Spectral Results for Perturbed Periodic Jacobi Operators

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Contents

Introduction

Discrete systems are a rich and interesting area of modern mathematics. One needs only consider a difference equation as elementary as the logistic map to see this: several of the most intellectually, and visually, invigorating concepts of the twentieth century (chaos theory, fractals, Mandelbrot sets, etc.) captured in its rudimentary structure. However, as valuable as these geometric properties are, they will not be the focus of this thesis. Instead, the subject of our detailed investigation will be the less famed, yet equally impressive, analytic theory of difference operators, specifically Jacobi matrices.

A Jacobi matrix, J, is a semi-infinite tri-diagonal matrix. Throughout this thesis we consider the Jacobi matrix as an operator acting on $l^2(\mathbb{N};\mathbb{C})$, with the exception of Chapter 5 where the operator acts on $l^2(\mathbb{N},\mathbb{R})$. For the sake of self-adjointness and simplicity (see Lemma 1.1.5 for more details) the off-diagonal entries are required to be positive and symmetric, and the diagonal entries real, consequently giving

$$J := \begin{pmatrix} b_1 & a_1 & & & & & & & & \\ a_1 & b_2 & a_2 & & & & & & & \\ & a_2 & b_3 & a_3 & & & & & & \\ & & a_3 & b_4 & a_4 & & & & & \\ & & & a_4 & b_5 & a_5 & & & & \\ & & & & \ddots & \ddots & \ddots & & & \\ & & & & & a_{n-1} & b_n & a_n & & \\ & & & & & & a_n & b_{n+1} & a_{n+1} & \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad (1)$$

where $a_i, b_i \in \mathbb{R}$ and $a_i > 0$ for all i.

The history of Jacobi operators is a difficult one to chart. Many trace their origins to Jacobi's 1845 paper on solving the least squares problem [28], although they bear only a thin resemblance, in structure and style, to how we know them today. Later, in 1850, Jacobi published another paper that showed the eigenvalues of these rudimentary, finite Jacobi operators to be the solutions to particular continuous fractions [29]. From here subsequent mathematicians have established their association with three-term recurrence relations and, consequently, the study of orthogonal polynomials (see, for example, [1, 12]).

Still, the question arises why this class of discrete operators should be the subject of this thesis whilst others, like CMV matrices (see, for example, [73] or [74]), are not. There is no simple answer, only opinion and preference. One rather persuasive argument, however, is that the much celebrated time independent, one-dimensional Schrödinger wave equation,

$$\frac{h^2}{2m}\frac{d^2\psi(x)}{dx^2} + (E - V(x))\psi(x) = 0, \tag{2}$$

so ubiquitous in quantum mechanics, as well as causality and dispersion relations [66], has a discrete counterpart, aptly called the discrete Schrödinger operator with potential. This is a Jacobi matrix, and studying this simplified version can offer sought after information and insight into the original continuous problem. It is for this reason why even today the spectrum of discrete Schrödinger operators is still a subject of much study. The likes of [27,48,58,59] are but a few in the plethora of recent developments.

The same discretization techniques are a well-established tool for many other continuous problems, as well, including more general differential operators. However, as this thesis will illustrate, discretization does not always produce a more tractable problem; the lack of smoothness occasionally leading to the need for more elaborate constraints that were nowhere to be found in the original continuous setting (see the function $C(\lambda; T)$ in Section 3.2 for an example of this). It is these types of discrepancies — rare as they may be — that have made discrete operators such a rewarding and taxing field in their own right.

One particular type of continuous operator that is of significance, at least in the form of motivation, is a differential operator that is periodic due to the presence of periodic coefficients: coefficients that repeat after a certain interval. The Schrödinger equation with a periodic potential is a simple example, and with regards to this thesis, the most important one. It can be used to model the electrostatic action on an electron of the regular arrangement of atomic nuclei in a crystal [8]. Consequently, the spectrum corresponds to allowed energy levels of the electron and goes some way towards explaining the overall conducting and insulating properties of the crystal. In order to better understand this theory, one can discretize the periodic Schrödinger equation and study the discrete analogue, which is a periodic Jacobi matrix.

It will be seen later that the spectrum for a periodic Jacobi matrix exists in bands. An interesting problem, and one which we will explore in great detail, is to investigate whether it is possible to change the spectrum of these periodic Jacobi matrices with the introduction of eigenvalues into the operators' essential bands using sufficiently small perturbations to the potential. This creates what is known as embedded eigenvalues (as opposed to the isolated eigenvalues that might exist between the bands).

The first recorded instance of a technique to embed eigenvalues was the Wigner-von Neumann method [67]. Published in 1929, it provides a way of embedding eigenvalues into the absolutely continuous spectrum of a one-dimensional

Schrödinger operator with a suitable choice of a potential $V(x)$,

$$-\frac{d^2}{dx^2}+V(x), \quad V(x)=\frac{c\sin(2\omega x+\varphi)}{x}.$$

This causes the eigenvalue $E=\omega^2$ to become embedded in the interval $[0,\infty)$ of a.c. spectrum. Since then the method has been adapted to embed multiple eigenvalues $E_i=\omega_i^2$, $i=1,2,\ldots,N$ [70] using a single potential

$$\sum_{i=1}^{N}\frac{c_i\sin\left(2\omega_i x+\varphi_i\right)}{x}.$$

More recently, the technique has been employed on periodic Schrödinger operators which have several or infinitely many bands of absolutely continuous spectrum [47, 55, 64], in particular, the unperturbed operators having the form

$$-\frac{d^2}{dx^2}+V_{per}(x)$$

where $V_{per}(x+T)=V_{per}(x)$ for some period T. Another method based on the explicit solution of the inverse problem [3] also allows to embed multiple (but finitely many) eigenvalues into the essential spectrum with a potential similar to the above.

There are also other types of potentials with which to embed eigenvalues into the spectrum of Schrödinger operators, although these have met with varying success. Indeed, it was proven by Eastham and Kalf that for any potential $|V(x)|=o(x^{-1})$ there are no embedded eigenvalues in the essential spectrum [15]. Contrarily, in [62] Naboko showed how to construct a dense point spectrum on the interval $[0,\infty)$ for self-adjoint one-dimensional Schrödinger operators, with potential, $V(x)$, of order $O(C(x)/x)$ where $C(x)\to\infty$ as $r\to\infty$. Simon extended this result to embed any infinite sequence of eigenvalues into the interval $[0,\infty)$, this time with the potential $V(x)$ of a Wigner-von Neumann type construction, and where $|V(x)|\leq\frac{g(x)}{1+x}$ where $g(x)\to\infty$ as $x\to\infty$ [72]. In particular the aforementioned results for Naboko and Simon use potentials which decay arbitrarily slower than x^{-1}. Later, the authors in [41] employed the tool of Prüfer equations to establish that the eigenvalues embedded in the essential spectrum by perturbing the Schrödinger operator with a potential, $V(x)$, of Coulomb-type decay, i.e. $|V(x)|\leq\frac{c}{x}$ for some constant c, can only accumulate at 0. They also devised so-called Pearson potentials such that the absolutely continuous spectrum of the Schrödinger operator becomes singular continuous. Similarly, the authors in [42] used the tool of Prüfer equations to extend Naboko's denseness of eigenvalue results to Schrödinger operators, where the background potential is no longer zero, but possibly periodic. The discrete analogue of the Prüfer equations, EFGP equations, were also discussed here and applied to the discrete Schrödinger operator to show that much like in the continuous case, Pearson potentials could be found such that the absolutely continuous spectrum becomes singular continuous.

This illustrates that the subject of embedded eigenvalues has also been well studied for Jacobi matrices. For example, in [71], Simon showed that for a 2-sided discrete Schrödinger operator with a particular power decaying potential, $q_n \sim O(n^{-\frac{1}{2}+\epsilon})$, the operator has point spectrum almost everywhere in the interval of essential spectrum $[-2, 2]$. Later, in [65], Naboko and Yakovlev extended the former's result in [62] to the discrete Schrödinger operator, showing that for some potentials (q_n) of the form $|q_n| \leq \frac{C_n}{n}$ with C_n increasing arbitrarily slowly (and therefore q_n decaying slower than the Coulomb-type), the point spectrum is dense in the interval $[-2, 2]$. Published in 2002, Krutikov used the EFGP transformation to establish that the discrete Schrödinger operator with any Coulomb-type potential, and eigenvalues $\lambda_j \in (-2, 2)$, is such that the following inequality is always satisfied

$$\sum_{j=1}^{\infty} \left(1 - \frac{\lambda_j^2}{4}\right) \leq \frac{C^2 + 2}{2}.$$

Consequently, this gives that the set of eigenvalues inside the interval $[-2, 2]$ is at most countable with only two possible accumulation points -2 or 2 [46]. Interestingly, the strategy is the same as used as in the continuous case [41], however the details are considerably more complicated to calculate which attest to the fact that the results for the discrete setting are not always a simple corollary of the continuous counterpart. In [45] Krüger showed that for the discrete Schrödinger operator with parameterised boundary condition, H_β, and Wigner-von Neumann potential

$$q_n = k_n \sum_{j=1}^{d} b_j \cos(2\pi(\alpha_j n + x_j)),$$

for parameters α_j, b_j, x_j and d, where k_n is a sequence decaying monotonically to zero, that there exists $\beta \in \mathbb{R} \cup \{\infty\}$ such that H_β has an eigenvalue embedded in the absolutely continuous spectrum. More recently, Lukic and Ong have used EFGP equations for periodic Jacobi matrices with perturbations of a Wigner-von Neumann type, $q_n = \frac{\sin(n\beta)}{n^\gamma}, \gamma > 0$, to show that such perturbations leave the a.c. spectrum of the periodic background operator unchanged (see [56] for the discrete setting and [55] for the continuous).

Although no immediate applications have been found for embedded eigenvalues in this discrete case, in the continuous settings the physical motivations are quite surprising. Originally, the oscillating potentials and diffractive interference involved were considered mere oddities with little physical applications, however, due to the Schrödinger equation's key role in quantum mechanics, it has since been suggested that these bound states might be found in particular molecular and atomic systems [13, 18, 25, 76–78]. Indeed, physical evidence has actually been recorded to support these assertions in semi-conductor heterostructures [9].

The purpose of this thesis is to explore various techniques with which to embed eigenvalues into the essential spectrum of Hermitian periodic Jacobi operators. This will involve generalising some approaches already established for the discrete Schrödinger operator with zero potential (DSO) to the more arbitrary periodic analogue, and investigating what additional conditions these demand. The discrete analogue of Wigner-von Neumann type perturbations and elaborate geometric constructions will also be employed to achieve this. However, before embarking on these more original aspects of the thesis, we gather together a collection of results on the spectrum of unperturbed Hermitian periodic Jacobi operators to serve as a foundation. Indeed, with some noticeable exceptions, such as Stolz [80], Teschl [82] and Elaydi [17], there is little in the way of easily available introductory material to this field, and therefore by reproducing the background theory so thoroughly in the early chapters, it is hoped that this thesis can serve as another helpful monograph on the subject.

The structure of the thesis is as follows:

In Chapter 1 the spectrum of the DSO is analysed. By exploring these elementary, established results certain key concepts are discussed, such as the discrete and absolutely continuous spectrum, Weyl-sequences, Gilbert-Pearson theory, and the transfer matrix approach to solving recurrence relations. These provide a gentle introduction to the area and a sound basis on which to build the more advanced, original arguments that constitute the heart of the thesis.

In Chapter 2 some basic results of Hermitian period-T Jacobi operators are considered. However, although already accepted by those working in the field, there are several results whose proofs are hard to come by in the literature, and so original alternatives have been provided. The importance of this chapter rests in establishing the general absence of point spectrum for periodic Jacobi operators without a potential, and the conditions on the potential for which we are unable to embed an eigenvalue. This provides the inspiration, and also conceptual starting point, for investigating techniques that will perturb the operator enough to embed an eigenvalue into one of its intervals of essential spectrum. It is this aim that governs the next few chapters.

In Chapter 3 the first of the embedding techniques is proposed. The Wigner-von Neumann method, previously used for perturbing continuous Schrödinger operators, is here applied to their discrete counterparts. In particular, we consider perturbations of Hermitian period-T Jacobi matrices, and unlike other methods in the literature assume a simple structure for the eigenvector a priori. From this we infer the structure of the potential (which is more complicated than the eigenvector) but asymptotically resembles a Wigner-von Neumann type. The asymptotic behaviour of the subordinate solutions is investigated, as too are their initial components, together giving a general technique for embedding eigenvalues into the operator's absolutely continuous spectrum. The finitely many elements of the a.c. spectrum for which this method fails are also described. Additional details surrounding some of the arguments can be found in the appendix.

In Chapter 4 another embedding technique is discussed. Whereas we assumed the structure of the eigenvector from the outset in the previous chapter, we now assume the structure of the perturbation – a Wigner-von Neumann type potential to be precise – and use discrete Levinson techniques to embed eigenvalues into the absolutely continuous spectrum of a Hermitian period-T

Jacobi operator. Note that discrete Levinson tools were also used to study the spectrum of Jacobi matrices in [31] and [32], but the operators here were unbounded and the main focus was on investigating the asymptotic behaviour of solutions to the formal spectral equation without consideration to explicit eigenvalues. Contrarily, we discuss how to devise subordinate solutions for an arbitrary (possibly infinite) set, and consider the special case of embedding a single eigenvalue in some detail. An analogue of the quantisation conditions from the continuous case also appears, relating the frequency of the oscillation of the potential to the quasi-momentum associated with the eigenvalue. This argument can be seen as the discrete analogue of [47].

In Chapter 5 the geometric method devised by Naboko and Yakovlev, previously used to embed eigenvalues into the essential spectrum of the discrete Schrödinger operator [65], is here applied to Hermitian period-T Jacobi operators. Moreover, the rational dependence conditions from [65] on the quasi-momenta, $\theta(\lambda)$, are relaxed so that with the exception of points satisfying $\theta(\lambda) = \frac{\pi}{2}$ any individual element in the generalised interior of the essential spectrum (see Definition 1.3.7) is embeddable. We also explore the conditions that permit the simultaneous embedding of finitely many eigenvalues whose quasi-momenta are (possibly) rationally dependent with each other and π, alongside infinitely many of those with rationally independent quasi-momenta. Unlike in the previous two chapters where either the eigenvector or potential was explicit, here we have an explicit formula for neither.

In Chapter 6, we discuss the ansatz approach which deals with devising potentials and eigenvectors for specific elements of the generalised interior of the essential spectrum, namely those λ which have quasi-momentum rationally dependent with π. In Chapter 7 conditions are discussed for the potential that if satisfied either prohibit or guarantee the existence of a single embedded eigenvalue in an interval of the essential spectrum for a Hermitian period-T Jacobi operator.

Finally, we conclude the thesis with a brief discussion of the respective benefits and disadvantages of each of the three main techniques (Wigner von Neumann, discrete Levinson, geometric) to embed eigenvalues. Note that due to its limited scope, we ignore the ansatz method.

For the reader who still has reservations about the utility of difference operators, this section is concluded with a quote by the revered mathematician, and pedagogue, Gilbert Strang [81]:

"*Calculus is mostly about one special operation (the derivative) and its inverse (the integral). Of course I admit that calculus could be important. But so many applications of mathematics are discrete rather than continuous, digital rather than analog. The century of data has begun!*"

Notation

Throughout this thesis an infinite vector $\underline{x} := (x_i)_{i=1}^{\infty}$ belongs to the sequence space $l^1(\mathbb{N};\mathbb{R})$ iff $x_i \in \mathbb{R}$ for all i and

$$\sum_{i=1}^{\infty} |x_i| < \infty. \tag{3}$$

Similarly, $\underline{x} \in l^1(\mathbb{N},\mathbb{C})$ iff $x_i \in \mathbb{C}$ for all i and satisfies (3). The sequence $\underline{x}$ belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$ iff $x_i \in \mathbb{R}$ for all i and

$$\|\underline{x}\| := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}} < \infty. \tag{4}$$

Likewise, $\underline{x}$ belongs to the sequence space $l^2(\mathbb{N};\mathbb{C})$ iff $x_i \in \mathbb{C}$ and satisfies (4). The sequence $\underline{x}$ belongs to the sequence space $l^{\infty}(\mathbb{N};\mathbb{R})$ iff $x_i \in \mathbb{R}$ and

$$\|\underline{x}\|_{\infty} := \sup_{i\in\mathbb{N}} |x_i| < \infty. \tag{5}$$

Similarly, the sequence $\underline{x}$ belongs to the sequence space $l^{\infty}(\mathbb{N};\mathbb{C})$ iff $x_i \in \mathbb{C}$ and satisfies (5).

We will also deal with the inner product on the space $l^2(\mathbb{N};\mathbb{C})$, $\langle\cdot,\cdot\rangle$ where

$$\langle\underline{x},\underline{y}\rangle = \sum_{i=0}^{\infty} x_i\overline{y}_i, \qquad \underline{x},\underline{y} \in l^2(\mathbb{N};\mathbb{C}),$$

and where $\overline{y}_i$ denotes taking the complex conjugate of element y_i. The inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}^n} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ will always denote the standard complex inner product, i.e. for $\underline{x} := (x_1,\ldots,x_n)^T, \underline{y} := (y_1,\ldots,y_n)^T \in \mathbb{C}^n$

$$\langle\underline{x},\underline{y}\rangle_{\mathbb{C}^n} = \sum_{i=1}^{n} x_i\overline{y}_i.$$

If the context is obvious, we will for simplicity let $\langle\cdot,\cdot\rangle$ denote $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$.

Many arguments will involve taking the matrix product, say $\prod\limits_{i=1}^{n} M_i$. It should be stressed that this will always be the ordered matrix product starting from the right, i.e.

$$\prod_{i=1}^{n} M_i = M_n M_{n-1}\ldots M_2 M_1.$$

In later chapters we will investigate and discuss the asymptotic behaviour of certain sequences, and this will understandably involve the usage of several symbols from the asymptotic branch of mathematics, which we define now. For functions $f(x), g(x)$ we have that

$$f(x) = O(g(x)), x \to \infty, \text{ if and only if } |f(x)| < k_1 |g(x)|,$$

for all $x \geq x_0$ and for some $k_1 > 0$ and $x_0 \in \mathbb{R}$. Similarly,

$$f(x) = o(g(x)), x \to \infty, \text{ if and only if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

$$f(x) \sim g(x), x \to \infty, \text{ if and only if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$

and

$$f(x) \asymp g(x), x \to \infty, \text{ if and only if } k_1 \left|g(x)\right| \leq \left|f(x)\right| \leq k_2 \left|g(x)\right|,$$

for all $x > x_0$ and $k_1, k_2 > 0$ and $x_0 \in \mathbb{R}$.

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Chapter 1

Jacobi operators and the discrete Schrödinger operator (DSO)

The Schrödinger equation is widely used, and studied, in the field of quantum mechanics, modelling the evolution of a quantum system with respect to time; the solution $\psi(\underline{x}, t)$ to the equation giving the likelihood of a particle appearing with position $\underline{x}$ at time t. In this chapter, the spectral properties of its discrete analogue, aptly named the discrete Schrödinger operator (DSO), are studied where the DSO is defined to be the Jacobi operator, J_0, with $b_i = 0$ and $a_i = 1$ for all i, i.e.

$$J_0 := \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1.1}$$

More generally, the purpose of this chapter is to introduce some of the concepts and theorems that will be used to produce the more original and innovative parts of the thesis. For this reason, several of the results presented in this section will be superseded by other theorems, valid for more a general class of Jacobi matrices, later on.

1.1 Preliminary results for Jacobi operators

Before discussing the DSO, it is best to introduce some of the more elementary results known for Hermitian Jacobi operators, J, with entries $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ and structure as in (1).

Lemma 1.1.1. *Let* $\underline{a} := (a_n)_{n\geq 1} \in l^\infty(\mathbb{N};\mathbb{R}), \underline{b} := (b_n)_{n\geq 1} \in l^\infty(\mathbb{N};\mathbb{R})$. *Then, for the Jacobi operator,* J, *we have the estimate*

$$\|J\| \leq 2\|\underline{a}\|_\infty + \|\underline{b}\|_\infty,$$

where $\|J\|$ *is the operator norm of* J *as an operator in* l^2.

Proof. Let $\underline{x} := (x_n)_{n\geq 1}$ be an arbitrary element in $l^2(\mathbb{N};\mathbb{C})$. Then

$$\begin{aligned}
\|J\underline{x}\|^2 &= |b_1x_1 + a_1x_2|^2 + \sum_{i=2}^{\infty} |a_{i-1}x_{i-1} + b_ix_i + a_ix_{i+1}|^2 \\
&\leq (\|\underline{b}\|_\infty|x_1| + \|\underline{a}\|_\infty|x_2|)^2 + \sum_{i=2}^{\infty}\Big(\|\underline{a}\|_\infty(|x_{i-1}| + |x_{i+1}|) + \|\underline{b}\|_\infty|x_i|\Big)^2 \\
&\leq \|\underline{b}\|_\infty^2 \sum_{i=1}^{\infty}|x_i|^2 + 2\|\underline{a}\|_\infty\|\underline{b}\|_\infty\left(2\sum_{i=2}^{\infty}|x_{i-1}x_i|\right) \\
&+ \|\underline{a}\|_\infty^2\left(\sum_{i=1}^{\infty}|x_i|^2 + \sum_{i=2}^{\infty}|x_i|^2 + 2\sum_{i=2}^{\infty}|x_{i-1}x_{i+1}|\right)
\end{aligned}$$

Then, using the Cauchy-Schwarz inequality (see, for example, Lemma 1.7 in [22]) on the series with $|x_ix_{i-1}|$ and $|x_{i-1}x_{i+1}|$ terms, as well as introducing additional $|x_1|^2, |x_2|^2$ terms we obtain

$$\begin{aligned}
\|J\underline{x}\|^2 &\leq \|\underline{b}\|_\infty^2 \sum_{i=1}^{\infty}|x_i|^2 + 2\|\underline{a}\|_\infty\|\underline{b}\|_\infty\left(2\sum_{i=1}^{\infty}|x_i|^2\right) \\
&+ \|\underline{a}\|_\infty^2\left(\sum_{i=1}^{\infty}|x_i|^2 + \sum_{i=2}^{\infty}|x_i|^2 + 2\sum_{i=1}^{\infty}|x_i|^2\right) \\
&\leq \left(4\|\underline{a}\|_\infty^2 + 4\|\underline{a}\|_\infty\|\underline{b}\|_\infty + \|\underline{b}\|_\infty^2\right)\|\underline{x}\|^2 = (2\|\underline{a}\|_\infty + \|\underline{b}\|_\infty)^2\|\underline{x}\|^2.
\end{aligned}$$

This implies

$$\|J\| \leq (2\|\underline{a}\|_\infty + \|\underline{b}\|_\infty).$$

□

Remark Later, in Lemma 1.4.1, we establish that for the discrete Schrödinger operator, J_0, $\sigma(J_0) = [-2, 2]$. Therefore, using that the spectral radius for a self-adjoint operator is equal to its norm, we have that $\|J_0\| = 2$ and thus the inequality in Lemma 1.1.1 is sharp.

In fact J is bounded if precisely $\underline{a}$ and $\underline{b}$ are bounded.

Lemma 1.1.2. *A Jacobi operator,* J, *is bounded iff* $\underline{a} := (a_n)_{n\geq 1}$ *and* $\underline{b} := (b_n)_{n\geq 1}$ *are bounded.*

Proof. ($\Leftarrow$) If $\underline{a}$ and $\underline{b}$ are both bounded then the operator is bounded with an operator norm which can be estimated by Lemma 1.1.1.

($\Rightarrow$) Assume either $\underline{a}$ or $\underline{b}$ is unbounded and define the sequence

$$\underline{e}_i := (0,0,\dots,0,1,0,\dots)^T$$

which has 1 in the i-th component and $0's$ everywhere else. Then $\|\underline{e}_i\| = 1$ and

$$J\underline{e}_i = a_{i-1}\underline{e}_{i-1} + b_i\underline{e}_i + a_i\underline{e}_{i+1},$$

which implies

$$\|J\underline{e}_i\|^2 = a_{i-1}^2 + b_i^2 + a_i^2.$$

Moreover, since either $\underline{a}$ or $\underline{b}$ is unbounded, we obtain for a subsequence

$$\|J\underline{e}_i\| \to \infty$$

as $i \to \infty$. We see that J is unbounded, and by contraposition the result is proven. $\square$

Corollary 1.1.3. *The discrete Schrödinger operator, J_0, is bounded.*

Note that in the introduction it was stated that the Jacobi operators considered in this thesis have positive off-diagonal entries for the sake of self-adjointness and simplicity. However, a Jacobi operator with off-diagonal entries that are complex, but are such that

$$\widetilde{J} := \begin{pmatrix} b_1 & a_1 & & & & & & & \\ \overline{a}_1 & b_2 & a_2 & & & & & & \\ & \overline{a}_2 & b_3 & a_3 & & & & & \\ & & \overline{a}_3 & b_4 & a_4 & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \overline{a}_{n-1} & b_n & a_n & & \\ & & & & & \overline{a}_n & b_{n+1} & a_{n+1} & \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.2}$$

can also be self-adjoint, as illustrated by the following result.

Lemma 1.1.4. *For $\underline{a} := (a_n)_{n\geq 1} \in l^\infty(\mathbb{N};\mathbb{C})$ and $\underline{b} := (b_n)_{n\geq 1} \in l^\infty(\mathbb{N};\mathbb{R})$ the Jacobi operator, $\widetilde{J}$, is self-adjoint.*

Proof. Define $a_0 := x_0 := y_0 := 0$. Then for $\underline{x} = (x_n)_{n\geq 1}, \underline{y} = (y_n)_{n\geq 1}$ both in l^2 and possibly complex, the following holds

$$\begin{aligned}\langle \widetilde{J}\underline{x}, \underline{y}\rangle &= \sum_{i=1}^{\infty} \left(\overline{a}_{i-1}x_{i-1} + b_i x_i + a_i x_{i+1}\right)\overline{y}_i \\ &= \sum_{i=1}^{\infty} \overline{a}_{i-1}x_{i-1}\overline{y}_i + \sum_{i=1}^{\infty} b_i x_i \overline{y}_i + \sum_{i=1}^{\infty} a_i x_{i+1}\overline{y}_i \\ &= \sum_{j=0}^{\infty} x_j \overline{a}_j \overline{y}_{j+1} + \sum_{j=1}^{\infty} x_j b_j \overline{y}_j + \sum_{j=0}^{\infty} x_j a_{j-1}\overline{y}_{j-1} \\ &= \sum_{j=1}^{\infty} x_j \left(a_{j-1}\overline{y}_{j-1} + b_j\overline{y}_j + \overline{a}_j\overline{y}_{j+1}\right) = \langle \underline{x}, \widetilde{J}\underline{y}\rangle. \end{aligned}$$

$\square$

Remark It should be noted that the last line of the calculation holds because (b_n) is real, in particular $\overline{b}_j = b_j$. Thus for self-adjointness it is a necessary condition for the diagonal entries to be real, as well as the off-diagonal being complex conjugates as described in (1.2).

The next lemma will show that in our situation we can reduce our consideration to those Jacobi operators with positive off-diagonal entries.

Lemma 1.1.5. *Let* $\underline{a}_n := (a_n)_{n\geq 1} \in l^\infty(\mathbb{N};\mathbb{C})$, $\underline{b} := (b_n)_{n\geq 1} \in l^\infty(\mathbb{N};\mathbb{R})$, $\widetilde{J}$ *as in* (1.2) *and* $J_{|\cdot|}$ *such that*

$$J_{|\cdot|} := \begin{pmatrix} b_1 & |a_1| & & & & & & & \\ |a_1| & b_2 & |a_2| & & & & & & \\ & |a_2| & b_3 & |a_3| & & & & & \\ & & |a_3| & b_4 & |a_4| & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & |a_{n-1}| & b_n & |a_n| & & \\ & & & & & |a_n| & b_{n+1} & |a_{n+1}| & \\ & & & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then the Jacobi matrices $\widetilde{J}, J_{|\cdot|}$ *are unitarily equivalent.*

Proof. Define the diagonal matrix D with entries, $(d_n)_{n=1}^\infty$, where

$$d_1 := 1, d_2 := \frac{|a_1|}{a_1}, d_3 := \frac{|a_2||a_1|}{a_2 a_1}, d_4 := \frac{|a_3||a_2||a_1|}{a_3 a_2 a_1}, \ldots,$$

$$d_n := \frac{\prod\limits_{i=1}^{n-1} |a_i|}{\prod\limits_{i=1}^{n-1} a_i}.$$

Then $|d_i|^2 = 1$ for all i, implying D is bounded and unitary. Moreover

$$\begin{aligned} D^*\widetilde{J}D &= \begin{pmatrix} |d_1|^2 b_1 & \overline{d}_1 d_2 a_1 & & & & & \\ d_1 \overline{d}_2 \overline{a}_1 & |d_2|^2 b_2 & \overline{d}_2 d_3 a_2 & & & & \\ & d_2 \overline{d}_3 \overline{a}_2 & |d_3|^2 b_3 & \overline{d}_3 d_4 a_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & d_{n-1}\overline{d}_n \overline{a}_{n-1} & |d_n|^2 b_n & \overline{d}_n d_{n+1} a_n & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \\ &= J_{|\cdot|}. \end{aligned}$$

□

Remark If two operators are unitarily equivalent, then their spectra are identical (see, for example, Section 73 of Chapter VI in [2]). Therefore to study complex self-adjoint Jacobi operators it is sufficient to only study their positive off-diagonal counterparts. It is because of Lemma 1.1.5 why only Jacobi operators with positive off-diagonal entries are discussed in this thesis.

The Jacobi matrices featured throughout are self-adjoint, consequently permitting the Spectral Theorem for self-adjoint operators [44] along with the Lebesgue Decomposition Theorem [11] to be employed. This gives the following decomposition for the spectrum, σ, of the self-adjoint Jacobi operator, J:

$$\sigma(J) = \overline{\sigma_p(J)} \cup \sigma_{a.c.}(J) \cup \sigma_{s.c.}(J) \tag{1.3}$$

where $\sigma_p, \sigma_{a.c.}, \sigma_{s.c.}$ denote the point spectrum, absolutely continuous spectrum and singular continuous spectrum, respectively (see, for example, Section VII.2 in [69] for more detail). However, this decomposition of the spectrum is not necessarily disjoint. For one that is consider:

$$\sigma(J) = \sigma_{disc}(J) \cup \sigma_{ess}(J), \tag{1.4}$$

where $\sigma_{disc}, \sigma_{ess}$ denote the discrete and essential spectrum, respectively (see, for example, Section VII.4 in [69] for more detail). The essential spectrum contains the absolutely continuous and singular spectra from (1.3) and the eigenvalues of the point spectrum that have infinite (geometric) multiplicity as well as any accumulation points; whilst the discrete spectrum contains only isolated eigenvalues with a finite (geometric) multiplicity.

Remark Note that for a Jacobi operator there are no eigenvalues of infinite multiplicity. This follows from the fact that when an eigenvector is prescribed the first term, u_1, determines all the subsequent ones uniquely. Indeed, each eigenvalue is of multiplicity one.

Remark The motivation behind taking the closure of the point spectrum follows from the fact that in general operator theory it is possible to have eigenvalues accumulate. However, this is not necessary here due to the fact that in most cases we will be dealing with finitely many eigenvalues, and where there are infinitely many eigenvalues, they will be embedded and therefore already belong to the a.c. spectrum (which is closed). Thus, for the particular Jacobi operators J we deal with in the thesis (periodic self-adjoint or periodic self-adjoint with a compact perturbation) we have the relation

$$\sigma(J) = \sigma_p(J) \cup \sigma_{a.c.}(J) \cup \sigma_{s.c.}(J).$$

1.2 The essential spectrum of the DSO

First let us introduce a way to characterise the essential spectrum of an operator without the usual, and constructively unhelpful, reference to Fredholm operators.

Proposition 1.2.1. *If for a particular $\lambda \in \mathbb{C}$ there is a sequence, $(\underline{x}_n)_{n\in\mathbb{N}}$, in the domain of the operator J such that*

1. $(J - \lambda)\underline{x}_n \to \underline{0}$ *as* $n \to \infty$,

2. $\|\underline{x}_n\| = 1$, *for all* n,

3. $\underline{x}_n \overset{weak}{\to} \underline{0}$, *as* $n \to \infty$,

then $(\underline{x}_n)_{n\in\mathbb{N}}$ *is called a singular sequence and* λ *belongs to the essential spectrum of the operator* J.

Remark Note that Conditions 1 and 2 together are equivalent to

$$\frac{\|(J-\lambda)\underline{x}_n\|}{\|\underline{x}_n\|} \to 0$$

as $n \to \infty$, and we can use this instead. Also, if Conditions 1 and 2 are satisfied then λ is in either the essential or point spectrum of the operator. However, if Condition 3 is also satisfied then we know for sure that λ is in the essential spectrum.

In order to calculate the essential parts of the DSO spectrum the following lemma will be needed:

Lemma 1.2.2. *Let* $\lambda \in (-2, 2)$. *Then there exists a sequence* $\{P_n(\lambda)\}_{n=1}^{\infty}$ *and* $c_0(\lambda) \in (0, 1]$ *such that for large enough* N

$$\|\{P_n(\lambda)\}_{n=1}^{N}\| \geq c_0(\lambda)\sqrt{N},$$

where $\{P_n(\lambda)\}_{n=1}^{\infty}$ *is the solution to the recurrence relation*

$$P_{n-1}(\lambda) - \lambda P_n(\lambda) + P_{n+1}(\lambda) = 0, n \geq 2 \tag{1.5}$$

with $P_1(\lambda) = 1$ *and* $P_2(\lambda) = \lambda$.

Proof. (Step One) We first calculate the general structure a vector solution must have in order to satisfy the recurrence relationship (1.5) and also the initial conditions. Assume $\underline{x} := (x_n)_{n\geq 1}$ is an element of $l^2(\mathbb{N};\mathbb{C})$ such that (1.5) is satisfied as well as the initial conditions. Then without loss of generality $x_1 = 1$ and $\underline{x}$ satisfies the following two conditions:

$$x_2 = \lambda x_1 \tag{1.6}$$

and

$$x_{n-1} + x_{n+1} = \lambda x_n, \tag{1.7}$$

for all $n \geq 2$.

Using the ansatz $x_n(\lambda) = \mu^n(\lambda)$, the recurrence relationship can be solved to obtain

$$\begin{aligned} \mu^{n-1}(\lambda) + \mu^{n+1}(\lambda) = \lambda\mu^n(\lambda) &\iff 1 + \mu^2(\lambda) = \lambda\mu(\lambda) \\ &\iff \mu^2(\lambda) - \lambda\mu(\lambda) + 1 = 0 \\ &\iff \mu(\lambda) = \mu_\pm(\lambda), \end{aligned} \tag{1.8}$$

where $\mu_{\pm}(\lambda) := \frac{\lambda \pm \sqrt{\lambda^2-4}}{2}$ and noting that $|\mu_+(\lambda)| = |\mu_-(\lambda)| = 1, \mu_+(\lambda)\mu_-(\lambda) = 1$ for $\lambda \in (-2,2)$. (Indeed the last relation holds for all $\lambda \in \mathbb{C}$).

Since the sequences $\left(\mu_+^n\right)_{n\in\mathbb{N}}$ and $\left(\mu_-^n\right)_{n\in\mathbb{N}}$ are linearly independent, the general solution of (1.7) is

$$x_n(\lambda) = C(\lambda)\mu_+^n(\lambda) + D(\lambda)\mu_-^n(\lambda), \tag{1.9}$$

where $C(\lambda)$ and $D(\lambda)$ are constants, in λ, determined by substituting (1.9) into (1.6). This produces

$$x_1(\lambda) = 1 \iff C(\lambda)\mu_+(\lambda) + D(\lambda)\mu_-(\lambda) = 1, \tag{1.10}$$

$$x_2(\lambda) = \lambda \iff C(\lambda)\mu_+^2(\lambda) + D(\lambda)\mu_-^2(\lambda) = \lambda\left(C(\lambda)\mu_+(\lambda) + D(\lambda)\mu_-(\lambda)\right). \tag{1.11}$$

Consequently,

$$C(\lambda) = \frac{1}{2\mu_+(\lambda)}\left(1 + \frac{\lambda}{\sqrt{\lambda^2-4}}\right), D(\lambda) = \frac{1}{2\mu_-(\lambda)}\left(1 - \frac{\lambda}{\sqrt{\lambda^2-4}}\right).$$

(Step Two) We now find a lower bound for the sum of squares of the first N components from the above vector solution, i.e. a lower-bound for the norm $\|\{P_n(\lambda)\}_{n=1}^N\|$. For $\lambda \in (-2,2)$ we obtain,

$$\begin{aligned} P_n(\lambda) &= C(\lambda)\mu_+^n(\lambda) + D(\lambda)\mu_-^n(\lambda) \\ &= \mu_-^n(\lambda)\left(C(\lambda)\left(\frac{\mu_+(\lambda)}{\mu_-(\lambda)}\right)^n + D(\lambda)\right) \end{aligned} \tag{1.12}$$

Then,

$$\begin{aligned} |P_n(\lambda)| &= |C(\lambda)z^n(\lambda) + D(\lambda)| \\ &= |C(\lambda)|\left|z^n(\lambda) - \frac{D(\lambda)}{C(\lambda)}\right|, \end{aligned}$$

where $z(\lambda) := \frac{\mu_+(\lambda)}{\mu_-(\lambda)}$. Recalling that $|\mu_+(\lambda)| = |\mu_-(\lambda)| = 1$ for $\lambda \in (-2,2)$ we see that $z(\lambda)$ is on the unit circle. Then, there are a total of two distinct cases depending on the location of $-D(\lambda)/C(\lambda)$:

Case A: If $-D(\lambda)/C(\lambda)$ is outside or inside the unit circle, $\mathbb{D}$, then geometrically it can be seen that

$$|P_n(\lambda)| \geq c_0(\lambda),$$

where

$$c_0(\lambda) := |C(\lambda)|\text{dist}\left(\mathbb{D}, \frac{-D(\lambda)}{C(\lambda)}\right).$$

Therefore

$$\|\{P_n(\lambda)\}_{n=1}^N\| \geq c_0(\lambda)\sqrt{N}.$$

Case B: If $-D(\lambda)/C(\lambda)$ is on the unit circle then using $z_0(\lambda) := -\frac{D(\lambda)}{C(\lambda)}$ with $|z_0(\lambda)| = 1$,

$$\|\{P_n(\lambda)\}_{n=1}^N\|^2 = |C(\lambda)|^2 \sum_{n=1}^N |z^n(\lambda) + z_0(\lambda)|^2 \tag{1.13}$$

$$= |C(\lambda)|^2 \sum_{n=1}^N (z^n(\lambda) + z_0(\lambda))(\overline{z(\lambda)}^n + \overline{z_0(\lambda)})$$

$$= |C(\lambda)|^2 \sum_{n=1}^N (|z(\lambda)|^{2n} + z_0(\lambda)\overline{z(\lambda)}^n + \overline{z_0(\lambda)}z^n(\lambda) + |z_0(\lambda)|^2)$$

$$= |C(\lambda)|^2 \sum_{n=1}^N (2 + z_0(\lambda)\overline{z(\lambda)}^n + \overline{z_0(\lambda)}z^n(\lambda))$$

$$= |C(\lambda)|^2 \left(2N + \frac{z_0(\lambda)\overline{z(\lambda)}(1-\overline{z(\lambda)}^N)}{1-\overline{z(\lambda)}} + \frac{\overline{z_0(\lambda)}z(\lambda)(1-z^N(\lambda))}{1-z(\lambda)}\right). \tag{1.14}$$

Now observe that

$$\begin{aligned}\frac{|z_0(\lambda)\overline{z(\lambda)}(1-\overline{z(\lambda)}^n)|}{|1-\overline{z(\lambda)}|} &\leq \frac{|z_0(\lambda)||\overline{z(\lambda)}||1-\overline{z(\lambda)}^n|}{|1-\overline{z(\lambda)}|} = \frac{|1-\overline{z(\lambda)}^n|}{|1-\overline{z(\lambda)}|} \\ &\leq \frac{1+|z(\lambda)|^n}{|1-\overline{z(\lambda)}|} = \frac{2}{|1-\overline{z(\lambda)}|}\end{aligned}$$

and is thus bounded for all $n \in \mathbb{N}$. Using that $\frac{\overline{z_0(\lambda)}z(\lambda)(1-z^n(\lambda))}{1-z(\lambda)}$ is the conjugate of the above, we see that it too is bounded for all $n \in \mathbb{N}$. Consequently, for any N sufficiently large, and some $K(\lambda) \in \mathbb{R}$, we have

$$2N + \frac{z_0(\lambda)\overline{z(\lambda)}(1-\overline{z(\lambda)}^N)}{1-\overline{z(\lambda)}} + \frac{\overline{z_0(\lambda)}z(\lambda)(1-z^N(\lambda))}{1-z(\lambda)} \geq 2N - K(\lambda),$$

so

$$\|\{P_n(\lambda)\}_{n=1}^N\| \geq |C(\lambda)|\sqrt{2N-K(\lambda)} \geq |C(\lambda)|\sqrt{N} \tag{1.15}$$

for sufficiently large N. □

The following result is necessary to prove the next lemma for the DSO.

Proposition 1.2.3. *The sequence* $\underline{u}_1, \underline{u}_2, \underline{u}_3, \ldots, \underline{u}_n, \ldots,$ *where* $\underline{u}_n = (u_i^{(n)})_{i=1}^\infty$, *weakly converges to* $\underline{0}$ *in* $l^2(\mathbb{N};\mathbb{C})$ *iff (a)* $\|\underline{u}_n\| \leq \alpha$ *for some* α, *and (b)* $u_i^{(n)} \to 0$ *as* $n \to \infty$ *for each* $i \in \mathbb{N}$.

Proof. See result 4.8-6 in [44] for details. □

Now we can state the following important lemma for the DSO:

Lemma 1.2.4. *Let $\lambda \in (-2, 2)$ and $P(\lambda)$ be as in Lemma 1.2.2. The sequence*

$$\underline{u}_n(\lambda) = \frac{1}{\sqrt{P_1^2(\lambda) + \cdots + P_n^2(\lambda)}} \left(P_1(\lambda), \ldots, P_n(\lambda), 0, \ldots\right)^T$$

is a singular sequence for $J_0 - \lambda$.

Proof. From Proposition 1.2.1 there are three conditions to be satisfied. Observe Condition 2 is immediately obeyed. To fulfill the others apply Lemma 1.2.2 to obtain the relation

$$\begin{aligned}
&\|(J-\lambda)\underline{u}_n(\lambda)\|^2 = \frac{1}{P_1^2(\lambda) + \cdots + P_n^2(\lambda)} \Big(|P_{n-1}(\lambda) - \lambda P_n(\lambda)|^2 + |P_n(\lambda)|^2 \Big) \\
&\leq \frac{1}{P_1^2(\lambda) + \cdots + P_n^2(\lambda)} \Big(|P_{n-1}(\lambda)|^2 + (-\lambda) P_n(\lambda) \overline{P_{n-1}(\lambda)} \\
&\qquad\qquad + \overline{(-\lambda)P_n(\lambda)} P_{n-1}(\lambda) + |\lambda|^2 |P_n(\lambda)|^2 + |P_n(\lambda)|^2 \Big) \\
&\leq \frac{1}{P_1^2(\lambda) + \cdots + P_n^2(\lambda)} \Big((|C(\lambda)| + |D(\lambda)|)^2 + 2|\lambda|(|C(\lambda)| + |D(\lambda)|)^2 \\
&\qquad\qquad\qquad + |\lambda|^2 (|C(\lambda)| + |D(\lambda)|)^2 + (|C(\lambda)| + |D(\lambda)|)^2 \Big) \\
&\leq \frac{1}{c_0^2(\lambda) n} \left((|C(\lambda)| + |D(\lambda)|)^2 (2 + 2|\lambda| + |\lambda|^2) \right) \longrightarrow 0,
\end{aligned}$$

as $n \longrightarrow \infty$. Thus the first condition is satisfied.

By invoking Proposition 1.2.3, the third somewhat intimidating condition is broken into two more manageable ones. The first criterion, (a), requires that the sequence $\|\underline{u}_n\| < \alpha$ for some $\alpha > 0$ and is obviously satisfied since $\|u_n\| = 1$. The second, (b), requires that $(u_i^{(n)}) \to 0$ as $n \to \infty$ and is satisfied by the fact that the i-th component of the n-th element of the sequence, $u_i^{(n)}(\lambda)$, is such that for large enough n,

$$|u_i^{(n)}(\lambda)| = \left| \frac{P_i(\lambda)}{\sqrt{P_1^2(\lambda) + \cdots + P_n^2(\lambda)}} \right| \leq \frac{|P_i(\lambda)|}{c_0(\lambda)\sqrt{n}},$$

by Lemma 1.2.2, which tends to zero as $n \longrightarrow \infty$. This completes the proof. □

This gives the following description for the essential spectrum of the DSO:

Corollary 1.2.5. *For the DSO we have that*

$$[-2, 2] \subseteq \sigma_{ess}(J_0).$$

Proof. By Proposition 1.2.1 and Lemma 1.2.4 we have that the open interval $(-2, 2)$ belongs to the essential spectrum. The closure follows from the fact that the essential spectrum is always closed (see, for example, Theorem VII.9 in [69]). □

It will be shown later in Section 1.4 that the closed interval $[-2, 2]$ is indeed the whole of the essential spectrum for the DSO. But, first, we present an alternative technique that will reproduce the results of this section.

1.3 Gilbert-Pearson theory

In the previous section the essential spectrum of the DSO was discussed. However, the techniques employed there, although valid, are too specific to be generalized to arbitrary periods, which will be one of the aims of this thesis. Thus, before moving onto any other aspect of the DSO spectrum, here we explore an alternative approach to describing the essential spectrum of a Jacobi operator which will also hold for the Hermitian periodic Jacobi operator, J_T where

$$J_T := \begin{pmatrix} b_1 & a_1 & & & & & & & & & & & \\ a_1 & b_2 & a_2 & & & & & & & & & & \\ & a_2 & b_3 & a_3 & & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & & \\ & & & a_{T-1} & b_T & a_T & & & & & & & \\ & & & & a_T & b_1 & a_1 & & & & & & \\ & & & & & a_1 & b_2 & a_2 & & & & & \\ & & & & & & \ddots & \ddots & \ddots & & & & \\ & & & & & & & a_{T-1} & b_T & a_T & & & \\ & & & & & & & & a_T & b_1 & a_1 & & \\ & & & & & & & & & a_1 & b_2 & a_2 & \\ & & & & & & & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.16}$$

with $a_i > 0, b_i \in \mathbb{R}$ for all i. It should be stressed that when we refer to periodic Jacobi operators we are specifically referring to Hermitian operators like J_T even if the Hermitian requirement is not explicitly stated.

Firstly, consider the theory of transfer matrices. As has already been seen, the study of eigenvectors of Jacobi matrices is inextricably linked with three-term recurrence relations, specifically the initial condition

$$(b_1 - \lambda)u_1 + a_1 u_2 = 0, \tag{1.17}$$

and the formal spectral equation

$$a_{n-1}u_{n-1} + (b_n - \lambda)u_n + a_n u_{n+1} = 0, n \geq 2. \tag{1.18}$$

Transfer matrices provide a way to capitalise on this relationship and permit the computation of the next component in the solution (u_n) if the preceding two are already known. In particular, (1.18) is equivalent to

$$\begin{aligned}\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{a_{n-1}}{a_n} & \frac{\lambda-b_n}{a_n} \end{pmatrix}\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \\ &= B_n(\lambda)\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \\ &= B_n(\lambda)B_{n-1}(\lambda)B_{n-2}(\lambda)\dots B_2(\lambda)B_1(\lambda)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}\end{aligned} \tag{1.19}$$

for some matrices $B_i(\lambda)$.

Remark By defining $u_0 := 0$ we see that both (1.17) and (1.18) are satisfied by the transfer matrices. Moreover, to obtain a non-zero solution $(u_n)_{n\geq 1}$ to recurrence relations (1.17) and (1.18), it is a necessary condition that $u_1 \neq 0$, and we also observe that changing the size of u_1 by a non-zero factor, C, simply changes all the other components in the vector solution by the same factor C. Thus, without loss of generality set $u_1 := 1$. However, in many cases we will consider (1.18) without (1.17) and therefore not make the assumption that $u_0 = 0, u_1 = 1$.

In the case of the DSO

$$B_i(\lambda) \equiv \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}$$

for all i, and so (1.19) can be more explicitly stated as

$$\begin{aligned}\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}\begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}^n \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.\end{aligned} \tag{1.20}$$

Remark It should be stressed that the transfer matrix technique does not guarantee the decay that we need for an $l^2(\mathbb{N};\mathbb{C})$ solution, or, in fact, any decay at all; it only produces a vector that solves the system of recurrence-relationships demanded by a candidate eigenvector.

Now that we understand the mechanisms with which we may find a solution to the recurrence relations, we define what it means for a solution to (1.18) to be subordinate:

Definition 1.3.1. *Let $\underline{u} := (u_n)_{n\geq 1}$ be a non-trivial solution to* (1.18) *for a given $\lambda \in \mathbb{R}$. Then (u_n) is said to be subordinate if and only if*

$$\lim_{N\to\infty} \frac{\|\underline{u}\|_N}{\|\underline{v}\|_N} = 0, \text{ where } \|\underline{x}\|_N = \sqrt{\sum_{n=1}^{N} |x_n|^2},$$

for any solution $\underline{v} := (v_n)_{n\geq 1}$ of Equation (1.18) *not a constant multiple of $\underline{u}$.*

Remark Simply because the solution decays does not mean that it inhabits the sequence space l^2, which will be needed for the subordinate solution to become an eigenvector and the corresponding λ an eigenvalue. Indeed, for the subordinate solution to become an eigenvector the initial condition (1.17) must also be satisfied. We will see in Chapters 3 and 4 (where we deal with embedded eigenvalues) the difficulties that arise when trying to impose the initial conditions on subordinate solutions. Note that in these chapters we do not encode the initial conditions into the transfer by setting $u_0 := 0$ since the method involves first understanding the asymptotic behaviour of a solution and then working back through the series of entries to the initial components, rearranging the recurrence relationships to compute the necessary values.

According to Gilbert-Pearson theory [21,39,40] a detailed description of the spectral structure for periodic Jacobi operators can be inferred from the existence or non-existence of subordinate solutions to the formal spectral equation (1.18). We reproduce an abridged version of this result tailored to our situation now (see, for example, Theorem 3 in [40] for more detail).

Theorem 1.3.2. *Let J_T be a Hermitian periodic Jacobi operator and*

$$\mathscr{S}_{ac} = \{\lambda \in \mathbb{R} : \textit{ no subordinate solution of the recurrence relation } (1.18) \textit{ exists}\}.$$

Then

$$\sigma_{a.c.}(J_T) = \overline{\mathscr{S}_{a.c.}}$$

where $\sigma_{a.c.}$ is the absolutely continuous spectrum.

To illustrate the above let us consider again the DSO. We now see that we could have instead diagonalised the matrix in Equation (1.20) to obtain

$$\begin{aligned} \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} &= \left(V(\lambda) \begin{pmatrix} \mu_+(\lambda) & 0 \\ 0 & \mu_-(\lambda) \end{pmatrix} V^{-1}(\lambda) \right)^n \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ &= V(\lambda) \begin{pmatrix} \mu_+^n(\lambda) & 0 \\ 0 & \mu_-^n(\lambda) \end{pmatrix} V^{-1}(\lambda) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{aligned} \tag{1.21}$$

where $\mu_\pm(\lambda) = \frac{\lambda \pm \sqrt{\lambda^2-4}}{2}$ and the columns of $V(\lambda)$ are the eigenvectors corresponding to $\mu_+(\lambda), \mu_-(\lambda)$ respectively. Consequently, for $\lambda \in (-2, 2)$,

$$\mu_-(\lambda) = \overline{\mu_+(\lambda)} = \mu_+(\lambda)^{-1}, |\mu_\pm(\lambda)| = 1,$$

i.e the eigenvalues, $\mu_\pm(\lambda)$, of the transfer matrix are complex conjugates and reside on the unit circle. They will now be denoted as $\mu(\lambda), \overline{\mu(\lambda)}$. Then for any two arbitrary non-zero solutions to the spectral equation (1.18), $\underline{u} := (u_n), \underline{v} := (v_n)$, we have

$$u_n(\lambda) = c_1(\lambda)\mu^n(\lambda) + c_2(\lambda)\overline{\mu^n(\lambda)}, v_n(\lambda) = c_3(\lambda)\mu^n(\lambda) + c_4(\lambda)\overline{\mu^n(\lambda)},$$

for complex constants $c_1(\lambda), c_2(\lambda), c_3(\lambda), c_4(\lambda)$ where $|c_1(\lambda)| + |c_2(\lambda)| > 0$ and $|c_3(\lambda)| + |c_4(\lambda)| > 0$. Clearly, since $|\mu^n(\lambda)| = |\overline{\mu^n(\lambda)}| = 1$, we have that

$$|u_n(\lambda)| \leq |c_1(\lambda)| + |c_2(\lambda)|, |v_n(\lambda)| \leq |c_3(\lambda)| + |c_4(\lambda)| =: k_1(\lambda), \tag{1.22}$$

for all n, and furthermore by (1.21) we have

$$\begin{aligned}
\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} &= \left(V(\lambda)\begin{pmatrix} \mu_+(\lambda) & 0 \\ 0 & \mu_-(\lambda) \end{pmatrix} V^{-1}(\lambda)\right)^{-n} \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \\
&= V(\lambda)\begin{pmatrix} \overline{\mu}^n(\lambda) & 0 \\ 0 & \mu^n(\lambda) \end{pmatrix} V^{-1}(\lambda) \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix}
\end{aligned}$$

which implies

$$\left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\| \leq \|V(\lambda)\| \|V^{-1}(\lambda)\| \left\| \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \right\|.$$

Therefore,

$$\left\| \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} \right\| \geq \|V(\lambda)\|^{-1} \|V^{-1}(\lambda)\|^{-1} \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|,$$

in particular $k_2(\lambda) \leq |u_n|^2 + |u_{n+1}|^2$ for all n where $k_2(\lambda)$ is strictly positive providing u_0, u_1 are not both 0.

This implies that

$$\lim_{N\to\infty} \frac{\|\underline{u}\|_{2N}}{\|\underline{v}\|_{2N}} \geq \lim_{N\to\infty} \sqrt{\frac{Nk_2(\lambda)}{Nk_1(\lambda)}} \tag{1.23}$$

for non-zero constants $k_1(\lambda)$(from (1.22)) and $k_2(\lambda)$, and so (1.23) cannot tend to zero as n tends to infinity. Therefore, for the DSO, there are no subordinate solutions for $\lambda \in (-2, 2)$, and so by Theorem 1.3.2

$$[-2, 2] \subseteq \sigma_{a.c.}(J_0). \tag{1.24}$$

Note that the closed interval appeared above, rather than the open, because the a.c. spectrum is always a closed set.

Remark Given that an eigenvector is just a special kind of subordinate solution, the calculations above which establish the absence of any subordinate solutions in the interval $(-2, 2)$ also tells us that there are no eigenvalues here either.

Let $M(\lambda)$ denote the monodromy matrix for a T-periodic Jacobi operator, which is defined to be the product of the corresponding periodic transfer matrices, i.e. $M(\lambda) := B_T(\lambda) \dots B_1(\lambda)$. We obtain the following characterisation for $\lambda \in \mathbb{C}$ depending on the eigenvalues of the monodromy matrix, $M(\lambda)$.

Definition 1.3.3. *Let J_T be a periodic Jacobi operator and $M(\lambda)$ be the associated monodromy matrix with $\lambda \in \mathbb{C}$. The hyperbolic points are those λ that produce a monodromy matrix with two real eigenvalues, μ_1, μ_2 where $|\mu_1| > 1$ and $|\mu_2| < 1$. The elliptic points are those λ that produce a monodromy matrix with two distinct complex eigenvalues, $\mu, \overline{\mu}$ of modulus one. The parabolic points are those λ that produce a monodromy matrix with one eigenvalue, equal to either 1 or -1, with algebraic multiplicity 2.*

We can also use the trace of the monodromy matrix to characterise the hyperbolic, elliptic, and parabolic points.

Lemma 1.3.4. *For $\lambda \in \mathbb{C}$ we can distinguish the hyperbolic, elliptic and parabolic points by $|\operatorname{Tr}(M(\lambda)| > 2, |\operatorname{Tr}(M(\lambda))| < 2, |\operatorname{Tr}(M(\lambda))| = 2$, respectively.*

Proof. Given that the determinant of a matrix is equal to the product of its eigenvalues and

$$\begin{aligned}\det M(\lambda) &= \det\left(B_T(\lambda)B_{T-1}\dots B_2(\lambda)B_1(\lambda)\right)\\ &= \det(B_T(\lambda))\det(B_{T-1}(\lambda))\dots\det(B_2(\lambda))\det(B_1(\lambda))\\ &= \frac{a_{T-1}}{a_T}\frac{a_{T-2}}{a_{T-1}}\dots\frac{a_1}{a_2}\frac{a_T}{a_1} = 1\end{aligned}$$

for all $\lambda \in \mathbb{C}$, we have that the product of the eigenvalues of $M(\lambda)$ is always equal to 1. In particular, if μ_1, μ_2 are the eigenvalues of $M(\lambda)$, then $\mu_1\mu_2 = 1$ and therefore $\mu_2 = \mu_1^{-1}$. If $\mu_1 \in \mathbb{C} \setminus \mathbb{R}$ then recalling that μ_1 is an eigenvalue and therefore a root of a quadratic equation in λ (with real coefficients since other than λ the entries in $M(\lambda)$ are real) we have that μ_2 must be the complex conjugate of μ_1 with modulus 1. Consequently, from the fact that the trace of a matrix gives the sum of its eigenvalues, we have that in this case $|\operatorname{Tr}(M(\lambda))| = |2\operatorname{Re}(\mu_1)| < 2$. If μ_1, μ_2 are real then there are two cases. Either, $|\mu_1| = |\mu_2| = 1$ with $\mu_1 = \mu_2$, in which case $|\operatorname{Tr}(M(\lambda))| = |\mu_1 + \mu_2| = 2$; or $|\mu_1| > 1, |\mu_2| < 1$ which implies $|\operatorname{Tr}(M(\lambda))| = |\mu_1 + \frac{1}{\mu_1}| > 2$. ☐

Consequently, for those λ that are elliptic and parabolic, i.e. $|\operatorname{Tr}(M(\lambda))| \leq 2$, we can use a technique similar to the one that was used to obtain (1.24) and show that λ lies in the absolutely continuous spectrum, $\sigma_{a.c.}(J_T)$. For those λ that are hyperbolic, i.e. $|\operatorname{Tr}(M(\lambda))| > 2$, we have that λ lies in the resolvent or the point spectrum, $\rho(J_T) \cup \sigma_p(J_T)$. Indeed as $\lambda \mapsto \operatorname{Tr} M(\lambda)$ is continuous, the a.c. spectrum for a period-T Jacobi operator appears in bands:

$$\sigma_{a.c.}(J_T) = \{\lambda \in \mathbb{R} \mid \operatorname{Tr}(M(\lambda)) \leq 2\}. \tag{1.25}$$

Observe that the parabolic points belong to $\sigma_{a.c.}(J_T)$ by Theorem 1.3.2. However, it should be stressed that we may have subordinate solutions here, although if we do they are non-decaying (see Section 1.4 for a more detailed discussion of this in a particular example). Consequently, since an eigenvalue demands a decaying subordinate solution we see that an eigenvalue can only occur inside an interval of hyperbolic points for a periodic Jacobi operator. This leads to the following:

Lemma 1.3.5. *For a Hermitian periodic Jacobi operator, J_T, there exist at most two eigenvalues in any interval of hyperbolic points and each has geometric multiplicity one.*

Proof. The result rests on the fact that we can construct a 2-sided periodic Jacobi operator, J_d, where

$$J_d := \begin{pmatrix} \ddots & \ddots & \ddots & & & & & \\ & a_{T-2} & b_{T-1} & a_{T-1} & & & & \\ & & a_{T-1} & b_T & a_T & & & \\ & & & a_T & b_1 & a_1 & & \\ & & & & a_1 & b_2 & a_2 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

i.e. an extended Jacobi operator which acts on $l^2(\mathbb{Z};\mathbb{C})$ rather than $l^2(\mathbb{N};\mathbb{C})$, from the Hermitian periodic operator J_T (its contribution in blue) using the so-called Glazman procedure. In particular,

$$J_d = J_T \oplus J_u + R_2,$$

where J_u is unitarily equivalent to a (1-sided) Hermitian periodic Jacobi operator (its contribution in red) and R_2 is a Hermitian self-adjoint rank-2 perturbation (its contribution in black). Clearly, J_d has no eigenvalues since any solution to the eigenvalue equation would be decaying in one direction, but expanding in the other, and so $\sigma_p(J_d) = \emptyset$. Moreover,

$$\sigma_p(J_T \oplus J_u) = \sigma_p(J_T) \cup \sigma_p(J_u),$$

which implies

$$\sigma_p(J_d - R_2) = \sigma_p(J_T) \cup \sigma_p(J_u).$$

Also, as the hyperbolic intervals are gaps in the essential spectrum, a rank-n perturbation can at most add n eigenvalues along any interval of hyperbolic points (see, for example, Theorem 3 of Chapter 9 in [7]). Therefore, since the hyperbolic intervals of J_u and J_T are the same (due to their respective monodromy matrices being inverses of each other) we have that on any hyperbolic interval of J_T there are at most 2 eigenvalues. The details of the geometric multiplicity follow from the fact that due to the recurrence relationships involved as soon as the first component of the eigenvector is defined so are all subsequent ones. $\square$

Remark Although it is not actually proved in [40], it does follows from their working that for a Hermitian period-T Jacobi operator, J_T,

$$\sigma_{s.c.}(J_T) = \emptyset,$$

where $\sigma_{s.c.}$ denotes the singular continuous spectrum. Specifically, we have from Theorem 3 in [40] that the essential support for the singular measure, $\mathscr{S}_s$, is composed of those $\lambda \in \mathbb{R}$ such that a subordinate solution of the recurrence relation (1.18) exists and satisfies the initial condition (1.17). Then, by (1.21), we have that the only possibility for subordinate solutions is either at the end points of the bands of a.c. spectrum or in the intervals of hyperbolic points (see Definition 1.3.3). However, we see in the remark to Theorem 2.1.9 that there are at most $2T$ bands of absolutely continuous spectrum (which gives $4T$ end points with possible subordinate solutions) and also at most $2T+2$ eigenvalues from the collection of hyperbolic intervals. This means that $\mathscr{S}_s$ is finite (at most $6T+2$ entries in total); however, a necessary condition for the singular continuous spectrum to be non-empty is that the cardinality of the essential support for the singular measure be uncountable (see, for example, the discussion following Theorem 3 in [21]). Thus, we deduce that $\sigma_{s.c.}(J_T) = \emptyset$.

This leads to the following important corollary.

Corollary 1.3.6. *For a Hermitian periodic Jacobi operator, J_T, we have*

$$\sigma_{a.c.}(J_T) = \sigma_{ess}(J_T).$$

Proof. The absolutely continuous spectrum, singular continuous spectrum, eigenvalues of infinite multiplicity as well as the accumulation points of any eigenvalues comprise the essential spectrum. By the remark following Lemma 1.3.5 we know there is no singular spectrum, and by Lemma 1.3.5 there are no eigenvalues of infinite multiplicity. □

It is because of the above corollary why we will often use the terms 'essential spectrum' and 'absolutely continuous spectrum' interchangeably. We now introduce the final definition of the section.

Definition 1.3.7. *For a Hermitian periodic Jacobi operator, J_T, we define the generalised interior of the essential spectrum, $\sigma_{ell}(J_T)$, to be the set of elliptic points, i.e.*

$$\sigma_{ell}(J_T) = \{\lambda \in \mathbb{R} \mid \operatorname{Tr} M(\lambda) < 2\}.$$

For $\lambda \in \sigma_{ell}(J_T)$, the eigenvalues $\mu_\pm$ of the monodromy matrix, $M(\lambda)$, are such that $\mu_\pm = e^{\pm i\theta(\lambda)}$, where the function $\theta(\lambda)$ is called the quasi-momentum, and has range $(0, \pi)$.

This brings us to the following proposition which easily follows from Equation 3.10 in [16] and which we will use often throughout the thesis.

Proposition 1.3.8. *The quasi-momentum, $\theta(\lambda)$, is a strictly monotonic function on each band of the generalised interior of the essential spectrum.*

1.4 Absence of point spectrum for the DSO

In this section we describe the spectrum of the DSO and show that it is purely absolutely continuous. We also illustrate the discussion in Section 1.3 about the absence of decaying subordinate solutions in the closed interval [-2,2] by showing that though we obtain subordinate solutions at the parabolic points $\{-2, 2\}$, they are, indeed, non-decaying.

Lemma 1.4.1. *The DSO has spectrum*

$$\sigma(J_0) = [-2, 2]$$

and is purely absolutely continuous. In particular, the point spectrum is empty.

Proof. By Lemmas 1.1.1 and 1.1.4, J_0 is Hermitian and bounded (with $\|J_0\| \leq 2$) and therefore $\sigma(J_0)$ is a closed region of $[-2, 2]$. By Lemmas 1.2.5 and 1.3.6 we have that

$$\sigma_{a.c.}(J_0) = \sigma_{ess}(J_0) = [-2, 2]$$

and therefore $\sigma(J_0)$ must be the entire closed interval, i.e.

$$\sigma(J_0) = [-2, 2].$$

Additionally, by arguments in Section 1.3 which prove the absence of any subordinate solutions in the interval $(-2, 2)$, we know there are no eigenvalues in $(-2, 2)$. We now show that there are no decaying subordinate solutions at the parabolic points 2 and -2 and therefore no eigenvalues anywhere in the spectrum.

First consider the case when $\lambda = 2$ and define the matrix

$$B := \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

Then from the recurrence relations

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}^n \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

However, elementary calculations show that the eigenvalue 1 of matrix B has algebraic multiplicity 2 and geometric multiplicity 1, meaning that the matrix cannot be diagonalised. Here, though, the Jordan-Normal form will suffice, (see, for example, Section 3.3.2 in [17]) requiring the following two relations to be satisfied

$$(B - I)\vec{v}_1 = 0, \tag{1.26}$$

$$(B - I)\vec{f}_1 = \vec{v}_1, \tag{1.27}$$

where $\vec{v}_1$ is the existing eigenvector and $\vec{f}_1$ is the root vector we are trying to infer. Then, rather than diagonalising B, one can instead obtain

$$\begin{aligned} B &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}, \quad \text{so} \\ B^n &= \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \right)^n \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1-n & n \\ -n & n+1 \end{pmatrix}. \end{aligned}$$

Consequently, there are two linearly independent solutions, $u_n = 1, v_n = n$, to the formal spectral equation, (1.18), for the DSO when $\lambda = 2$. Clearly, $u_n = 1$ is the only subordinate solution; however, it is non decaying and therefore there is no possibility for an eigenvector. Alternatively, we could have recalled that without loss of generality the initial components can be set to $u_0 = 0, u_1 = 1$, which would have given the particular non-decaying solution $u_n = n$. Thus, $\lambda = 2$ is not an eigenvalue

When $\lambda = -2$ a similar method shows that the linearly independent solutions to the formal spectral equation are $u_n = (-1)^n, v_n = (-1)^{n-1}n$. Again, there is only one subordinate solution, and it is non-decaying so there is no possibility for an eigenvalue here. $\square$

Remark The argument used in Lemma 1.4.1 can be extended and applied to the parabolic points of any Hermitian period-T Jacobi operator to establish that there are no decaying subordinate solutions there. Indeed, the parabolic points by definition are those λ such that the eigenvalues of the monodromy matrix, $M(\lambda)$, are both 1 or -1. Then, if the eigenvalue of the monodromy matrix has geometric multiplicity 1 we repeat the procedure outlined in the proof of Lemma 1.4.1 and obtain similar results. If the eigenvalue of the monodromy matrix, instead, has geometric multiplicity 2 then the monodromy matrix is equivalent to either I, the identity matrix, or $-I$ and we obtain two non-decaying linearly independent solutions. Therefore in this case no subordinate solutions exist.

1.5 Rank-one perturbations for the DSO

Here the effect of including a single entry along the diagonal is considered on the spectrum of the DSO. This produces a new operator J_b where

$$J_b := \begin{pmatrix} b & 1 & & & & & \\ 1 & 0 & 1 & & & & \\ & 1 & 0 & 1 & & & \\ & & 1 & 0 & 1 & & \\ & & & 1 & 0 & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1.28}$$

and $b \in \mathbb{R}$.

Lemma 1.5.1. *The essential spectrum for the new operator, J_b is the same as that of J_0, but there is now a new eigenvalue $\lambda = b + \frac{1}{b}$, providing $|b| > 1$.*

Proof. (Step One) The perturbation acting on the DSO will not affect the essential spectrum. This follows from the fact that the perturbation is finite-rank and therefore compact, and the result that states the addition of a compact operator has no effect on the essential spectrum (see, for example, Theorem 5.35 in [39]).

(Step Two) We now consider the effect of the perturbation on points, λ, outside of the essential spectrum and consider if λ satisfies the conditions to be an eigenvalue. Firstly, consider $\lambda > 2$. If this is an eigenvalue the exact solution needs to inhabit the sequence space $l^2(\mathbb{N};\mathbb{C})$. From (1.7) and (1.9)

$$u_n(\lambda) = C(\lambda)\mu_+^n(\lambda) + D(\lambda)\mu_-^n(\lambda), \quad \mu_\pm(\lambda) := \frac{\lambda \pm \sqrt{\lambda^2-4}}{2},$$

and $C(\lambda), D(\lambda)$ are determined using the new initial conditions:

1. $u_1 = 1$,
2. $b + u_2 = \lambda$.

Furthermore, since $\mu_+^n(\lambda) \to \infty$ as $n \to \infty$ and $\mu_-^n(\lambda) = \frac{1}{\mu_+^n}$ (i.e. there is no chance of $\mu_-^n(\lambda)$ cancelling out the growth in $\mu_+^n(\lambda)$) this means for $u(\lambda)$ to possibly reside in $l^2(\mathbb{N};\mathbb{C})$, the coefficient $C(\lambda)$ must be zero, and therefore $D(\lambda) = \frac{1}{\mu_-(\lambda)}$. Then,

$$\begin{aligned} b + u_2 = \lambda &\iff b + \left(\frac{\lambda - \sqrt{\lambda^2-4}}{2}\right) = \lambda \\ &\iff 2b - \sqrt{\lambda^2-4} = \lambda, \end{aligned}$$

i.e.

$$2b > \lambda. \tag{1.29}$$

Proceeding with the simplification gives

$$\begin{aligned} 2b - \sqrt{\lambda^2-4} = \lambda \Rightarrow (2b-\lambda)^2 &= \lambda^2 - 4 \\ \iff \lambda^2 - 4b\lambda + 4b^2 &= \lambda^2 - 4 \\ \iff \lambda &= \frac{1}{b} + b. \end{aligned}$$

Additionally $\lambda > 2$ implies by (1.29) that $b > 1$. Thus, since $C(\lambda) = 0$ we have that $u(\lambda) \in l^2(\mathbb{N};\mathbb{R})$ and is the eigenvector corresponding to the eigenvalue $\lambda = \frac{1}{b} + b$.

(Step Three) Consider the second interval, $\lambda < -2$. This is similar to Step Two except now $D(\lambda) = 0$. Then

$$\begin{aligned} b + A_+ = \lambda &\iff b + \left(\frac{\lambda + \sqrt{\lambda^2-4}}{2}\right) = \lambda \\ &\iff \sqrt{\lambda^2-4} = \lambda - 2b, \end{aligned}$$

i.e.

$$\lambda > 2b. \tag{1.30}$$

Proceeding with the simplification gives

$$\begin{aligned} \sqrt{\lambda^2-4} = \lambda - 2b \Rightarrow \lambda^2 - 4 &= (\lambda - 2b)^2 \\ \iff \lambda^2 - 4 &= \lambda^2 - 4b\lambda + 4b^2 \\ \iff \lambda &= \frac{1}{b} + b. \end{aligned}$$

Additionally, $\lambda < -2$ implies by (1.30) that $b < -1$. Thus, since $D(\lambda) = 0$ we have that $u(\lambda) \in l^2(\mathbb{N}; \mathbb{R})$ and is the eigenvector corresponding to the eigenvalue $\lambda = \frac{1}{b} + b$. $\square$

Remark The essential spectrum remaining unchanged also follows from the fact that although the operator is no longer periodic, it is periodic after the first row and therefore the unperturbed monodromy matrix can still be used to find a solution to the recurrence relations for all u_n where $n \geq 2$. Consequently, the asymptotics remain unchanged; in particular the diagonalisation of the monodromy matrix in the elliptic interval is unaffected and therefore the theory of Gilbert-Pearson (Theorem 1.3.2) is still applicable.

Remark All finite-rank perturbations are compact and therefore had the diagonal entries ranged from $b_1, \dots, b_n$ the essential spectrum would have still remained unchanged. However, as will be seen in Chapters 3, 4, and 5 this is not necessarily the case when the perturbations are no longer of finite rank.

1.6 Absence of embedded eigenvalues for the perturbed DSO

One of the objectives of this thesis is to explore new ways in which we can change the spectrum of periodic Jacobi operators by using a perturbation. More specifically, we wish to embed an eigenvalue into one of the operator's bands of essential spectrum with the addition of a potential down the diagonal; this particular element of the spectrum belonging to both $\sigma_{a.c.}$ and σ_p, simultaneously. Without a potential these conditions are mutually exclusive since by Theorem 1.3.2, the a.c. spectrum appears in bands, i.e. for a purely periodic operator, J_T, $\lambda \in \sigma_{a.c.}(J_T) \Rightarrow \operatorname{Tr} M(\lambda) \leq 2$ and $\lambda \in \sigma_p(J_T) \Rightarrow \operatorname{Tr} M(\lambda) > 2$. However, with the presence of the perturbation, the new operator, J_q, now ceases to be periodic and therefore there may be some points in the closure of the a.c. interval which do possess a subordinate solution, and therefore possibly an eigenvector.

The following result gives a condition for the absence of embedded eigenvalues for the perturbed DSO.

Theorem 1.6.1. *Define J_q as the Discrete-Schrödinger operator with potential (q_n), i.e.*

$$J_q := \begin{pmatrix} q_1 & 1 & & & & & \\ 1 & q_2 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & q_n & 1 & & \\ & & & 1 & q_{n+1} & 1 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then for $(q_n) \in l^1(\mathbb{N};\mathbb{C})$,

$$\sigma_p(J_q) \cap (-2,2) = \emptyset.$$

The proof of this result requires some subsidiary lemmas which will be stated now.

Lemma 1.6.2. *Let* $\lambda \in (-2,2)$*. Define* $V_n(\lambda)$ *and* $V(\lambda)$ *as the matrix of eigenvectors for the transfer matrices* $B(\lambda - q_n) := \begin{pmatrix} 0 & 1 \\ -1 & \lambda - q_n \end{pmatrix}$ *and* $B(\lambda) := \begin{pmatrix} 0 & 1 \\ -1 & \lambda \end{pmatrix}$*, respectively, where* $B(\lambda - q_n)$ *corresponds to the perturbed and* $B(\lambda)$*, the unperturbed Jacobi operator, with* $q_n \to 0$ *as* $n \to \infty$*. Then* $V(\lambda)$ *is invertible and, for sufficiently large* n*, we have*

$$V_n(\lambda) = V(\lambda) + R_n(\lambda)$$

where $\|R_n(\lambda)\| \leq C(\lambda)|q_n|$*,* $C(\lambda)$ *is uniform in* n*.*

Proof. The eigenvector matrix associated with $B(\lambda)$ is

$$V(\lambda) = \begin{pmatrix} 1 & 1 \\ \frac{\lambda+\sqrt{\lambda^2-4}}{2} & \frac{\lambda-\sqrt{\lambda^2-4}}{2} \end{pmatrix}.$$

The determinant of this matrix is $-\sqrt{\lambda^2-4}$ which is not equal to zero for all $\lambda \in (-2,2)$ and $V(\lambda)$ is therefore invertible.

Similarly, for sufficiently large n, the eigenvector matrix associated with $B(\lambda - q_n)$ is

$$V_n(\lambda) = \begin{pmatrix} 1 & 1 \\ \frac{(\lambda-q_n)+\sqrt{(\lambda-q_n)^2-4}}{2} & \frac{(\lambda-q_n)-\sqrt{(\lambda-q_n)^2-4}}{2} \end{pmatrix}.$$

Thus

$$V_n(\lambda) = V(\lambda) + R_n(\lambda)$$

where $R_n(\lambda) := \begin{pmatrix} 0 & 0 \\ \frac{-q_n-\sqrt{\lambda^2-4}+\sqrt{(\lambda-q_n)^2-4}}{2} & \frac{-q_n+\sqrt{\lambda^2-4}-\sqrt{(\lambda-q_n)^2-4}}{2} \end{pmatrix}$.

Now observe that

$$\begin{aligned}\sqrt{(\lambda-q_n)^2-4} &= \sqrt{\lambda^2-4-2\lambda q_n+q_n^2} \\ &= \sqrt{\lambda^2-4}\sqrt{1+\frac{q_n^2-2\lambda q_n}{\lambda^2-4}}\end{aligned}$$

and, also,

$$\sqrt{1+x} = 1 + \frac{x}{2} + O\left(x^2\right),$$

as $x \to 0$, using the Taylor expansion for the function $f(z) = \sqrt{z}$ about 1. Consequently, using this and the fact that $q_n \to 0$ we obtain

$$\sqrt{1 + \frac{q_n^2 - 2\lambda q_n}{\lambda^2 - 4}} = 1 - \frac{\lambda q_n}{\lambda^2 - 4} + O(q_n^2).$$

Note that the radius of convergence for the Taylor expansion is of no concern here since the error term only explodes when x approaches -1 which, by choosing n large enough, isn't an issue.

Thus

$$\sqrt{(\lambda - q_n)^2 - 4} = \sqrt{\lambda^2 - 4} - \frac{\lambda q_n}{\sqrt{\lambda^2 - 4}} + O(q_n^2)$$

and so

$$R_n(\lambda) = \begin{pmatrix} 0 & 0 \\ -\left(\frac{1}{2} + \frac{\lambda}{\sqrt{\lambda^2 - 4}}\right) q_n + O(q_n^2) & \left(-\frac{1}{2} + \frac{\lambda}{\sqrt{\lambda^2 - 4}}\right) q_n + O(q_n^2). \end{pmatrix} \tag{1.31}$$

Furthermore, setting

$$\|R_n(\lambda)\| := \max\{|R_{11}(\lambda)|, |R_{12}(\lambda)|, |R_{21}(\lambda)|, |R_{22}(\lambda)|\},$$

then

$$\begin{aligned} \|R_n(\lambda)\| &\leq \left(\frac{1}{2} + \left|\frac{\lambda}{\sqrt{\lambda^2 - 4}}\right|\right) |q_n| + \widetilde{C}(\lambda)|q_n|^2 \\ &\leq \left(\frac{1}{2} + \left|\frac{\lambda}{\sqrt{\lambda^2 - 4}}\right| + k\widetilde{C}(\lambda)\right) |q_n| \end{aligned}$$

where $\widetilde{C}(\lambda)$ is independent in n and, k is the upper bound for (q_n) (which exists since $q_n \to 0$). Choosing

$$C(\lambda) := \frac{1}{2} + \left|\frac{\lambda}{\sqrt{\lambda^2 - 4}}\right| + k\widetilde{C}(\lambda)$$

gives the result. □

Corollary 1.6.3. *Letting $q_n \to 0$, as $n \to \infty$, and using the same notation as in Lemma 1.6.2, we have that $V_n^{-1}(\lambda)$ exists for n sufficiently large, and*

$$V_n^{-1}(\lambda) = V^{-1}(\lambda) + R'_n(\lambda),$$

where $\|R'_n(\lambda)\| \leq C'(\lambda)|q_n|$, and $C'(\lambda)$ is uniform in n.

Proof. (Step One) By Lemma 1.6.2 it is known that for n sufficiently large

$$V_n(\lambda) = V(\lambda) + R_n(\lambda)$$

which is equivalent to

$$V_n(\lambda) = (I + R_n(\lambda)V^{-1}(\lambda))V(\lambda).$$

Then this implies

$$V_n^{-1}(\lambda) = V^{-1}(\lambda)(I + R_n(\lambda)V^{-1}(\lambda))^{-1}.$$

The expression $(I+R_n(\lambda)V^{-1}(\lambda))$ is invertible when $\|R_n(\lambda)V^{-1}(\lambda)\| < 1$, which happens for large enough n since, by Lemma 1.6.2, $\|R_n(\lambda)\| \leq C(\lambda)|q_n|$ and q_n goes to zero as $n \to \infty$. Then, by the theory of Neumann series

$$\left(I + R_n(\lambda)V^{-1}(\lambda)\right)^{-1} = \sum_{i=0}^{\infty} \left(-R_n(\lambda)V^{-1}(\lambda)\right)^i$$

for large enough n and so $V_n(\lambda)$ is invertible for large enough n.

(Step Two) It still remains to be shown that $V_n^{-1}(\lambda)$ can be expressed in the form as stated in the corollary. From the previous step we see that

$$\begin{aligned} V_n^{-1}(\lambda) &= V^{-1}(\lambda)\left(I + R_n(\lambda)V^{-1}(\lambda)\right)^{-1} \\ &= V^{-1}(\lambda) + R'_n(\lambda), \end{aligned}$$

where $R'_n(\lambda) := V^{-1}(\lambda) \sum_{i=1}^{\infty} \left(-R_n(\lambda)V^{-1}(\lambda)\right)^i$.

(Step Three) Finally, a bound on $R'_n(\lambda)$ needs to be established:

$$\begin{aligned} \|R'_n(\lambda)\| &= \left\| -V^{-1}(\lambda)R_n(\lambda)V^{-1}(\lambda) \sum_{i=0}^{\infty} \left(-R_n(\lambda)V^{-1}(\lambda)\right)^i \right\| \\ &\leq \|V^{-1}(\lambda)\|C(\lambda)|q_n|\|V^{-1}(\lambda)\| \frac{1}{1 - C(\lambda)|q_n|\|V(\lambda)\|} \\ &\leq C'(\lambda)|q_n|, \end{aligned}$$

where $C'(\lambda) := 2C(\lambda)\|V^{-1}(\lambda)\|^2$, and n is chosen large enough such that

$$1 - C(\lambda)|q_n|\|V^{-1}(\lambda)\| \geq \frac{1}{2}.$$

Note that because of Lemma 1.6.2 the requirement for n to be sufficiently large still holds. Moreover, n may be even larger in order for $R_n(\lambda)$ to be small enough so that the Neumann series exists. □

Corollary 1.6.4. *Letting $q_n \to 0$, as $n \to \infty$, and using the same notation as in Lemma 1.6.2, we have*

$$V_n^{-1}(\lambda)V_{n-1}(\lambda) = I + R''_n(\lambda)$$

where $\|R''_n(\lambda)\| \leq C''(\lambda)(|q_n| + |q_{n-1}|)$ for sufficiently large n, and $C''(\lambda)$ is uniform in n.

Proof. Using Lemma 1.6.2 and Corollary 1.6.3 gives
$V_{n-1}(\lambda) = V(\lambda) + R_{n-1}(\lambda)$ and $V_n^{-1}(\lambda) = V^{-1}(\lambda) + R_n'(\lambda)$. Then

$$\begin{aligned}
V_n^{-1}(\lambda)V_{n-1}(\lambda) &= (V^{-1}(\lambda) + R_n'(\lambda))(V(\lambda) + R_{n-1}(\lambda)) \\
&= I + V^{-1}(\lambda)R_{n-1}(\lambda) + R_n'(\lambda)V(\lambda) + R_n'(\lambda)R_{n-1}(\lambda) \\
&= I + R_n''(\lambda),
\end{aligned}$$

where $R_n'' := V^{-1}(\lambda)R_{n-1}(\lambda) + R_n'(\lambda)V + R_n'(\lambda)R_{n-1}(\lambda)$.

To show that $R_n''(\lambda)$ is bounded by q_n and q_{n-1}, use that for sufficiently large n:

$$\begin{aligned}
\|R_n''(\lambda)\| &= \|V^{-1}(\lambda)R_{n-1}(\lambda) + R_n'(\lambda)V(\lambda) + R_n'(\lambda)R_{n-1}(\lambda)\| \\
&\leq \|V^{-1}(\lambda)\|\|R_{n-1}(\lambda)\| + \|R_n'(\lambda)\|\|V(\lambda)\| + \|R_n'(\lambda)\|\|R_{n-1}(\lambda)\| \\
&\leq \|V^{-1}(\lambda)\|C(\lambda)|q_{n-1}| + C'(\lambda)|q_n|\|V(\lambda)\| + C'(\lambda)C(\lambda)|q_n||q_{n-1}| \\
&\leq C''(\lambda)(|q_{n-1}| + |q_n|),
\end{aligned}$$

where $C''(\lambda) := C(\lambda)\|V^{-1}(\lambda)\| + C'(\lambda)\|V(\lambda)\| + C'(\lambda)C(\lambda)$. □

Before stating Lemma 1.6.6, the following result is required:

Lemma 1.6.5. *For $(\alpha_k)_{k=1}^{\infty}$ in $l^2(\mathbb{N};\mathbb{C})$, $|\alpha_k| < 1$ $k \in \mathbb{N}$, we have*

$$\prod_{k=1}^{n}(1+\alpha_k) = (\widetilde{c} + o(1))\, e^{\sum\limits_{k=1}^{n}\alpha_k}$$

as $n \to \infty$ for some real constant $\widetilde{c}$ uniform in n. Moreover,

$$\prod_{k=m}^{n}(1+\alpha_k) = e^{\sum\limits_{k=m}^{n}\alpha_k(1+o(1))},$$

as $m \to \infty$.

Proof. Using the Taylor expansion of the function $\log(1+x)$ for small x we obtain the sequence of relations

$$\begin{aligned}
\prod_{k=1}^{n}(1+\alpha_k) &= e^{\log\left(\prod\limits_{k=1}^{n} 1+\alpha_k\right)} = e^{\sum\limits_{k=1}^{n}\log(1+\alpha_k)} \\
&= e^{\sum\limits_{k=1}^{n}\alpha_k + O\left(\alpha_k^2\right)} = (\tilde{c} + o(1))\, e^{\sum\limits_{k=1}^{n}\alpha_k}
\end{aligned}$$

using that fact that since α_k is square-summable over k, then $f_k = O(\alpha_k^2)$ is also summable. Similarly, using that α_k is square-summable and therefore tends to zero we obtain:

$$e^{\sum\limits_{k=m}^{n}\alpha_k + O\left(\alpha_k^2\right)} = e^{\sum\limits_{k=1}^{n}\alpha_k(1+o(1))}$$

as $m \to \infty$. □

Note that it is the following lemma that demands the l^1-condition stated in Theorem 4.4.

Lemma 1.6.6. *Let U_n, T_n be sequences of $N \times N$ matrices where U_n is unitary, $(\|T_n\|) \in l^1$ and $\|T_n\| < 1$ for all n. Then*

$$U_n(I+T_n)U_{n-1}(I+T_{n-1})\dots U_1(I+T_1)$$

is uniformly bounded in n.

Proof. First note that unitary matrices are distance preserving and therefore $\|U_k x\| = \|x\|$. This implies $\|U_k\| = 1$ for all k. Thus

$$\begin{aligned}\|U_n(I+T_n)\dots U_1(I+T_1)\| &\le \left(\prod_{k=1}^{n}\|U_k\|\right)\left(\prod_{j=1}^{n}\|I+T_j\|\right)\\ &\le \left(\prod_{k=1}^{n}\|U_k\|\right)\left(\prod_{j=1}^{n}(1+\|T_j\|)\right) = \prod_{j=1}^{n}(1+\|T_j\|)\,. \qquad (1.32)\end{aligned}$$

Then, since $(\|T_n\|) \in l^1$, we can apply Lemma 1.6.5 to (1.32) to obtain

$$\begin{aligned}\|U_n(I+T_n)\dots U_1(I+T_1)\| &\le (\widetilde{c}+o(1))\exp\left(\sum_{j=1}^{n}\|T_j\|\right)\\ &\le (\widetilde{c}+o(1))\exp\left(\sum_{j=1}^{\infty}\|T_j\|\right),\end{aligned}$$

which is an upper-bound independent of n. □

With these lemmas and corollaries, Theorem 1.6.1 can now be proven:

Proof of Theorem 1.6.1. Recalling that $(q_n) \in l^1$, we see that the discriminant of the eigenvalue equation for the matrix $B(\lambda - q_n)$, which was defined in Lemma 1.6.2, is $(\lambda^2 - 4) + O(q_n)$. This means that as n gets very large, for $\lambda \in (-2, 2)$, we will be in the elliptic case for $B(\lambda - q_n)$, in particular there will be two distinct non-real eigenvalues on the unit circle. Thus, for large n,

$$B(\lambda - q_n) = V_n(\lambda)D_n(\lambda)V_n^{-1}(\lambda),$$

where $D_n(\lambda)$ is unitary. Moreover, Corollary 1.6.4 implies

$$\begin{aligned}&B(\lambda-q_n)B(\lambda-q_{n-1})\dots B(\lambda-q_N)\\ &= \left(V_n(\lambda)D_n(\lambda)V_n^{-1}(\lambda)\right)\left(V_{n-1}(\lambda)D_{n-1}(\lambda)V_{n-1}^{-1}(\lambda)\right)\dots(V_N(\lambda)D_N(\lambda)V_N^{-1}(\lambda))\\ &= V_n(\lambda)D_n(\lambda)(V_n^{-1}(\lambda)V_{n-1}(\lambda))D_{n-1}(\lambda)(V_{n-1}^{-1}(\lambda)V_{n-2}(\lambda))\dots D_N(\lambda)V_N^{-1}(\lambda)\\ &= V_n(\lambda)D_n(\lambda)(I+R_n''(\lambda))D_{n-1}(\lambda)(I+R_{n-1}''(\lambda))\dots(I+R_{N+1}''(\lambda))D_N(\lambda)V_N^{-1}(\lambda)\end{aligned} \qquad (1.33)$$

for large enough N and $n > N$. Consequently, using Lemma 1.6.2 to give a uniform bound on $V_n(\lambda)$ for large enough n, and Lemma 1.6.6 to give a uniform bound on the product

$$D_n(\lambda)(I + R''_n(\lambda))D_{n-1}(\lambda)(I + R''_{n-1}(\lambda)) \dots (I + R''_{N+1}(\lambda))D_N(\lambda)$$

for large enough N, (we recall that (q_n) belongs to the sequence space l^1 and so by Corollary 1.6.4 $(\|R''_n(\lambda)\|)_{n\in\mathbb{N}} \in l^1$) we can deduce from Equation (1.33) that the norm of the product of the matrices $B(\lambda - q_n)B(\lambda - q_{n-1}) \dots B(\lambda - q_1)$ is bounded above as $n \to \infty$.

Furthermore, for large enough n, V_n is invertible, D_n has a unitary inverse, and by Neumann series

$$(I + R''_n)^{-1} = I - R''_n + (R''_n)^2 - (R''_n)^3 + \dots,$$

with

$$\begin{aligned} \| - R''_n + (R''_n)^2 - (R''_n)^3 + \dots \| &\leq \|R''_n\| \sum_{i=0}^{\infty} \|R''_n\|^i \\ &\leq 2C''(\lambda)\left(|q_n| + |q_{n-1}|\right) \end{aligned}$$

where the last inequality was derived using Corollary 1.6.4 and choosing n large enough such that $\|R''_n\| \leq \frac{1}{2}$. Recalling that $(q_n) \in l^1$, this implies that $(I + R''_n)^{-1} = I + T_n$ where $(\|T_n\|) \in l^1$. Thus we can take the inverse of (1.33) and use Corollary 1.6.3 (to give a uniform bound on $V_n^{-1}(\lambda)$) and Lemma 1.6.6 to show that the norm of the inverse of the product of the matrices is also bounded from above. Consequently, the norm of the product is bounded both above and away from zero uniformly in n. Then, for all $\vec{x} \in \mathbb{C}^2 \setminus \{(0,0)\}$,

$$0 < k_x^{(1)} \leq \|B(\lambda - q_n)B(\lambda - q_{n-1}) \dots B(\lambda - q_2)B(\lambda - q_1)\vec{x}\| \leq k_x^{(2)}$$

for all n, with $k_x^{(1)}, k_x^{(2)} \in \mathbb{R}^+$. For arbitrary solutions to the spectral equation, $\underline{u} := (u_n)_{n=1}^{\infty}, \underline{v} := (v_n)_{n=1}^{\infty}$, we then have that there exists $k_1, k_2 \in \mathbb{R}^+$ such that

$$k_1 \leq |u_n|^2 + |u_{n+1}|^2 \text{ and } |v_n|^2 + |v_{n+1}|^2 \leq k_2, \forall n.$$

This implies

$$\lim_{N\to\infty} \frac{\|\underline{u}\|_{2N}}{\|\underline{v}\|_{2N}} \geq \lim_{N\to\infty} \sqrt{\frac{Nk_1}{Nk_2}} > 0$$

and so by Definition 1.3.1 there are no subordinate solutions in the interval $(-2, 2)$ for the operator J_q. Thus, there can be no eigenvalues in this interval, either. $\square$

Chapter 2

Periodic Jacobi operators

Linear second-order differential equations with periodic coefficients abound in nature, one of many such examples being Hill's work on the lunar perigee [26]. Innovative investigations by Floquet [19] and Lyapunov [57] explored many of the fundamental principles governing this rich and still flourishing area of mathematics, and laid the groundwork that generations of other researchers have balanced their breakthroughs on. In this chapter, the discrete analogue of these types of equations is discussed. Here the Jacobi operators, J_T, we study will be Hermitian and T-periodic and thus have the same form as in (1.16)

2.1 The spectrum of a period-T Jacobi operator

Before discussing more general Hermitian periodic Jacobi operators, the absolutely continuous spectrum of the period-2 Jacobi operator, J_2, with zero diagonal (i.e. $b_i \equiv 0$) will be described. This information will be used in later chapters to illustrate certain techniques with numerical examples.

Lemma 2.1.1. *The periodic Jacobi operator (with zero diagonal) J_2 has the property*

$$[-(a_1 + a_2), -|a_1 - a_2|] \cup [|a_1 - a_2|, (a_1 + a_2)] = \sigma_{a.c.}(J_2).$$

Proof. Let $\underline{u} = (u_n)_{n \geq 1}$ be such that

$$(J_2 - \lambda)\underline{u} = \underline{0}.$$

Then using the theory of transfer matrices and monodromy matrices (see Section 1.3) it can be deduced that

$$\begin{aligned}
\begin{pmatrix} u_{2n} \\ u_{2n+1} \end{pmatrix} &= (B_2(\lambda)B_1(\lambda))^n \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\
&= \left(\begin{pmatrix} 0 & 1 \\ -\frac{a_1}{a_2} & \frac{\lambda}{a_2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{a_2}{a_1} & \frac{\lambda}{a_1} \end{pmatrix} \right)^n \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{-a_2}{a_1} & \frac{\lambda}{a_1} \\ \frac{-\lambda}{a_1} & \frac{-a_1}{a_2} + \frac{\lambda^2}{a_1 a_2} \end{pmatrix}^n \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},
\end{aligned}$$

where $u_0 := 0$ and without loss of generality $u_1 := 1$. Simple calculations show that the eigenvalues, $\mu_\pm(\lambda)$, of the monodromy matrix, $M(\lambda) := B_2(\lambda)B_1(\lambda)$, are

$$\mu_\pm(\lambda) := \frac{-(a_1^2 + a_2^2 - \lambda^2) \pm \sqrt{(a_1^2 + a_2^2 - \lambda^2)^2 - 4a_2^2 a_1^2}}{2a_1 a_2}. \tag{2.1}$$

Then

$$\begin{aligned}
&|\mathrm{Tr}\, M(\lambda)| = |\mu_+(\lambda) + \mu_-(\lambda)| \leq 2 \\
\iff\ & (a_1^2 + a_2^2 - \lambda^2)^2 - 4a_2^2 a_1^2 \leq 0 \\
\iff\ & \lambda \in [-a_1 - a_2, -|a_1 - a_2|] \cup [|a_1 - a_2|, a_1 + a_2].
\end{aligned}$$

By (1.25) the result is obtained. □

Henceforth in this thesis, we consider arbitrary periodic Jacobi operators with possible non-zero diagonal entries ($b_1, b_2, \ldots, b_T \in \mathbb{R}$). Here we discuss some results regarding the structure of an arbitrary T-periodic Jacobi operator's spectrum, although first it is necessary to restrict ourselves to the case when $b_i \equiv 0$. We will generalise this result in Lemma 2.1.3

Lemma 2.1.2. *When $b_i \equiv 0$ the spectrum of a Hermitian period-T Jacobi operator, J_T, is such that*

$$\sigma(J_T) \subseteq [-\max\{a_T + a_1, \ldots, a_{T-1} + a_T\}, \max\{a_T + a_1, \ldots, a_{T-1} + a_T\}].$$

Moreover the operator is self-adjoint and its spectrum is symmetric about the origin.

Proof. The self-adjointness of J_T follows from Lemma 1.1.4, and therefore the spectrum is real. Moreover, the symmetry of the spectrum follows from observing that

$$J_T = D^{-1}(-J_T)D$$

where

$$D := \begin{pmatrix} 1 & 0 & & & & \\ 0 & -1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & 0 & -1 & 0 & \\ & & & 0 & 1 & 0 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

In particular, J_T is unitarily equivalent to $-J_T$ and so $\sigma(J_T) = \sigma(-J_T)$. Consequently, by invoking the Spectral Mapping Theorem (see, for example, Theorem 7.4-2 in [44]) the result is obtained.

We now consider the numerical range of the periodic Jacobi operator J_T, i.e. the set $W(J_T) = \{\langle J_T\underline{x}, \underline{x}\rangle, \underline{x} \in l^2(\mathbb{N};\mathbb{C}), \|\underline{x}\| = 1\}$. Letting

$$\underline{x} := (x_1, x_2, \dots, x_n, x_{n+1}, \dots)^T,$$

with $x_0 := 0$, simple calculations give that

$$\begin{aligned} |\langle J_T\underline{x}, \underline{x}\rangle| &\leq \left|\sum_{i=1}^{\infty} (a_{i-1}x_{i-1} + a_i x_{i+1})\,\overline{x}_i\right| \\ &\leq \sum_{i=1}^{\infty} (a_{i-1}|x_{i-1}||x_i| + a_i|x_{i+1}||x_i|) \\ &\leq \frac{1}{2}\sum_{i=1}^{\infty} \left(a_{i-1}|x_{i-1}|^2 + a_{i-1}|x_i|^2 + a_i|x_i|^2 + a_i|x_{i+1}|^2\right) \end{aligned}$$

where the last inequality was obtained using the inequality $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$. Recalling that $x_0 = 0$, we collect terms and re-label indices to obtain

$$\begin{aligned} |\langle J_T\underline{x}, \underline{x}\rangle| &\leq \frac{1}{2}\sum_{i=1}^{\infty}(a_{i-1} + a_i)|x_i|^2 + \frac{1}{2}\sum_{i=1}^{\infty}(a_{i-1} + a_i)|x_i|^2 \\ &\leq \sum_{i=1}^{\infty}(a_{i-1} + a_i)|x_i|^2 \leq \max\{a_T + a_1, \dots, a_{T-1} + a_T\}\sum_{i=1}^{\infty}|x_i|^2, \end{aligned}$$

where the final inequality was obtained using the fact that the a_i coefficients are periodic. Then using this and the result that states the closure of the numerical range contains the spectrum of J_T (see, for example Theorem 1.2-1 in [23]), we see that for $\lambda \in \sigma(J_T)$, $|\lambda| \leq \max\{a_T + a_1, \dots, a_{T-1} + a_T\}$. Furthermore, since the spectrum is real and symmetric about zero, we obtain the result

$$\sigma(J_T) \subset [-\max\{a_T + a_1, \dots, a_{T-1} + a_T\}, \max\{a_T + a_1, \dots, a_{T-1} + a_T\}]. \qquad \square$$

Remark Since J_T is similar to $-J_T$ we also have that the point spectrum (the set of eigenvalues) of J_T is symmetric about the origin. This follows from the fact that if λ is an eigenvalue of J_T with eigenvector $\underline{v}$,

$$D^{-1}(-J_T)D\underline{v} = \lambda\underline{v} \iff J_T D\underline{v} = -\lambda D\underline{v},$$

i.e. $-\lambda$ is an eigenvector of J_T. A similar argument implies that the essential spectrum is also symmetric.

Now we can state a result describing the spectrum of an arbitrary period-T Jacobi operator, i.e. a Jacobi operator with possible non-zero periodic diagonal.

Lemma 2.1.3. *The spectrum of a Hermitian period-T Jacobi matrix, J_T, is such that*

$$\sigma(J_T) \subseteq [-\max\{a_T + a_1, \ldots, a_{T-1} + a_T\} + \min\{b_1, \ldots, b_T\}, \\ \max\{a_T + a_1, \ldots, a_{T-1} + a_T\} + \max\{b_1, \ldots, b_T\}].$$

Moreover, the operator is self-adjoint (although the spectrum can now no longer be guaranteed to be symmetric about zero.)

Proof. The self-adjointness of J_T follows from Lemma 1.1.4. Now, let $J_T := J_{zero} + B$, where J_{zero} is the periodic Jacobi operator with zero diagonal and B is the infinite diagonal matrix with periodic entries, $b_1, \ldots, b_T$. Moreover, define $J_{max} = J_{zero} + diag(b_{max})$ and $J_{min} := J_{zero} + diag(b_{min})$, where $diag(b_{max})$ and $diag(b_{min})$ are infinite diagonal matrices with entries b_{max} (the maximum entry from B) and b_{min} (the minimum entry from B), respectively. Then

$$J_{min} \leq J \leq J_{max},$$

i.e. $\langle J_{min}u, u\rangle \leq \langle Ju, u\rangle \leq \langle J_{max}u, u\rangle$ for all $u \in l^2(\mathbb{N}; \mathbb{C})$. Thus by the fact that the end points of the closure of the numerical range are included in the spectrum (see, for example, Theorem 1.2-4 in [23]) this implies

$$\min(\sigma(J)) \geq \min(\sigma(J_{min})), \max(\sigma(J)) \leq \max \sigma(J_{max}),$$

where $\min \sigma(X)$ and $\max \sigma(X)$ denote the minimum and maximum entries of the spectrum of the bounded self-adjoint operator X, respectively. Finally, by employing the Spectral Mapping Theorem we see that

$$\sigma(J_{min}) = \sigma(J_{zero}) + b_{min}, \sigma(J_{max}) = \sigma(J_{zero}) + b_{max}$$

and invoking Lemma 2.1.2 (to give the maximum and minimum on the spectral values of J_{zero}) gives the result. $\square$

The next lemma gives information about the entries of the monodromy matrix associated to J_T.

Lemma 2.1.4. *Let*

$$A_s = \begin{pmatrix} 0 & 1 \\ c_s & \frac{\lambda - b_s}{a_s} \end{pmatrix}$$

with a_s, b_s, c_s real and $a_s, c_s \neq 0$ for all $s \in \mathbb{N}$. Let $m \in \mathbb{N}$ and

$$A(\lambda) = \begin{pmatrix} a_{11}(\lambda) & a_{12}(\lambda) \\ a_{21}(\lambda) & a_{22}(\lambda) \end{pmatrix} = \prod_{s=1}^{m} A_s.$$

Then, for $m \geq 1$,

$$a_{11}(\lambda) = c_1 \frac{\lambda^{m-2}}{\prod_{s=2}^{m-1} a_s} + P_{m-3}(\lambda), \qquad a_{12}(\lambda) = \frac{\lambda^{m-1}}{\prod_{s=1}^{m-1} a_s} + P_{m-2}(\lambda),$$

$$a_{21}(\lambda) = c_1 \frac{\lambda^{m-1}}{\prod_{s=2}^{m} a_s} + \widetilde{P}_{m-2}(\lambda), \qquad \textit{and} \qquad a_{22}(\lambda) = \frac{\lambda^{m}}{\prod_{s=1}^{m} a_s} + P_{m-1}(\lambda),$$

where $P_{m-1}(\lambda), P_{m-2}(\lambda), \widetilde{P}_{m-2}(\lambda)$ *and* $P_{m-3}(\lambda)$ *are real polynomials in* λ *of degree less than or equal to* $m-1, m-2, m-2$ *and* $m-3$*, respectively,* $P_k(\lambda) := 0$.

Proof. We use induction on m, $m \geq 2$. For the base case of $m = 1$ the result clearly holds. For the base case $m = 2$ we have $a_{11} = c_1, a_{12} = \frac{\lambda - b_1}{a_1}, a_{21} = c_1 \frac{\lambda - b_2}{a_2}$ and $a_{22} = c_2 + \frac{(\lambda - b_2)(\lambda - b_1)}{a_2 a_1}$. Immediately, they satisfy the hypothesis. Now assume the result holds up to $m = k$. To prove the result for $m = k + 1$, observe that by induction,

$$\prod_{s=1}^{k+1} \begin{pmatrix} 0 & 1 \\ c_s & \frac{\lambda - b_s}{a_s} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c_{k+1} & \frac{\lambda - b_{k+1}}{a_{k+1}} \end{pmatrix} \prod_{s=1}^{k} \begin{pmatrix} 0 & 1 \\ c_s & \frac{\lambda}{a_s} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ c_{k+1} & \frac{\lambda - b_{k+1}}{a_{k+1}} \end{pmatrix} \begin{pmatrix} c_1 \frac{\lambda^{k-2}}{\prod_{s=2}^{k-1} a_s} + P_{k-3} & \frac{\lambda^{k-1}}{\prod_{s=1}^{k-1} a_s} + P_{k-2} \\ c_1 \frac{\lambda^{k-1}}{\prod_{s=2}^{k} a_s} + \widetilde{P}_{k-2} & \frac{\lambda^{k}}{\prod_{s=1}^{k} a_s} + P_{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \frac{\lambda^{k-1}}{\prod_{s=2}^{k} a_s} + \widetilde{P}_{k-2} & \frac{\lambda^{k}}{\prod_{s=1}^{k} a_s} + P_{k-1} \\ c_1 \frac{\lambda^{k}}{\prod_{s=2}^{k+1} a_s} + \widetilde{P}_{k-1} & \frac{\lambda^{k+1}}{\prod_{s=1}^{k+1} a_s} + P_{k} \end{pmatrix}. \quad \square$$

Remark We alert the reader to the fact that the polynomials $P_{m-1}(\lambda)$, $P_{m-2}(\lambda)$, $P_{m-3}(\lambda)$ in Lemma 2.1.4 also depend on m so that in general $P_{(m+1)-2}(\lambda) \neq P_{m-1}(\lambda)$.

Corollary 2.1.5. *Let* M *be the monodromy matrix for an arbitrary Hermitian period-*T *operator,* J_T*, i.e.*

$$M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} := B_T(\lambda) B_{T-1}(\lambda) \ldots B_1(\lambda)$$

and $B_i(\lambda)$ *are the transfer matrices given by*

$$B_i(\lambda) := \begin{pmatrix} 0 & 1 \\ \frac{-a_{i-1}}{a_i} & \frac{\lambda - b_i}{a_i} \end{pmatrix}, \lambda \in \mathbb{C}, \tag{2.2}$$

where $i = 1, 2, \ldots, T$*, with* $a_0 := a_T$*. Then,* $det(M(\lambda)) = 1$ *and for all* $T \geq 1$

we have

$$m_{11}(\lambda) = -a_T \frac{\lambda^{T-2}}{\prod_{s=1}^{T-1} a_s} + P_{T-3}(\lambda), \qquad m_{12}(\lambda) = \frac{\lambda^{T-1}}{\prod_{s=1}^{T-1} a_s} + P_{T-2}(\lambda),$$

$$m_{21}(\lambda) = -\frac{\lambda^{T-1}}{\prod_{s=1}^{T-1} a_s} + \widetilde{P}_{T-2}(\lambda), \qquad m_{22}(\lambda) = \frac{\lambda^{T}}{\prod_{s=1}^{T} a_s} + P_{T-1}(\lambda),$$

where $P_{T-1}(\lambda), P_{T-2}(\lambda), \widetilde{P}_{T-2}(\lambda)$ *and* $P_{T-3}(\lambda)$ *are real polynomials in* λ *of degree less than or equal to* $T-1, T-2, T-2$ *and* $T-3$*, respectively.*

Remark For the case of $T = 2, 3, 4, 5$ the value of $m_{2,2}(\lambda)$ is

$$\frac{\lambda^2 - a_1^2}{a_1 a_2}, \frac{\lambda^3 - \lambda(a_1^2 + a_2^2)}{a_1 a_2 a_3}, \frac{\lambda^4 - \lambda^2(a_1^2 + a_2^2 + a_3^2)}{a_1 a_2 a_3 a_4},$$

and

$$\frac{\lambda^5 - \lambda^3(a_1^2 + a_2^2 + a_3^2 + a_4^2) + \lambda(a_1^2 a_3^2 + a_1^2 a_4^2 + a_2^2 a_4^2)}{a_1 a_2 a_3 a_4 a_5},$$

respectively.

We now show that all non-real λ belong to the hyperbolic region for an arbitrary period-T Jacobi operator.

Lemma 2.1.6. *All points in* $\mathbb{C}^+ \cup \mathbb{C}^-$ *for an arbitrary Hermitian* T*-periodic Jacobi operator,* J_T*, belong to the hyperbolic region. In particular, all parabolic and elliptic points are real.*

Proof. Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$, which is clearly in the resolvent of the self-adjoint operator J_T, and consider the Weyl vector, f_λ, defined as

$$f_\lambda := (J_T - \lambda)^{-1} e_1$$

where $e_1 = (1, 0, 0, \dots)$. Since λ belongs to the resolvent of J_T, we have that $(J_T - \lambda)^{-1}$ is bounded and therefore the vector f_λ belongs to l^2; in particular, the vector f_λ is decaying. Then

$$(J_T - \lambda) f_\lambda = e_1,$$

which implies that the vector f_λ satisfies the three term recurrence relationship, (1.18), for $n \geq 2$. If λ belongs to either the elliptic or parabolic regions then trivial analysis of the powers of the monodromy matrix leads to the fact that no decaying solution of the recurrence relation (1.18) exists. Thus, since for non-real λ there exists a decaying solution to the recurrence relation we have that λ cannot be an elliptic or parabolic point, and therefore must be a hyperbolic point. $\square$

Remark The above proof also shows that the resolvent of the periodic operator J_T, $\rho(J_T)$, belongs to the hyperbolic region.

Later arguments will involve comparing the order of certain rational functions. We formally define what this means now.

Definition 2.1.7. *Consider a rational function in the variable x of the form $\frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ are polynomials. The order of the rational function is defined to be the difference in degree of the polynomials $P(x)$ and $Q(x)$: $\deg P \; - \; \deg Q$.*

The following result will not surprise specialists in the area, but to the best of our knowledge there is no proof in the literature. Of course, it is a folklore-type result.

Lemma 2.1.8. *We consider a family of Hermitian period-T Jacobi operators*

$$J_{\epsilon,\eta} := \begin{pmatrix} b_1+\epsilon & a_1+\eta & & & & & & \\ a_1+\eta & b_2 & a_2 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & a_{T-1} & b_T & a_T & & & \\ & & & a_T & b_1+\epsilon & a_1+\eta & & \\ & & & & a_1+\eta & b_2 & a_2 & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.3}$$

depending on the two parameters ϵ and η. Then there exists an open dense set D in $\mathbb{R}^2$ such that for all $(\epsilon,\eta) \in D$ the essential spectrum of $J_{\epsilon,\eta}$ consists of T distinct real intervals.

Proof. We will refer to the situation that $\sigma_{ess}(J_{\epsilon,\eta})$ consists of T distinct real intervals as the non-degenerate case. The proof will consist of two parts: We show that non-degeneracy is stable under small perturbations, while on the other hand the degenerate case is not stable. We initially introduce some notation.

Define the transfer matrices for $J_{\epsilon,\eta}$ as

$$B_i(\lambda) := \begin{pmatrix} 0 & 1 \\ \frac{-a_{i-1}}{a_i} & \frac{\lambda-b_i}{a_i} \end{pmatrix},$$

for $i = 3, 4, \dots, T$, and

$$B_{1,\epsilon,\eta}(\lambda) := \begin{pmatrix} 0 & 1 \\ \frac{-a_T}{a_1+\eta} & \frac{\lambda-b_1-\epsilon}{a_1+\eta} \end{pmatrix}, \quad B_{2,\epsilon,\eta}(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{a_1+\eta}{a_2} & \frac{\lambda-b_2}{a_2} \end{pmatrix}.$$

Let M_0 be the monodromy matrix for $J_{0,0}$, i.e. $M_0 := B_T B_{T-1} \dots B_{2,0,0} B_{1,0,0}$, and $M_{\epsilon,\eta}$ the monodromy matrix for $J_{\epsilon,\eta}$, i.e.

$$M_{\epsilon,\eta} := B_{T,0,0} B_{T-1,0,0} \dots B_{3,0,0} B_{2,\epsilon,\eta} B_{1,\epsilon,\eta} = M_0 B_{1,0,0}^{-1} B_{2,0,0}^{-1} B_{2,\epsilon,\eta} B_{1,\epsilon,\eta}. \tag{2.4}$$

By Corollary 2.1.5 we have

$$M_0(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix}, \tag{2.5}$$

where $m_{11}(\lambda), m_{12}(\lambda), m_{21}(\lambda)$ and $m_{22}(\lambda)$ are real polynomials in λ of order $T-2$, $T-1, T-1$ and T respectively. Then, as $\lambda \mapsto (Tr(M_0(\lambda)) \pm 2)$ are two polynomials each of degree T in λ, there are at most $2T$ real zeros of these functions, providing at most T intervals of a.c. spectrum. Recall from Lemma 2.1.6 that all of the parabolic points for J_T are real.

(Step One) It needs to be shown that if $J_{0,0}$ is non-degenerate, then adding sufficiently small ϵ, η to the operator does not cause two previously distinct parabolic points to overlap. The argument is simple: For each pair of distinct parabolic points (λ_j, λ_k) of $J_{0,0}$ there exists $\delta_{j,k} > 0$ such that for the corresponding parabolic points of $J_{\epsilon,\eta}$ we have $\lambda_j(\epsilon, \eta) \neq \lambda_k(\epsilon, \eta)$ for $|\epsilon|, |\eta| < \delta_{j,k}$, using the fact that the roots of a polynomial depend continuously on its coefficients. (See, for example, Appendix A in [68].) Then, since there are at most $2T$ parabolic points in total, we can define

$$\delta := \min_{j,k} \delta_{j,k} > 0$$

which implies

$$\lambda_m(\epsilon, \eta) \neq \lambda_n(\epsilon, \eta)$$

for all $|\epsilon|, |\eta| < \delta, m, n \in \{1, \ldots, 2T\}$, $m \neq n$. This shows that the non-degenerate case is stable.

(Step Two) We now show that the case where two of the intervals of essential spectrum of $J_{0,0}$ overlap is unstable. Let λ_0 be a parabolic point for $J_{0,0}$. We will only consider the case when $\mathrm{Tr}(M_0(\lambda_0)) = 2$, the case $\mathrm{Tr}(M_0(\lambda_0)) = -2$ can be dealt with similarly. Assume that $\lambda_0 \notin \partial\sigma_{ess}(J_{0,0})$. Then $\frac{d}{d\lambda}\mathrm{Tr}(M_0(\lambda_0)) = 0$, otherwise $\mathrm{Tr}(M_0(\lambda)) - 2$ would change sign at λ_0 and λ_0 would separate the elliptic and hyperbolic regions, implying $\lambda_0 \in \partial\sigma_{ess}(J_{0,0})$.

We now show that in most cases a diagonal perturbation is sufficient to split the overlapping intervals. Assume that

$$|m_{21}(\lambda_0)| + |m'_{21}(\lambda_0)| \neq 0, \tag{2.6}$$

where m'_{21} denotes the derivative of the polynomial m_{21} with respect to λ. Let λ (depending on ϵ) be a degenerate parabolic point for some $J_{\epsilon,0}$, i.e.

$$\mathrm{Tr}(M_{\epsilon,0}(\lambda)) = 2 \quad \text{and} \quad \frac{d}{d\lambda}\mathrm{Tr}(M_{\epsilon,0}(\lambda)) = 0. \tag{2.7}$$

We assume for contradiction that (2.7) holds for sufficiently small $\lambda - \lambda_0$ and ϵ. Due to continuous dependence of the roots on the small parameter ϵ, there is no need to consider the case $\mathrm{Tr}(\mathrm{M}_{\epsilon,0}(\lambda)) = -2$. Also, (2.6) will hold (for the same polynomial m_{21} from M_0) with λ_0 replaced by λ in a sufficiently small neighbourhood of λ_0. Noting that $B_{2,\epsilon,0}$ is independent of ϵ we get that

$$\begin{aligned} M_{\epsilon,0} &= B_T B_{T-1} \dots B_{2,0,0}\Bigg(B_{1,0,0} - \frac{\epsilon}{a_1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Bigg) \\ &= M_0\Bigg(I - \frac{\epsilon}{a_1} B_{1,0,0}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Bigg). \end{aligned}$$

As

$$B_{1,0,0}^{-1}(\lambda) = \begin{pmatrix} \frac{\lambda - b_1}{a_T} & -\frac{a_1}{a_T} \\ 1 & 0 \end{pmatrix},$$

we get

$$\mathrm{Tr}(M_{\epsilon,0}(\lambda)) = m_{11}(\lambda) + m_{22}(\lambda) + \frac{\epsilon m_{21}(\lambda)}{a_T}. \tag{2.8}$$

Now, Equations (2.8) and (2.7) combine to give the new conditions

$$m_{11}(\lambda) + m_{22}(\lambda) + \frac{\epsilon m_{21}(\lambda)}{a_T} = 2 \tag{2.9}$$

and

$$m'_{11}(\lambda) + m'_{22}(\lambda) + \frac{\epsilon m'_{21}(\lambda)}{a_T} = 0. \tag{2.10}$$

If Equations (2.9) and (2.10) are both satisfied then we obtain

$$(2 - m_{11}(\lambda) - m_{22}(\lambda))m'_{21}(\lambda) + (m'_{12}(\lambda) + m'_{22}(\lambda))m_{21}(\lambda) = 0. \tag{2.11}$$

We observe that the product of polynomials on the left hand side equals

$$2m'_{21}(\lambda) + \left(\frac{m_{11}(\lambda) + m_{22}(\lambda)}{m_{21}(\lambda)}\right)' m_{21}^2(\lambda). \tag{2.12}$$

By Corollary 2.1.5 the term $2m'_{21}(\lambda)$ is a polynomial of degree $(T-2)$, and the term $m_{21}^2(\lambda)$ is a polynomial of degree $2(T-1)$. Moreover, since

$$\begin{aligned} \frac{m'_{22}(\lambda)m_{21}(\lambda) - m_{22}(\lambda)m'_{21}(\lambda)}{m_{21}^2(\lambda)} &= \left(-\frac{a_T}{\prod_{s=1}^{T} a_s^2}\right)\left(\prod_{s=1}^{T-1} a_s^2\right) + O\left(\lambda^{2T-3}\right) \\ &= -\frac{1}{a_T} + O\left(\frac{1}{\lambda}\right), \end{aligned}$$

the rational function $\left(\frac{m_{11}(\lambda)+m_{22}(\lambda)}{m_{21}(\lambda)}\right)'$ is of order 0. Combining these observations we have that the entire last term of the expression in (2.12) is a polynomial of degree $2(T-1)$. Since $2(T-1)$ is greater than $(T-2)$ the whole expression has degree $2(T-1)$. Clearly, this is not identically zero. Furthermore, this means there are at most $2T-2$ roots, say $\mu_1, \ldots, \mu_{2T-2}$, which are independent of ϵ. Then, since under our assumptions $|m_{21}(\lambda)| + |m_{21}'(\lambda)| \neq 0$, we calculate the valid values for ϵ (valid in the sense that (2.7) is not contradicted) by substituting $\lambda := \mu_i$ into either Equation (2.9) or (2.10), and, so, there are at most $2T-2$ valid values for ϵ. In particular, for any sufficiently small $\epsilon \neq 0$, the value λ cannot be a degenerate parabolic point for $J_{\epsilon,0}$. Therefore, we have a contradiction (i.e. (2.7) cannot hold for sufficiently small $\lambda - \lambda_0, \epsilon$) and all degenerate parabolic points satisfying (2.6) will be split into non-degenerate points for $|\epsilon| \neq 0$ sufficiently small.

It remains to deal with the exceptional case $m_{21}(\lambda_0) = m_{21}'(\lambda_0) = 0$. In this case, we use a perturbation with $\epsilon = 0$, $\eta \neq 0$. Note that since $\mathrm{Tr}(M_0(\lambda_0)) = 2$ and $\det M_0(\lambda_0) = 1$, we have that $m_{21}(\lambda_0) = 0$ implies $m_{11}(\lambda_0) = m_{22}(\lambda_0) = 1$. Then

$$B_{1,0,0}^{-1} B_{2,0,0}^{-1} B_{2,0,\eta} B_{1,0,\eta} = \begin{pmatrix} \frac{a_1}{a_1+\eta} & \frac{\lambda - b_1}{a_T}\left(\frac{a_1+\eta}{a_1} - \frac{a_1}{a_1+\eta}\right) \\ 0 & \frac{a_1+\eta}{a_1} \end{pmatrix}$$

and using (2.4) we get

$$\mathrm{Tr} M_{0,\eta}(\lambda) = m_{11}(\lambda)\frac{a_1}{a_1+\eta} + m_{21}(\lambda)\frac{\lambda - b_1}{a_T}\left(\frac{a_1+\eta}{a_1} - \frac{a_1}{a_1+\eta}\right) + m_{22}(\lambda)\frac{a_1+\eta}{a_1}.$$

Evaluating at λ_0, we get

$$\mathrm{Tr} M_{0,\eta}(\lambda_0) = \frac{a_1}{a_1+\eta} + \frac{a_1+\eta}{a_1} = \frac{a_1^2 + (a_1+\eta)^2}{a_1(a_1+\eta)} = 2 + \frac{\eta^2}{a_1(a_1+\eta)} > 2,$$

for all $|\eta| \neq 0$, so λ_0 is a hyperbolic point for $J_{0,\eta}$ for $\eta \neq 0$. Choosing $|\eta|$ sufficiently small such that no non-degenerate parabolic points can degenerate (see Step One), this implies that replacing $J_{0,0}$ by $J_{0,\eta}$ the total degeneracy of the roots must have decreased by at least one.

Repeating the procedure finitely many times, we can ensure that all roots of $\mathrm{Tr} M_{\epsilon,\eta}(\lambda) - 2$ are simple for sufficiently small non-zero (ϵ, η). Note that in each step, ϵ or η may be chosen arbitrarily small. This shows that in any neighbourhood of $(0,0)$ we can find an (ϵ, η) in D.

By suitably modifying a_1, b_1, the same same argument shows that in all neighbourhoods of any $(\epsilon_0, \eta_0) \in \mathbb{R}^2$ there exists $(\epsilon, \eta) \in D$. Thus, D is dense in $\mathbb{R}^2$. $\square$

As a consequence of the above and using that $\sigma_{ess} = \sigma_{a.c.}$ (by Corollary 1.3.6) we obtain the following:

Theorem 2.1.9. *For a generic choice of parameters $(a_1, \ldots, a_T, b_1, \ldots, b_T) \in (\mathbb{R}^+)^T \times \mathbb{R}^T$ the essential spectrum (which equals the absolutely continuous spectrum) of the associated Hermitian T-periodic Jacobi matrix consists of T distinct real intervals.*

Remark The above theorem gives that at most there are only ever T bands of essential spectrum for a Hermitian period-T Jacobi operator. Consequently, by this and Lemma 1.3.5 (which states that eigenvalues can only appear in the hyperbolic intervals and at most 2 in any one interval) we have that there are at most $2T+2$ eigenvalues for a Hermitian period-T Jacobi operator.

2.2 Absence of embedded eigenvalues for perturbed period-T Jacobi operators

Here we establish the conditions on the potential that prohibit any eigenvalue being embedded in the generalized interior of the essential spectrum.

Theorem 2.2.1. *Define J_q as a periodic Jacobi operator with potential, (q_n), i.e.*

$$J_q := \begin{pmatrix} b_1+q_1 & a_1 & & & & & \\ a_1 & b_2+q_2 & a_2 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_T & b_1+q_{T+1} & a_1 & & \\ & & & a_1 & b_2+q_{T+2} & a_2 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Then for $(q_n) \in l^1(\mathbb{N};\mathbb{C})$,

$$\sigma_p(J_q) \cap \sigma_{ell}(J_T) = \emptyset,$$

where J_T is the unperturbed period-T Jacobi operator from (1.16).

Proof. Firstly, letting $n = Tk$, we have

$$\begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} = M_k(\lambda)M_{k-1}(\lambda)\dots M_1(\lambda)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \tag{2.13}$$

where

$$M_j(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-1}}{a_T} & \frac{\lambda-b_T-q_{jT+T}}{a_T} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\frac{a_{T-2}}{a_{T-1}} & \frac{\lambda-b_2-q_{jT+T-1}}{a_{T-1}} \end{pmatrix}\dots\begin{pmatrix} 0 & 1 \\ -\frac{a_T}{a_1} & \frac{\lambda-b_1-q_{jT+1}}{a_1} \end{pmatrix}. \tag{2.14}$$

Moreover,

$$\begin{aligned} M_j(\lambda) &= \left(\begin{pmatrix} 0 & 1 \\ -\frac{a_{T-1}}{a_T} & \frac{\lambda-b_T}{a_T} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{jT+T}}{a_T} \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 1 \\ -\frac{a_{T-2}}{a_{T-1}} & \frac{\lambda-b_{T-1}}{a_{T-1}} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{jT+T-1}}{a_{T-1}} \end{pmatrix}\right) \\ &\qquad \dots\left(\begin{pmatrix} 0 & 1 \\ -\frac{a_T}{a_1} & \frac{\lambda-b_1}{a_1} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{jT+1}}{a_1} \end{pmatrix}\right) \\ &= \left(B_T(\lambda) - \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{jT+T}}{a_T} \end{pmatrix}\right)\left(B_{T-1}(\lambda) - \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{jT+T-1}}{a_{T-1}} \end{pmatrix}\right)\dots\left(B_1(\lambda) - \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{jT+1}}{a_1} \end{pmatrix}\right) \\ &= M(\lambda) - \Sigma_j(\lambda) + O\left(\frac{1}{j^2}\right), \end{aligned} \tag{2.15}$$

with $M(\lambda)$ equal to the monodromy matrix of the unperturbed Jacobi operator, J_T, and

$$\Sigma_j(\lambda) := \sum_{i=0}^{T-1} B_T(\lambda) B_{T-1}(\lambda) \dots B_{T-(i-1)}(\lambda) \times \begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{T(j+1)-i}}{a_{T-i}} \end{pmatrix} B_{T-(i+1)}(\lambda) \dots B_1(\lambda).$$

Then, for $\lambda \in \sigma_{ell}(J_T)$, with k, N large enough, $N < k$ we have by (2.15)

$$\begin{aligned} \left[\prod_{j=N}^{k} M_j(\lambda)\right] &= \prod_{j=N}^{k} \left[M(\lambda) - \Sigma_j(\lambda) + O\left(\frac{1}{j^2}\right)\right] \\ &= V(\lambda) \left[\prod_{j=N}^{k} D(\lambda)(I - R_j(\lambda))\right] V^{-1}(\lambda) + O\left(\frac{1}{N^2}\right), \end{aligned}$$

where $R_j(\lambda) := V^{-1}(\lambda)\Sigma_j(\lambda)V(\lambda)$, $\|R_j(\lambda)\| \leq \max_{i \in \{0,\dots,T-1\}} |q_{T(j+1)-i}| C(\lambda)$ for some constant $C(\lambda)$, uniform in j, and

$$D(\lambda) := \begin{pmatrix} e^{i\theta(\lambda)} & 0 \\ 0 & e^{-i\theta(\lambda)} \end{pmatrix}.$$

Then, by Lemma 1.6.6, Equation (2.13) is uniformly bounded from above for $\lambda \in \sigma_{ell}(J_T)$ as $k \to \infty$. Furthermore, $D(\lambda), V$ and $(I - R_j(\lambda))$ are invertible (the last by Neumann series, for large enough j) and therefore we can use similar techniques to establish that the inverse of the product $M_k(\lambda)M_{k-1}(\lambda)\dots M_N(\lambda)$ is uniformly bounded from above for large enough k, N, for $\lambda \in \sigma_{ell}(J_T)$. Thus, the original product of matrices in (2.13) is uniformly bounded away from zero for $\lambda \in \sigma_{ell}(J_T)$ when $k \to \infty$, and so there are no subordinate solutions for the operator J_q here, and therefore no eigenvalues (see the proof of Theorem 1.6.1 for more details on this). $\square$

Chapter 3

Embedded eigenvalues using the Wigner-von Neumann approach

The purpose of this chapter is to extend the Wigner-von Neumann method [67] from the continuous case to the discrete analogue of periodic Jacobi operators, and produce a new technique for embedding a single eigenvalue into one of the bands of the periodic operator's essential spectrum. Although there exist several other approaches for embedding eigenvalues into the essential spectrum of both Schrödinger operators and Jacobi matrices the big advantage of the Wigner-von Neumann method is that it gives an explicit, and relatively simple, formula for the potential and eigenvector, even for the periodic case.

The chapter is structured as follows. We make an ansatz for a possible eigenvector (introduced in Section 3.3) and then establish the asymptotics of the potential needed to realize this eigenvector (Section 3.4). Additionally, we must confirm that the subordinate solution constructed in this way also satisfies the initial equations encoded within the Jacobi matrix (Section 3.5), thus giving an embedded eigenvalue. Section 3.1 contains some preliminary results for the construction whilst in Section 3.2 we introduce the function $C(\lambda; T)$, which must be non-zero for the technique to work, and analyze its properties, in particular that it is a rational function of λ.

3.1 Solutions to period-T difference equations

In this section, and the next, the subsidiary functions our eigenvector will depend upon are defined.

Recall $B_j(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{a_{j-1}}{a_j} & \frac{\lambda - b_j}{a_j} \end{pmatrix}$, where $j \in \{1, \dots, T\}$, and

$$M(\lambda) = B_T(\lambda) B_{T-1}(\lambda) \dots B_1(\lambda).$$

Then, if $\lambda \in \sigma_{ell}(J_T)$ we have that $\sigma(M(\lambda)) = \{e^{i\theta(\lambda)}, e^{-i\theta(\lambda)}\}$ for some real-valued function $\theta(\lambda)$ (the quasi-momentum). Therefore there exists an invertible matrix V such that $M = V^{-1} \begin{pmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{pmatrix} V$, where

$$\mu(\lambda) = e^{i\theta(\lambda)}. \tag{3.1}$$

Now, using a discrete analogue of a method used by Stolz (see Theorem 6 in [79]), we obtain the following lemma.

Lemma 3.1.1. *Let $\lambda \in \sigma_{ell}(J_T)$. Then, for any non-zero solution, $(\psi_n)_{n\geq 1}$, to the period-T difference equation, $a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, n > 1$, we have*

$$(\mathrm{Im}(\psi_n))^2 = \eta_s(\lambda) \sin(2(k-1)\theta(\lambda) + \phi_s(\lambda)) + \eta_s(\lambda),$$

where $n = T(k-1)+s$, $s \in \{0, \dots, T-1\}$ and η_s and ϕ_s are both real-functions independent of k, and $\theta(\lambda)$ is given by Equation (3.1).

Proof. Since ψ_n satisfies the difference equation, and $n = T(k-1)+s$ with $s \in \{0, \dots, T-1\}$ we have that for any $\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \in \mathbb{C}^2 \setminus \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} \psi_n \\ \psi_{n+1} \end{pmatrix} &= B_s \dots B_1 M^{k-1} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \\ &= B_s \dots B_1 V \begin{pmatrix} \mu^{k-1}(\lambda) & 0 \\ 0 & \overline{\mu}^{k-1}(\lambda) \end{pmatrix} V^{-1} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \\ &= \begin{pmatrix} \alpha_s(\lambda) e^{i(k-1)\theta(\lambda)} + \beta_s(\lambda) e^{-i(k-1)\theta(\lambda)} \\ \kappa_s(\lambda) e^{i(k-1)\theta(\lambda)} + \chi_s(\lambda) e^{-i(k-1)\theta(\lambda)} \end{pmatrix}, \end{aligned}$$

for some functions $\alpha_s, \beta_s, \kappa_s, \chi_s$ of $\lambda \in \sigma_{ell}(J_T)$ and s. In the case of $s = 0$ we interpret $B_0 \dots B_1$ to equal the identity, and $B_1 \dots B_1 = B_1$. Consequently,

$$\psi_n = \psi_{T(k-1)+s} = \alpha_s(\lambda) e^{i(k-1)\theta(\lambda)} + \beta_s(\lambda) e^{-i(k-1)\theta(\lambda)}.$$

Thus,

$$\mathrm{Im}(\psi_n) = \widetilde{\alpha}_s(\lambda) \sin((k-1)\theta) + \widetilde{\beta}_s(\lambda) \cos((k-1)\theta), \tag{3.2}$$

where $\widetilde{\alpha}_s(\lambda) := \mathrm{Re}(\alpha_s(\lambda)) - \mathrm{Re}(\beta_s(\lambda))$, $\widetilde{\beta}_s(\lambda) := \mathrm{Im}(\alpha_s(\lambda)) + \mathrm{Im}(\beta_s(\lambda))$ are real-valued functions of λ.

Furthermore, using the double-angle formulae, $\sin(2x) = 2\sin(x)\cos(x)$ and $\cos(2x) = \cos^2(x) - \sin^2(x)$, we have

$$\begin{aligned}
(\mathrm{Im}(\psi_n))^2 &= \widetilde{\alpha}_s^2 \sin^2((k-1)\theta) + \widetilde{\beta}_s^2 \cos^2((k-1)\theta) \\
&\quad + \widetilde{\alpha}_s \widetilde{\beta}_s \left(2\sin((k-1)\theta)\cos((k-1)\theta)\right) \\
&= \left(\widetilde{\beta}_s^2 - \widetilde{\alpha}_s^2\right)\cos^2((k-1)\theta) + \widetilde{\alpha}_s\widetilde{\beta}_s \sin(2(k-1)\theta) + \widetilde{\alpha}_s^2 \\
&= \left(\frac{\widetilde{\beta}_s^2 - \widetilde{\alpha}_s^2}{2}\right)\cos(2(k-1)\theta) + \widetilde{\alpha}_s\widetilde{\beta}_s \sin(2(k-1)\theta) + \frac{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}{2} \\
&= \eta_s \sin(2(k-1)\theta + \phi_s) + \eta_s, \qquad (3.3)
\end{aligned}$$

where η_s, ϕ_s are real-valued functions, such that $\eta_s := \frac{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}{2}$ and $\sin\phi_s = \frac{\widetilde{\beta}_s^2 - \widetilde{\alpha}_s^2}{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}$ and $\cos\phi_s = \frac{2\widetilde{\alpha}_s\widetilde{\beta}_s}{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}$. $\square$

Remark By choosing at least one of ψ_0, ψ_1 to belong to $\mathbb{C} \setminus \mathbb{R}$ we get that the vector $(\eta_0, \ldots, \eta_{T-1})$ is non-trivial.

Given that the operator J_T is periodic, we now introduce into our technique the discrete analogue of Floquet solutions (see, for example, Section 1.3 in [8]).

Lemma 3.1.2. *Let $\lambda \in \sigma_{ell}(J_T)$. Then, there exists a particular non-zero solution, $(\varphi_n)_{n\geq 1}$, to the period-T difference equation $a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} = \lambda u_n, n > 1$ which has the property*

$$\varphi_n(\lambda) = \varphi_s(\lambda) e^{i(k-1)\theta(\lambda)} \qquad (3.4)$$

for some non-trivial set of functions $(\varphi_s)_{s=0}^{T-1}$, where $n = T(k-1) + s, s \in \{0, \ldots, T-1\}$.

Proof. Recall from above that $M(\lambda)$ has eigenvalues $e^{\pm i\theta(\lambda)}$, i.e.

$$M(\lambda)\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = e^{i\theta(\lambda)}\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix},$$

for some φ_0, φ_1 not both 0. Define $\varphi_2, \ldots, \varphi_{T-1}$ by

$$\begin{pmatrix} \varphi_s \\ \varphi_{s+1} \end{pmatrix} := B_s B_{s-1} \ldots B_1 \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}.$$

Then using the notation $n = T(k-1) + s$,

$$\begin{aligned}
\begin{pmatrix} \varphi_n \\ \varphi_{n+1} \end{pmatrix} &= B_s \ldots B_1 M^{k-1} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \\
&= B_s \ldots B_1 e^{(k-1)i\theta(\lambda)} \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \\
&= e^{(k-1)i\theta(\lambda)} \begin{pmatrix} \varphi_s \\ \varphi_{s+1} \end{pmatrix}.
\end{aligned}$$

Consequently,

$$\varphi_n = \varphi_s e^{(k-1)i\theta(\lambda)}. \quad \square$$

Remark Henceforth, the eigenvector of the monodromy matrix will be normalized with $\varphi_0 = 1$. Subsequent calculations in Lemma 3.2.2 will confirm the validity of this choice for almost every λ.

3.2 The function $C(\lambda; T)$

In this section we introduce a new, analytic function of λ, $C(\lambda; T)$. This will play an important role in the asymptotic expansion of our eigenvector, (u_n). Its zeros will give values of λ where our construction fails. Here we explore its properties and structure.

Definition 3.2.1. *For $\lambda \in \sigma_{ell}(J_T)$, let $C(\lambda; T) := \mathrm{Re}\left(\sum_{s=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda)\right)$, where φ_s are as in Lemma 3.1.2 and $\varphi_T = e^{i\theta(\lambda)}\varphi_0$.*

Note that Definition 3.2.1 is invariant w.r.t. the choice of branches μ and $\overline{\mu}$ on $\sigma_{ell}(J_T)$. Indeed, since $\lambda \in \sigma_{ell}(J_T) \subset \mathbb{R}$, all matrix elements of $B_s(\lambda), s = 1, 2, \ldots, T$ and $M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix}$ are real polynomials, $\varphi_0 = 1$ and

$$\varphi_1(\lambda) = (\mu - m_{11}(\lambda))\, m_{12}^{-1}(\lambda) \tag{3.5}$$

changes under the transformation $\mu \mapsto \overline{\mu}$ to the complex conjugate function $\varphi_1(\lambda) \mapsto \overline{\varphi}_1(\lambda), \lambda \in \sigma_{ell}(J_T)$. For the last fact the inclusion $\lambda \in \sigma_{ell}(J_T)$ is essential. Hence for all $s = 1, \ldots, T$ $\varphi_s(\lambda) \mapsto \overline{\varphi}_s(\lambda)$ and the expression for $C(\lambda; T)$, $\lambda \in \sigma_{ell}(J_T)$, transforms into

$$\mathrm{Re}\left(\sum_{i=1}^{T} \overline{\varphi}_s(\lambda)\varphi_{s-1}(\lambda)\right) = \mathrm{Re}\left(\overline{\sum_{i=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda)}\right) = C(\lambda; T).$$

Recall, from Definition 2.1.7, what the order of a rational function is defined to be. We use this concept in the following lemma.

Lemma 3.2.2. *The function $C(\lambda; T)$ is a rational function on $\sigma_{ell}(J_T)$ and can be extended, uniquely, as a rational function to $\mathbb{C}$.*

Proof. Let $\lambda \in \sigma_{ell}(J_T)$. First, the special cases of $T = 1$ and $T = 2$ must be considered separately. For $T = 1$, $\varphi_0 = 1$ and $\varphi_1 = \mu$, so

$$C(\lambda; 1) = \mathrm{Re}\,(\varphi_1\overline{\varphi}_0) = \mathrm{Re}(\mu) = \frac{\mathrm{Tr}(M(\lambda))}{2} = \frac{\lambda - b_1}{2a_1}.$$

For $T = 2$ we have

$$M(\lambda) = \begin{pmatrix} -\frac{a_2}{a_1} & \frac{\lambda - b_1}{a_1} \\ -\frac{\lambda - b_2}{a_1} & \frac{(\lambda - b_1)(\lambda - b_2)}{a_1 a_2} - \frac{a_1}{a_2} \end{pmatrix}.$$

By defining $\varphi_0 := 1, \varphi_1$ is such that

$$M(\lambda)\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = e^{i\theta(\lambda)}\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$$

and so

$$\varphi_1 = \frac{a_1\mu + a_2}{\lambda - b_1}.$$

Furthermore, since

$$\mu\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = M(\lambda)\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ \varphi_3 \end{pmatrix},$$

we obtain $\varphi_2 = \mu$. Then, using $\mu\overline{\mu} = 1$,

$$\begin{aligned} C(\lambda; 2) &= \mathrm{Re}\left(\varphi_2\overline{\varphi}_1 + \varphi_1\overline{\varphi}_0\right) \\ &= \mathrm{Re}\left(\mu\frac{a_1\overline{\mu} + a_2}{\lambda - b_1} + \frac{a_1\mu + a_2}{\lambda - b_1}\right) \\ &= \frac{(a_1 + a_2)}{\lambda - b_1}\left(1 + \mathrm{Re}(\mu)\right) = \frac{(a_1 + a_2)}{\lambda - b_1}\left(1 + \frac{\mathrm{Tr}(M(\lambda))}{2}\right) \\ &= \frac{(a_1 + a_2)}{2(\lambda - b_1)a_1a_2}\left((\lambda - b_1)(\lambda - b_2) - (a_1^2 + a_2^2) + 2a_1a_2\right) \\ &= \frac{(a_1 + a_2)}{2(\lambda - b_1)a_1a_2}\left((\lambda - b_1)(\lambda - b_2) - |a_1 - a_2|^2\right). \end{aligned} \tag{3.6}$$

Thus, the assertion holds for both of these cases.

For $T \geq 3$ we define $\varphi_0 := 1$ and follow a similar technique to the case for $T = 2$. Here we see that the normalisation $\varphi_0 = 1$ is valid unless $m_{12}(\lambda) = 0$. Consequently, for $m_{12}(\lambda) \neq 0$,

$$\varphi_1 = \frac{\mu - m_{11}}{m_{12}},$$

where m_{11}, m_{12} are as described in Corollary 2.1.5. Throughout the proof P_k will denote a polynomial of at most degree k, while $R_k, \widetilde{R}_k$ will denote rational functions of order at most k. Using Lemma 2.1.5, again, and a similar calculation as in the case of Lemma 3.1.2, for $s = 2, \ldots, T$ we obtain

$$\varphi_s = \left(\frac{\lambda^{s-1}}{\prod_{j=1}^{s-1} a_j} + \ P_{s-2}\right)\frac{\mu - m_{11}}{m_{12}} + \left(-\frac{a_T\lambda^{s-2}}{\prod_{j=1}^{s-1} a_j} + \ P_{s-3}\right),$$

where P_{-1} and $\widetilde{P}_{-1}$ are both identically zero. However, since $\mu\overline{\mu} = 1$ and

$$\mathrm{Re}(\mu) = \mathrm{Re}(\overline{\mu}) = \frac{\mathrm{Tr}(M(\lambda))}{2},$$

$\mathrm{Re}\left(\varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda)\right)$ is clearly a rational function of λ, so $C(\lambda; T)$ is also a rational function of λ, $\lambda \in \sigma_{ell}(J_T)$. Now we see that $C(\lambda; T)$ is well-defined as an analytic function not only on $\sigma_{ell}(J_T)$, but everywhere on $\mathbb{C}$ except at the roots of $m_{12}(\lambda)$. $\square$

Remark The function $C(\lambda;T)$ only fails to be defined at the zeros of the polynomial $m_{12}(\lambda)$, defined in Corollary 2.1.5. For $\lambda \in \sigma_{ell}(J_T)$ we have $m_{12}(\lambda) \neq 0$ since if $m_{12}(\lambda) = 0$ then the eigenvalues of the monodromy matrix for real λ are real, and as usual their product is 1. Indeed, since $m_{12}(\lambda) = 0$, we have that the monodromy matrix is lower-triangular and therefore $m_{11}(\lambda)$ and $m_{22}(\lambda)$ are the (real) eigenvalues. Thus, λ is either in the hyperbolic or parabolic case, contradicting that $\lambda \in \sigma_{ell}(J_T)$, and so the denominator has no roots in $\sigma_{ell}(J_T)$.

Remark Note that $\mathrm{Im}(\mu)$ is an algebraic but not rational function of λ. Indeed, $\mathrm{Tr}(M(\lambda))$ is a polynomial in λ and $\det(M(\lambda)) = 1$, therefore $\mathrm{Im}(\mu)$ is the square-root of $1 - \left(\frac{\mathrm{Tr}(M(\lambda))}{2}\right)^2$, and so assuming $\mathrm{Im}(\mu) = \frac{p(\lambda)}{q(\lambda)}$ where $p(\lambda), q(\lambda)$ are real polynomials in λ, we obtain

$$\frac{p(\lambda)}{q(\lambda)} = \sqrt{1 - \left(\frac{\mathrm{Tr}(M(\lambda))}{2}\right)^2} \iff \frac{p^2(\lambda)}{q^2(\lambda)} + \left(\frac{\mathrm{Tr}(M(\lambda))}{2}\right)^2 = 1.$$

Thus as λ goes to infinity the leading terms of $(\mathrm{Tr}(M(\lambda))^2$ and $\frac{p^2(\lambda)}{q^2(\lambda)}$ must cancel. However, since the leading terms of each have positive coefficients then this is a contradiction unless both are constants. Therefore, $\mathrm{Im}(\mu)$ is a rational function iff $\mathrm{Tr}\,(M(\lambda))$ is equal to a constant function for $\lambda \in \sigma_{ell}(J_T)$, which we know from Lemma 2.1.5 is never the case.

Our technique for embedding eigenvalues fails for values λ when the function $C(\lambda;T) = 0$. It is important to understand when this situation arises.

Lemma 3.2.3. *Let $b_i = 0$ for all i. Then for all odd T we have the relationship*

$$C(0;T) = 0.$$

Proof. Observe that the transfer matrices are now off-diagonal. Thus, the product of any two transfer matrices is diagonal. In particular,

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix} = \begin{pmatrix} ad & 0 \\ 0 & bc \end{pmatrix}.$$

Furthermore, the product of diagonal matrices is also diagonal:

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} d_3 & 0 \\ 0 & d_4 \end{pmatrix} = \begin{pmatrix} d_1 d_3 & 0 \\ 0 & d_2 d_4 \end{pmatrix}.$$

Consequently, for odd T, we have

$$\begin{aligned} M(\lambda) &= \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-1}}{a_T} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-2}}{a_{T-1}} & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\frac{a_T}{a_1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-1}}{a_T} & 0 \end{pmatrix} \begin{pmatrix} \left(-\frac{a_{T-3}}{a_{T-2}}\right)\left(-\frac{a_{T-5}}{a_{T-4}}\right)\cdots\left(-\frac{a_T}{a_1}\right) & 0 \\ 0 & \left(-\frac{a_{T-2}}{a_{T-1}}\right)\left(-\frac{a_{T-4}}{a_{T-3}}\right)\cdots\left(-\frac{a_1}{a_2}\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \left(-\frac{a_{T-2}}{a_{T-1}}\right)\left(-\frac{a_{T-4}}{a_{T-3}}\right)\cdots\left(-\frac{a_1}{a_2}\right) \\ \left(-\frac{a_{T-1}}{a_T}\right)\left(-\frac{a_{T-3}}{a_{T-2}}\right)\left(-\frac{a_{T-5}}{a_{T-4}}\right)\cdots\left(-\frac{a_2}{a_3}\right)\left(-\frac{a_T}{a_1}\right) & 0 \end{pmatrix}. \end{aligned}$$

Then, the standard normalisation of $\varphi_0 := 1$ and the fact that $\mu = i$ here implies

$$\varphi_1 = \frac{1}{i}\left(-\frac{a_{T-1}}{a_T}\right)\left(-\frac{a_{T-3}}{a_{T-2}}\right)\left(-\frac{a_{T-5}}{a_{T-4}}\right)\cdots\left(-\frac{a_2}{a_3}\right)\left(-\frac{a_T}{a_1}\right);$$
$$\varphi_2 = -\frac{a_T}{a_1}; \varphi_3 = -\frac{a_1}{a_2}\varphi_1; \ldots ; \varphi_j = -\frac{a_{j-2}}{a_{j-1}}\varphi_{j-2},$$

and so

$$\varphi_j = \begin{cases} 1, & \text{if } j = 0, \\ \frac{1}{i}\left(-\frac{a_{T-1}}{a_T}\right)\left(-\frac{a_{T-3}}{a_{T-2}}\right)\cdots\left(-\frac{a_T}{a_1}\right), & \text{if } j = 1, \\ \left(-\frac{a_{j-2}}{a_{j-1}}\right)\left(-\frac{a_{j-4}}{a_{j-3}}\right)\cdots\left(-\frac{a_T}{a_1}\right), & \text{if } j \geq 2, \text{ even}, \\ \left(-\frac{a_{j-2}}{a_{j-1}}\right)\left(-\frac{a_{j-4}}{a_{j-3}}\right)\cdots\left(-\frac{a_1}{a_2}\right)\varphi_1, & \text{if } j \geq 2, \text{ odd}. \end{cases}$$

Thus, letting $C_i, D_i, i \in \{1, \ldots, T\}$, denote monomials with real coefficients in $a_j, j \in \{1, \ldots, T\}$, then

$$C(0;T) = \operatorname{Re}\left(\varphi_T\overline{\varphi}_{T-1} + \ldots \varphi_1\overline{\varphi}_0\right) = \operatorname{Re}\left(\frac{C_T}{iD_T} - \frac{C_{T-1}}{iD_{T-1}} + \cdots - \frac{C_1}{iD_1}\right)$$

and so, clearly, $C(\lambda;T) = 0$. □

Remark Again, for the sake of simplicity, let $b_i = 0$ for all i. Then for the case $T = 1$, the function $C(\lambda;1)$ has only one root at $\lambda = 0$. From Equation (3.6) we know that for the case $T = 2$ the function $C(\lambda;2)$ has no zeros for $\lambda \in \sigma_{ell}(J_2)$ as its two roots, $\lambda_\pm = \pm|a_1 - a_2|$, are parabolic points. For the case $T = 3$ the function

$$C(\lambda;3) = \frac{\lambda\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)}{2(\lambda^2 - a_1^2)}\left(\lambda^2 - (a_1^2 + a_2^2 + a_3^2) + \frac{2(a_1 + a_2 + a_3)}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}}\right) \quad (3.7)$$

has a zero at $\lambda = 0$. In order to preclude any other roots in the generalised interior of the a.c. spectrum it is sufficient to establish that $|\mathrm{Tr}(M(\lambda))| \geq 2$ whenever $C(\lambda;3) = 0$. A simple calculation shows that this is equivalent to

$$\begin{aligned} g(a_1,a_2) := (a_1^3 + a_1^3a_2 + a_2^3 + a_1a_2^3 + a_2 + a_1 - a_1^2a_2 - a_1a_2^2 - a_1a_2)(a_1 + a_2 + 1)^2 \\ - (a_1 + a_2 + a_1a_2)^3 \geq 0, \end{aligned}$$

where, by homogeneity, w.l.o.g $a_3 = 1$. Numerical calculations of the roots of g suggest that this function is non-negative for $a_1, a_2 > 0$. More generally, we believe that for even T the function $C(\lambda;T)$ has no zeros in the generalized interior of the a.c. spectrum, and for odd T there is a single solution at $\lambda = 0$.

Now, for the sake of thoroughness, we consider a different formula for $C(\lambda;T)$ having a "symplectic character". Using it one can easily deduce, in a slightly different way, the rationality of $C(\lambda;T)$. Introducing the indefinite matrix $\hat{\mathcal{J}} :=$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $\mathbb{C}^2$ one can rewrite the expression for $C(\lambda;T)$ in the following form (we assume below that $\lambda \in \sigma_{ell}(J_T)$):

$$\begin{aligned} C(\lambda;T) &= \frac{1}{2}\sum_{s=1}^{T}\left(\varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda) + \varphi_{s-1}(\lambda)\overline{\varphi}_s(\lambda)\right) \\ &= \frac{1}{2}\sum_{s=1}^{T}\left\langle \hat{\mathcal{J}}\left(\prod_{k=1}^{s-1} B_k(\lambda)\right)\begin{pmatrix} \varphi_0 \\ \varphi_1(\lambda) \end{pmatrix}, \left(\prod_{k=1}^{s-1} B_k(\lambda)\right)\begin{pmatrix} \varphi_0 \\ \varphi_1(\lambda) \end{pmatrix}\right\rangle_{\mathbb{C}^2} \end{aligned}$$

We now use that since $\lambda \in \sigma_{ell}(J_T) \subset \mathbb{R}$ we have $\lambda = \overline{\lambda}$. Making this substitution in the righthand side of the inner product (and also those expressions emanating from it) will serve to simplify the final expression (3.8) when it appears so that no conjugacy occurs when the inner product is evaluated, since this operation does not preserve analyticity. Thus,

$$\begin{aligned} C(\lambda;T) &= \frac{1}{2}\sum_{s=1}^{T}\left\langle \left(\prod_{k=1}^{s-1} B_k(\overline{\lambda})\right)^{*}\hat{\mathcal{J}}\left(\prod_{k=1}^{s-1} B_k(\lambda)\right)\begin{pmatrix} \varphi_0 \\ \varphi_1(\lambda) \end{pmatrix}, \begin{pmatrix} \varphi_0 \\ \varphi_1(\overline{\lambda}) \end{pmatrix}\right\rangle_{\mathbb{C}^2} \\ &= \sum_{s=1}^{T}\left\langle F_s(\lambda)\left[\begin{pmatrix} 1 \\ -m_{11}(\lambda)m_{12}^{-1}(\lambda) \end{pmatrix} + \mu(\lambda)\begin{pmatrix} 0 \\ m_{12}^{-1}(\lambda) \end{pmatrix}\right],\right. \\ &\qquad\qquad \left.\left[\begin{pmatrix} 1 \\ -m_{11}(\overline{\lambda})m_{12}^{-1}(\overline{\lambda}) \end{pmatrix} + \mu(\overline{\lambda})\begin{pmatrix} 0 \\ m_{12}^{-1}(\overline{\lambda}) \end{pmatrix}\right]\right\rangle_{\mathbb{C}^2} \end{aligned}$$

where we recalled the standard normalisation $\varphi_0 = 1$, which by (3.5) gives $\varphi_1 = -m_{11}(\lambda)m_{12}^{-1}(\lambda) + \mu(\lambda)m_{12}^{-1}(\lambda)$, and we denoted the real matrix polynomials

$$\frac{1}{2}\left(\prod_{k=1}^{s-1} B_k(\overline{\lambda})\right)^{*}\hat{\mathcal{J}}\left(\prod_{k=1}^{s-1} B_k(\lambda)\right)$$

by $F_s(\lambda)$, $s > 1$ and $F_1(\lambda) := \frac{\hat{\mathcal{J}}}{2}$. Therefore

$$\begin{aligned} C(\lambda;T) = \sum_{s=1}^{T}\Bigg\{ &\left\langle F_s(\lambda)\begin{pmatrix} 1 \\ -m_{11}(\lambda)m_{12}^{-1}(\lambda) \end{pmatrix}, \begin{pmatrix} 1 \\ -m_{11}(\overline{\lambda})m_{12}^{-1}(\overline{\lambda}) \end{pmatrix}\right\rangle_{\mathbb{C}^2} \\ &+ \mathrm{Tr}\left(M(\lambda)\right)\left\langle F_s(\lambda)\begin{pmatrix} 1 \\ -m_{11}(\lambda)m_{12}^{-1}(\lambda) \end{pmatrix}, \begin{pmatrix} 0 \\ m_{12}^{-1}(\overline{\lambda}) \end{pmatrix}\right\rangle_{\mathbb{C}^2} \\ &+ \left\langle F_s(\lambda)\begin{pmatrix} 0 \\ m_{12}^{-1}(\lambda) \end{pmatrix}, \begin{pmatrix} 0 \\ m_{12}^{-1}(\overline{\lambda}) \end{pmatrix}\right\rangle_{\mathbb{C}^2}\Bigg\} \end{aligned} \tag{3.8}$$

where we used that $\mathrm{Tr}(M(\lambda)) = \mu(\lambda) + \overline{\mu}(\lambda), |\mu(\lambda)| = 1, \lambda \in \sigma_{ell}(J_T)$. From the last expression, taking into consideration that $F_s(\lambda), m_{11}(\lambda), m_{12}(\lambda), \mathrm{Tr}\left(M(\lambda)\right)$ are polynomials in λ, we see immediately that $C(\lambda;T)$ is a rational function of λ on $\sigma_{ell}(J_T)$ and therefore admits unique analytic continuation as a rational function to the whole of $\mathbb{C}$, apart from zeros of $m_{12}(\lambda)$, given by Formula 3.8.

Moreover, using the last formula one can give an upper bound for the order of $C(\lambda;T)$ as a rational function, but in the next theorem we will present an explicit calculation of the order.

Theorem 3.2.4. *The function $C(\lambda;T)$ is a rational function of λ of order 1. Moreover, its asymptotic expansion is given by*

$$C(\lambda;T) \sim \frac{1}{2}\left(a_1^{-1}+\cdots+a_T^{-1}\right)\lambda, \lambda \to \infty.$$

Proof. <u>(Step One)</u> For $\lambda \in \sigma_{ell}(J_T)$ we have

$$\begin{aligned} C(\lambda;T) &= \sum_{s=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda) - i\,\mathrm{Im}\left(\sum_{s=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda)\right) \\ &= \sum_{s=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda) - \frac{1}{2}\sum_{s=1}^{T}\left(\varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda) - \varphi_{s-1}(\lambda)\overline{\varphi}_s(\lambda)\right). \end{aligned} \tag{3.9}$$

Note that since both $\varphi_s(\lambda)$ and $\overline{\varphi}_s(\lambda)$ are solutions to the recurrence relations, for $\lambda \in \sigma_{ell}(J_T)$, we have that the discrete Wronskian, $W(s)$, is such that

$$\begin{aligned} W(s) &:= a_s\left(\varphi_{s+1}\overline{\varphi}_s - \varphi_s\overline{\varphi}_{s+1}\right) \\ &= \left((\lambda-b_s)\varphi_s - a_{s-1}\varphi_{s-1}\right)\overline{\varphi}_s - \varphi_s\left((\lambda-b_s)\overline{\varphi}_s - a_{s-1}\overline{\varphi}_{s-1}\right) \\ &= a_{s-1}\left(\varphi_s\overline{\varphi}_{s-1} - \varphi_{s-1}\overline{\varphi}_s\right) = W(s-1). \end{aligned}$$

This implies

$$a_s(\varphi_{s+1}(\lambda)\overline{\varphi}_s(\lambda) - \varphi_s(\lambda)\overline{\varphi}_{s+1}(\lambda)) = a_{s-1}(\varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda) - \varphi_{s-1}(\lambda)\overline{\varphi}_s(\lambda)), \tag{3.10}$$

$s = 1, 2, \ldots, T-1$ where $a_0 := a_T$.

Then, applying Equation (3.10) to (3.9) one obtains

$$\begin{aligned} C(\lambda;T) \;\; = \;\; &\left(\sum_{s=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\lambda)\right) \\ &-\frac{1}{2}\left(a_1^{-1}+\cdots+a_T^{-1}\right)a_T\left(\varphi_1(\lambda)\overline{\varphi}_0(\lambda) - \varphi_0(\lambda)\overline{\varphi}_1(\lambda)\right). \end{aligned} \tag{3.11}$$

Now, since

$$\begin{aligned} \varphi_1(\lambda)\overline{\varphi}_0(\lambda) - \varphi_0(\lambda)\overline{\varphi}_1(\lambda) &= \varphi_1(\lambda) - \overline{\varphi_1}(\lambda) \\ &= \left(\mu(\lambda) - m_{11}(\lambda)m_{12}^{-1}(\lambda)\right) - \left(\overline{\mu}(\lambda) - m_{11}(\lambda)m_{12}^{-1}(\lambda)\right) \\ &= \left(\mu(\lambda) - \overline{\mu}(\lambda)\right)m_{12}^{-1}(\lambda) \end{aligned}$$

it admits analytic continuation from $\sigma_{ell}(J_T)$ to $\mathbb{C} \setminus \sigma_{ess}(J_T)$ as an analytic (algebraic, but not rational) function $\left(\mu(\lambda) - \mu^{-1}(\lambda)\right)m_{12}^{-1}(\lambda)$. Using Corollary 2.1.5, we see that this asymptotically behaves like

$$-\mathrm{Tr}(M(\lambda))m_{12}^{-1}(\lambda) \sim -\lambda^T\left(\prod_{s=1}^{T} a_s\right)^{-1}\left(\lambda^{T-1}\left(\prod_{s=1}^{T-1} a_s\right)^{-1}\right)^{-1} = -\frac{\lambda}{a_T}$$

assuming that the branch of the analytic function $\mu(\lambda)$ has been chosen so that $\mu(\lambda) \to 0$, as $\lambda \to \infty$. Note that in $\mathbb{C}^+ \cup \mathbb{C}^-$ we are in the hyperbolic situation (Lemma 2.1.6) so the eigenvalues of $M(\lambda)$ are $\mu(\lambda), \mu^{-1}(\lambda)$ (as $\det(M(\lambda)) \equiv 1$) with one of them (at our choice) behaving at infinity like $\mu(\lambda) \sim \left(\lambda^T / \left(\prod_{s=1}^{T} a_s\right)\right)^{-1}$ and the other one like $\mu^{-1}(\lambda) \sim \left(\lambda^T / \prod_{s=1}^{T} a_s\right)$.

Therefore the second term in Equation (3.11) admits the asymptotics

$$\frac{1}{2}\left(a_1^{-1} + \cdots + a_T^{-1}\right)\lambda$$

as $\lambda \to \infty$ according to our choice of the branch $\mu(\lambda)$.

(Step Two) We next start to analyse the asymptotics at infinity of the first term in Formula (3.11). Using Corollary 2.1.5 and that $\varphi_0 = 1$, we obtain

$$\varphi_1(\lambda) = (\mu(\lambda) - m_{11}(\lambda)) m_{12}^{-1}(\lambda) = O\left(\lambda^{-1}\right),$$

as $\mu(\lambda) = O\left(\lambda^{-T}\right), \lambda \to \infty$. For the function $\varphi_s(\lambda)$ we have

$$\begin{aligned}
\begin{pmatrix} \varphi_s(\lambda) \\ \varphi_{s+1}(\lambda) \end{pmatrix} &= B_s(\lambda) \dots B_1(\lambda) \begin{pmatrix} \varphi_0 \\ \varphi_1(\lambda) \end{pmatrix} \\
&= \left(B_{s+1}^{-1}(\lambda) \dots B_T^{-1}(\lambda)\right) M(\lambda) \begin{pmatrix} \varphi_0 \\ \varphi_1(\lambda) \end{pmatrix} \\
&= \mu \left(B_{s+1}^{-1}(\lambda) \dots B_T^{-1}(\lambda)\right) \begin{pmatrix} \varphi_0 \\ \varphi_1(\lambda) \end{pmatrix} \\
&= \left[\mu \left(B_{s+1}^{-1}(\lambda) \dots B_T^{-1}(\lambda)\right)\right] \begin{pmatrix} 1 \\ O\left(\frac{1}{\lambda}\right) \end{pmatrix} \\
&= \begin{pmatrix} O\left(\lambda^{(T-s)-T}\right) \\ O\left(\lambda^{(T-s)-T}\right) \end{pmatrix} = O\left(\lambda^{-s}\right)
\end{aligned}$$

since obviously the matrix function

$$\mu(\lambda) B_{s+1}^{-1}(\lambda) \dots B_T^{-1}(\lambda) = O\left(\lambda^{(T-s)-T}\right),$$

as

$$B_j^{-1}(\lambda) = \frac{a_j}{a_{j-1}} \begin{pmatrix} \frac{\lambda}{a_j} & -1 \\ \frac{a_{t-1}}{a_j} & 0 \end{pmatrix} = O(\lambda)$$

and $\mu(\lambda) = O\left(\lambda^{-T}\right), \lambda \to \infty$. Hence

$$\varphi_s(\lambda) = O\left(\lambda^{-s}\right), \lambda \to \infty,$$

$s = 1, 2, \dots, T$.

(Step Three) Recall that on $\sigma_{ell}(J_T)$, one obtains $\overline{\varphi}_s$ from φ_s by replacing μ by $\overline{\mu}$. We now analyse the asymptotics of the analytic continuation of $\overline{\varphi}_s(\lambda) \equiv$

$\overline{\varphi_s(\overline{\lambda})}$. Note that this is an analytic continuation of the function due to the 'anti-analyticity' of the conjugacy being ultimately cancelled out (since it happens twice). Taking the complex conjugate for $\lambda \in \sigma_{ell}(J_T)$ we get

$$\begin{pmatrix} \overline{\varphi}_s(\lambda) \\ \overline{\varphi}_{s+1}(\lambda) \end{pmatrix} = B_s(\lambda)\dots B_1(\lambda) \begin{pmatrix} 1 \\ \left(\mu^{-1}(\lambda) - m_{11}(\lambda)\right) m_{12}^{-1}(\lambda) \end{pmatrix}.$$

Since $\mu^{-1}(\lambda) = O\left(\lambda^T\right), \lambda \to \infty$, then by Corollary 2.1.5, this gives for the analytic continuation to $\mathbb{C}$

$$\begin{pmatrix} \overline{\varphi}_s(\lambda) \\ \overline{\varphi}_{s+1}(\lambda) \end{pmatrix} = B_s(\lambda)\dots B_1(\lambda) \begin{pmatrix} 1 \\ O(\lambda) \end{pmatrix},$$

where the matrix polynomial $B_s(\lambda)\dots B_1(\lambda) = O\left(\lambda^s\right), \lambda \to \infty$. So,

$$\overline{\varphi_{s+1}(\overline{\lambda})} = O\left(\lambda^{s+1}\right),$$

$\lambda \to \infty, s = 0, 1, \dots, T-1$.

<u>(Step Four)</u> Combining both asymptotic formulas for $\varphi_s(\lambda)$ and $\overline{\varphi}_s(\overline{\lambda})$ we finally obtain

$$\begin{aligned} \sum_{s=1}^{T} \varphi_s(\lambda)\overline{\varphi}_{s-1}(\overline{\lambda}) &= \sum_{s=1}^{T} O\left(\lambda^{-s}\right) \cdot O\left(\lambda^{s-1}\right) \\ &= \sum_{s=1}^{T} O\left(\lambda^{-1}\right) = O\left(\lambda^{-1}\right) \end{aligned}$$

as $\lambda \to \infty$, which leads to the formula

$$C(\lambda; T) = \frac{1}{2}\left(a_1^{-1} + \dots + a_T^{-1}\right)\lambda + O(1), \lambda \to \infty. \quad \Box$$

Remark In the proof of Theorem 3.2.4, the opposite choice of the branch for μ changes the sign of the second term in (3.11). This alone is quite innocuous, however the change in branch also leads to a sophisticated calculation of the first term, which we are not able to produce here. Thus, the correct choice of the branch of $\mu(\lambda)$ (despite invariance of the definition of $C(\lambda; T)$ under that choice) is crucial for our proof.

As a corollary we obtain that the function $C(\lambda; T)$ is always of order exactly 1 and is therefore never identically zero.

3.3 The ansatz for the eigenvector and its asymptotics

In this section we plan to elaborate on the explicit construction of the eigenvector associated with the eigenvalue embedded in the a.c. spectrum of the Jacobi matrix with a diagonal perturbation of Coulomb-type decay.

The following classical result will be used in the next lemma.

Proposition 3.3.1. (see [83]). *Assume $\alpha, \gamma, \widetilde{c} \in \mathbb{R}$ and $\widetilde{c} \geq 0, \gamma > 0$, then the following estimate holds:*

$$\sum_{k=n}^{\infty} \frac{e^{ik\alpha}}{k^\gamma + \widetilde{c}} = O\left(\frac{1}{n^\gamma}\right), n \to \infty, \qquad \iff \frac{\alpha}{2\pi} \notin \mathbb{Z}.$$

We will now introduce the function ω_n which is an important part of the eigenvector of the embedded eigenvalue.

Lemma 3.3.2. *Let $\lambda \in \sigma_{ell}(J_T), \alpha > 1$ and*

$$\omega_n(\lambda) := \sum_{m=n+1}^{\infty} m^{-\alpha} \operatorname{Im}(\varphi_m(\lambda)) \operatorname{Im}(\varphi_{m-1}(\lambda)), \tag{3.12}$$

where (φ_n) is defined as in (3.4)*. Then*

$$\omega_n = \frac{C(\lambda; T)}{2(\alpha-1)Tn^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right), n \to \infty. \tag{3.13}$$

Moreover, $\omega_n \in l^2$ for $\alpha > \frac{3}{2}$.

Remark Formula (3.13) shows that at zeros of $C(\lambda;T)$ the asymptotics for the function ω_n change drastically. This proves the importance of our analysis in Section 3.2.

Proof of Lemma 3.3.2. The proof is divided into two cases.

Case 1 If $n = T(k-1)$ then by Lemma 3.1.2 we obtain the relation

$$\begin{aligned}
\omega_n &= \sum_{j=k-1}^{\infty} \sum_{s=1}^{T} (Tj+s)^{-\alpha} \operatorname{Im}(\varphi_{Tj+s}) \operatorname{Im}(\varphi_{Tj+s-1}) \\
&= \sum_{j=k-1}^{\infty} (Tj)^{-\alpha} \left(\sum_{s=1}^{T-1} \left[\operatorname{Im}\left(e^{ij\theta}\varphi_s\right) \operatorname{Im}\left(e^{ij\theta}\varphi_{s-1}\right)\right] \right. \\
&\qquad\qquad \left. + \operatorname{Im}\left(e^{i(j+1)\theta}\varphi_0\right) \operatorname{Im}\left(e^{ij\theta}\varphi_{T-1}\right) \right) + O(k^{-\alpha}) \\
&= \sum_{j=k-1}^{\infty} (Tj)^{-\alpha} \Bigg(\left[\sum_{s=1}^{T-1} \frac{-1}{4} \left(e^{ij\theta}\varphi_s - e^{-ij\theta}\overline{\varphi}_s\right)\left(e^{ij\theta}\varphi_{s-1} - e^{-ij\theta}\overline{\varphi}_{s-1}\right) \right] \\
&\qquad - \frac{1}{4}\left(e^{i(j+1)\theta}\varphi_0 - e^{-i(j+1)\theta}\overline{\varphi}_0\right)\left(e^{ij\theta}\varphi_{T-1} - e^{-ij\theta}\overline{\varphi}_{T-1}\right) \Bigg) + O(k^{-\alpha}).
\end{aligned}$$

Then, $\theta(\lambda) \notin \pi\mathbb{Z}$ as $\lambda \in \sigma_{ell}(J_T)$, so by Proposition 3.3.1

$$\begin{aligned}
\omega_n &= \frac{T^{-\alpha}}{4} \sum_{j=k-1}^{\infty} \frac{1}{j^\alpha} \left(\sum_{s=1}^{T-1} \left(\varphi_s\overline{\varphi}_{s-1} + \overline{\varphi}_s\varphi_{s-1}\right) + e^{i\theta}\varphi_0\overline{\varphi}_{T-1} + e^{-i\theta}\overline{\varphi}_0\varphi_{T-1} \right) \\
&\quad + O(k^{-\alpha}) \\
&= T^{-\alpha} \sum_{j=k-1}^{\infty} \frac{j^{-\alpha}}{2} C(\lambda;T) + O(k^{-\alpha}).
\end{aligned}$$

Thus we can apply the Integral Test and obtain

$$\omega_n = \frac{C(\lambda;T)}{2(\alpha-1)Tn^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right).$$

Finally, if $C(\lambda;T) \neq 0$,

$$\omega_n \asymp n^{1-\alpha} \in l^2 \iff \alpha > \frac{3}{2}.$$

This proves the result for Case 1.

Case 2 If $n = T(k-1) + s_n$ with $s_n \in \{1, \ldots, T-1\}$. Then

$$\omega_n = \sum_{j=k}^{\infty} \sum_{s=1}^{T} (Tj+s)^{-\alpha} \operatorname{Im}(\varphi_{Tj+s}) \operatorname{Im}(\varphi_{Tj+s-1}) + F(n)$$

where, noting that $s_n + 1 \geq 2$,

$$\begin{aligned} F(n) &:= \sum_{s=s_n+1}^{T} (T(k-1)+s)^{-\alpha} \operatorname{Im}\left(\varphi_{T(k-1)+s}\right) \operatorname{Im}\left(\varphi_{T(k-1)+s-1}\right) \\ &= \sum_{s=s_n+1}^{T} (T(k-1)+s)^{-\alpha} \operatorname{Im}\left(e^{i(k-1)\theta}\widetilde{\varphi}_s\right) \operatorname{Im}\left(e^{i(k-1)\theta}\widetilde{\varphi}_{s-1}\right) \\ &= O(k^{-\alpha}) = O\left(n^{-\alpha}\right). \end{aligned}$$

Thus, the remainder can be absorbed in the error term. □

We now make an ansatz for the eigenvector of the embedded eigenvalue, $\lambda \in \sigma_{ell}(J_T)$, in the form

$$u_n = \operatorname{Im}(\varphi_n)\omega_n.$$

Theorem 3.3.3. *Let $\lambda \in \sigma_{ell}(J_T)$. The sequence, (u_n), has the asymptotic form*

$$u_n = \frac{\widetilde{\eta}_s \sin(n\theta/T + \widetilde{\zeta}_s)}{n^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right),$$

where $\widetilde{\eta}_s$ and $\widetilde{\zeta}_s$ are real functions, $\alpha > 1, n = T(k-1)+s$ with $s \in \{0, \ldots, T-1\}$ and $\theta(\lambda)$ is as in Equation (3.1). Moreover, the vector $(\widetilde{\eta}_s)_{s=0}^{T-1}$ is equal to the product of $C(\lambda;T)$ with some non-null vector. Therefore, the vector $(\widetilde{\eta}_s)_{s=0}^{T-1}$ can only equal $\underline{0}$ when the function $C(\lambda;T)$ vanishes.

Proof. For this calculation, assume $(\widetilde{\alpha}_s)_{s=0}^{T-1} \neq \underline{0}, (\widetilde{\beta}_s)_{s=0}^{T-1} \neq \underline{0}$. We show later in this proof why this is the case. Then, by Equation (3.2)

$$\begin{aligned} \operatorname{Im}(\varphi_n) &= \widetilde{\alpha}_s \sin((k-1)\theta) + \widetilde{\beta}_s \cos((k-1)\theta) \\ &= \sqrt{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2} \left(\frac{\widetilde{\alpha}_s}{\sqrt{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}} \sin((k-1)\theta) + \frac{\widetilde{\beta}_s}{\sqrt{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}} \cos((k-1)\theta) \right) \\ &= \eta_s' \sin((k-1)\theta + \phi_s') \end{aligned} \tag{3.14}$$

where $\eta'_s := \sqrt{\widetilde{\alpha}_s^2 + \widetilde{\beta}_s^2}$ and ϕ'_s are real functions of λ. Then, using Lemma 3.3.2, we obtain

$$\begin{aligned} u_n &= \mathrm{Im}(\varphi_n)\omega_n \\ &= (\eta'_s \sin((k-1)\theta + \phi'_s)) \left(\frac{2C(\lambda;T)}{(\alpha-1)Tn^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right)\right) \\ &= \frac{\widetilde{\eta}_s \sin((k-1)\theta + \phi'_s)}{n^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right), \end{aligned}$$

where $\widetilde{\eta}_s := \frac{C(\lambda;T)\eta'_s}{2(\alpha-1)T}$. If $C(\lambda;T) \neq 0$, then $(\widetilde{\eta}_s)_{s=0}^{T-1} \neq \underline{0}$. Otherwise this would imply $\widetilde{\alpha}_s = \widetilde{\beta}_s = 0$ for all s, and recalling that $\widetilde{\alpha}_s := \mathrm{Re}(\alpha_s) - \mathrm{Re}(\beta_s) = 0$, and $\widetilde{\beta}_s := \mathrm{Im}(\alpha_s) + \mathrm{Im}(\beta_s) = 0$, gives $\overline{\alpha_s} = \beta_s$ and therefore $\psi_n = 2\,\mathrm{Re}\left(\alpha_s e^{i(k-1)\theta}\right)$ for all n, implying φ_n is also real for all n. Then, by Equation (3.4), $\varphi_n = 0$ for all n and thus we have a contradiction, since φ_n was assumed to be non-zero.

Finally, we wish to express our eigenvector in terms of n. Thus,

$$\begin{aligned} u_n &= \frac{\widetilde{\eta}_s \sin((k-1)\theta + \phi'_s)}{n^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right) \\ &= \frac{\widetilde{\eta}_s \sin(n\theta/T + \widetilde{\zeta}_s)}{n^{\alpha-1}} + O\left(\frac{1}{n^\alpha}\right), \end{aligned}$$

where $\widetilde{\zeta}_s := \phi'_s - s\theta/T$. □

3.4 The structure of the potential and its asymptotics

The following theorem gives an explicit formula for the potential, and the eigenvector, in terms of the solutions φ_n of the periodic problem, λ and the parameter α.

Theorem 3.4.1. *Let $\lambda \in \sigma_{ell}(J_T)$ with $C(\lambda;T) \neq 0$. Define $\omega_n(\lambda)$ as in (3.12) and $\varphi_n(\lambda)$ as in (3.4) and let $\alpha > \frac{3}{2}, n = T(k-1)+s, s \in \{0,\dots,T-1\}$,*

$$q_n = -a_{n-1}(\mathrm{Im}(\varphi_{n-1}))^2 \left(\frac{n^{-\alpha}}{\omega_n}\right) + a_n(\mathrm{Im}(\varphi_{n+1}))^2 \left(\frac{(n+1)^{-\alpha}}{\omega_n}\right). \tag{3.15}$$

Then

$$u_n(\lambda) = \omega_n(\lambda)\,\mathrm{Im}\left(\varphi_n(\lambda)\right) \tag{3.16}$$

satisfies

$$a_{n-1}u_{n-1} + a_n u_{n+1} + (q_n + b_n - \lambda)u_n = 0 \tag{3.17}$$

for $n \geq 2$. Moreover, q_n has the following asymptotic behaviour:

$$q_n = \frac{1}{n}\left(\rho_s(\lambda)\sin\left(2n\theta(\lambda)/T + \zeta_s(\lambda)\right) + \delta_s(\lambda)\right) + O\left(\frac{1}{n^2}\right), \tag{3.18}$$

where ρ_s, ζ_s and δ_s are real functions.

Remark In Formula (3.15) we assume without loss of generality that $\omega_n \neq 0\ \forall n = 1, 2, \ldots$. Indeed, due to the condition that $C(\lambda; T) \neq 0$ and Formula (3.13) we see that $\omega_n(\lambda) \neq 0\ \forall n \geq L$, where L is sufficiently large. If $\omega_n(\lambda)$ vanishes for some $n < L$, then one can change the ansatz for $\omega_n(\lambda)$, (3.12), by introducing into the sum over m an extra factor c_m, where $c_m = 1\ \forall\ m \geq L$. The values $c_1, c_2, \ldots, c_{L-1}$ can be chosen in a suitable way such that $\omega_1(\lambda), \omega_2(\lambda), \ldots, \omega_{L-1}(\lambda)$ are not equal to zero. In order for the corresponding u_n to still solve (3.17) it will be necessary to adapt the corresponding q_n slightly, introducing coefficients c_{n-1} and c_n into the first and second terms of (3.15), respectively. The asymptotic behaviour described by (3.18) will remain the same.

Remark Using one of the Janas-Moszynski results (see Theorem 3.1 in [31]), it can be shown that if $C(\lambda; T) \neq 0$ no eigenvalues can be embedded in $\sigma_{ell}(J_T)$ using a potential (q_n), where $q_n = \frac{\delta_s}{n} + O(\frac{1}{n^2})$. Since our technique for embedding eigenvalues works whenever $C(\lambda; T) \neq 0$, the potential must have an oscillating term and therefore

$$\sum_{s=0}^{T-1} |\rho_s(\lambda)|^2 > 0.$$

Proof of Theorem 3.4.1.
<u>(Step One)</u> We check that $u_n(\lambda)$ in (3.16) satisfies (3.17). For $n \geq 2$,

$$\begin{aligned}
& a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} - \lambda u_n = -q_n u_n \\
\iff\ & a_{n-1}\omega_{n-1} \operatorname{Im}(\varphi_{n-1}) + b_n \omega_n \operatorname{Im}(\varphi_n) + a_n \omega_{n+1} \operatorname{Im}(\varphi_{n+1}) \\
& \quad - \lambda \omega_n \operatorname{Im}(\varphi_n) = -q_n \omega_n \operatorname{Im}(\varphi_n) \\
\iff\ & \operatorname{Im}(a_{n-1}\varphi_{n-1} + b_n \varphi_n + a_n \varphi_{n+1} - \lambda \varphi_n)\omega_n + \operatorname{Im}(\varphi_{n-1})\, a_{n-1}(\omega_{n-1} - \omega_n) \\
& \qquad\qquad + a_n \operatorname{Im}(\varphi_{n+1})(\omega_{n+1} - \omega_n) = -q_n \operatorname{Im}(\varphi_n)\, \omega_n \\
\iff\ & \operatorname{Im}(\varphi_{n-1})\, a_{n-1}(\omega_{n-1} - \omega_n) \\
& \qquad\qquad + a_n \operatorname{Im}(\varphi_{n+1})(\omega_{n+1} - \omega_n) = -q_n \operatorname{Im}(\varphi_n)\, \omega_n
\end{aligned} \tag{3.19}$$

where we have used that φ_n satisfies the three-term recurrence relation (3.17). Observing that

$$\omega_{n-1} - \omega_n = n^{-\alpha} \operatorname{Im}(\varphi_n) \operatorname{Im}(\varphi_{n-1}), \tag{3.20}$$

we can choose q_n as in (3.15) to guarantee the equality (3.19).

<u>(Step Two)</u> We now prove (3.18). Since $\lambda \in \sigma_{ell}(J_T)$, by Lemma 3.1.1 we have that $a_{n-1}(\operatorname{Im}(\varphi_{n-1}))^2$ and $a_n(\operatorname{Im}(\varphi_{n+1}))^2$ are periodic in k and by Lemma 3.1.2 are not all zero. Then one can expect growth or decay in q_n to come from the components $\left(\frac{n^{-\alpha}}{\omega_n}\right)$ and $\left(\frac{(n+1)^{-\alpha}}{\omega_n}\right)$. By Lemma 3.3.2 we have the relation $\omega_n \asymp n^{1-\alpha}$, and so we obtain

$$\frac{(n+1)^{-\alpha}}{\omega_n} \asymp n^{-1},$$

which gives a Coulomb-type decay for q_n.

Using Lemmas 3.1.1 and 3.3.2 for $n = T(k-1)+s, s \in \{1, \dots, T-2\}$, we obtain

$$
\begin{aligned}
q_n &= -a_{s-1}\left(\eta_{s-1}\sin(2(k-1)\theta + \phi_{s-1}) + \eta_{s-1}\right)\left(\frac{2(\alpha-1)T}{nC(\lambda;T)} + O\left(\frac{1}{n^2}\right)\right) \\
&\quad + a_s\left(\eta_{s+1}\sin(2(k-1)\theta + \phi_{s+1}) + \eta_{s+1}\right)\left(\frac{2(\alpha-1)T}{nC(\lambda;T)} + O\left(\frac{1}{n^2}\right)\right) \\
&= \frac{2(\alpha-1)T}{nC(\lambda;T)}\Bigg(\left(-\eta_{s-1}a_{s-1}\cos\phi_{s-1} + \eta_{s+1}a_s\cos\phi_{s+1}\right)\sin(2(k-1)\theta) \\
&\qquad + \left(\eta_{s+1}a_s\sin\phi_{s+1} - \eta_{s-1}a_{s-1}\sin\phi_{s-1}\right)\cos(2(k-1)\theta) \\
&\qquad - \eta_{s-1}a_{s-1} + \eta_{s+1}a_s\Bigg) + O\left(\frac{1}{n^2}\right)
\end{aligned}
$$

Then using a similar technique as in (3.3) and (3.14) gives

$$
q_n = \frac{1}{n}\left(\left[\rho_s(\lambda)\sin(2(k-1)\theta(\lambda) + \zeta'_s(\lambda))\right] + \delta_s\right) + O\left(\frac{1}{n^2}\right), \tag{3.21}
$$

for real functions ζ'_s of λ,

$$
\begin{aligned}
\rho_s^2 := \frac{4T^2(\alpha-1)^2}{C(\lambda;T)^2}\Bigg(&\left(\eta_{s+1}a_s\cos\phi_{s+1} - \eta_{s-1}a_{s-1}\cos\phi_{s-1}\right)^2 \\
&+ \left(\eta_{s+1}a_s\sin\phi_{s+1} - \eta_{s-1}a_{s-1}\sin\phi_{s-1}\right)^2\Bigg),
\end{aligned}
$$

and $\delta_s := \frac{2T(\alpha-1)}{C(\lambda;T)}\left(-\eta_{s-1}a_{s-1} + \eta_{s+1}a_s\right)$.

For the special cases of $s \in \{0, T-1\}$ we must be careful because $n-1$ and $n+1$ will produce different values in the parameter k to those contained in the n-th element. When $s = 0$ (i.e. $n = (k-1)T$):

$$
\begin{aligned}
q_n &= -a_{T-1}\left(\eta_{T-1}\sin(2(k-2)\theta + \phi_{T-1}) + \eta_{T-1}\right)\left(\frac{2(\alpha-1)T}{nC(\lambda;T)} + O\left(\frac{1}{n^2}\right)\right) \\
&\qquad + a_T\left(\eta_1\sin(2(k-1)\theta + \phi_1) + \eta_1\right)\left(\frac{2(\alpha-1)T}{nC(\lambda;T)} + O\left(\frac{1}{n^2}\right)\right) \\
&= \frac{2(\alpha-1)T}{nC(\lambda;T)}\Bigg(-\eta_{T-1}a_{T-1}\sin(2(k-1)\theta + \phi_{T-1} - 2\theta) \\
&\qquad + \eta_1 a_0\sin(2(k-1)\theta + \phi_1) - a_{T-1}\eta_{T-1} + a_T\eta_1\Bigg) + O\left(\frac{1}{n^2}\right).
\end{aligned}
$$

This is of the same form as (3.21). Consequently,

$$
q_n = \frac{1}{n}\left[\rho_0(\lambda)\sin\left(2(k-1)\theta(\lambda) + \zeta'_0\right) + \delta_0(\lambda)\right] + O\left(\frac{1}{n^2}\right),
$$

for functions $\zeta_0', \delta_0 := \frac{2T(\alpha-1)}{C(\lambda;T)}\left(-\eta_{T-1}a_{T-1} + \eta_1 a_T\right)$ and

$$\rho_0^2 := \frac{4T^2(\alpha-1)^2}{C(\lambda;T)^2}\big(\left(\eta_1 a_T \cos\phi_1 - \eta_{T-1}a_{T-1}\cos(\phi_{T-1} - 2\theta)\right)^2 + \left(\eta_1 a_T \sin\phi_1 - \eta_{T-1}a_{T-1}\sin(\phi_{T-1} - 2\theta)\right)^2\big).$$

Similarly, when $s = T-1$ (i.e. $n = kT - 1$):

$$\begin{aligned} q_n &= -a_{T-2}\left(\eta_{T-2}\sin(2(k-1)\theta + \phi_{T-2}) + \eta_{T-2}\right)\left(\frac{2(\alpha-1)T}{nC(\lambda;T)} + O\left(\frac{1}{n^2}\right)\right) \\ &\qquad + a_T\left(\eta_0 \sin(2k\theta + \phi_0) + \eta_0\right)\left(\frac{2(\alpha-1)T}{nC(\lambda;T)} + O\left(\frac{1}{n^2}\right)\right) \\ &= \frac{1}{n}\left(\rho_{T-1}(\lambda)\sin\left(2(k-1)\theta(\lambda) + \zeta_{T-1}'\right) + \delta_{T-1}(\lambda)\right) + O\left(\frac{1}{n^2}\right), \end{aligned}$$

for functions $\zeta_{T-1}', \delta_{T-1} := \frac{2T(\alpha-1)}{C(\lambda;T)}\left(-\eta_{T-2}a_{T-2} + \eta_0 a_{T-1}\right)$ and

$$\rho_{T-1}^2 := \frac{4T^2(\alpha-1)^2}{C(\lambda;T)^2}\big(\left(\eta_0 a_{T-1}\cos(\phi_0 + 2\theta) - \eta_{T-2}a_{T-2}\cos\phi_{T-2}\right)^2 + \left(\eta_0 a_{T-1}\sin(\phi_0 + 2\theta) - \eta_{T-2}a_{T-2}\sin\phi_{T-2}\right)^2\big).$$

Thus, for all $s \in \{0, \dots, T-1\}$, we have the result:

$$q_n = \frac{1}{n}\left[\rho_s(\lambda)\sin\left(2(k-1)\theta(\lambda) + \zeta_s'\right) + \delta_s(\lambda)\right] + O\left(\frac{1}{n^2}\right).$$

However, we still wish to express our potential in terms of the variable n. This follows simply from defining the new function ζ_s, where $\zeta_s := \zeta_s' - s\theta/T$. $\square$

Remark Concerning the roots of $C(\lambda;T)$ for $\lambda \in \sigma_{ell}(J_T)$, we may say the following. As has been stated in the theorem, using a sufficiently slowly decaying potential, $q_n = O\left(\frac{1}{n}\right)$, it is possible to introduce a subordinate l^2-solution for any fixed $\lambda \in \sigma_{ell}(J_T)$, except at roots of $C(\lambda;T)$. However we believe that at any root of $C(\lambda;T)$, $\lambda \in \sigma_{ell}(J_T)$, the existence of the subordinate l^2-solution can still be obtained by using a potential, $q_n = O\left(\frac{1}{n}\right)$, $n \to \infty$. See Section 6.1 for more details.

3.5 Embedded eigenvalues

Theorem 3.4.1 guarantees a subordinate solution of the recurrence relation (3.17), which lies in l^2, but does not guarantee an embedded eigenvalue since it still remains to be seen if the first-row equation of the Jacobi matrix is satisfied, i.e.

$$(q_1 + b)u_1 + a_1 u_2 = \lambda u_1.$$

The next result shows that it is always possible to make λ an eigenvalue by suitably modifying the potential, slightly.

Theorem 3.5.1. *Assume $\lambda \in \sigma_{ell}(J_T), C(\lambda;T) \neq 0$ and $\alpha > \frac{3}{2}$. Let u_n be given by (3.16) for $n \geq 2$ and q_n by (3.15) for $n \geq 3$. Then it is possible to choose $u_1, q_1, q_2 \in \mathbb{R}$ such that $\lambda \in \sigma_p(J_T + Q)$, where Q is an infinite diagonal matrix with entries (q_n).*

Remark Given that Q can be approximated by finite rank operators in the operator norm, it is clearly compact. Thus, by the classical Weyl Theorem [39] the essential spectrum of J_T and J_T+Q coincide. This means that λ is embedded in the essential spectrum of J_T+Q. Furthermore, we have by Theorem 3 in [56] that for potentials of the Wigner-von Neumann type $\sigma_{a.c.}(J_T) = \sigma_{a.c.}(J_T+Q)$, and therefore λ is embedded into the a.c. spectrum of J_T+Q.

Proof. By Theorem 3.4.1 we have

$$a_{n-1}u_{n-1} + a_n u_{n+1} + (q_n + b_n - \lambda)u_n = 0$$

for $n \geq 3$. However, we also need to satisfy

$$(q_1 + b_1)u_1 + a_1 u_2 = \lambda u_1 \tag{3.22}$$

and

$$a_1 u_1 + (q_2 + b_2 - \lambda)u_2 + a_2 u_3 = 0. \tag{3.23}$$

We have two cases:

1. If $u_2 \neq 0$ then defining $q_2 := \frac{-\lambda u_2 - a_2 u_3 - a_1 u_1 - b_2 u_2}{u_2}$ with $u_1 := -\frac{a_1 u_2}{q_1 + b_1 - \lambda}$, with q_1 as a free parameter and not equal to $\lambda - b_1$, ensures all conditions are satisfied.

2. If $u_2 = 0$ then defining $u_1 := -\frac{a_2 u_3}{a_1}$ and $q_1 := \lambda - b_1$, with q_2 as a free parameter, ensures all conditions are satisfied. $\square$

Chapter 4

Spectral results using the discrete Levinson technique

The simplest classical Levinson result roughly states that for the initial value problem

$$y'' + (\lambda^2 - P(x))y = 0, x \in \mathbb{R}^+, y(0,\lambda) = 0, y'(0,\lambda) = 1,$$

the solution will approach a sine function as x tends to infinity, providing λ is real and $P(x)$ obeys certain conditions, such as non-negativity and tending to zero quickly enough [47, 68]. In 1987, Benzaid and Lutz [67] adapted and applied the theory to the study of discrete systems of the form

$$x(n+1) = A(n+1)x(n), n \geq n_0 \tag{4.1}$$

where $x = (x(n))_{n \geq n_0}$ is a sequence of $\mathbb{C}^d$ vectors and $A = (A(n))_{n \geq n_0}$, a sequence of $d \times d$ complex matrices. Since then the assumptions on A have been varied and investigated to better determine the effects on the spectrum of Jacobi matrices (see, for example, [5, 14, 17, 30–33, 35–38, 60, 61]). In this chapter we explore Levinson-type techniques and use a diagonal perturbation to produce subordinate solutions and to embed eigenvalues into the essential spectrum of periodic Jacobi matrices, J_T. Again, we will be using a Wigner-von Neumann potential [67], however unlike in the previous chapter where we deduced it from an ansatz we made for the eigenvector, we here assume the Wigner-von Neumann structure a priori. It should be stressed that the class of potentials in Chapter 3 is slightly larger since the coefficients, ρ_s, ϕ_s are allowed to change across the period, whereas the coefficients here, ρ, ϕ, will be seen to be fixed.

The first main theorem (recorded below) states that for any individual λ in the generalised interior of the essential spectrum of an arbitrary period-T Jacobi operator, J_T, a potential, (q_n), can be contrived such that the new diagonally-perturbed Jacobi operator has a subordinate solution at λ. Recall that in our case this is a solution, $\underline{u} := (u_n)_{n\geq 1}$, to

$$a_{n-1}u_{n-1} + (b_n + q_n)u_n + a_n u_{n+1} = \lambda u_n, n \geq 2 \tag{4.2}$$

that decays.

Theorem 4.0.1. *For $\lambda \in \sigma_{ell}(J_T)$, let $e^{\pm i\theta(\lambda)}$ be the eigenvalues of $M(\lambda)$, where $\theta(\lambda)$ is the quasi-momentum. For any $\lambda \in \sigma_{ell}(J_T)$ outside an explicitly described finite set, we can choose ω s.t. $\omega T + 2\theta(\lambda) \in 2\pi\mathbb{Z}$ or $\omega T - 2\theta(\lambda) \in 2\pi\mathbb{Z}$, and*

$$q_n = \frac{c\sin(n\omega + \phi)}{n} \tag{4.3}$$

for some $c \in \mathbb{R} \setminus \{0\}, \phi \in \mathbb{R}$, such that there exists a subordinate solution $\underline{u} := (u_n)_{n\geq 1}$ to Equation (4.2). In this case, there exists a $\delta > 0$ s.t. for $|c| > \delta$ the subordinate solution resides in l^2.

Remark We stress that the values of $\lambda \in \sigma_{ell}(J_T)$ for which the theorem holds are defined explicitly using the functions $E(\lambda), \widetilde{E}(\lambda), \widetilde{\widetilde{E}}(\lambda)$, given later in (4.30),(4.37) and (4.41), respectively.

Remark Later, in Theorem 4.2, we deal with the initial conditions and establish explicit $\underline{u}$ such that

$$(J_T + Q)\underline{u} = \lambda\underline{u},$$

where Q is a diagonal matrix with entries (q_n) described by Equation (4.3) with a suitable correction for q_1, q_2.

The proof of the result is separated into five steps (Sections 4.1 to 4.5). Once the result regarding the subordinate solution for a single candidate eigenvector has been expounded, the initial conditions are discussed (Section 4.5) so that the value λ becomes a formal eigenvalue. Then, in Section 4.6, the technique is adapted to construct a collection of subordinate solutions corresponding to (possibly infinitely many) values of the spectral parameter in the generalised interior of the essential spectrum and in the special case of two candidate eigenvalues the conditions that must be satisfied in order for these two subordinate solutions to become eigenvectors (Section 4.6).

4.1 Variation of parameters

In this section we adopt a suitable change of discrete variables with the aim of simplifying the analysis of the transfer matrix product.

Recall from Section 2.2 that when $n = kT$, for any solution (u_n) to (4.2), we have

$$\begin{pmatrix} u_{n+T} \\ u_{n+T+1} \end{pmatrix} = \vec{u}_{n+T} = M_k(\lambda)\vec{u}_n \tag{4.4}$$

where

$$M_k(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-1}}{a_T} & \frac{\lambda - b_T - q_{n+T}}{a_T} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-2}}{a_{T-1}} & \frac{\lambda - b_2 - q_{n+T-1}}{a_{T-1}} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -\frac{a_T}{a_1} & \frac{\lambda - b_1 - q_{n+1}}{a_1} \end{pmatrix},$$

the perturbed monodromy matrix. Now considering only n such that $n = kT$, define a new parameter $\vec{f}_k$ such that

$$\vec{u}_n = M^k(\lambda)\vec{f}_k, \tag{4.5}$$

where $M(\lambda)$ is the unperturbed monodromy matrix (3.16). Substituting (4.5) into (4.4) gives

$$\vec{u}_{n+T} = M_k(\lambda)M^k(\lambda)\vec{f}_k. \tag{4.6}$$

Recall

$$\Sigma_k(\lambda) := \sum_{j=0}^{T-1} B_T(\lambda)B_{T-1}(\lambda)\dots B_{T-(j-1)}(\lambda)\begin{pmatrix} 0 & 0 \\ 0 & \frac{q_{T(k+1)-j}}{a_{T-j}} \end{pmatrix} B_{T-(j+1)}(\lambda)\dots B_1(\lambda),$$

with $B_i(\lambda)$ as in (2.2), and if the order of decreasing indices is formally violated we understand the corresponding product to be the identity matrix. By combining the information in Equations (4.5) and (4.6), and noting that $q_n = O\left(\frac{1}{n}\right)$, we obtain for $\lambda \in \sigma_{ell}(J_T)$

$$\begin{aligned}
&\vec{f}_{k+1} = M^{-k-1}(\lambda)M_k(\lambda)M^k(\lambda)\vec{f}_k \\
&= M^{-k-1}(\lambda)\left(M(\lambda) - \Sigma_k(\lambda) + O\left(\frac{1}{k^2}\right)\right)M^k(\lambda)\vec{f}_k \\
&= \left(I - M^{-k-1}(\lambda)\Sigma_k(\lambda)M^k(\lambda) + O\left(\frac{1}{k^2}\right)\right)\vec{f}_k \\
&= \left(I - V(\lambda)\begin{pmatrix} \mu^{-k-1} & 0 \\ 0 & \overline{\mu}^{-k-1} \end{pmatrix} V^{-1}(\lambda)\Sigma_k(\lambda)V(\lambda)\begin{pmatrix} \mu^k & 0 \\ 0 & \overline{\mu}^k \end{pmatrix} V^{-1}(\lambda) + O\left(\frac{1}{k^2}\right)\right)\vec{f}_k,
\end{aligned} \tag{4.7}$$

where we go from the first to the second line of the above calculation using (2.15); the second to the third using that $\|M^m(\lambda)\|$ is uniformly bounded for fixed $\lambda \in \sigma_{ell}(J_T)$ in $m \in \mathbb{Z}$; $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ are the (conjugate) eigenvalues (of modulus 1) of the unperturbed transfer matrix, $M(\lambda)$, with $\lambda \in \sigma_{ell}(J_T)$; and $V(\lambda), V^{-1}(\lambda)$ are the matrices that diagonalise $M(\lambda)$.

Remark It is sufficient to consider only n of the type $n = kT$ for the general asymptotic analysis of our solution. This follows from the fact that for $n = kT + s$ where $s \in \{1, \dots, T-1\}$, we have the relation

$$\begin{aligned}
\vec{u}_{kT+s} &= (B_s(\lambda - q_{kT+s})\dots B_1(\lambda - q_{kT+1}))M_{k-1}(\lambda)\dots M_1(\lambda)\vec{u}_0 \\
&= (B_s(\lambda - q_{kT+s})\dots B_1(\lambda - q_{kT+1}))\vec{u}_{Tk}
\end{aligned} \tag{4.8}$$

and since the $B_j(\lambda - q_{kT+j})$ are invertible and q_n tends to zero we have that for k sufficiently large

$$\|B_j(\lambda - q_{kT+j})\| \le K_1; \|B_j^{-1}(\lambda - q_{kT+j})\| \le K_2$$

for all $j \in \{1, \ldots, T-1\}$, and finite K_1, K_2. Then

$$\|\vec{u}_{kT+s}\| \leq K_1^s \|\vec{u}_{kT}\|, \|\vec{u}_{kT+s}\| \geq K_2^{-s} \|\vec{u}_{kT}\|,$$

so

$$K_2^{-s} \|\vec{u}_{kT}\| \leq \|\vec{u}_{kT+s}\| \leq K_1^s \|\vec{u}_{kT}\|$$

and this with (4.8), together with the fact $B_s(\lambda - q_{kT+s}) \to B_s(\lambda)$ as $k \to \infty$, gives that the asymptotic behaviour of $\vec{u}_{kT}$ uniquely determines the asymptotic behaviour of $\vec{u}_{kT+s}$ for any $s \in \{0, \ldots, T-1\}$. In other words, we can interpolate the asymptotic behaviour for $n = kT$ to arbitrary values of n.

4.2 Preparation for the Harris-Lutz procedure

In this section we apply variation of parameters again (this time on $\vec{f}_k$) and continue simplifying the expression down into something to which we can apply the Harris-Lutz procedure [17, 24, 34, 42]. The Harris-Lutz procedure gives us a way to remove all the terms that do not affect the asymptotics from our analysis.

Recall from Lemma 2.1.5 that the unperturbed monodromy matrix, $M(\lambda)$, has the form

$$M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix},$$

where $m_{11}(\lambda), m_{12}(\lambda), m_{21}(\lambda), m_{22}(\lambda)$ are real polynomials in λ. Also, observe for $j \in \{0, \ldots, T-1\}$,

$$B_T(\lambda) \ldots B_{T-(j-1)}(\lambda) = \begin{pmatrix} \alpha_1^{(j)}(\lambda) & \alpha_2^{(j)}(\lambda) \\ \alpha_3^{(j)}(\lambda) & \alpha_4^{(j)}(\lambda) \end{pmatrix};$$

$$B_{T-(j+1)}(\lambda) \ldots B_1(\lambda) = \begin{pmatrix} \tilde{\alpha}_1^{(j)}(\lambda) & \tilde{\alpha}_2^{(j)}(\lambda) \\ \tilde{\alpha}_3^{(j)}(\lambda) & \tilde{\alpha}_4^{(j)}(\lambda) \end{pmatrix},$$

where $\alpha_i^{(j)}(\lambda), \widetilde{\alpha}_i^{(j)}(\lambda)$ are also real polynomials in λ. Using Lemma 2.1.4, we obtain the following result on the form of $\alpha_i^{(j)}, \widetilde{\alpha}_i^{(j)}$ for $j \in \{0, \ldots, T-1\}$:

Lemma 4.2.1. *For $B_i(\lambda), i \in \{1, \ldots, T\}$, as described in Equation* (3.15)*, we* *have that for* $j \in \{0, \dots, T-1\}$

$$\alpha_1^{(j)}(\lambda) = \begin{cases} 1, \ j = 0, \\ 0, \ j = 1, \\ -a_{T-j} \frac{\lambda^{j-2}}{\prod\limits_{s=T-j+1}^{T-1} a_s} + P_{j-3}(\lambda), \ j \geq 2, \end{cases}$$

$$\alpha_2^{(j)}(\lambda) = \begin{cases} 0, \ j = 0, \\ \frac{\lambda^{j-1}}{\prod\limits_{s=T-j+1}^{T-1} a_s} + P_{j-2}(\lambda), \ j \geq 1, \end{cases}$$

$$\alpha_3^{(j)}(\lambda) = \begin{cases} 0, \ j = 0, \\ -a_{T-j} \frac{\lambda^{j-1}}{\prod\limits_{s=T-j+1}^{T} a_s} + Q_{j-2}(\lambda), \ j \geq 1, \end{cases}$$

$$\alpha_4^{(j)}(\lambda) = \frac{\lambda^j}{\prod\limits_{s=T-j+1}^{T} a_s} + P_{j-1}(\lambda),$$

where $P_{j-1}(\lambda), P_{j-2}(\lambda), Q_{j-2}(\lambda)$ *and* $P_{j-3}(\lambda)$ *are real polynomials in* λ *of degree less than or equal to* $j-1, j-2, j-2$ *and* $j-3$*, respectively, and* $P_k(\lambda) = 0 = Q_k(\lambda)$ *for* $k < 0$*.*

Similarly, for $j \in 0, \dots, T-1$*, we have*

$$\widetilde{\alpha}_1^{(j)}(\lambda) = \begin{cases} 1, \ j = T-1, \\ 0, \ j = T-2, \\ -a_T \frac{\lambda^{T-j-3}}{\prod\limits_{s=1}^{T-j-2} a_s} + \widetilde{P}_{T-j-4}(\lambda), \ j \leq T-3, \end{cases}$$

$$\widetilde{\alpha}_2^{(j)}(\lambda) = \begin{cases} 0, \ j = T-1, \\ \frac{\lambda^{T-j-2}}{\prod\limits_{s=1}^{T-j-2} a_s} + \widetilde{P}_{T-j-3}(\lambda), \ j \leq T-2, \end{cases}$$

$$\widetilde{\alpha}_3^{(j)}(\lambda) = \begin{cases} 0, \ j = T-1, \\ -a_T \frac{\lambda^{T-j-2}}{\prod\limits_{s=1}^{T-j-1} a_s} + \widetilde{Q}_{T-j-3}(\lambda), \ j \leq T-2, \end{cases}$$

$$\widetilde{\alpha}_4^{(j)}(\lambda) = \frac{\lambda^{T-j-1}}{\prod\limits_{s=1}^{T-j-1} a_s} + \widetilde{P}_{T-j-2}(\lambda),$$

where $\widetilde{P}_{T-j-2}(\lambda), \widetilde{P}_{T-j-3}(\lambda), \widetilde{Q}_{T-j-3}(\lambda)$ *and* $\widetilde{P}_{T-j-4}(\lambda)$ *are real polynomials in* λ *of degree less than or equal to* $T-j-2, T-j-3, T-j-3$ *and* $T-j-4$*, respectively, and* $\widetilde{P}_k(\lambda) = 0 = \widetilde{Q}_k$ *for* $k < 0$*.*

Now using the (small) freedom we have in diagonalising matrices $V(\lambda)$, for $M(\lambda), \lambda \in \sigma_{ell}(J_T)$, we can construct them such that the entries in the second row are equal to 1. To show that this is always possible assume for contradiction that the eigenvector, $\vec{v}_1$, has second component zero, i.e. $\vec{v}_1 = \begin{pmatrix} a \\ 0 \end{pmatrix}, a \neq 0$. Then

$$\begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \mu \begin{pmatrix} a \\ 0 \end{pmatrix}.$$

Thus

$$am_{11}(\lambda) = \mu a \Rightarrow a(m_{11}(\lambda) - \mu) = 0,$$

which gives $m_{11}(\lambda) = \mu$. However, $m_{11}(\lambda) \neq \mu$ because $m_{11}(\lambda)$ is real and, since $\lambda \in \sigma_{ell}(J_T)$, μ is non-real. A simple calculation shows that

$$V(\lambda) = \begin{pmatrix} \frac{m_{12}(\lambda)}{\mu - m_{11}(\lambda)} & \frac{m_{12}(\lambda)}{\overline{\mu} - m_{11}(\lambda)} \\ 1 & 1 \end{pmatrix}.$$

Note that the same reasoning as above shows that there are no eigenvectors of $M(\lambda)$, $\lambda \in \sigma_{ell}(J_T)$, with zero first component. Therefore, $V(\lambda)$ is always invertible for $\lambda \in \sigma_{ell}(J_T)$, as μ is non-real and $m_{12}(\lambda) \neq 0$. (The latter follows from the fact that if $m_{12}(\lambda) = 0$ then the monodromy matrix is lower triangular and therefore the eigenvalues are the diagonal entries, which in this case are real.)

Define a new sequence, $\vec{g}_k$, such that

$$\vec{g}_k := V^{-1}(\lambda)\vec{f}_k. \tag{4.9}$$

In terms of $\vec{g}_k$, Equation (4.7) becomes

$$\begin{aligned}
\vec{g}_{k+1} &= \left(I - \begin{pmatrix} \overline{\mu}^{k+1} & 0 \\ 0 & \mu^{k+1} \end{pmatrix} V^{-1}(\lambda)\Sigma_{k+1}(\lambda)V(\lambda) \begin{pmatrix} \mu^k & 0 \\ 0 & \overline{\mu}^k \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right)\vec{g}_k \\
&= \left(I - \sum_{j=0}^{T-1} \frac{q_{(k+1)T-j}}{a_{T-j}} \begin{pmatrix} \overline{\mu}^{k+1} & 0 \\ 0 & \mu^{k+1} \end{pmatrix} V^{-1}(\lambda) B_T(\lambda) \dots B_{T-(j-1)}(\lambda) \right. \\
&\times \left. \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} B_{T-(j+1)}(\lambda) \dots B_1(\lambda) V(\lambda) \begin{pmatrix} \mu^k & 0 \\ 0 & \overline{\mu}^k \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right)\vec{g}_k \\
&= \left(I - \sum_{j=0}^{T-1} \frac{q_{(k+1)T-j}}{a_{T-j}} \begin{pmatrix} \overline{\mu}^{k+1} & 0 \\ 0 & \mu^{k+1} \end{pmatrix} \begin{pmatrix} \frac{m_{12}}{\mu - m_{11}} & \frac{m_{12}}{\overline{\mu} - m_{11}} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \alpha_1^{(j)} & \alpha_2^{(j)} \\ \alpha_3^{(j)} & \alpha_4^{(j)} \end{pmatrix} \right. \\
&\times \left. \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_1^{(j)} & \tilde{\alpha}_2^{(j)} \\ \tilde{\alpha}_3^{(j)} & \tilde{\alpha}_4^{(j)} \end{pmatrix} \begin{pmatrix} \frac{m_{12}}{\mu - m_{11}} & \frac{m_{12}}{\overline{\mu} - m_{11}} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mu^k & 0 \\ 0 & \overline{\mu}^k \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right)\vec{g}_k \\
&= \left(I + \frac{1}{i \sin\theta(\lambda)} \sum_{j=0}^{T-1} \frac{q_{(k+1)T-j}}{a_{T-j}} \left\{ \begin{pmatrix} -\overline{C_j(\lambda)} & 0 \\ 0 & C_j(\lambda) \end{pmatrix} + \begin{pmatrix} 0 & -\overline{D_j(\lambda)}\overline{\mu}^{2k} \\ D_j(\lambda)\mu^{2k} & 0 \end{pmatrix} \right\} \right. \\
&\qquad \left. + O\left(\frac{1}{k^2}\right) \right)\vec{g}_k,
\end{aligned} \tag{4.10}$$

where the explicit calculation of the product of the seven matrices in (4.10) gives

$$C_j(\lambda) := \frac{|\mu - m_{11}(\lambda)|^2\mu}{2m_{12}(\lambda)} \left(\tilde{\alpha}_3^{(j)}(\lambda)\frac{m_{12}(\lambda)}{\overline{\mu} - m_{11}(\lambda)} + \tilde{\alpha}_4^{(j)}(\lambda)\right) \times \left(-\alpha_2^{(j)}(\lambda) + \alpha_4^{(j)}(\lambda)\frac{m_{12}(\lambda)}{\mu - m_{11}(\lambda)}\right) \quad (4.11)$$

and

$$D_j(\lambda) := \frac{|\mu - m_{11}(\lambda)|^2\mu}{2m_{12}(\lambda)} \left(\tilde{\alpha}_3^{(j)}(\lambda)\frac{m_{12}(\lambda)}{\mu - m_{11}(\lambda)} + \tilde{\alpha}_4^{(j)}(\lambda)\right) \times \left(-\alpha_2^{(j)}(\lambda) + \alpha_4^{(j)}(\lambda)\frac{m_{12}(\lambda)}{\mu - m_{11}(\lambda)}\right). \quad (4.12)$$

Remark Observe that $C_j(\lambda) \neq 0, D_j(\lambda) \neq 0$ for $\lambda \in \sigma_{ell}(J_T)$. Indeed, we prove this property only for $C_j(\lambda)$ (the argument is similar for $D_j(\lambda)$). As μ is non-real, the first two sets of brackets in the definition of $C_j(\lambda)$ are non-zero. Now considering only the third set of brackets we see that if $\widetilde{\alpha}_3^{(j)}(\lambda)$ is non-zero then the bracket is non-vanishing, since $\overline{\mu}$ is non-real. Otherwise, for this bracket to vanish means $\widetilde{\alpha}_4^{(j)}(\lambda)$ should also be zero, and if so the determinant of the matrix product $B_T(\lambda)\dots B_{T-j-1}(\lambda)$ is zero, but this never happens. That the fourth, and final, set of brackets is non-zero follows similarly.

4.3 Application of the Harris-Lutz procedure

Here we employ the Harris-Lutz procedure which will permit the removal of the matrix with components C_j from the expression defining $\vec{g}_{k+1}$. As the C_j term contains no oscillation it cannot cancel the oscillation from the potential and therefore this term can be eliminated using a suitable Harris-Lutz transformation.

Recall that

$$q_n = \frac{c\sin(n\omega + \phi)}{n}$$

for some $c \in \mathbb{R} \setminus \{0\}, \phi \in \mathbb{R}$. Clearly, $q_n = O\left(\frac{1}{n}\right)$. Moreover, by assumption either $\omega T + 2\theta(\lambda) \in 2\pi\mathbb{Z}$ or $\omega T - 2\theta(\lambda) \in 2\pi\mathbb{Z}$, which implies $\omega T \notin 2\pi\mathbb{Z}$ (since $0 < \theta(\lambda) < \pi$).

The Harris-Lutz technique can now be employed to simplify the recurrence equation in (4.10). First, define $\vec{h}_k$ such that

$$\vec{g}_k = (I + G_k)\vec{h}_k \quad (4.13)$$

for some $G_k = O\left(\frac{1}{k}\right) \in \mathbb{C}^{2\times 2}$ that has yet to be defined. Then Equation (4.10) becomes

$$\vec{h}_{k+1} = (I+G_{k+1})^{-1}\Bigg(I + \frac{1}{i\sin\theta(\lambda)}\sum_{j=0}^{T-1}\frac{q_{(k+1)T-j}}{a_{T-j}}\Bigg\{\begin{pmatrix} -\overline{C_j(\lambda)} & 0 \\ 0 & C_j(\lambda)\end{pmatrix}$$
$$+\begin{pmatrix} 0 & -\overline{D_j(\lambda)}\overline{\mu}^{2k} \\ D_j(\lambda)\mu^{2k} & 0\end{pmatrix}\Bigg\} + O\left(\frac{1}{k^2}\right)\Bigg)(I+G_k)\vec{h}_k, \quad (4.14)$$

and by Neumann series

$$(I+G_{k+1})^{-1} = I - G_{k+1} + O\left(\frac{1}{k^2}\right),$$

providing $\|G_{k+1}\| \leq \frac{1}{2}$, strictly less than one. Generally, this condition need not be true, however we may assume this without loss of generality. Indeed, for large values of k the condition is true, and one can rearrange the formula for G_k putting $G_k = 0$, for $k = 1, 2, 3, \dots, N$ for N sufficiently large. It is clear this correction will serve the same goal for the Harris-Lutz transformation satisfying the smallness condition. In what follows we will use this idea every time we use the Harris-Lutz transformation without especially mentioning it.

Define the functions

$$T_1(k) := \frac{1}{i\sin\theta(\lambda)}\sum_{j=0}^{T-1}\frac{q_{(k+1)T-j}}{a_{T-j}}\begin{pmatrix} -\overline{C_j(\lambda)} & 0 \\ 0 & C_j(\lambda)\end{pmatrix}$$

and

$$T_2(k) := \frac{1}{i\sin\theta(\lambda)}\sum_{j=0}^{T-1}\frac{q_{(k+1)T-j}}{a_{T-j}}\begin{pmatrix} 0 & -\overline{D_j(\lambda)}\overline{\mu}^{2k} \\ D_j(\lambda)\mu^{2k} & 0\end{pmatrix}.$$

As $T_1(k)$ and $T_2(k)$ are of order k^{-1}, we see that

$$\begin{aligned}\vec{h}_{k+1} &= (I-G_{k+1})(I+T_1(k)+T_2(k))(I+G_k)\vec{h}_k + O\left(\frac{1}{k^2}\right)\vec{h}_k \\ &= \left(I + G_k - G_{k+1} + T_1(k) + T_2(k) + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k.\end{aligned}$$

Furthermore, setting $G_k := -\sum\limits_{l=0}^{\infty} T_1(k+l)$ (which is well-defined by Proposition 3.3.1), $\kappa := \frac{1}{i\sin\theta(\lambda)}$, and letting $F_j := \frac{1}{a_{T-j}}\begin{pmatrix} -\overline{C_j(\lambda)} & 0 \\ 0 & C_j(\lambda)\end{pmatrix}$ gives

$$T_1(k) = \kappa\sum_{j=0}^{T-1} F_j q_{(k+1)T-j}.$$

Then,

$$
\begin{aligned}
G_k &= -\kappa \sum_{l=0}^{\infty} \sum_{j=0}^{T-1} F_j q_{(k+1)T-j+lT} \\
&= -\kappa \sum_{l=0}^{\infty} \sum_{j=0}^{T-1} F_j \operatorname{Im} \frac{ce^{i(((k+1)T-j+lT)\omega+\phi)}}{(k+1)T-j+lT} = O\left(\frac{1}{k}\right),
\end{aligned}
$$

using Proposition 3.3.1, and consequently G_k has the desired decay properties. Then,

$$
G_{k+1} - G_k = -\sum_{l=1}^{\infty} T_1(k+l) + \sum_{l=0}^{\infty} T_1(k+l) = T_1(k) = O\left(\frac{1}{k}\right), \tag{4.15}
$$

and the Harris-Lutz procedure is successful meaning that Equation (4.10) can now be written as

$$
\vec{h}_{k+1} = \left(I + T_2(k) + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k. \tag{4.16}
$$

In the next section, we will use the Harris-Lutz procedure to get rid of the $T_2(k)$ term stated above, for almost every value $\lambda \in \sigma_{ell}(J_T)$, specifically those that do not satisfy the so-called quantisation conditions.

4.4 The necessity of quantisation conditions

Here the effects of the Harris-Lutz procedure applied previously are seen. Moreover, in its new form the recurrence equation for $\vec{h}_k$ can be rearranged, again, to clarify the role of the potential and the conditions for resonance seen; specifically, what values of $\theta(\lambda)$ prohibit another application of the Harris-Lutz procedure to the entire expression.

So, in the aftermath of the Harris-Lutz procedure, we have

$$
\begin{aligned}
&\vec{h}_{k+1} = \left(I + T_2(k) + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k \\
&= \left(I + \frac{1}{i\sin\theta(\lambda)} \sum_{j=0}^{T-1} \frac{c\sin(((k+1)T-j)\omega+\phi)}{a_{T-j}((k+1)T-j)} \begin{pmatrix} 0 & -\overline{D_j(\lambda)}\overline{\mu}^{2k} \\ D_j(\lambda)\mu^{2k} & 0 \end{pmatrix} \right. \\
&\left. + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k \\
&= \left(I + \frac{1}{i\sin\theta(\lambda)} \sum_{j=0}^{T-1} \frac{ce^{i(((k+1)T-j)\omega+\phi)} - ce^{-i(((k+1)T-j)\omega+\phi)}}{2ia_{T-j}((k+1)T-j)} \right. \\
&\left. \times \begin{pmatrix} 0 & -\overline{D_j(\lambda)}e^{-2ik\theta} \\ D_j(\lambda)e^{i2k\theta} & 0 \end{pmatrix} + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k.
\end{aligned}
$$

Then, using the relation

$$\frac{1}{(k+1)T-j} = \frac{1}{kT} + O\left(\frac{1}{k^2}\right),$$

we obtain

$$\vec{h}_{k+1} = \left(I + \frac{c}{k\sin\theta(\lambda)}\sum_{j=0}^{T-1}\left\{\left(\begin{smallmatrix} 0 & -\overline{a_j(\lambda)}+\overline{b_j(\lambda)} \\ -a_j(\lambda)+b_j(\lambda) & 0 \end{smallmatrix}\right) + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k, \tag{4.17}$$

where

$$a_j(\lambda) := E_j(\lambda)e^{i(k(2\theta(\lambda)+\omega T)+(T-j)\omega+\phi)},$$
$$b_j(\lambda) := E_j(\lambda)e^{-i(k(\omega T-2\theta(\lambda))+(T-j)\omega+\phi)}$$

and

$$E_j(\lambda) := \frac{D_j(\lambda)}{2Ta_{T-j}}. \tag{4.18}$$

It is natural to ask whether the Harris-Lutz technique can be applied, again, in order to further simplify the recurrence relation. The next result shows that this can be done whenever the so-called quantisation conditions

$$\omega T \pm 2\theta(\lambda) \in 2\pi\mathbb{Z} \tag{4.19}$$

are not satisfied and that, for λ not satisfying the quantisation conditions, we only have solutions bounded from above and towards zero. However, decay is needed for a subordinate solution.

Remark The quantisation formula gives the only possible location for eigenvalues in the a.c. spectrum involving integer parameters in the style of the Bohr-Sommerfield condition (see, for example, Lemma 2.2 [49]). For the (continuous) periodic Schrödinger operator case this appears in [47].

Theorem 4.4.1. *Assume $\lambda \in \sigma_{ell}(J_T)$ does not satisfy either of the quantisation conditions in* (4.19). *Then there is no subordinate solution to the perturbed recurrence relations in* (4.2). *Moreover, each non-zero solution of the relation in* (4.2) *is bounded above and towards zero, exactly like the solution to the unperturbed system described by* (1.18).

Proof. We introduce a new sequence of vectors $\vec{l}_k$ such that

$$\vec{h}_k = (I + H_k)\vec{l}_k \tag{4.20}$$

and where H_k will be defined below to satisfy $H_k = O\left(\frac{1}{k}\right)$. Then, (4.16) implies

$$\begin{aligned}\vec{l}_{k+1} &= (I + H_{k+1})^{-1}\left(I + T_2(k) + O\left(\frac{1}{k^2}\right)\right)(I + H_k)\,\vec{l}_k \\ &= \left(I + H_k - H_{k+1} + T_2(k) + O\left(\frac{1}{k^2}\right)\right)\vec{l}_k.\end{aligned}$$

By Proposition 3.3.1 we have

$$\sum_{m=k}^{\infty}\sum_{j=0}^{T-1}\frac{e^{i(m(\omega T\pm 2\theta(\lambda))+(T-j)\omega+\phi)}}{m}=O\left(\frac{1}{k}\right),$$

since both $\omega T\pm 2\theta(\lambda)\notin 2\pi\mathbb{Z}$ and choosing

$$H_k=-\sum_{j=k}^{\infty}T_2(j)=O\left(\frac{1}{k}\right)$$

with

$$H_{k+1}-H_k=T_2(k)$$

we obtain

$$\vec{l}_{k+1}=\left(I+O\left(\frac{1}{k^2}\right)\right)\vec{l}_k.$$

Without loss of generality we assume the matrices $\left(I+O\left(\frac{1}{k^2}\right)\right)$ are invertible for all $k\in\mathbb{N}$. Moreover, using an elementary result (for example Lemma 2.1 in [33]) we have

$$\vec{l}_k=(C+o\,(1))\vec{l}_1, \tag{4.21}$$

where $C\in\mathbb{C}^{2\times 2}$ is invertible and $\lim_{k\to\infty}\vec{l}_k=C\vec{l}_1$. Then, substituting (4.21) into (4.20) we obtain

$$\vec{h}_k=(I+H_k)(C+o(1))\vec{l}_1 \tag{4.22}$$

and substituting this into (4.13) gives

$$\vec{g}_k=(I+G_k)(I+H_k)(C+o(1))\vec{l}_1. \tag{4.23}$$

Substituting (4.23) into (4.9) gives

$$\vec{f}_k=V(\lambda)(I+G_k)(I+H_k)(C+o(1))\vec{l}_1 \tag{4.24}$$

and, in turn, substituting this into (4.6) we obtain

$$\vec{u}_{kT}=M^k(\lambda)V(\lambda)(I+G_k)(I+H_k)(C+o(1))\vec{l}_1. \tag{4.25}$$

Finally, recalling that $H_k,G_k\to 0$ and $B_j(\lambda+q_{kT+j})\to B_j(\lambda)$ as $k\to\infty$ we have

$$\vec{u}_{kT+s}=B_s(\lambda)B_{s-1}(\lambda)\dots B_1(\lambda)M^k(\lambda)\left(\vec{r}(\lambda)+o\,(1)\right) \tag{4.26}$$

for $s\in\{0,\dots,T-1\}$ and where $\vec{r}:=V(\lambda)C\vec{l}_1\in\mathbb{C}^2$ which is arbitrary since $\vec{l}_1$ is arbitrary. Consequently, the solution to the perturbed system, (4.2), behaves like the solution to the unperturbed system, (1.18). Moreover, the solutions are bounded from above and therefore there are no subordinate solutions by the generalised Behnke-Stolz Lemma (see [33]). $\square$

Remark The set of λ satisfying the quantisation conditions is discrete and since the intervals of a.c. spectrum are closed the theorem shows that

$$\sigma_{a.c.}(J_T) \subseteq \sigma_{a.c.}(J_T + Q).$$

Moreover, since the a.c. spectrum always belongs to the essential spectrum, and the Weyl-Theorem gives that the essential spectrum for the perturbed periodic Jacobi operator is the same as for the unperturbed periodic Jacobi operator, we also have

$$\sigma_{a.c.}(J_T + Q) \subseteq \sigma_{ess}(J_T + Q) = \sigma_{ess}(J_T).$$

Finally, by Corollary 1.3.6 we have that

$$\sigma_{ess}(J_T) = \sigma_{a.c.}(J_T).$$

All together these give

$$\sigma_{a.c.}(J_T) = \sigma_{a.c.}(J_T + Q).$$

4.5 Resonance cases and asymptotic behaviour of subordinate solutions

In this section, the final steps of the method are carried out. Indeed, each of the quantisation conditions are considered, giving three resonance cases in total. In each of the resonance cases, various techniques are employed (including the Harris-Lutz transformation, again, although not to the entire expression which the resonance cases prohibit) so that ultimately it is established that up to a few exceptions, regardless of what resonance case we are in, a decaying solution exists.

Without loss of generality, in the consideration below we confine ourselves to one band of $\sigma_{ell}(J_T)$. Choose ω such that $0 < \omega < 2\pi$. All the resonance cases can be described as follows:

$$\text{Case 1} : 2\theta(\lambda) + \omega T = 2k_+\pi, \text{ where } k_+ \in \{1, \ldots, T\}, \ \omega T \notin \pi\mathbb{Z}.$$

This range of k_+ is a consequence of $0 < \omega T + 2\theta(\lambda) < 2\pi(T + 1)$.

$$\text{Case 2} : 2\theta(\lambda) - \omega T = -2k_-\pi, \text{ where } k_- \in \{0, \ldots, T - 1\}, \ \omega T \notin \pi\mathbb{Z}.$$

The range of k_- follows as a similar consideration to k_+. And, finally, the special case where both first conditions in Cases 1 and 2 are satisfied.

$$\text{Case 3} : 2\theta(\lambda) + \omega T = 2k_+\pi, 2\theta(\lambda) - \omega T = -2k_-\pi,$$

where $k_+ \in \{1, \ldots, T\}, k_- \in \{0, \ldots, T-1\}$. The range of k_+, k_- follow similarly to before. Indeed, by considering $\theta(\lambda)$ we see that here $k_- = k_+ - 1, \theta(\lambda) = \frac{\pi}{2}$ (which corresponds to the generalised 'midpoint' of one band of $\sigma_{ell}(J_T)$) and

$\omega T = (k_+ + k_-)\pi$. Note that according to Theorem 4.4.1 one of these three cases will need to hold to obtain a subordinate solution.

Furthermore, since we will be discussing the asymptotics of recurrences we introduce an equivalence on the set of recurrences, two recurrences being equivalent when their solutions have the same asymptotic behaviour. Specifically, we say $(a) \sim_r (b)$ where

$$\begin{aligned} (a): \quad & a_{k+1} = A_k a_k \quad \forall k \in \mathbb{N} \\ (b): \quad & b_{k+1} = B_k b_k \quad \forall k \in \mathbb{N} \end{aligned}$$

with $A_k, B_k \in \mathbb{C}^{2\times 2}$ and invertible for all $k > N$, for some $N \in \mathbb{N}$, whenever for any solution (a_n) of (a) there exists a solution (b_n) of (b) such that $a_n = C_n b_n$, for all $n > N$ and where $\lim_{n\to\infty} C_n$ exists and is invertible.

Before we start discussing the separate cases we state and prove a lemma that will be used in the following arguments.

Lemma 4.5.1. *Let $A, \widetilde{c}$ be complex constants. The recurrence*

$$(u): \quad \vec{u}_{k+1} = \left(I + \frac{\widetilde{c}}{k} \begin{pmatrix} 0 & -\overline{A} \\ -A & 0 \end{pmatrix} + O\left(\frac{1}{k^2}\right) \right) \vec{u}_k \quad \forall k \tag{4.27}$$

is equivalent to the recurrence

$$(v): \quad \vec{v}_{k+1} = \left(I + \frac{\widetilde{c}}{k} \begin{pmatrix} 0 & -\overline{A} \\ -A & 0 \end{pmatrix} \right) \vec{v}_k \quad \forall k,$$

i.e. $(u) \sim_r (v)$.

Proof. The case $\widetilde{c} = 0$ is trivial. For the case $\widetilde{c} \neq 0$, the result follows from a generalisation of the Janas-Moszynski result (see Theorem 2 in [63]), however we must check that the following conditions are satisfied. We need to write (4.27) in the following form

$$\vec{u}_{k+1} = (I + p_k V_k + R_k)\, \vec{u}_k,$$

where

1. $p_k \geq 0, p_k \to 0$ and $\sum\limits_{k=1}^{\infty} p_k = \infty$,
2. $\{R_k\}$ is a sequence of 2×2 matrices each matrix element belonging to the sequence space l^1,
3. $\{V_k\}$ such that $\sum\limits_{k=1}^{\infty} \|V_{k+1} - V_k\| < \infty$ with $\operatorname{disc} V_k > 0$ and satisfying $\operatorname{disc}(\lim_{k\to\infty} V_k) \neq 0$, where $\operatorname{disc} V_k := (\operatorname{Tr}(V_k))^2 - 4\det(V_k)$.

Defining $p_k := \frac{1}{k}$, the first condition is satisfied. Then defining R_k as the error term of matrices of order $O\left(\frac{1}{k^2}\right)$ we see that the second condition is also satisfied. Finally, defining

$$V_k := \widetilde{c} \begin{pmatrix} 0 & -\overline{A} \\ -A & 0 \end{pmatrix},$$

we see that V_k is just a constant matrix sequence and immediately satisfies the first constraint in condition 3, the other two following from the conjugate entries of the matrix. $\square$

We now resume our discussion of the different cases.

Case 1 Here, $\omega T + 2\theta(\lambda) = 2k_+\pi, \quad \omega T \notin \pi\mathbb{Z}$. We have from (4.17) that

$$\vec{h}_{k+1} = \left(I + \frac{c}{k\sin\theta(\lambda)}\sum_{j=0}^{T-1}\left\{\begin{pmatrix} 0 & -\overline{E_j(\lambda)}e^{-i((T-j)\omega+\phi)} \\ -E_j(\lambda)e^{i((T-j)\omega+\phi)} & 0 \end{pmatrix}\right.\right.$$
$$\left.\left. + \begin{pmatrix} 0 & \overline{E_j(\lambda)}e^{i(2k\omega T+(T-j)\omega+\phi)} \\ E_j(\lambda)e^{-i(2k\omega T+(T-j)\omega+\phi)} & 0 \end{pmatrix}\right\} + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k.$$

Then the Harris-Lutz procedure (i.e. a substitution of the form $\vec{h}_k = (I + \hat{H}_k)\vec{m}_k$) can be used again to get rid of the third term (as $\omega T \notin \pi\mathbb{Z}$). Then, removing the error term using Lemma 4.5.1, and for a suitable choice of $\hat{H}_k$ (similar to the proof of Theorem 4.4.1), we have

$$\vec{m}_{k+1} = \left(I + \frac{c}{k\sin\theta(\lambda)}\sum_{j=0}^{T-1}\begin{pmatrix} 0 & -\overline{E_j(\lambda)}e^{-i((T-j)\omega+\phi)} \\ -E_j(\lambda)e^{i((T-j)\omega+\phi)} & 0 \end{pmatrix}\right. \tag{4.28}$$
$$\left. + O\left(\frac{1}{k^2}\right)\right)\vec{m}_k$$
$$\sim_r \vec{m}_{k+1} = \left(I + \frac{c}{k\sin\theta(\lambda)}\begin{pmatrix} 0 & -\overline{E(\lambda;k_+)} \\ -E(\lambda;k_+) & 0 \end{pmatrix}\right)\vec{m}_k, \tag{4.29}$$

where

$$E(\lambda;k_+) := \sum_{j=0}^{T-1} E_j(\lambda)e^{i((T-j)\omega+\phi)}. \tag{4.30}$$

Observing that $\omega = \omega(\lambda;k_+) = \frac{-2\theta(\lambda)+2k_+\pi}{T}$ we see

$$e^{i(T-j)\omega} = \mu^{-2}(\lambda)\left(\mu^{\frac{2}{T}}(\lambda)e^{-\frac{i2k_+\pi}{T}}\right)^j.$$

By Corollary 2.1.5,

$$\operatorname{Tr} M(\lambda) = \mu(\lambda) + \frac{1}{\mu(\lambda)} \sim \frac{\lambda^T}{\prod_{s=1}^{T} a_s} \text{ as } \lambda \to \infty.$$

We choose the branch of the square-root so that $\mu(\lambda)$ is decreasing as $\lambda \to \infty$ and thus it follows that $\mu(\lambda) \sim \frac{\prod_{s=1}^{T} a_s}{\lambda^T}$. Also, there exists an appropriate branch of $\left(\mu^2(\lambda)\right)^{\frac{1}{T}}$ such that

$$\mu^{\frac{2}{T}}(\lambda) = e^{\frac{i2\pi l_+}{T}}\left(\mu_g^2(\lambda)\right)^{\frac{1}{T}},$$

for some $l_+ \in \{1, \dots, T\}$, with $\left(\mu_g^2(\lambda)\right)^{\frac{1}{T}} \sim \frac{\prod_{s=1}^{T} a_s^{\frac{2}{T}}}{\lambda^2}$ as $\lambda \to \infty$. Then,

$$\begin{aligned} e^{i(T-j)\omega} &= \mu^{-2}(\lambda)\left(\left(\mu_g^2(\lambda)\right)^{\frac{1}{T}} e^{\frac{i2\pi l_+}{T}} e^{-\frac{i2k_+\pi}{T}}\right)^j \\ &= \mu^{-2}(\lambda)\left(\mu_g^2(\lambda)\right)^{\frac{j}{T}}, \end{aligned}$$

if k_+ is chosen such that $k_+ = l_+$. This particular choice of k_+, which will vary depending on λ, ensures that there is no oscillation occurring in the expression $e^{i(T-j)\omega}$ between different values of j. We denote

$$E(\lambda) := E(\lambda; l_+). \tag{4.31}$$

Then $E(\lambda) = e^{i\phi}\mu^{-2}(\lambda) \sum_{j=0}^{T-1} E_j(\lambda)\left(\mu_g^2(\lambda)\right)^{\frac{j}{T}}$.

Lemma 4.5.2. *The function $E(\lambda)$ is algebraic and is not identically zero.*

Proof. From the explicit formula (4.18) for $E_j(\lambda)$, and $\frac{1}{\mu(\lambda)}$, we see that $E(\lambda)$ is algebraic. To show that the function is not identically zero, we consider the case $T = 1$ first. Here, $E(\lambda)$ is just a non-negative multiple of D_0, from (4.12), and therefore, by the remark at the end of Section 4.2, the function is not only non-trivial, but also non-zero for $\lambda \in \sigma_{ell}(J_T)$.

Next, consider $T \geq 2$. Letting $\lambda \to \infty, \mu \to 0$ we show that the highest-order term does not cancel. Note that from Corollary 2.1.5

$$m_{11}(\lambda) \sim -\frac{a_T \lambda^{T-2}}{\prod_{s=1}^{T-1} a_s}, m_{12}(\lambda) \sim \frac{\lambda^{T-1}}{\prod_{s=1}^{T-1} a_s},$$

and using

$$\begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix} \begin{pmatrix} \widetilde{\alpha}_1^{(j)}(\lambda) & \widetilde{\alpha}_2^{(j)}(\lambda) \\ \widetilde{\alpha}_3^{(j)}(\lambda) & \widetilde{\alpha}_4^{(j)}(\lambda) \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_1^{(j)}(\lambda) & \alpha_2^{(j)}(\lambda) \\ \alpha_3^{(j)}(\lambda) & \alpha_4^{(j)}(\lambda) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{a_{T-j-1}}{a_{T-j}} & \frac{\lambda}{a_{T-j}} \end{pmatrix}$$

we see from the (1,1) entry

$$\begin{aligned} \widetilde{\alpha}_3^{(j)} m_{12} + \widetilde{\alpha}_4^{(j)}(\mu - m_{11}) &= \frac{a_{T-j-1}}{a_{T-j}} \alpha_2^{(j)} \left(\widetilde{\alpha}_1^{(j)}\widetilde{\alpha}_4^{(j)} - \widetilde{\alpha}_2^{(j)}\widetilde{\alpha}_3^{(j)}\right) + \mu\widetilde{\alpha}_4^{(j)} \\ &= \frac{a_{T-j-1}}{a_{T-j}} \alpha_2^{(j)} \det\left(B_{T-j-1}(\lambda) \dots B_1(\lambda)\right) + \mu\widetilde{\alpha}_4^{(j)} \\ &= \frac{a_T}{a_{T-j}} \alpha_2^{(j)} + \mu\widetilde{\alpha}_4^{(j)}, \quad j \in \{0, \dots, T-1\}, \end{aligned} \tag{4.32}$$

where we have used $\det B_j(\lambda) = \frac{a_{j-1}}{a_j}$.

Then, recalling (4.12) and Lemma 4.2.1, for $j \notin \{0, T-2, T-1\}$ we have that the leading term of $D_j(\lambda)e^{i(T-j)\omega}$ as λ tends to infinity is

$$\frac{1-\mu(\lambda)m_{11}(\lambda)}{2(\mu(\lambda)-m_{11}(\lambda))m_{12}(\lambda)}\left(\frac{a_T a_{T-j}\lambda^{3T-2}}{\left(\prod_{s=1}^{T} a_s^{\frac{3T-2j}{T}}\right)\left(\prod_{l=T-j}^{T-1} a_l^2\right)}\right). \tag{4.33}$$

For $j = 0$ the leading term of $D_j(\lambda)e^{i(T-j)\omega}$, as λ tends to infinity, is

$$\frac{1-\mu(\lambda)m_{11}(\lambda)}{2(\mu(\lambda)-m_{11}(\lambda))m_{12}(\lambda)}\left(\frac{a_T^2\lambda^{3T-2}}{\prod_{s=1}^{T} a_s^3}\right). \tag{4.34}$$

For $j = T-2$ the leading term of $D_j(\lambda)e^{i(T-j)\omega}$, as λ tends to infinity, is

$$\frac{1-\mu(\lambda)m_{11}(\lambda)}{2(\mu(\lambda)-m_{11}(\lambda))m_{12}(\lambda)}\left(\frac{a_T^3 a_1^2 a_2\lambda^{3T-2}}{\prod_{s=1}^{T} a_s^{\frac{3T+4}{T}}}\right). \tag{4.35}$$

For $j = T-1$ the leading term of $D_j(\lambda)e^{i(T-j)\omega}$, as λ tends to infinity, is

$$\frac{1-\mu(\lambda)m_{11}(\lambda)}{2(\mu(\lambda)-m_{11}(\lambda))m_{12}(\lambda)}\left(\frac{a_1 a_T^3\lambda^{3T-2}}{\prod_{s=1}^{T} a_s^{\frac{2+3T}{T}}}\right). \tag{4.36}$$

Since for each possible j the leading term, up to an identical complex non-zero constant, is positive there is no chance of their cancellation in the sum that comprises $E(\lambda)$, and therefore the function is not identically zero. $\square$

Remark The function $E(\lambda)$ is algebraic, and therefore only has finitely many roots. Moreover, for $T = 1$ there are no roots in $\sigma_{ell}(J_T)$. For the case $T = 2$ with zero diagonal ($b_i = 0$) we see by explicit calculation that

$$E(\lambda; k_+) = \frac{e^{i\phi}(a_1 + a_2\mu)}{8\lambda(a_1\mu + a_2)}\left[\frac{\mu\lambda^2 e^{i2\omega(\lambda;k_+)}}{a_1 a_2} + \left(\frac{\lambda^2 - a_2^2}{a_1 a_2} - \mu\right)\left(\frac{a_2}{a_1} + \mu\right)e^{i\omega(\lambda;k_+)}\right].$$

Then, we have that $E(\lambda; 2) = 0$ if and only if $\lambda = \pm|a_1 - a_2|$ which do not belong to the generalised interior. Similarly, we have that $E(\lambda; 1) = 0$ if and only if $\lambda = \pm(a_1 + a_2)$, which again do not belong to the elliptic interval. Indeed, the points $\pm|a_1 - a_2|, \pm(a_1 + a_2)$ lie on the boundary.

We now continue with a matrix transform. $\Gamma(\lambda) := \begin{pmatrix} 0 & -\frac{\overline{E(\lambda)}}{|E(\lambda)|} \\ -\frac{E(\lambda)}{|E(\lambda)|} & 0 \end{pmatrix}$, for $\lambda \in \sigma_{ell}(J_T)$ with $E(\lambda) \neq 0$, is Hermitian, has trace zero and determinant equal to -1. This information dictates that $\Gamma(\lambda)$ has eigenvalues 1 and -1 and is thus diagonalisable, i.e.

$$\Gamma(\lambda) = W(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W^{-1}(\lambda),$$

where $W(\lambda)$ is the 2×2 matrix whose columns are the eigenvectors of Γ. Consequently, we see that choosing $k_+ = l_+$, (4.29) becomes

$$\begin{aligned} \vec{m}_{k+1} &= W(\lambda) \left[\prod_{t=1}^{k} \left(I + \frac{c|E(\lambda)|}{t \sin\theta(\lambda)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right] W^{-1}(\lambda) \vec{m}_1 \\ &= W(\lambda) \begin{pmatrix} \prod_{t=1}^{k} \left(1 + \frac{c|E(\lambda)|}{t \sin\theta(\lambda)}\right) & 0 \\ 0 & \prod_{t=1}^{k} \left(1 - \frac{c|E(\lambda)|}{t \sin\theta(\lambda)}\right) \end{pmatrix} W^{-1}(\lambda) \vec{m}_1 \\ \sim_r \vec{m}_{k+1} &= \begin{pmatrix} (\widetilde{c}_1 + o(1))\, k^{\frac{c|E(\lambda)|}{\sin\theta(\lambda)}} & 0 \\ 0 & (\widetilde{c}_2 + o(1))\, k^{-\frac{c|E(\lambda)|}{\sin\theta(\lambda)}} \end{pmatrix} \vec{m}_1, \end{aligned}$$

for some non-zero constants $\widetilde{c}_1, \widetilde{c}_2$ depending on $\lambda \in \sigma_{ell}(J_T)$ and where Lemma 1.6.5 and the integral test were used to obtain

$$\prod_{t=1}^{k} \left(1 + \frac{c|E(\lambda)|}{t \sin\theta(\lambda)}\right) = (\widetilde{c} + o(1))\, e^{\frac{c|E(\lambda)|}{\sin\theta(\lambda)} \sum_{t=1}^{k} \frac{1}{t}} = (\widetilde{c} + o(1)) e^{\frac{c|E(\lambda)| \ln k}{\sin\theta(\lambda)}}.$$

Retracing the steps back to the original u_n (as in the case of Theorem 4.4.1), we see that this implies there exists a subordinate solution of the final system, (4.2), asymptotically equivalent to $k^{-\left|\frac{cE(\lambda)}{\sin\theta(\lambda)}\right|}$. This is in $l^2(\mathbb{N};\mathbb{C})$ if c is large enough:

$$\left| \frac{cE(\lambda)}{\sin\theta(\lambda)} \right| > \frac{1}{2},$$

where the value of $E(\lambda)$ is assumed to be non-zero. This completes the analysis for Case 1.

Case 2 Here, $2\theta(\lambda) - \omega T = -2k_-\pi, \omega T \notin \pi\mathbb{Z}$. We have from (4.17) that

$$\begin{aligned} \vec{h}_{k+1} = \Bigg(I + \frac{c}{k \sin\theta(\lambda)} \sum_{j=0}^{T-1} \Bigg\{ &\begin{pmatrix} 0 & \overline{E_j(\lambda)} e^{i((T-j)\omega+\phi)} \\ E_j(\lambda) e^{-i((T-j)\omega+\phi)} & 0 \end{pmatrix} \\ - &\begin{pmatrix} 0 & \overline{E_j(\lambda)} e^{-i(2Tk\omega+(T-j)\omega+\phi)} \\ E_j(\lambda) e^{i(2Tk\omega+(T-j)\omega+\phi)} & 0 \end{pmatrix} \Bigg\} + O\left(\frac{1}{k^2}\right) \Bigg) \vec{h}_k. \end{aligned}$$

Then the Harris-Lutz procedure can be used again (i.e. a substitution of the from $\vec{h}_k = (I + \widetilde{H}_k)\vec{m}_k$) to get rid of the oscillating second term (as $\omega T \notin \pi\mathbb{Z}$). Then, removing the error term using Lemma 4.5.1, and for a suitable choice of $\widetilde{H}_k$ (similar to the proof of Theorem 4.4.1), we have

$$\begin{aligned}
\vec{m}_{k+1} &= \left(I + \frac{c}{k\sin\theta(\lambda)} \sum_{j=0}^{T-1} \begin{pmatrix} 0 & \overline{E_j(\lambda)} e^{i((T-j)\omega+\phi)} \\ E_j(\lambda) e^{-i((T-j)\omega+\phi)} & 0 \end{pmatrix} \right. \\
&\left. + O\left(\frac{1}{n^2}\right)\right)\vec{m}_k \\
\sim_r \vec{m}_{k+1} &= \left(I + \frac{c|\widetilde{E}(\lambda;k_-)|}{k\sin\theta(\lambda)} \begin{pmatrix} 0 & \frac{\overline{\widetilde{E}(\lambda;k_-)}}{|\widetilde{E}(\lambda;k_-)|} \\ \frac{\widetilde{E}(\lambda;k_-)}{|\widetilde{E}(\lambda;k_-)|} & 0 \end{pmatrix} \right)\vec{m}_k,
\end{aligned}$$

where

$$\widetilde{E}(\lambda;k_-) := \sum_{j=0}^{T-1} E_j(\lambda) e^{-i((T-j)\omega+\phi)}. \tag{4.37}$$

Observing that $\omega = \omega(\lambda;k_-) = \frac{2\theta(\lambda)+2k_-\pi}{T}$ we see

$$e^{-i(T-j)\omega} = \mu^{-2}\left(\mu^{\frac{2}{T}}(\lambda) e^{\frac{i2k_-\pi}{T}}\right)^j.$$

As in Case 1 we can choose the branch such that $\mu(\lambda) \sim \frac{\prod_{s=1}^{T} a_s}{\lambda^T}$ as $\lambda \to \infty$ and there exists an appropriate branch of $\left(\mu^2(\lambda)\right)^{\frac{1}{T}}$ such that

$$\mu^{\frac{2}{T}} = e^{\frac{-i2\pi l_-}{T}} \left(\mu_g^2(\lambda)\right)^{\frac{1}{T}},$$

for some $l_- \in \{0, \dots, T-1\}$. Then

$$\begin{aligned}
e^{i(T-j)\omega} &= \mu^{-2}(\lambda)\left(\left(\mu_g^2(\lambda)\right)^{\frac{1}{T}}(\lambda) e^{\frac{-i2\pi l_-}{T}} e^{\frac{ik_-\pi}{T}}\right)^j \\
&= \mu(\lambda)^{-2}\left(\mu_g^2(\lambda)\right)^{\frac{j}{T}},
\end{aligned}$$

if k_- is chosen such that $k_- = l_-$. We denote

$$\widetilde{E}(\lambda) := \widetilde{E}(\lambda; l_-). \tag{4.38}$$

Lemma 4.5.3. *The function $\widetilde{E}(\lambda)$ is algebraic and is not identically zero. Moreover we have that*

$$\widetilde{E}(\lambda) = e^{-i2\phi} E(\lambda).$$

Proof. We see that

$$E(\lambda) = e^{i\phi} \sum_{i=0}^{T-1} E_j(\lambda) e^{\frac{i2(T-j)}{T}}; \quad \widetilde{E}(\lambda) = e^{-i\phi} \sum_{j=1}^{T-1} E_j(\lambda) e^{\frac{i2(T-j)}{T}} = e^{-i2\phi} E(\lambda).$$

That $\widetilde{E}(\lambda)$ is algebraic follows from the corresponding result for $E(\lambda)$. $\square$

Remark For $T=1$ we have that

$$E(\lambda;k_+) = \frac{e^{i\phi}}{4a_1\mu^2},\ \widetilde{E}(\lambda;k_-) = \frac{e^{-i\phi}}{4a_1\mu^2}.$$

If $T>1$ then the functions $E(\lambda;k_+), \widetilde{E}(\lambda;k_-)$, besides the trivial dependence on the parameter ϕ, depend on the frequency, ω, of the perturbation, (q_n), through the integer quantisation parameters, k_+, k_-, respectively. For $T=2$ we have that $E(\lambda;2)=e^{i\phi}A(\lambda), \widetilde{E}(\lambda;0)=e^{-i\phi}A(\lambda)$ where

$$A(\lambda) := \frac{a_1+a_2\mu}{8a_1^2a_2\lambda(a_1\mu+a_2)}\left[\frac{1}{\mu}(\lambda^2(a_1+a_2)-a_2^3)-\mu a_1^2a_2-2a_2^2a_1+\lambda^2a_1\right],$$

and $E(\lambda;1)=e^{i\phi}B(\lambda), \widetilde{E}(\lambda;1)=e^{-i\phi}B(\lambda)$ where

$$B(\lambda) := \frac{a_1+a_2\mu}{8a_1^2a_2\lambda(a_1\mu+a_2)}\left[\frac{1}{\mu}(\lambda^2(a_1-a_2)+a_2^3)+\mu a_1^2a_2+2a_2^2a_1-\lambda^2a_1\right].$$

The simple relationships between $E(\lambda;k_+)$ and $\widetilde{E}(\lambda;k_-)$ in general do not hold for $T>2$. This can be seen from the fact that

$$e^{-i((T-j)\omega+\phi)} = e^{-i(T-j)\left(\frac{2\theta+2\pi k_-}{T}\right)-i\phi} = \mu^{\frac{2(T-j)}{T}}e^{\frac{2ij\pi k_-}{T}}e^{-i\phi}, \tag{4.39}$$

for $k_- \in \{0,\dots,T-1\}$ and

$$e^{i((T-j)\omega+\phi)} = e^{i(T-j)\left(\frac{2\pi k_+-2\theta}{T}\right)+i\phi} = \mu^{\frac{2(T-j)}{T}}e^{-\frac{2ij\pi k_+}{T}}e^{i\phi}, \tag{4.40}$$

for $k_+ \in \{1,\dots,T\}$.

By using the same diagonalisation argument as described in Case 1 we see that (4.29) becomes

$$\begin{aligned}
\vec{m}_{k+1} &= W(\lambda)\left[\prod_{t=1}^{k}\left(I+\frac{c|\widetilde{E}(\lambda)|}{t\sin\theta(\lambda)}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\right]W^{-1}(\lambda)\vec{m}_1\\
&= W(\lambda)\begin{pmatrix}\prod_{t=1}^{k}\left(1+\frac{c|\widetilde{E}(\lambda)|}{t\sin\theta(\lambda)}\right) & 0\\ 0 & \prod_{t=1}^{k}\left(1-\frac{c|\widetilde{E}(\lambda)|}{t\sin\theta(\lambda)}\right)\end{pmatrix}W^{-1}(\lambda)\vec{m}_1\\
\sim_r \vec{m}_{k+1} &= \begin{pmatrix}(\widetilde{c}_3+o(1))\,k^{\frac{c|\widetilde{E}(\lambda)|}{\sin\theta(\lambda)}} & 0\\ 0 & (\widetilde{c}_4+o(1))\,k^{-\frac{c|\widetilde{E}(\lambda)|}{\sin\theta(\lambda)}}\end{pmatrix}\vec{m}_1,
\end{aligned}$$

for some non-zero constants $\widetilde{c}_3, \widetilde{c}_4$ depending on $\lambda \in \sigma_{ell}(J_T)$. As in Case 1, this implies there exists a subordinate solution of the final system, (4.2), asymptotically equivalent to $k^{-\left|\frac{c\widetilde{E}(\lambda)}{\sin\theta(\lambda)}\right|}$. This is in $l^2(\mathbb{N};\mathbb{C})$ if c is large enough:

$$\left|\frac{c\widetilde{E}(\lambda)}{\sin\theta(\lambda)}\right| > \frac{1}{2},$$

where the value of $\widetilde{E}(\lambda)$ is assumed to be non-zero. This completes the analysis for Case 2.

Case 3 Here $2\theta(\lambda) + \omega T = 2k_+\pi, 2\theta(\lambda) - \omega T = -2k_-\pi$ which implies $\theta(\lambda) = \frac{\pi}{2}, k_- = k_+ - 1$ and $\omega T = (2k_+ - 1)\pi$, $k_+ \in \{1, \dots, T\}$. Thus, there are no oscillating terms and another application of the Harris-Lutz procedure is not needed. Then, from (4.17) and removing the error term using Lemma 4.5.1, we have

$$\vec{h}_{k+1} = \left(I - \frac{c}{k} \sum_{j=0}^{T-1} \begin{pmatrix} 0 & \overline{E_j(\lambda)}\left(e^{-i((T-j)\omega+\phi)} - e^{i((T-j)\omega+\phi)}\right) \\ E_j(\lambda)\left(e^{i((T-j)\omega+\phi)} - e^{-i((T-j)\omega+\phi)}\right) & 0 \end{pmatrix} \right.$$
$$\left. + O\left(\frac{1}{k^2}\right) \right) \vec{h}_k$$
$$\sim_r \vec{h}_{k+1} = \left(I - \frac{c|\widetilde{\widetilde{E}}(k_+)|}{k} \sum_{j=0}^{T-1} \begin{pmatrix} 0 & \frac{\overline{\widetilde{\widetilde{E}}(k_+)}}{|\widetilde{\widetilde{E}}(k_+)|} \\ \frac{\widetilde{\widetilde{E}}(k_+)}{|\widetilde{\widetilde{E}}(k_+)|} & 0 \end{pmatrix} \right) \vec{h}_k,$$

where

$$\widetilde{\widetilde{E}}(k_+) := \sum_{j=0}^{T-1} E_j\left(\theta^{-1}\left(\frac{\pi}{2}\right)\right)\left(e^{i(j\omega+\phi)} - e^{i(-j\omega+\phi)}\right) \tag{4.41}$$

with the simplification $e^{i\omega T} = -1$ and the error term was removed using Lemma 4.5.1.

Remark Due to the very specific relations between k_+ and k_- for this case, it is not possible to imitate the technique employed in $E(\lambda), \widetilde{E}(\lambda)$, to eliminate the roots of unity that arise from the expression $e^{\pm i(T-j)\omega}$, and show that the expression, $\widetilde{\widetilde{E}}(\lambda)$, is not identically zero simply by looking at the signs of the leading terms. However, since this case only arises when $\theta(\lambda) = \frac{\pi}{2}$, then by the strict monotonicity of $\theta(\lambda)$ on the elliptic interval (see Proposition 1.3.8) there is only one λ in each band of essential spectrum that is possibly excluded. Indeed, for the case $T = 1$ we see that $\omega = \pi$ and

$$\widetilde{\widetilde{E}}(k_+) = \frac{-1}{4a_1 e^{i\phi}}\left(e^{2i\phi} - 1\right),$$

which implies that providing $\phi \not\equiv 0 \mod \pi$ the function $\widetilde{\widetilde{E}}(\lambda)$ is non-zero and the technique is therefore applicable. Note that in the current case if $\phi \equiv 0 \mod \pi$, then $q_n \equiv 0$, so there are no subordinate solutions.

Now, using the same diagonalisation argument described in Case 1, and defining $W := W(\theta^{-1}(\frac{\pi}{2}))$ we obtain

$$\vec{h}_{k+1} = W\left[\prod_{t=1}^{k}\left(I + \frac{c|\widetilde{\widetilde{E}}(k_+)|}{t}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right]W^{-1}\vec{h}_1$$

$$= W\begin{pmatrix} \prod_{t=1}^{k}\left(1 + \frac{c|\widetilde{\widetilde{E}}(k_+)|}{t}\right) & 0 \\ 0 & \prod_{t=1}^{k+1}\left(1 - \frac{c|\widetilde{\widetilde{E}}(k_+)|}{t}\right) \end{pmatrix}W^{-1}\vec{h}_1$$

$$\sim_r \vec{h}_{k+1} = \begin{pmatrix} (\widetilde{c}_5 + o(1))\, k^{c|\widetilde{\widetilde{E}}(k_+)|} & 0 \\ 0 & (\widetilde{c}_6 + o(1))\, k^{-c|\widetilde{\widetilde{E}}(k_+)|} \end{pmatrix}\vec{h}_1,$$

for non-zero constants $\widetilde{c}_5, \widetilde{c}_6$ depending on $\lambda \in \sigma_{ell}(J_T)$. As in Cases 1 and 2, this implies there exists a subordinate solution of the final system asymptotically equivalent to $k^{-c|\widetilde{\widetilde{E}}(k_+)|}$. This is in $l^2(\mathbb{N};\mathbb{C})$ if c is large enough:

$$\left|c\widetilde{\widetilde{E}}(k_+)\right| > \frac{1}{2},$$

where $\widetilde{\widetilde{E}}(k_+)$ is assumed to be non-zero.

Thus regardless of the case, there always exists a subordinate solution, providing a suitable Wigner-von Neumann potential is chosen and the corresponding value of $E(\lambda;k_+), \widetilde{E}(\lambda;k_-), \widetilde{\widetilde{E}}(k_+)$ is non-zero. The subordinate solution is in l^2 if we choose the constant c large enough. This proves Theorem 4.0.1.

Remark We expect that any $\lambda \in \sigma_{ell}(J_T)$ is not simultaneously a root of $E(\lambda;k_+), \widetilde{E}(\lambda;k_-)$, for all $k_+ \in \{1,\dots,T\}, k_- \in \{0,\dots,T-1\}$. Therefore, whenever the quantisation condition is satisfied a subordinate solution should exist.

The above result, however, gives only the chance to prove that a potential of the form, $q_n = \frac{c\sin(n\omega+\phi)}{n}$ embeds an eigenvalue, rather than produce only a subordinate solution. In order for it to be a true eigenvector, the initial conditions encoded in the periodic Jacobi operator must also be satisfied. This leads to:

Theorem 4.5.4. *Let $\lambda \in \sigma_{ell}(J_T)$. If $\theta(\lambda) \neq \frac{\pi}{2}$ choose ω s.t. $\omega T + 2\theta(\lambda) = 2\pi k_+$ and assume $E(\lambda;k_+) \neq 0$. Let $q_n = \frac{c\sin(n\omega+\phi)}{n}, n \geq 3$, for arbitrary $\phi \in \mathbb{R}$. Then there exist c and real values q_1, q_2 such that*

$$\lambda \in \sigma_p(J_T + Q)$$

where σ_p is the point spectrum and Q is the diagonal matrix with entries (q_n).

Remark Similar results can be proved if one of the other quantisation conditions, (4.19), is satisfied.

Proof. Theorem 4.0.1 gives $(u_n), (q_n)$ such that

$$a_{n-1}u_{n-1} + a_n u_{n+1} + (q_n + b_n - \lambda)u_n = 0$$

for $n \geq 3$. There are two cases:

1. If $u_2 \neq 0$ then defining $q_2 := \frac{-\lambda u_2 - a_2 u_3 - a_1 u_1 - b_2 u_2}{u_2}$ with $u_1 := -\frac{a_1 u_2}{q_1 + b_1 - \lambda}$, with q_1 as a free parameter and not equal to $\lambda - b_1$, ensures all conditions are satisfied.

2. If $u_2 = 0$ then defining $u_1 := -\frac{a_2 u_3}{a_1}$ and $q_1 := \lambda - b_1$, with q_2 as a free parameter, ensures all conditions are satisfied.

See the proof of Theorem 3.5.1 for more details. □

4.6 Multiple subordinate solutions

Here we extend Theorem 4.0.1 to construct subordinate solutions for a (possibly infinite) set of spectral parameters belonging to the generalised interior of the essential spectrum.

Theorem 4.6.1. *Let $S \subseteq \mathbb{N}$ and $(\lambda_i)_{i \in S}$ be a sequence of numbers belonging to $\sigma_{ell}(J_T)$. Assume $\theta(\lambda_i) \neq \frac{\pi}{2}$ and $E(\lambda_i) \neq 0$ for all $i \in S$, with*

$$q_n^{(i)} := \frac{\sin(n\omega_i + \phi_i)}{n}, \tag{4.42}$$

where ω_i is such that $T\omega_i + 2\theta(\lambda_i) = 2k_+^{(i)}\pi$, for a suitably chosen integer $k_+^{(i)}$. Then, there exists a real strictly positive sequence $(c_i)_{i \in S}$ belonging to $l^1(\mathbb{N})$ such that for the potential, (q_n),

$$q_n := \sum_{i \in S} c_i q_n^{(i)} = \sum_{i \in S} \frac{c_i \sin(n\omega_i + \phi_i)}{n}, \tag{4.43}$$

for arbitrary $\phi_i \in \mathbb{R}$, there are subordinate solutions, $\underline{u}^{(i)} := \left(u_n^{(i)}\right)_{n \geq 1}$, to the recurrence equations

$$a_{n-1}u_{n-1}^{(i)} + (b_n + q_n)u_n^{(i)} + a_n u_{n+1}^{(i)} = \lambda_i u_n^{(i)}, n \geq 2, i \in S.$$

Remark The reader should observe that there is no rational dependence condition between the $\theta(\lambda), \lambda \in (\lambda_i)_{i \in S}$, like in some results (see, for example, Theorem 1 in [65]). Indeed, our only constraint is that $E(\lambda_i) \neq 0$ and since the function is algebraic there are only finitely many roots and therefore finitely many points in the generalised interior of the essential spectrum where the technique fails. (For the periods $T = 1$ and $T = 2$ we have seen that the function $E(\lambda)$ has no roots in $\sigma_{ell}(J_T)$ and therefore there are no restrictions for these two cases.) Moreover, the frequency, ω_i used to define the potential has no dependency on any other λ_i than that with which it satisfies the resonance conditions.

Remark To simplify notation, unless explicitly mentioned we will assume $S = \mathbb{N}$ as this is the most general and interesting case. All other cases can be proven in the same way. Later, in Theorem 4.6.6, for the case $S = \{1, 2\}$, we deal with the initial conditions and establish explicit $\underline{u}^{(i)} \in l^2, (q_n)$ such that

$$(J_T + Q)\underline{u}^{(i)} = \lambda_i \underline{u}^{(i)}$$

for each $i \in \{1, 2\}$ and where Q is a diagonal matrix with entries q_n.

The aim is to consider an arbitrary $\lambda_t \in (\lambda_i)_{i \in S}$ and show that the new perturbation, (q_n), still produces a subordinate solution for λ_t. Note that each λ_i will now be associated to an eigenvalue, $\mu(\lambda_i)$, of the monodromy matrix, where $\mu(\lambda_i) = e^{i\theta(\lambda_i)}$. Moreover, since the explicit nature of (q_n) in the single eigenvalue case is not discussed until the section dealing with the Harris-Lutz procedure in the proof of Theorem 4.0.1, this means that the results of Sections 4.1 and 4.2 are still applicable here. Moreover, we see that by choosing $(c_l)_{l \in S}$ such that $\sum\limits_{l=1}^{\infty} c_l < \infty$ then

$$|q_n| = \left| \sum_{l=1}^{\infty} c_l \frac{\sin(n\omega_l + \phi_l)}{n} \right| \leq \sum_{l=1}^{\infty} \frac{c_l}{n} = O\left(\frac{1}{n}\right).$$

The details at the end of Section 4.3 follow similarly to before except now we must use a more detailed version of Proposition 3.3.1:

Lemma 4.6.2. *Let $\alpha \in \mathbb{R}, \alpha \notin 2\pi\mathbb{Z}$. Then*

$$\left| \sum_{k=n}^{\infty} \frac{e^{ik\alpha}}{k} \right| \leq \frac{2}{n\,|e^{i\alpha} - 1|} = \frac{1}{n\left|\sin(\frac{\alpha}{2})\right|}.$$

Proof. Define $\widetilde{g}_n := \sum\limits_{k=n}^{\infty} \frac{e^{ik\alpha}}{k}$, which exists by Proposition 3.3.1. Consequently,

$$\widetilde{g}_n e^{i\alpha} = \sum_{k=n}^{\infty} \frac{e^{i(k+1)\alpha}}{k} = \sum_{k=n}^{\infty} \frac{e^{i(k+1)\alpha}}{k+1} + \sum_{k=n}^{\infty} \frac{e^{i(k+1)\alpha}}{k(k+1)} = \widetilde{g}_{n+1} + \sigma_n, \tag{4.44}$$

where $\sigma_n := \sum\limits_{k=n}^{\infty} \frac{e^{i(k+1)\alpha}}{k(k+1)}$ and $|\sigma_n| \leq \frac{1}{n}$. Then, by (4.44)

$$(e^{i\alpha} - 1)\widetilde{g}_n = \widetilde{g}_{n+1} - \widetilde{g}_n + \sigma_n = -\frac{e^{in\alpha}}{n} + \sigma_n,$$

which implies $|e^{i\alpha} - 1||\widetilde{g}_n| \leq \frac{2}{n}$. Thus

$$|\widetilde{g}_n| = \left| \sum_{k=n}^{\infty} \frac{e^{ik\alpha}}{k} \right| \leq \frac{2}{|e^{i\alpha} - 1|}. \quad \square$$

This gives the following corollary which will be used repeatedly throughout this section of the chapter.

Corollary 4.6.3. *Let $\alpha \in \mathbb{R}, \alpha \notin 2\pi\mathbb{Z}$. Then for $n_1, n_2 \in \mathbb{N}, n_2 > n_1$*

$$\left| \sum_{k=n_1}^{n_2} \frac{e^{ik\alpha}}{k} \right| \leq \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \frac{1}{|\sin\left(\frac{\alpha}{2}\right)|}.$$

Proof. Observe that

$$\begin{aligned} \left| \sum_{k=n_1}^{n_2} \frac{e^{ik\alpha}}{k} \right| &= \left| \sum_{k=n_1}^{\infty} \frac{e^{ik\alpha}}{k} - \sum_{k'=n_2+1}^{\infty} \frac{e^{ik'\alpha}}{k'} \right| \leq \left| \sum_{k=n_1}^{\infty} \frac{e^{ik\alpha}}{k} \right| + \left| \sum_{k'=n_2+1}^{\infty} \frac{e^{ik'\alpha}}{k'} \right| \\ &\leq \frac{1}{n_1 \left|\sin\left(\frac{\alpha}{2}\right)\right|} + \frac{1}{n_2 \left|\sin\left(\frac{\alpha}{2}\right)\right|} = \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \frac{1}{|\sin\left(\frac{\alpha}{2}\right)|}, \end{aligned}$$

where the final inequality is a consequence of Lemma 4.6.2. □

To apply the Harris-Lutz transformation in this case we define $\vec{h}_k$ such that $\vec{g}_k = (I + G_k)\vec{h}_k$ with $\|G_k\| = O\left(\frac{1}{k}\right)$. From the analogue of (4.14) this gives

$$\vec{h}_{k+1} = \left(I - G_{k+1} + G_k + T_1(k) + T_2(k) + O\left(\frac{1}{k^2}\right) \right) \vec{h}_k, \tag{4.45}$$

where

$$T_1(k) := \frac{1}{i \sin\theta(\lambda_t)} \sum_{l=1}^{\infty} \sum_{j=0}^{T-1} \frac{c_l q^{(l)}_{(k+1)T-j}}{a_{T-j}} \begin{pmatrix} -\overline{C_j(\lambda_t)} & 0 \\ 0 & C_j(\lambda_t) \end{pmatrix}, \tag{4.46}$$

$$T_2(k) := \frac{1}{i \sin\theta(\lambda_t)} \sum_{l=1}^{\infty} \sum_{j=0}^{T-1} \frac{c_l q^{(l)}_{(k+1)T-j}}{a_{T-j}} \begin{pmatrix} 0 & -\overline{D_j(\lambda_t)}\ \overline{\mu(\lambda_t)}^{2k} \\ D_j(\lambda_t)\mu(\lambda_t)^{2k} & 0 \end{pmatrix} \tag{4.47}$$

with $C_j(\lambda_t), D_j(\lambda_t)$ as defined in (4.11) and (4.12). Then $T_1(k), T_2(k) = O\left(\frac{1}{k}\right)$ due to the condition that $(c_l)_{l\in S} \in l^1$. In addition, define $G_k^N := -\sum_{m=k}^{N} T_1(m)$. Consequently, for N_1, N_2 large enough with $N_2 > N_1 > k$ we have

$$\begin{aligned} \left\| G_k^{N_2} - G_k^{N_1} \right\| &= \left\| \sum_{m=N_1}^{N_2} T_1(m) \right\| = \left| \sum_{m=N_1}^{N_2} \frac{1}{\sin\theta(\lambda_t)} \sum_{l=1}^{\infty} \sum_{j=0}^{T-1} c_l \frac{q^{(l)}_{(m+1)T-j}}{a_{T-j}} C_j(\lambda_t) \right| \\ &= \frac{1}{|\sin\theta(\lambda_t)|} \left| \sum_{l=1}^{\infty} c_l \sum_{j=0}^{T-1} \frac{C_j(\lambda_t)}{a_{T-j}} \sum_{m=N_1}^{N_2} \frac{\sin(((m+1)T-j)\omega_l + \phi_l)}{(m+1)T-j} \right|. \end{aligned}$$

Then, using that

$$\frac{1}{(m+1)T-j} \leq \frac{1}{(m+1)T} + \frac{j}{m^2T^2}$$

and

$$\sum_{m=N_1}^{N_2} \frac{1}{m^2} \leq \sum_{m=N_1}^{\infty} \frac{1}{m^2} \leq \frac{1}{N_1},$$

for some non-zero real constant C, we obtain

$$\begin{aligned}\left\|G_k^{N_2} - G_k^{N_1}\right\| &\leq \frac{K_t}{N_1} + \frac{1}{|\sin\theta(\lambda_t)|}\sum_{l=1}^{\infty} c_l \left|\sum_{j=0}^{T-1} \frac{C_j(\lambda_t)}{a_{T-j}} \operatorname{Im} \sum_{m=N_1}^{N_2} \frac{e^{i(((m+1)T-j)\omega_l+\phi_l)}}{(m+1)T}\right| \\ &\leq \frac{K_t}{N_1} + \widetilde{K}_t \sum_{l=1}^{\infty} c_l \left|\sum_{m'=N_1+1}^{N_2+1} \frac{e^{im'T\omega_l}}{m'}\right| \leq \frac{K_t}{N_1} + \frac{2\widetilde{K}_t}{TN_1}\sum_{l=1}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_l}{2}\right)\right|},\end{aligned} \tag{4.48}$$

where $K_t := T\widetilde{K}_t \sum_{l=1}^{\infty} c_l$, $\widetilde{K}_t := \frac{\max_{j\in\{0,\dots,T-1\}} |C_j(\lambda_t)|}{|\sin\theta(\lambda_t)| \min_{j\in\{0,\dots,T-1\}} a_j}$, and the final inequality follows from Corollary 4.6.3 and the strictly positive sequence $(c_l)_{l\in S}$ being chosen such that

$$\sum_{l=1}^{\infty} \frac{c_l}{|\sin\left(\frac{T\omega_l}{2}\right)|} < \infty. \tag{4.49}$$

Clearly, $\left(G_k^N\right)_N$ is a Cauchy-sequence and therefore the limit $G_k := \lim_{N\to\infty} G_k^N$ exists.

For N sufficiently large we can employ similar techniques to establish that

$$\left\|G_k^N\right\| \leq \frac{K_t}{k} + \frac{2\widetilde{K}_t}{Tk}\sum_{l=1}^{\infty} c_l \frac{1}{\left|\sin\left(\frac{T\omega_l}{2}\right)\right|} = O\left(\frac{1}{k}\right)$$

is uniformly bounded in N for any value of the parameter t. Consequently, the limit G_k is also bounded for any value of the parameter t, providing the strictly positive sequence $(c_l)_{l\in S}$ is chosen to decay fast enough.

Remark It should be stressed that any real strictly positive sequence which satisfies (4.49) will suffice to make the above Harris-Lutz procedure valid.

The Harris-Lutz procedure is well-defined and we remove the $T_1(k)$ term like in the single eigenvalue case (i.e. $G_{k+1} - G_k = T_1(k)$). The analogue of Section 4.4 becomes

$$
\begin{aligned}
\vec{h}_{k+1} &= \left(I + T_2(k) + O\left(\frac{1}{k^2}\right)\right)\vec{h}_k \\
&= \Bigg(I + \frac{1}{i\sin\theta(\lambda_t)} \sum_{j=0}^{T-1}\sum_{l=1}^{\infty} \frac{c_l \sin(((k+1)T-j)\omega_l+\phi_l)}{a_{T-j}((k+1)T-j)} \times \\
&\left(\begin{smallmatrix} 0 & -\overline{D_j(\lambda_t)}\ \overline{\mu(\lambda_t)}^{2k} \\ D_j(\lambda_t)\mu^{2k}(\lambda_t) & 0 \end{smallmatrix}\right) + O\left(\frac{1}{k^2}\right)\Bigg)\vec{h}_k \\
&= \Bigg(I + \frac{1}{i\sin\theta(\lambda_t)} \sum_{j=0}^{T-1}\sum_{l=1}^{\infty} \frac{c_l e^{i(((k+1)T-j)\omega_l+\phi_l)} - c_l e^{-i(((k+1)T+j)\omega_l+\phi_l)}}{2ia_{T-j}((k+1)T+j)} \\
&\times \left(\begin{smallmatrix} 0 & -\overline{D_j(\lambda_t)}\overline{\mu}^{2k}(\lambda_t) \\ D_j(\lambda_t)\mu^{2k}(\lambda_t) & 0 \end{smallmatrix}\right) + O\left(\frac{1}{k^2}\right)\Bigg)\vec{h}_k \\
&= \Bigg(I + \frac{1}{(k+1)\sin\theta(\lambda_t)} \sum_{l=1}^{\infty} c_l \Big\{ \left(\begin{smallmatrix} 0 & -\overline{d_l(\lambda_t)}+\overline{f_l(\lambda_t)} \\ -d_l(\lambda_t)+f_l(\lambda_t) & 0 \end{smallmatrix}\right) + O\left(\frac{1}{k^2}\right)\Bigg)\vec{h}_k
\end{aligned}
\tag{4.50}
$$

where

$$d_l(\lambda_t) := e^{ik[2\theta(\lambda_t)+T\omega_l]} \sum_{j=0}^{T-1} E_j(\lambda_t) e^{i((T-j)\omega_l+\phi_l)}$$

and

$$f_l(\lambda_t) := e^{-ik[T\omega_l-2\theta(\lambda_t)]} \sum_{j=0}^{T-1} E_j(\lambda_t) e^{-i((T-j)\omega_l+\phi_l)}$$

with $E_j(\lambda_t)$ as defined previously.

To establish that the perturbation affects the asymptotics, we observe that for $l = t$ there is resonance between the frequency of the oscillation and the quasi-momentum. In particular, for a suitable choice of $k_+^{(i)}$ (see Section 4.5) we have

$$c_t \sum_{j=0}^{T-1} E_j(\lambda_t) e^{i((T-j)\omega_l+\phi_l)} = c_t E(\lambda_t),$$

which is non-zero by the conditions assumed in the statement of the theorem. Furthermore, it is possible for other resonance to occur when $l \neq t$ and this is discussed in the following lemma.

Lemma 4.6.4. *Let $\lambda_\alpha \sim \lambda_\beta$ denote when $2\theta(\lambda_\alpha) + T\omega_\beta \in 2\pi\mathbb{Z}$ or $2\theta(\lambda_\alpha) - T\omega_\beta \in 2\pi\mathbb{Z}$ for $\lambda_\alpha, \lambda_\beta$ from the set $(\lambda_i)_{i\in S}$ given in Theorem 4.6.1 and where ω_i is such that $2\theta(\lambda_i)+T\omega_i \in 2\pi\mathbb{Z}$. Then $\sim$ is an equivalence relation; in particular, the set $\{\lambda_i \mid i \in S\}$ can be partitioned into equivalence classes. Moreover, each class has at most $2T$ elements.*

Proof. First observe that

$$\begin{aligned}\lambda_\alpha \sim \lambda_\beta &\iff 2\theta(\lambda_\alpha) + T\omega_\beta \in 2\pi\mathbb{Z} \text{ or } 2\theta(\lambda_\alpha) - T\omega_\beta \in 2\pi\mathbb{Z} \\ &\iff \theta(\lambda_\alpha) - \theta(\lambda_\beta) \in \pi\mathbb{Z} \text{ or } \theta(\lambda_\alpha) + \theta(\lambda_\beta) \in \pi\mathbb{Z},\end{aligned}$$

using that $T\omega_i = 2z_1\pi - 2\theta(\lambda_i)$ for some $z_1 \in \mathbb{Z}$ by the conditions assumed in the theorem. As $\theta(\lambda) \in (0, \pi)$, this implies

$$\lambda_\alpha \sim \lambda_\beta \iff \theta(\lambda_\alpha) - \theta(\lambda_\beta) = 0 \text{ or } \theta(\lambda_\alpha) + \theta(\lambda_\beta) = \pi,$$

which is clearly an equivalence relation. Moreover, as $\theta(\lambda)$ is strictly monotonic on each band of $\sigma_{ell}(J_T)$ (see, for example, 3.10 in [16]) each band of essential spectrum contributes at most 2 elements to the equivalence class. Given that there are at most T bands of essential spectrum for a period-T Jacobi operator, then there at most $2T$ elements for the equivalence class in total. See Figure 4.1 for an illustration. $\square$

Figure 4.1: In this example, for a period-3 Jacobi operator, the three thick horizontal lines denote the three bands of essential spectrum. The possible resonant cases, λ_i, for a particular point λ in the generalised interior of the essential spectrum are represented by X and $\widetilde{X}$, for those λ_i such that $\theta(\lambda_i) = \theta(\lambda)$, and $\theta(\lambda_i) = \pi - \theta(\lambda)$, respectively. Note that the six points are only candidate elements of the equivalence class, since it still remains to check for each whether it is also an element of the sequence $(\lambda_i)_{i\in S}$.

From Lemma 4.6.4 we see there are only finitely many resonating terms for each fixed t. Later, it will be shown that for an appropriate choice of $(c_j)_{j\in S}$ these finitely many resonance terms cannot cancel, but for now we focus on removing the infinitely many non-resonant terms from the consideration of the asymptotics for (4.50) using the Harris-Lutz technique.

Define, for each $t \in \mathbb{N}$,

$$I_t^+ := \{n \in \mathbb{N} | 2\theta(\lambda_t) + T\omega_n \in 2\pi\mathbb{Z}\}, \quad I_t^- := \{n \in \mathbb{N} | 2\theta(\lambda_t) - T\omega_n \in 2\pi\mathbb{Z}\}.$$

For the case that $2\theta(\lambda_t) + T\omega_t \in 2\pi\mathbb{Z}$, by Corollary 4.6.3 we have for any M

$$\left| \sum_{k=1}^{M} \sum_{\substack{l=1,\\ l\notin I_t^+}}^{\infty} c_l \frac{e^{-ik[2\theta(\lambda_t)+T\omega_l]}}{k} \right| \leq \sum_{\substack{l=1,\\ l\notin I_t^+}}^{\infty} c_l \left| \sum_{k=1}^{M} \frac{e^{ikT[\omega_t-\omega_l]}}{k} \right| \leq 2 \sum_{\substack{l=1,\\ l\notin I_t^+}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t - T\omega_l}{2}\right)\right|}.$$

Thus

$$\left| \sum_{k=1}^{\infty} \sum_{\substack{l=1,\\ l\notin I_t^+}}^{\infty} c_l \frac{e^{-ik[2\theta(\lambda_t)+T\omega_l]}}{k} \right| \leq 2 \sum_{\substack{l=1,\\ l\notin I_t^+}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t - T\omega_l}{2}\right)\right|}. \tag{4.51}$$

Similarly

$$\left| \sum_{k=1}^{\infty} \sum_{\substack{l=1, \\ l \notin I_t^-}}^{\infty} c_l \frac{e^{ik[T\omega_l - 2\theta(\lambda_t)]}}{k} \right| \leq 2 \sum_{\substack{l=1, \\ l \notin I_t^-}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t + T\omega_l}{2}\right)\right|}. \tag{4.52}$$

Now an upper-bound needs to be established for (4.51) and (4.52) that is uniform in the parameter t. This is achieved by defining a new positive sequence

$$b_k := \min\left\{ \min_{j \in \{1,\dots,k-1\} \setminus I_k^+} \left|\sin\left(\frac{T\omega_k - T\omega_j}{2}\right)\right|, \min_{j \in \{1,\dots,k-1\} \setminus I_k^-} \left|\sin\left(\frac{T\omega_k + T\omega_j}{2}\right)\right| \right\}, \tag{4.53}$$

$k > 2T$, and observing that for all $t \in \mathbb{N}$ we have $\left|\sin\left(\frac{T\omega_t - T\omega_l}{2}\right)\right| \geq b_l$, for $l > \max\{t, 2T\}, l \notin I_t^+$, and $\left|\sin\left(\frac{T\omega_t + T\omega_l}{2}\right)\right| \geq b_l$ for $l > \max\{t, 2T\}, l \notin I_t^-$.

Remark The motivation behind letting $k > 2T$ follows from the fact that the formal definition of b_k requires taking the minimum over a set that is non-empty. Since there are at most $2T$ possible j where resonance occurs, and these instances are excluded from our consideration, we must ensure that we are taking the minimum over a set that has more than $2T$ entries to guarantee at least one entry in the set.

The initial terms of the series being finite, we focus on the tail and see that

$$\sum_{\substack{l=2T+1, \\ l \notin I_t^+}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t - T\omega_l}{2}\right)\right|} \leq \sum_{\substack{l=2T+1 \\ l \notin I_t^+}}^{\infty} \frac{c_l}{b_l}, \tag{4.54}$$

which is convergent providing the sequence $(c_j)_{j \in S}$ is chosen such that

$$\sum_l \frac{c_l}{b_l} < \infty. \tag{4.55}$$

Similarly, we have

$$\sum_{\substack{l=2T+1, \\ l \notin I_t^-}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t + T\omega_l}{2}\right)\right|} \leq \sum_{\substack{l=2T+1, \\ l \notin I_t^-}}^{\infty} \frac{c_l}{b_l}, \tag{4.56}$$

which is also finite providing $(c_j)_{j \in S}$ satisfies the same conditions as in (4.55). If instead $2\theta(\lambda_t) - \omega_t T \in 2\pi\mathbb{Z}$ we obtain similar estimates.

It is now possible to remove the non-oscillating terms using a well-defined Harris-Lutz transformation, i.e. a substitution of the form $\vec{h}_{k+1} = (I + H_k)\vec{m}_k$ with $\|H_k\| = O\left(\frac{1}{k}\right)$, which gives

$$\vec{m}_{k+1} = \left(I - H_{k+1} + H_k + T_3(k) + T_4(k) + O\left(\frac{1}{k^2}\right)\right)\vec{m}_{k+1},$$

where

$$T_3(k) = \frac{-1}{(k+1)\sin\theta(\lambda_t)}\left\{ \begin{pmatrix} 0 & \sum\limits_{l=1, l\notin I_t^+}^{\infty} c_l \overline{d_l(\lambda_t)} \\ \sum\limits_{l=1, l\notin I_t^+}^{\infty} c_l d_l(\lambda_t) & 0 \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} 0 & \sum\limits_{l=1, l\notin I_t^-}^{\infty} c_l \overline{f_l(\lambda_t)} \\ \sum\limits_{l=1, l\notin I_t^-}^{\infty} c_l f_l(\lambda_t) & 0 \end{pmatrix} \right\},$$

$$T_4(k) = \frac{1}{(k+1)\sin\theta(\lambda_t)}\left\{ \begin{pmatrix} 0 & \sum\limits_{l\in I_t^+} c_l \overline{d(\lambda_t)} \\ \sum\limits_{l\in I_t^+}^{\infty} c_l d_l(\lambda_t) & 0 \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} 0 & \sum\limits_{l\in I_t^-} c_l \overline{f_l(\lambda_t)} \\ \sum\limits_{l\in I_t^-} c_l f_l(\lambda_t) & 0 \end{pmatrix} \right\},$$

and $H_k = -\sum\limits_{r=k}^{\infty} T_3(r)$, with

$$\|H_k\| \le \frac{T \max\limits_{j\in\{0,\ldots,T-1\}} |E_j(\lambda_t)|}{k\sin\theta(\lambda_t)} \left(\sum_{\substack{l=1,\\ l\notin I_t^+}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t - T\omega_l}{2}\right)\right|} + \sum_{\substack{l=1,\\ l\notin I_t^-}}^{\infty} \frac{c_l}{\left|\sin\left(\frac{T\omega_t + T\omega_l}{2}\right)\right|} \right)$$

which by (4.54) and (4.56) is convergent for any value of the parameter t, providing the sequence $(c_j)_{j\in S}$, is chosen so that it satisfies (4.55), and in which case $\|H_k\| = O\left(\frac{1}{k}\right)$. Then, since $H_{k+1} - H_k = T_3(k)$, Equation (4.50) becomes

$$\vec{m}_{k+1} = \left(I + T_4(k) + O\left(\frac{1}{k^2}\right)\right)\vec{m}_k. \tag{4.57}$$

It is now shown that the sequence $(c_j)_{j\in S}$ can be chosen such that the finitely many resonating terms appearing in $T_4(k)$ do not cancel. This involves the following elementary lemma.

Lemma 4.6.5. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix where all the diagonal entries are non-zero. Then for any vector $\vec{f} \in \mathbb{C}^n$ with all entries positive there exists a vector $\vec{f'}$ arbitrarily close to $\vec{f}$ so that $A\vec{f'} = \vec{v}$ where the entries of $\vec{v}$ are all non-zero.*

Proof. Given $\vec{f}$ with all entries positive, if there are no zero entries in $\vec{v} = A\vec{f}$ then the result is already proven. Otherwise, consider the first non-zero entry located at $\vec{v}_{j_1}, j_1 \in \{1,\ldots,n\}$. Then, alter the vector $\vec{f}$ to become $\vec{f'}$ so that $\vec{f'_i} := \vec{f_i}$ for all $i \ne j_1$ and $\vec{f'_{j_1}} := \vec{f}_{j_1} + \epsilon$, where $\epsilon > 0$ is sufficiently small so that the first $j_1 - 1$ entries in the new vector $\vec{v}' = A\vec{f}'$ are non-zero. From the fact that the diagonal entries of A are non-zero, the entry $\vec{v}'_{j_1}$ is also non-zero. We repeat the procedure for each subsequent zero entry in the vector $\vec{v}'$, and since the vector is finite-dimensional this completes the proof. $\square$

By Lemma 4.6.4 we have that we can partition the sequence $(\lambda_i)_{i\in S}$ into finite disjoint sets. The same is true for the associated sequence $(c_i)_{i\in S}$. Consequently, the c_i which appear in (4.57) in $T_4(k)$ all belong to the same equivalence class. Then, to establish that there exists a sequence $(c_i)_{i\in S}$ such that $T_4(k) \neq 0$, we partition the c_i elements into their equivalence classes, each of finite size, and to each class apply Lemma 4.6.5: the elements, c_i, comprising the entries of of the vector $\vec{f}$, whilst each row of the matrix A encodes the exponential and $E(\lambda_i)$ relations that form the sum when resonance occurs. Assuming the positive sequence $(c_l)_{l\in S}$ is chosen to satisfy both (4.49) and (4.55) one can use Lemma 4.6.5 to vary the sequence, $(c_l)_{l\in S}$ slightly so that resonance appears and the convergence of (4.49) and (4.55) remain unchanged. Then, by applying the generalised Janas-Moszynski theorem [63] to eliminate the error term of order k^{-2} we see that the solution of (4.57) behaves asymptotically like the solution of

$$\vec{m}_{k+1} = \left(I + \frac{1}{k\sin\theta(\lambda_t)} \begin{pmatrix} 0 & \overline{Y(\lambda_t)} \\ Y(\lambda_t) & 0 \end{pmatrix} \right) \vec{m}_k, \tag{4.58}$$

where

$$Y(\lambda_t) := \sum_{l\in I_t^-} c_l \sum_{j=0}^{T-1} E_j(\lambda_t) e^{-i((T-j)\omega_l+\phi_l)} + \sum_{l\in I_t^+} c_l \sum_{j=0}^{T-1} E_j(\lambda_t) e^{i((T-j)\omega_l+\phi_l)} \neq 0.$$

Much like in the individual eigenvalue case, we need to diagonalise the matrix in (4.58). The matrix is already Hermitian and has trace zero, and therefore by removing the term $|Y(\lambda_t)|$ as a factor we observe that the new matrix has eigenvalues 1 and -1 and determinant -1. Thus, (4.58) is equivalent to

$$\begin{aligned}
\vec{m}_{k+1} &= W \left[\prod_{r=1}^{k} \left(I + \frac{|Y(\lambda_t)|}{r\sin\theta(\lambda_t)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \right] W^{-1}\vec{m}_1 \\
&= W \begin{pmatrix} \prod_{r=1}^{k} \left(1 + \frac{|Y(\lambda_t)|}{r\sin\theta(\lambda_t)}\right) & 0 \\ 0 & \prod_{r=1}^{k} \left(1 - \frac{|Y(\lambda_t)|}{r\sin\theta(\lambda_t)}\right) \end{pmatrix} W^{-1}\vec{m}_1 \\
\sim_r \vec{m}_{k+1} &= \begin{pmatrix} (\widetilde{c}_7 + o(1))\, k^{\left|\frac{Y(\lambda_t)}{\sin\theta(\lambda_t)}\right|} & 0 \\ 0 & (\widetilde{c}_8 + o(1))\, k^{-\left|\frac{Y(\lambda_t)}{\sin\theta(\lambda_t)}\right|} \end{pmatrix} \vec{m}_1,
\end{aligned}$$

for non-zero constants $\widetilde{c}_7, \widetilde{c}_8$ depending on $\lambda_t \in \sigma_{ell}(J_T)$.

Returning to the original recurrence relation, this implies that for λ_t there exists a decaying (subordinate) solution, $u_k(\lambda_t) \sim k^{-\left|\frac{Y(\lambda_t)}{\sin\theta(\lambda_t)}\right|}$. Since λ_t was an arbitrary element of the sequence $(\lambda_i)_{i\in S}$ this concludes the argument.

Remark For the case of only finitely many $\lambda_i, i \in \{1, \ldots, n\}$, it is possible to amend the proof of Theorem 4.6.1 so that we compute subordinate solutions that reside in the sequence space $l^2(\mathbb{N}; \mathbb{C})$. Replacing the potential (q_k) in the theorem by (cq_k) replaces $Y(\lambda_t)$ in (4.58) with $cY(\lambda_t)$. Then, choosing c sufficiently large such that $\left|\frac{cY(\lambda_t)}{\sin\theta(\lambda_t)}\right| > \frac{1}{2}$ for all $t \in \{1, \ldots, n\}$ the subordinate solutions will all lie in l^2.

The previous theorem proves only that the sum of the potentials simultaneously produces subordinate solutions associated to all $(\lambda_i)_{i\in S}$. It has not been shown that a potential of this structure simultaneously satisfies the initial conditions encoded in the periodic Jacobi operator necessary for an eigenvalue to exist for each $\lambda_i, i \in \{1, \ldots, n\}$. This, in general, leads to solving a system of non-linear equations. For the case of $n = 2$ we explicitly solve this system.

Theorem 4.6.6. *Let $\lambda_1, \lambda_2 \in \sigma_{ell}(J_T)$, $\theta(\lambda_1) \neq \frac{\pi}{2} \neq \theta(\lambda_2)$ and assume $E(\lambda_i) \neq 0$ for $i \in \{1, 2\}$. Then for q'_n given by (4.43) for $n \geq 5$, and suitably chosen q'_1, q'_2, q'_3, q'_4 we have that $\{\lambda_1, \lambda_2\} \subseteq \sigma_p(J_T + Q' + R)$ where Q' is a diagonal matrix with entries q'_n of a Wigner-von Neumann structure and order $\frac{1}{n}$ as $n \to \infty$, and*

$$R := \begin{pmatrix} 0 & r & 0 & \ldots \\ r & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

for some $r \in \mathbb{R}$. For a generic choice of J_T we can choose $r = 0$.

Proof. Choose q'_4 as given by (4.43). By the previous remark there exist non-zero solutions $v_n^{(1)}$, $v_n^{(2)}$ in l^2 such that (4.2) is satisfied. Set $u_n^{(1)} = v_n^{(1)}$, $u_n^{(2)} = v_n^{(2)}$ for $n \geq 3$. Then,

$$a_{n-1}u_{n-1}^{(1)} + a_n u_{n+1}^{(1)} + (q'_n + b_n - \lambda_1)u_n^{(1)} = 0, \text{ for } n \geq 4$$

and

$$a_{n-1}u_{n-1}^{(2)} + a_n u_{n+1}^{(2)} + (q'_n + b_n - \lambda_2)u_n^{(2)} = 0 \text{ for } n \geq 4.$$

Thus, $u_n^{(1)}$, $u_n^{(2)}$ for $n \geq 3$ and q'_n for $n \geq 4$ are now given. However, for λ_1, λ_2 to be embedded eigenvalues, the following system of equations still needs to be satisfied:

$$\begin{aligned}
a_2u_2^{(1)} + (q'_3 + b_3)u_3^{(1)} + a_3u_4^{(1)} &= \lambda_1 u_3^{(1)} \\
a_2u_2^{(2)} + (q'_3 + b_3)u_3^{(2)} + a_3u_4^{(2)} &= \lambda_2 u_3^{(2)} \\
a_1u_1^{(1)} + (q'_2 + b_2)u_2^{(1)} + a_2u_3^{(1)} &= \lambda_1 u_2^{(1)} \\
a_1u_1^{(2)} + (q'_2 + b_2)u_2^{(2)} + a_2u_3^{(2)} &= \lambda_2 u_2^{(2)} \\
(q'_1 + b_1)u_1^{(1)} + a_1u_2^{(1)} &= \lambda_1 u_1^{(1)} \\
(q'_1 + b_1)u_1^{(2)} + a_1u_2^{(2)} &= \lambda_2 u_1^{(2)},
\end{aligned}$$

where $u_1^{(1)}$, $u_2^{(1)}$, $u_1^{(2)}$, $u_2^{(2)}$, q_1', q_2', q_3' are presently undetermined, while $u_3^{(1)}$, $u_3^{(2)}$, $u_4^{(1)}$, $u_4^{(2)}$ are already defined. We consider two cases depending on the values of $u_3^{(1)}$ and $u_3^{(2)}$.

(Case One) If $u_3^{(i)} \neq 0$, for some i, then without loss of generality let $i = 1$ and we set $r := 0$. Then, we set

$$q_1' = \lambda_1 - b_1; q_2' = \frac{(\lambda_2 - b_2)u_2^{(2)} - a_2 u_3^{(2)} - a_1 u_1^{(2)}}{u_2^{(2)}}; q_3' = \frac{(\lambda_1 - b_3)u_3^{(1)} - a_3 u_4^{(1)}}{u_3^{(1)}}.$$

Choosing $u_1^{(1)} = -\frac{a_2 u_3^{(1)}}{a_1}$; $u_2^{(1)} = 0$; $u_1^{(2)} = -\frac{a_1 u_2^{(2)}}{\lambda_1 - \lambda_2}$; $u_2^{(2)} = \frac{\lambda_2 u_3^{(2)} - a_3 u_4^{(2)} - (q_3' + b_3) u_3^{(2)}}{a_2}$ satisfies the six equations listed above, providing the $u_2^{(2)}$ we chose above is non-zero, i.e. $\left(\lambda_2 - \lambda_1 + \frac{a_3 u_4^{(1)}}{u_3^{(1)}}\right) u_3^{(2)} - a_3 u_4^{(2)} \neq 0$.

If $\left(\lambda_2 - \lambda_1 + \frac{a_3 u_4^{(1)}}{u_3^{(1)}}\right) u_3^{(2)} - a_3 u_4^{(2)} = 0$ then this implies $u_3^{(2)} \neq 0$. We further subdivide into two cases. If $a_1 \leq \frac{1}{2} |\lambda_1 - \lambda_2|$ then set

$$x := \frac{\lambda_2 - \lambda_1 + \sqrt{(\lambda_2 - \lambda_1)^2 - 4a_1^2}}{2} \neq 0$$

and instead choose

$$q_1' = \lambda_1 - b_1 + x; q_2' = -\frac{a_1^2 \epsilon}{a_2 x}; q_3' = \frac{(\lambda_1 - b_3)u_3^{(1)} - a_3 u_4^{(1)}}{u_3^{(1)}} + \epsilon$$

and set $u_1^{(1)} = \frac{u_3^{(1)}(-a_2^2 + \epsilon(q_2' + b_2 - \lambda_1))}{a_2^2}$; $u_2^{(1)} = -\frac{\epsilon u_3^{(1)}}{a_2}$; $u_1^{(2)} = \frac{u_3^{(2)}(-a_2^2 + \epsilon(q_2' + b_2 - \lambda_2))}{a_2^2}$; $u_2^{(2)} = -\frac{\epsilon u_3^{(2)}}{a_2}$ for any non-zero constant ϵ chosen such that $u_1^{(1)} \neq 0, u_1^{(2)} \neq 0$. With this choice it is easy to check the six equations listed above are satisfied. If instead $a_1 > \frac{1}{2} |\lambda_1 - \lambda_2|$ then we choose r such that $0 < a_1 + r \leq \frac{1}{2} |\lambda_1 - \lambda_2|$ and return to the start of case one.

(Case Two) Here we have $u_3^{(1)} = u_3^{(2)} = 0$. Then, since this implies $u_4^{(1)}$ and $u_4^{(2)}$ are both non-zero, it is possible to add an arbitrary perturbation to q_4' so that by continuity $u_3^{(j)}$ becomes non-zero for both j. The conditions for case one are then satisfied. $\square$

Remark Theorem 4.6.6 gives an illustrative example of two embedded eigenvalues constructed by a Wigner-von Neumann type perturbation. There exists another technique that succeeds in embedding infinitely many eigenvalues into the essential spectrum of a period-T Jacobi operator, and we will explore this in the following chapter. However, this other technique does not give as explicit a formula for the potential.

Chapter 5

Embedded eigenvalues using the geometric method

In 1991 a new technique was published by Naboko and Yakovlev [65] which embedded infinitely many eigenvalues into the essential spectrum of the DSO. In this chapter, we extend the method to arbitrary period-T Jacobi operators, J_T, essentially by treating the monodromy matrix, $M(\lambda)$, as analogous to a rotation matrix. Indeed, for $\lambda \in \sigma_{ell}(J_T)$ the monodromy matrix is similar to a diagonal matrix with conjugate entries on the unit circle, which is unitarily equivalent to a rotation matrix. Thus we obtain the following sequence of equations

$$\begin{aligned} M(\lambda) &= B_T(\lambda) \dots B_1(\lambda) \\ &= V(\lambda) \begin{pmatrix} \mu(\lambda) & 0 \\ 0 & \overline{\mu(\lambda)} \end{pmatrix} V^{-1}(\lambda) \\ &= [V(\lambda)U(\lambda)]\, R(\theta(\lambda))\, [V(\lambda)U(\lambda)]^{-1} \end{aligned} \tag{5.1}$$

for $\lambda \in \sigma_{ell}(J_T)$, where $\mu(\lambda) = e^{i\theta(\lambda)}$, the columns of $V(\lambda)$ are composed of the eigenvectors for $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ respectively, $U(\lambda)$ is a unitary matrix and $R(\theta(\lambda))$ is a rotation matrix of angle $\theta(\lambda)$. Additionally, for products of unperturbed monodromy matrices, we have

$$\begin{aligned} M(\lambda)^k &= \left([V(\lambda)U(\lambda)]R(\theta(\lambda))[V(\lambda)U(\lambda)]^{-1}\right)^k \\ &= [V(\lambda)U(\lambda)]\, R^k(\theta(\lambda))\, [V(\lambda)U(\lambda)]^{-1} \\ &= [V(\lambda)U(\lambda)]\, R\left(k\theta(\lambda)\right) [V(\lambda)U(\lambda)]^{-1} . \end{aligned}$$

This suggests that it is possible to interpret the solution to the recurrence relations encoded in the Jacobi matrix (when an eigenvalue exists) as simply rotations of the initial components; in particular, there is no change in size for the subsequent components. Of course, for the solution $(u_n)_{n\geq 1}$ to be an eigenvector, it must also shrink sufficiently fast to be in the sequence space, $l^2(\mathbb{N};\mathbb{R})$.

However, at the very least, the above relations offer motivation to investigate a new way to embed eigenvalues for a period-T Jacobi operator. Its success depends upon finding an approach to introduce sufficient shrinkage into the solution, and we show that this can be done with the application of a potential and providing that the components of the solution are not constantly rotated into four arbitrarily narrow bad cones.

The structure of the chapter is as follows. First, we expound the geometric approach to embed a single eigenvalue into the essential spectrum of a periodic Jacobi operator (Section 5.1). However, unlike the original Naboko/Yakovlev technique which is only valid for the special case of the DSO and those λ in the generalised interior of the essential spectrum rationally independent with π, our technique works for an arbitrary period-T Jacobi operator and for any element in the generalised interior of the essential spectrum with quasi-momentum not equal to $\frac{\pi}{2}$. We then reproduce the mechanisms devised by Naboko and Yakovlev to embed infinitely many eigenvalues, simultaneously, into the essential spectrum, providing their quasi-momenta are rationally independent with each other and π (Section 5.2), although we adapt the argument to work for arbitrary periodic Jacobi operators. These ideas are then developed further to permit the embedding of a single λ whose quasi-momentum is rationally dependent with π, simultaneously, with infinitely many other eigenvalues whose quasi-momenta are rationally independent with each other and π (Section 5.3). Similar results are given for the case of two eigenvalues, λ_1, λ_2 whose quasi-momenta are rationally dependent with π (Section 5.4), and also the case for arbitrarily (but finitely) many, λ_i, with quasi-momentum rationally dependent with π, providing certain co-prime conditions are satisfied by the denominators of the quasi-momenta (Section 5.5).

5.1 The single eigenvalue case

Throughout this chapter the perturbed monodromy matrix, $M(\lambda-q_k)$, is defined to be

$$\begin{aligned}
M(\lambda - q_k) &:= B_T(\lambda)B_{T-1}(\lambda)\dots B_1(\lambda - q_k)\\
&= B_T(\lambda)\dots B_2(\lambda)\left(B_1(\lambda) - \frac{q_k}{a_1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)\\
&= M(\lambda) - \frac{q_k}{a_1}A(\lambda) = W(\lambda)(R(\theta(\lambda)) - \frac{q_k}{a_1}W^{-1}(\lambda)A(\lambda)W(\lambda))W^{-1}(\lambda),
\end{aligned}$$

where $A(\lambda) := M(\lambda)B_1^{-1}(\lambda)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, W(\lambda) := V(\lambda)U(\lambda)$.

The following lemma will be needed:

Lemma 5.1.1. *The invertible matrix $W(\lambda)$ can always be chosen to be real.*

Proof. The matrix $W(\lambda) = V(\lambda)U(\lambda)$, where both $U(\lambda)$ and $V(\lambda)$ are invertible, is also invertible. However, the matrix $W(\lambda)$ may not always be real, but we do have a choice in what $U(\lambda), V(\lambda)$ we use to define it. All that is needed is that $W(\lambda)$ satisfies the relation

$$M(\lambda)W(\lambda) = W(\lambda)R(\theta(\lambda)). \tag{5.2}$$

Indeed, there are many possibilities for a real $W(\lambda)$; for instance, we could just take the $\mathrm{Re}(W(\lambda))$ or $\mathrm{Im}(W(\lambda))$ of any $W(\lambda)$ that satisfies (5.2), however, this does not guarantee the invertibility condition. So, assume for contradiction that $\mathrm{Re}(W(\lambda)) + \beta\,\mathrm{Im}(W(\lambda))$ is not invertible for any $\beta \in \mathbb{R}$. Then, define the function

$$f(\beta) := \det\left[\mathrm{Re}(W(\lambda)) + \beta\,\mathrm{Im}(W(\lambda))\right],$$

and by our assumption $f(\beta) = 0$ for all $\beta \in \mathbb{R}$. But the function f is just a polynomial in β and is therefore analytic in β, and can be extended analytically to the complex plane. Thus, it is zero everywhere on the plane. However, if we choose $\beta = i$ then we get the original expression for $W(\lambda)$, which is always invertible, and thus we have a contradiction. This means that there exists at least one $\beta \in \mathbb{R}$ such that the matrix $\mathrm{Re}(V(\lambda)U(\lambda)) + \beta\,\mathrm{Im}(V(\lambda)U(\lambda))$ is both invertible and real. We choose this matrix to be $W(\lambda)$. □

Throughout this chapter $\lambda \in \sigma_{ell}(J_T)$ and therefore $M(\lambda), B_j(\lambda)$ are real. Moreover, by Lemma 5.1.1 we may assume from now on that $W(\lambda)$ is real. Consequently, since all the matrices have real entries, we have for $\vec{f} \in \mathbb{R}^2$

$$\|\left(R(\theta(\lambda)) - \frac{q_k}{a_1}W^{-1}(\lambda)A(\lambda)W(\lambda)\right)\vec{f}\|^2$$

$$\begin{aligned}
&= \left\langle \left(R^*(\theta(\lambda)) - \frac{q_k}{a_1}W^*(\lambda)A^*(\lambda)\left(W^{-1}(\lambda)\right)^*\right)\right.\\
&\qquad\qquad \left.\times\left(R(\theta(\lambda)) - \frac{q_k}{a_1}W^{-1}(\lambda)A(\lambda)W(\lambda)\right)\vec{f}, \vec{f}\right\rangle\\
&= \left\langle \left(I - \frac{2q_k}{a_1}\mathrm{Re}\left(R^*(\theta(\lambda))W^{-1}(\lambda)A(\lambda)W(\lambda)\right) + O(q_k^2)\right)\vec{f}, \vec{f}\right\rangle\\
&= \|\vec{f}\|^2 + O(q_k^2)\|\vec{f}\|^2 - \frac{2q_k}{a_1}\langle R(-\theta(\lambda))W^{-1}(\lambda)A(\lambda)W(\lambda)\vec{f}, \vec{f}\rangle, \qquad (5.3)
\end{aligned}$$

and where the second equality was obtained using that for an arbitrary real matrix B, $\mathrm{Re}(B) = \frac{1}{2}(B + B^*)$. Additionally, observing that the matrix $A(\lambda)$

is rank one yields:

$$
\begin{aligned}
&\langle R(-\theta(\lambda))W^{-1}(\lambda)A(\lambda)W(\lambda)\vec{f},\vec{f}\rangle = \langle A(\lambda)W(\lambda)\vec{f},(W^{-1}(\lambda))^{*}R(-\theta(\lambda))^{*}\vec{f}\rangle\\
&= \langle M(\lambda)B_1^{-1}(\lambda)\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}W(\lambda)\vec{f},\left(W^{-1}(\lambda)\right)^{*}R^{*}(-\theta(\lambda))\vec{f}\rangle\\
&= \langle M(\lambda)B_1^{-1}(\lambda)\langle W(\lambda)\vec{f},\vec{e}_2\rangle\vec{e}_2,(W^{-1}(\lambda))^{*}R^{*}(-\theta(\lambda))\vec{f}\rangle\\
&= \langle W(\lambda)\vec{f},\vec{e}_2\rangle\langle M(\lambda)B_1^{-1}(\lambda)\vec{e}_2,(W^{-1}(\lambda))^{*}R^{*}(-\theta(\lambda))\vec{f}\rangle\\
&= \langle\vec{f},W^{*}(\lambda)\vec{e}_2\rangle\langle R(-\theta(\lambda))W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2,\vec{f}\rangle\\
&= \|\vec{f}\|^2\|W^{*}(\lambda)\vec{e}_2\|\,\left\|W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2\right\|\\
&\quad\times\cos(\vec{f},W^{*}(\lambda)\vec{e}_2)\cos(R(-\theta(\lambda))W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2,\vec{f}) \qquad (5.4)
\end{aligned}
$$

where (x,y) means the angle between x and y, and $\vec{e}_1=(1,0)^T$, $\vec{e}_2=(0,1)^T$. Now by combining Equations (5.3) and (5.4) we obtain

$$
\begin{aligned}
&\|\left(R(\theta(\lambda))-q_kW^{-1}(\lambda)A(\lambda)W(\lambda)\right)\vec{f}\|^2\\
&\qquad= \|\vec{f}\|^2\Bigg(1-2\frac{q_k}{a_1}\|W^{*}(\lambda)\vec{e}_2\|\,\left\|W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2\right\|\\
&\times\cos(\vec{f},W^{*}(\lambda)\vec{e}_2)\cos(R(-\theta(\lambda))W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2,\vec{f})+O(q_k^2)\Bigg) \qquad (5.5)
\end{aligned}
$$

The next stage of the argument is to establish for what $\vec{f}$ we can reduce the expression on the righthand-side of Equation (5.5). In order to do this we must investigate the relationship between the vectors $R(-\theta(\lambda))W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2$ and $W^{*}(\lambda)\vec{e}_2$.

Lemma 5.1.2. *Let $\lambda\in\sigma_{ell}(J_T)$, where J_T is an arbitrary period-T Jacobi operator and $\theta(\lambda)$, abbreviated to θ, is the quasi-momentum. Then the vectors $W^{*}(\lambda)\vec{e}_2$ and $R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2$, where $W(\lambda)$ is defined as previously, are always orthogonal with respect to the standard complex inner product.*

Proof. By observing that

$$M^{-1}(\lambda)=(W(\lambda)R(\theta)W^{-1}(\lambda)))^{-1}=W(\lambda)R(-\theta)W^{-1}(\lambda),$$

the result clearly follows from the explicit calculation:

$$\langle W^{*}(\lambda)\vec{e}_2,R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2\rangle$$

$$
\begin{aligned}
&= \langle\vec{e}_2,W(\lambda)R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2\rangle\\
&= \langle\vec{e}_2,M^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2\rangle\\
&= \langle\vec{e}_2,B_1^{-1}(\lambda)\vec{e}_2\rangle=\langle\vec{e}_2,-\frac{a_1}{a_T}\vec{e}_1\rangle=0. \qquad \square
\end{aligned}
$$

With the knowledge that the vectors are always orthogonal (for any $\lambda \in \sigma_{ell}(J_T)$) the following lemma implies that with the exception of a collection of arbitrarily narrow cones around $\pm W^*(\lambda)\vec{e}_2, \pm R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2$ of opening angle ϵ, we have for any vector $\vec{f}$ the estimate

$$\| \left(R(\theta) - q_k W^{-1}(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} W(\lambda) \right) \vec{f} \|^2 \leq \|\vec{f}\|^2 \left(1 - C(\lambda)|q_k| + O\left(q_k^2\right)\right)$$

for some $C(\lambda) > 0$, providing that the sign of q_k is chosen appropriately. (Otherwise the corresponding components of the solution won't decrease in size, but increase.)

Lemma 5.1.3. *Consider $\vec{h}_0, \vec{h}_1 \neq \underline{0} \in \mathbb{R}^2$. Then for arbitrarily small $0 < \epsilon < \frac{\pi}{2}$ we have*

$$|\sin(\vec{h}_0, \vec{f}) \sin(\vec{h}_1, \vec{f})| \geq \sin^2(\epsilon)$$

for $\vec{f} \in \mathbb{R}^2$ lying outside the four cones with central axis $\pm\vec{h}_0, \pm\vec{h}_1$ and opening angles less than or equal to ϵ.

Proof. Using that $(\vec{h}_0, \vec{f}) > \epsilon, (\vec{h}_1, \vec{f}) > \epsilon$ and that $\sin(x)$ is a monotonic increasing function on the range $(0, \frac{\pi}{2})$ yields the result. □

Consequently, by defining $\vec{h}_0 := \vec{v}_0$ where $\vec{v}_0 \perp W^*(\lambda)\vec{e}_2$, and $\vec{h}_1 := \vec{v}_1$ where $\vec{v}_1 \perp R(-\theta)W^{-1}M(\lambda)B_1^{-1}(\lambda)\vec{e}_2$ then

$$\begin{aligned} |\cos(W^*(\lambda)\vec{e}_2, \vec{f}) \cos(R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2, \vec{f})| &= |\sin(\vec{h}_0, \vec{f}) \sin(\vec{h}_1 \vec{f})| \\ &\geq \sin^2(\epsilon) > 0 \end{aligned}$$

for all $\vec{f}$ lying outside the four cones with central axis $\pm\vec{h}_0, \pm\vec{h}_1$ and opening angles less than or equal to ϵ.

We will use the following definition throughout this section.

Definition 5.1.4. *We define the set S_ϵ to be the four orthogonal cones about the vectors $\pm W^*(\lambda)\vec{e}_2$ and $\pm R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2$ each with opening angle ϵ (see Figure 5.1 for more details).*

Thus, it has been established that for any $\lambda \in \sigma_{ell}(J_T)$ and for any $\vec{f} \in \mathbb{R}^2 \backslash S_\epsilon$ we have the relations

$$\begin{aligned} \left\| M(\lambda - q_k)\vec{f} \right\|^2 &= \left\| W(\lambda) \left(R(\theta) - q_k W^{-1}(\lambda)A(\lambda)\right) W^{-1}(\lambda) \right\|^2 \\ &\leq \|W(\lambda)\|^2 \|W^{-1}(\lambda)\|^2 \|\vec{f}\|^2 \left(1 - C(\lambda)|q_k| + O\left(q_k^2\right)\right) \end{aligned} \quad (5.6)$$

for some constant $C(\lambda) > 0$, i.e. for a small enough potential the vector $\vec{f}$ shrinks. However it still remains to show that any arbitrary initial vector can be moved into the region $\mathbb{R}^2 \setminus S_\epsilon$, to undergo shrinkage, regardless of where in the plane it is initially and the value of the quasi-momentum, $\theta(\lambda)$. Lemma 5.1.6 will establish this, however the following proposition is needed first.

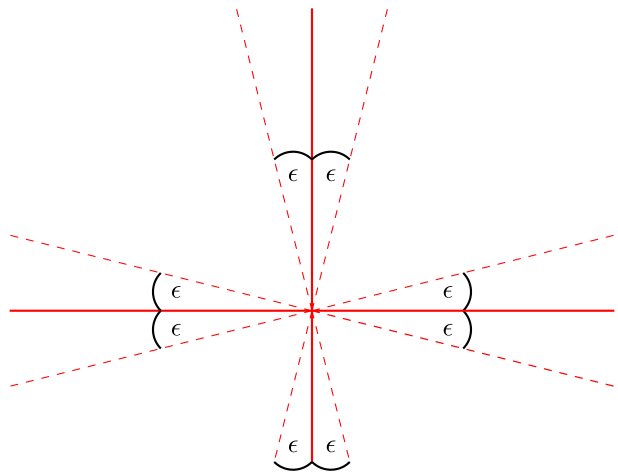

Figure 5.1: The four solid red lines represent the central axes of the four bad cones comprising S_ϵ whilst the dashed lines represent the width of the respective cones, each with opening angle ϵ.

Proposition 5.1.5. [43, 62] *Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_N$ and $\vec{e}'_1, \vec{e}'_2, \ldots, \vec{e}'_N$ be two arbitrary collections of unit vectors in $\mathbb{R}^2$, and $\{\pi, \theta_1, \ldots, \theta_N\}$ a collection of rationally independent numbers. If $R(\theta)$ is the operator of rotation through an angle θ around the origin of coordinates in $\mathbb{R}^2$, then for any $\gamma > 0$ there is a number $m \in \mathbb{N}$ such that the angle between the vectors $\vec{e}'_i$ and $R^m(\theta_i)\vec{e}_i$ is smaller than γ simultaneously for all $i = 1, 2, \ldots, N$. Here $m \leq R(N, \gamma)$, where the constant R depends only on N and γ.*

Consequently, for $\lambda \in \sigma_{ell}(J_T)$ such that $\theta(\lambda) \notin \pi\mathbb{Q}$, Proposition 5.1.5 gives an upper bound on the number of applications of the rotation matrix, $R(\theta(\lambda))$, required to rotate an arbitrary vector, $\vec{f} \in \mathbb{R}^2$, into an acceptable region of the plane, $\mathbb{R}^2 \setminus S_\epsilon$.

The following gives an analogous result for when $\theta(\lambda)$ is not rationally independent with π, although we do have to exclude from our consideration the case when $\theta(\lambda) = \frac{\pi}{2}$.

Lemma 5.1.6. *Let $\lambda \in \sigma_{ell}(J_T)$ with $\theta(\lambda) = \frac{p\pi}{q} \neq \frac{\pi}{2}$ where $\gcd(p, q) = 1$. Then, for any real vector, $\vec{f} \in \mathbb{R}^2$ and any $\epsilon \in \left(0, \frac{\pi}{2q}\right)$ there exists some $k \in \{0, 1, 2\}$ such that $R(k\theta)\vec{f} \notin S_\epsilon$.*

Proof. If $\frac{\theta(\lambda)}{\pi} = \frac{p}{q}$, where $\gcd(p, q) = 1$, then there are two cases:
(Case One) If $\vec{f} \notin S_\epsilon$ then take $k = 0$.
(Case Two) Otherwise $\vec{f} \in S_\epsilon$ and it needs to be shown that after a certain number of rotations the vector $\vec{f}$ no longer inhabits any of the four orthogonal bad cones. Define ξ to be the angle between the vector $\vec{f}$ and the central axis

of the cone containing $\vec{f}$ (see Figure 5.2 for more details). Clearly, we have the relation

$$|\xi| < \epsilon. \tag{5.7}$$

Moreover, since $2\epsilon < \frac{\pi}{q} < \theta(\lambda)$, we also have that one application of the rotation matrix, $R(\theta)$, ensures that the vector $\vec{f}$ is moved to outside the first cone. We now establish that the vector $\vec{f}$ is not moved into the bad cone opposite under the same single application of $R(\theta)$, i.e. $|\pi - \left(\xi + \frac{p\pi}{q}\right)| \geq \epsilon$. Recalling that $\epsilon < \frac{\pi}{2q}$ and (5.7), we assume for contradiction

$$\begin{aligned}
\left|\pi - \left(\xi + \frac{p\pi}{q}\right)\right| < \epsilon \Rightarrow \left|\pi - \frac{p\pi}{q}\right| < 2\epsilon \\
\iff |q\pi - p\pi| < \frac{2\pi}{2q} \cdot q \\
\Rightarrow |q - p|\,\pi < \pi.
\end{aligned} \tag{5.8}$$

This implies $q = p$, which is a contradiction. Thus the condition $\epsilon < \frac{\pi}{2q}$ guarantees that a rotation will move the vector, $\vec{f}$, outside of the first cone and not into the 'third', i.e. the mirror image of the first.

It still remains to consider

$$\left|\frac{\pi}{2} - \left(\xi + \frac{p\pi}{q}\right)\right| < \epsilon, \tag{5.9}$$

which describes the instance of applying the rotation matrix, $R(\theta)$, once, but only succeeding to move the vector from the first bad cone into the (orthogonal) 'second'. Clearly, if this doesn't happen then one rotation is enough to move $\vec{f}$ outside of S_ϵ as $\theta < \pi$. Thus, we assume this is not the case and that (5.9) happens. We then consider the case of applying another rotation and moving from the second cone into the third. This is described by

$$\left|\pi - \left(\xi + \frac{2p\pi}{q}\right)\right| < \epsilon. \tag{5.10}$$

Then, by recalling that $\epsilon < \frac{\pi}{2q}$ and combining Equations (5.7) and (5.10) we see that

$$\begin{aligned}
\left|\pi - \left(\xi + \frac{2p\pi}{q}\right)\right| < \epsilon \Rightarrow \left|\pi - \frac{2p\pi}{q}\right| < 2\epsilon \\
\iff |q\pi - 2p\pi| < \frac{\pi}{2q} \cdot 2q \\
\Rightarrow |q - 2p|\,\pi < \pi.
\end{aligned}$$

This implies $q = 2p$ which is a contradiction. This tells us that (5.10) can never happen; in particular, after at most two rotations the vector $\vec{f}$ will be rotated out of the set S_ϵ. □

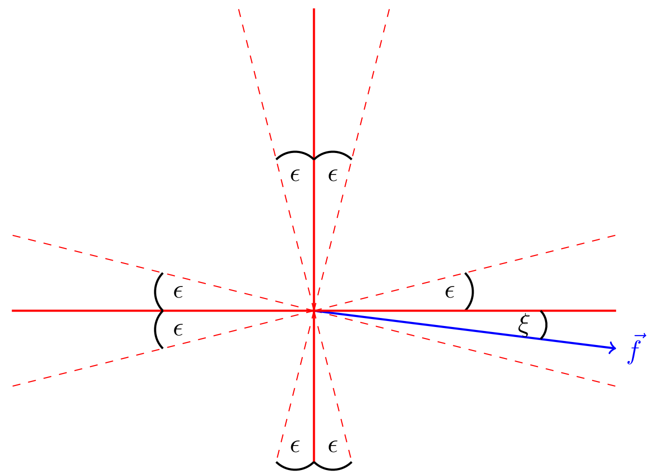

Figure 5.2: The four solid red lines represent the central axes of the four bad cones comprising S_ϵ whilst the dashed lines represent the width of the respective cones, each with opening angle ϵ. The vector $\vec{f}$ in this particular example resides in the rightmost bad cone, and is of angle ξ from the central axis of the nearest bad cone.

Remark The geometric difficulty of those λ with $\theta(\lambda) = \frac{\pi}{2}$ arises from the orthogonality of the bad cones. Thus if for these λ the vector $\vec{f}$ should fall into one of the bad cones, no amount of rotating will ever relocate the vector into a shrinkable area of the domain (i.e. a region outside the four arbitrarily small bad cones) since the vector will just move from one bad cone into another.

Remark It should be stressed not only for this section, but also the ones that follow, that the only truly bad regions in the plane are along the central axes of the bad cones. Thus, when proving certain results (i.e. Lemmas 5.1.6, 5.4.1, 5.5.2 and 5.5.3) on the feasibility of manoeuvring certain vectors into shrinkable regions of the plane we only focus on avoiding the central axis of the bad-cones. This is because regardless of how close the vector $\vec{f}$ is to the central axis, providing the vector is not on it, the opening angle, ϵ, can simply be reduced so that the bad cone is narrow enough for the vector, $\vec{f}$, to avoid the forbidden region altogether. However, this should not be misconstrued as a statement advocating the possibility of bad-lines, rather than bad-cones (arbitrarily small as they may be). The need for some opening angle, ϵ, about the bad cones follows from Lemma 5.1.3 which tells us that if ϵ is zero then it is possible to choose $\vec{f}$ arbitrarily near to the central axis such that

$$|\cos(W^*(\lambda)\vec{e}_2, \vec{f}) \cos(R(-\theta)W^{-1}(\lambda)M(\lambda)B_1^{-1}(\lambda)\vec{e}_2, \vec{f})| \geq |\sin(\epsilon)|^2 = 0.$$

In which case we only obtain the result

$$\| \left(R(\theta) - q_k W^{-1}(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} W(\lambda) \right) \vec{f} \|^2 \leq \|\vec{f}\|^2 + O\left(q_k^2\right),$$

in particular there is no shrinking of $\|\vec{f}\|^2$ by factor $C(\lambda)|q_k|$.

It follows from Proposition 5.1.5 and Lemma 5.1.6 that for all pairs $(\theta(\lambda), \vec{f})$, with the condition $\theta(\lambda) \neq \frac{\pi}{2}$, $\exists\ n \leq n_0(\lambda;\epsilon)$, for ϵ small enough, such that $R(n\theta)\vec{f} \notin S_\epsilon$, and thus by Lemma 5.1.3 the vector, $\vec{f}$, can be diminished in size by the application of a perturbed monodromy matrix with an appropriate potential (q_n). Note that the potential will take the form

$$(q_n) = (0, \dots, 0, \widetilde{q}_1, 0, \dots, 0, \widetilde{q}_2, 0, \dots, \dots, \widetilde{q}_i, 0, \dots, 0, \widetilde{q}_{i+1}, \dots),$$

i.e. a vector characterised mostly by zeros (to account for when the monodromy matrix is analogous to a rotation and the fact that only one of the T transfer matrices is perturbed) and where the non-zero entries are denoted by an overhead tilde. In particular, $\widetilde{q}_i$ represents the i-th non-zero entry of the sequence (q_n).

We now wish to show that the candidate eigenvector, $\underline{u}$, is in the sequence space $l^2(\mathbb{N};\mathbb{R})$. This will use the following lemma.

Lemma 5.1.7. *Let $\lambda \in \sigma_{ell}(J_T)$ such that $\theta(\lambda) \neq \frac{\pi}{2}$. Then, for $\widetilde{q}_n \ll 1$ and $\widetilde{q}_n \in l^2$, the candidate eigenvector $\underline{u} := (u_n)_{n\geq 1}$, obtained by the procedure in this section, satisfies the estimate*

$$\|\underline{u}\|^2 \leq K_1(\lambda) \sum_{i=0}^{\infty} \left[\prod_{j=0}^{i} (1 - C(\lambda)|\widetilde{q}_j|) \right] (r_{i+1} + 1) \\ + K_2(\lambda) \sum_{t=1}^{\infty} \prod_{s=1}^{t-1} (1 - C(\lambda)|\widetilde{q}_s|) + K_3(\lambda) \quad (5.11)$$

for some $K_1(\lambda), K_2(\lambda), K_3(\lambda) \in \mathbb{R}$, where $\widetilde{q}_0 := 0$ and r_i represents the i-th interval of rotation, i.e. the number of rotations required to rotate the components following the $i-1$-th shrinkage into the acceptable region, $\mathbb{R}^2 \setminus S_\epsilon$, so that the perturbed monodromy matrix $M(\lambda - \widetilde{q}_i)$ can be applied.

Remark The first component of the expression comes from calculating the contribution at every T-th term, as well as all those interim components that arise from an incomplete unperturbed monodromy matrix, i.e. those u_{kT+s} such that

$$\begin{pmatrix} u_{kT+s} \\ u_{kT+s+1} \end{pmatrix} = B_s(\lambda) \dots B_1(\lambda) \begin{pmatrix} u_{kT} \\ u_{kT+1} \end{pmatrix}$$

for some $k \in \mathbb{N}, 1 \leq s < T$. The second term estimates those components that arise from an incomplete perturbed monodromy matrix, i.e. those u_{kT+s} such that

$$\begin{pmatrix} u_{kT+s} \\ u_{kT+s+1} \end{pmatrix} = B_s(\lambda) \dots B_1(\lambda - \widetilde{q}_t) \begin{pmatrix} u_{kT} \\ u_{kT+1} \end{pmatrix}$$

for some $k, t \in \mathbb{N}, 1 \leq s < T$ and uses the fact that $\|B_1(\lambda - \widetilde{q}_t)\|$ is uniformly bounded in t. Finally, the third term factors in the $O(\widetilde{q}_k^2)$ contributions (see Appendix B for more details).

Finally, setting $\widetilde{q}_n := \frac{c_0}{n}$ for some constant c_0, and invoking Proposition 5.1.5 and Lemma 5.1.6 to establish that there exists some R such that $r_i \leq R$ for all i gives:

$$\left[\prod_{j=0}^{i}(1-|\widetilde{q}_j|C(\lambda))\right](r_{i+1}+1) \leq (R+1)e^{\sum\limits_{j=1}^{i}\log(1-|\widetilde{q}_j|C(\lambda))}$$

$$= (R+1)e^{-\sum\limits_{j=1}^{i}|\widetilde{q}_j|C(\lambda)+O(\widetilde{q}_j^2)} \asymp e^{-C(\lambda)\sum\limits_{j=1}^{i}\frac{c_0}{j}} \sim e^{-C(\lambda)\cdot c_0 \ln i} = \frac{1}{i^{C(\lambda)c_0}}$$

which is in $l^1(\mathbb{N};\mathbb{C})$ as $i \to \infty$, providing $C(\lambda)c_0 > 1$. Thus, we need $c_0 > \frac{1}{C(\lambda)}$. This concludes the argument that $\underline{u} \in l^2$.

We can summarise the above result in the following theorem.

Theorem 5.1.8. *Let J_T be an arbitrary period-T Jacobi operator and $\lambda \in \sigma_{ell}(J_T) \setminus \{x|\theta(x) = \frac{\pi}{2}\}$. Then there exists a potential $q_n = O\left(\frac{1}{n}\right)$, with $q_n \neq o\left(\frac{1}{n}\right)$, and a vector $(u_n) \in l^2(\mathbb{N};\mathbb{R})$ such that*

$$(J_T + Q)(u_n) = \lambda(u_n)$$

where Q is an infinite diagonal matrix with entries (q_n).

Remark The Coulomb-type decay of (q_n) gives that the potential is a compact perturbation and therefore the essential spectrum of the operators J_T and J_T+Q coincide. Consequently, the eigenvalue λ in Theorem 5.1.8 is embedded in the essential spectrum of the operator $J_T + Q$.

5.2 Infinitely many eigenvalues — Case One

In this section we adapt the method from [65] to embed infinitely many eigenvalues, simultaneously, into the essential spectrum using a single potential, q_n. Indeed, the problem that arises whenever one contemplates the embedding of multiple eigenvalues is that not only must all the two dimensional vectors,

$$\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n, \ldots,$$

corresponding to the respective candidate eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots,$$

inhabit shrinkable areas of the plane, but also that the respective $\vec{f}_i$ are in compatible shrinkable areas, since the reduction factor

$$(1 - C(\lambda)|\widetilde{q}_k|)\|\vec{f}\|^2$$

used in the single eigenvalue case depended on choosing the sign of the potential-component, $\widetilde{q}_k$, correctly. Otherwise as the components of one candidate eigenvector are decreased, the components for the other eigenvector will possibly be increased. Consequently, in this and the following three sections, it is necessary to impose restrictions on the set of eigenvalues being embedded to ensure it is possible that the initial components, $\vec{f_i}$, can all be manoeuvred simultaneously into compatible, shrinkable regions of the real plane regardless of where they first lie. Throughout this section, we insist that any finite selection of eigenvalues from the infinite set have quasi-momenta rationally independent with each other and π.

The following definition will be needed throughout the rest of this chapter.

Definition 5.2.1. *Let $S_\epsilon^{(i)}$ denote the four arbitrarily small orthogonal cones each with opening angle ϵ about the vectors $\pm R(-\theta(\lambda_i))W^{-1}(\lambda_i)M(\lambda_i)B_1^{-1}(\lambda_i)\vec{e}_2$ and $\pm W^*(\lambda_i)\vec{e}_2$.*

Without loss of generality, the orthogonal bad cones, $S_\epsilon^{(i)}$, corresponding to each distinct λ_i to be embedded can be standardised/rotated so that the quadrants (i.e. those regions between bad cones) where the potential, $\widetilde{q}_k$, needs to be positive in order for the respective eigenvector to shrink are the same quadrants as for the other eigenvectors to be embedded (see Figure 5.3 for more details). This results in rotating the vector with the initial components $(u_0, u_1)^T$ about the plane; however, since all the lemmas in this chapter are independent of the location of any of the starting vectors then the results remain valid. Consequently, we will assume all $S_\epsilon^{(i)}$ are equal to some standard S_ϵ and then it is possible to visualise the various rotations by different angles $\theta(\lambda_i)$, acting in the same plane to avoid the same cone, S_ϵ. This implies that for all candidate eigenvector solutions to simultaneously receive sufficient shrinkage, it is sufficient to rotate the relevant vector components into the same quadrant of the plane as those for the other eigenvalues, or into quadrants diametrically opposite. This leads to the following definition for the acceptable region, A_ϵ, which will be used throughout the rest of this chapter.

Definition 5.2.2. *Let $\vec{f_i} \in \mathbb{R}^2$ for $i \in \{1, \dots, n\}$. The collection of vectors $(\vec{f_1}, \dots, \vec{f_n}) \in A_\epsilon$ with $A_\epsilon \subseteq \mathbb{R}^{2n}$ iff for all $i, j \in \{1, \dots, n\}$ we have $f_i \notin S_\epsilon$ and $Q\vec{f_i} \equiv Q\vec{f_j} \mod 2$, where $Q : \mathbb{R}^2 \mapsto \{1, 2, 3, 4\}$, where $\{1, 2, 3, 4\}$ correspond to the quadrants in Figure 5.4. Informally, a collection of vectors belong to A_ϵ if they reside in compatible regions of the plane.*

As an example consider the vectors $\vec{f_1}, \vec{f_2}, \vec{f_3}, \vec{f_4}$ in Figure 5.4. The collection $\left(\vec{f_1}, \vec{f_3}\right) \in A_\epsilon$, as the vector $\vec{f_1}$ inhabits quadrant 1 and $\vec{f_3}$ inhabits quadrant 3, and $1 \equiv 3 \mod 2$. For similar reasons $\left(\vec{f_2}, \vec{f_4}\right) \in A_\epsilon$. However the collection $\left(\vec{f_1}, \vec{f_2}\right) \notin A_\epsilon$ as $\vec{f_1}$ inhabits quadrant 1 and $\vec{f_2}$ inhabits quadrant 2 and $1 \not\equiv 2 \mod 2$. The collection $\left(\vec{f_3}, \vec{f_4}\right)$ does not belong to A_ϵ by a similar argument.

Theorem 5.2.3. *Let J_T be an arbitrary period-T Jacobi operator and $\{\lambda_i\}_{i=1}^\infty$ a sequence of numbers belonging to $\sigma_{ell}(J_T)$, where any finite collection*

$$\{\pi, \theta(\lambda_1), \theta(\lambda_2), \dots, \theta(\lambda_n)\}, n \geq 1$$

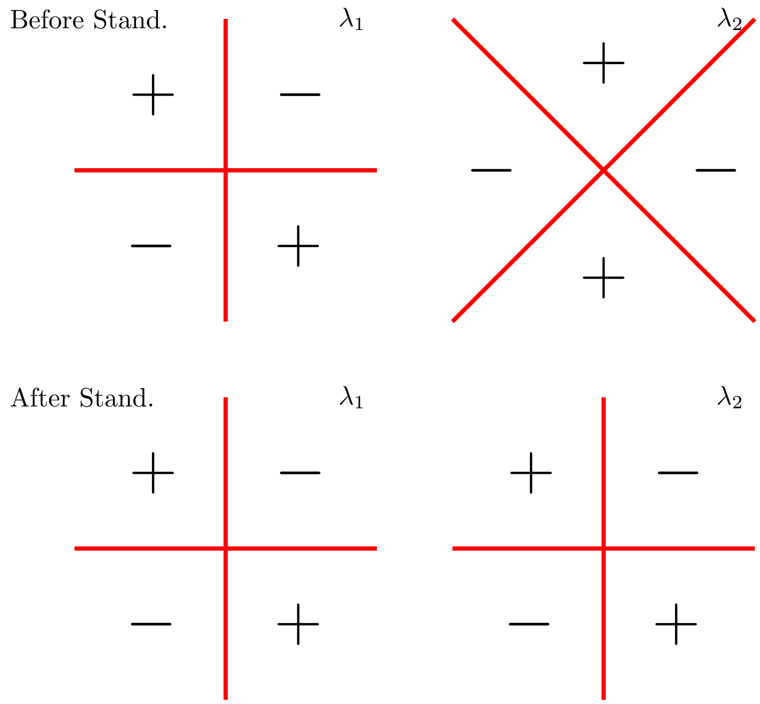

Figure 5.3: The thick red axes illustrate the arbitrarily narrow cones to be avoided for two candidate eigenvalues λ_1, λ_2 respectively, with the first row representing the situation before any attempt at standardization has been made, and the second afterwards, specifically once the orthogonal bad cones corresponding to λ_2 have been appropriately rotated. The $+, -$ signs in each acceptable quadrant indicate the sign that the relevant non-zero component of the potential, $\widetilde{q}_k$, must take in order for the candidate eigenvector to shrink sufficiently. Observe that after the standardization has been made (i.e. the second row of axes) the same quadrant can be chosen for both λ_1 and λ_2 and it will be consistent in terms of the sign required of the potential component, $\widetilde{q}_k$, to produce shrinkage.

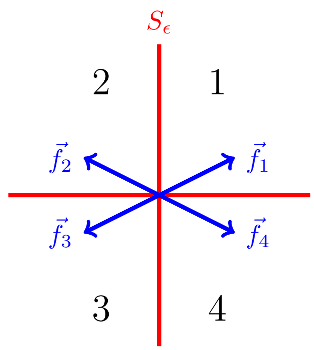

Figure 5.4: The thick red axes illustrate S_ϵ, the arbitrarily narrow cones to be avoided. The numbers $1, 2, 3, 4$ label the acceptable quadrants between the cones and are used in Definition 5.2.2, whilst the vectors $\vec{f}_1, \vec{f}_2, \vec{f}_3, \vec{f}_4$ are employed in the discussion that follows the definition.

is rationally independent. Then there exists a potential $q_n = O\left(\frac{1}{n}\right)$*, with* $q_n \neq o\left(\frac{1}{n}\right)$*, and vectors* $(u_{n,i}) \in l^2(\mathbb{N};\mathbb{R})$ *such that*

$$(J_T + Q)(u_{n,i}) = \lambda_i(u_{n,i})$$

where Q *is an infinite diagonal matrix with entries* (q_n)*.*

Proof. The ability to embed infinitely many eigenvalues using a result (Proposition 5.1.5) that is only valid for finitely many eigenvalues follows by breaking up the set $\{\lambda_i\}_{i=1}^{\infty}$ to be embedded into an increasing sequence of finite subsets, N_k, such that

$$N_1 \subset N_2 \subset N_3 \subset \cdots \subset N_k \subset \ldots$$

and

$$\bigcup_{k=1}^{\infty} N_k = \{\lambda_i\}_{i=1}^{\infty}.$$

Then at each particular stage we are only ever dealing with finitely many eigenvalues (with quasi-momenta rationally dependent with each other and π) which Proposition 5.1.5 is sufficient to deal with. For instance, consider a possible sequence N_k where

$$N_1 = \{\lambda_1\}, N_2 = \{\lambda_1, \lambda_2\}, N_3 = \{\lambda_1, \lambda_2, \lambda_3\}, \ldots.$$

Then for the first stage of our calculation we are concerned only with embedding one eigenvalue, λ_1, so that we deal with rotating the initial components corresponding to this candidate eigenvector, $(u_{n,1})$ into a shrinkable area of the plane, and applying an appropriate potential $\widetilde{q}_1$ once there for the reduction to take effect. (As in the single eigenvalue case we denote the number of rotations necessary by r_1). Then move onto $N_2 = \{\lambda_1, \lambda_2\}$ and use Proposition 5.1.5 to find a bound, r_2, on the number of rotations necessary to manoeuvre the appropriate components of both candidate eigenvectors, $(u_{n,1}), (u_{n,2})$, simultaneously into consistent shrinkable areas of the plane, A_ϵ, and applying a potential $\widetilde{q}_2$. The process is then continued by considering the set N_3, et cetera.

Consequently, the proof rests on showing that an eigenvector, $\underline{u}$, corresponding to an arbitrary $\lambda \in \{\lambda_i\}_{i=1}^{\infty}$, belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$; the idea being somewhat similar to the three-step single eigenvalue technique. However, now the eigenvalue corresponding to this eigenvector might not necessarily be in the set N_1 (the set of eigenvalues whose components are rotated into their respective cones straight away). Instead, we assume $\lambda \in N_i$ for all $i \geq k$ and $\lambda \notin N_j$ for all $j < k$ where $k \in \mathbb{N}$. In particular λ first appears in the set N_k, and we denote the contribution to the square of the norm from the initial components as A_k. Then continuing as before in the single eigenvalue case one uses Proposition 5.1.5 to ensure that all the vectors $\vec{f_i}$ corresponding to the elements of N_j are simultaneously rotated into shrinkable regions of the plane.

Thus, the norm of the eigenvector solution can be estimated by

$$\|\underline{u}(\lambda)\|^2 \leq A_k(\lambda) + D_k(\lambda) \sum_{i=k}^{\infty} \left[\prod_{j=k}^{i} (1 - C(\lambda)|\widetilde{q}_j|) \right] (R(N_{i+1}) + 1)$$
$$+ F_k(\lambda) \sum_{t=k}^{\infty} \left[\prod_{s=k}^{t-1} (1 - C(\lambda)|\widetilde{q}_s|) \right],$$

where $A_k(\lambda), D_k(\lambda), F_k(\lambda) \in \mathbb{R}, \widetilde{q}_0 := 0$ and $R(N_i)$ corresponds to the number of rotations necessary to simultaneously rotate the collection of vectors corresponding to elements of the set N_i into consistent acceptable regions of the plane, A_ϵ.

It remains to show that this sum is bounded; in particular that the vector, $\underline{u}$, belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$. Unlike in the single eigenvalue case, the term $R(N_k)$ is now no longer constant and in fact grows. However, by choosing the sets N_i appropriately the term $R(N_k)$ can be controlled so as to be of order $\sqrt{k}$ (see the remark below for more details). Thus, choosing $\widetilde{q}_n = \frac{c_0}{n}$ and from ideas already expounded for the single eigenvalue case, we have that

$$\left[\prod_{j=k}^{i} (1 - |\widetilde{q}_j| C(\lambda)) \right] (R(N_{i+1}) + 1) \leq (1 + \sqrt{i}) e^{\sum_{j=k}^{i} \log(1 - |\widetilde{q}_j| C(\lambda))} \asymp \frac{1 + \sqrt{i}}{i^{C(\lambda) c_0}}$$

which is in $l^1(\mathbb{N};\mathbb{C})$ as $i \to \infty$ providing $C(\lambda)c_0 > \frac{3}{2}$. Thus, we need $c_0 > \frac{3}{2C(\lambda)}$. □

Remark The exact mechanism by which we can slow the rate of growth of $R(N_k)$ to be of order $O(\sqrt{k})$ is by repeating the sets, N_i sufficiently often, i.e. letting $N_i = N_{i+1} = N_{i+2}$ for sufficiently many steps.

5.3 Infinitely many eigenvalues — Case Two

Here we explore how to embed infinitely many eigenvalues (any finite selection of which have quasi-momenta that are rationally independent with each other and π) simultaneously with a single eigenvalue, λ_1, whose quasi-momentum, $\theta(\lambda_1)$, is specified to be rationally dependent with π, but not equal to $\frac{\pi}{2}$.

The following lemma will be needed first.

Lemma 5.3.1. *Let $\theta(\lambda_1) = \frac{p\pi}{q}$ where $p, q \in \mathbb{N}$ and $\gcd(p,q) = 1$, and let*

$$\{\pi, \theta(\lambda_2), \theta(\lambda_3), \dots, \theta(\lambda_n)\}$$

be rationally independent. Then for any collection of non-zero real vectors $\{\vec{f_1}, \dots, \vec{f_n}\}$ there exists $t \in \mathbb{N}$ such that the collection

$$\Big(R(t\theta(\lambda_1))\vec{f_1}, \dots, R(t\theta(\lambda_n))\vec{f_n}\Big) \in A_\epsilon,$$

where A_ϵ is as defined in Definition 5.2.2. In particular, the vectors, $\vec{f_1}, \dots, \vec{f_n}$ can be simultaneously rotated into the same quadrant or diametrically opposite ones.

Proof. First consider $\theta(\lambda_1)$. By Lemma 5.1.6 there exists a number, k, such that the new vector $R(k\theta(\lambda_1))\vec{f_1}$ is in $\mathbb{R}^2 \setminus S_\epsilon$. Now, since the angle, $\theta(\lambda_1)$, associated with this vector is of the form $\frac{p\pi}{q}$, every subsequent $2q$ rotations will return us to the same point in the plane. Thus, create new angles, $\widetilde{\theta}(\lambda_i) := 2q\theta(\lambda_i)$ for all $i \in \{2, \dots, n\}$. These new angles are still rationally independent, and consequently Proposition 5.1.5 can be applied to give an upper bound, r, on the number of rotations necessary to move all the $R(k\widetilde{\theta}(\lambda_2))\vec{f_2}, \dots, R(k\widetilde{\theta}(\lambda_n))\vec{f_n}$ into the same quadrant where $R(k\theta(\lambda_1))\vec{f_1}$ resides. Note that the purpose of these new angles is to ensure that every new step is $2q$ old steps, implying that the vector $R(k\theta(\lambda_1))\vec{f_1}$ remains fixed under subsequent rotations. The upper bound for the number of rotations is $k + 2qr$. Thus, for some $t \leq k + 2qr$, we have $\Big(R(t\theta(\lambda_1))\vec{f_1}, \dots, R(t\theta(\lambda_n))\vec{f_n}\Big) \in A_\epsilon$. □

Theorem 5.3.2. *Let J_T be an arbitrary period-T Jacobi operator and $\{\lambda_i\}_{i=2}^{\infty}$ a sequence of numbers belonging to $\sigma_{ell}(J_T)$ where any collection*

$$\{\pi, \theta(\lambda_2), \theta(\lambda_3), \dots, \theta(\lambda_n)\}, \quad n \geq 2,$$

is rationally independent and $\lambda_1 \in \sigma_{ell}(J_T)$ such that $\theta(\lambda) = \frac{p\pi}{q} \neq \frac{\pi}{2}$ with $\gcd(p,q) = 1$*. Then there exists a potential $q_n = O\left(\frac{1}{n}\right)$, with $q_n \neq o\left(\frac{1}{n}\right)$, and vectors $(u_{n,i}) \in l^2(\mathbb{N}; \mathbb{R})$ such that*

$$(J_T + Q)(u_{n,i}) = \lambda_i(u_{n,i})$$

for all i and where Q is an infinite diagonal matrix with entries (q_n).

Proof. We wish to show that an arbitrary eigenvector, $\underline{u}(\lambda)$, belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$. Moreover, it is required that λ_1 appear in the set N_1, where $\bigcup_{i=1}^{\infty} N_i = \{\lambda_i\}_{i=1}^{\infty}$ and $N_i \subseteq N_{i+1}$ as before. This means that at every stage we are dealing with embedding one eigenvalue with a quasi-momentum rationally dependent with π, along with finitely many others eigenvalues whose quasi-momenta are rationally independent with each other and π. Lemma 5.3.1 is now invoked instead of the Proposition 5.1.5 to ensure that all relevant vectors associated to the elements of the set N_i are simultaneously rotated into shrinkable areas of the domain. Thus, we construct a candidate eigenvector solution whose norm can be estimated by

$$\|\underline{u}(\lambda)\|^2 \leq A_k(\lambda) + D_k(\lambda)\sum_{i=k}^{\infty}\left[\prod_{j=k}^{i}(1 - C(\lambda)|\widetilde{q}_j|)\right](R(N_{i+1}) + 1) + F_k(\lambda)\sum_{t=1}^{\infty}\left[\prod_{s=k}^{t-1}(1 - C(\lambda)|\widetilde{q}_s|)\right].$$

We bound the growth of $R(N_k) = O\left(\sqrt{k}\right)$ by increasing the sets N_k sufficiently slowly, and then take logarithms and exponentials as in Section 5.2 to show that the candidate eigenvector belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$. □

5.4 Infinitely many eigenvalues — Case Three

Here we discuss the case of embedding two eigenvalues, λ_1, λ_2, first where $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta_1 < \theta_2 < \pi$, and then where $0 < \theta(\lambda_1) < \frac{\pi}{2} < \theta_2(\lambda_2) < \pi$.

Lemma 5.4.1. *Let $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta_1 < \theta_2 < \pi$. Then, for any pair of non-zero real vectors $\{\vec{f_1}, \vec{f_2}\}$ in the plane, there exists k such that the collection*

$$\left(R(k\theta_1)\vec{f_1}, R(k\theta_2)\vec{f_2}\right) \in A_\epsilon,$$

where A_ϵ is as defined in Definition 5.2.2. In particular, the vectors, $\vec{f_1}, \vec{f_2}$ can be simultaneously rotated into the same quadrant or diametrically opposite ones.

Proof. First consider the case when $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$. Since $\theta_2 > \theta_1$ then eventually there will be an 'overtaking'. This means that the vector $\vec{f_2}$ will overtake the vector $\vec{f_1}$ at some stage. In the step preceding this overtaking the two vectors $\vec{f_1}, \vec{f_2}$ are either in the same quadrant or not. If they're in the same quadrant, then the objective is already achieved. If they're not then the vector $\vec{f_2}$ must be in the quadrant 'behind' the quadrant $\vec{f_1}$ is in or in the cone; however after one more rotation the vector $\vec{f_2}$ overtakes $\vec{f_1}$, and since $\theta_2 < \frac{\pi}{2}$ the two vectors must then be in the same quadrant (see Figure 5.5 for more details) and outside S_ϵ for sufficiently small ϵ.

When $\frac{\pi}{2} < \theta_1 < \theta_2 < \pi$ the argument is similar. Again, the only bad situation is when $\vec{f_2}$ is immediately 'behind' $\vec{f_1}$. Then, in the next step when

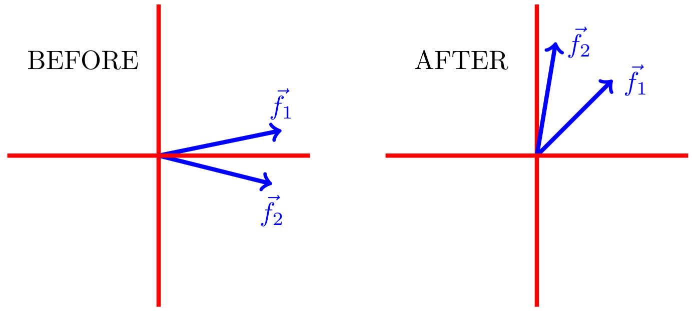

Figure 5.5: The diagram illustrates one particular example of 'overtaking', specifically when $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ and $\vec{f_1}, \vec{f_2}$ do not inhabit the same quadrant before the overtaking. The two images denote the location of the vectors before and after the threshold rotation has been applied. As can be seen, since $\theta_2 < \frac{\pi}{2}$ there is no threat of $\vec{f_2}$ overshooting, and landing in the next (inconsistent) quadrant instead. Note that the thick red axes denote the arbitrarily narrow bad cones to be avoided.

the 'overtaking' happens the vector $\vec{f_1}$ must move into the next quadrant, where it is joined by the vector $\vec{f_2}$. It is not possible for the vector $\vec{f_2}$ to 'overshoot' and land in the inconsistent quadrant beyond $\vec{f_1}$ as this would require $\vec{f_2}$ to move more than two whole quadrants in one rotation, and we already assume $\theta_2 < \pi$ (see Figure 5.6 for more details). $\square$

Theorem 5.4.2. *Let J_T be an arbitrary period-T Jacobi operator and $\{\lambda_i\}_{i=3}^{\infty}$ a sequence of numbers belonging to $\sigma_{ell}(J_T)$ where any collection*

$$\{\pi, \theta(\lambda_3), \theta(\lambda_4), \dots, \theta(\lambda_n)\}, \quad n \geq 3$$

is rationally independent, and $\{\lambda_1, \lambda_2\} \subseteq \sigma_{ell}(J_T)$ where either $0 < \theta(\lambda_1) < \theta(\lambda_2) < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta(\lambda_1) < \theta(\lambda_2) < \pi$. Then there exists a potential $q_n = O\left(\frac{1}{n}\right)$, with $q_n \neq o\left(\frac{1}{n}\right)$, and vectors $(u_{n,i}) \in l^2(\mathbb{N}; \mathbb{R})$ such that

$$(J_T + Q)(u_{n,i}) = \lambda_i(u_{n,i})$$

for all i and where Q is an infinite diagonal matrix with entries (q_n).

Proof. For the case when only one of $\theta(\lambda_1), \theta(\lambda_2)$ is rationally dependent with π, we already have the result (see Theorem 5.3.2). Thus the only outstanding case is when $\theta(\lambda_1) = \frac{p_1\pi}{q_1}$ and $\theta(\lambda_2) = \frac{p_2\pi}{q_2}$ with $\gcd(p_1, q_1) = 1 = \gcd(p_2, q_2)$.

The argument follows similarly to the proof of Theorem 5.3.2 by dividing up the set of eigenvalues to be embedded into sets $N_1 \subset N_2 \subset N_3 \subset \dots$ except now $\{\lambda_1, \lambda_2\} \subseteq N_1$. The 'overtaking' argument used in Lemma 5.4.1 can be applied again to establish that a finite number of rotations will be enough to simultaneously manoeuvre the initial vectors in the plane corresponding to the

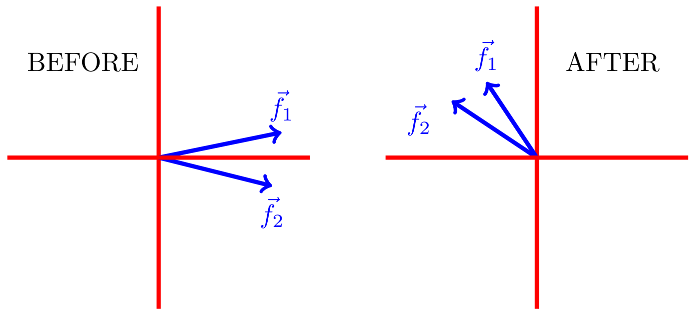

Figure 5.6: The diagram illustrates another example of 'overtaking', specifically when $\frac{\pi}{2} < \theta_1 < \theta_2 < \pi$ and $\vec{f}_1, \vec{f}_2$ do not inhabit the same quadrant before the overtaking. The two images denote the location of the vectors before and after the threshold rotation has been applied. As can be seen, since $\frac{\pi}{2} < \theta_2$ the vector $\vec{f}_2$ lands in the next diametrically opposite quadrant. Note that the thick red axes denote the arbitrarily narrow bad cones to be avoided.

candidate eigenvalues, λ_1, λ_2, into shrinkable regions. Moreover, for subsequent sets, N_k, it will become necessary to simultaneously rotate finitely many other initial vector components using quasi-momenta $\theta(\lambda_{n_1}), \ldots, \theta(\lambda_{n_k})$, rationally independent with each other and π. Specifically, we first set about moving the vectors, $\vec{f}_1, \vec{f}_2$ corresponding to the two quasi-momenta rationally dependent with π into shrinkable regions of the plane, just as in the N_1 case, with the upper bound n_0 on the number of rotations required. Then, observing that for every subsequent $t := \mathrm{LCM}(q_1, q_2)$ rotations the vectors $\vec{f}_1, \vec{f}_2$ remain in the shrinkable region of the plane, create new quasi-momenta $\widetilde{\theta}(\lambda_i) = t\theta(\lambda_i)$ for $i \in \mathbb{N} \setminus \{1, 2\}$, rationally independent with each other and π. Then, Proposition 5.1.5 can again be applied to the vectors $\vec{f}_i$ for $i \in I_k$, where $I_k := \{i \in \mathbb{N} \setminus \{1, 2\} : \lambda_i \in N_k\}$, to rotate them all simultaneously into the same shrinkable region as $\vec{f}_1, \vec{f}_2$ but now using the new angles $\widetilde{\theta}_i$ since this ensures that at every time a rotation is applied the vectors $\vec{f}_1, \vec{f}_2$ remain in the same shrinkable region. This technique is applicable for all sets N_k.

Thus, for each $\lambda \in \{\lambda\}_{i=1}^{\infty}$, we can construct a candidate eigenvector solution, $\underline{u}(\lambda)$ that satisfies

$$\|\underline{u}(\lambda)\|^2 \leq A_k(\lambda) + D_k(\lambda) \sum_{i=k}^{\infty} \left[\prod_{j=k}^{i} (1 - C(\lambda)|\widetilde{q}_j|) \right] (R(N_{i+1}) + 1) + F_k(\lambda) \sum_{t=1}^{\infty} \left[\prod_{s=k}^{t-1} (1 - C(\lambda)|\widetilde{q}_s|) \right], \quad (5.12)$$

Again we slow the growth of $R(N_k) = O\left(\sqrt{k}\right)$ and then take logarithms and exponentials as before to show that the candidate eigenvector belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$. $\square$

We now continue to look at embedding two eigenvalues, simultaneously, but this time consider the more sophisticated case when the quasi-momenta are such that $0 < \theta(\lambda_1) < \frac{\pi}{2} < \theta(\lambda_2) < \pi$.

Lemma 5.4.3. *Let $0 < \theta_1 < \frac{\pi}{2} < \theta_2 < \pi$, but where $\pi - \theta_2 \neq \theta_1$. Then for any pair of non-zero real vectors $\{\vec{f_1}, \vec{f_2}\}$ in the plane there exists k such that the collection*

$$\left(R(k\theta_1)\vec{f_1}, R(k\theta_2)\vec{f_2}\right) \in A_\epsilon,$$

where A_ϵ is as defined in Definition 5.2.2. In particular, the vectors $\vec{f_1}, \vec{f_2}$ can be simultaneously rotated into the same quadrant or diametrically opposite ones.

Proof. The technique uses the 'overtaking' argument, like Lemma 5.4.1. However, due to the comparative size of the angles involved there is now the threat that before and after the overtaking happens the vectors $\vec{f_1}$ and $\vec{f_2}$ are still in inconsistent quadrants. This is the only case that needs to be considered. Define the new angle $\widetilde{\theta}_2 := \theta_2 - \pi$ and without loss of generality consider the situation when $|\widetilde{\theta}_2| < \theta_1$. (The case $|\widetilde{\theta}_2| > \theta_1$ follows a similar argument, whilst the case $\left|\widetilde{\theta}_2\right| = \theta_1$ has been excluded by the acceptable conditions of the lemma.) Since we may identify opposite quadrants we can assume $\vec{f_2}$ is rotated by $\widetilde{\theta}_2$ rather than θ_2.

For the sake of simplicity we skip straight to the step before the overtaking is to happen and assume the two vectors inhabit inconsistent quadrants (otherwise there is no problem). Moreover, after the overtaking we assume the vectors reside in inconsistent quadrants (otherwise again there is no problem). By identifying consistent quadrants, without loss of generality the problematic case occurs when $\xi_1 \leq 0, \xi_2 \geq 0, \xi_1 + \theta_1 \geq 0, \xi_2 + \theta_2 \leq 0$, where $\xi_i := \arg(f_i)$. We aim to show that since $\theta_1 > |\widetilde{\theta}_2|$ there exists a $k_1 \in \mathbb{N}$ such that

$$\xi_1 + k_1\theta_1 > \frac{\pi}{2} \quad \text{and} \quad \xi_2 + k_1\widetilde{\theta}_2 > -\frac{\pi}{2}, \tag{5.13}$$

i.e. after k_1 steps $\vec{f_1}$ has been rotated into the second quadrant, while $\vec{f_2}$ is still in the consistent fourth quadrant (see Figure 5.7 for more details). These are satisfied if

$$(k_1 - 1)\theta_1 > \frac{\pi}{2} \text{ and } k_1 < -\frac{\pi}{2\widetilde{\theta}_2},$$

respectively. For this to occur, it is sufficient for $\theta_1, \widetilde{\theta}_2$ be such that

$$\frac{\pi}{2\theta_1} + 1 < k_1 < -\frac{\pi}{2\widetilde{\theta}_2} - 1.$$

This implies that

$$\begin{aligned} &\pi\widetilde{\theta}_2 + 4\theta_1\widetilde{\theta}_2 > -\pi\theta_1 \\ \iff &\pi\left(\theta_1 + \widetilde{\theta}_2\right) > -4\theta_1\widetilde{\theta}_2 \end{aligned}$$

Note that we subtracted 1 from the upper-bound since not only must k exist it must also be a natural number. Indeed, if $\theta_1, \widetilde{\theta}_2$ are such that the relations hold for the new (reduced) upper bound then they will also hold for the original bound and since the difference between the two upper bounds is 1 there must be some natural number k in the range.

If these conditions aren't met then there is still the chance that the same 'consistent-coinciding', happens later, just in another quadrant. This happens when both are in the lower left quadrant, i.e. if there exists $k_2 \in \mathbb{N}$ such that

$$\xi_1 + k_2\theta_1 > \pi \quad \text{and} \quad \xi_2 + k_2\widetilde{\theta}_2 > -\pi. \tag{5.14}$$

These are satisfied if

$$k_2 - 1 > \frac{\pi}{\theta_1} \text{ and } k_2 < \frac{-\pi}{\widetilde{\theta}_2},$$

respectively. Then for this to occur it is sufficient for $\theta_1, \widetilde{\theta}_2$ to be such that

$$1 + \frac{\pi}{\theta_1} < k_2 < \frac{\pi}{-\widetilde{\theta}_2} - 1.$$

This implies that

$$\begin{aligned} &2\theta_1\widetilde{\theta}_2 + \pi(\theta_1 + \widetilde{\theta}_2) > 0 \\ \iff &\pi(\theta_1 + \widetilde{\theta}_2) > -2\theta_1\widetilde{\theta}_2. \end{aligned}$$

Note again that we have subtracted 1 from the upper-bound. If such a k_2 does not exist then we can keep repeating the process. For k_n we need $\theta_1, \widetilde{\theta}_2$ such that

$$1 + \frac{n(\frac{\pi}{2})}{\theta_1} < \frac{n(\frac{\pi}{2})}{-\widetilde{\theta}_2} - 1 \iff \frac{n\pi}{2}(\theta_1 + \widetilde{\theta}_2) > -2\theta_1\widetilde{\theta}_2.$$

Then for any $|\widetilde{\theta}_2| < \theta_1$ this condition is satisfied for any n large enough. For the case $|\widetilde{\theta}_2| > \theta_1$ the same expression is obtained. □

Theorem 5.4.4. *Let J_T be an arbitrary period-T Jacobi operator and $\{\lambda_i\}_{i=3}^{\infty}$ a sequence of numbers belonging to $\sigma_{ell}(J_T)$ where any collection*

$$\{\pi, \theta(\lambda_3), \theta(\lambda_4), \ldots, \theta(\lambda_n)\}, \quad n \geq 3$$

is rationally independent, and $\{\lambda_1, \lambda_2\} \subseteq \sigma_{ell}(J_T)$ where $0 < \theta(\lambda_1) < \frac{\pi}{2} < \theta(\lambda_2) < \pi$ and $\theta(\lambda_1) \neq \pi - \theta(\lambda_2)$. Then there exists a potential $(q_n) = O\left(\frac{1}{n}\right)$, with $q_n \neq o\left(\frac{1}{n}\right)$, and vectors $(u_{n,i}) \in l^2(\mathbb{N}; \mathbb{R})$ such that

$$(J_T + Q)(u_{n,i}) = \lambda_i(u_{n,i})$$

for all i and where Q is an infinite diagonal matrix with entries (q_n).

Proof. The argument is the same as in Theorem 5.4.2, except now Lemma 5.4.1 is replaced by Lemma 5.4.3. □

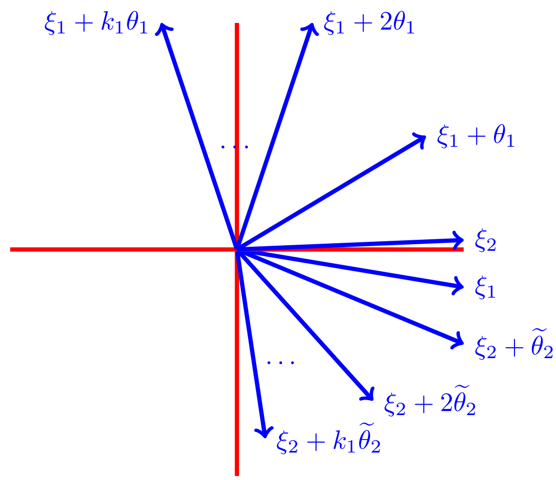

Figure 5.7: The diagram illustrates the possibility of the vector $\vec{f_1}$ being rotated into the top-left quadrant whilst the vector $\vec{f_2}$ is still tarrying in the bottom-right. The expression $\xi_i + z\theta_i$ corresponds to the angle of the vector being described, whilst the thick red axes are as in Figure 5.5.

5.5 Infinitely many eigenvalues — Case Four

Here, alongside infinitely many eigenvalues whose quasi-momenta are rationally independent with each other and π, we consider an arbitrary (but finite) selection of eigenvalues, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\theta(\lambda_1) = \frac{p_1\pi}{q_1}, \theta(\lambda_2) = \frac{p_2\pi}{q_2}, \ldots, \theta(\lambda_n) = \frac{p_n\pi}{q_n},$$

$\gcd(p_i, q_i) = 1, \gcd(q_i, q_j) = 1$ for all $i \neq j$ and $\theta(\lambda_i) \neq \frac{\pi}{2}$ for all i.

The following elementary result will be needed:

Proposition 5.5.1. *Let $\vec{f}$ be an arbitrary vector in the plane and $\theta = \frac{p\pi}{q}$ where $\gcd(p,q) = 1$. Then, if $2|p$ the orbit of $\vec{f}$ has q distinct elements under the rotation matrix $R(\theta)$, all separated evenly by an angle of $\frac{2\pi}{q}$. If $2 \nmid p$ then the orbit of $\vec{f}$ has $2q$ distinct elements under the rotation matrix $R(\theta)$, all uniformly distributed with separating angle $\frac{\pi}{q}$.*

We can now prove the following lemma which will be used in the main theorem of this subsection.

Lemma 5.5.2. *Let $\vec{f}$ be an arbitrary non-zero vector in the plane. Then for $\theta = \frac{p\pi}{q} \notin \{\frac{\pi}{2}, \frac{2\pi}{3}\}$ where $\gcd(p,q) = 1$, every quadrant will feature at least once in the orbit of $\vec{f}$ under the action of $R(\theta)$.*

Proof. First consider the case when $2 \nmid p$. Then, by Proposition 5.5.1, the orbit of $\vec{f}$ has $2q$ distinct elements, evenly distributed about the circle of radius $|\vec{f}|$.

Thus, if $q \geq 4$ then even if all the axes are hit, there will still be at least four other points in the orbit evenly distributed within the quadrants. If $q = 3$ then there are two options: either no axis is hit, in which case there are six points of the orbit evenly distributed about the circle of radius $|\vec{f}|$, four of which must visit every quadrant; or at least one axis is hit by the orbit. However, if one axis is hit then the opposite axis is hit 3 rotations later (or earlier), thus meaning at least two two axes are hit. However, no other axis can be hit because this would be at a distance of $\frac{\pi}{2}$ from either axis, and this is not a multiple of $\frac{\pi}{3}$. Thus, there are four points left in the orbit, meaning every one of the four quadrants is visited.

Secondly, consider the case when $2|p$. Then, by Proposition 5.5.1, the orbit of $\vec{f}$ has q distinct elements, evenly distributed about the circle of radius $|\vec{f}|$. Thus, for $q \geq 8$ even if all four axes are hit, there are still four orbits left with which to visit every quadrant. Now since $q \in \{2, 3, 4, 6\}$ are not valid options, we direct our attentions to $q = 5$. The problem becomes tantamount to looking at a regular pentagon and seeing that at most only one vertex can lie on the axes, thus leaving four to visit every quadrant. Finally, for $q = 7$, it is sufficient to look at a regular heptagon and observe that at most one vertex can reside on the axes. $\square$

Remark The invalidness of the result for the angle $\theta = \frac{\pi}{2}$ follows from the fact that if $\vec{f_i}$ begins on the central-axis of a bad cone then no quadrant is visited over its orbit. Similarly, for $\theta = \frac{2\pi}{3}$ the result fails because at most three quadrants are visited (since its orbit only has three entries) and at worst two (when the one of the orbit includes the axis).

Lemma 5.5.3. *Let $\vec{f_1}, \ldots, \vec{f_n}$ be any non-zero collection of vectors in the plane and $\theta_1 = \frac{p_1\pi}{q_1}, \theta_2 = \frac{p_2\pi}{q_2}, \ldots, \theta_n = \frac{p_n\pi}{q_n}$ where $\gcd(p_i, q_i) = 1$, $\gcd(q_i, q_j) = 1$ for all $i \neq j$ and $\theta_i \neq \frac{\pi}{2}$ for any i. Then there exists $k \in \mathbb{N}$ such that the collection*

$$\left(R(k\theta_1)\vec{f_1}, \ldots, R(k\theta_n)\vec{f_n}\right) \in A_\epsilon,$$

where A_ϵ is as defined in Definition 5.2.2.

Proof. (Case One) For all $\theta_i \neq \frac{2\pi}{3}$, we know that from Lemma 5.5.2 there exists some a_i such that $R(a_i\theta_i)\vec{f_i} \notin S_\epsilon$. What we are now aiming to do is rotate all vectors $\vec{f_i}$ into the same quadrant, i.e. we wish to find some x such that

$$\begin{aligned}
x &\equiv a_1 \mod \alpha_1 q_1, \\
x &\equiv a_2 \mod \alpha_2 q_2, \\
&\vdots \\
x &\equiv a_n \mod \alpha_n q_n,
\end{aligned}$$

where $\alpha_i = 1$ if $2|p_i$ and $\alpha_i = 2$ if $2 \not| p_i$. Now if at most only one p_i is divisible by 2 then we can apply the Chinese Remainder Theorem to obtain the result.

However, if $2|p_j$ for all $j \in A \subset \{1,\dots,n\}$ where $|A| \geq 2$, then the Chinese Remainder Theorem is no longer immediately applicable since the moduli are not co-prime. Instead, we break the system down into co-prime factors, that is $x \equiv a_j \mod 2q_j$ implies

$$x \equiv a_j \mod 2 \qquad \text{and} \qquad x \equiv a_j \mod q_j,$$

for all $j \in A$. This new system can be solved using the Chinese Remainder Theorem, but only providing there are no inconsistencies with regards $x \equiv a_j \mod 2$; in particular, for all $j \in A$, a_j must have the same parity. If there is an inconsistency, then we observe that due to the co-prime conditions on q_i there is at most one $j_0 \in A$ where q_{j_0} is even, and we choose to make every other a_j of the same parity. This is achieved by adding q_j (which must be odd) to all those a_j whose parity is different to a_{j_0}, and thus all the parities are now the same. This follows from the fact that

$$q_j \times \frac{p_j \pi}{q_j} = p_j \pi,$$

which, since $2 \nmid p_j$, is an odd multiple of π and so

$$R(\theta_j(a_j + q_j))\vec{f_j} = R(\theta_j a_j)\vec{f_j} + \pi.$$

Thus by eliminating any inconsistencies in the parity, we have only moved the vector $\vec{f_j}$ from its current quadrant into the diametrically opposite one. The Chinese Remainder Theorem can now be applied.

(Case Two) If $\theta_t = \frac{2\pi}{3}$ for some $t \in \{1,\dots,n\}$ then Lemma 5.5.2 is no longer valid for this particular quasi-momentum. However, the result still follows, since the two quadrants the vector $\vec{f_t}$ does visit under the action of the rotation matrix $R(\theta(\lambda_t))$ are indeed enough. This is because by Lemma 5.5.2 the vectors, $\vec{f_i}, i \in \{1,\dots,n\} \setminus \{t\}$, can be moved to one of these two quadrants under the corresponding action of $R(3\theta_i)$ (the new angle $3\theta_i$ ensuring $\vec{f_t}$ remains fixed in the correct quadrant whilst the other vectors are being moved about). $\square$

We now state the main theorem of this subsection.

Theorem 5.5.4. *Let J_T be an arbitrary period-T Jacobi operator and $\{\lambda_i\}_{i=n+1}^{\infty}$ a sequence of numbers belonging to $\sigma_{ell}(J_T)$ where any collection*

$$\{\pi, \theta(\lambda_{n+1}), \theta(\lambda_{n+2}), \dots, \theta(\lambda_{n+k})\}, \quad k \geq 1,$$

is rationally independent, and $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subseteq \sigma_{ell}(J_T)$ where

$$\theta(\lambda_1) = \frac{p_1\pi}{q_1}, \theta(\lambda_2) = \frac{p_2\pi}{q_2}, \dots, \theta(\lambda_n) = \frac{p_n\pi}{q_n},$$

$\gcd(p_i, q_i) = 1, \gcd(q_i, q_j) = 1$ *for all $i \neq j$ and $\theta(\lambda_i) \neq \frac{\pi}{2}$ for all i. Then there exists a potential $q_n = O\left(\frac{1}{n}\right)$, with $q_n \neq o\left(\frac{1}{n}\right)$, and vectors $(u_{n,i}) \in l^2(\mathbb{N};\mathbb{R})$ such that*

$$(J_T + Q)(u_{n,i}) = \lambda_i(u_{n,i})$$

for all i and where Q is an infinite diagonal matrix with entries (q_n).

Proof. The proof is the same as that used in Theorem 5.4.2, except now when we divide up the eigenvalues into sets we have $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq N_1$, and replace Lemma 5.4.1 with Lemma 5.5.3 to simultaneously rotate the vectors $\vec{f}_1, \ldots, \vec{f}_n$ into shrinkable regions. $\square$

Remark Clearly, Theorem 5.5.4 covers all cases in Theorem 5.3.2. However, Theorem 5.5.4 does not replace Theorems 5.4.4 or 5.4.2 since the pair of eigenvalues with quasi-momenta rationally dependent with π considered in these cases, say $\theta(\lambda_1) = \frac{p_1\pi}{q_1}, \theta(\lambda_2) = \frac{p_2\pi}{q_2}$, could be such that q_1, q_2 are not co-prime and for Theorem 5.5.4 we demand that all q_i be pairwise co-prime.

Chapter 6

Embedded eigenvalues using the ansatz method

With the exception of a few points, the techniques discussed in previous chapters were devised for arbitrary $\lambda \in \sigma_{ell}(J_T)$. We now change this philosophy somewhat and devise eigenvectors and potentials for more explicit elements of the essential spectrum, usually those that have been excluded from other techniques; this explicitness in the eigenvalues we're embedding making the ansatz for the eigenvector somewhat more transparent and the calculations themselves simpler. In the first section we focus on the DSO and establish methods to embed the point $\lambda = 0$ which is excluded under the Wigner-von Neumann and geometric methods, and also the parabolic points 2 and -2. For the latter pair we actually show that a weaker potential of the order n^{-2} suffices, which does not contradict Theorem 1.6.1, since the parabolic points do not belong to the elliptic interval. In the second, and last, section we focus only on those elliptic points whose quasi-momenta share rational dependence with π, complementing the techniques employed in the geometric method.

6.1 Exceptional points for the DSO

Recall that the spectrum of the DSO is the closed interval $[-2, 2]$ and that $C(\lambda; 1) = \frac{\lambda}{2}$ (see the proof of Lemma 3.2.2). Thus, the only element of the generalised interior of the essential spectrum for which the Wigner-von Neumann method is invalid is $\lambda = 0$. We now consider alternative methods to embed eigenvalues here as well as at the boundary points.

Lemma 6.1.1. *The discrete Schrödinger operator with a potential* $q_n = \frac{2n(-1)^n}{n^2-1}$, *for* $n \geq 2$ *and* $q_1 = \frac{1}{2}$ *has an embedded eigenvalue at* $\lambda = 0$ *and associated eigenvector* $u_n = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{n}$ *for* $n \geq 1$.

Proof. Clearly $u_2 + q_1 u_1 = 0$. Moreover, for this choice of (u_n)

$$\begin{aligned}
u_{n+1} + u_{n-1} = -q_n u_n &\iff (-1)^{\lfloor \frac{n+1}{2} \rfloor} \left(\frac{1}{n+1} - \frac{1}{n-1} \right) = -q_n \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{n} \\
&\iff q_n = \frac{2n}{n^2-1} (-1)^{\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor} \\
&\iff q_n = \frac{2n(-1)^n}{n^2-1}.
\end{aligned}$$

Thus, the recurrence equations

$$\frac{(-1)^{\lfloor \frac{n+1}{2} \rfloor}}{n+1} + \frac{(-1)^{\lfloor \frac{n-1}{2} \rfloor}}{n-1} + \frac{2n(-1)^n}{n^2-1} \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{n} = 0,$$

are also satisfied for $n \geq 2$, and the eigenvalue $\lambda = 0$ becomes embedded in the a.c. spectrum of the operator. □

Alternatively, there is also the following lemma which uses a different potential and different eigenvector to embed the same value.

Lemma 6.1.2. *The discrete Schrödinger operator with potential*

$$q_n = \begin{cases} \frac{12}{\pi^2} & n = 1 \\ \frac{1}{n} & n = 2k, k \in \mathbb{N} \\ -\frac{2}{(n^2-1)} \left(\sum\limits_{l=n+1}^{\infty} \frac{1}{l^2} \right)^{-1} & n = 2k+1, k \in \mathbb{N} \end{cases}$$

has an embedded eigenvalue at $\lambda = 0$ and associated eigenvector

$$u_n = A_n \sin\left(\frac{n\pi}{2}\right) + B_n \cos\left(\frac{n\pi}{2}\right),$$

where $A_n := \sum\limits_{k=n+1}^{\infty} q_k B_k$ and

$$B_n := \begin{cases} \frac{1}{n} & n = 2k, k \in \mathbb{N}, \\ 0 & n = 2k+1, k \in \mathbb{N}. \end{cases}$$

Proof. Substituting $u_n = A_n \cos(\frac{n\pi}{2}) + B_n \sin(\frac{n\pi}{2})$ for A_n, B_n undefined at the moment, gives

$$u_{n-1} + u_{n+1} = -q_n u_n$$

$$\begin{aligned}
&\iff A_{n+1} \sin\left((n+1)\frac{\pi}{2}\right) + B_{n+1} \cos\left((n+1)\frac{\pi}{2}\right) + A_{n-1} \sin\left((n-1)\frac{\pi}{2}\right) \\
&+ B_{n-1} \cos\left((n-1)\frac{\pi}{2}\right) = -q_n \left(A_n \sin\left(\frac{n\pi}{2}\right) + B_n \cos\left(\frac{n\pi}{2}\right)\right) \\
&\iff (A_{n+1} - A_{n-1}) \sin\left((n+1)\frac{\pi}{2}\right) + (B_{n+1} - B_{n-1}) \cos\left((n+1)\frac{\pi}{2}\right) \\
&\qquad = -q_n \left(A_n \sin\left(\frac{n\pi}{2}\right) + B_n \cos\left(\frac{n\pi}{2}\right)\right).
\end{aligned}$$

Furthermore when n is even

$$\begin{aligned}(A_{n+1}-A_{n-1})\sin\left((n+1)\frac{\pi}{2}\right) &= -q_nB_n\cos\left(n\frac{\pi}{2}\right)\\ \iff (A_{n+1}-A_{n_1})\cos\left(n\frac{\pi}{2}\right) &= -q_nB_n\cos\left(n\frac{\pi}{2}\right)\\ \iff (A_{n+1}-A_{n-1}) &= -q_nB_n, \end{aligned} \tag{6.1}$$

using $\sin\left(\alpha+\frac{\pi}{2}\right)=\cos(\alpha)$, and $\cos(k\pi)\in\{1,-1\}, k\in\mathbb{Z}$. When n is odd

$$\begin{aligned}(B_{n+1}-B_{n-1})\cos((n+1)\frac{\pi}{2}) &= -q_nA_n\sin(n\frac{\pi}{2})\\ \iff -(B_{n+1}-B_{n-1})\sin(n\frac{\pi}{2}) &= -q_nA_n\sin(n\frac{\pi}{2})\\ \iff (B_{n+1}-B_{n-1}) &= q_nA_n, \end{aligned} \tag{6.2}$$

using $\cos(\alpha+\frac{\pi}{2})=-\sin(\alpha)$. Thus, even-indexed B_i are related to odd indexed A_i. Hence, define B_n as in the lemma and substituting into Equation (6.1) gives for n even

$$A_{n+1}-A_{n-1}=-q_nB_n \Leftarrow A_n=\sum_{k=n+1}^{\infty} q_nB_n \tag{6.3}$$

by observing that $A_{n+1}-A_{n-1}$ is the discrete derivative. Similarly, substituting the definition of B_n into Equation (6.2) gives for n odd

$$\begin{aligned}(B_{n+1}-B_{n-1})=q_nA_n \iff & \frac{1}{n+1}-\frac{1}{n-1}=q_nA_n\\ \iff & q_n=\left(\sum_{k=n+1}^{\infty} q_kB_k\right)^{-1}\left(\frac{1}{n+1}-\frac{1}{n-1}\right).\end{aligned}$$

The initial relationship $u_2+q_1u_1=0$ is also satisfied, using that $\sum_{k=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$. $\square$

The boundary points of the generalised interior of the essential spectrum $\{-2,2\}$ are also excluded from the Wigner-von Neumann technique for the DSO, as well as from the geometric and discrete Levinson method. This is due to the inability to diagonalize the monodromy matrix at these particular values of λ. However, as with the case of $\lambda=0$, other approaches exist to embed eigenvalues here.

Lemma 6.1.3. *The discrete Schrödinger operator with potential*

$$q_n=\begin{cases}\frac{2(1-3n^2)}{(n^2-1)^2} & n\geq 2\\ \frac{7}{4} & n=1\end{cases}$$

has an embedded eigenvalue at $\lambda=2$ and associated eigenvector $u_n=\frac{1}{n^2}$.

Proof. For this choice of (u_n)

$$\begin{aligned} u_{n+1} + q_n u_n - u_{n-1} = 2u_n &\iff \frac{1}{(n+1)^2} + \frac{q_n}{n^2} - \frac{1}{(n-1)^2} = \frac{2}{n^2} \\ &\iff q_n = 2\left(1 - \frac{n^2(n^2+1)}{(n^2-1)^2}\right) \\ &\iff q_n = \frac{2(1-3n^2)}{(n^2-1)^2}, \end{aligned}$$

for $n \geq 2$. Moreover $q_1 = \frac{7}{4}$ ensures the initial conditions are satisfied, and the eigenvalue $\lambda = 2$ becomes embedded in the a.c. spectrum of the operator. □

Lemma 6.1.4. *The discrete Schrödinger operator with potential*

$$q_n = \begin{cases} \frac{6n^2-1}{(n^2-1)^2} & n \geq 2 \\ -1 & n = 1 \end{cases}$$

has an embedded eigenvalue at $\lambda = -2$ and associated eigenvector $u_n = \frac{(-1)^n}{n^2}$.

Proof. For this choice of (u_n)

$$\begin{aligned} u_{n+1} + q_n u_n - u_{n-1} = -2u_n &\iff \frac{(-1)^{n+1}}{(n+1)^2} + \frac{q_n(-1)^n}{n^2} + \frac{(-1)^{n-1}}{(n-1)^2} = \frac{-2(-1)^n}{n^2} \\ &\iff q_n = 2\left(-1 + \frac{n^4+n^2}{(n^2-1)^2}\right) \\ &\iff q_n = \frac{6n^2-1}{(n^2-1)^2}, \end{aligned}$$

for $n \geq 2$. Moreover $q_1 = -1$ ensures the initial conditions are satisfied, and the eigenvalue $\lambda = -2$ becomes embedded in the a.c. spectrum of the operator. □

Remark Note that with $\lambda \in \{-2, 2\}$ it was possible to embed an eigenvalue using a potential only of order $\frac{1}{n^2}$. Our conjecture is that for any Hermitian periodic operator, J_T, a potential weaker than the standard $\frac{1}{n}$ is sufficient to embed an eigenvalue at the boundary points of the generalised interior. Theorem 2.2.1 prohibits this from being true for $\lambda \in \sigma_{ell}(J_T)$.

6.2 The ansatz approach for period-T Jacobi operators

The focus is now turned to dealing with Hermitian period-T Jacobi operators, J_T, and embedding those $\lambda \in \sigma_{ell}(J_T)$, with a quasi-momentum that is not rationally independent with π. The idea rests on defining a candidate eigenvector (u_n) for the perturbed problem

$$(b_1+q_1)u_1+a_1u_2=\lambda u_1, \tag{6.4}$$
$$a_{n-1}u_{n-1}+(b_n+q_n)u_n+a_nu_{n+1}=\lambda u_n, \quad n\geq 2, \tag{6.5}$$

from the components

$$\vec{u}_{Tk}:=\begin{pmatrix} u_{Tk} \\ u_{Tk+1} \end{pmatrix}=\frac{1}{k}M^k(\lambda)\begin{pmatrix} 1 \\ \alpha \end{pmatrix}, k\in\mathbb{N}, \tag{6.6}$$

for some α and where $M(\lambda)$ is the monodromy matrix. In particular, we define $u_n=\frac{x_n}{k}$ for $n=kT+s, s\in\{0,1,\dots,T-1\}$, and where (x_n) is the solution to the unperturbed problem

$$a_{n-1}x_{n-1}+b_nx_n+a_nx_{n+1}=\lambda x_n, \quad n\geq 1, \tag{6.7}$$

with initial components $x_0:=1, x_1:=\alpha$. Then, since $\lambda\in\sigma_{ell}(J_T)$ and the solutions to the unperturbed problem (6.7) are bounded from above we have that $u_{kT}=O\left(\frac{1}{kT}\right)$. By the remark in Section 4.1 we obtain that $u_n=O\left(\frac{1}{n}\right)$ for all $n\in\mathbb{N}$. Furthermore, from the involvement of the monodromy matrix it can be seen that, without loss of generality letting $n=Tk$ and $T>1$, we have

$$\begin{aligned}
a_{n-1}u_{n-1}+(b_n-\lambda)u_n+a_nu_{n+1}&=\frac{a_{n-1}x_{n-1}}{k-1}+\frac{b_nx_n}{k}+\frac{a_nx_{n+1}}{k}-\frac{\lambda x_n}{k}\\
&=\frac{a_{n-1}x_{n-1}}{k-1}+\frac{b_nx_n}{k}+\frac{a_nx_{n+1}}{k}\\
&\quad-\frac{a_{n-1}x_{n-1}+b_nx_n+a_nx_{n+1}}{k}\\
&=O\left(\frac{1}{k^2}\right)=O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Consequently, if $u_n\asymp\frac{1}{n}$ then

$$q_n'=\frac{\lambda u_n-a_{n-1}u_{n-1}-b_nu_n-a_nu_{n+1}}{u_n}=O\left(\frac{1}{n}\right) \tag{6.8}$$

which will solve (6.4) and (6.5) for $q_n:=q_n'$ for $n\geq 2$ and $q_1:=q_1'-\frac{a_T}{\alpha}$.

Remark Note that if we'd instead defined $q_n:=q_n'$ for all n only (6.5) would be satisfied. The need for q_1 to be defined differently to the other q_n follows from the need to satisfy the initial condition (6.4). Setting the initial component $u_0:=0$ (see Section 1.3 for more details) would also have achieved this, however, if we wish to use Lemma 6.2.1, as we plan to do, then it is necessary that the initial component of the vector equal 1.

The definition for q_n' rests on finding an α such that $u_n\asymp\frac{1}{n}$. We use the following result to achieve this.

Lemma 6.2.1. *Let $A_1, \dots, A_T$ be a sequence of invertible 2×2 matrices and $D := A_T \dots A_1$, a finite product of invertible matrices such that $D^k = I$ for some $k \in \mathbb{N}$. Then with the exception of at most $2kT - 1$ values of α, the vectors*

$$A_1 \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, A_2 A_1 \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \dots, A_{T-1} \dots A_2 A_1 \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, D \begin{pmatrix} 1 \\ \alpha \end{pmatrix},$$
$$A_1 D \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \dots, A_{T-1} \dots A_1 D \begin{pmatrix} 1 \\ \alpha \end{pmatrix}, \dots, A_{T-1} \dots A_1 D^{k-1} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \tag{6.9}$$

have both coordinates non-zero.

Proof. (Step One) Consider the invertible matrix A_i with entries w_i, x_i, y_i, z_i acting on a column vector of the form $(1, \widetilde{\alpha})^T$. Then,

$$\begin{pmatrix} w_i & x_i \\ y_i & z_i \end{pmatrix} \begin{pmatrix} 1 \\ \widetilde{\alpha} \end{pmatrix} = \begin{pmatrix} w_i + x_i \widetilde{\alpha} \\ y_i + z_i \widetilde{\alpha} \end{pmatrix}.$$

Now since the matrix A_i is invertible this also means that

$$w_i z_i - x_i y_i \neq 0.$$

In particular if $w_i = 0$ or $z_i = 0$ then both x_i and y_i are non-zero. Similarly, if $x_i = 0$ or $y_i = 0$ then both w_i and z_i are non-zero. Thus, the elements of the product of the matrix A_i with the column vector are non-zero providing that $\widetilde{\alpha} \notin \{0, \frac{-w_i}{x_i}, \frac{-y_i}{z_i}\}$ and assuming that w_i, x_i, y_i, z_i are non-zero. If any of the latter are zero, then omit the elements from the exclusion-set for α where the denominator is zero. For example, if $x_i = 0$ then $\alpha \notin \{0, \frac{-y_i}{z_i}\}$.

(Step Two) Now consider the arbitrary product

$$\begin{pmatrix} \rho \\ \sigma \end{pmatrix} := T_N T_{N-1} \dots T_1 \begin{pmatrix} 1 \\ \alpha \end{pmatrix}. \tag{6.10}$$

We wish to show that $\sigma \neq 0 \neq \rho$ for all α except a finite number. The product of invertible operators $T_N T_{N-1} \dots T_1$ is an invertible matrix, say T. Then by applying Step One to this matrix we see that ρ, σ are non-zero providing α does not belong to a finite set with at most three elements. However, if values of α have already been excluded so that the results holds for example for T_1, then there are most two additional values to exclude for the product T as $\alpha = 0$ is already out of consideration.

Consequently, it can be seen that for the result to hold for the collection in (6.9) there are at most $2kT - 1$ values that α cannot take: three for A_1 and two for each of the other $kT - 2$ elements in the collection. □

The relevance of Lemma 6.2.1 to our problem is not immediately obvious since it demands that the monodromy matrix, $M(\lambda)$, to the unperturbed problem be such that $M^j(\lambda) = I$ for some $j \in \mathbb{N}$. Indeed, given that we are only interested in $\lambda \in \sigma_{ell}(J_T)$ such that $\theta(\lambda) = \frac{p\pi}{q}$ for some $p, q \in \mathbb{N}, \gcd(p, q) = 1$

the monodromy matrix has this property. This follows from the fact that for λ as above the monodromy matrix $M(\lambda)$ has distinct eigenvalues $e^{\pm i\theta(\lambda)}$ with corresponding eigenvectors $\vec{v}_\pm(\lambda)$ and so

$$\begin{aligned} M^j(\lambda)\vec{v}_\pm(\lambda) &= M^{j-1}(\lambda)\left(M(\lambda)\vec{v}_\pm(\lambda)\right) \\ &= e^{\pm 2i\theta(\lambda)}M^{j-1}(\lambda)\vec{v}_\pm(\lambda) \\ &= e^{\pm 2i\theta(\lambda)}M^{j-2}(\lambda)\left(M(\lambda)\vec{v}_\pm(\lambda)\right) = \dots \\ &= e^{\pm 2\pi i}\vec{v}_\pm = \vec{v}_\pm, \end{aligned}$$

where j is such that $\frac{jp\pi}{q} = 0 \mod 2\pi$. In particular, the matrix $M^j(\lambda)$ has eigenvalue 1 with geometric and algebraic multiplicity 2 and is consequently the identity matrix.

Thus, by applying Lemma 6.2.1 to the transfer matrices and monodromy matrix, $M(\lambda)$, associated to the unperturbed T-periodic Jacobi operator, where $\theta(\lambda)$ is rationally dependent with π, it can be established that for appropriate initial elements each component of the solution to the unperturbed problem, (x_n), is non-zero and only equal to possibly finitely many values. Thus $x_n \asymp 1$, and therefore $u_n \asymp \frac{1}{n}$, implying that the potential, (q_n), can be defined by Equation (6.8). This results in the following theorem.

Theorem 6.2.2. *Let J_T be a T-periodic Jacobi operator. Then, for $\lambda \in \sigma_{ell}(J_T)$ such that $\theta(\lambda) = \frac{p\pi}{q}$, $p, q \in \mathbb{N}, \gcd(p,q) = 1$, there exists a potential $(q_n) \in l^2(\mathbb{C})$ such that*

$$(J_T + Q)(u_n) = \lambda(u_n)$$

where Q is an infinite diagonal matrix with entries (q_n) and (u_n) belongs to the sequence space $l^2(\mathbb{N};\mathbb{R})$.

Remark This technique complements the geometric technique since the mechanisms involved are somewhat mutually exclusive. Consider the explicit case of $T = 1$ and the task of showing $B^n(\lambda)\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \neq \begin{pmatrix} 0 \\ \beta \end{pmatrix}$ for all n, non-zero $\beta \in \mathbb{R}$. Assume it does happen then

$$B^n(\lambda)\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ \mu(\lambda) & \overline{\mu(\lambda)} \end{pmatrix}\begin{pmatrix} \mu^n(\lambda) & 0 \\ 0 & \overline{\mu(\lambda)}^n \end{pmatrix}\begin{pmatrix} 1 & 1 \\ \mu(\lambda) & \overline{\mu(\lambda)} \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \mu(\lambda) & \overline{\mu(\lambda)} \end{pmatrix}\begin{pmatrix} \mu^n(\lambda) & 0 \\ 0 & \overline{\mu(\lambda)}^n \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 \\ \mu(\lambda) & \overline{\mu(\lambda)} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \frac{\beta}{\overline{\mu(\lambda)} - \mu(\lambda)}\begin{pmatrix} -\overline{\mu(\lambda)}^n + \mu^n(\lambda) \\ -\overline{\mu(\lambda)}^{n-1} + \mu^{n-1}(\lambda) \end{pmatrix}$$

$$\iff \alpha = \frac{\overline{\mu(\lambda)}^{n-1} - \mu^{n-1}(\lambda)}{\overline{\mu(\lambda)}^n - \mu^n(\lambda)}$$

by taking ratios, and where $\mu(\lambda) = e^{i\theta(\lambda)}$. This is equivalent to

$$\alpha = \frac{\sin((n-1)\theta(\lambda))}{\sin(n\theta(\lambda))} = \cos\theta(\lambda) + \sin\theta(\lambda)\cot((n-1)\theta(\lambda)). \tag{6.11}$$

It remains to be seen if for a particular λ an α be chosen so that this equality never happens. For $\theta(\lambda)$ rationally independent with π, $(n-1)\theta(\lambda) \mod 2\pi$ is dense in the interval $[-\pi, \pi]$ which means $\cot((n-1)\theta(\lambda))$ is dense in $(-\infty, \infty)$. Thus, for $\theta(\lambda)$ rationally independent with π, we have that for all $\epsilon > 0$ and α there exists n such that

$$\left| \alpha - \frac{\sin(n\theta)}{\sin((n+1)\theta)} \right| < \epsilon.$$

This implies $|x_n| < C\epsilon$ for some C and therefore $|u_n| \leq \frac{C\epsilon}{k}$ for $n = kT + s$; in particular, $u_n \not\asymp \frac{1}{n}$ since ϵ is arbitrary. Of course, if $\sin\theta(\lambda) = 0$ then the contribution of $\cot((n-1)\theta(\lambda))$ is cancelled out, but this only happens for $\theta(\lambda) = 0$ or $\theta(\lambda) = \pi$ which are not rationally independent with π in the first place. However, for $\theta(\lambda)$ rationally dependent with π then the expression $\frac{\sin((n-1)\theta(\lambda))}{\sin(n\theta(\lambda))}$ only assumes a finite number of values, as $n \to \infty$ thus making it possible to select a value of α not equal to any of these. (Note that investigating when $B^n(\lambda)\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \neq \begin{pmatrix} \beta \\ 0 \end{pmatrix}$ for all n, non-zero $\beta \in \mathbb{R}$ produces the same result). Consequently, the properties of $\theta(\lambda)$, rationally dependent with π, that were so problematic in the geometric approach are now, ironically, what make these rationally dependent points so useful.

Chapter 7

Conditions for the absence and existence of embedded eigenvalues

It has already been established that for Hermitian period-T Jacobi operators a potential, (q_n), belonging to the sequence space $l^1(\mathbb{N};\mathbb{R})$ is too small to embed any eigenvalues into the background operator's essential spectrum (see Theorem 2.2.1). Naturally one then asks what other conditions exist to preclude the embedding of particular eigenvalues and if such conditions can be rearranged to describe the 'embeddability' of other values in the same band of essential spectrum, providing the condition is satisfied. This question was investigated for the DSO in [65] and where additional conditions for the guarantee of embedded eigenvalues were also discussed. For example, if the sequence $q_n \to 0$ for $\lambda \in (0,2)$ and

$$g(\lambda;q) := \sum_{n=1}^{\infty} \exp\left\{-\frac{1}{\sqrt{1-(\lambda/2)^2}}\sum_{k=1}^{n}|q_k|\right\} = \infty$$

then the DSO with (q_n) has no eigenvalues in the interval $(-\lambda,\lambda)$, whilst if $g(\lambda;q) < \infty$ then it is possible to devise a potential, $(q_n) \sim \left(\frac{1}{n}\right)$ to embed any element in the set $(-2,-\lambda)\cup(\lambda,2)$. From these results we see that for the DSO it is 'harder' to embed λ the further it is from the boundary of $\sigma_{ell}(J_T)$ and therefore require a stronger potential.

In this chapter we adopt similar techniques to obtain analogous results for the arbitrary period-T Jacobi operator case, however we stress that whereas Theorem 2.2.1 held for a potential of any structure, only potentials of the sort discussed in Chapter 5 will be valid here, namely, where the perturbed monodromy matrix is of the form

$$M(\lambda-q_i) := B_T(\lambda)B_{T-1}(\lambda)\dots B_1(\lambda-q_i).$$

7.1 Conditions for the absence of embedded eigenvalues

Observe that for $\lambda \in \sigma_{ell}(J_T)$ and $\vec{u}_i = (u_i, u_{i+1})^T$, we have

$$\begin{aligned}
\vec{u}_{n+T} &= M(\lambda - q_n)\vec{u}_n = M(\lambda) - \frac{q_n}{a_1} M(\lambda) B_1^{-1}(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \vec{u}_n \\
&= V(\lambda) \left(D(\lambda) - \frac{q_n}{a_1} V^{-1}(\lambda) M(\lambda) B_1^{-1}(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} V(\lambda) \right) V^{-1}(\lambda)\vec{u}_n \\
&= V(\lambda)(D(\lambda) + Q_n(\lambda))V^{-1}(\lambda)\vec{u}_n, \qquad (7.1)
\end{aligned}$$

where $M(\lambda) = \begin{pmatrix} m_{11}(\lambda) & m_{12}(\lambda) \\ m_{21}(\lambda) & m_{22}(\lambda) \end{pmatrix}$, $D(\lambda) := \begin{pmatrix} \mu(\lambda) & 0 \\ 0 & \overline{\mu(\lambda)} \end{pmatrix}$, $\mu(\lambda) = e^{i\theta(\lambda)}$, $V(\lambda) := \begin{pmatrix} \frac{m_{12}(\lambda)}{\mu - m_{11}(\lambda)} & \frac{m_{12}(\lambda)}{\overline{\mu} - m_{11}(\lambda)} \\ 1 & 1 \end{pmatrix}$ and $\alpha_n(\lambda) := \frac{q_n|\mu - m_{11}(\lambda)|^2}{a_T m_{12}(\lambda)(\overline{\mu} - \mu)}$, with

$$Q_n(\lambda) := -\alpha_n(\lambda) \begin{pmatrix} -m_{11}(\lambda) + \frac{m_{12}(\lambda)m_{21}(\lambda)}{\overline{\mu} - m_{11}(\lambda)} & -m_{11}(\lambda) + \frac{m_{12}(\lambda)m_{21}(\lambda)}{\overline{\mu} - m_{11}(\lambda)} \\ m_{11}(\lambda) - \frac{m_{12}(\lambda)m_{21}(\lambda)}{\mu - m_{11}(\lambda)} & m_{11}(\lambda) - \frac{m_{12}(\lambda)m_{21}(\lambda)}{\mu - m_{11}(\lambda)} \end{pmatrix}.$$

Lemma 7.1.1. *For all $\vec{e} \in \mathbb{C}^2$, $q_n = O\left(\frac{1}{n}\right)$ and n sufficiently large*

$$\|(D(\lambda) + Q_n(\lambda))\vec{e}\|^2 \geq \left(1 - \frac{|q_n||m_{21}(\lambda)|}{a_T \sin\theta(\lambda)}\right) \|\vec{e}\|^2.$$

Proof. For the sake of space unless defining a new operator we will suppress the dependency of functions on λ throughout this proof. Let $T_n(\lambda) := I + D^*(\lambda)Q_n(\lambda)$. Then, since D is unitary

$$\begin{aligned}
\|(D + Q_n)\vec{e}\|^2 &= \langle (D + Q_n)\vec{e}, (D + Q_n)\vec{e} \rangle \\
&= \langle D(I + D^*Q_n))\vec{e}, D(I + D^*Q_n)\vec{e} \rangle \\
&= \langle T_n^*T_n\vec{e}, \vec{e} \rangle.
\end{aligned}$$

Indeed, since the operator $T_n^*T_n$ is linear and self-adjoint, its eigenvectors can be chosen to be orthonormal, and therefore the operator can be diagonalised with unitary matrix $\widetilde{V}_n(\lambda)$ (see, for example, Theorem 13.11 in [52]). Then, by defining $\vec{f}(\lambda) := \widetilde{V}_n^{-1}(\lambda)\vec{e}$

$$\begin{aligned}
\|(D + Q_n)\vec{e}\|^2 &= \langle T_n^*T_n\widetilde{V}_n\vec{f}, \widetilde{V}_n\vec{f} \rangle = \langle \widetilde{V}_n^*T_n^*T_n\widetilde{V}_n\vec{f}, \vec{f} \rangle \\
&= \lambda_n^{min}|f_1|^2 + \lambda_n^{max}|f_2|^2 \geq \lambda_n^{min}\|\vec{f}\|^2 = \lambda_n^{min}\|\vec{e}\|^2, \qquad (7.2)
\end{aligned}$$

where $\lambda_n^{min}, \lambda_n^{max}$ denote the minimum and maximum eigenvalues of the operator $T_n^*T_n$, respectively.

Simple computations show that

$$D^*Q_n = \alpha_n \begin{pmatrix} \overline{\mu x} & \overline{\mu x} \\ -\mu x & -\mu x \end{pmatrix}$$

where $\alpha_n(\lambda)$ is as used in (7.1) and $x(\lambda) := m_{11}(\lambda) - \frac{m_{12}(\lambda)m_{21}(\lambda)}{\mu(\lambda)-m_{11}(\lambda)}$. Moreover, recalling that $m_{11}(\lambda)m_{22}(\lambda)-m_{12}(\lambda)m_{21}(\lambda) = 1$ and $m_{11}(\lambda)+m_{22}(\lambda) = 2\cos\theta$ we see that

$$-\left|\mu - m_{11}\right|^2 = 2m_{11}\cos\theta - m_{11}^2 - 1 = m_{12}m_{22}$$

and therefore

$$\mu x = \mu m_{11} - \frac{\mu m_{12}m_{21}}{\mu - m_{11}} = \mu m_{11} + \frac{\mu|\mu - m_{11}|^2}{\mu - m_{11}} = \mu m_{11} + \mu(\overline{\mu} - m_{11}) = 1.$$

Thus,

$$T_n = \begin{pmatrix} 1+\alpha_n & \alpha_n \\ -\alpha_n & 1-\alpha_n \end{pmatrix}$$

and so

$$\begin{aligned} T_n^* T_n &= \begin{pmatrix} 1+\overline{\alpha}_n & -\overline{\alpha}_n \\ \overline{\alpha}_n & 1-\overline{\alpha}_n \end{pmatrix}\begin{pmatrix} 1+\alpha_n & \alpha_n \\ -\alpha_n & 1-\alpha_n \end{pmatrix} \\ &= \begin{pmatrix} 1+2|\alpha_n|^2 & 2|\alpha_n|^2+2\alpha_n \\ 2|\alpha_n|^2+2\overline{\alpha}_n & 1+2|\alpha_n|^2 \end{pmatrix} \end{aligned}$$

using that q_n is real and therefore α_n is purely imaginary, $\overline{\alpha}_n = -\alpha_n$. Consequently, the smallest eigenvalue, λ_n^{min}, for $T_n^*T_n$ is

$$\lambda_n^{min} = 1 + 2|\alpha_n|^2 - 2|\alpha_n|\sqrt{1+|\alpha_n|^2}.$$

Now, by applying a Taylor expansion to the function $f(z) = \sqrt{1+z}$ about $z = 0$, and using that $\alpha_n = O(n^{-1})$ (since $q_n = O\left(\frac{1}{n}\right)$) we obtain, for n sufficiently large,

$$\begin{aligned} \lambda_n^{min} &= 1 + 2|\alpha_n|^2 - 2|\alpha_n|\left(1 + \frac{1}{2}|\alpha_n|^2 + O\left(n^{-4}\right)\right) \\ &= 1 - 2|\alpha_n| + 2|\alpha_n|^2 + O(n^{-3}) \\ &\geq 1 - 2|\alpha_n|. \end{aligned} \tag{7.3}$$

Finally, by observing that

$$|\alpha_n| = \left|\frac{q_n|\mu - m_{11}|^2}{a_T m_{12}(\overline{\mu}-\mu)}\right| = \left|\frac{-q_n m_{12}m_{21}}{2a_T m_{12}\sin\theta}\right| = \frac{q_n|m_{21}|}{2a_T\sin\theta} \tag{7.4}$$

we can substitute (7.4) and (7.3) into (7.2) to produce the result. ☐

Theorem 7.1.2. *Let $\lambda \in \sigma_{ell}(J_T)$. If (q_n) is a real sequence such that $q_n \to 0$ and*

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{|m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}\sum_{k=1}^{n}|q_k|\right\} = \infty \tag{7.5}$$

then the operator J_T with potential (q_n) has no eigenvalues for all $\lambda' \in \sigma_{ell}(J_T)$ such that

$$\frac{|m_{21}(\lambda')|}{\sin\theta(\lambda')} < \frac{|m_{21}(\lambda)|}{\sin\theta(\lambda)}, \tag{7.6}$$

where $m_{21}(\lambda)$ is the bottom-left entry in the unperturbed monodromy matrix, $M(\lambda)$.

Remark For the particular case of $T = 2$ (with zero diagonal) we have that the function

$$g(\lambda) := \frac{m_{21}(\lambda)}{a_T \sin\theta(\lambda)}$$

satisfies

$$g''(\lambda) = \frac{4\lambda^7 + 4\lambda^5(a_1^2 + a_2^2) - 11\lambda^3(a_1^2 - a_2^2)^2 + 12\lambda(a_1^2 - a_2^2)^2(a_1^2 + a_2^2)}{(-\lambda^4 + 2\lambda^2(a_1^2 + a_2^2) - (a_1^2 - a_2)^2)^{\frac{5}{2}}},$$

recalling that $m_{21}(\lambda) = -\frac{\lambda}{a_1}$ and $\sin\theta(\lambda) = \sqrt{1 - \left(\frac{\lambda^2 - (a_1^2 + a_2^2)}{2a_1a_2}\right)^2}$. Moreover, since $m_{21}(0) = 0$ for $T = 2$ we obviously have that $g(0) = 0$. Also, for $\lambda > 0$, $g''(\lambda) > 0$ and for all $\lambda \in \mathbb{R}$ $g''(\lambda) = -g(-\lambda)$. Thus,

$$g_{mod}(\lambda) := \frac{|m_{21}(\lambda)|}{a_T \sin\theta(\lambda)}$$

is convex. Then, for each band of essential spectrum, Condition (7.6) can instead be re-phrased to state that the result holds for all λ' between λ and $\widetilde{\lambda}$, where $\widetilde{\lambda}$ is such that $g(\widetilde{\lambda}) = g(\lambda)$ and resides in the same band as λ. However for arbitrary periods we do not have an explicit proof of this convexity, although numerical calculations do suggest that the property holds.

Proof. By Equation 7.1 and Lemma 7.1.1 we have that for m sufficiently large:

$$\begin{aligned} \left\|V^{-1}(\lambda')\vec{u}_{nT}\right\|^2 &\geq \left(\prod_{k=m}^{n-1}\left(1 - \frac{|q_k||m_{21}(\lambda')|}{a_T \sin\theta(\lambda')}\right)\right)\left\|V^{-1}(\lambda')\vec{u}_{mT}\right\|^2 \\ \Rightarrow \left\|\underline{u}(\lambda')\right\|^2 &\geq A_m(\lambda')\sum_{n=m}^{\infty}\prod_{k=m}^{n-1}\left(1 - \frac{|q_k||m_{21}(\lambda')|}{a_T \sin\theta(\lambda')}\right) + F_m(\lambda'), \end{aligned} \tag{7.7}$$

where $\underline{u} := (u_n)_{n=1}^{\infty}$, $A_m(\lambda) := \left(\left\|V^{-1}(\lambda)\right\|^{-1}\left\|V^{-1}(\lambda)\vec{u}_{mT}\right\|\right)^2$ and $F_m(\lambda) := \sum_{i=0}^{m}|u_{iT}|^2$. The value of A_m in (7.7) follows from the fact that

$$\left\|V^{-1}(\lambda')\right\|^2\left\|\vec{u}_j\right\|^2 \geq \left\|V^{-1}(\lambda')\vec{u}_j\right\|^2$$

and so

$$\sum_{n=m+1}^{\infty} \|\vec{u}_{nT}\|^2 \geq \sum_{n=m+1}^{\infty} \left(\|V^{-1}(\lambda')\|^{-1}\|V^{-1}(\lambda')\vec{u}_{nT}\|\right)^2$$
$$\geq \sum_{n=m}^{\infty} \prod_{k=m}^{n} \left(1 - \frac{|q_k||m_{21}(\lambda)|}{a_T \sin\theta(\lambda')}\right) \left(\|V^{-1}(\lambda')\|^{-1}\|V^{-1}(\lambda')\vec{u}_{mT}\|\right)^2.$$

Indeed, it is unnecessary to to include the other $|u_{iT+j}|$ components, where $1 < j < T-1$, since we are only interested in a lower bound. Similarly, it is for this reason why we would have been entitled to exclude the constant $F_m(\lambda')$ in (7.7); however, for the sake of thoroughness it was included.

Now continuing with the proof, we use Lemma 1.6.5 to deduce that

$$\prod_{k=m}^{n-1} \left(1 - \frac{|q_k||m_{21}(\lambda')|}{a_T \sin\theta(\lambda')}\right) = \exp\left\{\sum_{k=m}^{n-1} -\frac{|q_k||m_{21}(\lambda')|}{a_T \sin\theta(\lambda')}(1+o(1))\right\}$$
$$\geq \exp\{\sum_{k=1}^{n-1} -\frac{|q_k||m_{21}(\lambda')|}{a_T \sin\theta(\lambda')}(1+\epsilon)\} \tag{7.8}$$

for $\epsilon > 0$ and m sufficiently large.

For those λ' that satisfy Inequality (7.6) we obtain

$$\exp\left\{-\frac{|m_{21}(\lambda)|}{a_T \sin\theta(\lambda)} \sum_{k=1}^{n} |q_k|\right\} \leq \exp\left\{-\frac{|m_{21}(\lambda')|(1+\epsilon)}{a_T \sin\theta(\lambda')} \sum_{k=1}^{n} |q_k|\right\},$$

where ϵ is the same as in (7.8), providing m was chosen sufficiently large. Finally, by invoking Condition (7.5) and Inequality (7.7) we see that $\underline{u}(\lambda)$ cannot reside in the sequence space $l^2(\mathbb{N};\mathbb{C})$ and thus obtain the result. $\square$

Remark Consider Inequality (7.8) and recall that since the objective is to bound from below the worst case is when $o(1)$ is positive (when $o(1)$ is negative can simply be dealt with by ignoring the term). However, even when positive the problem is overcome by choosing an m large enough so that $o(1)$ can be bounded by a small enough ϵ. It is important to note that this technique can only be employed because we have the luxury of starting from an arbitrary m; if the exponential exponent were instead

$$\left\{\sum_{k=1}^{n} -\frac{|q_k||m_{21}(\lambda')|}{a_T \sin\theta(\lambda')}(1+o(1))\right\}$$

we would have to resort to other means.

Corollary 7.1.3. *Let $\lambda \in \sigma_{ell}(J_T)$. If the sequence (q_n) satisfies the inequality $\sum_{n=1}^{\infty} |q_n|^2 < \infty$ and (7.5), then J_T with (q_n) has no eigenvalues for λ' where λ' are elements of $\sigma_{ell}(J_T)$ such that*

$$\frac{|m_{21}(\lambda')|}{\sin\theta(\lambda')} \leq \frac{|m_{21}(\lambda)|}{\sin\theta(\lambda)},$$

where $m_{21}(\lambda)$ is the bottom-left entry in the unperturbed monodromy matrix, $M(\lambda)$.

Proof. Observe, that by employing Lemma 1.6.5, again, except this time expanding up to the x^2 term (instead of x) we obtain

$$
\begin{aligned}
&\prod_{k=m}^{n-1}\left(1-\frac{|q_k||m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}\right)\\
&=\exp\left\{\sum_{k=m}^{n-1}\left(-\frac{|q_k||m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}-\frac{|q_k|^2|m_{21}(\lambda)|^2}{a_T^2\sin^2\theta(\lambda)}(1+o(1))\right)\right\}\\
&\geq\exp\left\{-\sum_{k=m}^{n-1}\frac{|q_k||m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}-2\sum_{k=1}^{\infty}\frac{|q_k|^2|m_{21}(\lambda)|^2}{a_T^2\sin^2\theta(\lambda)}\right\} \qquad (7.9)\\
&=\exp\left\{\frac{-2K|m_{21}(\lambda)|^2}{a_T^2\sin^2\theta(\lambda)}\right\}\exp\left\{-\sum_{k=m}^{n-1}\frac{|q_k||m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}\right\},
\end{aligned}
$$

providing m is sufficiently large and where $K := \sum_{n=1}^{\infty}|q_n|^2$. Thus, if Equation (7.5) is also satisfied we see that by Inequality (7.7) that $\|\underline{u}(\lambda)\|$ is not finite and therefore the value λ is not embeddable. Coupling this with Theorem 7.1.2 the result is obtained. $\square$

Remark Indeed, it should be stressed that Equation (7.9) was deduced using the same technique as in the previous remark, (bounding $o(1)$ with ϵ) except this time choosing ϵ to be explicitly 1, rather than some sufficiently small number.

We conclude this subsection by discussing what the recent results mean for the potentials with Coulomb-type decay which were used in Chapter 5.

Corollary 7.1.4. *If $|q_n| \leq \frac{C}{n}$ for some $C > 0$ then for all $\lambda \in \sigma_{ell}(J_T)$ such that*

$$\frac{|Cm_{21}(\lambda)|}{a_T\sin\theta(\lambda)} \leq 1 \qquad (7.10)$$

λ is not an eigenvalue of the operator $J_T + Q$ where Q is the diagonal matrix with entries (q_n).

Proof. By substituting the explicit upper bound for q_k we obtain that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\exp\left\{-\frac{|m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}\sum_{k=1}^{n}|q_k|\right\} &\geq \sum_{n=1}^{\infty}\exp\left\{-\frac{C|m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}\sum_{k=1}^{n}\frac{1}{k}\right\}\\
&\geq \sum_{n=1}^{\infty}\exp\left\{-\frac{C|m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}(\log n+1)\right\}\\
&= \exp\left\{-\frac{C|m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}\right\}\sum_{n=1}^{\infty}n^{-\frac{C|m_{21}(\lambda)|}{a_T\sin\theta(\lambda)}}
\end{aligned}
$$

using that $\sum\limits_{k=2}^{n} \frac{1}{k} \leq \log n$. This is infinite if (7.10) is satisfied. Then condition (7.5) is met and, since q_n is also square-summable, Corollary 7.1.3 gives that λ is not an eigenvalue of $J_T + Q$. $\square$

7.2 Conditions for the existence of embedded eigenvalues

Before stating the main result of this section we will need the following three lemmas.

Lemma 7.2.1. *Let $V(\lambda)$ be as used in (7.1) and $\lambda \in \sigma_{ell}(J_T)$. If the vector $(u,v)^T \in \mathbb{C}^2$ is such that $\overline{u}v \in \mathbb{R}$ then the vector $(\widetilde{u},\widetilde{v})^T := V^{-1}(u,v)^T$ satisfies the equality $|\widetilde{u}| = |\widetilde{v}|$.*

Proof. Clearly,

$$\begin{aligned} V^{-1}(\lambda)\begin{pmatrix} u \\ v \end{pmatrix} &= -\frac{|\mu(\lambda)-m_{11}(\lambda)|^2}{2im_{12}(\lambda)\sin\theta(\lambda)}\begin{pmatrix} 1 & -\frac{m_{12}(\lambda)}{\overline{\mu(\lambda)}-m_{11}(\lambda)} \\ -1 & \frac{m_{12}(\lambda)}{\mu(\lambda)-m_{11}(\lambda)} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= -\frac{|\mu(\lambda)-m_{11}(\lambda)|^2}{2im_{12}(\lambda)\sin\theta(\lambda)}\begin{pmatrix} u - \frac{m_{12}(\lambda)v}{\overline{\mu(\lambda)}-m_{11}(\lambda)} \\ -u + \frac{m_{12}(\lambda)v}{\mu(\lambda)-m_{11}(\lambda)} \end{pmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| u - \frac{m_{12}(\lambda)v}{\overline{\mu(\lambda)}-m_{11}(\lambda)} \right|^2 = |u|^2 + \frac{1}{|\mu(\lambda)-m_{11}(\lambda)|^2}\Big(& m_{12}^2(\lambda)|v|^2 \\ & - 2\,\mathrm{Re}\,((\mu(\lambda)-m_{11}(\lambda))(m_{12}(\lambda)v\overline{u}))\Big) \end{aligned}$$

and

$$\begin{aligned} \left| -u + \frac{m_{12}(\lambda)v}{\mu(\lambda)-m_{11}(\lambda)} \right|^2 = |u|^2 + \frac{1}{|\mu(\lambda)-m_{11}(\lambda)|^2}\Big(& m_{12}^2(\lambda)|v|^2 \\ & - 2\,\mathrm{Re}\left((\overline{\mu(\lambda)}-m_{11}(\lambda))(m_{12}(\lambda)v\overline{u})\right)\Big). \end{aligned}$$

Recalling that $v\overline{u} \in \mathbb{R}$ and $m_{11}(\lambda), m_{12}(\lambda)$ are real concludes the proof. $\square$

Lemma 7.2.2. *Let $\lambda \in \sigma_{ell}(J_T)$. If the complex numbers u and v are such that $|u| = |v|$ then the vector $(\widetilde{u},\widetilde{v})^T$, where*

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} := V^{-1}(\lambda)M(\lambda - q_n)V(\lambda)\begin{pmatrix} u \\ v \end{pmatrix}$$

also satisfies $|\widetilde{u}| = |\widetilde{v}|$.

Proof. Simple calculations give that

$$\begin{aligned}\begin{pmatrix} \widetilde{u} \\ \widetilde{v} \end{pmatrix} &= V^{-1}(\lambda)\left(M(\lambda) - \frac{q_n}{a_1}M(\lambda)B_1^{-1}(\lambda)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)V(\lambda)\begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} \mu(\lambda)u - \frac{q_n|\mu(\lambda)-m_{11}(\lambda)|^2\mu(\lambda)(u+v)}{2ia_T m_{12}(\lambda)\sin\overline{\theta(\lambda)}} \\ \overline{\mu(\lambda)}v + \frac{q_n(|\mu(\lambda)-m_{11}(\lambda)|^2)\overline{\mu(\lambda)}(u+v)}{2ia_T m_{12}(\lambda)\sin\theta(\lambda)} \end{pmatrix}.\end{aligned}$$

Then

$$|\widetilde{u}|^2 = |u|^2 + \left|\frac{q_n|\mu(\lambda)-m_{11}(\lambda)|^2(u+v)}{2a_T m_{12}(\lambda)\sin\theta(\lambda)}\right|^2 + \frac{q_n|\mu(\lambda)-m_{11}(\lambda)|^2}{a_T m_{12}(\lambda)\sin\theta(\lambda)}\operatorname{Re}(iv\overline{u}) \qquad (7.11)$$

and

$$|\widetilde{v}|^2 = |v|^2 + \left|\frac{q_n|\mu(\lambda)-m_{11}(\lambda)|^2(u+v)}{2a_T m_{12}(\lambda)\sin\theta(\lambda)}\right|^2 - \frac{q_n|\mu(\lambda)-m_{11}|^2}{a_T m_{12}\sin\theta(\lambda)}\operatorname{Re}(i+u\overline{v}). \qquad (7.12)$$

Finally, using

$$\operatorname{Re}(iu\overline{v}) = -\operatorname{Im}(u\overline{v}) = \operatorname{Im}(\overline{u}v) = -\operatorname{Re}(i\overline{u}v)$$

and that $|u|^2 = |v|^2$ the result is obtained. □

Lemma 7.2.3. *Let $\lambda \in \sigma_{ell}(J_T)$. If $\vec{e} = (u,v)^T \in \mathbb{C}^2$ is a vector such that $|u|^2 = |v|^2 = \frac{1}{2}, \alpha := \arg(u), \beta := \arg(v)$ then the norm of the vector $\vec{f}(\lambda) := V^{-1}(\lambda)M(\lambda - q_n)V(\lambda)\vec{e}$ has as $q_n \to 0$ the asymptotic expression*

$$\|\vec{f}(\lambda)\|^2 = 1 - \frac{q_n m_{21}(\lambda)\cos(\frac{\pi}{2}+\alpha-\beta)}{a_T \sin\theta(\lambda)} + O\left(q_k^2\right).$$

Proof. Adding together Equations (7.11) and (7.12) gives

$$\|\vec{f}(\lambda)\|^2 = 1 + \frac{2q_n|\mu(\lambda)-m_{11}(\lambda)|^2\operatorname{Im}(v\overline{u})}{a_T m_{12}(\lambda)\sin\theta(\lambda)} + O\left(q_n^2\right).$$

Then, using

$$\operatorname{Im}(v\overline{u}) = \operatorname{Im}\left(\frac{1}{2}e^{(\alpha-\beta)i}\right) = \frac{\cos(\frac{\pi}{2}+\alpha-\beta)}{2}$$

and

$$|\mu(\lambda) - m_{11}(\lambda)|^2 = 1 - 2m_{11}(\lambda)\cos\theta + m_{11}^2(\lambda) = -m_{12}(\lambda)m_{21}(\lambda)$$

the result is obtained. □

Remark Observe that the expression only guarantees shrinkage if $m_{21}(\lambda)q_n$ and $\cos(\frac{\pi}{2}+\alpha-\beta)$ are of the same sign. In subsequent calculations we will see that α and β can be chosen such that this is always the case.

From the above lemmas it now becomes possible to adapt a variant of the geometric technique, presented in Section 5.1, to embed a single λ from the generalised interior, whose quasi-momentum, $\theta(\lambda)$, is rationally independent with π, for an arbitrary period-T Jacobi operator. We now assume these same conditions for the new technique and note that it is more explicit in terms of the shrinkage that is occurring at each step. We also recall that without loss of generality $u_0 = 0, u_1 = 1$, r_i represents the number of rotations required to rotate an arbitrary vector into an acceptable region of the plane for the i-th shrinkage, and the term $\widetilde{q}_i$ denoting the i-th non-zero entry in the potential, (q_n). Thus, if for example $n = T(r_1 + r_2 + 2)$, say, we obtain

$$\begin{aligned} \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix} &= M(\lambda - \widetilde{q}_2)M_2^r(\lambda)M(\lambda - \widetilde{q}_1)M^r(\lambda)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= M(\lambda - \widetilde{q}_2)V(\lambda)D^r(\lambda)V^{-1}(\lambda)M(\lambda - \widetilde{q}_1)V(\lambda)D^r(\lambda)V^{-1}(\lambda)\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{7.13}$$

Now, we consider an arbitrary n. Invoking Lemma 7.2.1 gives that $(u, v)^T := V^{-1}(\lambda)(0, 1)^T$ is such that $|u| = |v|$. Then, applying the diagonal matrix, $D(\lambda)$ (with entries, $e^{\pm i\theta(\lambda)}$) won't change the size of the components u, v. Indeed, if $D(\lambda)$ is applied r_i times, $D^{r_i}(\lambda)(u, v)^T =: (\widetilde{u}, \widetilde{v})^T$ then

$$\arg(\widetilde{u}) = \arg(u) + r_i\theta(\lambda), \quad \arg(\widetilde{v}) = \arg(v) - r_i\theta(\lambda)$$

which implies

$$\alpha := \frac{\pi}{2} + \arg(\widetilde{u}) - \arg(\widetilde{v}) = \frac{\pi}{2} + \arg(u) - \arg(v) + 2r_i\theta(\lambda). \tag{7.14}$$

If $\widetilde{q}_n m_{21}(\lambda)$ is positive then by Proposition 5.1.5 an upper-bound, $\widetilde{r}_1$, can be found independent of the starting arguments, u, v such that the sum, α, in Equation (7.14) is less than some $\gamma < \frac{\pi}{6}$. If $\widetilde{q}_k m_{21}(\lambda)$ is negative, the Proposition 5.1.5 can still be applied to find an upper bound, $\widetilde{r}_2$, for the sum, α, although this time to guarantee an α such that $\frac{5\pi}{6} \leq \gamma < \alpha < \pi$. Then, applying $V^{-1}(\lambda)M(\lambda - \widetilde{q}_k)V(\lambda)$ provides a shrinkage to the vector $(\widetilde{u}, \widetilde{v})^T$ of factor

$$1 - \frac{|\widetilde{q}_i m_{21}(\lambda)||\cos(\alpha)|}{a_T \sin\theta(\lambda)} \leq 1 - \frac{|\widetilde{q}_i m_{21}(\lambda)||\cos(\gamma)|}{a_T \sin\theta(\lambda)}.$$

The process is repeated for subsequent components, bearing in mind that $r := \max\{\widetilde{r}_1, \widetilde{r}_2\}$ will be fixed for each λ, i.e. $r_i \leq r$ for all i. Then

$$\|\underline{u}(\lambda)\|^2 \leq A_k(\lambda)\sum_{n=k}^{\infty}\left(\prod_{m=k}^{n}\left(1 - \frac{|\widetilde{q}_m m_{21}(\lambda)||\cos\gamma|}{a_T \sin\theta(\lambda)} + O\left(\widetilde{q}_m^2\right)\right)\right) + F_k(\lambda), \tag{7.15}$$

where

$$\begin{aligned} A_k(\lambda) &:= \|V(\lambda)\|^2(r+1)(|u_k|^2 + |u_{k+1}|^2)\times \\ &\qquad \left(1 + \sum_{j=1}^{T-1}\prod_{i=1}^{j}\|B_i(\lambda)\|^2 + K\sum_{j=2}^{T-1}\prod_{i=1}^{j}\|B_i(\lambda)\|^2\right), \end{aligned}$$

with K such that $\|B_1(\lambda - \widetilde{q}_n)\|^2 < K$, for all n and $F_k(\lambda) := \sum\limits_{n=1}^{k-1} |u_n|^2$.

Remark The presence of $V(\lambda)$ in the definition of the constant $A_k(\lambda)$ can be understood from the example in (7.13) and generalised to arbitrary n. Here, we see the expression $M(\lambda - \widetilde{q}_2)V(\lambda)$ on the righthand-side of $(u_n, u_{n+1})^T$. We multiply the former by $V(\lambda)V^{-1}(\lambda)$ so that the shrinkage factor $1 - \frac{|\widetilde{q}_2 m_{21}(\lambda)||\cos\gamma|}{a_T \sin\theta(\lambda)}$ can be applied, leaving a remaining factor $V(\lambda)$.

Furthermore, by applying Lemma 1.6.5 we obtain

$$\begin{aligned}
&\prod_{m=k}^{n} \left(1 - \frac{|\widetilde{q}_m m_{21}(\lambda)||\cos\gamma|}{a_T \sin\theta} + O\left(\widetilde{q}_m^2\right)\right) \\
&\le \exp\left(\sum_{m=k}^{n} -\frac{|\widetilde{q}_m m_{21}(\lambda)||\cos\gamma|}{a_T \sin\theta(\lambda)}(1 + o(1))\right) \\
&\le \widetilde{A}_k \exp\left(\sum_{m=1}^{n} -\frac{|\widetilde{q}_m m_{21}(\lambda)|}{a_T \sin\theta(\lambda)}(1 - \epsilon)\right) \qquad (7.16)
\end{aligned}$$

for some $\epsilon > 0$ where k is chosen to be sufficiently large and

$$\widetilde{A}_k := \exp\left(\sum_{m=1}^{k} \frac{|\widetilde{q}_m m_{21}(\lambda)|}{a_T \sin\theta(\lambda)}(1 - \epsilon)\right).$$

Remark The ϵ term follows from the $o(1)$ and $\cos(\gamma)$ expressions, and can be made smaller by choosing the γ at the outset to be either closer to 0 or π (depending on whether $\widetilde{q}_n m_{21}(\lambda)$ is positive or negative). This will affect the value of r.

Substituting (7.16) into (7.15) it is not difficult to see that $\underline{u}(\lambda) \in l^2(\mathbb{N};\mathbb{C})$ if for some $\epsilon > 0$ we have:

$$\sum_{n=1}^{\infty} \exp\left(-\frac{|m_{21}(\lambda)|(1-\epsilon)}{a_T \sin\theta(\lambda)} \sum_{m=1}^{n} |\widetilde{q}_m|\right) < \infty. \qquad (7.17)$$

Before stating the main result the following definition will be needed.

Definition 7.2.4. *We call two potentials, $(q_n)_{n=1}^{\infty}$ and $(q'_n)_{n=1}^{\infty}$ equivalent if*

$$\frac{\sum\limits_{m=1}^{n} |q_m|}{\sum\limits_{m=1}^{n} |q'_m|} \to 1 \qquad (7.18)$$

as $n \to \infty$.

One of the motivations for defining two potentials, (q_n) and (q'_n) as equivalent is that in the event they do satisfy (7.18) we have that $\sum\limits_{m=1}^{n} |q_m| \sim \sum\limits_{m=1}^{n} |q'_m|$ and so

$$\sum_{m=1}^{n} |q_m| = (1 + o(1)) \sum_{m=1}^{n} |q'_m|.$$

This implies

$$\begin{aligned} \exp\left(-\sum_{m=1}^{n} |q_m|\right) &= \exp\left(-(1+o(1))\sum_{m=1}^{n} |q'_m|\right) \\ &\geq \exp\left(-(1-\epsilon)\sum_{m=1}^{n} |q'_m|\right), \end{aligned} \tag{7.19}$$

for some $\epsilon > 0$ providing n is chosen large enough. This will be used in the proof of the next result.

We now conclude the chapter with the following theorem, which establishes conditions for the existence of an embedded eigenvalue for a perturbed period-T Jacobi operator.

Theorem 7.2.5. *If the sequence $q_n \to 0$ is such that for some $\lambda \in \sigma_{ell}(J_T)$*

$$\sum_{n=1}^{\infty} \exp\left\{-\frac{|m_{21}(\lambda)|}{a_T \sin\theta(\lambda)} \sum_{m=1}^{n} |q_k|\right\} < \infty \tag{7.20}$$

then for any $\lambda' \in \sigma_{ell}(J_T)$ such that

$$\frac{|m_{21}(\lambda')|}{\sin\theta(\lambda')} > \frac{|m_{21}(\lambda)|}{\sin\theta(\lambda)}$$

under the condition that $\theta(\lambda')$ is rationally independent with π there is a real potential (q'_n) equivalent to (q_n) such that λ' is an eigenvalue of the operator J_T with potential (q'_n).

Proof. Clearly, for any λ' such that $\frac{|m_{21}(\lambda')|}{\sin\theta(\lambda')} > \frac{|m_{21}(\lambda)|}{\sin\theta(\lambda)}$ there is an $\epsilon > 0$ such that

$$\exp\left(-\frac{|m_{21}(\lambda)|}{a_T \sin\theta(\lambda)} \sum_{m=1}^{n} |q_m|\right) \geq \exp\left(-\frac{|m_{21}(\lambda')|(1-\epsilon)}{a_T \sin\theta(\lambda')} \sum_{m=1}^{n} |q_m|\right). \tag{7.21}$$

Since $\theta(\lambda')$ and π are rationally independent, γ in (7.16) can be chosen appropriately so that the ϵ appearing in (7.17) is the same as in (7.21). Consequently, the sufficient condition for an eigenvector to exist for λ' becomes

$$\sum_{n=1}^{\infty} \exp\left(-\frac{|m_{21}(\lambda')|(1-\epsilon)}{a_T \sin\theta(\lambda')} \sum_{m=1}^{n} |\widetilde{q}_m|\right) < \infty \tag{7.22}$$

for some potential which is mostly zeros (corresponding to the rotations taking place) and occasional non-zero entries $\widetilde{q}_k$. However, this condition is not obviously satisfied as the potential here is not the same potential, (q_n), as given in (7.20). This problem is overcome by constructing the k-th non-zero entry, $\widetilde{q}_k$, of the former so that it is the sum of the components of the potential, (q_n), corresponding to the interval of rotation r_k, i.e. letting $s_i := T \sum_{k=1}^{i} r_k$ then

$$\widetilde{q}_1 = \sum_{i=1}^{s_1+1} q_i, \quad \widetilde{q}_2 = \sum_{i=s_1+2}^{s_2+1} q_i, \quad \widetilde{q}_3 = \sum_{i=s_2+2}^{s_3+1} q_i, \ldots, \widetilde{q}_n = \sum_{i=s_{n-1}+2}^{s_n+1} q_i,$$

recalling that r_i denotes the number of rotations necessary to move the initial components after the $(i-1)$-th shrinkage into an acceptable region so that the i-th shrinkage can be applied. Clearly, this potential for the construction of the eigenvector is equivalent to the potential, (q_n), given in (7.20) and so using (7.21) we see that

$$\sum_{n=k}^{\infty} \exp\left(-\frac{|m_{21}(\lambda')|(1-\epsilon)}{a_T \sin\theta(\lambda')} \sum_{m=1}^{n} |\widetilde{q}_m|\right) \leq \sum_{n=k}^{\infty} \exp\left(-\frac{|m_{21}(\lambda)|}{a_T \sin\theta(\lambda)} \sum_{m=1}^{n} |q_m|\right) < \infty,$$

with k chosen large enough. Thus, the sufficient condition, (7.22), for J_T to have an eigenvalue λ' with potential (q'_n) (with non-zero entries $\widetilde{q}_n$) is satisfied. $\square$

Conclusion

Now we conclude the thesis with a brief discussion on the relative merits and difficulties of each of the main embedding techniques. First, we consider the Wigner-von Neumann method. The primary advantage of this approach is its explicitness for the eigenvector associated with the embedded eigenvalue; a feature that both the discrete Levinson and geometric techniques lack. However, there are several setbacks to this explicitness, namely the ability to only embed one eigenvalue at a time into the essential spectrum and the condition that such a λ is not a root of the rational function $C(\lambda;T)$. Regarding the latter, we suspect that this function has no roots in the generalised interior of the essential spectrum for even periods, and for odd periods (with zero diagonal) only has one root at $\lambda = 0$. We have shown this to be true for the cases $T = 2$, $T = 3$, although have yet to prove the conjecture more generally.

Secondly, we discuss the discrete Levinson technique to embed eigenvalues. The immediate advantage to this approach over the Wigner-von Neumann method is its ability to embed at least two eigenvalues simultaneously with a single potential. However, unlike the Wigner-von Neumann method, which has an explicit eigenvector, this method instead has an explicit formula for the potential, a Wigner-von Neumann structure to be precise, and which we assume a priori. The method also suffers a similar disadvantage as the Wigner-von Neumann technique in that it fails for those $\lambda \in \sigma_{ell}(J_T)$ that are simultaneously the roots of $E(\lambda;k_+), \widetilde{E}(\lambda;k_-)$ for all $k_+ \in \{1,\dots,T\}, k_- \in \{0,\dots,T-1\}$; we conjecture that no such λ satisfies all these conditions and for $T = 1, T = 2$ have proven this to be the case. A final remark about this method is that its focus is less on embedded eigenvalues and more on subordinate solutions, which possess a greater stability, since these are, in fact, stable with respect to an l^1-perturbation whereas eigenvalues are unstable with respect to arbitrarily small rank-1 perturbations (see, for example, [51]). Using the discrete Levinson method we manage to simultaneously devise subordinate solutions corresponding to a possibly infinite set of spectral parameters, but do not establish how these subordinate solutions can be made into formal eigenvectors.

Lastly, we consider the geometric approach for embedding eigenvalues, which can be used to embed any single eigenvalue, λ, into the essential spectrum providing $\theta(\lambda) \neq \frac{\pi}{2}$, or to embed infinitely many eigenvalues simultaneously with various conditions on the quasi-momenta. However, this capacity for infinite embeddings comes at the expense of explicitness for both the eigenvector and the potential.

This suggests a certain incomparability when analysing the different embedding methods, since the usefulness of each depends upon the particular end one has in mind. For instance, if one were tasked with embedding a single, specific eigenvalue into an interval of a.c. spectrum, then the discrete Levinson approach would be preferable; if one were tasked instead with embedding multiple or infinitely many eigenvalues, then the geometric approach would be best; and, finally, if one were tasked with exploring the differences between the discrete and the continuous case, then the Wigner-von Neumann technique would be more effective, since the very property that made it less appealing before, i.e. the $C(\lambda; T)$ function, is now a selling point. This is because, to the best of our knowledge, such a function does not exist in the continuous setting, and so its presence here has deep philosophical implications, namely that the discrete area of mathematics is not simply a tame backyard of the more forbidding continuous landscape: it is a wild tundra in its own right, with its own rules and its own equally fascinating challenges.

Appendix A

The function $C(\lambda; 3)$

For a zero-diagonal period-3 Jacobi operator, J_3, the monodromy matrix, $M(\lambda)$, is such that

$$M(\lambda) := B_3(\lambda)B_2(\lambda)B_1(\lambda) = \begin{pmatrix} -\frac{\lambda a_3}{a_1 a_2} & -\frac{a_1}{a_2} + \frac{\lambda^2}{a_1 a_2} \\ \frac{a_2}{a_1} - \frac{\lambda^2}{a_1 a_2} & -\frac{\lambda a_1}{a_2 a_3} - \frac{\lambda a_2}{a_1 a_3} + \frac{\lambda^3}{a_1 a_2 a_3} \end{pmatrix}.$$

Moreover, the standardised eigenvector of $M(\lambda)$ with normalised first component is

$$\begin{pmatrix} 1 \\ \frac{\lambda a_3 + \mu a_1 a_2}{\lambda^2 - a_1^2} \end{pmatrix}.$$

Consequently,

$$\varphi_0 := 1, \varphi_1 := \frac{\lambda a_3 + \mu a_1 a_2}{\lambda^2 - a_1^2}, \varphi_2 := \frac{\lambda \mu a_2 + a_1 a_3}{\lambda^2 - a_1^2}, \varphi_3 := \mu.$$

This gives

$$\begin{aligned}
C(\lambda; 3) &:= \operatorname{Re}\left(\sum_{i=1}^{3} \varphi_i \overline{\varphi}_{i-1}\right) = \operatorname{Re}\left(\varphi_1 \overline{\varphi}_0 + \varphi_2 \overline{\varphi}_1 + \varphi_3 \overline{\varphi}_2\right) \\
&= \operatorname{Re}\left(\frac{\lambda a_3 + \mu a_1 a_2}{\lambda^2 - a_1^2} + \left(\frac{\lambda \mu a_2 + a_1 a_3}{\lambda^2 - a_1^2}\right)\left(\frac{\lambda a_3 + \overline{\mu} a_1 a_2}{\lambda^2 - a_1^2}\right) + \mu \frac{\lambda \overline{\mu} a_2 + a_1 a_3}{\lambda^2 - a_1^2}\right) \\
&= \frac{\lambda}{(\lambda^2 - a_1^2)^2}\Bigg[(\lambda^2 - a_1^2)(a_3 + a_2) + a_1(a_2^2 + a_3^2) \\
&\quad + \frac{\left(\lambda^2 - (a_1^2 + a_2^2 + a_3^2)\right)}{2 a_1 a_2 a_3}\left((\lambda^2 - a_1^2)(a_1 a_2 + a_1 a_3) + \lambda^2 a_2 a_3 + a_1^2 a_2 a_3\right)\Bigg],
\end{aligned}$$

using that μ is on the unit circle, and therefore $\operatorname{Tr}(M(\lambda)) = 2\operatorname{Re}(\mu)$. Indeed further simplifications give:

$$C(\lambda;3)=\frac{\lambda\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)}{2(\lambda^2-a_1^2)}\left(\lambda^2-(a_1^2+a_2^2+a_3^2)+\frac{2(a_1+a_2+a_3)}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}}\right). \tag{A.1}$$

Evidently, there are three roots

$$0,\lambda_\pm:=\pm\sqrt{(a_1^2+a_2^2+a_3^2)-\frac{2(a_1+a_2+a_3)}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}}}.$$

Now, since

$$\mathrm{Tr}\,(M(0))=0\leq 2$$

this means $\lambda=0\in\sigma_{a.c.}(J_3)$ which is completely consistent with our earlier conjecture. It is the location of the other roots, $\lambda_\pm$, that is less clear. In order to investigate $\lambda_\pm$ and determine whether they belong to the spectrum of $\sigma_{a.c.}(J_T)$ it is sufficient to to see if

$$|\,\mathrm{Tr}(M(\lambda_+))|\geq 2.$$

Recall that the symmetry of the spectrum guarantees that λ_- will behave identically. Thus,

$$\mathrm{Tr}(M(\lambda_+))-2\geq 0$$

$$\begin{aligned}
\iff\ &\sqrt{a_1^2+a_2^2+a_3^2-\frac{2(a_1+a_2+a_3)}{\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}}}\,\frac{a_1+a_2+a_3}{\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)a_1a_2a_3}-1\geq 0\\
\iff\ &\sqrt{\frac{a_1^2}{a_2}+\frac{a_1^2}{a_3}+\frac{a_2^2}{a_1}+\frac{a_2^2}{a_3}+\frac{a_3^2}{a_1}+\frac{a_3^2}{a_2}-a_1-a_2-a_3}\,(a_1+a_2+a_3)\\
&\qquad\geq a_1a_2a_3\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)^{\frac{3}{2}}\\
\iff\ &\left(\frac{a_1^2}{a_2}+\frac{a_1^2}{a_3}+\frac{a_2^2}{a_1}+\frac{a_2^2}{a_3}+\frac{a_3^2}{a_1}+\frac{a_3^2}{a_2}-a_1-a_2-a_3\right)(a_1+a_2+a_3)^2\\
&\qquad\geq a_1^2a_2^2a_3^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}\right)^3\\
\iff\ &(a_1^3+a_1^3a_2+a_2^3+a_1a_2^3+a_2+a_1-a_1^2a_2-a_1a_2^2-a_1a_2)(a_1+a_2+1)^2\\
&\qquad-(a_1+a_2+a_1a_2)^3\geq 0
\end{aligned}$$

by stating, without loss of generality, that $a_3=1$.

This yields a continuous function in two variables, say $g(a_1,a_2)$. It now suffices to investigate the roots of this function. There are three instances:

1. $a_1=1,a_2=-1$.

2. $a_1 = -1, a_2 = -1$.

3. a_2 is free and a_1 is a root of the following quintic polynomial:

$$h(a_2) := (a_2+1)Z^5 + (2a_2^2 + 3a_2 + 2)Z^4 + (a_2^3 - 3a_2^2 - 3a_2 + 1)Z^3 \\ + (2a_2^4 - 3a_2^3 - 12a_2^2 - 3a_2 + 2)Z^2 + (a_2^5 + 3a_2^4 - 3a_2^3 - 3a_2^2 + 3a_2 + 1)Z \\ + a_2^5 + 2a_2^4 + a_2^3 + 2a_2^2 + a_2.$$

The first two can be dismissed because they do not involve positive a_1, a_2 and thus the quintic equation must be solved. MAPLE produces five polynomial solutions (which it can't explicitly state), and of these three always produce real values. Figure A.1 illustrates this, and we see that all the a_1 are non-positive here. Consequently, these three polynomial solutions can be dismissed, too.

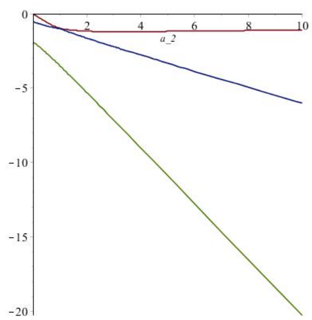

Figure A.1: Graph of three real solutions of a_1

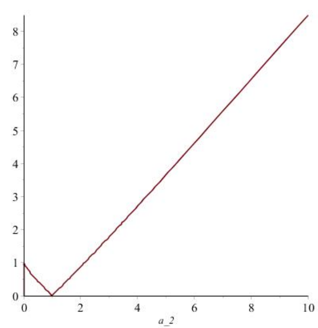

Figure A.2: Graph of imaginary part of solutions of a_1

In order to consider the complex pair of conjugate polynomial solutions and determine whether they produce any real values of a_1 it suffices just to consider one. Thus, plotting the graph of a_2 against $\mathrm{Im}(a_1)$, Figure A.2 illustrates that the only possibility of a real solution is at $a_2 = 1$. Inspecting the graph of $\mathrm{Re}(a_1)$ against a_2 for this particular complex polynomial solution we see that $a_2 = 1$

implies $a_1 = 1$. This means that the original function $g(a_1, a_2)$ only equals zero at one point $(1, 1)$ in the positive a_1, a_2 plane. However, since $g(a_1, a_2)$ is a polynomial and therefore continuous it suffices to show that $g(a_1, a_2)$ is positive for any point outside (1,1) to show that $g(a_1, a_2)$ is positive everywhere outside (1,1), and indeed it is established that at $(1, 2)$ the function $g(1, 2) = 99 > 0$. This concludes the argument, but admittedly we have not shown that $g(a_1, a_2) \neq 0$ for $a_2 > 10$ with hallmark mathematical rigour. However, at the very least we have an idea.

Appendix B

Additional details for the geometric method

Here we provide additional details about the derivation of (5.11) at the end of Section 5.1, concerning a bound on the candidate eigenvector $\underline{u} := (u_n)_{n\geq 1}$ for $\lambda \in \sigma_{ell}(J_T)$, where J_T is a Hermitian period-T Jacobi operator. There are three steps: first, consider the first Tr_1 components formed by the unperturbed transfer matrices, $M(\lambda)$, acting on the initial entries $u_0 := 0, u_1$ where r_1 is defined as the number of times it is necessary to apply the monodromy matrix before the initial components are rotated into an acceptable/shrinakble area of the plane. In particular,

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= B_1(\lambda) \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \\ \Rightarrow \left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|^2 &\leq \|B_1(\lambda)\|^2 \left\| \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \right\|^2 \\ \iff |u_1|^2 + |u_2|^2 &\leq \|B_1(\lambda)\|^2 |u_1|^2, \end{aligned}$$

using the 2-dimensional l^2-norm. Then, for $1 \leq j \leq T$, we have

$$\begin{pmatrix} u_j \\ u_{j+1} \end{pmatrix} = B_j(\lambda) \dots B_1(\lambda) \begin{pmatrix} 0 \\ u_1 \end{pmatrix}$$

$$\Rightarrow |u_{j+1}|^2 \leq \|B_j(\lambda) \dots B_1(\lambda)\|^2 |u_1|^2.$$

By labelling $c_j(\lambda) := \|B_j(\lambda) \dots B_2(\lambda) B_1(\lambda)\|^2$ and $c_0 := 1$ this implies

$$\sum_{i=0}^{T-1} |u_i|^2 \leq \left(\sum_{j=0}^{T-2} c_j(\lambda) \right) |u_1|^2.$$

Similarly, for any $k \in \mathbb{N}$

$$\sum_{i=0}^{T-1} |u_{kT+i}|^2 \leq \left(\sum_{j=0}^{T-2} c_j(\lambda) \right) \left(|u_{kT}|^2 + |u_{kT+1}|^2 \right). \tag{B.1}$$

Moreover, using that

$$\begin{pmatrix} u_{kT} \\ u_{kT+1} \end{pmatrix} = M(\lambda)^k \begin{pmatrix} 0 \\ u_1 \end{pmatrix} = V(\lambda) D^k(\lambda) V^{-1}(\lambda) \begin{pmatrix} 0 \\ u_1 \end{pmatrix},$$

with $V(\lambda)$ as in (5.1) and $D(\lambda)$ is the diagonal matrix with conjugate entries on the unit circle, produces the relations

$$|u_{kT}|^2 \leq \|V(\lambda)\|^2 \|V^{-1}(\lambda)\|^2 |u_1|^2, \quad |u_{kT+1}|^2 \leq \|V(\lambda)\|^2 \|V^{-1}(\lambda)\|^2 |u_1|^2 \tag{B.2}$$

for $\lambda \in \sigma_{ell}(J_T)$. Combining (B.1) and (B.2) yields

$$\sum_{i=0}^{T-1} |u_{kT+i}|^2 \leq \widetilde{K}_1(\lambda) |u_1|^2,$$

for $1 \leq k \leq r_1$, and where $\widetilde{K}_1(\lambda) := 2 \left(\sum_{j=0}^{T-2} c_j(\lambda) \right) \|V(\lambda)\|^2 \left\|V^{-1}(\lambda)\right\|^2$. Thus, setting $u_1 = 1$, the contribution for the first Tr_1 components is less than or equal to $r_1 \widetilde{K}_1(\lambda)$, i.e.

$$\sum_{i=0}^{Tr_1 - 1} |u_i|^2 \leq r_1 \widetilde{K}_1(\lambda).$$

Secondly, we consider what happens to the eigenvector at every T-th component after the first shrinkage in the cone. Recall that in general r_i denotes the i-th interval of rotation, corresponding to the number of rotations necessary to rotate the initial components of the $(i-1)$th shrinkage into a shrinkable region of the real plane. Then, we have

$$\begin{aligned}
&|u_{(r_1+1)T}|^2 + |u_{(r_1+1)T+1}|^2 \leq \|M(\lambda - \widetilde{q}_1) M^{r_1}(\lambda)\|^2 \left(|u_0|^2 + |u_1|^2 \right) \\
&= \|W(\lambda) \left(R(\theta(\lambda)) - \widetilde{q}_1 W^{-1}(\lambda) A(\lambda) W(\lambda) \right) R^{r_1}(\theta(\lambda)) W^{-1}(\lambda)\|^2 \\
&\leq \|W(\lambda)\|^2 \|W^{-1}(\lambda)\|^2 \left(1 - C(\lambda)|\widetilde{q}_1| + O\left(\widetilde{q}_1^2\right) \right)
\end{aligned}$$

using $u_0 := 0, u_1 := 1$, $\|R^k(\theta(\lambda))\| = \|D^k(\lambda)\| = 1$ and (5.6).

Similarly, for $1 \leq j \leq r_2$, we have

$$\begin{aligned}
&|u_{(r_1+1+j)T}|^2 + |u_{(r_1+1+j)T+1}|^2 \leq \|M^{r_2}(\lambda) M(\lambda - \widetilde{q}_1) M^{r_1}(\lambda)\|^2 \left(|u_0|^2 + |u_1|^2 \right) \\
&= \|W(\lambda) R^{r_2}(\theta(\lambda)) \left(R(\theta(\lambda)) - \widetilde{q}_1 W^{-1}(\lambda) A(\lambda) W \right) R^{r_1}(\theta(\lambda)) W^{-1}(\lambda)\|^2 \\
&\leq \|W(\lambda)\|^2 \|W^{-1}(\lambda)\|^2 \left(1 - C(\lambda)|\widetilde{q}_1| + O\left(\widetilde{q}_1^2\right) \right).
\end{aligned}$$

Thus, at regular T-intervals there is a total contribution of

$$(r_2+1)\|W(\lambda)\|^2\|W^{-1}(\lambda)\|^2\left(1-C(\lambda)|\widetilde{q}_1|+O\left(\widetilde{q}_1^2\right)\right)$$

before the second non-trivial component of the potential is introduced. Extending the idea to before the N-th non-trivial component of the potential and incorporating the contributions in between the regular T-intervals using techniques mentioned in the first step produces

$$\widetilde{K}_1(\lambda)\sum_{i=0}^{N-1}\left[\prod_{j=0}^{i}(1-C(\lambda)|\widetilde{q}_j|)\right](r_{i+1}+1)+\widetilde{K}_3(\lambda), \tag{B.3}$$

where $\widetilde{K}_3(\lambda)$ accounts for all $O\left(q_k^2\right)$ contributions and $\widetilde{q}_0=0$.

Thirdly the contribution from the perturbed transfer matrix on the intermediate components needs to be calculated. Observe that if

$$\begin{pmatrix} u_{kT+1} \\ u_{kT+2} \end{pmatrix} = B_1(\lambda-\widetilde{q}_t)\begin{pmatrix} u_{kT} \\ u_{kT+1} \end{pmatrix}$$

then

$$\begin{aligned}|u_{kT+2}|^2 &\le \|B_1(\lambda-\widetilde{q}_t)\|^2\left(|u_{kT}|^2+|u_{kT+1}|^2\right)\\ &\le \|B_1(\lambda-\widetilde{q}_t)\|^2\|W\|^2\|W^{-1}\|^2\prod_{s=0}^{t-1}\left(1-C(\lambda)|\widetilde{q}_s|+O\left(\widetilde{q}_s^2\right)\right)\end{aligned}$$

and so, for $2\le j\le T-1$,

$$\begin{aligned}|u_{kT+j}|^2 \le{}& \|B_{j-1}(\lambda)\dots B_2(\lambda)B_1(\lambda-\widetilde{q}_t)\|^2\\ &\times\|W(\lambda)\|^2\|W^{-1}(\lambda)\|^2\prod_{s=0}^{t-1}\left(1-C(\lambda)|\widetilde{q}_s|+O\left(\widetilde{q}_s^2\right)\right).\end{aligned}$$

This implies

$$\sum_{j=1}^{T-1}|u_{kT+j}|^2\le\widetilde{K}_2(\lambda)\|B_1(\lambda-q_t)\|^2\prod_{s=0}^{t-1}\left(1-C(\lambda)|\widetilde{q}_s|+O\left(\widetilde{q}_s^2\right)\right)$$

where $\widetilde{K}_2(\lambda):=\|W(\lambda)\|^2\|W^{-1}(\lambda)\|^2\sum_{j=2}^{T-1}\left\|\prod_{i=2}^{j-1}B_i(\lambda)\right\|^2$ and $\prod_{i=2}^{1}B_i:=I$. This gives the second term of the total sum of the components of the candidate eigenvector to be estimated by

$$\widetilde{K}_2(\lambda)\sum_{t=1}^{N}\|B_1(\lambda-\widetilde{q}_t)\|^2\prod_{s=0}^{t-1}(1-C(\lambda)|\widetilde{q}_s|)+\widetilde{K}_4(\lambda) \tag{B.4}$$

as $N\to\infty$, where $\widetilde{K}_4(\lambda)$ accounts for all $O(\widetilde{q}_k^2)$ contributions.

Denoting that $\|X\|_{\infty\to\infty} := \sup_{\vec{f}\in\mathbb{R}^2} \frac{\|X\vec{f}\|_\infty}{\|\vec{f}\|_\infty}$, then

$$\|B_1(\lambda - \widetilde{q}_s)\|_{\infty\to\infty} \leq \max\left\{1, \frac{a_T + |\lambda| + |b_1| + |\widetilde{q}_s|}{a_1}\right\}$$

and since $q \ll 1$ then we can replace $|\widetilde{q}_s|$ with 1 and get a bound uniform in $\widetilde{q}_n$. Consequently, using that all matrix norms are equivalent we obtain

$$\begin{aligned}\|\underline{u}\|^2 \leq \widetilde{K}_1(\lambda) \sum_{i=0}^{\infty} \left[\prod_{j=0}^{i} (1 - C(\lambda)|\widetilde{q}_j|)\right] (r_{i+1} + 1) \\ + K_2(\lambda) \sum_{t=1}^{\infty} \prod_{s=1}^{t-1} (1 - C(\lambda)|\widetilde{q}_s|) + K_3(\lambda)\end{aligned}$$

where $K_3(\lambda) := \widetilde{K}_3(\lambda) + \widetilde{K}_4(\lambda)$ and for some $K_2(\lambda) \in \mathbb{R}$.

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