate p and the distribution of T.
- **Model 3**. This model assumes that during the sales delay period the products lifetime distribution is given by $F_1(t)$, and during the operating period the products lifetime distribution is given by $F_2(t|t_s)$, where t_s is the sales delay time.

Based on the above assumptions, the expected number of failures in month t for each model is as follows.

Model 1:

$$\mu_t = \sum_{i=1}^{t} s_i(F_0(t) - F_0(t-i)). \tag{17}$$

where s_i is the number of products sold in month i. The expected number of failures, μ_t, consists of the expected number of failures for products sold in month t and previous months with appropriate ages. Here, we assume that products that have been recorded to have failed in each month have been operated for the whole of the month. That is, the products are assumed to be sold in the beginning of each month.

Model 2:

$$\mu_t = ps_t + \sum_{i=1}^{t-1} s_i(F_0(t) - F_0(t-1)). \tag{18}$$

The first term represent the expected number of products that are found in the failed state at the time of sale. The second term represents the expected number of failures in month t for products sold in previous months with appropriate ages. This model assumes that products are sold at the end of each month.

Model 3:

$$\mu_t = s_t F_1(t) + \sum_{i=1}^{t-1} s_i (F_2(t-i|i) - F_2(t-i-1|i)). \tag{19}$$

The first term represents the expected number of products found failed at the time of sale. The second term represents the sum of the expected number of products that failed in month t from sales in previous months excluding the current month. As the case with the previous model, this model assumes that the products are sold at the end of each month.

Previous studies such as Majeske [16], Wang et al. [3], Karim et al. [17] have estimated warranty claims using a non-homogeneous Poisson process (NHPP) model. Here, we adopt a similar strategy and model our data using the NHPP model where the probability of failures in month t is given by a Poisson distribution with mean μ_t. Thus, the log-likelihood function for each of the above models is given by:

$$\ln L = \sum_{i=1}^{m} (n_i \ln \mu_i - \mu_i - \ln n_i!) \tag{20}$$

where m is the number of months used for fitting the models, in this case 24 months. The models are fitted by maximising their respective log-likelihood functions. The maximisation is done using numerical methods with multiple starting points.

4.1 Simulation

The sales delay times are simulated from a lognormal distribution with the mean and variance of the associated normal distribution given by $\mu = 2.05$, and $\sigma^2 = 0.06$. This means that the expected sales delay is about 8 months and that 95% of products are sold within the first 12 months.

Both $F_1(t)$ and $F_2(t)$ are derived from a Weibull distribution with scale parameter $\eta = 75$ and shape parameter $\alpha = 1.2$, given by equations (14) - (16). The scaling parameters for the hazard function are $\lambda = 0.03$ and $\nu = 0.15$. This means that during the sales delay time the average failure time is 1175 months and during the operating time with average sales delay of 8 months, the expected failure time is around 69 months.

The simulated data is the average of 1000 runs of 10000 products. If the products fail during the sales delay, the failure time is taken to be the sales delay time, as failures discovered only at the point of sale. The failure time for products that survive beyond the sales delay time are taken to be the sales delay time plus the failure time generated from

$F_2(t)$. As mentioned previously, in this paper we focus on only on times to first failure. Thus, the renewals of newly replaced products are not generated. This will be done in our future works.

Thus, we have artificially generated data from a process where products fail during the sales delay time. However, it is not possible do discern from the data the number of failures that occurred during the sales delay period. The only data available is the records of monthly failures. This data was generated in a way that matches the data available for the case study, which is discussed in the next subsection.

The results of fitting the models to the simulation data are presented in Table 4, where columns headed with K represent prediction horizons of 12, 24, and 36 months with their respective means squared errors. Predictions were done based on $\hat{\mu}_t$ for $t = m + 1, m + 2, ..., m + K$. Table 5 presents the estimated parameters for each model. It is clear from Table 4 that Model 3 has the highest likelihood, however, its AIC is bigger than the AIC of Model 2. Nevertheless, Model 3 seems to be more preferable as it has much better prediction accuracy. In practise, the estimation of the reliability of the products is often done for purposes of improving product quality and predicting the expected number of failures. Thus, the prediction accuracy is an important issue (Wu and Akbarov [18], Fredette and Lawless [19], and Wasserman [20]).

[Insert Table 4 here]

[Insert Table 5 here]

4.2 Case study

The products under consideration are electronic products, specifically, Internet networking equipment that have lifetime warranty. Such products can fail during the sales delay time due to reasons such as damp storage conditions and damage due to poor handling during transportation and storage. In relation to such products, Yang et al. [21] note that up to 66% of the products found failed at the time of sale can be attributed to damage during the sales delay period.

The available data also includes products that are in the failed state due to manufacturing process. Thus, the failure data also includes information about manufacturing quality. However, in practise, it is not always possible to distinguish between products that fail during the sales delay and product that are poorly manufactured.

The available data represents a collection of records over a period of time categorised into calendar months. It consists of the following three pieces of information: the month of manufacture, j, the shipment amount in the month of manufacture, N_j, and the number of products returned in month i that were manufactured in month j, n_{ji}. The shipment amount in month j consists of products manufactured in month j and products manufactured in previous months. However, for the purposes of this study, we assume that the shipment amounts, N_j, adequately represent the number of products manufactured in month j. The data used for fitting the models discussed earlier and prediction consists of the aggregate data for 10 production batches. That is, we have the total number of failures for each month, $n_t = \sum_{j=1}^{10} n_{jt}$, for the period of 60 months along with the total number of shipments for 10 production batches, N. The data used in this study is given in Table 6 and plotted in Figure 6. The sales amounts for each month, s_t, were estimated from subjective data, which represents monthly sales as a percentage of the monthly shipments.

[Inset Table 6 here]

[Insert Figure 6 here]

Tables 7 and 8 show the results of applying the models to the case study data. As the case with the simulated data, it can be seen that Model 3 has bigger AIC than Model 2. However, Model 3 has better prediction accuracy.

5 Conclusions

Many manufacturers offer warranty on their products from the date of sale to a pre-specified point in time. For the cases where products spend prolonged periods of time before being sold it is necessary to take these periods into account as they can have a significant impact on the expected number of warranty claims.

In this study we have achieved the following. The expected number of warranty claims for products with several states under both non-renewing warranty and renewing warranty have been formulated and derived. A numerical example to examine the methods proposed has been demonstrated. This paper also considers three different models applied to simulated data and data from electronics industry. The results show that the models that take into account failures during the sales delay period result in better predictions. We also show that longer sales delays result in larger numbers of warranty claims reinforcing the remark made by Robinson and McDonald [10].

In the future work we can consider the following issues:

- Consider the expected number of warranty claims in a more general framework that takes into account different costs associated with inventory holding, replacements and so on, in the same line as some of the recent studies on warranty analysis.

- Consider the role of human factors in the sales delay such as the ones consider by Wu[22].
- Consider the case of extended warranties and how the sales delay can impact the formulation of extended warranty policies .

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Table 1: Parameters (Three state model)

η	α	λ	ν	w	N	σ	μ
100	1.5	0.2	0.2	24	1000	1.5	0.7

Table 2: Parameters (Multistate model)

γ	w	μ	σ	N	t_1	t_2	t_3	t_4	α_1	α_2	α_3	α_4
0.04	12	1.5	0.7	100	2	3	2	5	0.008	0.016	0.024	0.032

Table 3: Number of warranty claims versus λ (Multistate model)

λ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
S_3	3.336	5.504	7.596	9.615	11.567	13.453	15.279	17.045	18.756	20.415
S_4	3.342	5.510	7.602	9.621	11.573	13.459	15.284	17.051	18.762	20.412

Table 4: Simulation results. K-prediction horizon columns show mean squared errors.

	lnL	AIC	$K = 12$	$K = 24$	$K = 36$
Model 1	5300.4	-10596.8	1.89	2.33	2.25
Model 2	5303.5	-10601.0	14.48	20.32	22.62
Model 3	5303.9	-10599.8	0.10	0.12	0.17

Table 5: Simulation results. Estimated parameters.

	$\hat{\eta}$	$\hat{\alpha}$	$\hat{p}$	$\hat{\lambda}$	$\hat{\nu}$
M1	73.98	1.19	-	-	-
M2	79.31	1.12	0.0048	-	-
M3	75.05	1.19	-	0.13	0.11

Table 7: Case study results. K-prediction horizon columns show mean squared errors.

	lnL	AIC	$K = 12$	$K = 24$	$K = 36$
Model 1	51308	-102612	2033	1688	1619
Model 2	51339	-102672	1131	821	754
Model 3	51333	-102658	371	461	639

Table 8: Case study results. Estimated parameters.

	$\hat{\eta}$	$\hat{\alpha}$	$\hat{p}$	$\hat{\lambda}$	$\hat{\nu}$
M1	467.85	0.80	-	-	-
M2	540.82	0.75	0.0028	-	-
M3	874.34	0.56	-	0.078	1.057

Table 6: Case study data.

t	n_t	t	n_t	t	n_t	t	n_t
1	51	16	391	31	291	46	273
2	163	17	380	32	288	47	254
3	299	18	384	33	269	48	246
4	484	19	370	34	262	49	244
5	597	20	333	35	265	50	224
6	662	21	312	36	277	51	280
7	671	22	332	37	272	52	253
8	623	23	308	38	257	53	235
9	552	24	333	39	258	54	241
10	530	25	317	40	255	55	215
11	501	26	287	41	282	56	264
12	460	27	345	42	251	57	240
13	462	28	325	43	240	58	211
14	413	29	262	44	260	59	210
15	447	30	276	45	273	60	214

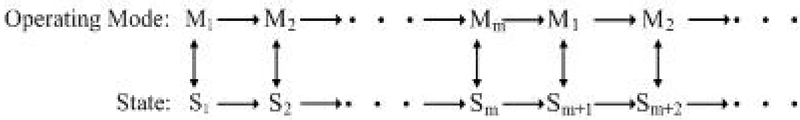

Figure 1: Operating modes and their corresponding states

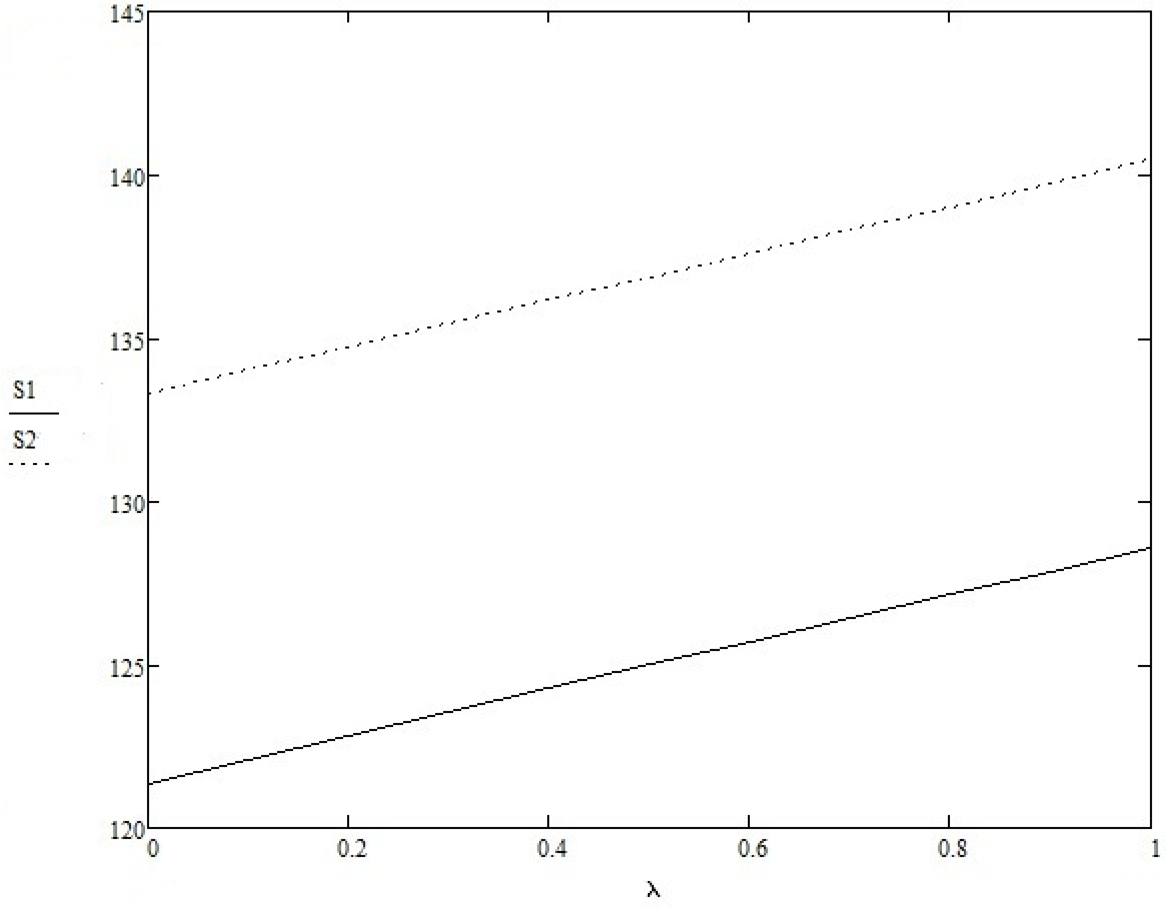

Figure 2: Expected number of warranty claims versus λ.

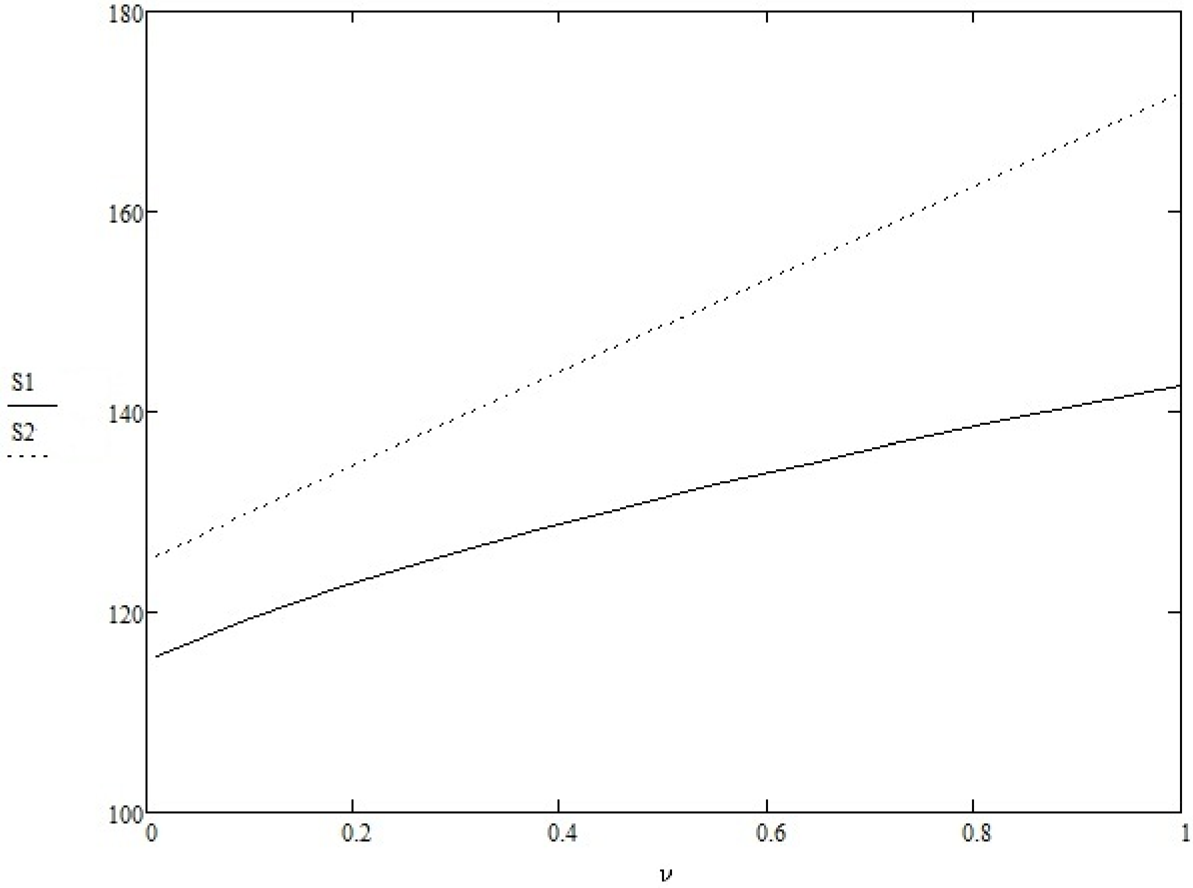

Figure 3: Expected number of warranty claims versus ν.

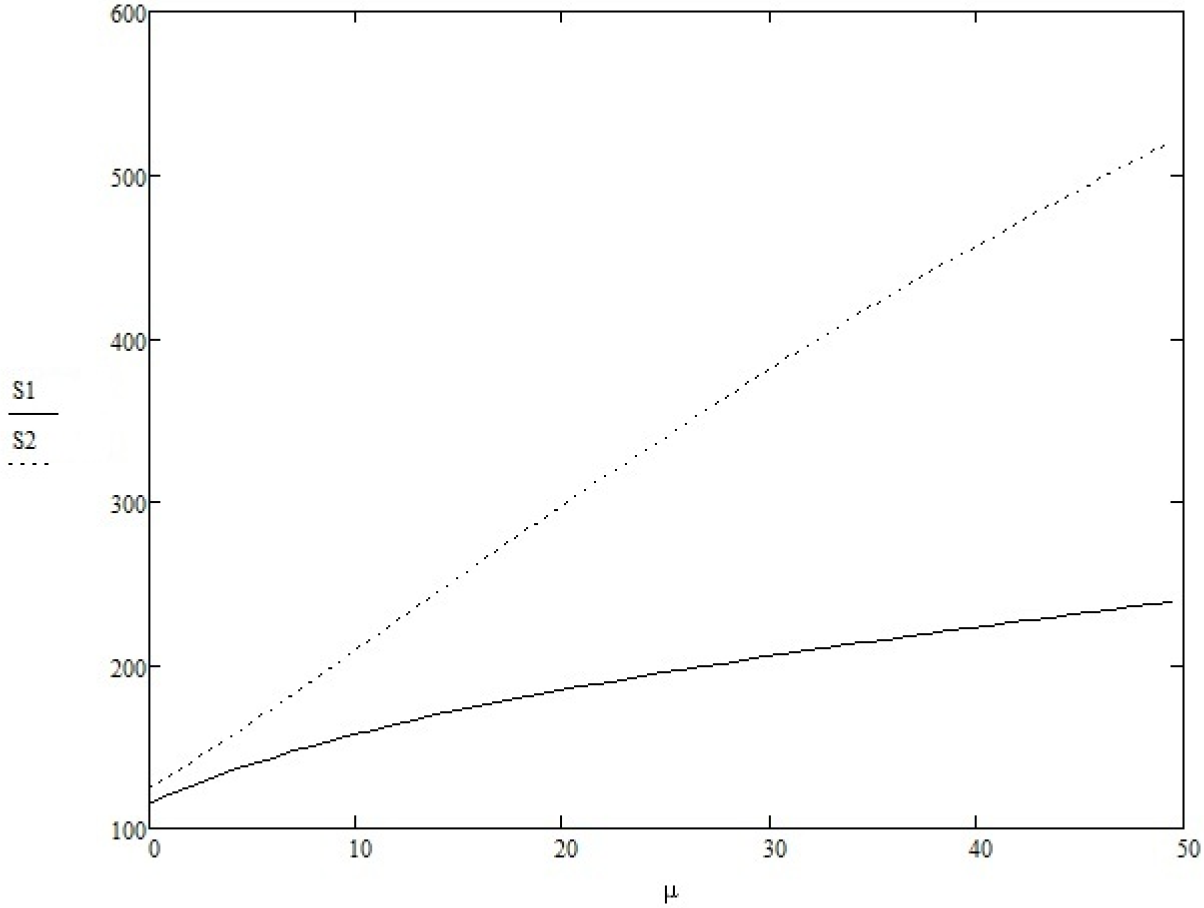

Figure 4: Expected number of warranty claims versus the mean of sales delay distribution, μ.

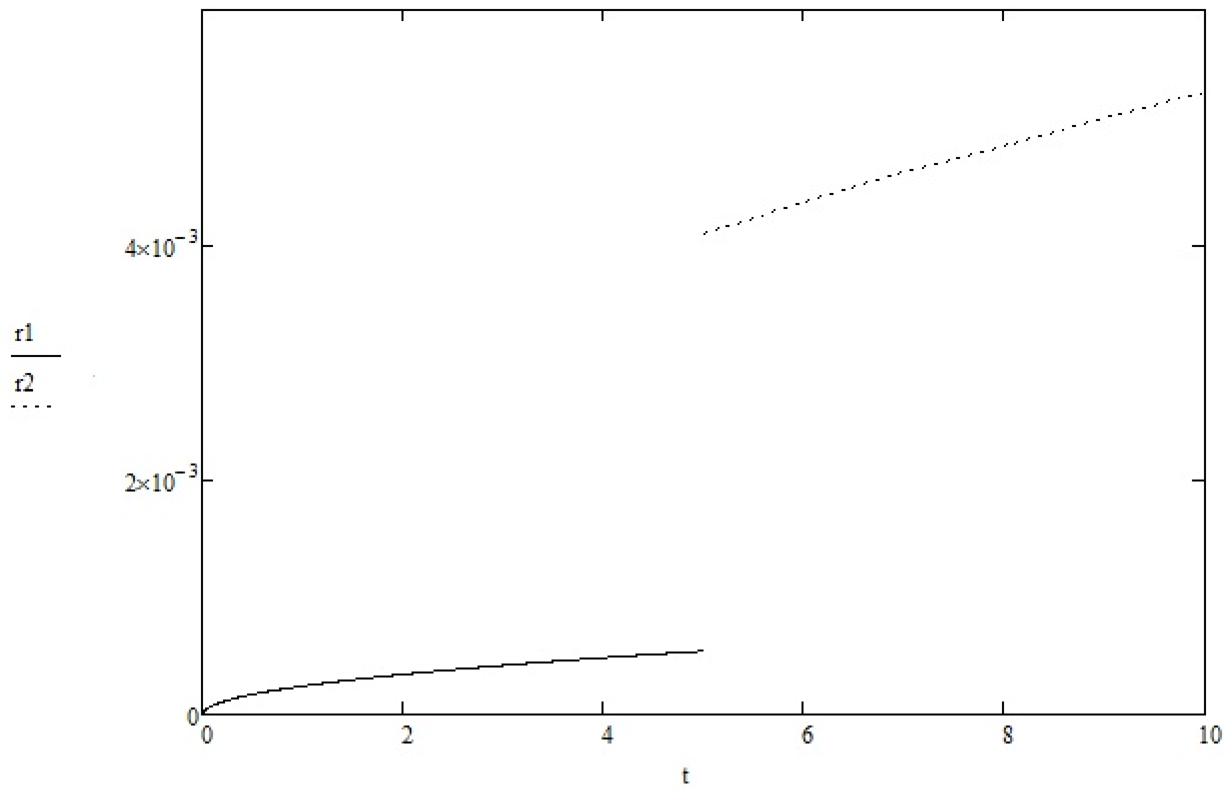

Figure 5: Failure rate during the sales delay and operating states for a product sold at time $t = 5$.

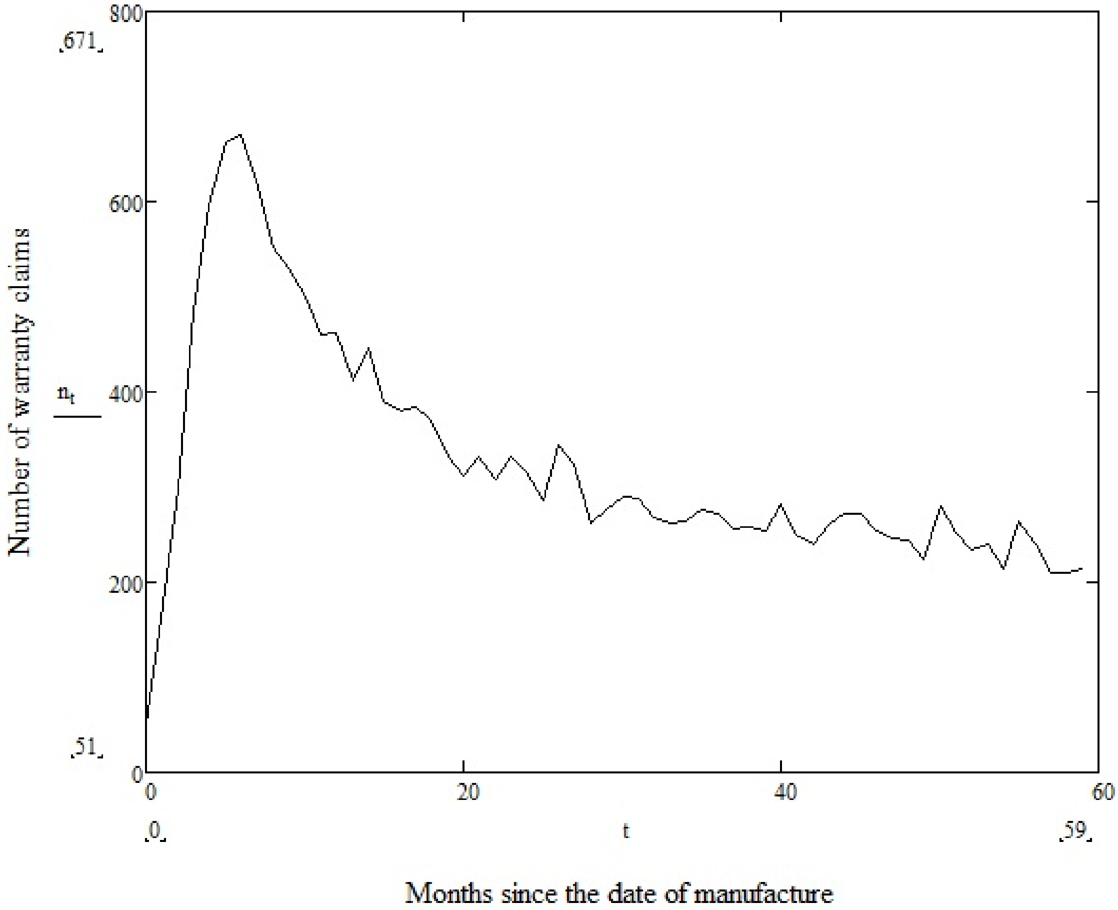

Figure 6: Data used the case study.