Optimizing replacement policy for a cold standby system with waiting repair times

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Abstract

This paper presents the formulas of the expected long-run cost per unit time for a cold standby system having two identical components with perfect switching. When a component fails, a repairman will be called in to bring the component back to a certain state. The time to repair is composed of two different time periods: waiting time and real repair time. The waiting time starts from the failure of a component to the start of repair, and the real repair time is the time between the start to repair and the completion of the repair. We also assume that the time to repair can either include only real repair time with a probability *p*, or include waiting time and real repair time with a probability 1-*p*. Special cases are discussed when both the working times and real repair times are assumed to be a type of stochastic processes: *geometric processes*, and the waiting time is assumed to be a renewal process. The expected long-run cost per unit time is derived and a numerical example is given to demonstrate the usefulness of the derived expression.

Keywords: Geometric process, Cold standby system, Long-run cost per unit time, Replacement policy, Maintenance policy

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1. Introduction

A two-component cold standby system is composed of a primary component and a backup component, where the backup component is only called upon when the primary component fails. Cold

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standby systems are commonly used for non-critical applications. However, cold standby systems are one of most important structures in the reliability engineering and have been widely applied in reality. An example of such a system is the data backup system in computer networks.

The reliability analysis and maintenance policy optimisation for cold standby systems has attracted attentions from many researchers. Zhang and Wang (2006, 2007) and Zhang et al (2006) derived the expected long run cost per unit time for a repairable system consisting of two identical components and one repairman when a geometric process for working times is assumed or for cold standby systems. Utkin (2003) proposed imprecise reliability models of cold standby systems when he assumed that arbitrary probability distributions of the component time to failure are possible and they are restricted only by available information in the form of lower and upper probabilities of some events. Coit (2001) described a solution methodology to optimal design configurations for non-repairable series–parallel systems with cold-standby redundancy when he assumed non-constant component hazard functions and imperfect switching. Yu et al. (2007) considers a framework to optimally design a maintainable previous term cold-stand by next term system, and determine the maintenance policy and the reliability character of the components.

Due to various reasons, repair might start immediately after a component fails. In some scenarios, from the failure of a component to the completion of repair, there might be two periods: waiting time and real repair time. The waiting time starts from the failure of the component to the start of repair; and the real repair time is the time between the start to repair and the completion of the repair. This is especially true for cold standby systems as they are not critical enough for a standby repairman be equipped for it. For example, when a component fails to work, its owner will call its contracted maintenance company or return the component to its suppler for repair. After a time period, which can be the time spent by repairmen from their working place to the place where the component fails, or the time on delivering the failed component to its supplier. This time period is called waiting time in what follows. Usually, the waiting time can be seen as a random variable independent of the age of the component, whereas the real repair time can become longer and longer when the component becomes older. On the other hand, the working time of the component can become shorter and shorter due to various reasons such as ageing, and deterioration. Such working time patterns and real repair time patterns can be depicted by geometric processes as many authors have studied (Lam 1988).

The geometric processes introduced by Lam (1988) define an alternative to the non-homogeneous Poisson processes: a sequence of random variables $\{X_k, k=1,2,...\}$ is a geometric process if the distribution function of X_k is given by $F(a^{k-1}t)$ for k=1,2,... and a is a positive constant. Wang and Pham (1996) later refer the geometric process as a quasi-renewal process. Finkelstein (Finkelstein 1993) develops a very similar model: he defines a general deteriorating renewal process such that $F_{k+1}(t) \leq F_k(t)$. Wu and Clements-Croome (2006) extend the geometric process by replacing its parameter a^{k-1} with $a_1 a^{k-1} + b_1 b^{k-1}$, where $a>1$ and $0<b<1$. The geometric process has been applied in reliability analysis and maintenance policy optimisation for various systems by many authors; for example, see Wang, Pham (1996), and Wu and Clements-Croome (2005).

This paper presents the formulations of the expected long-run cost per unit time for a cold standby system that consists of two identical components with perfect switching. When a component fails, a repairman will be called in to bring the component back to a certain state. The time to repair is composed of two different time periods: waiting time and real repair time. The waiting time starts from the component failure to the start to repair, and the real repair time is the time between the start to repair and the completion of the repair. Both the working times and real repair times are assumed to be a type of stochastic processes: *geometric processes*, and the waiting time is assumed to be a renewal process. We also assume that the time to repair can either include only real repair time with a probability p, or include waiting time and real repair time with a probability 1-p. The expected long-run cost per unit time is derived and a numerical example is given to demonstrate the usefulness of the derived expression.

The paper is structured as follows. The coming section introduces geometric processes defined by Lam (1988), denotation and assumptions. Section 3 discusses special cases. Section 4 offers numerical examples. Concluding remarks are offered in the last section.

2. Definitions and Model Assumptions

This section first borrows the definition of geometric process from Lam (1988), and then makes assumptions for the paper.

2.1 Definition

Definition 1 Assume ξ, η are the two random variables. For arbitrary real number α, there is

$$P(\xi \geq \alpha) > P(\eta \geq \alpha)$$

then ξ is called stochastically bigger than η**.** Similarly, if ξ stochastically smaller than η**.**

Definition 2 (Lam 1988) Assume that $\{X_n, n=1,2,\ldots\}$ is a sequence of independent non-negative random variables. If the distribution function of X_n is $F(a^{n-1}t)$, for some a>0 and all, n=1,2, …, then $\{X_n, n=1,2,\ldots\}$ is called a geometric process**.**

Obviously,

if $a>1$, then $\{X_n, n=1,2,\ldots\}$ is stochastically decreasing,

if $a<1$, then $\{X_n, n=1,2,\ldots\}$ is stochastically increasing, and

if $a=1$, $\{X_n, n=1,2,\ldots\}$ is a renewal process**.**

2.2 Assumptions and Denotation

The following assumptions are assumed to hold in what follows.

A. At the beginning, the two components are both new, component 1 is first working and component 2 is under cold standby.

B. When both of the two components are in good condition, one is working and the other is under cold standby**.** When the working component fails, a repairman repairs the failed component immediately with probability *p*, or repairs it with a waiting time with probability 1-*p*. As soon as the working component fails, the standby one will start to work**.** Assume the switching is perfect. After a failed one has been repaired, it is either put in use if another one fails or put in standby if another one is working. If one fails while the other is still under repair, the failed one must wait for repair until the repair for another one is completed.

C. The time interval from the completion of the (*n*-1)th repair to that of the *n*th repair of component i is called the *n*th cycle of component *i*, where $i=1,2$**;** $n=1,2,\ldots$. Denote the working time and the repair time of component i in the nth cycle ($i=1, 2$**;** $n=1,2,\ldots$) as $X_n^{(i)}$ and $Y_n^{(i)}$, respectively. Denote the waiting time of component i ($i=1, 2$) in the *n*th cycle as $\{Z_n^{(i)}, n=1,2,\cdots\}$. Denote the cumulative distribution functions of $X_n^{(i)}$, $Y_n^{(i)}$ and $Z_n^{(i)}$, as $F_n(x)$ $G_n(x)$, and $S(x)$, respectively.

D. $X_n^{(i)}$, $Y_n^{(i)}$, and $Z_n^{(i)}$ (*i*=1,2**,** and $n=1,2,\ldots$) are statistically independent.

E. When a replacement is required, a brand new but identical component will be used to replace, and the replacement time is negligible.

F. Denote the repair cost per unit time of two components as C_m, the working reward per unit time as C_w, the replacement cost as C_r.

3. Expected cost under replacement policy N

Figure 1 shows a typical scenario, given the above-mentioned assumption. In what follows, we consider a replacement policy *N*, where a replacement is carried out if the number of failures reaches *N* for the component 1.

Fig.1 a possible progressive figure of the system

Denote the time between the (n-1)th replacement and the nth replacement of the system as T_n. Obviously, $\{T_1, T_2, \ldots\}$ forms a renewal process.

Let $C(N)$ be the expected long run cost per unit time of the system under the policy *N*. Because $\{T_1, T_2, \ldots\}$ is a renewal process, the interval time between two consecutive replacements is a renewal cycle. Then, according to renewal reward theorem, we can know that the long run average cost per unit time is given by

$$C(N) = \frac{\text{Expected cost incurred in a cycle}}{\text{Expected length of a cycle}}. \tag{1}$$

Let W be the length of a renewal cycle of the system, then

$$W = X_1^{(1)} + \sum_{i=1}^{N-1}[\max\{Z_i^{(1)} + Y_i^{(1)}, X_i^{(2)}\}I\{A_i^{(1)}\} + \max\{Y_i^{(1)}, X_i^{(2)}\}I\{B_i^{(1)}\}] +$$

$$\sum_{i=1}^{N-2}[\max\{Z_i^{(2)} + Y_i^{(2)}, X_{i+1}^{(1)}\}I\{A_i^{(2)}\} + \max\{Y_i^{(2)}, X_{i+1}^{(1)}\}I\{B_i^{(2)}\}] + X_N^{(1)}.$$

The expected length of a renewal cycle is

$$E(W) = E[X_1^{(1)}] + E[X_N^{(1)}] + \sum_{i=1}^{N-1}E[\max\{Z_i^{(1)} + Y_i^{(1)}, X_i^{(2)}\}I\{A_i^{(1)}\} + \max\{Y_i^{(1)}, X_i^{(2)}\}I\{B_i^{(1)}\}]$$

$$+ \sum_{i=1}^{N-2}E[\max\{Z_i^{(2)} + Y_i^{(2)}, X_{i+1}^{(1)}\}I\{A_i^{(2)}\} + \max\{Y_i^{(2)}, X_{i+1}^{(1)}\}I\{B_i^{(2)}\}]\cdot \tag{2}$$

Let *C* be the cost of a renewal cycle of the system under the policy *N*, then

$$C = C_r + C_m\{\sum_{i=1}^{N-1}Y_i^{(1)} + \sum_{i=1}^{N-2}Y_i^{(2)} + [Y_{N-1}^{(2)}I\{A\} + (X_N^{(1)} - Z_{N-1}^{(2)})I\{\overline{A}\}]I(A_{N-1}^{(2)}) + [Y_{N-1}^{(2)}I\{B\} + X_N^{(1)}I\{\overline{B}\}]I(B_{N-1}^{(2)})\}$$

$$-C_\omega[\sum_{i=1}^{N}X_i^{(1)} + \sum_{i=1}^{N-1}X_i^{(2)}], \tag{3}$$

where $A = \{X_N^{(1)} - Z_{N-1}^{(2)} > Y_{N-1}^{(2)}\}, \overline{A} = \{X_N^{(1)} - Z_{N-1}^{(2)} < Y_{N-1}^{(2)}\}, B = \{X_N^{(1)} - Y_{N-1}^{(2)} > 0\}$, and $\overline{B} = \{X_N^{(1)} - Y_{N-1}^{(2)} < 0\}$.

If X and Y are two independent non-negative random variables and their cumulative distribution functions are $F(x)$ and $G(x)$, respectively, we have following three lemmas.

Denote $E(C)$ as the expected value of C. By substituting the numerator and denominator of Eq. (1) with $E(C)$ and $E(W)$, respectively, we have

$$C(N) = \frac{E(C)}{E(W)} \tag{4}$$

Then the optimal replacement number can be obtained by minimising the value of $C(N)$ in Eq. (4).

Lemma 1

$$E(\max\{X, Y\}) = EX + \int_0^\infty F(x)[1 - G(x)]dx \tag{5}$$

$$= EY + \int_0^\infty G(x)[1 - F(x)]dx . \tag{6}$$

The proof of Lemma 1 is given in Appendix.

Lemma 2

$$E[I\{Y > X\}X] + E[I\{0 < Y < X\}Y] = \int_0^\infty [1 - F(x)][1 - G(x)]dx . \tag{7}$$

The proof of Lemma 2 is given in Appendix.

Similarly, we have

Lemma 3

$$E[(Y - X)I\{Y - X > 0\}] = \int_0^\infty [1 - G(x)]F(x)\,dx . \tag{8}$$

4. Special Cases and Discussion

Denote the distributions of $X_n^{(i)}$ and $Y_n^{(i)}$ as $F(a^{n-1}t)$ and $G(b^{n-1}t)$, respectively, where $a > 1, 0 < b < 1$. $\{Y_n^{(i)}, n=1,2,\ldots\}$ constitutes an increasing geometric process, whereas $\{X_n^{(i)}, n=1,2,\ldots\}$ constitutes a decreasing geometric process. Then we have the following Theorem.

Theorem Assume $F_n(t)=F(a^{n-1}t)=1-\exp(-\frac{a^{n-1}}{\lambda}t)$, $G_n(t)=G(b^{n-1}t)=1-\exp(-\frac{b^{n-1}}{\mu}t)$, and $S(t)=1-\exp(-\frac{t}{\gamma})$, $(t\ge 0)$, respectively. Then the expected length of a renewal cycle is given by

$$
\begin{aligned}
E(W) &= \lambda+\frac{\lambda}{a^{N-1}}+(2N-3)(1-p)\gamma+2\sum_{i=1}^{N-2}\frac{\mu}{b^{i-1}}+\frac{\mu}{b^{N-2}} \\
&+(1-p)\lambda^2\left(\frac{1}{a^{N-2}(\gamma a^{N-2}+\lambda)}-\frac{\mu^2}{(a^{N-2}\mu+b^{N-2}\lambda)(2b^{N-2}\gamma\lambda+\mu\lambda+a^{N-2}\mu\gamma)}\right)+\frac{p\lambda^2 b^{N-2}}{a^{N-2}(b^{N-2}\lambda+a^{N-2}\mu)} \\
&+\sum_{i=1}^{N-2}\left((1-p)\lambda^2\left(\frac{1}{a^i(\gamma a^i+\lambda)}+\frac{1}{a^{i-1}(\gamma a^{i-1}+\lambda)}\right)-\mu^2\left(\frac{1}{(a^i\mu+b^{i-1}\lambda)(2b^{i-1}\gamma\lambda+\mu\lambda+a^i\mu\gamma)}\right.\right. \\
&\left.\left.+\frac{1}{(a^{i-1}\mu+b^{i-1}\lambda)(2b^{i-1}\gamma\lambda+\mu\lambda+a^{i-1}\mu\gamma)}\right)+p\left(\frac{\lambda(1+a)}{a^i}-\frac{\lambda\mu(a^i\mu+a^{i-1}\mu+2b^{i-1}\lambda)}{(b^{i-1}\lambda+a^i\mu)(b^{i-1}\lambda+a^{i-1}\mu)}\right)\right), \qquad (9)
\end{aligned}
$$

and the expected cost is a cycle is

$$
\begin{aligned}
E(C) &= C_m\left(2\sum_{i=1}^{N-2}\frac{\mu}{b^{i-1}}+\frac{\mu}{b^{N-2}}+(1-p)\left(\frac{2\mu}{b^{N-2}}-\frac{a^{N-1}\mu\gamma^2}{(a^{N-1}\gamma+\lambda)(b^{N-2}\gamma-\mu)}-\frac{\lambda^2\,\mu}{(a^{N-1}\gamma+\lambda)(b^{N-2}\lambda+a^{N-1}\mu)}\right)\right. \\
&\left.+\frac{\lambda\mu}{b^{N-2}\lambda+a^{N-1}\mu}\,p\right)+C_r-C_\omega\left(2\sum_{i=1}^{N-1}\frac{\lambda}{a^{i-1}}+\frac{\lambda}{a^{N-1}}\right), \qquad (10)
\end{aligned}
$$

and the expected long run cost per unit time is given by

$$C(N)=\frac{E(C)}{E(W)} \qquad (11)$$

If one sets a=1 and b=1, the above results $E(W)$ and $E(C)$ will be the situations where the components can be repaired as good as new.

5. Numerical Example

5.1 Parameter set 1

If we set $a=1.8, b=0.98$, $\lambda=100$, $\mu=10, \gamma=5, C_{\omega}=500, C_{m}=20, C_{r}=5000$, and $p=0.8$, then the optimum number for a replacement will be N=6, and the corresponding expected long run cost per unit time is -433.41. The expected long-run cost per unit time is shown in Table 1, which corresponds to Figure 1.

Fig. 2 The change of $C(N)$ ***over N for parameter set 1***

5.2 Parameter set 2

If we set $a=1.1, b=0.98$, $\lambda=100$, $\mu=1, \gamma=0.2, C_{\omega}=500, C_{m}=20, C_{r}=5000$, and $p=0.8$, then the optimum number for a replacement will be N=35, and the corresponding expected long run cost per unit time is -491.85. The expected long-run cost per unit time is shown in Table 2, which corresponds to Figure 2.

Fig. 3 The change of $C(N)$ ***over N for parameter set 2***

Compare Figures 2 and 3, we can find that the optimum replacement time becomes longer in the second situation. In both situations, we can easily find an optimum replacement time point. However, due to the complexity of Eq. (11), we are not able to prove that there exists a unique optimal value *N*.

6. Conclusions

Cold standby systems are a category of important reliability structure in engineering. Searching an optimal replacement point for such systems is of interest and important. This paper derived the expected long run cost per unit time for a cold standby system when time to repair is composed of two time periods: waiting time and real repair time. We also considered a special scenario where the working times and real repair times are geometric processes. Numerical examples are given to demonstrate the usefulness of the derived expression.

References

Coit, D.W. 2001, "Cold-standby redundancy optimization for nonrepairable systems", *IIE Transactions (Institute of Industrial Engineers),* vol. 33, no. 6, pp. 471-478.

Finkelstein, M.S. 1993, "A scale model of general repair", *Microelectronics Reliability,* vol. 33, no. 1, pp. 41-44.

Lam, Y. 1988, "A note on the optimal replacement problem", *Advances in Applied Probability,* vol. 20, no. 2, pp. 479-482.

Utkin, L.V. 2003, "Imprecise reliability of cold standby systems", *Int J Qual Reliab Mgmt,* vol. 20, no. 6, pp. 722-739.

Wang, H. & Pham, H. 1996, "A quasi renewal process and its applications in imperfect maintenance", *International Journal of Systems Science,* vol. 27, no. 10, pp. 1055-1062.

Wu, S. & Clements-Croome, D. 2006, "A novel repair model for imperfect maintenance", *IMA Journal Management Mathematics,* vol. 17, no. 3, pp. 235-243.

Wu, S. & Clements-Croome, D. 2005, "Optimal maintenance policies under different operational schedules", *IEEE Transactions on Reliability,* vol. 54, no. 2, pp. 338-346.

Yu, H., Yalaoui, F., Châtelet, Ė & Chu, C. 2007, "Optimal design of a maintainable cold-standby system", *Reliability Engineering & System Safety,* vol. 92, no. 1, pp. 85-91.

Zhang, Y. L., Wang, G. J. and Ji, Z. C. 2006, “Replacement problems for a cold standby repairable system,” *International Journal of Systems Science*, 37(1), pp 17-25.

Zhang, Y. L. and Wang, G. J. 2006, A bivariate optimal repair- replacement model using geometric process for cold standby repairable system. *Engineering Ootimization*, 38(6), pp 609-619.

Zhang, Y.L. & Wang, G.J. 2007, "A deteriorating cold standby repairable system with priority in use", *European Journal of Operational Research*, vol. 183, no. 1, pp. 278-295.

Appendix

Proof of Lemma 1.

***Proof*:** Because X , Y are two independent random variables, therefore

$$E(\max\{X,Y\})$$

$$=\iint_D \max\{x,y\}\cdot f(x)g(y)dxdy$$

$$=\iint_{x\le y} yf(x)g(y)dxdy+\iint_{x>y} xf(x)g(y)dxdy$$

$$=\int_0^\infty\int_0^y yf(x)g(y)dxdy+\int_0^\infty\int_0^x xf(x)g(y)dydx$$

$$=\int_0^\infty yg(y)F(y)dy+\int_0^\infty xG(x)f(x)dx$$

$$=-\int_0^\infty xF(x)d[1-G(x)]+\int_0^\infty xG(x)f(x)dx$$

$$=\int_0^\infty [xf(x)[1-G(x)]+xG(x)f(x)]dx+\int_0^\infty F(x)[1-G(x)]dx$$

$$=EX+\int_0^\infty F(x)[1-G(x)]dx$$

where $f(x)=\dfrac{dF(x)}{dx}$ and $g(y)=\dfrac{dG(y)}{dy}$.

Proof of Lemma 2.

Proof: As X and Y are two independent non-negative random variables,

$$E[I\{Y>X\}X]=\iint_{x<y} xf(x)g(y)dxdy$$

$$=\int_0^\infty\left(\int_x^\infty xf(x)g(y)dy\right)dx$$

$$=\int_0^\infty xf(x)[1-G(x)]dx$$

$$=-\int_0^\infty x[1-G(x)]d[1-F(x)]$$

$$= \int_0^\infty [1-G(x)-xg(x)][1-F(x)]dx$$

$$E[I\{0<Y<X\}Y] = \iint_{0<y<x} yf(x)g(y)dxdy$$

$$= \int_0^\infty yg(y)[1-F(y)]dy$$

$$= \int_0^\infty xg(x)[1-F(x)]dx$$

and

$$E[I\{Y>X\}X]+E[I\{0<Y<X\}Y] = \int_0^\infty [1-F(x)][1-G(x)]dx \, .$$

Proof of Theorem.

Proof.

According to the above theorems and formula (2) (3), we have

$$E(W) = E[X_1^{(1)}] + E[X_N^{(1)}] + \sum_{i=1}^{N-1} E[\max\{Z_i^{(1)}+Y_i^{(1)}, X_i^{(2)}\} I\{A_i^{(1)}\} + \max\{Y_i^{(1)}, X_i^{(2)}\} I\{B_i^{(1)}\}]$$

$$+ \sum_{i=1}^{N-2} E[\max\{Z_i^{(2)}+Y_i^{(2)}, X_{i+1}^{(1)}\} I\{A_i^{(2)}\} + \max\{Y_i^{(2)}, X_{i+1}^{(1)}\} I\{B_i^{(2)}\}]$$

$$= E[X_1^{(1)}] + E[X_N^{(1)}] + \sum_{i=1}^{N-1} \{E[\max\{Z_i^{(1)}+Y_i^{(1)}, X_i^{(2)}\}](1-p) + E[\max\{Y_i^{(1)}, X_i^{(2)}\}]p\}$$

$$+ \sum_{i=1}^{N-2} \{E[\max\{Z_i^{(2)}+Y_i^{(2)}, X_{i+1}^{(1)}\}](1-p)\} + E[\max\{Y_i^{(2)}, X_{i+1}^{(1)}\}]p\}$$

$$= E[X_1^{(1)}] + E[X_N^{(1)}] + \{\sum_{i=1}^{N-1} E[\max\{Z_i^{(1)}+Y_i^{(1)}, X_i^{(2)}\}] + \sum_{i=1}^{N-2} E[\max\{Z_i^{(2)}+Y_i^{(2)}, X_{i+1}^{(1)}\}]\}(1-p)$$

$$+ \{\sum_{i=1}^{N-1} E[\max\{Y_i^{(1)}, X_i^{(2)}\}] + \sum_{i=1}^{N-2} E[\max\{Y_i^{(2)}, X_{i+1}^{(1)}\}]\}p$$

$$= E[X_1^{(1)}] + E[X_N^{(1)}] + \{\sum_{i=1}^{N-1} E[Z_i^{(1)}+Y_i^{(1)}] + \sum_{i=1}^{N-1} \int_0^\infty H_i(t)[1-F_i(t)]dt + \sum_{i=1}^{N-2} E[Z_i^{(2)}+Y_i^{(2)}]$$

$$+\sum_{i=1}^{N-2}\int_0^\infty H_i(t)[1-F_{i+1}(t)]dt\}(1-p)+\{\sum_{i=1}^{N-1}E[Y_i^{(1)}]+\sum_{i=1}^{N-2}E[Y_i^{(2)}]+\sum_{i=1}^{N-1}\int_0^\infty G_i(t)[1-F_i(t)]dt$$

$$+\sum_{i=1}^{N-2}\int_0^\infty G_i(t)[1-F_{i+1}(t)]dt\}p$$

$$=\lambda+\frac{\lambda}{a^{N-1}}+(2N-3)(1-p)\gamma+2\sum_{i=1}^{N-2}\frac{\mu}{b^{i-1}}+\frac{\mu}{b^{N-2}}+\sum_{i=1}^{N-1}\int_0^\infty [(1-p)H_i(t)+pG_i(t)][1-F_i(t)]dt$$

$$+\sum_{i=1}^{N-2}\int_0^\infty [(1-p)H_i(t)+pG_i(t)][1-F_{i+1}(t)]dt$$

$$=\lambda+\frac{\lambda}{a^{N-1}}+(2N-3)(1-p)\gamma+2\sum_{i=1}^{N-2}\frac{\mu}{b^{i-1}}+\frac{\mu}{b^{N-2}}+\int_0^\infty[(1-p)H_{N-1}(t)+pG(b^{N-2}t)][1-F(a^{N-2}t)]dt$$

$$+\sum_{i=1}^{N-2}\int_0^\infty [(1-p)H_i(t)+pG(b^{i-1}t)][2-F(a^i t)-F(a^{i-1}t)]dt$$

and

$$EL=C_m[\sum_{i=1}^{N-1}\frac{\mu}{b^{i-1}}+\sum_{i=1}^{N-2}\frac{\mu}{b^{i-1}}+(1-p)\int_0^\infty [1-R_N(t)][1-G(b^{N-2}t)]dt+p\int_0^\infty [1-G(b^{N-2}t)][1-F(a^{N-1}t)]dt]$$

$$+C_r-C_\omega[\sum_{i=1}^{N}\frac{\lambda}{a^{i-1}}+\sum_{i=1}^{N-1}\frac{\lambda}{a^{i-1}}]$$

$$=C_m\{2\sum_{i=1}^{N-2}\frac{\mu}{b^{i-1}}+\frac{\mu}{b^{N-2}}+(1-p)\int_0^\infty [1-R_N(t)][1-G(b^{N-2}t)]dt+p\int_0^\infty [1-G(b^{N-2}t)][1-F(a^{N-1}t)]dt\}$$

$$+C_r-C_\omega[2\sum_{i=1}^{N-1}\frac{\lambda}{a^{i-1}}+\frac{\lambda}{a^{N-1}}]$$

where $t\ge 0$, $H_i(t)$ and $R_N(x)$ represent the cumulative distribution functions of the random variables $Z_i^{(i)}+Y_i^{(i)}$ and $X_N^{(1)}-Z_{N-1}^{(2)}$, respectively. Hence we have $H_i(x)=S(t)*G_i(t)$, and $R_N(x)=F_N(t)*[1-S(-t)]$, where "*" indicates convolution, and

$$H_i(t)=S(t)*G(b^{i-1}t)=\int_0^t \{1-\exp[-\frac{b^{i-1}}{\mu}(t-u)]\}\,d\,[1-\exp(-\frac{u}{\gamma})]$$

$$=1-\exp(-\frac{t}{\gamma})-\frac{\mu}{\gamma b^{i-1}+\mu}\{\exp(-\frac{b^{i-1}}{\mu}t)-\exp[-(\frac{2b^{i-1}}{\mu}+\frac{1}{\gamma})t]\}$$

$$R_N(t) = F(a^{N-1}t) * [1 - S(-t)] = \int_0^t \{1 - \exp[-\frac{a^{N-1}}{\lambda}(t-u)]\} d\exp(\frac{u}{\gamma})$$

$$= \exp(\frac{t}{\gamma}) - 1 + \frac{\lambda}{\lambda + a^{N-1}\gamma}[\exp(-\frac{a^{N-1}}{\lambda}t) - \exp(\frac{t}{\gamma})]$$

where

$$\int_0^\infty [(1-p)H_{N-1}(t) + pG(b^{N-2}t)][1 - F(a^{N-2}t)]dt$$

$$= (1-p)\lambda^2[\frac{1}{a^{N-2}(\gamma a^{N-2} + \lambda)} - \frac{\mu^2}{(a^{N-2}\mu + b^{N-2}\lambda)(2b^{N-2}\gamma\lambda + \mu\lambda + a^{N-2}\mu\gamma)}] + \frac{p\lambda^2 b^{N-2}}{a^{N-2}(b^{N-2}\lambda + a^{N-2}\mu)},$$

$$\int_0^\infty [(1-p)H_i(t) + pG(b^{i-1}t)][2 - F(a^i t) - F(a^{i-1}t)]dt$$

$$= (1-p)\lambda^2\{[\frac{1}{a^i(\gamma a^i + \lambda)} + \frac{1}{a^{i-1}(\gamma a^{i-1} + \lambda)}] - \mu^2[\frac{1}{(a^i\mu + b^{i-1}\lambda)(2b^{i-1}\gamma\lambda + \mu\lambda + a^i\mu\gamma)} +$$

$$\frac{1}{(a^{i-1}\mu + b^{i-1}\lambda)(2b^{i-1}\gamma\lambda + \mu\lambda + a^{i-1}\mu\gamma)}]\} + p[\frac{\lambda(1+a)}{a^i} - \frac{\lambda\mu(a^i\mu + a^{i-1}\mu + 2b^{i-1}\lambda)}{(b^{i-1}\lambda + a^i\mu)(b^{i-1}\lambda + a^{i-1}\mu)}],$$

$$\int_0^\infty [1 - R_N(t)][1 - G(b^{N-2}t)]dt$$

$$= \int_0^\infty \{2 - \exp(\frac{t}{\gamma}) - \frac{\lambda}{\lambda + a^{N-1}\gamma}[\exp(-\frac{a^{N-1}}{\lambda}t) - \exp(\frac{t}{\gamma})]\} \cdot \exp\{-\frac{b^{N-2}}{\mu}t\} dt$$

$$= \frac{2\mu}{b^{N-2}} - \frac{a^{N-1}\mu\gamma^2}{(a^{N-1}\gamma + \lambda)(b^{N-2}\gamma - \mu)} - \frac{\lambda^2\mu}{(a^{N-1}\gamma + \lambda)(b^{N-2}\lambda + a^{N-1}\mu)},$$

and

$$\int_0^\infty [1 - G(b^{N-2}t)][1 - F(a^{N-1}t)]dt$$

$$= \int_0^\infty \exp(-\frac{a^{N-1}t}{\lambda}) \cdot \exp\{-\frac{b^{N-2}}{\mu}t\} dt = \frac{\lambda\mu}{b^{N-2}\lambda + a^{N-1}\mu}.$$

Times	Cost rate	Times	Cost rate	Times	Cost rate	Times	Cost rate	Times	Cost rate
1	-88.74	15	-305.11	29	-178.66	43	-111.84	57	-71.56
2	-107.92	16	-292.59	30	-172.52	44	-108.29	58	-69.32
3	-279.47	17	-280.78	31	-166.64	45	-104.87	59	-67.14
4	-389.09	18	-269.62	32	-161	46	-101.56	60	-65.02
5	-428.42	19	-259.06	33	-155.6	47	-98.37	61	-62.97
6	-433.41	20	-249.07	34	-150.42	48	-95.28	62	-60.97
7	-424.96	21	-239.6	35	-145.44	49	-92.29	63	-59.04
8	-411.2	22	-230.61	36	-140.66	50	-89.4	64	-57.15
9	-395.37	23	-222.07	37	-136.06	51	-86.6	65	-55.33
10	-378.99	24	-213.94	38	-131.63	52	-83.89	66	-53.55
11	-362.83	25	-206.21	39	-127.38	53	-81.27	67	-51.82
12	-347.25	26	-198.84	40	-123.27	54	-78.73	68	-50.14
13	-332.41	27	-191.8	41	-119.32	55	-76.27	69	-48.5
14	-318.37	28	-185.08	42	-115.51	56	-73.88	70	-46.91

Table 1. The expected long-run cost per unit time versus replacement times for parameter set 1.

Times	Cost rate	Times	Cost rate	Times	Cost rate	Times	Cost rate	Times	Cost rate
1	-17.37	15	-384.14	29	-489.22	43	-489.36	57	-477.22
2	-18.96	16	-403.59	30	-490.15	44	-488.78	58	-476.05
3	-37.01	17	-420.14	31	-490.84	45	-488.15	59	-474.85
4	-58.53	18	-434.04	32	-491.32	46	-487.48	60	-473.61
5	-83.54	19	-445.6	33	-491.63	47	-486.76	61	-472.34
6	-111.78	20	-455.12	34	-491.8	48	-485.99	62	-471.03
7	-142.75	21	-462.91	35	-491.85	49	-485.18	63	-469.7
8	-175.68	22	-469.24	36	-491.79	50	-484.33	64	-468.33
9	-209.62	23	-474.35	37	-491.65	51	-483.43	65	-466.93
10	-243.55	24	-478.46	38	-491.43	52	-482.5	66	-465.5
11	-276.47	25	-481.74	39	-491.13	53	-481.52	67	-464.05
12	-307.51	26	-484.34	40	-490.77	54	-480.5	68	-462.56
13	-336.03	27	-486.39	41	-490.36	55	-479.44	69	-461.05
14	-361.63	28	-487.99	42	-489.88	56	-478.35	70	-459.5

Table 2. The expected long-run cost per unit time versus replacement times for parameter set 2.

Note: *times* in Table 1 and Table 2 stands for replacement times; *Cost rate* stands for the expected long-run cost per unit time