Parameter Estimation Risk in Asset Pricing and Risk Management: A Bayesian Approach

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Abstract

Parameter estimation risk is non-trivial in both asset pricing and risk management. We adopt a Bayesian estimation paradigm supported by the Markov Chain Monte Carlo inferential techniques to incorporate parameter estimation risk in financial modelling. In option pricing activities, we find that the Merton's Jump-Diffusion (MJD) model outperforms the Black-Scholes (BS) model both in-sample and out-of-sample. In addition, the construction of Bayesian posterior option price distributions under the two well-known models offers a robust view to the influence of parameter estimation risk on option prices as well as other quantities of interest in finance such as probabilities of default. We derive a VaR-type parameter estimation risk measure for option pricing and we show that parameter estimation risk can bring significant impact to Greeks' hedging activities. Regarding the computation of default probabilities, we find that the impact of parameter estimation risk increases with gearing level, and could alter important risk management decisions.

Key words: parameter estimation risk, Bayesian option pricing, Greeks, probability of default

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1 Introduction

Since the financial crisis of 2008, model risk has attracted more attention in academic research. Andrikopoulos (2015) argues that the "true" asset value in the financial world is quite often model dependent. Portfolio selection, trading strategies and corporate finance decisions are consequentially made based on these "true" values as well as the difference between the market value and the "true" value. Therefore, risk stems from financial modelling can have a substantial impact on financial quantities such as option prices, hedging ratios or default probabilities.

Model risk was linked to a long series of significant events in the financial markets, see Jacque (2015). In 1987, Merrill Lynch reported losses of $300 millions on stripped mortgage-backed securities caused by an incorrect pricing model. In 1992, J.P. Morgan lost about $200 millions in the mortgage-backed securities market due to the inadequate modelling of prepayments. Later in 1997, the New York subsidiary of the Bank of Tokyo/Mitsubishi lost $83 millions because their internal pricing model overvalued OTM Bermudan swaptions (Dowd, 2002). A Deutsche Bank subsidiary in Japan traded electronically using some smart models which went out of control in June 2010. One model went into an infinite loop and took out a $183 billions stock position (Tunaru, 2015). More recently in 2013, J.P. Morgan revealed a trading loss of more than $6.2 billions, which was indirectly caused by the underestimation of risk level by their value-at-risk (VaR) model. Model risk is also identified by the Basel Committee on Banking Supervision in the old and new Basel frameworks; see Basel Committee on Banking Supervision (2006), Basel Committee on Banking Supervision (2011), Detering and Packham (2016). Financial institutions must gauge their model risk to fulfil regulatory requirements (Kerkhof et al., 2010).

Focused on parameter estimation risk, in this study, we adopt a Bayesian approach to financial modelling with applications to option pricing, including hedging ratios, and to default probabilities calculations. We advocate using distributions of any quantity of interest to the investor. The distributions are generated by the Markov Chain Monte Carlo (MCMC) inferential process. Existing literature eloquently explains how to implement the Bayesian estimation framework to option pricing models, yet empirical applications are very limited. We advocate an improved methodology for investigating the impact of parameter estimation risk towards option pricing, hedging and risk management activities. Moreover, we propose a VaR-type technique to measure parameter estimation risk in option pricing. Interestingly, our results indicate that model risk may not be symmetric for the buyer and the seller in a derivative contract.

Parameter estimation risk is an important source of model risk (Glasserman and Xu, 2014; Tunaru, 2015). It refers to the uncertainty of estimating the correct parameter values given a model structure. Standard practices in financial modelling are based on point estimations of parameters, with standard estimation errors being ignored completely in both asset pricing and risk management computations. Any estimation method would induce a certain level of parameter estimation risk. There is no justification for neglecting any of these errors.

While most market makers and researchers agree that liquidly traded option prices of major stocks and indices are determined by market supply and demand, less liquid options such as exotic options do not have available market prices and depend heavily on models to determine their values (Cont, 2006; Jacquier and Jarrow, 2000; Dahlbokum, 2010). Therefore, parameter estimation risk in option pricing is of great interest to many market participants. Jacquier and Jarrow (2000) carried out a study to incorporate parameter estimation risk of the Black-Scholes (BS) model and its non-parametric extensions into option pricing using the Bayesian estimation approach. They found that even upon consideration of parameter estimation risk, these models cannot deliver promising results in forecasting due to rigid model assumptions. They suggest that further study should be extended to use models with parameters capturing missing time varying dynamics (e.g. jump process). Later studies also confirm the capability of Bayesian econometrics in capturing model uncertainty as well as the feasibility of implementing it in financial practices, but most of this literature focuses on describing the methodology; see for example Bunnin et al. (2002), Jacquier and Polson (2010), Johannes and Polson (2010). Other contributions to the literature emphasise the advantage of extracting latent parameters using the Bayesian estimation approach, but paying little attention to its application in dealing with parameter estimation risk in practices; see for example Eraker et al. (2003), Yu et al. (2011), Kaeck and Alexander (2013).

The Bayesian method advances in its ability of delivering the joint posterior distribution of parameters, which contains all possible value of the parameters given the model and the observed data, so that shapes of the distributions as well as credibility intervals can be obtained easily (Laurini and Hotta, 2010). Therefore, under the Bayesian framework, all parameters are stochastic, accounting for the uncertainty in their estimation. We highlight our proposed methodology using two well-known models as vehicles of research, the Black-Scholes (BS) model and the Merton Jump-Diffusion model (MJD). The MJD model was developed by Merton (1976) as a key alternative model to the BS model, capable of generating kurtosis and skewness in line with empirical literature on stock returns; see Bakshi et al. (1997) and Dahlbokum

(2010). Nevertheless, this model has been omitted from most of the related literature which provides empirical tests (Jacquier and Jarrow, 2000; Eraker et al., 2003; Gupta and Reisinger, 2014; Kaeck and Alexander, 2013; Yu et al., 2011), except for Frey (2013) which adopts the model in pricing CO_2 options.

We show how to construct the distributions of option prices directly from the posterior distributions of parameters for the two models investigated. Another key contribution of the study is that we show how to measure parameter estimation risk, and how significant it is in empirical practices. We apply a VaR-type measure for parameter estimation risk exposure in option pricing, and we show that model risk is asymmetric to buyers and sellers of options. By applying Bayesian MCMC techniques to the Greek parameters, we show that the impact of parameter estimation risk can be very significant to hedging activities. Finally, we describe how to apply this Bayesian approach to the Merton's Credit Risk model with a focus on investigating the impact of parameter estimation risk in computing the probability of default.

The rest of the paper presents as follow: Section 2 provides a summary of literature review; Section 3 reviews briefly the MJD model; Section 4 introduces the Bayesian econometrics and MCMC simulation techniques; Section 5 shows the empirical application results of both the BS and MJD models under the Bayesian estimation approach and a VaR-type measure for parameter estimation risk in option pricing; Section 6 demonstrates the application of Bayesian econometrics in the Merton's Credit Risk model; and Section 7 provides summary conclusions.

2 Literature Review

There has been an array of evidence that the BS model is not consistent with empirical data (Das and Sundaram, 1999; Merton, 1976; Jorion, 1988; Drost, Nijman and Werker, 1998; Backus, Foresi and Wu, 2004; Batten and Ellis, 2005). The model suggests a normal distribution of stock return, whereas empirical evidence, as we know it, generally shows excessive kurtosis and negative skewness.

The MJD model developed by Merton (1976) is a key extension of the BS model. Several studies suggest that the anomalies of market return could be a result of jump events, and large price jumps are observed in market return data; see Das and Sundaram (1999), Drost et al. (1998), Jarrow and Rosenfeld (1984), Kim et al. (1994) and Maekawa et al. (2008). Burger and Kliaris (2013) argue that while the diffusion process captures the volatility generated by trading activities, the jump component captures more significant changes of stock prices generated by new information. The jump component also generates skewness and kurtosis to the stock return distribution as discussed by Das and Sundaram (1999), Gardoń (2011), Bates (1996).

Estimating the parameters of the MJD model is not a straightforward exercise because under this model the stock return distribution is an infinite mixture of normal distributions. Even under the simplified Bernoulli-Jump Diffusion setting proposed by Ball and Torous (1985), in which a maximum of one jump can occur during one unit time interval, the likelihood function is still unbounded and may have many local modes. This leads to the difficulty in estimating the parameter values using the maximum likelihood method (Kostrzewski, 2014). Due to this issue, many empirical studies of the MJD model show unreasonable large number of jumps: 162 per annum (Hanson and Westman, 2002), 179 per annum (Ramezani and Zeng, 1998), 142 per annum (Honore, 1998). Honore (1998) suggests that this issue can be circumvented by treating the jump magnitude as a constant input to the model. However, the option pricing results under such strict constraints can hardly show any improvement compared to the BS model pricing results. Frühwirth-Schnatter (2006) and Kostrzewski (2014) show that MCMC Bayesian econometrics framework can provide a better solution to this calibration problem.

Parameter estimation risk is never a trivial problem in financial modelling. Focused on asset pricing and risk management, Chung et al. (2013) account for parameter estimation risk in equity pricing models by calculating the Bayesian posterior standard deviation of parameters, and they conclude that parameter uncertainty is sufficient to explain the price discrepancy between Chinese A- and H-share prices. Jacquier et al. (1994), Bunnin et al. (2002) and Gupta and Reisinger (2014) also emphasise the importance of parameter estimation risk in option pricing and suggest the Bayesian estimation approach through MCMC computational techniques as a solution. Butler and Schachter (1997), Christoffersen and Gonçalves (2005), and Kerkhof et al. (2010) report value-at-risk (VaR) with upper and lower bounds to account for parameter uncertainty. Tarashev (2010) shows that ignoring parameter uncertainty would lead to substantial underestimation of risk level. Rodríguez et al. (2015) present the advantage of Bayesian estimation methods in capturing parameter estimation risk in structural credit risk models.

The distinct advantage of MCMC methods is the ability of delivering not only point estimation values, but also the entire posterior distributions of parameters by the inference of the observed data and prior beliefs of the parameters (Stanescu et al., 2014). Therefore, the final option price can be seen as the expectation of the model price over the distributions of unknown parameters (Jacquier and Polson, 2010). This argument shows a shift in the philosophy of the traditional estimation approach. Gupta and Reisinger (2014) assert that finding a best-fit solution in model calibration makes the problem ill-posed as all estimation errors are simply neglected in this manner. Any best-fit parameter point value cannot be good enough to underpin the correct model form and inference should be shifted onto exploring a distribution of solutions, which sufficiently captures all possible parameter values given the observed data. Bunnin et al. (2002) were among the first to show how Bayesian posterior distributions of parameters can capture and reflect parameter estimation risk in option pricing. However, they only applied the method to a single European at-the-money call option. For simplicity, they ignored any dividend impact on the underlying asset and fixed the stock return rate at 10% without any justification.

Jacquier and Jarrow (2000) calibrate model parameters using a Bayesian approach on call option data of TOYSR US between 4-Dec- and 15-Dec-1989. They found that the standard BS model leads to a narrow model price distribution, whereas its static non-parametric extension models tend to produce more disperse model prices. Extended models show improvements over the BS model in in-sample performance but the BS model is superior in out-of-sample tests. Therefore, the non-parametric extended models do not fully overcome the shortcomings of the BS model. The authors suggest that models with additional parameters capturing missing time varying dynamics (e.g. jump process) might improve the pricing performance. We based the research in this paper on this suggestion. Yu et al. (2011) and Kaeck and Alexander (2013) recently use the Bayesian estimation approach through ~~the~~ MCMC techniques to extract inference on parameters and latent volatility/jump variables of the Lévy jump models, but they do not take into consideration its application in dealing with parameter estimation risk.

Gupta and Reisinger (2014) and Johannes and Polson (2010) provide a step-by-step instruction of how to calibrate the BS and MJD models using MCMC techniques. Prior distributions of parameters are carefully derived to ensure appropriate posterior distributions of all parameters. Gupta and Reisinger (2014) also suggest the application of a Bayesian approach in option hedging activities as a potential future research direction. We follow this methodology in our paper and carry out an extensive option pricing exercise to see whether the Bayesian approach can effectively generate posterior distributions of option prices that contain the realised future market values under the two models investigated.

3 Merton's Jump-Diffusion Model

The Merton's Jump-Diffusion model is a continuous-time model described by the stochastic differential equation [1]:

$$dS_t = (\mu - \delta - \lambda k)S_t dt + \sigma S_t dz_t + (\mu_t^J - 1)S_t dI_t \qquad (1)$$

where μ is the expected rate of return; δ is the dividend yield, σ is the volatility of stock return; z_t is a standard Wiener Process; λ is the intensity of jump events per unit time interval; I_t is a Poisson process with intensity λ; μ_t^J is the jump size of stock price; and $k = E(\mu_t^J - 1)$.

Under the assumption that $\ln(\mu_t^J)$ is normally distributed $\ln(\mu_t^J) \sim N(a, \zeta^2)$, the probability density of log stock return $R_{t+\Delta} = ln\left(\frac{S_{t+\Delta}}{S_t}\right)$ is the weighted average of normal densities by the probability that i jumps would occur:

$$p(R_{t+\Delta}) = \sum_{i=0}^{\infty} \frac{e^{-\lambda\Delta}[\lambda\Delta]^i}{i!} N(R_{t+\Delta}; (\mu - \delta - \frac{\sigma^2}{2} - \lambda k)\Delta + ia, \sigma^2\Delta + i\zeta^2) \qquad (2)$$

Under a risk-neutral measure, Merton (1976) proved that the European option pricing formula (for both call and put options) under the MJD model is a weighted average of the BS option prices V_t^{BS} by the probability that i jumps would occur (Matsuda, 2004):

$$V_t^{Merton} = \sum_{i=0}^{\infty} \frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!} V_t^{BS}(S_i, \sigma_i, \delta, T-t, r, K)$$

$$S_i \equiv S_t \exp\left\{ ia + \frac{i\zeta^2}{2} - \lambda(e^{a+\frac{\zeta^2}{2}} - 1)(T-t) \right\} \qquad (3)$$

$$\sigma_i \equiv \sqrt{\sigma^2 + \frac{i\zeta^2}{T-t}}$$

where r is the risk-free return, K is the strike price of the option, and $T-t$ is the time-to-maturity. To obtain values of the model parameters, one can either estimate them using the historical return data with equation (2) in an *estimation* exercise or imply them from market options data using equation (3) in a *calibration* exercise. The distinction is important since risk managers use the former while traders employ the later, *both* ignoring the error caused by using

[1] Please refer to Matsuda (2004); McDonald et al. (2006); Merton (1976) for more details of the MJD model.

a specific dataset.

4 Integrating Parameter Estimation Risk under Bayesian Econometrics

For a model M with a parameter θ and given the observed data y_{t-1}, Bayes' formula gives

$$p(\theta \mid y_{t-1}) = \frac{p(y_{t-1} \mid \theta) p(\theta)}{p(y_{t-1})} \tag{4}$$

where $p(\theta \mid y_{t-1})$ is the posterior distribution of θ given y_{t-1}; $p(y_{t-1} \mid \theta)$ is the conditional likelihood under the model given θ; $p(\theta)$ is the prior marginal distribution of θ and $p(y_{t-1})$ is the marginal distribution of y_{t-1}.

In the MJD model, with the parameter vector $\Theta = \{\lambda, \sigma, a, \zeta, \delta\}$ [2], using the joint posterior distribution of all parameters $p(\Theta \mid y_{t-1})$ is the key way to consider parameter estimation risk in all asset pricing and risk management calculations. From a computational point of view, the option price V_t^{Merton} is just a function of S_t, t and Θ, hence the ergodic results underpinning ~~the~~ MCMC inference provide a direct mechanism to extract the posterior distribution of model price $p(V_t^{Merton} \mid S_t, t, \Theta)$. ~~The~~ MCMC simulation methods can efficiently sample from the posterior distribution of $p(V_t^{Merton} \mid S_t, t, \Theta)$ without knowing its mathematical analytical form; hence the mean, standard deviation, quantiles and shape of the distribution can be easily assessed. In each MCMC iteration, a joint draw of parameter values is obtained from $p(\Theta \mid y_{t-1})$. Then for each joint draw of parameters, option price V_t^{Merton} can be calculated following formula (3). This yields a sample of draw from $p(V_t^{Merton} \mid S_t, t, \Theta)$. Further details behind ~~the~~ MCMC algorithms[3] are explained eloquently by Johannes and Polson (2010), Gelman et al. (2014) and Tunaru (2015).

[2] Parameter $\lambda, \sigma, a, \zeta$ are estimated from historical return data or calibrated from option prices, whereas parameter δ is estimated separately from historical dividend yield data. More details are presented in section 5.2.

[3] The open source software OpenBUGS provides a readily written stable programme to run MCMC simulations even on complex models. Conveniently, the OpenBUGS programme will automatically choose the most feasible updater according to the model setting.

5 Option Pricing with Parameter Estimation Risk: BS vs MJD

5.1 Data

All empirical data used in this study is obtained from Bloomberg. Empirical tests are carried out using daily S&P 500 index log-return data from 31/07/2012 to 29/08/2014, and S&P 500 index European call and put option data in August 2014. The option data set excludes[4] options with $\mid K/S_t - 1 \mid > 0.2$, options with time-to-maturities shorter than 7 days or longer than 500 days, options with market prices less than \$0.5, options with zero trading volume, and options which do not fulfil the non-arbitrage profit conditions. Furthermore, on a trading day, if there are less than 5 options for a maturity date or the available option data set for a maturity date does not cover all types of moneyness (i.e. ITM, ATM and OTM), options of this maturity date will be dropped out from the data set of that particular trading day. This is to ensure that there are enough data points to calibrate the implied parameter values in the calibration exercise of Section 5.4. Our final option sample contains 2871 pairs of option contracts. The option data is classified as in-the-money (ITM) for call option and out-of-the-money (OTM) for put option if $K/S_t < 0.99$; at-the-money (ATM) if $0.99 \leqslant K/S_t \leqslant 1.01$; OTM for call option and ITM for put option if $K/S_t > 1.01$.

5.2 Parameter Inference

Under the MJD model, the log stock return distribution is an infinite mixture of normal distributions. To circumvent this problem, we follow Ball and Torous (1985) [5] which approximated the MJD model with the Bernoulli-Jump Diffusion model, using the assumption that only one jump is allowed to happen per unit time interval. When the time interval Δ is short (e.g.daily), the jump intensity $\lambda\Delta$ will be very small, and the Bernoulli-Jump Diffusion model will be a good proxy for the Merton's model.

The equity index return is impacted by dividend payments. Among all studies on index option pricing, many of them ignore dividend impacts; see Eraker et al. (2003), Johannes and

[4] We follow standard empirical option pricing filtering methodology described in Bakshi et al. (1997); Dahlbokum (2010).

[5] Another widely used proxy is the M-Jump Diffusion model described in (Kostrzewski, 2014; Burger and Kliaris, 2013). This model refers to a specification of a cutting point M, so that the model takes into account a maximum of M jumps. Researchers need to ensure that the probability of observing more than M jumps is extremely small, hence neglecting it would not bring any significant impact to the results.

Polson (2010), Jacquier and Jarrow (2000), Maekawa et al. (2008) and Bauwens and Lubrano (2002). However, this may bias the results. Ferreira and Gama (2005) and Bakshi et al. (1997) stress the importance of considering dividend impact in calculating option prices. In our study, a posterior predictive dividend yield value $\widehat{\delta}$ is estimated separately from historical dividend yield data and used in option pricing as an estimate of δ in equation (3). If the historical dividend yield data δ_t follows a gamma distribution:

$$\delta_t \delta \sim Gamma(\alpha, \beta)$$

The posterior predictive distribution of dividend yield can be computed by:

$$\begin{aligned} p(\widehat{\delta} \mid \delta_t \delta) &= \int p(\widehat{\delta}, \alpha, \beta) d\alpha d\beta \\ &= \int p(\widehat{\delta} \mid \delta_t \delta, \alpha, \beta) p(\alpha, \beta \mid \delta) d\alpha d\beta \end{aligned}$$

Statistical results of the posterior predictive dividend yield are tabulated in Table 1:

Table 1: Statistical Results of the Posterior Predictive Distribution of Dividend Yield

	in %
	2.0490
$\widehat{\delta}$	(0.0954)
	[1.8650 - 2.2410]

Note: statistics shown above are posterior predictive mean of dividend yield with the standard deviation in ( ) and 95% credibility interval in [ ]. Parameter estimation data: S&P 500 index daily annualised dividend yield data 31/07/2012 - 31/07/2014

For the MJD model, prior distributions of parameters should be selected carefully. Uninformative prior distributions are not suggested as they could make convergence difficult to achieve (Eraker et al., 2003; Johannes and Polson, 2010). The selection of prior distributions is also critical to control for the size of jump intensity (Kostrzewski, 2014). Similarly, Jacquier and Polson (2010) emphasise that uninformative priors could result in an unbounded likelihood for the jump-diffusion model and useful priors shall be given to "impose constraints on the parameter space". Following Jacquier and Polson (2010) and Eraker et al. (2003), informative priors *Beta*(2, 40) and *Gamma*(10, 0.01) are assigned to the daily jump intensity λ and jump size precision $1/\zeta^2$. Uninformative normal distribution N (0,100) is assigned to the stock return drift μ' ($\mu' = \mu - \delta$) and the jump size drift a, whereas the prior of stock precision $1/\sigma^2$ takes the form of *Gamma*(0.0001, 0.0001).

Parameters of both the BS and MJD models are estimated from the S&P 500 index daily log-return data. After checking for the convergence of MCMC chains, inferential results are presented in Table 2.

Table 2: Parameter Estimation Results of the Black-Scholes and Merton's Jump-Diffusion Models

Parameters	Models	
	BS	MJD
μ' (in %)	17.2872 (7.9128) [1.7892-32.9112]	16.9495 (9.2207) [-1.3159-34.8516]
σ (in %)	11.1300 (0.3533) [10.4700-11.8500]	10.0500 (0.4406) [9.2510-10.9300]
λ	- - -	8.3790 (3.6263) [2.4792-16.5514]
a (in %)	- - -	-0.8350 (0.9035) [-2.8170-0.7450]
ζ (in %)	- - -	2.7160 (0.4009) [2.0450-3.5930]

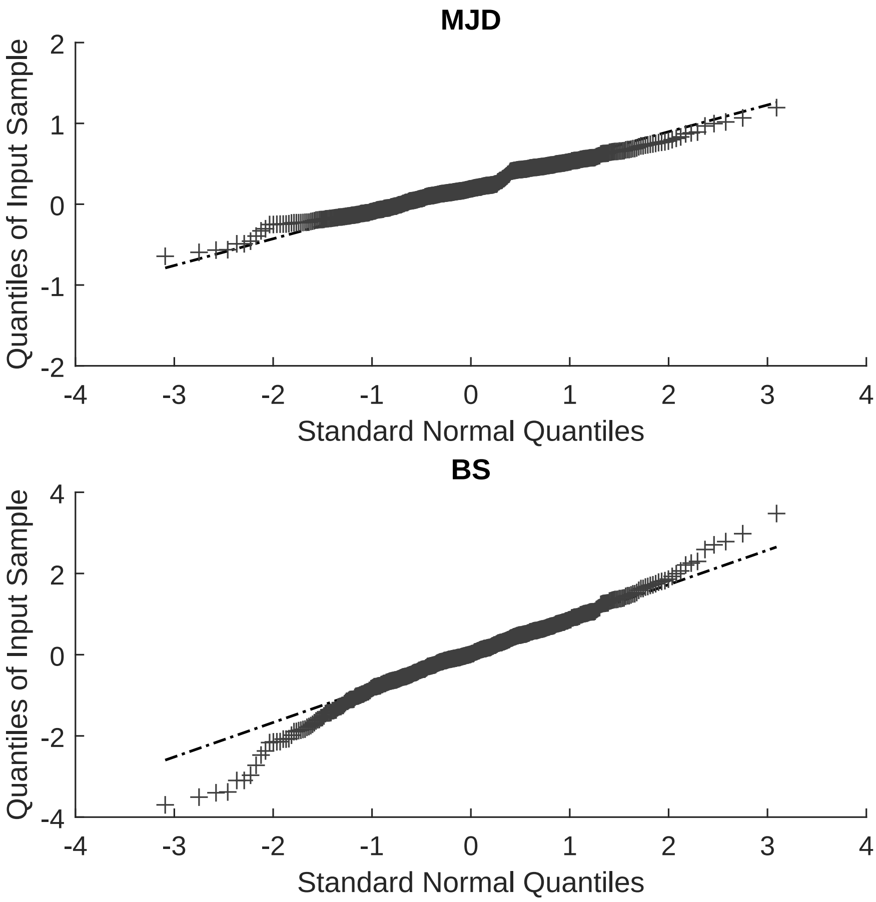
Note: statistics shown above are posterior mean results of parameters with the standard deviation in ( ) and 95% confidence interval in [ ]. Statistics of μ', σ and λ are annualised results. $\mu' = \mu - \delta$. Parameter estimation data: S&P 500 index daily log-return data 31/07/2012 - 31/07/2014.

The results indicate that, under the BS model, the annual return drift $\mu' = \mu - \delta$ can range from 1.79% to 32.91% at the 95% credibility level with a mean value of 17.29%, and the annual stock return volatility reports at a mean value of 11.13% with a 95% credibility interval of $[10.47\%, 11.85\%]$. With the impact of jumps, the annual return drift μ' of the MJD model is estimated at a mean value of 16.95%. The 95% credibility interval of μ' is enlarged to $[-1.32\%, 34.85\%]$. The jump magnitude a is estimated at an average level of -0.84% with a 95% credibility interval of $[-2.82\%, 0.75\%]$, meaning that both positive and negative jumps are feasible. The inference results of jump intensity λ indicate an average of 8.38 jumps per annum, with the rate varying between 2.48 jumps to 16.55 jumps. With the introduction of jumps, a part of the stock return volatility is captured by the jump process, thus the estimated stock volatility σ of the MJD model is 10.05%, which is smaller than the σ under the BS model 11.13%. The volatility of jump size ζ has an estimated posterior mean of 2.72%, and a 95% credibility interval of $[2.05\%, 3.59\%]$.

5.2.1 QQ-plot

Goodness-of-fit tests are often ignored in options pricing literature. Here we carried out a goodness-of-fit analysis for both the BS and MJD models, as this is an important step before proceeding on drawing conclusions on empirical exercises. Standardised Pearson residuals are computed for each model using the posterior mean of parameters. The QQ-plots of both models are displayed in Figure 1, and they indicate that the MJD model provides better data fitting performance than the BS model. The MJD model exhibits a smaller deviation from the standard normal quantiles compared with the BS model. Our results indicate that the MJD model better captures the leptokurticity of the data.

Figure 1: QQ-plots of the Standardised Pearson Residuals of the Black-Scholes and Merton's Jump-Diffusion Models



Note: the Standardised Pearson residuals are calculated based on the estimated parameter posterior means of both the BS and MJD models. Parameter estimation data: S&P 500 index daily log-return data 31/07/2012 - 31/07/2014.

5.2.2 DIC comparison of the two models

The Deviance Information Criterion (DIC), introduced by Spiegelhalter et al. (2002), provides a robust yardstick for Bayesian model comparison:

$$DIC = \overline{D} + pD \tag{5}$$

where $\overline{D}$ is the posterior mean deviance (a measure of fit), $D = -2\,log\,p(y \mid \theta)$; and pD is the the effective number of parameters (a measure of model complexity). The smaller the DIC value, the better the model.[6] DIC results of the two models are tabulated in Table 3. Model comparison and selection are based on the difference of DICs between the two models, not the absolute values of DICs. There is no universal standard on what should be treated as an important difference in DIC values (Spiegelhalter et al., 2002). In this study, we follow Lunn et al. (2012): a DIC difference of 10 can definitely rule out the higher DIC model; a DIC difference of 5 is still substantial; a DIC difference of less than 5 is trivial, and neither model shall be ruled out.

Table 3: DIC Results for the Black-Scholes and Merton's Jump-Diffusion Models

	DIC
BS	-3563
MJD	-3580
Difference	17

Note: Difference is calculated as $DIC^{BS} - DIC^{MJD}$. DICs are computed based on the models' fitting results of the S&P 500 index daily log-return data 31/07/2012 - 31/07/2014.

The DIC results again suggest a better in-sample fitting of the MJD model compared with the BS model. The DIC^{MJD} is less than the DIC^{BS} by 17, indicating superiority of the MJD model over the BS model.

5.2.3 Bayesian p-value

The Bayesian p-value measures the discrepancy between the model replicative simulation data and the observed data via selected test statistics (Gelman et al., 2014). The chosen test statistics can be some summary statistics of the observed data. Denoting generically a test statistic by $T(y)$, it is clear that $T(y)$ can be easily calculated given the observed data set. On the other hand, when a series of replicated data y^{rep} is simulated using the posterior predictive distribution, the

[6] In the case when the likelihood $p(y \mid \theta)$ is greater than 1, DIC could be legitimately negative, and this does not affect its validity in model comparison (Spiegelhalter, 2006).

test statistics $T(y^{rep})$ can also be calculated consequently. If the model fits the observed data well, the distribution of $T(y^{rep})$ will be concentrated around $T(y)$. This is assessed by the Bayesian p-value: $Pr(T(y^{rep}) \geqslant T(y) \mid y)$. A p-value of 0.5 means the probability of obtaining $T(y^{rep}) \geqslant T(y)$ is 50%, so that the model is likely to generate the observed data series. In contrast, obtaining an extreme p-value of $\leqslant 0.01$ or $\geqslant 0.99$ indicates a likely misfit between the model and the observed data (Gelman et al., 2014).

Four test statistics, including sample mean, variance, skewness and kurtosis (excess kurtosis), are chosen to test the ability of the two models in reproducing the observed sample data series. Parameter posterior distributions of each model are used to simulate the replicated data. Mean(y^{rep}), Variance(y^{rep}), Skewness(y^{rep}) and Kurtosis(y^{rep}) are computed consequently. The Bayesian p-values of test statistics are reported in Table 4.

Table 4: Bayesian P-values of the Black-Scholes and Merton's Jump-Diffusion Models

	Sample	BS		MJD	
Test Statistics	$T(y)$	$T(y_{BS}^{rep})$	p-value	$T(y_{MJD}^{rep})$	p-value
Mean	0.1663	0.1648	0.4585	0.1628	0.4618
	-	(0.1119)	-	(0.1320)	-
Variance	0.0123	0.0124	0.6019	0.0170	0.9267
	-	(0.0011)	-	(0.0043)	-
Skewness	-0.3824	0.0005	1.0000	-0.8165	0.3936
	-	(0.1077)	-	(1.4121)	-
Kurtosis	1.4176	-0.02411	0.0000	11.7800	0.9330
	-	(0.2145)	-	(9.1570)	-

Note: Bayesian p-values account for $Pr(T(y^{rep}) \geqslant T(y))$. $T(y)$: test statistics of the observed S&P 500 index daily log-return data; $T(y^{rep})$: test statistics of replicated data simulated using the posterior results of parameters, standard deviation is reported in (). Parameter estimation data: S&P 500 index daily log-return data 31/07/2012 - 31/07/2014.

While the BS model tends to successfully replicate the distribution location and dispersion of the observed log-return data, its p-values of Skewness and Kurtosis reject the assumption of normal distributed stock return explicitly. Consistent with past empirical findings (Das and Sundaram, 1999; Merton, 1976; Jorion, 1988; Drost et al., 1998; Backus et al., 2004), the observed S&P 500 log-return data shows a negative skewness of -0.3824 and a excessive kurtosis of 1.4176. The p-values of $Skewness(y_{BS}^{rep})$ and $Kurtosis(y_{BS}^{rep})$ are strictly equal to 1 and 0, indicating that the skewness of the replicated data of the BS model is always higher than the sample skewness and the kurtosis of the replicated data is always smaller than the observed kurtosis.

According to the p-value results, the MJD model replicates the location of the observed log-return distribution successfully. It also seems to correctly reproduce the negative skewness of the observed data. Nevertheless, its ability in reproducing the observed variance and kurtosis is not as competent as expected. With p-values of 0.9267 and 0.9330 for variance and kurtosis respectively, the MJD model seems to produce too much variance and kurtosis in general. Although these results do not exceed the critical boundary of 0.99, they suggest that, over 90% of time, the model is generating higher variance and kurtosis than the empirical data.

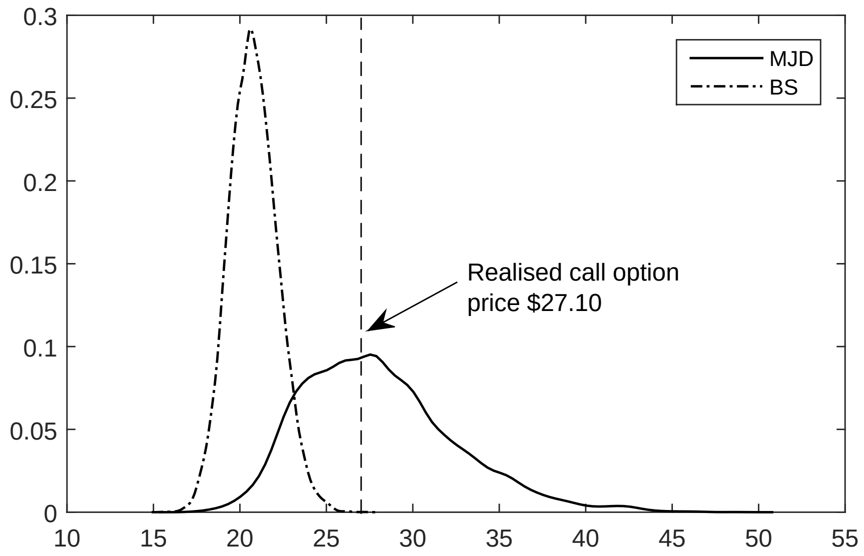
The Bayesian p-values show that, although the MJD model is superior to the BS model, it also has some new problems. Even though the p-values indicate a certain level of imperfect fitting for both models, any single p-value shall not act as the evidence to reject a model (Gelman et al., 2014). The main purpose of the Bayesian p-values is to check and understand the limitation of a model's ability in replicating the observed data. Practical feasibility of a model shall also be evaluated based on its performance in applications. An out-of-sample option pricing performance analysis of the two models is presented next.

5.3 Out-of-sample Option Pricing

Out-of-sample option pricing tests are carried out using a rolling window approach. The estimation window contains 2 years of daily return data, estimated parameter results are used to price options one trading day ahead, and the estimation window rolls one day forward each time to price all option contracts in August 2014. An example of posterior density plots of an OTM S&P 500 European call option on 01/08/2014 with strike price $2000 and time-to-maturity 141 days under both of the models are shown in Figure 2. As reflected in the plots, MCMC methods successfully draws samples from the posterior distributions of model prices. In this way, all parameter estimation errors are incorporated.

The posterior price distribution range is wider under the MJD model than under the BS model, indicating that the MJD model, having more parameters, leads to a higher level of uncertainty in computing the final option price. However, this does not necessarily mean that the model with less parameter estimation risk is the better one to use. A narrower price range might in fact give a false security. For the data analysed here, the posterior distribution of the MJD model option price captures the realised market option value very close to the centre of the distribution. The BS model price, on the other hand, has the realised future market price positioned at the very end of its right tail area. It seems that the BS model is still far away from predicting the underlying option price successfully even when taking the parameter estimation risk into consideration. However, this is just one instance of option pricing, a more informed

Figure 2: Posterior Density Plots of a European Call Option under the Black-Scholes and Merton's Jump-Diffusion Models



Note: the figure shows posterior density plots of an OTM S&P 500 index European call option contract on 01/08/2014 with strike price $2000 and time-to-maturity 141 days. Parameter estimation data: S&P 500 index daily log-return data 31/07/2012-31/07/2014.

view of the overall out-of-sample option pricing performance of the two models is presented below.

In addition to paying attention to traditional performance evaluation statistics, such as pricing errors or absolute pricing errors, we are particularly interested in whether the posterior distributions, which incorporate parameter estimation risk, can cover the realised observations; if not, how far are the realised prices from the model price distributions. The overview of coverage performance is tabulated in Table 5, and pricing error performance resutls are shown in Table 6 and 7.

Overall, posterior distribution coverage results of the MJD model are better than the results of the BS model for both call and put options as it can be seen in Table 5. In the tested sample period, the MJD model price distributions (i.e. 95% credibility intervals) cover an extra of 5.63% call option mid-quotes and an extra of 13.72% put option mid-quotes compared with the BS model results. The mid-quote coverage rates of the MJD model are also higher than the BS model across different moneyness and time-to-maturities, except for ITM call options and ATM put options. More importantly, the MJD model shows great improvements in both call and put OTM option pricing performance compared with the BS model (with improvements of 26% for OTM call options and 22.33% for OTM put options respectively). The bid/ask-quote coverage statistics show the percentages of options with both bid- and ask-quotes lying within the 95%

Table 5: Out-of-sample S&P 500 Index European Option Pricing Coverage Performance of the Black-Scholes and Merton's Jump-Diffusion Models

	Call Option		Put Option	
	Mid-quote Coverage Difference	Bid/Ask-quote Coverage Difference	Mid-quote Coverage Difference	Bid/Ask-quote Coverage Difference
All Options	5.63%	7.91%	13.72%	16.27%
ITM	-1.33%	0.49%	13.72%	11.16%
ATM	5.28%	7.13%	-2.07%	0.23%
OTM	26.00%	29.83%	22.33%	21.89%
<60 days	2.70%	5.77%	13.07%	14.79%
60 - 180 days	8.87%	10.32%	12.85%	16.92%
> 180 days	14.6%	13.87%	28.47%	28.47%

Note: out-of-sample option pricing period: 01/08/2014 - 31/08/2014; rolling window estimation method with S&P 500 index daily log-return data 31/07/2012 - 29/08/2014; estimation window rolls one day ahead after each estimation. Estimated posterior parameter results are used to price options one day ahead. Mid-quote Coverage and Bid/Ask-quote Coverage account for the % of realised market option prices covered by the 95% credibility intervals of posterior model price distributions. The table reports the differences of coverage rates between the MJD and the BS model (i.e. MJD coverage rate - BS coverage rate).

credibility intervals of the posterior model price distributions. The MJD model outperforms the BS model on this measure across all moneyness and time-to-maturities in both call and put options. These evidences reflect that the narrow model price distributions produced by the BS model have more difficulty in capturing the bid-ask spreads of market prices.

Pricing Error (PE), Absolute Pricing Error (APE), and Root-mean-square Pricing Error (RMSPE) are calculated respectively to measure the differences between the posterior mean of model prices and the market prices (mid-quote). On the other hand, Outside Pricing Error (OPE), Absolute Outside Pricing Error (AOPE) and Root-mean-square Outside Pricing Error (RMSOPE) statistics aim to evaluate the distances between the closest 95% credibility interval bound values and the market prices (mid-quote) when the mid-quotes are not covered by the intervals. The BS model outperforms the MJD model with respect to these six statistical performance measures in general. The MJD model seems to produce results that can match the BS model in OTM options and longer term options (i.e. time-to-maturity $>$ 180 days). However, for either model, options with time-to-maturity greater than 180 days remain the most challenging type of options to price.

The BS model price distributions capture market prices of OTM call options more effectively than ITM call options, and the advantage in pricing OTM call options is also reflected in the PE, APE, RMSPE, OPE, AOPE and RMSOPE results, which is consistent with the finding

Table 6: Out-of-sample S&P 500 Index European Call Option Pricing Error Performance of the Black-Scholes and Merton's Jump-Diffusion Models

Panel A. BS - S&P 500 Index European Call Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	-5.32	6.68	9.76	-4.45	5.25	8.29
ITM	-7.82	7.95	11.21	-6.57	6.66	9.74
ATM	-2.81	5.19	7.38	-2.04	3.31	5.59
OTM	0.13	4.04	5.98	0.02	2.52	4.42
$<$60 days	-1.85	3.66	4.49	-1.61	2.76	3.66
60-180 days	-7.92	8.77	10.61	-6.47	6.85	8.88
$>$180 days	-25.60	25.72	29.14	-21.89	21.91	25.40

Panel B. MJD - S&P 500 Index European Call Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	-26.82	27.82	37.60	-22.64	23.06	33.70
ITM	-39.46	39.99	46.73	-34.27	34.68	42.20
ATM	-11.08	11.65	14.15	-6.25	6.46	9.19
OTM	-1.26	3.94	5.75	-0.47	1.02	2.33
$<$60 days	-25.83	27.12	37.55	-23.26	23.92	34.91
60-180 days	-27.80	28.46	37.46	-22.16	22.26	32.40
$>$180 days	-30.62	30.99	39.17	-19.19	19.20	29.09

Note: out-of-sample option pricing period: 01/08/2014 - 31/08/2014; rolling window estimation method with S&P 500 index daily log-return data 31/07/2012 - 29/08/2014; estimation window rolls one day ahead after each estimation. Estimated posterior parameter results are used to price options one day ahead. PE (Pricing Error): the average of (posterior mean price - market mid-quote); APE: Absolute Pricing Error; RMSPE: Root-mean-square Pricing Error; OPE (Outside Pricing Error): the average of differences between the posterior 2.5% or 97.5% credibility interval bound values and market prices (mid-quote) when the market prices are not covered by the 95% intervals of model prices; AOPE: Absolute Outside Pricing Error; RMSOPE: Root-mean-square Outside Pricing Error.

of Jacquier and Jarrow (2000). This pattern is reversed in its put option pricing performance. According to Backus et al. (2004), the anomalies in stock return distribution decrease as time interval increases. Hence, one could expect the BS model to perform better in longer-term options. However, our results of the S&P 500 option pricing do not support this statement. Both of the APE and RMSPE statistics increase with time-to-maturities, and the coverage performance measures also exhibit no advantage for the BS model in pricing longer-term options.

Regarding the MJD model results, Das and Sundaram (1999) argue that the MJD model can effectively reproduce skewness and kurtosis to stock return distribution in short period, but the ability declines as time length increases. They suggest that although the decline of anomalies is

Table 7: Out-of-sample S&P 500 Index European Put Option Pricing Error Performance of the Black-Scholes and Merton's Jump-Diffusion Models

Panel A. BS - S&P 500 Index European Put Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	-4.21	6.16	8.89	-3.61	4.86	7.58
ITM	1.12	4.98	6.38	0.92	3.28	4.64
ATM	-1.86	5.51	7.20	-1.56	3.62	5.40
OTM	-6.63	6.72	9.95	-5.68	5.71	8.76
$<$60 days	-1.43	3.69	4.65	-1.31	2.87	3.85
60-180 days	-6.13	7.72	9.46	-5.13	6.01	7.97
$>$180 days	-21.85	22.88	26.03	-18.80	19.20	22.70

Panel B. MJD - S&P 500 Index European Put Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	14.39	18.82	23.13	13.62	15.30	21.19
ITM	29.46	32.07	42.90	28.35	28.92	40.92
ATM	27.48	30.75	38.31	26.05	26.70	35.53
OTM	6.02	11.36	16.88	5.53	7.84	13.67
$<$60 days	15.62	18.13	28.59	14.75	15.79	27.08
60-180 days	14.32	19.41	27.92	13.11	14.84	24.68
$>$180 days	0.31	22.25	28.85	4.33	13.14	22.52

Note: out-of-sample option pricing period: 01/08/2014 - 31/08/2014; rolling window estimation method with S&P 500 index daily log-return data 31/07/2012 - 31/07/2014; estimation window rolls one day ahead after each estimation. Estimated posterior parameter results are used to price options one day ahead. PE (Pricing Error): the average of (posterior mean price - market mid-quote); APE: Absolute Pricing Error; RMSPE: Root-mean-square Pricing Error; OPE (Outside Pricing Error): the average of differences between the posterior 2.5% or 97.5% credibility interval bound values and market prices (mid-quote) when the market prices are not covered by the 95% intervals of model prices; AOPE: Absolute Outside Pricing Error; RMSOPE: Root-mean-square Outside Pricing Error.

consistent with empirical stock return evidence, these anomalies generated by the MJD model disappear much quicker than they should be as suggested by empirical data. As a result, the MJD model is expected to perform worse in longer term option pricing but better in shorter term option pricing. The APE and RMSPE statistics of the MJD model for both call and put options confirm this conclusion as they increase with time-to-maturities.

Over-pricing is defined as the case when the market price (mid-quote) is not covered by the 95% credibility interval of model price and located to the left of the distribution (positive OPE); Under-pricing is defined as the case when the market price is not covered by the 95% credibility interval of model price and located to the right of the distribution (negative OPE). Focused on the BS model results, consistent with Batten and Ellis (2005), under-pricing seems to have the dominant effect across all types of moneyness and maturities in both call and put option pricing. The only two exceptions are found in OTM call and ITM put options. It can be observed that the mis-pricing tendency moves from under-pricing to over-pricing with increases in the strike price. Moreover, the under-pricing magnitude tends to increase with time-to-maturities (see AOPE and RMSOPE results).

The MJD model, on the other hand, tends to under-price call options, but over-price put options. For both types of options, we find that the magnitude of mis-pricing decreases when option moneyness moves from ITM to OTM. Furthermore, the increase in time-to-maturities also tends to decrease the magnitude of mis-pricing (see AOPE and RMSOPE results). These results contradict with the trends we observed in the BS model, and indicate that when the model price distributions failed to capture the realised market data, the shortest distances between the market prices and the MJD model price distributions are found in OTM and longer-term options.

Yun (2014) argues that when parameter values are implied from market options prices (i.e. calibrated), the option pricing performance is better than when parameter values are estimated from historical stock return data. In the next section, we further investigate the out-of-sample option pricing performance of the BS and MJD models using implied parameter values.

5.4 Out-of-sample Option Pricing with Implied Parameter Values

Implied parameter values are calibrated under the Bayesian framework using option market prices (mid-quotes). Options with different strike prices but the same maturity date are used to calibrate posterior parameter distributions $p(\lambda, \sigma, a, \zeta \mid y_{t-1})$, which are then used to price the options with the same maturity date in the next trading day. Posterior distributions of parameters estimated using the historical daily log-return data in Section 5.3 are used as prior distributions of parameters in calibrating the implied parameter values.

Jacquier and Jarrow (2000) provide a thorough discussion on the necessity of allowing model error ε_t when calibrating implied parameter values from option price data; see also Jacquier and Polson (2010) and Johannes and Polson (2010). Ideally, observed market prices shall coincide with model prices. Nevertheless, this requires perfect synchronisation when recording the option price C_t and the underlying stock price S_t, which is empirically difficult to achieve. Even small non-synchronisation errors could result in the deviation between market and model option prices. Secondly, market prices could sometimes depart from their equilibrium values due to trading noises and market imperfection. The introduced error term could effectively capture the impact of such market errors. Finally, the deviation between market and model option prices could also stem from model structure uncertainty. Models are only approximations of the underlying asset pricing problem. Ignorance of the model imperfection issue could result in over-fitting, and consequently lead to the deterioration in model out-of-sample pricing performance as indicated by Dumas et al. (1998). Therefore, the model error term ε_t can also act as a vehicle to deal with model structure risk. Following Jacquier and Jarrow (2000), a Gaussian model error term is introduced in a multiplicative manner to ensure the non-negativity of option prices:

$$log(C_t) = log(m_t) + \varepsilon_t \tag{6}$$

where C_t is the market price of a European option, m_t is the model option price, and ε_t is a normally distributed error term with distribution $N(0, \sigma_\varepsilon^2)$.

Table 8: Implied Parameters - Out-of-sample S&P 500 Index European Option Pricing Coverage Performance Difference of the Black-Scholes and Merton's Jump-Diffusion Models

	Call Option		Put Option	
	Mid-quote Coverage Difference	Bid/Ask-quote Coverage Difference	Mid-quote Coverage Difference	Bid/Ask-quote Coverage Difference
All Options	14.98%	10.38%	10.38%	8.05%
ITM	27.42%	18.24%	-8.29%	-9.09%
ATM	0.46%	3.22%	21.61%	17.01%
OTM	-10.85%	-7.34%	14.15%	11.83%
<60 days	12.09%	6.45%	9.09%	7.06%
60 - 180 days	18.10%	15.38%	10.59%	8.05%
> 180 days	24.09%	16.79%	24.09%	19.71%

Note: out-of-sample test period: 01/08/2014 - 31/08/2014; Option market price (mid-quote) data: S&P 500 index European call and put options 01/08/2014 - 31/08/2014. Parameters are calibrated using the option data with same maturity date; calibrated posterior parameter results are used to price options with the same maturity date one day ahead. Mid-quote Coverage and Bid/Ask-quote Coverage account for the % of realised market option prices covered by the 95% credibility intervals of posterior model price distributions. The table reports the differences of coverage rates between the MJD and the BS model (i.e. MJD coverage rate - BS coverage rate).

The statistics of the out-of-sample option pricing performance are tabulated in Table 8, 9 and 10. Comparing with the results in Section 5.3, the results by calibration show improvements in both Mid-quote coverage and Bid/Ask-quote coverage for both of the models overall. A remarkable change is that, for call options, the advantage of the MJD model is now more

Table 9: Implied Parameters - Out-of-sample S&P 500 Index European Call Option Pricing Error Performance of the Black-Scholes and Merton's Jump-Diffusion Models

Panel A. BS - S&P 500 Index European Call Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	-4.80	6.21	9.05	-3.42	4.28	6.80
ITM	-7.48	7.58	10.55	-5.38	5.41	8.02
ATM	-2.12	3.93	6.03	-1.32	2.29	4.23
OTM	1.08	3.84	5.32	0.79	2.38	3.68
<60 days	-1.76	3.46	4.16	-1.22	2.33	3.16
60-180 days	-6.97	8.11	9.79	-4.94	5.54	7.32
>180 days	-23.45	23.71	27.17	-17.28	17.34	20.44

Panel B. MJD - S&P 500 Index European Call Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	0.38	4.68	5.91	0.56	2.28	3.46
ITM	-2.70	4.07	5.38	-1.10	1.62	2.76
ATM	4.55	4.77	5.77	2.49	2.50	3.67
OTM	6.37	6.38	7.30	4.00	4.00	4.84
<60 days	1.23	3.43	4.34	1.07	1.91	2.94
60-180 days	-0.61	5.89	6.87	-0.04	2.59	3.74
>180 days	-1.88	9.80	11.30	-0.77	4.05	5.89

Note: out-of-sample test period: 01/08/2014 - 31/08/2014; Option market price (mid-quote) data: S&P 500 index European call options 01/08/2014 - 31/08/2014. Parameters are calibrated using the option data with same maturity date; calibrated posterior parameter results are used to price options with the same maturity date one day ahead. PE (Pricing Error): the average of (posterior mean price - market mid-quote); APE: Absolute Pricing Error; RMSPE: Root-mean-square Pricing Error; OPE (Outside Pricing Error): the average of differences between the posterior 2.5% or 97.5% credibility interval bound values and market prices (mid-quote) when the market prices are not covered by the 95% intervals of model prices; AOPE: Absolute Outside Pricing Error; RMSOPE: Root-mean-square Outside Pricing Error.

significant in pricing ITM options, whereas the advantage in ATM and OTM options remains for put options. Among the three different time-to-maturities, the MJD results improved most significantly over the BS results in pricing longer term options. The highest coverage rate of the longer term options is observed in the MJD put option pricing.

Moreover, the PE, APE, RMSPE, OPE, AOPE and RMSOPE statistics indicate significant improvements for the MJD model, suggesting that the MJD model works better with calibration. Unlike the results shown in Section 5.3, the MJD model now has a similar or even better performance indicated by these measures in all option categories. In addition, the mis-pricing tendencies across different option categories of the BS model remain the same as in Section

Table 10: Implied Parameters - Out-of-sample S&P 500 Index European Put Option Pricing Error Performance of the Black-Scholes and Merton's Jump-Diffusion Models

Panel A. BS - S&P 500 Index European Put Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	-4.03	5.76	8.36	-3.44	4.43	6.93
ITM	1.62	4.51	5.76	1.14	2.43	3.55
ATM	-1.74	4.29	6.08	-1.19	2.53	4.15
OTM	-6.53	6.55	9.51	-5.57	5.58	8.23
$<$60 days	-1.56	3.34	4.09	-1.45	2.63	3.41
60-180 days	-5.62	7.35	8.96	-4.75	5.57	7.41
$>$180 days	-20.52	21.76	24.85	-16.60	16.64	20.62

Panel B. MJD - S&P 500 Index European Put Option

	PE	APE	RMSPE	OPE	AOPE	RMSOPE
All Options	-1.05	3.98	5.18	-0.88	2.24	3.29
ITM	4.82	5.22	6.34	2.58	2.64	3.53
ATM	1.56	2.68	3.23	0.50	0.85	1.44
OTM	-3.72	3.87	5.11	-2.41	2.43	3.52
$<$60 days	-0.60	2.50	3.03	-0.46	1.57	2.11
60-180 days	-1.59	5.47	6.37	-1.37	3.00	4.10
$>$180 days	-2.11	9.61	11.23	-1.95	4.00	6.17

Note: out-of-sample test period: 01/08/2014 - 31/08/2014; Option market price (mid-quote) data: S&P 500 index European put options 01/08/2014 - 31/08/2014. Parameters are calibrated using the option data with same maturity date; calibrated posterior parameter results are used to price options with the same maturity date one day ahead. PE (Pricing Error): the average of (posterior mean price - market mid-quote); APE: Absolute Pricing Error; RMSPE: Root-mean-square Pricing Error; OPE (Outside Pricing Error): the average of differences between the posterior 2.5% or 97.5% credibility interval bound values and market prices (mid-quote) when the market prices are not covered by the 95% intervals of model prices; AOPE: Absolute Outside Pricing Error; RMSOPE: Root-mean-square Outside Pricing Error.

5.3. On the other hand, the mis-pricing tendencies of the MJD model has changed to follow the similar trends as the BS model. Both under- and over-pricing are observed in different option categories, and the tendency moves from under-pricing to over-pricing with increases in strike prices. Furthermore, options with the longest time-to-maturities now have the largest mis-pricing magnitude.

Overall, after incorporating parameter estimation risk, the MJD model outperforms the BS model in all aspects of the out-of-sample option pricing tests when implied parameters are used. In the next section, we will discuss the measure of parameter estimation risk exposure.

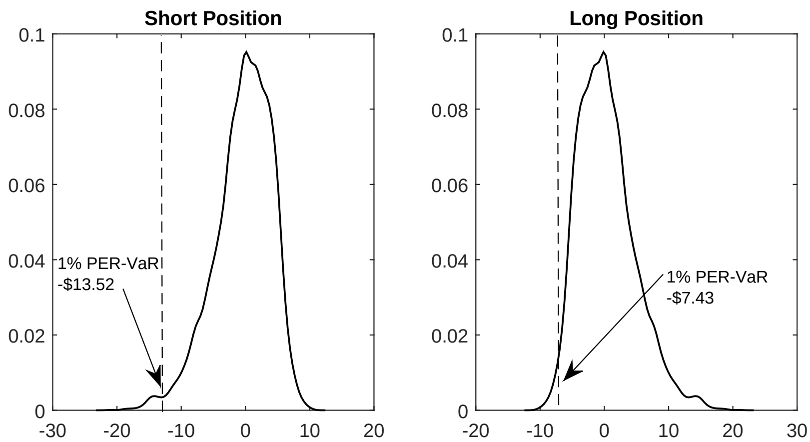
5.5 Measuring Parameter Estimation Risk

With the ability to obtain the entire posterior distributions of option prices, parameter estimation risk exposure to both long and short positions can be easily investigated. A simple VaR-type of measure to quantify parameter estimation risk is recently introduced by Tunaru (2015).

Regarding the MJD model, the posterior density of an option price V_t^{Merton} is obtained using MCMC techniques. The distributions of profit & loss for both long and short positions, given the trading price P_t, can be computed by simple linear loss functions shown below:

$$
\begin{aligned}
L^{long}(V_t^{Merton}) &= V_t^{Merton} - P_t \\
L^{short}(V_t^{Merton}) &= P_t - V_t^{Merton}
\end{aligned} \tag{7}
$$

Figure 3: An Example of PER-VaR of Parameter Estimation Risk Exposure under the Merton's Jump-Diffusion Option Pricing Model



Note: the figure shows 1% PER-VaR values for the parameter estimation risk exposure of both the short and long positions of a European Call option of the S&P 500 index under the MJD option pricing model, assuming that the mean of the posterior option price $27.67 is set as the trading price. The call option is an OTM option on 01/08/2014 with strike price 2000 and time-to-maturity 141 days. Parameter estimation data: S&P 500 index daily log-return data 31/07/2012 - 31/07/2014.

Denoting by $PER-VaR_{\eta}$, the measure of parameter estimation risk tells that the probability of having a loss beyond $PER-VaR_{\eta}$ due to parameter estimation risk is $\eta\%$. An example of the $PER-VaR_{1\%}$ for both long and short positions of a European call option computed using the posterior MJD model price distribution is given in Figure 3. It is important to notice that buyers and sellers do not have symmetric exposure towards parameter estimation risk. As illustrated in Figure 3, the magnitude of the $PER-VaR_{1\%}$ is smaller to the call option contract buyers than to the sellers. Furthermore, with the ability to visualised the entire distribution of potential loss caused by parameter estimation risk, market participants can easily assess the entire tail distribution beyond $PER - VaR_{\eta}$. In our example, a fatter tail is found in the parameter estimation risk exposure of short position holders, indicating that the potential loss which may be encountered by the sellers in extreme cases could be more tremendous. This is not a surprise as long position holders of European call options do not face downside variation in their final payoff.

Finally, it is worth highlighting that, in this one particular example, the $PER - VaR_{1\%}$ accounts for 48.86% and 26.85% of the trading price for option writers and buyers respectively. Therefore, parameter estimation risk can be substantial to all market participants. Neglecting parameter estimation risk when using either model in option pricing may place a blind spot in risk management analysis and result in great losses.

5.6 Greeks

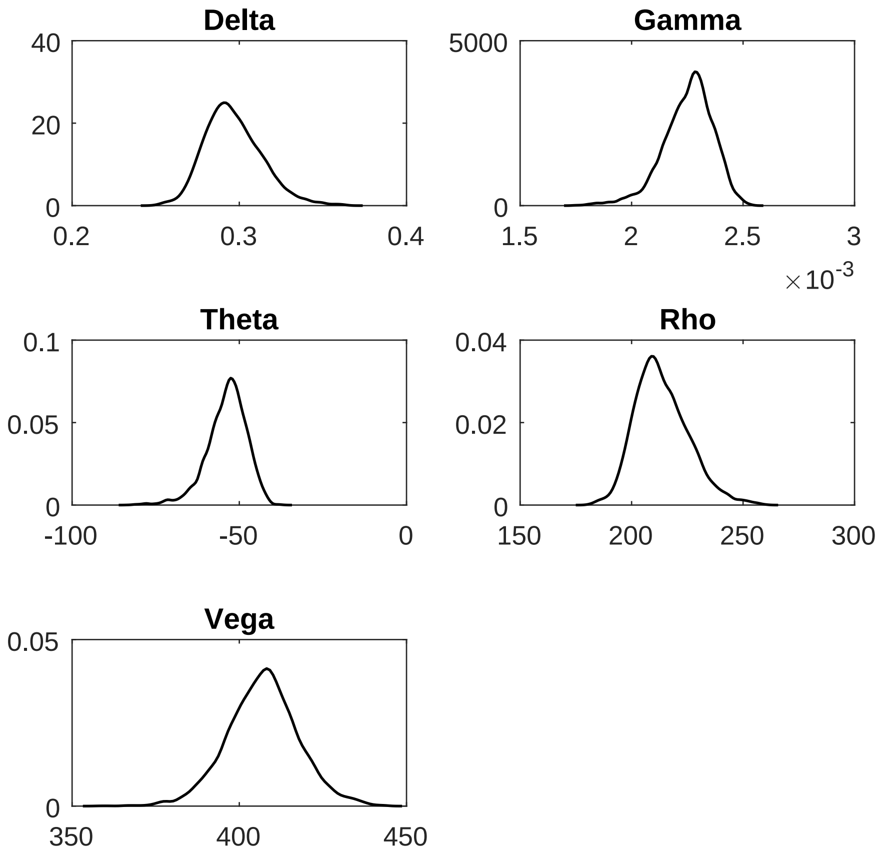
Parameter estimation risk can also bring significant impact to option hedging activities. Following a similar approach as formula (3), the Greeks under the MJD model can be derived as the weighted average of the BS Greeks by the probability that i jumps would occur before the maturity date (Merton, 1976).

$$\begin{aligned}
Delta_t^{Merton} &= \sum_{i=0}^{\infty} \frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!} Delta^{BS}(S_i, \sigma_i, \delta, T-t, r, K) \\
Gamma_t^{Merton} &= \sum_{i=0}^{\infty} \frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!} Gamma^{BS}(S_i, \sigma_i, \delta, T-t, r, K) \\
Theta_t^{Merton} &= \sum_{i=0}^{\infty} \frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!} Theta^{BS}(S_i, \sigma_i, \delta, T-t, r, K) \\
Rho_t^{Merton} &= \sum_{i=0}^{\infty} \frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!} Rho^{BS}(S_i, \sigma_i, \delta, T-t, r, K) \\
Vega_t^{Merton} &= \sum_{i=0}^{\infty} \frac{e^{-\lambda(T-t)}[\lambda(T-t)]^i}{i!} Vega^{BS}(S_i, \sigma_i, \delta, T-t, r, K)
\end{aligned} \tag{8}$$

$$S_i \equiv S_t exp\left\{ ia + \frac{i\zeta^2}{2} - \lambda(e^{a+\frac{\zeta^2}{2}} - 1)(T-t) \right\}, \quad \sigma_i \equiv \sqrt{\sigma^2 + \frac{i\zeta^2}{T-t}}$$

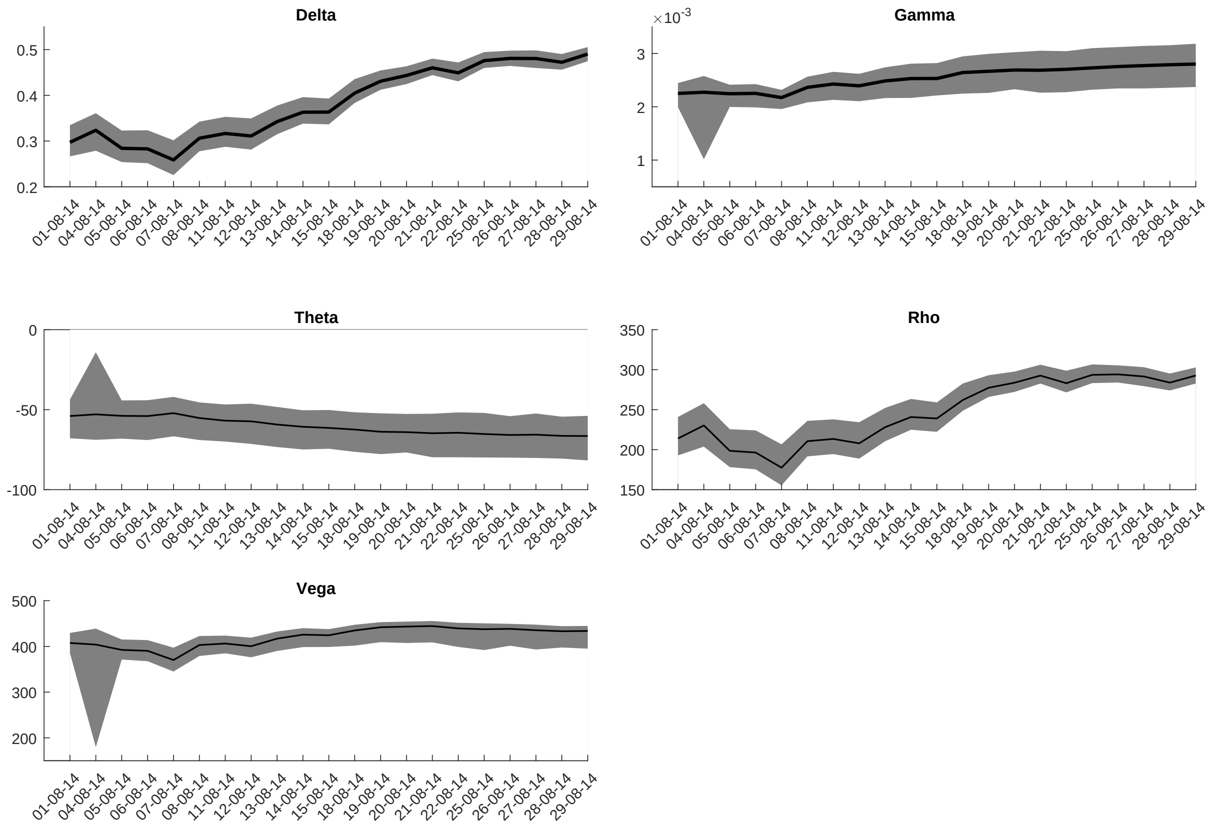
Given the estimated $p(\Theta \mid y_{t-1})$, the posterior distributions of Greeks can be easily obtained following the simulation techniques described in Section 4. Examples of Greeks' density plots of a S&P 500 European OTM call option are presented in Figure 4. Density plots and the ability of assessing the quantile values of Greeks provide very rich information to practitioners in hedging activities. While the posterior mean value of Delta could be adopted, practitioners shall also be aware that they are subject to parameter estimation risk when setting up their hedging strategy. The Bayesian posterior distribution of Delta depicts such risk exposure in detail. In our example a fatter right tail is observed showing a higher probability of realising higher Delta value. In the case when extra caution is needed towards the trading of this option, short position holders may hedge at the 75% or even the 97.5% upper quantile value of Delta to implement super hedging according to their capital availability. Similar discussions also apply to other Greeks hedging activities.

Figure 4: Posterior Distributions of Greeks of a European Call Option under the Merton's Jump-Diffusion Model



Note: the figure shows the posterior distributions of Delta, Gamma, Theta, Rho and Vega of a European Call option of the S&P 500 index given the estimated posterior distributions of parameters under the MJD model . The call option is an OTM option on 01/08/2014 with strike price 2000 and time-to-maturity 141 days. Parameter estimation data: S&P 500 index daily log-return data 31/07/2012 - 31/07/2014.

Figure 5: Movements of a European Call Option Greeks under the Merton's Jump-Diffusion Model during August 2014



Note: the figure shows the movements of posterior means, 95% credibility intervals of Delta, Gamma, Theta, Rho and Vega of a European Call option of the S&P 500 index given the estimated posterior distributions of parameters under the MJD model. The call option with strike price 2000 and maturity date 20/12/2014 is an OTM option on 01/08/2014, but becomes an ATM option on 19/08/2014 due to the increase in security spot price and remains ATM till the end of the test period. Parameter estimation data: S&P 500 index daily log-return data 31/07/2012 - 29/08/2014.

For risk management purposes, one can track the evolutions of the posterior means of Greeks as well as the upper and lower quantiles as illustrated in Figure 5. The potential risk or hedging errors generated by parameter estimation risk can be easily gauged with our approach. The 95% credibility interval widths are relatively stable for Delta and Rho throughout the test period. However, the Gamma, Theta and Vega plots all exhibit an increased uncertainty at the second estimation point. Moreover, the 95% credibility interval of Gamma is widened towards the end of the test period, while the distribution of Vega gradually become negative skewed. If only mean estimation value is considered under traditional estimation practices, traders might lose sight on such potential uncertainty and underestimate the movement of the Greeks.

6 Credit Risk Management with Parameter Estimation Risk

When using a model for risk management, adopting point estimation of parameters could result in under-estimation or over-estimation of risk. Lönnbark (2013) remarks that biases caused by parameter estimation risk could affect backtesting results of the model, and consequently affect regulation compliance. The Bayesian approach provides a direct way to deal with this issue. The probability of default (PD), the main concept in credit risk, can be reported with its full posterior distribution. We will demonstrate how to achieve this for the Merton's Credit Risk model.

In the Merton's Credit Risk model, the firm value F_t follows a Geometric Brownian Motion. If L is the notional of debt at maturity, the probability of default is given by $\Phi(-d_2)$

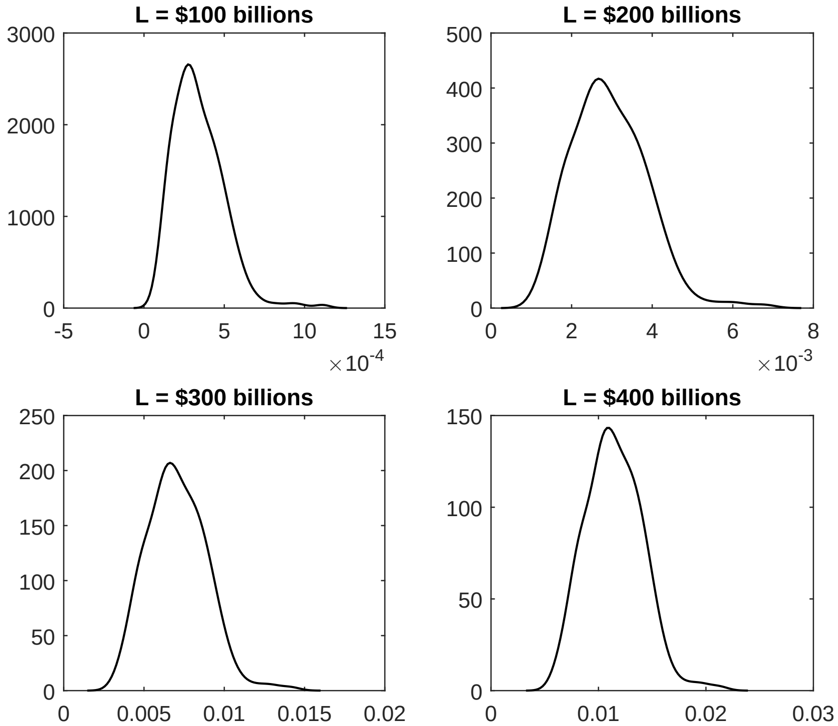
$$d_2 = \frac{ln(F_t/L) + (r - \sigma_f^2/2)(T-t)}{\sigma_f\sqrt{T-t}} \tag{9}$$

where r is the risk free rate and σ_f is the volatility of the firm value. While equity value is a function of the asset value, the equity volatility σ_e can be derived from the asset volatility σ_f using Itô's lemma. Consequently, F_t and σ_f can be calculated given the equity value and volatility; see Hull et al. (2004) and Merton (1974) for more details.

$$\begin{aligned} \sigma_e &= \frac{\sigma_f \Phi(d_1)}{\Phi(d_1) - A_t\Phi(d_2)} \\ A_t &= \frac{Le^{-r(T-t)}}{F_t} \end{aligned} \tag{10}$$

In the following example, the PD of Apple Inc. on 21/08/2014 is computed following the Bayesian approach. The equity value of Apple Inc. on the date is reported at \$589.40 billions. The parameters of the Merton's Credit Risk model are estimated using daily log-return data of Apple Inc. from 21/08/2012 to 21/08/2014. Figure 6 shows the PD of Apple Inc. for a time horizon of 5 years with L = \$100, \$200, \$300 or \$400 billions respectively. The density plots show that parameter estimation risk is well captured and incorporated into the computation of the PDs. The impact of parameter estimation risk becomes more critical when the company has its debt principal repayment equals to \$300 or \$400 billions. The PD of the company could vary from 0.25% to 1.5% when $L = \$300$ billions, and from 0.5% to 2.3% when $L = \$400$ billions. When practitioners only adopt point estimation values of the PDs (e.g. posterior means), they fail to gauge the uncertainty of PD values. This can result in biases and errors in counterparty credit risk calculations, regulatory capital computations, risk management and investment decision making towards the company.

Figure 6: Probability of Default Posterior Distributions of Apple Inc.



Note: the figure shows the posterior distributions of default probabilities for Apple Inc. given different notional values of debt L on 21/08/2014. The equity value of Apple Inc. on the date is $589.40 billions. Debt is assumed to have a time-to-maturity of 5 years. Parameter estimation data: Apple Inc. daily log-return data 21/08/2012 - 21/08/2014.

7 Conclusion

In this paper, we demonstrate how parameter estimation risk can be incorporated into option pricing and credit risk models via the application of Bayesian econometrics using MCMC techniques. The option pricing performance of the MJD and BS models are also investigated carefully when parameters are estimated or calibrated.

Parameter estimation risk is non-trivial, and adopting point estimations in both asset pricing and risk management can result in a narrow view about the underlying issue with a huge amount of information being neglected. It also prevents practitioners to receive early signal of market variation as reflected in the parameter uncertainty. As a result, actions towards trading, hedging, regulatory requirement and risk management can be delayed, which may trigger the loss one may suffer in a later stage.

Regarding the European option pricing performance of the MJD and BS models, it is found that the MJD model with posterior parameter values implied from market option prices outperforms the BS model in all performance measures, especially for longer term options. Nevertheless, the MJD model does generate more parameter estimation risk due to the increase in the number of parameters. This is reflected in the widened intervals of the MJD model price distributions compared with the BS model.

On the other hand, even when accounting for parameter estimation risk, neither of the models is able to provide full coverage of realised future market values. Such deviation could stem from model misspecification, or in other words, model structure risk. Furthermore, our Bayesian MCMC approach allows us to easily calculate Tunaru's measure of parameter estimation risk for European option pricing. Sellers of options have more exposure to this type of risk than buyers. In addition, we highlight the potential biases when ignoring parameter estimation risk in the calculation of default probabilities under the Merton's Credit Risk model.

Our methodology can be used by market-makers to define the bid-ask spread on assets that they need to contribute prices daily. If a chosen model provides the market-maker with a fair no-arbitrage price let say, then the confidence interval determined with the methodology highlighted in this paper can be used, at various confidence intervals, to determine a bid-ask price range containing the fair price. Economic theory places constraints on parameter values for many problems encountered in finance. Bayesian MCMC techniques provide an elegant solution to handle these problems that would be very cumbersome otherwise. While regulatory bodies require model risk to be accounted for as any other type of risk, practically speaking this is not easy to do with standard methods of inference. Constructing from model parameter estimation uncertainty an entire distribution of feasible price values of an asset allows a more insightful risk management to be performed. Our research could be expanded to a portfolio or multidimensional set-up and towards other important research questions such as the evolution of the risk premium in various markets.

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