Computer Science at Kent

Proof of New Decompositional Results for Generalized Symmetries

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Technical Report No: 05-06
Date: August 2006

Published by the Computing Laboratory,
University of Kent, Canterbury, Kent CT2 7NF, UK.

Abstract

This note provides proof of some new decompositional results for generalized symmetries in ROBDDs. These results permit generalized symmetries to be found with algorithms that are formulated as a series of passes one for each variable in the ROBDD, which is the basis of the anytime approach to symmetry detection.

Proposition 1. *An ROBDD f over a set of variables $X = \{x_1, \ldots, x_n\}$ is T_1-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *every ROBDD rooted at a node labeled x_i is T_1-symmetric in (x_i, x_j) and,*
- *every path from the root to a node labeled x_j passes through a node labeled x_i.*

Proof.

- Consider the *if* direction.
  - Since f is T_1-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ hence $g|_{x_i \leftarrow 1, x_j \leftarrow 0} = g|_{x_i \leftarrow 0, x_j \leftarrow 1}$.
  - Suppose for the sake of a contradiction that there exists a path from the root to a node labeled x_j that does not pass through a node labeled x_i. Thus, let $g = f(\mathbf{b}_1, 0, \mathbf{b}_2, x_j, \ldots, x_n) = f(\mathbf{b}_1, 1, \mathbf{b}_2, x_j, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$ and $\mathbf{b}_2 \in Bool^{j-i-1}$. Thus $g|_{x_j \leftarrow 0}(\mathbf{b}_3) = g|_{x_j \leftarrow 1}(\mathbf{b}_3)$ for all $\mathbf{b}_3 \in Bool^{n-j}$. Hence $g|_{x_j \leftarrow 0} = g|_{x_j \leftarrow 1}$ which is a contradiction since g is reduced.
- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$. Thus g is labeled x_i, hence there exists some $\mathbf{b}_2$ and $\mathbf{b}_3$ such that $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3) = 0$ as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) \neq h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ case follows analogously. □

Proposition 2. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_2-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *every ROBDD rooted at a node labeled x_i is T_2-symmetric in (x_i, x_j) and,*
- *every path from the root to a node labeled x_j passes through a node labeled x_i.*

Proof.

- Consider the *if* direction.
  - Since f is T_2-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ hence $g|_{x_i \leftarrow 0, x_j \leftarrow 0} = g|_{x_i \leftarrow 1, x_j \leftarrow 1}$.
  - Suppose for the sake of a contradiction that there exists a path from the root to a node labeled x_j that does not pass through a node labeled x_i. Thus, let $g = f(\mathbf{b}_1, 0, \mathbf{b}_2, x_j, \ldots, x_n) = f(\mathbf{b}_1, 1, \mathbf{b}_2, x_j, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$ and $\mathbf{b}_2 \in Bool^{j-i-1}$. Thus $g|_{x_j \leftarrow 0}(\mathbf{b}_3) = g|_{x_j \leftarrow 1}(\mathbf{b}_3)$ for all $\mathbf{b}_3 \in Bool^{n-j}$. Hence $g|_{x_j \leftarrow 0} = g|_{x_j \leftarrow 1}$ which is a contradiction since g is reduced.

- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$. Thus g is labeled x_i, hence there exists some $\mathbf{b}_2$ and $\mathbf{b}_3$ such that $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3) = 0$ as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) \neq h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  The $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ case follows analogously. □

Proposition 3. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_3-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *if whenever an ROBDD g occurs in f at a node labeled x_i then $g|_{x_i \leftarrow 0}$ does not contain a node labeled x_j and,*
- *every path from the root of f to a node labeled x_j passes through a node labeled x_i.*

Proof.

- Consider the *if* direction.
  - Since f is T_3-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ and $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n)$ thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$. Hence $h|_{x_j \leftarrow 0} = h|_{x_j \leftarrow 1}$, and thus it follows that the ROBDD h cannot be labeled with a variable x_j because h is reduced.
  - Suppose for the sake of a contradiction that there exists a path from the root of f to a node g labeled x_j that does not pass through a node labeled x_i. Since the path does not pass through x_i, let $g = f(\mathbf{b}_1, x_j, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{j-1}$. Since $f|_{x_i \leftarrow 0, x_j \leftarrow 0} = f|_{x_i \leftarrow 0, x_j \leftarrow 1}$ it follows that $g|_{x_j \leftarrow 0}(\mathbf{b}_2) = g|_{x_j \leftarrow 1}(\mathbf{b}_2)$ for all $\mathbf{b}_2 \in Bool^{n-j}$. Hence $g|_{x_j \leftarrow 0} = g|_{x_j \leftarrow 1}$, it follows that the ROBDD g cannot be labeled with a variable x_j because g is reduced, hence a contradiction follows.
- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$. Hence g is labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = 1$ and $h|_{x_j \leftarrow 1}(\mathbf{b}_3) = 0$. Hence $g|_{x_i \leftarrow 0}$ contains a node h labeled x_j as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = 1$ and $h|_{x_j \leftarrow 1}(\mathbf{b}_3) = 0$. Hence g contains a node labeled x_j and g is not labeled x_i as required.

  The $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ case follows analogously. □

Proposition 4. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_4-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *if whenever an ROBDD g occurs in f at a node labeled x_i then $g|_{x_i \leftarrow 1}$ does not contain a node labeled x_j and,*
- *every path from the root of f to a node labeled x_j passes through a node labeled x_i.*

Proof.

- Consider the *if* direction.
  - Since f is T_4-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ and $h = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$ thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$. Hence $h|_{x_j \leftarrow 0} = h|_{x_j \leftarrow 1}$, and thus it follows that the ROBDD h cannot be labeled with a variable x_j because h is reduced.
  - Suppose for the sake of a contradiction that there exists a path from the root of f to a node g labeled x_j that does not pass through a node labeled x_i. Since the path does not pass through x_i, let $g = f(\mathbf{b}_1, x_j, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{j-1}$. Since $f|_{x_i \leftarrow 1, x_j \leftarrow 0} = f|_{x_i \leftarrow 1, x_j \leftarrow 1}$ it follows that $g|_{x_j \leftarrow 0}(\mathbf{b}_2) = g|_{x_j \leftarrow 1}(\mathbf{b}_2)$ for all $\mathbf{b}_2 \in Bool^{n-j}$. Hence $g|_{x_j \leftarrow 0} = g|_{x_j \leftarrow 1}$, it follows that the ROBDD g cannot be labeled with a variable x_j because g is reduced, hence a contradiction follows.
- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$. Hence g is labeled x_i. Let $h = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = 1$ and $h|_{x_j \leftarrow 1}(\mathbf{b}_3) = 0$. Hence $g|_{x_i \leftarrow 0}$ contains a node h labeled x_j as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = 1$ and $h|_{x_j \leftarrow 1}(\mathbf{b}_3) = 0$. Hence g contains a node labeled x_j and g is not labeled x_i as required.

  The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ case follows analogously. □

Proposition 5. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_5-symmetric in the pair (x_i, x_j) and $i < j$ iff every ROBDD rooted at a node labeled x_i is T_5-symmetric in (x_i, x_j).*

Proof.

- Consider the *if* direction. Suppose g is an ROBDD rooted at a node labeled x_i. Then $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$. Since f is T_5-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$, thus $g|_{x_i \leftarrow 1, x_j \leftarrow 0} = g|_{x_i \leftarrow 0, x_j \leftarrow 0}$.
- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$. Suppose for the sake of a contradiction that $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Observe $g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$ and hence $g(0, \mathbf{b}_2, 0, \mathbf{b}_3) = g(1, \mathbf{b}_2, 0, \mathbf{b}_3)$ which is a contradiction. Therefore $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$ and g is labeled by x_i. Hence there exists some $\mathbf{b}_2$ and $\mathbf{b}_3$ such that $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3) = 0$ as required. The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ case follows analogously. □

Proposition 6. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_6-symmetric in the pair (x_i, x_j) and $i < j$ iff every ROBDD rooted at a node labeled x_i is T_6-symmetric in (x_i, x_j).*

Proof.

- Consider the *if* direction. Suppose g is an ROBDD rooted at a node labeled x_i. Then $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$. Since f is T_6-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$, thus $g|_{x_i \leftarrow 0, x_j \leftarrow 1} = g|_{x_i \leftarrow 1, x_j \leftarrow 1}$.

- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$. Suppose for the sake of a contradiction that $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Observe $g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$ and hence $g(0, \mathbf{b}_2, 1, \mathbf{b}_3) = g(1, \mathbf{b}_2, 1, \mathbf{b}_3)$ which is a contradiction. Therefore $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$ and g is labeled by x_i. Hence there exists some $\mathbf{b}_2$ and $\mathbf{b}_3$ such that $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3) = 0$ as required. The $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ case follows analogously. □

Proposition 7. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_7-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *every ROBDD rooted at a node labeled x_i is T_7-symmetric in (x_i, x_j) and,*
- *every path from the root of f to a node h labeled x_j which does not visit a node labeled x_i, satisfies the property that $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$.*

Proof.

- Consider the *if* direction.
  - Suppose g is an ROBDD rooted at a node labeled x_i. Then $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$. Since f is T_7-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = \neg f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$, hence $g|_{x_i \leftarrow 1, x_j \leftarrow 0} = \neg g|_{x_i \leftarrow 0, x_j \leftarrow 1}$.
  - Arguing by the contrapositive, suppose there exists a path from the root of f to a node h labeled x_j which does not visit a node labeled x_i and $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$. Thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) \neq \neg h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ for some $\mathbf{b}_3 \in Bool^{n-j}$. Let $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $a \in Bool$ such that $h = f(\mathbf{b}_1, a, \mathbf{b}_2, x_j, \ldots, x_n)$. Therefore, $f(\mathbf{b}_1, a, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, a, \mathbf{b}_2, 1, \mathbf{b}_3)$. Since the path does not visit x_i, $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3)$ as required.

- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$ and g is labeled by x_i. Hence there exists some $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3) = 1$ as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$ case follows analogously. □

Proposition 8. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_8-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *every ROBDD rooted at a node labeled x_i is T_8-symmetric in (x_i, x_j) and,*
- *every path from the root of f to a node h labeled x_j which does not visit a node labeled x_i, satisfies the property that $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$.*

Proof.

- Consider the *if* direction.
  - Suppose g is an ROBDD rooted at a node labeled x_i. Then $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$. Since f is T_8-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = \neg f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$, hence $g|_{x_i \leftarrow 0, x_j \leftarrow 0} = \neg g|_{x_i \leftarrow 1, x_j \leftarrow 1}$.
  - Arguing by the contrapositive, suppose there exists a path from the root of f to a node h labeled x_j which does not visit a node labeled x_i and $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$. Thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) \neq \neg h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ for some $\mathbf{b}_3 \in Bool^{n-j}$. Let $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $a \in Bool$ such that $h = f(\mathbf{b}_1, a, \mathbf{b}_2, x_j, \ldots, x_n)$. Therefore, $f(\mathbf{b}_1, a, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, a, \mathbf{b}_2, 1, \mathbf{b}_3)$. Since the path does not visit x_i, $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ as required.
- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$ and g is labeled by x_i. Hence there exists some $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3) = 1$ as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  The $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$ case follows analogously. □

Proposition 9. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_9-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *if whenever an ROBDD g occurs in f at a node labeled x_i then every path through $g|_{x_i \leftarrow 0}$ visits a node h labeled x_j such that $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$ and,*
- *every path from the root of f to a node h labeled x_j which does not visit a node labeled x_i, satisfies the property that $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$.*

Proof.

- Consider the *if* direction.
  - Since f is T_9-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = \neg f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ and $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n)$ thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = \neg h|_{x_j \leftarrow 1}(\mathbf{b}_3)$. Hence $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$.

– Arguing by the contrapositive, suppose there exists a path from the root of f to a node h labeled x_j which does not visit a node labeled x_i and $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$. Thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) \neq \neg h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ for some $\mathbf{b}_3 \in \mathit{Bool}^{n-j}$. Let $\mathbf{b}_1 \in \mathit{Bool}^{i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{j-i-1}$ and $a \in \mathit{Bool}$ such that $h = f(\mathbf{b}_1, a, \mathbf{b}_2, x_j, \ldots, x_n)$. Therefore, $f(\mathbf{b}_1, a, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, a, \mathbf{b}_2, 1, \mathbf{b}_3)$. Since the path does not visit x_i, $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3)$ as required.

- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in \mathit{Bool}^{i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{j-i-1}$ and $\mathbf{b}_3 \in \mathit{Bool}^{n-j}$ such that $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.

  – Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$. Hence g is labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n)$. Then $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  – Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ case follows analogously. □

Proposition 10. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_{10}-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *if whenever an ROBDD g occurs in f at a node labeled x_i then every path through $g|_{x_i \leftarrow 1}$ visits a node h labeled x_j such that $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$ and,*
- *every path from the root of f to a node h labeled x_j which does not visit a node labeled x_i, satisfies the property that $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$.*

Proof.

- Consider the *if* direction.

  – Since f is T_{10}-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = \neg f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_1 \in \mathit{Bool}^{i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{j-i-1}$ and $\mathbf{b}_3 \in \mathit{Bool}^{n-j}$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ and $h = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$ thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = \neg h|_{x_j \leftarrow 1}(\mathbf{b}_3)$. Hence $h|_{x_j \leftarrow 0} = \neg h|_{x_j \leftarrow 1}$.

  – Arguing by the contrapositive, suppose there exists a path from the root of f to a node h labeled x_j which does not visit a node labeled x_i and $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$. Thus $h|_{x_j \leftarrow 0}(\mathbf{b}_3) \neq \neg h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ for some $\mathbf{b}_3 \in \mathit{Bool}^{n-j}$. Let $\mathbf{b}_1 \in \mathit{Bool}^{i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{j-i-1}$ and $a \in \mathit{Bool}$ such that $h = f(\mathbf{b}_1, a, \mathbf{b}_2, x_j, \ldots, x_n)$. Therefore, $f(\mathbf{b}_1, a, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, a, \mathbf{b}_2, 1, \mathbf{b}_3)$. Since the path does not visit x_i, $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) \neq \neg f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ as required.

- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in \mathit{Bool}^{i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{j-i-1}$ and $\mathbf{b}_3 \in \mathit{Bool}^{n-j}$ such that $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.

  – Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$. Hence g is labeled x_i. Let $h = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Then $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

  – Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$. Hence g is not labeled x_i. Let $h = g(0, \mathbf{b}_2, x_j, \ldots, x_n) = g(1, \mathbf{b}_2, x_j, \ldots, x_n)$. Observe $h|_{x_j \leftarrow 0} \neq \neg h|_{x_j \leftarrow 1}$ since $h|_{x_j \leftarrow 0}(\mathbf{b}_3) = h|_{x_j \leftarrow 1}(\mathbf{b}_3)$ as required.

The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$ case follows analogously. $\square$

Proposition 11. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_{11}-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *every ROBDD rooted at a node labeled x_i is T_{11}-symmetric in (x_i, x_j) and,*
- *every path from the root of f passes through a node labeled x_i.*

Proof.

- Consider the *if* direction.
  - Suppose g is an ROBDD rooted at a node labeled x_i. Then $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$. Since f is T_{11}-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = \neg f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3)$ for all $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$, hence $g|_{x_i \leftarrow 1, x_j \leftarrow 0} = \neg g|_{x_i \leftarrow 0, x_j \leftarrow 0}$.
  - Suppose for the sake of a contradiction that there exists a path from the root of f to a node labeled x_j which does not pass through a node labeled x_i. Let $g = f(\mathbf{b}_1, x_j, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{j-1}$. Since $f|_{x_i \leftarrow 1, x_j \leftarrow 0} = \neg f|_{x_i \leftarrow 0, x_j \leftarrow 0}$ it follows that $g|_{x_j \leftarrow 0}(\mathbf{b}_2) = \neg g|_{x_j \leftarrow 0}(\mathbf{b}_2)$ for all $\mathbf{b}_2 \in Bool^{n-j}$. Hence $g|_{x_j \leftarrow 0} = \neg g|_{x_j \leftarrow 0}$ which is a contradiction.
- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 1$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$ and g is labeled by x_i. Hence there exists some $\mathbf{b}_2$ and $\mathbf{b}_3$ such that $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3) = 1$ as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$ then g is not labeled by x_i as required.

The $f(\mathbf{b}_1, 1, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 0, \mathbf{b}_2, 0, \mathbf{b}_3) = 0$ case follows analogously. $\square$

Proposition 12. *An ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_{12}-symmetric in the pair (x_i, x_j) and $i < j$ iff*

- *every ROBDD rooted at a node labeled x_i is T_{12}-symmetric in (x_i, x_j) and,*
- *every path from the root of f passes through a node labeled x_i.*

Proof.

- Consider the *if* direction.
  - Suppose g is an ROBDD rooted at a node labeled x_i. Then $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{i-1}$. Since f is T_{12}-symmetric in the variable pair (x_i, x_j), $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = \neg f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3)$ for all $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$, hence $g|_{x_i \leftarrow 0, x_j \leftarrow 1} = \neg g|_{x_i \leftarrow 1, x_j \leftarrow 1}$.
  - Suppose for the sake of a contradiction that there exists a path from the root of f to a node labeled x_j which does not pass through a node labeled x_i. Let $g = f(\mathbf{b}_1, x_j, \ldots, x_n)$ for some $\mathbf{b}_1 \in Bool^{j-1}$. Since $f|_{x_i \leftarrow 0, x_j \leftarrow 1} = \neg f|_{x_i \leftarrow 1, x_j \leftarrow 1}$ it follows that $g|_{x_j \leftarrow 1}(\mathbf{b}_2) = \neg g|_{x_j \leftarrow 1}(\mathbf{b}_2)$ for all $\mathbf{b}_2 \in Bool^{n-j}$. Hence $g|_{x_j \leftarrow 1} = \neg g|_{x_j \leftarrow 1}$ which is a contradiction.

- Consider the *only-if* direction, arguing by the contrapositive. Suppose there exists $\mathbf{b}_1 \in Bool^{i-1}$, $\mathbf{b}_2 \in Bool^{j-i-1}$ and $\mathbf{b}_3 \in Bool^{n-j}$ such that $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 1$. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$.
  - Suppose $g|_{x_i \leftarrow 0} \neq g|_{x_i \leftarrow 1}$ and g is labeled by x_i. Hence there exists some $\mathbf{b}_2$ and $\mathbf{b}_3$ such that $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3) = 1$ and $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3) = 1$ as required.
  - Suppose $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$ then g is not labeled by x_i as required.

  The $f(\mathbf{b}_1, 0, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$ and $f(\mathbf{b}_1, 1, \mathbf{b}_2, 1, \mathbf{b}_3) = 0$ case follows analogously. □

Lemma 1. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_5-symmetric in the pair (x_i, x_j) and $i < j$ then every ROBDD rooted at a node labeled x_i contains a node labeled x_j.*

Proof. For brevity, only the proof for the T_5 case is given. Suppose for the sake of a contradiction that there exists a node g labeled x_i that does not contain a node labeled x_j. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ such that g does not contain a node labeled x_j. Now $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3) = g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3)$ and since g does not include x_j it follows that $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3) = g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3)$ and $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3) = g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3)$. Thus $g|_{x_i \leftarrow 1}(\mathbf{b}_2) = g|_{x_i \leftarrow 0}(\mathbf{b}_2)$ for all $\mathbf{b}_2 \in Bool^{n-i}$. Hence $g|_{x_i \leftarrow 1} = g|_{x_i \leftarrow 0}$, it follows that the ROBDD g cannot be labeled with a variable x_i because g is reduced, hence a contradiction follows. □

Lemma 2. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_6-symmetric in the pair (x_i, x_j) and $i < j$ then every ROBDD rooted at a node labeled x_i contains a node labeled x_j.*

Proof. Suppose for the sake of a contradiction that there exists a node g labeled x_i that does not contain a node labeled x_j. Let $g = f(\mathbf{b}_1, x_i, \ldots, x_n)$ such that g does not contain a node labeled x_j. Now $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3) = g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3)$ and since g does not include x_j it follows that $g|_{x_i \leftarrow 0}(\mathbf{b}_2, 1, \mathbf{b}_3) = g|_{x_i \leftarrow 0}(\mathbf{b}_2, 0, \mathbf{b}_3)$ and $g|_{x_i \leftarrow 1}(\mathbf{b}_2, 1, \mathbf{b}_3) = g|_{x_i \leftarrow 1}(\mathbf{b}_2, 0, \mathbf{b}_3)$. Thus $g|_{x_i \leftarrow 0}(\mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_2)$ for all $\mathbf{b}_2 \in Bool^{n-i}$. Hence $g|_{x_i \leftarrow 0} = g|_{x_i \leftarrow 1}$, it follows that the ROBDD g cannot be labeled with a variable x_i because g is reduced, hence a contradiction follows. □

Lemma 3. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_7-symmetric in the pair (x_i, x_j) and $i < j$ then every ROBDD g rooted at a node labeled x_i satisfies the property that*

- *g contains a node labeled x_j or,*
- $g|_{x_i \leftarrow 0} = \neg g|_{x_i \leftarrow 1}$.

Proof. For brevity, only the proof for the T_7 case is given. To argue by contrapositive, suppose there exists a node g labeled x_i such that g does not contain a node labeled x_j and $g|_{x_i \leftarrow 0} \neq \neg g|_{x_i \leftarrow 1}$. Therefore $g|_{x_i \leftarrow 0}(\mathbf{b}_1, a, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, a, \mathbf{b}_2)$ for some $\mathbf{b}_1 \in Bool^{j-i-1}$, $\mathbf{b}_2 \in Bool^{n-j}$ and $a \in Bool$. Since g does not contain a node labeled x_j it follows that $g|_{x_i \leftarrow 1}(\mathbf{b}_1, 0, \mathbf{b}_2) = g|_{x_i \leftarrow 0}(\mathbf{b}_1, 1, \mathbf{b}_2)$. Thus f is T_7-asymmetric in the pair (x_i, x_j) as required. □

Lemma 4. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_8-symmetric in the pair (x_i, x_j) and $i < j$ then every ROBDD g rooted at a node labeled x_i satisfies the property that*

- *g contains a node labeled x_j or,*
- $g|_{x_i \leftarrow 0} = \neg g|_{x_i \leftarrow 1}$.

Proof. To argue by contrapositive, suppose there exists a node g labeled x_i such that g does not contain a node labeled x_j and $g|_{x_i \leftarrow 0} \neq \neg g|_{x_i \leftarrow 1}$. Therefore $g|_{x_i \leftarrow 0}(\mathbf{b}_1, a, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, a, \mathbf{b}_2)$ for some $\mathbf{b}_1 \in \mathit{Bool}^{j-i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{n-j}$ and $a \in \mathit{Bool}$. Since g does not contain a node labeled x_j it follows that $g|_{x_i \leftarrow 0}(\mathbf{b}_1, 0, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, 1, \mathbf{b}_2)$. Thus f is T_8-asymmetric in the pair (x_i, x_j) as required. □

Lemma 5. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_{11}-symmetric in the pair (x_i, x_j) and $i < j$ then every ROBDD g rooted at a node labeled x_i satisfies the property that*

- *g contains a node labeled x_j or,*
- $g|_{x_i \leftarrow 0} = \neg g|_{x_i \leftarrow 1}$.

Proof. To argue by contrapositive, suppose there exists a node g labeled x_i such that g does not contain a node labeled x_j and $g|_{x_i \leftarrow 0} \neq \neg g|_{x_i \leftarrow 1}$. Therefore $g|_{x_i \leftarrow 0}(\mathbf{b}_1, a, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, a, \mathbf{b}_2)$ for some $\mathbf{b}_1 \in \mathit{Bool}^{j-i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{n-j}$ and $a \in \mathit{Bool}$. Since g does not contain a node labeled x_j it follows that $g|_{x_i \leftarrow 0}(\mathbf{b}_1, 0, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, 0, \mathbf{b}_2)$. Thus f is T_{11}-asymmetric in the pair (x_i, x_j) as required. □

Lemma 6. *If an ROBDD f over a set of variables $\{x_1, \ldots, x_n\}$ is T_{12}-symmetric in the pair (x_i, x_j) and $i < j$ then every ROBDD g rooted at a node labeled x_i satisfies the property that*

- *g contains a node labeled x_j or,*
- $g|_{x_i \leftarrow 0} = \neg g|_{x_i \leftarrow 1}$.

Proof. To argue by contrapositive, suppose there exists a node g labeled x_i such that g does not contain a node labeled x_j and $g|_{x_i \leftarrow 0} \neq \neg g|_{x_i \leftarrow 1}$. Therefore $g|_{x_i \leftarrow 0}(\mathbf{b}_1, a, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, a, \mathbf{b}_2)$ for some $\mathbf{b}_1 \in \mathit{Bool}^{j-i-1}$, $\mathbf{b}_2 \in \mathit{Bool}^{n-j}$ and $a \in \mathit{Bool}$. Since g does not contain a node labeled x_j it follows that $g|_{x_i \leftarrow 0}(\mathbf{b}_1, 1, \mathbf{b}_2) = g|_{x_i \leftarrow 1}(\mathbf{b}_1, 1, \mathbf{b}_2)$. Thus f is T_{12}-asymmetric in the pair (x_i, x_j) as required. □