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UNIFORM BAHADUR REPRESENTATION FOR LOCAL POLYNOMIAL ESTIMATES OF M-REGRESSION AND ITS APPLICATION TO THE ADDITIVE MODEL

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We use local polynomial fitting to estimate the nonparametric M-regression function for strongly mixing stationary processes $\{(Y_i, \underline{X}_i)\}$. We establish a strong uniform consistency rate for the Bahadur representation of estimators of the regression function and its derivatives. These results are fundamental for statistical inference and for applications that involve plugging such estimators into other functionals where some control over higher order terms is required. We apply our results to the estimation of an additive M-regression model.

1. INTRODUCTION

In many contexts one wants to evaluate the properties of some procedure that is a functional of some given estimators. It is useful to be able to work with some plausible high level assumptions about those estimators rather than to rederive their properties for each different application. In a fully parametric (and stationary, weakly dependent data) context, it is quite common to assume that estimators are root-n consistent and asymptotically normal. In some cases this property suffices; in other cases one needs to be more explicit in terms of the linear expansion

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of these estimators, but in any case such expansions are quite natural and widely applicable. In a nonparametric context there is less agreement about the use of such expansions, and one often sees standard properties of standard estimators derived anew for a different purpose. It is our objective to provide results that can circumvent this. The types of applications we have in mind are estimations, of semiparametric models where the parameters of interest are explicit or implicit functionals of nonparametric regression functions and their derivatives; see Powell (1994), Andrews (1994), and Chen, Linton, and Van Keilegom (2003). Another class of applications includes estimations of structured nonparametric models like the additive models (Linton and Nielsen, 1995) or the generalized additive models (Linton, Sperlich, and Van Keilegom, 2008).

We motivate our results in a simple i.i.d. setting. Suppose we have a random sample $\{Y_i, X_i\}_{i=1}^n$ and consider the Nadaraya-Watson estimator of the regression function $m(x) = \mathrm{E}(Y_i | X_i = x)$,

$$\hat{m}(x) = \frac{\hat{r}(x)}{\hat{f}(x)} = \frac{n^{-1}\sum_{i=1}^n K_h(X_i - x)Y_i}{n^{-1}\sum_{i=1}^n K_h(X_i - x)},$$

where $K(.)$ is a symmetric density function, h is a bandwidth, and $K_h(.) = K(./h)/h$. Standard arguments (Härdle, 1990) show that under suitable smoothness conditions,

$$\hat{m}(x) - m(x) = h^2 b(x) + \frac{1}{nf(x)} \sum_{i=1}^n K_h(X_i - x)\varepsilon_i + R_n(x), \tag{1}$$

where $b(x) = \int u^2 K(u)\,du[m''(x) + 2m'(x)f'(x)/f(x)]/2$, while $f(x)$ is the covariate density function and $\varepsilon_i \equiv Y_i - m(X_i)$ is the error term. The remainder term $R_n(x)$ is of smaller order (almost surely) than the two leading terms. Such an expansion is sufficient to derive the central limit theorem for $\hat{m}(x)$ itself, but generally is not sufficient if $\hat{m}(x)$ is to be plugged into some semiparametric procedure. For example, suppose we estimate the parameter $\theta_0 = \int m(x)^2\,dx \neq 0$ by $\hat{\theta} = \int \hat{m}(x)^2\,dx$, where the integral is over some compact set $\mathcal{D}$; we would expect to find that $n^{1/2}(\hat{\theta} - \theta_0)$ is asymptotically normal. Based on expansion (1), the argument goes as follows.

First we obtain

$$n^{1/2}(\hat{\theta} - \theta_0) = 2n^{1/2} \int m(x)\{\hat{m}(x) - m(x)\}\,dx + n^{1/2} \int [\hat{m}(x) - m(x)]^2\,dx.$$

If it can be shown that $\hat{m}(x) - m(x) = o(n^{-1/4})$ a.s. uniformly in $x \in \mathcal{D}$ (such results are widely available; see, for example, Masry, 1996), we have

$$n^{1/2}(\hat{\theta} - \theta_0) = 2n^{1/2} \int m(x)\{\hat{m}(x) - m(x)\}\,dx + o(1) \quad \text{a.s.}$$

Note that the quantity on the right-hand side is the term in Assumption 2.6 of Chen et al. (2003), which is assumed to be asymptotically normal. It is the verification of this condition with which we are now concerned. We substitute in expansion (1) and obtain

$$n^{1/2}(\hat{\theta}-\theta_0) = 2n^{1/2}h^2\int m(x)b(x)\,dx + 2n^{1/2}\int n^{-1}\sum_{i=1}^{n}\varepsilon_i K_h(X_i-x)\frac{m(x)}{f(x)}\,dx$$

$$+2n^{1/2}\int m(x)R_n(x)\,dx + o(1) \quad \text{a.s.}$$

If $nh^4 \to 0$, then the first term (the smoothing bias term) is $o(1)$. The second term (the stochastic term) is a sum of independent random variables with mean zero, which can be rewritten, using a change of variables, as

$$n^{1/2}\int m(x)f^{-1}(x)n^{-1}\sum_{i=1}^{n}K_h(X_i-x)\varepsilon_i\,dx = n^{-1/2}\sum_{i=1}^{n}\xi_n(X_i)\varepsilon_i,$$

$$\xi_n(X_i) = \int m(X_i+uh)f^{-1}(X_i+uh)K(u)\,du,$$

and this term obeys the Lindeberg central limit theorem under standard conditions. The problem is with the third term, as equation (1) only guarantees that $\int m(x)R_n(x)\,dx = o(n^{-2/5})$ a.s. at best. In fact, it is possible to derive a more useful Bahadur representation (Bahadur, 1966) for the kernel estimator

$$\hat{m}(x)-m(x) = h^2 b_n(x) + \{\mathrm{E}\hat{f}(x)\}^{-1}n^{-1}\sum_{i=1}^{n}K_h(X_i-x)\varepsilon_i + R_n^*(x), \tag{2}$$

where $b_n(x)$ is deterministic and satisfies $b_n(x)\to b(x)$ and $\mathrm{E}\hat{f}(x)\to f(x)$ uniformly in $x\in\mathcal{D}$, while the remainder term now satisfies

$$\sup_{x\in\mathcal{D}}\left|R_n^*(x)\right| = O\left(\frac{\log n}{nh}\right) \quad \text{a.s.} \tag{3}$$

This property is a consequence of the uniform convergence rate of $\hat{f}(x)-\mathrm{E}\hat{f}(x)$, $n^{-1}\sum_{i=1}^{n}K_h(x-X_i)\{m(X_i)-m(x)\}-\mathrm{E}K_h(X_i-x)\{m(X_i)-m(x)\}$, and $n^{-1}\sum_{i=1}^{n}K_h(X_i-x)\varepsilon_i$ that follow from, for example Masry (1996). Clearly, by appropriate choice of the bandwidth h, $R_n^*(x)$ can be made $o(n^{-1/2})$ a.s. uniformly over $\mathcal{D}$ and thus $2n^{1/2}\int m(x)R_n^*(x)\,dx = o(1)$ a.s. Therefore, to derive asymptotic normality for $n^{1/2}(\hat{\theta}-\theta_0)$, one can just work with the two leading terms in (2). These terms are slightly more complicated than in the previous expansion but are still sufficiently simple for many purposes; in particular, $b_n(x)$ is uniformly bounded so that, provided $nh^4\to 0$, the smoothing bias term satisfies $h^2n^{1/2}\int m(x)b_n(x)\,dx \to 0$, while the stochastic term is a sum of zero mean independent random variables

$$n^{1/2}\int \frac{m(x)}{\overline{f}(x)} n^{-1}\sum_{i=1}^{n} K_h(X_i - x)\varepsilon_i\, dx = n^{-1/2}\sum_{i=1}^{n} \overline{\xi}_n(X_i)\varepsilon_i,$$

$$\overline{\xi}_n(X_i) = \int \frac{m(X_i + uh)}{\overline{f}(X_i + uh)} K(u)\, du,$$

and obeys the Lindeberg central limit theorem under standard conditions, where $\overline{f}(x) = \mathrm{E}\hat{f}(x)$. This argument shows the utility of Bahadur representation (2). There are many other applications of this result because a host of probabilistic results are available for random variables like $n^{-1}\sum_{i=1}^{n} K_h(X_i - x)\varepsilon_i$ and integrals thereof.

The one-dimensional Nadaraya-Watson estimator for i.i.d. data is particularly easy to analyze and the above arguments are well known. However, the limitations of this estimator are manyfold and there are good theoretical reasons for working instead with the local polynomial class of estimators (Fan and Gijbels, 1996). In addition, for many data, especially financial time series data, one may have concerns about heavy tails or outliers that point in the direction of using robust estimators like the local median or local quantile method, perhaps combined with local polynomial fitting. We examine a general class of (nonlinear) M-regression functions (that is, location functionals defined through minimization of a general objective function $\rho(.)$) and derivative estimators. We treat a general time series setting where the multivariate data are strongly mixing. Under mild conditions, we establish a uniform strong Bahadur representation like (2) and (3) with remainder term of order $(\log n/nh^d)^{3/4}$ almost surely, a rate that is almost optimal or in other words can't be improved further based on the results in Kiefer (1967) under i.i.d. setting. The leading terms are linear, and functionals of them can be analyzed simply. The remainder term can be made to be $o(n^{-1/2})$ a.s. under restrictions on the dimensionality in relation to the amount of smoothness possessed by the M-regression function.

The best convergence rate of unrestricted nonparametric estimators strongly depends on d, the dimension of the covariates. The rate decreases dramatically as d increases (Stone, 1982). This phenomenon is the so-called "curse of dimensionality." One approach to reduce the curse is by imposing model structure. A popular model structure is the additive model assuming that

$$m(x_1,\ldots,x_d) = c + m_1(x_1) + \cdots + m_d(x_d), \tag{4}$$

where c is an unknown constant and $m_k(.),\ k = 1,\ldots,d$ are unknown functions that have been normalized such that $\mathrm{E}m_k(\mathbf{x}_k) = 0$, $k = 1,\ldots,d$. In this case, the optimal rate of convergence is the same as in univariate nonparametric regression (Stone, 1986). An additive M-regression function is given by (4), where $m(x)$ is the M-regression function defined in (5) for some loss function $\rho(.;.)$. Previous work on additive quantile regression, for example, includes Linton (2001) and Horowitz and Lee (2005) for the i.i.d. case. An interesting application of the additive M-regression model is to combine (4) with the volatility model

$$Y_i = \sigma_i \varepsilon_i \quad \text{and} \quad \ln \sigma_i^2 = m(X_i),$$

where $X_i = (Y_{i-1}, \ldots, Y_{i-d})^\top$. We suppose that ε_i satisfies $\mathrm{E}[\varphi(\ln \varepsilon_i^2; 0)|X_i] = 0$ with $\varphi(.;.)$ the piecewise derivative of $\rho(.;.)$, whence $m(.)$ is the conditional M-regression of $\ln Y_i^2$ given X_i. Peng and Yao (2003) applied LAD estimation to parametric ARCH and GARCH models and showed the superior robustness property of this procedure over Gaussian QMLE with regard to heavy-tailed innovations. This heavy tail issue also arises in nonparametric regression models, and empirical evidences suggest that moderately high frequency financial data are often heavy tailed, which is why our procedures may be useful. We apply the Bahadur representations to the study of the marginal integration estimators (Linton and Nielsen, 1995) of the component functions in the additive M-regression model, in which case we only need the remainder term to be $o(n^{-p/(2p+1)})$ a.s., where p is a smoothness index.

Bahadur representations (Bahadur, 1966) have been widely studied and applied, with notable refinements in the i.i.d. setting by Kiefer (1967). A recent paper of Wu (2005) extends these results to a general class of dependent processes and provides a review. The closest paper to ours is Hong (2003), which establishes the Bahadur representation for essentially the same local polynomial M-regression estimator as ours. However, his results are (a) pointwise, i.e., for a single x only; (b) with a covariate that is univariate; and (c) for i.i.d. data. Clearly, this limits the range of applicability of his results, and specifically, the applications to semiparametric or additive models are perforce precluded.

2. THE GENERAL SETTING

Let $\{(Y_i, \underline{X}_i)\}$ be a jointly stationary process, where $\underline{X}_i = (\mathbf{x}_{i1}, \ldots, \mathbf{x}_{id})^\top$ with $d \geq 1$ and Y_i is a scalar. As dependent observations are considered in this paper, we introduce here the mixing coefficient. Let $\mathbf{F}_s^t$ be the σ-algebra of events generated by random variables $\{(Y_i, \underline{X}_i), s \leq i \leq t\}$. A stationary stochastic processes $\{(Y_i, \underline{X}_i)\}$ is strongly mixing if

$$\sup_{\substack{A \in \mathbf{F}_{-\infty}^0 \\ B \in \mathbf{F}_k^\infty}} |P[AB] - P[A]P[B]| = \gamma[k] \to 0, \quad \text{as } k \to \infty,$$

and $\gamma[k]$ is called the strong mixing coefficient.

Suppose $\rho(.;.)$ is a loss function. Our first goal is to estimate the multivariate M-regression function

$$m(x_1, \ldots, x_d) = \arg\min_\theta \mathrm{E}\{\rho(Y_i; \theta)|\underline{X}_i = (x_1, \ldots, x_d)\}, \tag{5}$$

and its partial derivatives based on observations $\{(Y_i, \underline{X}_i)\}_{i=1}^n$. An important example of the M-function is the qth $(0 < q < 1)$ quantile of Y_i given

$\underline{X}_i = (x_1, \ldots, x_d)^\top$, with loss function $\rho(y;\theta) = (2q-1)(y-\theta) + |y-\theta|$. Another example is the L_q criterion: $\rho(y;\theta) = |y-\theta|^q$ for $q > 1$, which includes the least square criterion $\rho(y;\theta) = (y-\theta)^2$ with $m(.)$ the conditional expectation of Y_i given $\underline{X}_i$. Yet another example is the celebrated Huber's function (Huber, 1973)

$$\rho(t) = t^2/2I\{|t| < k\} + (k|t| - k^2/2)I\{|t| \geq k\}. \tag{6}$$

Suppose $m(\underline{x})$ is differentiable up to order $p+1$ at $\underline{x} = (x_1, ..., x_d)^\top$. Then the multivariate pth order local polynomial approximation of $m(\underline{z})$ for any $\underline{z}$ close to $\underline{x}$ is given by

$$m(\underline{z}) \approx \sum_{0 \leq |\underline{r}| \leq p} \frac{1}{\underline{r}!} D^{\underline{r}} m(\underline{x})(\underline{z} - \underline{x})^{\underline{r}},$$

where $\underline{r} = (r_1, ..., r_d)$, $|\underline{r}| = \sum_{i=1}^d r_i$, $\underline{r}! = r_1! \times \cdots \times r_d!$, and

$$D^{\underline{r}} m(\underline{x}) = \frac{\partial^{\underline{r}} m(\underline{x})}{\partial x_1^{r_1} \cdots \partial x_d^{r_d}}, \qquad \underline{x}^{\underline{r}} = x_1^{r_1} \times \cdots \times x_d^{r_d}, \qquad \sum_{0 \leq |\underline{r}| \leq p} = \sum_{j=0}^{p} \sum_{r_1=0}^{j} \cdots \sum_{\substack{r_d=0 \\ r_1+\cdots+r_d=j}}^{j}. \tag{7}$$

Let $K(\underline{u})$ be a density function on R^d, h a bandwidth, and $K_h(\underline{u}) = K(\underline{u}/h)$. With observations $\{(Y_i, \underline{X}_i)\}_{i=1}^n$, we consider minimizing the following quantity with respect to $\beta_{\underline{r}}$, $0 \leq |\underline{r}| \leq p$:

$$\sum_{i=1}^{n} K_h(\underline{X}_i - \underline{x}) \rho\Big(Y_i; \sum_{0 \leq |\underline{r}| \leq p} \beta_{\underline{r}} (\underline{X}_i - \underline{x})^{\underline{r}}\Big). \tag{8}$$

Denote by $\hat{\beta}_{\underline{r}}(\underline{x})$, $0 \leq |r| \leq p$, the minima of (8). The M-regression function $m(\underline{x})$ and its partial derivatives $D^{\underline{r}} m(\underline{x})$, $1 \leq |\underline{r}| \leq p$ are then estimated, respectively, by

$$\hat{m}(\underline{x}) = \hat{\beta}_{\underline{0}}(\underline{x}) \quad \text{and} \quad \hat{D}^{\underline{r}} m(\underline{x}) = \underline{r}! \hat{\beta}_{\underline{r}}(\underline{x}), \qquad 1 \leq |\underline{r}| \leq p. \tag{9}$$

3. MAIN RESULTS

In Theorem 1 below we give our main result, the uniform strong Bahadur representation for the vector $\hat{\beta}_p(\underline{x})$. We first need to develop some notations to define the leading terms in the representation.

Let $N_i = \binom{i+d-1}{d-1}$ be the number of distinct $d-$tuples $\underline{r}$ with $|\underline{r}| = i$. Arrange these $d-$tuples as a sequence in a lexicographical order (with the highest priority given to the last position so that $(0, \ldots, 0, i)$ is the first element in the sequence and $(i, 0, \ldots, 0)$ the last element). Let τ_i denote this 1-to-1 mapping, i.e., $\tau_i(1) = (0, \ldots, 0, i), \ldots, \tau_i(N_i) = (i, 0, \ldots, 0)$. For each $i = 1, \ldots, p$, define an $N_i \times 1$ vector $\mu_i(\underline{x})$ with its kth element given by $\underline{x}^{\tau_i(k)}$ and write $\mu(\underline{x}) = (1, \mu_1(\underline{x})^\top, \cdots, \mu_p(\underline{x})^\top)^\top$, which is a column vector of length $N = \sum_{i=0}^{p} N_i$. Similarly define vectors $\beta_p(\underline{x})$ and $\underline{\beta}$ through the same lexicographical arrangement of $D^{\underline{r}} m(\underline{x})$ and $\beta_{\underline{r}}$ in (8) for $0 \le |\underline{r}| \le p$. Thus (8) can be rewritten as

$$\sum_{i=1}^{n} K_h(\underline{X}_i - \underline{x})\rho(Y_i; \mu(\underline{X}_i - \underline{x})^\top \underline{\beta}). \tag{10}$$

Suppose the minimizer of (10) is denoted as $\hat{\beta}_n(\underline{x})$. Let $\hat{\beta}_p(\underline{x}) = W_p \hat{\beta}_n(\underline{x})$, where W_p is a diagonal matrix with diagonal entries the lexicographical arrangement of $\underline{r}!$, $0 \le |\underline{r}| \le p$.

Let $\nu_{\underline{i}} = \int K(\underline{u})\underline{u}^{\underline{i}} d\underline{u}$. For $g(.)$ given in (A.7) in the Appendix, define

$$\nu_{n\underline{i}}(\underline{x}) = \int K(\underline{u})\underline{u}^{\underline{i}} g(\underline{x} + h\underline{u}) f(\underline{x} + h\underline{u})\, d\underline{u}.$$

For $0 \le j, k \le p$, let $S_{j,k}$ and $S_{n,j,k}(\underline{x})$ be two $N_j \times N_k$ matrices with their (l, m) elements, respectively, given by

$$\Big[S_{j,k}\Big]_{l,m} = \nu_{\tau_j(l)+\tau_k(m)}, \qquad \Big[S_{n,j,k}(\underline{x})\Big]_{l,m} = \nu_{n,\tau_j(l)+\tau_k(m)}(\underline{x}). \tag{11}$$

Now define the $N \times N$ matrices S_p and $S_{n,p}(\underline{x})$ by

$$S_p = \begin{bmatrix} S_{0,0} & S_{0,1} & \cdots & S_{0,p} \\ S_{1,0} & S_{1,1} & \cdots & S_{1,p} \\ \vdots & \ddots & \vdots & \\ S_{p,0} & S_{p,1} & \cdots & S_{p,p} \end{bmatrix},$$

$$S_{n,p}(\underline{x}) = \begin{bmatrix} S_{n,0,0}(\underline{x}) & S_{n,0,1}(\underline{x}) & \cdots & S_{n,0,p}(\underline{x}) \\ S_{n,1,0}(\underline{x}) & S_{n,1,1}(\underline{x}) & \cdots & S_{n,1,p}(\underline{x}) \\ \vdots & \ddots & \vdots & \\ S_{n,p,0}(\underline{x}) & S_{n,p,1}(\underline{x}) & \cdots & S_{n,p,p}(\underline{x}) \end{bmatrix}.$$

According to Lemma 8, $S_{n,p}(\underline{x})$ converges to $g(\underline{x}) f(\underline{x}) S_p$ uniformly in $\underline{x} \in \mathcal{D}$ almost surely. Hence for $|S_p| \neq 0$, we can define

$$\beta_n^*(\underline{x}) = -\frac{1}{nh^d} W_p S_{n,p}^{-1}(\underline{x}) H_n^{-1} \sum_{i=1}^{n} K_h(\underline{X}_i - \underline{x})\varphi(Y_i, \mu(\underline{X}_i - \underline{x})^\top \beta_p(\underline{x}))\mu(\underline{X}_i - \underline{x}), \tag{12}$$

where $\varphi(.;.)$ is the piecewise derivative of $\rho(.,.)$ as defined in Assumption A1 and H_n is a diagonal matrix with diagonal entries $h^{|\underline{r}|}$, $0 \le |\underline{r}| \le p$ in the aforementioned lexicographical order. The quantity $\beta_n^*(\underline{x})$ is the leading term in the Bahadur representation of $\hat{\beta}_p(\underline{x}) - \beta_p(\underline{x})$; it is the sum of a bias term, $\mathrm{E}\beta_n^*(\underline{x})$, and a stochastic term $\beta_n^*(\underline{x}) - \mathrm{E}\beta_n^*(\underline{x})$.

Denote the typical element of $\beta_n^*(\underline{x})$ by $\beta_{n\underline{r}}^*(\underline{x})$, $0 \le |\underline{r}| \le p$ and the probability density function of $\underline{X}$ by $f(.)$. The following results on $\mathrm{E}\beta_{n\underline{r}}^*(\underline{x})$ are an extension of Proposition 2.2 in Hong (2003) to the multivariate case.

PROPOSITION 1. *If $f(\underline{x}) > 0$ and Assumptions A1–A5 in the Appendix hold, then*

$$\mathrm{E}\beta_{n\underline{r}}^*(\underline{x}) = \begin{cases} -h^{p+1} e_{N(\underline{r})} W_p S_p^{-1} B_1 \mathbf{m}_{p+1}(\underline{x}) + o(h^{p+1}), & \text{for } p - |\underline{r}| \text{ odd,} \\ -h^{p+2} e_{N(\underline{r})} W_p S_p^{-1} \big[\{fg\}^{-1}(\underline{x}) \mathbf{m}_{p+1}(\underline{x}) \{\tilde{M}(\underline{x}) \\ \qquad - N_p S_p^{-1} B_1\} + B_2 \mathbf{m}_{p+2}(\underline{x})\big] + o(h^{p+2}), & \text{for } p - |\underline{r}| \text{ even,} \end{cases}$$

where $N(\underline{r}) = \tau_{|\underline{r}|}^{-1}(\underline{r}) + \sum_{k=0}^{|\underline{r}|-1} N_k$, e_i is an $N \times 1$ vector having 1 as the ith entry, with all other entries 0, *and* $B_1 = [S_{0,p+1}, S_{1,p+1}, \ldots, S_{p,p+1}]^\top$, $B_2 = [S_{0,p+2}, S_{1,p+2}, \ldots, S_{p,p+2}]^\top$.

We next present our main result, the Bahadur representation for the local polynomial estimates $\hat{\beta}_p(\underline{x})$.

THEOREM 1. *Suppose Assumptions A1–A7 in the Appendix hold with $\lambda_2 = (p+1)/2(p+s+1)$ for some $s \ge 0$, and $\mathcal{D}$ is any compact subset of R^d. Then*

$$\sup_{\underline{x} \in \mathcal{D}} |H_n\{\hat{\beta}_p(\underline{x}) - \beta_p(\underline{x})\} - \beta_n^*(\underline{x})| = O\left(\left\{\frac{\log n}{nh^d}\right\}^{\lambda(s)}\right) \quad \textit{almost surely,}$$

where $|.|$ is taken to be the sup norm and

$$\lambda(s) = \min\left\{\frac{p+1}{p+s+1}, \frac{3p+3+2s}{4p+4s+4}\right\}.$$

Remark 1. According to Theorem 1 in Kiefer (1967), the pointwise sharpest bound of the remainder term in the Bahadur representation of the sample quantiles is $(\log\log n/n)^{3/4}$. As $\lambda(0) = 3/4$, we could safely claim the results here could not be further improved for a general class of loss functions $\rho(.)$ specified by Assumptions A1 and A2. Nevertheless, it is possible to derive stronger results, if the concerned loss functions enjoy a higher degree of smoothness; e.g., (3), in which case $\rho(.)$ is the squared loss function. More specifically, suppose that $\varphi(.)$ is Lipschitz continuous and Assumptions A1–A7 in the Appendix hold with $\lambda_2 = 1/2$ and $\lambda_1 = 1$. Then we prove in the Appendix that

$$\sup_{\underline{x} \in \mathcal{D}} |H_n\{\hat{\beta}_p(\underline{x}) - \beta_p(\underline{x})\} - \beta_n^*(\underline{x})| = O\left(\frac{\log n}{nh^d}\right) \quad \text{almost surely.} \tag{13}$$

Remark 2. The dependence among the observations doesn't have any impact on the rate of uniform convergence, provided that the degree of the dependence, as measured by the mixing coefficient $\gamma[k]$, is weak enough such that (A.3) and (A.4) are satisfied. This is in accordance with the results in Masry (1996), where he proved that for a local polynomial estimator of the conditional mean function, the uniform convergence rate is $(nh^d/\log n)^{-1/2}$, the same as in the independent case.

Remark 3. It is of practical interest to provide an explicit rate of decay for the strong mixing coefficient $\gamma[k]$ of the form $\gamma[k] = O(1/k^c)$ for some $c > 0$ (to be determined) for Theorem 1 to hold. It is easy to see that, among all the conditions imposed on $\gamma[k]$, the summability condition (A.4) is the most restrictive. We assume that

$$h = h_n \sim (\log n/n)^{\bar{a}} \quad \text{for some } \frac{1}{2(p+s+1)+d} \le \bar{a} < \frac{1}{d}\left\{1 - \frac{4}{(1-\lambda_2)\nu_2 - 4\lambda_1 + 2(1+\lambda_2)}\right\},$$

whence (A.2) holds. Algebraic calculations show that (A.4) would be true if

$$c > \nu_2 \frac{(1-\bar{a}d)\{(1-\lambda_2)(4N+1)+8N\lambda_1\}+10+(4+8N)\bar{a}d}{2(1-\lambda_2)(1-\bar{a}d)\nu_2 - 8\bar{a}d + 4(1-\bar{a}d)(1-\lambda_2-2\lambda_1)} - 1 \equiv c(d, p, \nu_2, \bar{a}, \lambda_1, \lambda_2). \tag{14}$$

Note that we would need the condition

$$\nu_2 > 2 + \frac{4\{\bar{a}d + (1-\bar{a}d)\lambda_1\}}{(1-\bar{a}d)(1-\lambda_2)}$$

to secure a positive denominator for (14). As $c(d, p, \nu_2, \bar{a}, \lambda_1, \lambda_2)$ is decreasing in $\nu_2 (\le \nu_1)$, there is a trade-off between the order ν_1 of the moment $\mathrm{E}|\varphi(\varepsilon_i)|^{\nu_1} < \infty$ and the decay rate of the strong mixing coefficient $\gamma[k]$: The existence of higher order moments allows $\gamma[k]$ to decay more slowly.

Remark 4. It is trivial to generalize the result in Theorem 1 to functionals of the M-estimates $\hat{\beta}_p(\underline{x})$. Denote the typical elements of $\hat{\beta}_p(\underline{x})$ and $\beta_p(\underline{x})$ by $\hat{\beta}_{p\underline{r}}(\underline{x})$ and $\beta_{p\underline{r}}(\underline{x})$, $0 \le |\underline{r}| \le p$, respectively. Suppose $G(.) : R^d \to R$ satisfies that for any compact set $\mathcal{D} \subset R^d$, there exists some constant $C > 0$, such that $|G'(\beta_{p\underline{r}}(\underline{x}))| \le C$ and $|G''(\beta_{p\underline{r}}(\underline{x}))| \le C$ for all $\underline{x} \in \mathcal{D}$. Then, with probability 1,

$$\sup_{\underline{x}\in\mathcal{D}} \left| h^{|\underline{r}|}\{G(\hat{\beta}_{p\underline{r}}(\underline{x})) - G(\beta_{p\underline{r}}(\underline{x}))\} - G'(\beta_{p\underline{r}}(\underline{x}))\beta^*_{n\underline{r}}(\underline{x}) \right| = O\left(\left\{\frac{\log n}{nh^d}\right\}^{\lambda(s)}\right). \tag{15}$$

The following proposition follows from Theorem 1 and the uniform convergence of the sum of weakly dependent zero mean random variables.

COROLLARY 1. *Suppose conditions in Theorem 1 hold with* $s = 0$. *Then with probability* 1 *we have, uniformly in* $\underline{x} \in \mathcal{D}$,

$$H_n\{\hat{\beta}_p(\underline{x}) - \beta_p(\underline{x})\} - \mathrm{E}\beta_n^*(\underline{x}) - \frac{W_p H_n^{-1}}{nh^d} S_{np}^{-1}(\underline{x}) \sum_{i=1}^{n} K_h(\underline{X}_i - \underline{x})\varphi(\varepsilon_i)\mu(\underline{X}_i - \underline{x})$$

$$= O\Big(\Big\{\frac{\log n}{nh^d}\Big\}^{3/4}\Big).$$

4. M-ESTIMATION OF THE ADDITIVE MODEL

In this section, we apply our main result to derive the properties of a class of estimators in the additive M-regression model (4). In terms of estimating the component functions $m_k(.)$, $k = 1, \ldots, d$ in (4), the marginal integration method (Linton and Nielsen, 1995) is known to achieve the optimal rate under certain conditions. This involves estimating first the unrestricted M-regression function $m(.)$ and then integrating it over some directions. Partition $\underline{X}_i = (x_1, \ldots, x_d)$ as $\underline{X}_i = (\mathbf{x}_{1i}, \underline{X}_{2i})$, where $\mathbf{x}_{1i}$ is the one-dimensional direction of interest and $\underline{X}_{2i}$ is a $d-1$ dimensional nuisance direction. Let $\underline{x} = (x_1, \underline{x}_2)$ and define the functional

$$\phi_1(x_1) = \int m(x_1, \underline{x}_2) f_2(\underline{x}_2)\, d\underline{x}_2, \tag{16}$$

where $f_2(\underline{x}_2)$ is the joint probability density of $\underline{X}_{2i}$. Under the additive structure (4), $\phi_1(.)$ is $m_1(.)$ up to a constant. Replace $m(.)$ in (16) with $\hat{\beta}_0(x_1, \underline{x}_2) \equiv \hat{\beta}_0(\underline{x})$ given by (9), and $\phi_1(x_1)$ can thus be estimated by the sample version of (16):

$$\phi_{n1}(x_1) = n^{-1} \sum_{i=1}^{n} \hat{\beta}_0(x_1, \underline{X}_{2i}).$$

As noted by Linton and Härdle (1996) and Hengartner and Sperlich (2005), cautious choice of the bandwidth is crucial for $\phi_{n1}(.)$ to be asymptotically normal. They suggest different bandwidths be used for the direction of interest X_1 and the $d-1$ dimensional nuisance direction $\underline{X}_2$, say h_1 and h, respectively. Sperlich, Linton, and Härdle (1998) provides an extensive study of the small sample properties of the marginal integration estimators, including an evaluation of bandwidth choice.

The following corollary concerns the asymptotic properties of $\phi_{n1}(.)$.

COROLLARY 2. *Suppose the support of* $\underline{X}$ *is* $[0,1]^{\otimes d}$ *with strictly positive probability density function. Assume that conditions in Corollary 1 hold with* $\mathrm{T}_n \equiv \{r(n)/\min(h_1, h)\}^d$ *and the* h^d *replaced by* $h_1 h^{d-1}$ *in all the notations defined either in (A.1) or (A.2). If* $h_1 \propto n^{-1/(2p+3)}$, $h = O(h_1)$, *and (A.2) is modified as*

$$nh_1 h^{3(d-1)}/\log^3 n \to \infty, \qquad n^{-1}\{r(n)\}^{\nu_2/2} d_n \log n / M_n^{(2)} \to \infty, \tag{17}$$

then we have

$$(nh_1)^{1/2}\{\phi_{n1}(x_1)-\phi_1(x_1)\} \xrightarrow{L} N(e_1 W_p S_p^{-1} B_1 \mathbf{E}\mathbf{m}_{p+1}(x_1, \underline{X}_2), \tilde{\sigma}^2(x_1)),$$

where $\xrightarrow{L}$ *stands for convergence in distribution,*

$$\tilde{\sigma}^2(x_1) = \left\{\int_{[0,1]^{\otimes d-1}} \{fg^2\}^{-1}(x_1, \underline{X}_2) f_2^2(\underline{X}_2)\sigma^2(x_1, \underline{X}_2)\, d\underline{X}_2\right\} e_1 S_p^{-1} K_2 K_2^\top S_p^{-1} e_1^\top,$$

$\sigma^2(\underline{x}) = \mathrm{E}[\varphi^2(\varepsilon)|\underline{X} = \underline{x}]$, *and* $K_2 = \int_{[0,1]^{\otimes d}} K(\underline{v})\mu(\underline{v})\, d\underline{v}$. *In particular, for the additive quantile regression model, i.e.,* $\rho(y;\theta) = (2q-1)(y-\theta) + |y-\theta|$, *we have*

$$\tilde{\sigma}^2(x_1) = q(1-q)\left\{\int_{[0,1]^{\otimes d-1}} f^{-1}(x_1, \underline{X}_2) f_\varepsilon^{-2}(0|x_1, \underline{X}_2) f_2^2(\underline{X}_2)\, d\underline{X}_2\right\}$$
$$\times e_1 S_p^{-1} K_2 K_2^\top S_p^{-1} e_1^\top.$$

Remark 5. For conditions in Corollary 2 to hold, we would need $3d < 2p+5$, i.e., the order p of local polynomial approximation should increase with the dimension of the covariates $\underline{X}$. See also the discussion in Hengartner and Sperlich (2005).

Remark 6. Besides asymptotic normality, by applying Theorem 1 we could also develop Bahadur representations for $\phi_{n1}(x_1)$, like those assumed in Linton et al. (2008). Based on (15), similar results are also applicable to the generalized additive M-regression model, i.e., $G(m(x_1,\ldots,x_d)) = c + m_1(x_1) + \cdots + m_d(x_d)$ for some known smooth function $G(.)$, in which case the marginal integration estimator is defined as the sample average of $G(\hat{m}(x_1, \underline{X}_{2i}))$.

5. CONCLUSION

We have obtained an asymptotic expansion for a nonlinear local polynomial M-estimator of a conditional location functional for stationary weakly dependent processes. The approximations we have obtained are to a high enough order for many applications based on computing functionals of said estimators. The error from the omitted terms is established in two cases, the smooth case and the unsmooth case, and in both cases we achieve what appears to be the optimal rate.

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APPENDIX: Proofs

We will need the following notations: For any $\lambda_2 \in (0,1)$, $\lambda_1 \in (\lambda_2, (1+\lambda_2)/2]$, and $M > 2$, define

$$d_n = (nh^d/\log n)^{-(\lambda_1+\lambda_2/2)}(nh^d \log n)^{1/2}, \qquad r(n) = (nh^d/\log n)^{(1-\lambda_2)/2}, \quad \textbf{(A.1)}$$

$$M_n^{(1)} = M(nh^d/\log n)^{-\lambda_1}, \qquad M_n^{(2)} = M^{1/4}(nh^d/\log n)^{-\lambda_2}, \qquad \mathrm{T}_n = \{r(n)/h\}^d,$$

and L_n as the smallest integer such that $\log n(M/2)^{\mathrm{L}_n+1} > nM_n^{(2)}/d_n$. Let $\|.\|$ denote the Euclidean norm and C be a generic constant, which may take different values in each appearance. Let $\varepsilon_i \equiv Y_i - m(\underline{X}_i)$ and assume that the following hold.

Assumption A1. For each $y \in \mathcal{R}$, $\rho(y;\theta)$ is absolutely continuous in θ; i.e., there exists a function $\varphi(y;\theta) \equiv \varphi(y-\theta)$ such that for any $\theta \in \mathcal{R}$, $\rho(y;\theta) = \rho(y;0) + \int_0^\theta \varphi(y;t)\,dt$. The probability density function of ε_i is bounded with $\mathrm{E}|\varphi(\varepsilon_i)|^{\nu_1} < \infty$ for some $\nu_1 > 2$, and $\mathrm{E}\{\varphi(\varepsilon_i)|\underline{X}_i\} = 0$ almost surely.

Assumption A2. Assume that $\varphi(.)$ satisfies the Lipschitz condition in (a_j, a_{j+1}), $j = 0, \dots, m$, where $a_0 \equiv -\infty$, $a_{m+1} \equiv +\infty$ and $a_1 < \cdots < a_m$ are a finite number of jump discontinuity points of $\varphi(.)$.

Assumption A3. Assume $K(.)$ has a compact support, say $[-1,1]^{\otimes d}$, and $|H_{\underline{j}}(\underline{u}) - H_{\underline{j}}(\underline{v})| \le C\|u - v\|$ for all j with $0 \le |\underline{j}| \le 2p+1$, where $H_{\underline{j}}(u) = \underline{u}^{\underline{j}} K(\underline{u})$.

Assumption A4. The probability density function of $\underline{X}$, $f(.)$ is bounded with bounded first order derivatives. The joint probability density of $(\underline{X}_0, \underline{X}_l)$ satisfies $f(\underline{u}, \underline{v}; l) \le C < \infty$ for all $l \ge 1$.

Assumption A5. For $\underline{r}$ with $|\underline{r}| = p+1$, $D^{\underline{r}} m(\underline{x})$ is bounded with bounded first order derivatives.

Assumption A6. The bandwidth $h \to 0$, such that

$$nh^d/\log n \to \infty, \qquad nh^{d+(p+1)/\lambda_2}/\log n < \infty,$$

$$n^{-1}\{r(n)\}^{\nu_2/2} d_n \log n / M_n^{(2)} \to \infty, \tag{A.2}$$

for some $2 < \nu_2 \le \nu_1$ and the processes $\{(Y_i, \underline{X}_i)\}$ are strongly mixing with mixing coefficient $\gamma[k]$ satisfying

$$\sum_{k=1}^{\infty} k^a \{\gamma[k]\}^{1-2/\nu_2} < \infty \quad \text{for some } a > (p+d+1)(1-2/\nu_2)/d. \tag{A.3}$$

Moreover, the bandwidth h and $\gamma[k]$ should jointly satisfy the condition

$$\sum_{n=1}^{\infty} n^{3/2} \mathrm{T}_n \left\{\frac{M_n^{(1)}}{d_n}\right\}^{1/2} \frac{\gamma[r(n)(2^{\nu_2/2}/M)^{2\mathrm{L}_n/\nu_2}]}{r(n)(2^{\nu_2/2}/M)^{2\mathrm{L}_n/\nu_2}} \{4M^{2N}\}^{\mathrm{L}_n} < \infty,$$

$$\forall M > 0. \tag{A.4}$$

Assumption A7. The conditional density $f_{\underline{X}|Y}$ of $\underline{X}$ given Y exists and is bounded. The conditional density function $f_{(\underline{X}_1, \underline{X}_{l+1})|(Y_1, Y_{l+1})}$ of $(\underline{X}_1, \underline{X}_{l+1})$ given (Y_1, Y_{l+1}) exists and is bounded for all $l \ge 1$.

Remark 7. Conditions on $\varphi(.)$ as in Assumptions A1 and A2 are satisfied in almost all known robust and likelihood type regressions. For example, in the qth quantile regression, we have $\varphi(t) = 2qI\{t \ge 0\} + (2q-2)I\{t < 0\}$, while for the Huber's function (6), its piecewise derivative is given by

$$\varphi(t) = tI\{|t| < k\} + \mathrm{sign}(t)kI\{|t| \ge k\}.$$

Note that the condition $\mathrm{E}\{\varphi(\varepsilon_i)|\underline{X}_i\} = 0\; a.e.$ is necessary for model specification. Moreover, if the conditional density $f(y|\underline{x})$ of Y given $\underline{X}$ is also continuously differentiable with respect to y, then as shown in Hong (2003) there exists a constant $C > 0$, such that for all small t and $\underline{x}$,

$$\mathrm{E}\left[\left\{\varphi(Y;t+a)-\varphi(Y;a)\right\}^2|\underline{X}=\underline{u}\right] \le C|t| \tag{A.5}$$

holds for all $(a,\underline{u})$ in a neighborhood of $(m(\underline{x}),\underline{x})$. Define

$$G(t,\underline{u}) = \mathrm{E}\{\varphi(Y;t)|\underline{X}=\underline{u}\}, \qquad G_i(t,\underline{u}) = (\partial^i/\partial t^i)G(t,\underline{u}), \qquad i=1,2. \tag{A.6}$$

Then it holds that

$$g(\underline{x}) = G_1(m(\underline{x}),\underline{x}) \ge C > 0, \qquad G_2(t,\underline{x}) \text{ is bounded for all } \underline{x}\in\mathcal{D} \text{ and } t \text{ near } m(\underline{x}). \tag{A.7}$$

Assumptions A3–A7 are standard for nonparametric smoothing in multivariate time series analysis; see Masry (1996). For example, condition (A.3) is needed to bound the covariance of the partial sums of time series as in Lemma 5, while (A.4) plays a similar role to (4.7b) in Masry (1996). It guarantees that the dependence of the time series is weak enough such that the deviance caused by the approximation of dependent random variables by independent ones (through Bradley's strong approximation theorem) is negligible; see Lemma 4. Of course, (A.4) is more stringent than (4.7b) in Masry (1996), due to the nonlinear nature of the estimates obtained by using the loss function $\rho(.)$ instead of the method of least squares.

Proof of Proposition 1. Write $\beta_n^*(\underline{x}) = -W_p S_{n,p}^{-1}(\underline{x})\sum_{i=1}^n Z_{ni}(\underline{x})/n$, where

$$Z_{ni}(\underline{x}) = H_n^{-1}h^{-d}K_h(\underline{X}_i-\underline{x})\varphi(Y_i,\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x}))\mu(\underline{X}_i-\underline{x}).$$

We first focus on $\mathrm{E}Z_{ni}(\underline{x})$. Based on (A.6) and (A.7), we have

$$\begin{aligned}\mathrm{E}\{\varphi(Y_i,\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x}))|\underline{X}_i\} &= G(\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x}),\underline{X}_i)\\ &= -g(\underline{X}_i)\{m(\underline{X}_i)-\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x})\}\\ &\quad + G_2(\xi_i(x),\underline{X}_i)\{m(\underline{X}_i)-\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x})\}^2/2\end{aligned}$$

for some $\xi_i(x)$ between $\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x})$ and $m(\underline{X}_i)$. Apparently, if $\underline{X}_i = \underline{x}+h\underline{v}$, then

$$\begin{aligned}m(\underline{X}_i)-\mu(\underline{X}_i-\underline{x})^\top\beta_p(\underline{x}) &= h^{p+1}\sum_{|\underline{k}|=p+1}\frac{D^{\underline{r}}m(\underline{x})}{\underline{k}!}\underline{v}^{\underline{k}}\\ &\quad + h^{p+2}\sum_{|\underline{k}|=p+2}\frac{D^{\underline{r}}m(\underline{x})}{\underline{k}!}\underline{v}^{\underline{k}} + o(h^{p+2}).\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathrm{E}Z_{ni}(\underline{x}) &= h^{p+1}\int K(\underline{v}) fg(\underline{x}+h\underline{v})\mu(\underline{v}) \sum_{|\underline{k}|=p+1} \frac{D^{\underline{r}}m(\underline{x})}{\underline{k}!}\underline{v}^{\underline{k}}\,d\underline{v} \\
&\quad + h^{p+2}\int K(\underline{v}) fg(\underline{x}+h\underline{v})\mu(\underline{v}) \sum_{|\underline{k}|=p+2} \frac{D^{\underline{r}}m(\underline{x})}{\underline{k}!}\underline{v}^{\underline{k}}\,d\underline{v} + o(h^{p+2}) \\
&\equiv T_1 + T_2.
\end{aligned}
$$

Now arrange the N_{p+1} elements of the derivatives $D^{\underline{r}}m(\underline{x})/\underline{r}!$ for $|\underline{r}| = p+1$ as a column vector $\mathbf{m}_{p+1}(\underline{x})$ using the lexicographical order introduced earlier and define $\mathbf{m}_{p+2}(\underline{x})$ in a similar way. Let the $N \times N_{p+1}$ matrix $B_{n1}(\underline{x})$ and the $N \times N_{p+2}$ matrix $B_{n2}(\underline{x})$ be defined as

$$
B_{n1}(\underline{x}) = \begin{bmatrix} S_{n,0,p+1}(\underline{x}) \\ S_{n,1,p+1}(\underline{x}) \\ \vdots \\ S_{n,p,p+1}(\underline{x}) \end{bmatrix}, \qquad B_{n2}(\underline{x}) = \begin{bmatrix} S_{n,0,p+2}(\underline{x}) \\ S_{n,1,p+2}(\underline{x}) \\ \vdots \\ S_{n,p,p+2}(\underline{x}) \end{bmatrix},
$$

where $S_{n,i,p+1}(\underline{x})$ and $S_{n,i,p+2}(\underline{x})$ are as given by (11). Therefore, $T_1 = h^{p+1}B_{n1}(\underline{x})$ $\mathbf{m}_{p+1}(\underline{x})$, $T_2 = h^{p+2}B_{n2}(\underline{x})\mathbf{m}_{p+2}(\underline{x})$, and

$$
\begin{aligned}
\mathrm{E}\beta_n^*(\underline{x}) &= -W_p h^{p+1} S_{n,p}^{-1}(\underline{x}) B_{n1}(\underline{x})\mathbf{m}_{p+1}(\underline{x}) \\
&\quad - W_p h^{p+2} S_{n,p}^{-1}(\underline{x}) B_{n2}(\underline{x})\mathbf{m}_{p+2}(\underline{x}) + o(h^{p+2}).
\end{aligned}
$$

Let $\underline{e}_i$, $i = 1,\dots,d$ be the $d \times 1$ vector having 1 in the ith entry and all other entries 0. For $0 \le j \le p$, $0 \le k \le p+1$, let $N_{j,k}(\underline{x})$ be an $N_j \times N_k$ matrix with its (l,m) element given by

$$
\Big[N_{j,k}(\underline{x})\Big]_{l,m} = \sum_{i=1}^{d} D^{\underline{e}_i}\{fg\}(\underline{x}) \int K(\underline{u})\underline{u}^{\tau_j(l)+\tau_k(m)+\underline{e}_i}\,d\underline{u},
$$

and use these $N_{j,k}(\underline{x})$ to construct an $N \times N$ matrix $N_p(\underline{x})$ and an $N \times N_{p+1}$ matrix $\tilde{M}(\underline{x})$ via

$$
N_p(\underline{x}) = \begin{bmatrix} N_{0,0}(\underline{x}) & N_{0,1}(\underline{x}) & \cdots & N_{0,p}(\underline{x}) \\ N_{1,0}(\underline{x}) & N_{1,1}(\underline{x}) & \cdots & N_{1,p}(\underline{x}) \\ \vdots & & \ddots & \vdots \\ N_{p,0}(\underline{x}) & N_{p,1}(\underline{x}) & \cdots & N_{p,p}(\underline{x}) \end{bmatrix}, \qquad \tilde{M}(\underline{x}) = \begin{bmatrix} N_{0,p+1}(\underline{x}) \\ N_{1,p+1}(\underline{x}) \\ \vdots \\ N_{p,p+1}(\underline{x}) \end{bmatrix}.
$$

Then $S_{n,p}(\underline{x}) = \{fg\}(\underline{x})S_p + hN_p(\underline{x}) + O(h^2)$, $B_{n1}(\underline{x}) = \{fg\}(\underline{x})B_1 + h\tilde{M}(\underline{x}) + O(h^2)$, and $B_{n2}(\underline{x}) = \{fg\}(\underline{x})B_2 + O(h)$. As $S_{n,p}^{-1}(\underline{x}) = \{fg\}^{-1}(\underline{x})S_p^{-1} - h\{fg\}^{-2}(\underline{x})S_p^{-1}$

$N_p(\underline{x})S_p^{-1} + O(h^2)$, we have

$$
\begin{aligned}
-\mathrm{E}\beta_n^*(\underline{x}) = {} & W_p h^{p+1}\Big[\{fg\}^{-1}(\underline{x})S_p^{-1} - h\{fg\}^{-2}(\underline{x})S_p^{-1}N_p(\underline{x})S_p^{-1}\Big] \\
& \times \Big[\{fg\}(\underline{x})B_1 + h\tilde{M}(\underline{x})\Big]\mathbf{m}_{p+1}(\underline{x}) + W_p h^{p+2}\{fg\}^{-1}(\underline{x})S_p^{-1}\{fg\}(\underline{x}) \\
& \times B_2\mathbf{m}_{p+2}(\underline{x}) + o(h^{p+2}) \\
= {} & h^{p+1}W_p S_p^{-1}B_1\mathbf{m}_{p+1}(\underline{x}) + h^{p+2}W_p S_p^{-1} \\
& \times \Big[\{fg\}^{-1}(\underline{x})\mathbf{m}_{p+1}(\underline{x})\{\tilde{M}(\underline{x}) - N_p(\underline{x})S_p^{-1}B_1\} \\
& + B_2\mathbf{m}_{p+2}(\underline{x})\Big] + o(h^{p+2}).
\end{aligned}
$$

We claim that for elements $\mathrm{E}\beta_{n\underline{r}}^*(\underline{x})$ of $\mathrm{E}\beta_n^*(\underline{x})$ with $p - |\underline{r}|$ even, the h^{p+1} term will vanish. This means for any given $\underline{r}$ with $|\underline{r}| \le p$ and $\underline{r}_2$ with $|\underline{r}_2| = p+1$,

$$
\sum_{0\le|\underline{r}|\le p} \{S_p^{-1}\}_{N(\underline{r}_1),N(\underline{r})}\, \nu_{\underline{r}+\underline{r}_2} = 0. \tag{A.8}
$$

To prove this, first note that for any $\underline{r}_1$ with $0 \le |\underline{r}_1| \le p$ and $\underline{r}_2$ with $|\underline{r}_2| = p+1$,

$$
\sum_{0\le|\underline{r}|\le p} \{S_p^{-1}\}_{N(\underline{r}_1),N(\underline{r})}\, \nu_{\underline{r}+\underline{r}_2} = \int \underline{u}^{\underline{r}_2} K_{\underline{r}_1,p}(\underline{u})\, d\underline{u}, \tag{A.9}
$$

where $K_{\underline{r},p}(\underline{u}) = \{|M_{\underline{r},p}(\underline{u})|/|S_p|\}K(\underline{u})$ and $M_{\underline{r},p}(\underline{u})$ is the same as S_p, but with the $N(\underline{r})$ column replaced by $\mu(\underline{u})$. Let c_{ij} denote the cofactor of $\{S_p\}_{i,j}$, and expand the determinant of $M_{\underline{r},p}(\underline{u})$ along the $N(\underline{r})$ column. We can see that

$$
\int \underline{u}^{\underline{r}_2} K_{\underline{r},p}(\underline{u})\, d\underline{u} = |S_p|^{-1} \int \sum_{0\le|\underline{r}|\le p} c_{N(\underline{r}),N(\underline{r}_1)} \underline{u}^{\underline{r}_2+\underline{r}} K(\underline{u})\, d\underline{u},
$$

whence (A.9) follows, because $c_{N(\underline{r}),N(\underline{r}_1)}/|S_p| = \{S_p^{-1}\}_{N(\underline{r}_1),N(\underline{r})}$ from the symmetry of S_p and a standard result concerning cofactors. As a generalization of Lemma 4 in Fan, Heckman, and Wand (1995) to the multivariate case, we can further show that for any $\underline{r}_1$ with $0 \le |\underline{r}_1| \le p$ and $p - |\underline{r}_1|$ even,

$$
\int \underline{u}^{\underline{r}_2} K_{\underline{r},p}(\underline{u})\, d\underline{u} = 0, \quad \text{for any } |\underline{r}_2| = p+1,
$$

which together with (A.9) leads to (A.8). ∎

We proceed to prove Theorem 1. Define $\underline{X}_{ix} = \underline{X}_i - \underline{x}$, $\mu_{ix} = \mu(\underline{X}_{ix})$, $K_{ix} = K_h(\underline{X}_{ix})$, and $\varphi_{ni}(\underline{x};t) = \varphi(Y_i; \mu_{ix}^\top \beta_p(\underline{x}) + t)$. For any $\alpha,\ \beta \in \mathcal{R}^N$, define

$$
\begin{aligned}
\Phi_{ni}(\underline{x};\alpha,\beta) &= K_{ix}\Big\{\rho(Y_i;\mu_{ix}^\top(\alpha+\beta+\beta_p(\underline{x}))) - \rho(Y_i;\mu_{ix}^\top(\beta+\beta_p(\underline{x}))) - \varphi_i(\underline{x};0)\mu_{ix}^\top\alpha\Big\} \\
&= K_{ix}\int_{\mu_{ix}^\top\beta}^{\mu_{ix}^\top(\alpha+\beta)} \{\varphi_{ni}(\underline{x};t) - \varphi_{ni}(\underline{x};0)\}\, dt
\end{aligned}
$$

and $R_{ni}(\underline{x};\alpha,\beta) = \Phi_{ni}(\underline{x};\alpha,\beta) - \mathrm{E}\Phi_{ni}(\underline{x};\alpha,\beta)$.

LEMMA 1. *Under Assumptions A1–A6, we have for all large $M > 0$,*

$$\sup_{\underline{x}\in\mathcal{D}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^{n} R_{ni}(\underline{x};\alpha,\beta)\right| \le M^{3/2} d_n \quad \textit{almost surely}, \tag{A.10}$$

where $B_n^{(i)} \equiv \{\beta \in \mathcal{R}^N : |H_n\beta| \le M_n^{(i)}\}$, $i = 1, 2$.

Proof. Since $\mathcal{D}$ is compact, it can be covered by a finite number T_n of cubes $\mathcal{D}_k = \mathcal{D}_{n,k}$ with side length $l_n = O(\mathrm{T}_n^{-1/d}) = O\{h(nh^d/\log n)^{-(1-\lambda_2)/2}\}$ and centers $\underline{x}_k = \underline{x}_{n,k}$. Write

$$\begin{aligned}
&\sup_{\underline{x}\in\mathcal{D}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^{n} R_{ni}(\underline{x};\alpha,\beta)\right| \\
&\quad\le \max_{1\le k\le \mathrm{T_n}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^{n} \Phi_{ni}(\underline{x}_k;\alpha,\beta) - \mathrm{E}\Phi_{ni}(\underline{x}_k;\alpha,\beta)\right| \\
&\qquad + \max_{1\le k\le \mathrm{T}_n} \sup_{\underline{x}\in\mathcal{D}_k} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^{n} \left\{\Phi_{ni}(\underline{x}_k;\alpha,\beta) - \Phi_{ni}(\underline{x};\alpha,\beta)\right\}\right| \\
&\qquad + \max_{1\le k\le \mathrm{T}_n} \sup_{\underline{x}\in\mathcal{D}_k} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^{n} \left\{\mathrm{E}\Phi_{ni}(\underline{x}_k;\alpha,\beta) - \mathrm{E}\Phi_{ni}(\underline{x};\alpha,\beta)\right\}\right| \\
&\quad\equiv Q_1 + Q_2 + Q_3.
\end{aligned}$$

In Lemma 2, it is shown that $Q_2 \le M^{3/2} d_n/3$ almost surely and thus $Q_3 \le M^{3/2} d_n/3$.

It remains to bound Q_1. To this end, partition $B_n^{(i)}$, $i = 1, 2$, into a sequence of disjoint subrectangles $D_1^{(i)}, \ldots, D_{J_1}^{(i)}$, such that

$$|D_{j_1}^{(i)}| = \sup\left\{|H_n(\alpha-\beta)| : \alpha,\beta \in D_{j_1}^{(i)}\right\} \le 2M^{-1} M_n^{(i)}/\log n, \qquad 1 \le j_1 \le J_1.$$

Apparently, $J_1 \le (M\log n)^N$. For every $1 \le j_1 \le J_1, 1 \le k_1 \le J_1$, choose a point $\alpha_{j_1} \in D_{j_1}^{(1)}$ and $\beta_{k_1} \in D_{k_1}^{(2)}$. Then

$$\begin{aligned}
Q_1 \le{} & \max_{\substack{1\le k\le \mathrm{T}_n\\ 1\le j_1,k_1\le J_1}} \sup_{\substack{\alpha\in D_{j_1}^{(1)},\\ \beta\in D_{k_1}^{(2)}}} \left|\sum_{i=1}^{n} \{R_{ni}(\underline{x}_k;\alpha_{j_1},\beta_{k_1}) - R_{ni}(\underline{x}_k;\alpha,\beta)\}\right| \\
& + \max_{\substack{1\le k\le \mathrm{T}_n\\ 1\le j_1,k_1\le J_1}} \left|\sum_{i=1}^{n} R_{ni}(\underline{x}_k;\alpha_{j_1},\beta_{k_1})\right| = H_{n1} + H_{n2}.
\end{aligned} \tag{A.11}$$

We first consider H_{n1}. For each $j_1 = 1, \ldots, J_1$ and $i = 1, 2$, partition each rectangle $D^{(i)}_{j_1}$ further into a sequence of subrectangles $D^{(i)}_{j_1,1}, \ldots, D^{(i)}_{j_1,J_2}$. Repeat this process recursively as follows: Suppose after the lth round we get a sequence of rectangles $D^{(i)}_{j_1,j_2,\ldots,j_l}$ with $1 \le j_k \le J_k,\ 1 \le k \le l$; then in the $(l+1)$th round, each rectangle $D^{(i)}_{j_1,j_2,\ldots,j_l}$ is partitioned into a sequence of subrectangles $\{D^{(i)}_{j_1,j_2,\ldots,j_l,j_{l+1}}, 1 \le j_l \le J_l\}$ such that

$$
\begin{aligned}
|D^{(i)}_{j_1,j_2,\ldots,j_l,j_{l+1}}| &= \sup\left\{|H_n(\alpha-\beta)| : \alpha,\beta \in D^{(i)}_{j_1,j_2,\ldots,j_l,j_{l+1}}\right\} \\
&\le 2M^{(i)}_n/(M^l \log n), \qquad 1 \le j_{l+1} \le J_{l+1},
\end{aligned}
$$

where $J_{l+1} \le M^N$. End this process after the (L_n+1)th round, with L_n given at the beginning of Section 3. Let $D^{(i)}_l$, $i = 1, 2$, denote the set of all subrectangles of $D^{(i)}_0$ after the lth round of partition and a typical element $D^{(i)}_{j_1,j_2,\ldots,j_l}$ of $D^{(i)}_l$ is denoted as $D^{(i)}_{(j_l)}$. Choose a point $\alpha_{(j_l)} \in D^{(1)}_{(j_l)}$ and $\beta_{(j_l)} \in D^{(2)}_{(j_l)}$ and define

$$
V_l = \sum_{\substack{(j_l),\\(k_l)}} P\left\{\left|\sum_{i=1}^n \{R_{ni}(\underline{x}_k;\alpha_{j_l},\beta_{k_l}) - R_{ni}(\underline{x}_k;\alpha_{j_{l+1}},\beta_{k_{l+1}})\}\right| \ge \frac{M^{3/2}d_n}{2^l}\right\}, \quad 1 \le l \le \mathrm{L}_n,
$$

$$
Q_l = \sum_{\substack{(j_l),\\(k_l)}} P\left\{\sup_{\substack{\alpha\in D^{(1)}_{(j_l)},\\ \beta\in D^{(2)}_{(k_l)}}} \left|\sum_{i=1}^n \{R_{ni}(\underline{x}_k;\alpha_{j_l},\beta_{k_l}) - R_{ni}(\underline{x}_k;\alpha,\beta)\}\right| \ge \frac{M^{3/2}d_n}{2^l}\right\}, \quad 1 \le l \le \mathrm{L}_n+1.
$$

By Assumption A4 it is easy to see that, for any $\alpha \in D^{(1)}_{(j_{\mathrm{L}_n+1})} \in D^{(1)}_{\mathrm{L}_n+1}$ and $\beta \in D^{(2)}_{(k_{\mathrm{L}_n+1})} \in D^{(2)}_{\mathrm{L}_n+1}$,

$$
|R_{ni}(\underline{x}_k;\alpha,\beta) - R_{ni}(\underline{x}_k;\alpha_{j_{\mathrm{L}_n+1}},\beta_{k_{\mathrm{L}_n+1}})| \le \frac{CM^{(2)}_n}{M^{\mathrm{L}_n+1}\log n},
$$

which together with the choice of L_n implies that $Q_{\mathrm{L}_n+1} = 0$. As $Q_l \le V_l + Q_l,\ 1 \le l \le \mathrm{L}_n$,

$$
P\left(H_{n1} > \frac{M^{3/2}d_n}{2}\right) \le \mathrm{T}_n Q_1 \le \mathrm{T}_n \sum_{l=1}^{\mathrm{L}_n} V_l. \tag{A.12}
$$

To bound $V_l,\ l = 1, \ldots, \mathrm{L}_n$, let

$$
W_n = \sum_{i=1}^n Z_{ni}, \qquad Z_{ni} \equiv R_{ni}(\underline{x}_k;\alpha_{j_l},\beta_{k_l}) - R_{ni}(\underline{x}_k;\alpha_{j_{l+1}},\beta_{j_{l+1}}). \tag{A.13}
$$

Note that by Assumption A2 we have, uniformly in $\underline{x}$, α, and β, that

$$
|\Phi_{ni}(\underline{x};\alpha,\beta)| \le CM^{(1)}_n. \tag{A.14}
$$

Therefore, $|Z_{ni}| \le CM^{(1)}_n$. Using Lemma 6, we can apply Lemma 4 to each V_l with

$$
\begin{aligned}
&B_1 = C_1 M^{(1)}_n, \qquad B_2 = nh^d (M^{(1)}_n)^2 M^{(2)}_n \{M^l \log n\}^{-2/\nu_2}, \\
&r_n = r^l_n \equiv (2^{\nu_2/2}/M)^{2l/\nu_2} r(n), \qquad q = n/r^l_n, \qquad \eta = M^{3/2}d_n/2^l, \\
&\lambda_n = (2C_1 M^{(1)}_n r^l_n)^{-1}, \qquad \Psi(n) = Cq^{3/2}/\eta^{1/2}\gamma\,[r^l_n]\{r^l_n M^{(1)}_n\}^{1/2}.
\end{aligned}
$$

Note that $nM_n^{(1)}/\eta \to \infty$, $r_n^l \to \infty$ for all $1 \le l \le \mathrm{L}_n$ from (A.2) and

$$\lambda\eta = CM^{1/2}\log n M^{2l/\nu_2}/2^{2l}, \qquad \lambda^2 B_2 = C\log n^{1-2/\nu_2} M^{2l/\nu_2}/2^{2l} = o(\lambda\eta),$$

which hold uniformly for all $1 \le l \le \mathrm{L}_n$. Therefore,

$$V_l \le \Bigl(\prod_{j=1}^{l+1} J_j^2\Bigr) 4\exp\{-C_1 \log n (M/2^{\nu_2})^{2l/\nu_2}\} + C_2\tau_n^l,$$

where, because $J_1 \le 2(M\log n)^N$ and $J_l \le 2M^N$ for $2 \le l \le L_n$, τ_n^l is given by

$$\tau_n^l = 4^l M^{2N(l+1)}(\log n)^{2N} n^{3/2} \frac{\gamma[r_n^l]\{M_n^{(1)}\}^{1/2}}{r_n^l\{d_n\}^{1/2}}.$$

It is tedious but easy to check that, for M large enough,

$$\mathrm{T}_n \sum_{l=1}^{\mathrm{L}_n}\Bigl[\Bigl(\prod_{j=1}^{l+1} J_j^2\Bigr) 4\exp\{-C_1\log n (M/2^{\nu_2})^{2l/\nu_2}\}\Bigr] \quad \text{is summable over } n. \qquad \textbf{(A.15)}$$

As $\gamma[r_n^l]/r_n^l$ is increasing in l, we have

$$\mathrm{T}_n \sum_{l=1}^{\mathrm{L}_n} \tau_n^l \le \mathrm{T}_n (\log n)^{2N} n^{3/2} \frac{\{M_n^{(1)}\}^{1/2}}{\{d_n\}^{1/2}} \frac{\gamma[r_n^{\mathrm{L}_n}]}{r_n^{\mathrm{L}_n}} \prod_{l=1}^{\mathrm{L}_n} 4^l M^{2N(l+1)},$$

which is again summable over n according to (A.4). This along with (A.12) and (A.15) implies that $H_{n1} \le M^{3/2}d_n/2$ almost surely, using the Borel-Cantelli lemma.

For H_{n2}, first note that

$$P(H_{n2} > \eta) \le \mathrm{T}_n J_1^2 P\left(\left|\sum_{i=1}^n R_{ni}(\underline{x};\alpha_{j_1},\beta_{k_1})\right| > \eta\right). \qquad \textbf{(A.16)}$$

We apply Lemma 4 to quantify $P(|\sum_{i=1}^n R_{ni}(\underline{x};\alpha_{j_1},\beta_{k_1}| > \eta)$, with $r_n = r(n)$, $B_1 = 2C_1M_n^{(1)}$, $B_2 = C_2nh^d(M_n^{(1)})^2M_n^{(2)}$, $\lambda_n = \{r(n)M_n^{(1)}\}^{-1}/4C_1$, and $\eta = M^{3/2}d_n$. Then $nB_1/\eta \to \infty$ and

$$\lambda_n\eta/4 = (nh^d)^{(1-\lambda_2)/2}(\log n)^{(1+\lambda_2)/2}/\{16C_1 r(n)\} = M^{1/2}\log n/(16C_1),$$

$$\lambda_n^2 B_2 = M^{1/4}(nh^d)^{1-\lambda_2}(\log n)^{\lambda_2}/\{16C_1^2 r^2(n)\} = M^{1/4}\log n/(16C_1^2),$$

$$\Psi(n) \equiv q_n\{nB_1/\eta\}^{1/2}\gamma[r_n] = \mathrm{T}_n J_1^2 q(n)^{3/2}/\eta^{1/2}\gamma[r(n)]\{r(n)M_n^{(1)}\}^{1/2},$$

where $\Psi(n)$ is summable over n under condition (A.4). Therefore,

$$P(H_{n2} > \eta) \le 2\mathrm{T}_n J_1^2/n^b + \Psi(n), \qquad b = \frac{1}{16C_1}(M^{1/2} - M^{1/4}C_2/C_1). \qquad \textbf{(A.17)}$$

By selecting M large enough, we can ensure that the right-hand side of (A.17) is summable over n. Thus, for M large enough, $H_{n2} \le M^{3/2}d_n$ almost surely. By (A.39), we know that for large M, $Q_1 \le M^{3/2}d_n$ almost surely. ∎

The quantification of Q_2 is relatively more involved, so we put it as a separate lemma.

LEMMA 2. *Under conditions in Lemma 1, $Q_2 \le M^{3/2}d_n/3$ almost surely.*

Proof. Let $\underline{X}_{ik} = \underline{X}_i - \underline{x}_k$, $\mu_{ik} = \mu(\underline{X}_{ik})$, and $K_{ik} = K_h(\underline{X}_{ik})$. Write $\Phi_{ni}(\underline{x}_k;\alpha,\beta) - \Phi_{ni}(x;\alpha,\beta) = \xi_{i1} + \xi_{i2} + \xi_{i3}$, where

$$\xi_{i1} = \left(K_{ik}\mu_{ik} - K_{ix}\mu_{ix}\right)^\top \alpha \int_0^1 \left\{\varphi_{ni}(\underline{x}_k;\mu_{ik}^\top(\beta+\alpha t)) - \varphi_{ni}(\underline{x}_k;0)\right\} dt,$$

$$\xi_{i2} = K_{ix}\mu_{ix}^\top \alpha \int_0^1 \left\{\varphi_{ni}(\underline{x}_k;\mu_{ik}^\top(\beta+\alpha t)) - \varphi_{ni}(x;\mu_{ix}^\top(\beta+\alpha t))\right\} dt,$$

$$\xi_{i3} = K_{ix}\mu_{ix}^\top \alpha\{\varphi_{ni}(x;0) - \varphi_{ni}(\underline{x}_k;0)\}.$$

Then $P(Q_2 > M^{3/2}d_n/3) \le \mathrm{T}_n(P_{n1} + P_{n2} + P_{n3})$, with

$$P_{nj} \equiv \max_{1\le k\le \mathrm{T}_n} P\Big(\sup_{\underline{x}\in\mathcal{D}_k} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} |\sum_{i=1}^n \xi_{ij}| \ge M^{3/2}d_n/9\Big), \qquad j = 1,2,3.$$

Based on the Borel-Cantelli lemma, $Q_2 \le M^{3/2}d_n$ almost surely, if $\sum_n \mathrm{T}_n P_{nj} < \infty$, $j = 1,2,3$.

We first study P_{n1}. For any fixed $\alpha \in B_n^{(1)}$ and $\beta \in B_n^{(2)}$, let $I_{ik}^{\alpha,\beta} = 1$, if there exists some $t \in [0,1]$, such that there are discontinuity points of $\varphi(Y_i;\theta)$ between $\mu_{ik}^\top(\beta_p(\underline{x}_k) + \beta + \alpha t))$ and $\mu_{ik}^\top\beta_p(\underline{x}_k)$; and $I_{ik}^{\alpha,\beta} = 0$, otherwise. Write $\xi_{i1} = \xi_{i1}I_{ik}^{\alpha,\beta} + \xi_{i1}(1 - I_{ik}^{\alpha,\beta})$. Note that by Assumption A3, $|(K_{ik}\mu_{ik} - K_{ix}\mu_{ix})^\top\alpha| \le C_2M_n^{(1)}l_n/h$. Then by Assumption A2 and the fact that $|\mu_{ik}^\top(\beta+\alpha t)| \le CM_n^{(2)}$, we have $|\xi_{i1}(1 - I_{ik}^{\alpha,\beta})| \le CM_n^{(2)}M_n^{(1)}l_n/h$ uniformly in i,α,β, and $\underline{x} \in \mathcal{D}_k$. Define $U_{ik} = I\{|\underline{X}_{ik}| \le 2h\}$, whence $\xi_{i1} = \xi_{i1}U_{ik}$ since $l_n = o(h)$. Therefore,

$$P\Bigg(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \sup_{\underline{x}\in\mathcal{D}_k} \Big|\sum_{i=1}^n \xi_{i1}(1 - I_{ik}^{\alpha,\beta})\Big| > \frac{M^{3/2}d_n}{18}\Bigg)$$

$$\le P\Big(\sum_{i=1}^n U_{ik} > \frac{M^{1/4}nh^d}{18C}\Big) \le P\Big(|\sum_{i=1}^n U_{ik} - \mathrm{E}U_{ik}| > \frac{M^{1/4}nh^d}{36C}\Big), \tag{A.18}$$

where the second inequality follows from the fact that $\mathrm{Var}(\sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h) = O(nh^d)$ implied by Lemma 5. To quantify (A.18), we apply Lemma 4 with $B_1 = 1$, $\eta = M^{1/4}nh^d/(18C)$, $B_2 = nh^d$, $r_n = r(n)$. As $\lambda_n\eta = CM^{1/4}\log n(nh^d/\log n)^{(1+\lambda_2)/2}$, $\lambda_n^2 B_2 = o(\lambda_n\eta)$, and $\mathrm{T}_n\Psi_n$ is summable over n under condition (A.4), we know that

$$\mathrm{T}_n P\Bigg(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \Big|\sum_{i=1}^n \xi_{i1}(1 - I_{ik}^{\alpha,\beta})\Big| > M^{3/2}d_n/18\Bigg) \quad \text{is summable over } n, \tag{A.19}$$

whence $\sum_n \mathrm{T}_n P_{n1} < \infty$, is equivalent to

$$\mathrm{T}_n P\Bigg(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \Big|\sum_{i=1}^n \xi_{i1}I_{ik}^{\alpha,\beta}\Big| > M^{3/2}d_n/18\Bigg) \quad \text{is summable over } n. \tag{A.20}$$

To prove (A.20), first note that $I_{ik}^{\alpha,\beta} \le I\{\varepsilon_i \in S_{i;k}^{\alpha,\beta}\}$, where

$$\begin{aligned} S_{i;k}^{\alpha,\beta} &= \bigcup_{j=1}^{m} \bigcup_{t\in[0,1]} [a_j - A(\underline{X}_i, \underline{x}_k) + \mu_{ik}^\top(\beta+\alpha t), a_j - A(\underline{X}_i, \underline{x}_k)] \\ &\subseteq \bigcup_{j=1}^{m} [a_j - CM_n^{(2)}, a_j + CM_n^{(2)}] \equiv D_n, \quad \text{for some } C > 0, \end{aligned}$$

$$A(\underline{x}_1, \underline{x}_2) = (p+1) \sum_{|\underline{r}|=p+1} \frac{1}{\underline{r}!} (\underline{x}_1 - \underline{x}_2)^{\underline{r}} \int_0^1 D^{\underline{r}} m(\underline{x}_2 + w(\underline{x}_1 - \underline{x}_2))(1-w)^p \, dw,$$

where in the derivation of $S_{i;k}^{\alpha,\beta} \subseteq D_n$, we have used the fact that $|\underline{X}_{ik}| \le 2h$ and $A(\underline{X}_i, \underline{x}_k) = O(h^{p+1}) = O(M_n^{(2)})$ uniformly in i. As $I_{ik}^{\alpha,\beta} \le I\{\varepsilon_i \in D_n\}$, we have $|\xi_{i1}| I_{ik}^{\alpha,\beta} \le |\xi_{i1}| U_{ni}$, where $U_{ni} \equiv I(|\underline{X}_{ik}| \le 2h) I\{\varepsilon_i \in D_n\}$, which is independent of the choice of α and β. Therefore,

$$\begin{aligned} P\left(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left| \sum_{i=1}^n \xi_{i1} I_{ik}^{\alpha,\beta} \right| > M^{3/2} d_n/18 \right) &\le P\left(\sum_{i=1}^n U_{ni} > M^{1/2} n h^d M_n^{(2)}/(18C) \right) \\ &\le P\left(\sum_{i=1}^n (U_{ni} - \mathrm{E}U_{ni}) > \frac{M^{1/2} n h^d M_n^{(2)}}{36C} \right), \end{aligned} \tag{A.21}$$

where the first inequality is because $|\xi_{i1}| \le CM_n^{(1)} l_n/h$ and the second one is because $\mathrm{E}U_{ni} = O(h^d M_n^{(2)})$ by Assumption A1. As $\mathrm{E}U_{ni}^2 = \mathrm{E}U_{ni}$, by Lemma 5, we know that $\mathrm{Var}(\sum_{i=1}^n U_{ni}) = Cnh^d M_n^{(2)}$. We can then apply Lemma 4 to the last term in (A.21) with

$$B_2 = Cnh^d M_n^{(2)}, \qquad B_1 \equiv 1, \qquad r_n = r(n), \qquad \eta \equiv M^{1/2} n h^d M_n^{(2)}/(36C).$$

Apparently, $\lambda_n \eta = C \log n (nh^d/\log n)^{(1-\lambda_2)/2}$ and $\lambda_n^2 B_2 = o(\lambda_n \eta)$. As in this case $\mathrm{T}_n \Psi_n$ is still summable over n by (A.4), (A.20) follows.

For P_{n2}, first note that using the approach for P_{n1}, we can show that

$$\mathrm{T}_n P\left(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \sup_{\underline{x}\in\mathcal{D}_k} \left| \sum_{i=1}^n \{\xi_{i2} - \tilde{\xi}_{i2}\} \right| \ge M^{3/2} d_n/18 \right) \quad \text{is summable over } n,$$

where

$$\tilde{\xi}_{i2} = K_{ik} \mu_{ik}^\top \alpha \int_0^1 \left\{ \varphi_{ni}(\underline{x}_k; \mu_{ik}^\top(\beta+\alpha t)) - \varphi_{ni}(x; \mu_{ix}^\top(\beta+\alpha t)) \right\} dt.$$

Therefore, we would have $\sum \mathrm{T}_n P_{n2} < \infty$, if

$$\mathrm{T}_n P\left(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \sup_{\underline{x}\in\mathcal{D}_k} \left| \sum_{i=1}^n \tilde{\xi}_{i2} \right| \ge M^{3/2} d_n/18 \right) \quad \text{is summable over } n. \tag{A.22}$$

For any fixed $\alpha \in B_n^{(1)}$, $\beta \in B_n^{(2)}$, and $\underline{x} \in \mathcal{D}_k$, let $I_{i;k,x}^{\alpha,\beta} = 1$, if there exists some interval $[t_1, t_2] \subseteq [0, 1]$, such that

$$Y_i - \mu_{ik}^\top(\beta_p(\underline{x}_k) + \beta + \alpha t) \le a_j \le Y_i - \mu_{ix}^\top(\beta_p(\underline{x}) + \beta + \alpha t), \qquad \forall t \in [t_1, t_2], \tag{A.23}$$

with $a_j \in \{a_1, \ldots, a_m\}$; and $I_{i;k,x}^{\alpha,\beta} = 0$, otherwise. Write $\tilde{\xi}_{i2} = \tilde{\xi}_{i2} I_{i;k,x}^{\alpha,\beta} + \tilde{\xi}_{i2}(1 - I_{i;k,x}^{\alpha,\beta})$. Note that $K_{ik}\mu_{ik}^\top\alpha = O(M_n^{(1)})$ and $\varphi_{ni}(\underline{x}_k; \mu_{ik}^\top(\beta + \alpha t)) - \varphi_{ni}(x; \mu_{ix}^\top(\beta + \alpha t)) = O(M_n^{(2)} l_n/h)$ if $I_{i;k,x}^{\alpha,\beta} = 0$. Then again as $\tilde{\xi}_{i2} = \tilde{\xi}_{i2} I\{|\underline{X}_{ik}| \le 2h\}$, we have similar to (A.19) that

$$\mathrm{T}_n P\Bigg(\sup_{\substack{\alpha \in B_n^{(1)}, \\ \beta \in B_n^{(2)}}} \Big| \sum_{i=1}^n \tilde{\xi}_{i2}(1 - I_{i;k,x}^{\alpha,\beta}) \Big| > M^{3/2} d_n/18 \Bigg) \quad \text{is summable over } n.$$

Therefore, by (A.22), to show $\sum \mathrm{T}_n P_{n2} < \infty$, it is sufficient to show that

$$\mathrm{T}_n P\Bigg(\sup_{\substack{\alpha \in B_n^{(1)}, \\ \beta \in B_n^{(2)}}} \sup_{\underline{x} \in \mathcal{D}_k} \Big| \sum_{i=1}^n \tilde{\xi}_{i2} I_{i;k,x}^{\alpha,\beta} \Big| \ge M^{3/2} d_n/36 \Bigg) \text{ is summable over } n. \tag{A.24}$$

To this end, define $\epsilon_i = \varepsilon_i + A(\underline{X}_i, \underline{x}_k)$. Then $I_{i;k,x}^{\alpha,\beta} = 1$; i.e., (A.23) is equivalent to

$$A(\underline{X}_i, \underline{x}_k) - A(\underline{X}_i, \underline{x}) + \mu_{ix}^\top(\beta + \alpha t) \le \epsilon_i - a_j \le \mu_{ik}^\top(\beta + \alpha t), \qquad \forall t \in [t_1, t_2]. \tag{A.25}$$

Let $\delta_n \equiv M_n^{(2)} l_n/h$. Then $|A(\underline{X}_i, \underline{x}_k) - A(\underline{X}_i, \underline{x})| \le C\delta_n$, $|(\mu_{ik} - \mu_{ix})^\top \beta| \le C\delta_n$, and (A.25) thus implies that

$$-2C\delta_n + \mu_{ik}^\top(\beta + \alpha t) \le \epsilon_i - a_j \le \mu_{ik}^\top(\beta + \alpha t) + 2C\delta_n, \qquad \forall t \in [t_1, t_2]. \tag{A.26}$$

Without loss of generality, assume $\mu_{ik}^\top\alpha > 0$. Then from (A.26) we can see that

$$-2C\delta_n + \mu_{ik}^\top(\beta + \alpha t_2) \le \epsilon_i - a_j \le \mu_{ik}^\top(\beta + \alpha t_1) + 2C\delta_n, \tag{A.27}$$

which in turn means that if $I_{i;k,x}^{\alpha,\beta} = 1$, then $|\xi_{i2}| \le C(t_2 - t_1)|\mu_{ik}^\top\alpha| \le 4C\delta_n$ uniformly in i, $\alpha \in B_n^{(1)}$, $\beta \in B_n^{(2)}$, and $\underline{x} \in \mathcal{D}_k$. Therefore, as $\tilde{\xi}_{i2} = \tilde{\xi}_{i2} I\{|\underline{X}_{ik}| \le 2h\}$, we have

$$P\Bigg(\sup_{\substack{\alpha \in B_n^{(1)} \\ \beta \in B_n^{(2)}}} \sup_{\underline{x} \in \mathcal{D}_k} \Big| \sum_{i=1}^n \tilde{\xi}_{i2} I_{i;k,x}^{\alpha,\beta} \Big| \ge \frac{M^{3/2} d_n}{36} \Bigg)$$

$$\le P\Bigg(\sup_{\substack{\alpha \in B_n^{(1)} \\ \beta \in B_n^{(2)}}} \sup_{\underline{x} \in \mathcal{D}_k} \sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h\} I_{i;k,x}^{\alpha,\beta} \ge \frac{M^{5/4} n h^d M_n^{(1)}}{36C} \Bigg). \tag{A.28}$$

We will bound $I_{i;k,x}^{\alpha,\beta}$ by a random variable that is independent of the choice of $\alpha \in B_n^{(1)}$ and $\underline{x} \in D_k$. By the definition of $I_{i;k,x}^{\alpha,\beta}$ and (A.27), the necessary condition for $I_{i;k,x}^{\alpha,\beta} = 1$ is

$$\epsilon_i \in \bigcup_{j=1}^m [a_j + \mu_{ik}^\top\beta - 2M_n^{(1)}, \qquad a_j + \mu_{ik}^\top\beta + 2M_n^{(1)}] \equiv D_{ni}^\beta, \tag{A.29}$$

which is indeed independent of the choice of α and $\underline{x} \in \mathcal{D}_k$. Therefore,

$$P\left(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \sup_{\underline{x}\in\mathcal{D}_k} \sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h\} I_{i;k,x}^{\alpha,\beta} \ge \frac{M^{5/4} nh^d M_n^{(1)}}{36C}\right)$$
$$\le P\left(\sup_{\beta\in B_n^{(2)}} \sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h\} I\{\epsilon_i \in D_{ni}^{\beta}\} \ge \frac{M^{5/4} nh^d M_n^{(1)}}{36C}\right). \tag{A.30}$$

Now we partition $B_n^{(2)}$ into a sequence of subrectangles $S_1, \dots, S_m$, such that

$$|S_l| = \sup\left\{|H_n(\beta-\beta')| : \beta, \beta' \in S_l\right\} \le M_n^{(1)}, \qquad 1 \le l \le m.$$

Obviously, $m \le (M_n^{(2)}/M_n^{(1)})^N = M^{-3N/4}(nh^d/\log n)^{(\lambda_1-\lambda_2)N}$. Choose a point $\beta_l \in S_l$ for each $1 \le l \le m$, and thus

$$P\left(\sup_{\beta\in B_n^{(2)}} \sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h\} I\{\epsilon_i \in D_{ni}^{\beta}\} \ge \frac{M^{5/4} nh^d M_n^{(1)}}{36C}\right)$$
$$\le mP\left(\sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h\} I\{\epsilon_i \in D_{ni}^{\beta_l}\} \ge \frac{M^{5/4} nh^d M_n^{(1)}}{72C}\right)$$
$$+ mP\left(\sup_{\beta'\in S_l} \sum_{i=1}^n I\{|\underline{X}_{ik}| \le 2h\} |I\{\epsilon_i \in D_{ni}^{\beta_l}\} - I\{\epsilon_i \in D_{ni}^{\beta'}\}| \ge \frac{M^{5/4} nh^d M_n^{(1)}}{72C}\right)$$
$$\equiv m(T_1 + T_2). \tag{A.31}$$

We deal with T_1 first. Let

$$U_{ni}^j \equiv I\{|\underline{X}_{ik}| \le 2h\} I\{\epsilon_i \in D_{ni}^{\beta_l}\}. \tag{A.32}$$

Then by the definition of $D_{ni}^{\beta_j}$ given in (A.29), $\mathrm{E}U_{ni}^j = O(h^d M_n^{(1)}) < M^{5/4} h^d M_n^{(1)}/(144C)$ for large M, and we have

$$T_1 \le P\left(\sum_{i=1}^n (U_{ni}^j - \mathrm{E}U_{ni}^j) \ge \frac{M^{5/4} nh^d M_n^{(1)}}{144C}\right).$$

We can thus apply Lemma 4 to the quantity on the right-hand side with $B_1 \equiv 1$, B_2 given by (A.51), $r_n = r(n)$, $\eta \propto M^{5/4} nh^d M_n^{(1)}$, and $\lambda_n = 1/(2r_n)$. It follows that

$$\lambda_n \eta = CM^{5/4} \log n (nh^d/\log n)^{(1+\lambda_2)/2-\lambda_1}, \qquad \lambda_n^2 B_2 = C \log n (nh^d/\log n)^{-2(\lambda_1-\lambda_2)/\nu_2}.$$

As $(1+\lambda_2)/2 \ge \lambda_1$ and $\lambda_2 < \lambda_1$, we have $T_1 = O(n^{-b})$ for any $b > 0$.

For T_2, note that as $|\mu_{ik}^\top(\beta - \beta_l)| \le CM_n^{(1)}$ for any $\beta \in S_l$, $1 \le l \le m$, we have

$$|I\{\epsilon_i \in D_{ni}^{\beta_l}\} - I\{\epsilon_i \in D_{ni}^{\beta}\}|$$
$$= I\{\epsilon_i \in D_{ni}^{\beta_l} \setminus D_{ni}^{\beta}\}$$
$$\le I\left\{\epsilon_i \in \bigcup_{j=1}^m \left[a_j + \mu_{ik}^\top \beta_l - CM_n^{(1)}, a_j + \mu_{ik}^\top \beta_l + CM_n^{(1)}\right]\right\} \equiv U_{ni},$$

for some $C > 0$, which is independent of the choice of $\beta \in S_l$. Therefore,

$$T_2 \leq P\Big(\sum_{i=1}^{n} I\{|\underline{X}_{ik}| \leq 2h\}U_{ni} \geq \frac{M^{5/4}nh^d M_n^{(1)}}{72C}\Big),$$

which can be dealt with similarly as with T_1, and thus $T_2 = O(n^{-b})$ for any $b > 0$. Thus from (A.28), (A.30), and (A.31), we can claim that (A.24) is true and thus $\mathrm{T}_n P_{n2}$ is summable over n.

Dealing with P_{n3} is simpler, as no β is involved in ξ_{i3}. For any given $\underline{x} \in \mathcal{D}_k$, let $I_{i;k,x} = 1$, if there is a discontinuity point of $\varphi(Y_i;\theta)$ between $\mu_{ik}^{\top}\beta_p(\underline{x}_k)$ and $\mu_{ix}^{\top}\beta_p(\underline{x})$; and $I_{i;k,x} = 0$, otherwise. Write $\xi_{i3} = \xi_{i3}I_{i;k,x} + \xi_{i3}(1 - I_{i;k,x})$. Again by Assumption A2 and the fact that $|K_{ix}\mu_{ix}^{\top}\alpha| = O(M_n^{(1)})$ and $|\mu_{ik}^{\top}\beta_p(\underline{x}_k) - \mu_{ix}^{\top}\beta_p(\underline{x})| = |A(\underline{X}_i, \underline{x}_k) - A(\underline{X}_i, \underline{x})| = O(M_n^{(2)} l_n/h)$, we have, similar to (A.19), that

$$\mathrm{T}_n P\Big(\sup_{\substack{\alpha \in B_n^{(1)} \\ \underline{x} \in \mathcal{D}_k}} \Big|\sum_{i=1}^{n} \xi_{i3}(1 - I_{i;k,x})\Big| > M^{3/2} d_n/18\Big) \quad \text{is summable over } n.$$

It is easy to see that $I_{i;k,x} \leq I\{\varepsilon_i + A(\underline{X}_i, \underline{x}_k) \in S_{i;k,x}\}$, where

$$S_{i;k,x} = \bigcup_{j=1}^{m} \bigcup_{t\in[0,1]} \Big[a_j - |A(\underline{X}_i, \underline{x}_k) - A(\underline{X}_i, \underline{x})|, a_j + |A(\underline{X}_i, \underline{x}_k) - A(\underline{X}_i, \underline{x})|\Big]$$

$$\subseteq \bigcup_{j=1}^{m} [a_j - CM_n^{(2)} l_n/h, a_j + CM_n^{(2)} l_n/h] \equiv D_n, \quad \text{for some } C > 0.$$

Therefore, $|\xi_{i3}|I_{i;k,x} = |\xi_{i3}|I\{|\underline{X}_{ik}| \leq 2h\}I_{i;k,x} \leq U_{ni}$, with

$$U_{ni} \equiv M_n^{(1)} I\{|\underline{X}_{ik}| \leq 2h\} I\{\varepsilon_i + A(\underline{X}_i, \underline{x}_k) \in D_n\},$$

which is independent of the choice of $\alpha \in B_n^{(1)}$ and $\underline{x} \in \mathcal{D}_k$. Therefore,

$$\mathrm{T}_n P\Big(\sup_{\substack{\alpha \in B_n^{(1)} \\ \underline{x} \in \mathcal{D}_k}} \Big|\sum_{i=1}^{n} \xi_{i3} I_{i;k,x}\Big| > M^{3/2} d_n/18\Big) \leq \mathrm{T}_n P\Big(\sum_{i=1}^{n} [U_{ni} - \mathrm{E}U_{ni}] > M^{3/2} d_n/36\Big), \tag{A.33}$$

where we have used the fact that $\mathrm{E}U_{ni} = O(h^d M_n^{(1)} M_n^{(2)} l_n/h) = O(d_n/n)$. We will have $\sum \mathrm{T}_n P_{n3} < \infty$ if the right-hand side in (A.33) is summable over n; i.e.,

$$\mathrm{T}_n P\Big(\sum_{i=1}^{n} [U_{ni} - \mathrm{E}U_{ni}] > M^{3/2} d_n/36\Big) \quad \text{is summable over } n. \tag{A.34}$$

It is easy to check that Lemma 5 again holds with $\psi_{\underline{x}}(\underline{X}_i, Y_i)$ standing for U_{ni}. Applying Lemma 4 to (A.34) with $B_1 \equiv M_n^{(1)}$, $B_2 \equiv Cnh^d (M_n^{(1)})^2 M_n^{(2)} l_n/h$, $\eta \equiv M^{3/2} d_n/36$, and $r_n = r(n)$, we have (note that $nB_1/\eta \to \infty$ indeed)

$$\lambda_n \eta/4 = CM^{1/2} \log n, \qquad \lambda_n^2 B_2 = Cr_n^{-2/\nu_2} \log n = o(\lambda_n \eta).$$

Thus, $\mathrm{T}_n \Psi_n$ is again summable over n and (A.34) indeed holds. ∎

Proof of Theorem 1. Let $\lambda_1 = \lambda(s)$. Then according to Lemmas 1 and 9, we know that with probability 1, there exists some $C_1 > 1$, such that for all large $M > 0$,

$$\sup_{\underline{x}\in\mathcal{D}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left| \sum_{i=1}^{n} \Phi_{ni}(\underline{x};\alpha,\beta) - \frac{nh^d}{2}(H_n\alpha)^\top S_{np}(\underline{x}) H_n(\alpha+2\beta) \right|$$

$$\le C_1 M^{3/2}(d_{n1}+d_n) \le 2C_1 M^{3/2}(nh^d)^{1-2\lambda_1}(\log n)^{2\lambda_1} \quad \text{for large } n, \tag{A.35}$$

where $d_{n1} = (nh^d)^{1-\lambda_1-2\lambda_2}(\log n)^{\lambda_1+2\lambda_2}$. Note that based on (12), we can write

$$\sum_{i=1}^{n} K_{ni}\varphi(Y_i;\mu_{ni}^\top \beta_p(\underline{x}))\mu_{ni}^\top \alpha = nh^d \beta_n^*(\underline{x})^\top W_p^{-1} S_{np}(\underline{x}) H_n \alpha.$$

Replace $B_n^{(1)}$ in (A.35) with $B_{nk}^{(1)} = \left\{\alpha \in \mathcal{R}^N : k \le M^{-1}(nh^d/\log n)^{\lambda_1}|H_n\alpha| \le k+1\right\}$ and M with $(k+1)M$. We have, by the definition of $\Phi_{ni}(\underline{x};\alpha,\beta)$, that

$$\inf_{\underline{x}\in\mathcal{D}} \inf_{\substack{\alpha\in B_{nk}^{(1)},\\ \beta\in B_n^{(2)}}} \left\{ \sum_{i=1}^{n} \rho(Y_i;\mu_{ni}^\top(\alpha+\beta+\beta_p(\underline{x})))K_{ni} - \sum_{i=1}^{n} \rho(Y_i;\mu_{ni}^\top(\beta+\beta_p(\underline{x})))K_{ni} \right.$$

$$\left. + nh^d(W_p^{-1}\beta_n^*(\underline{x}) - H_n\beta)^\top S_{np}(\underline{x}) H_n\alpha \right\}$$

$$\ge \inf_{\underline{x}\in\mathcal{D}} \inf_{\alpha\in B_{nk}^{(1)}} \frac{nh^d}{2}(H_n\alpha)^\top S_{np}(\underline{x}) H_n\alpha - 2CM^{3/2}(nh^d)^{1-2\lambda_1}(\log n)^{2\lambda_1}$$

$$\ge \left\{C_3(kM)^2/2 - 2C_1(k+1)^{3/2}M^{3/2}\right\}(nh^d)^{1-2\lambda_1}(\log n)^{2\lambda_1}$$

$$\ge (8-2^{5/2})C_1C_4^{3/2}(nh^d)^{1-2\lambda_1}(\log n)^{2\lambda_1} > 0 \quad \text{almost surely}, \tag{A.36}$$

where the last term is independent of the choice of $k \ge 1$. The last inequality is derived as follows: As $S_p > 0$, suppose its minimum eigenvalue is $\tau_1 > 0$. As $S_{np}(\underline{x}) \to g(\underline{x})f(\underline{x})S_p$ uniformly in $\underline{x} \in \mathcal{D}$ by Lemma 8 and $g(\underline{x})f(\underline{x})$ is bounded away from zero by Assumption A5 and (A.7), there exists some constant $C_3 > 0$, such that for all $\underline{x} \in \mathcal{D}$, the minimum eigenvalue of $S_{np}(\underline{x})$ is greater than C_3. The last inequality thus holds if $M \ge C_4 = (16C_1/C_3)^2$. Note that

$$\bigcup_{k=1}^{\infty} B_{nk}^{(1)} = \left\{\alpha| \in \mathcal{R}^N : \left(\frac{nh^d}{\log n}\right)^{\lambda_1}|H_n\alpha| \ge M\right\} := B_n^N. \tag{A.37}$$

Therefore, from (A.36) and (A.37), we have

$$\inf_{\underline{x}\in\mathcal{D}} \inf_{\substack{\alpha\in B_n^N,\\ \beta\in B_n^{(2)}}} \left\{ \sum_{i=1}^{n} \rho(Y_i;\mu_{ni}^\top(\alpha+\beta+\beta_p(\underline{x})))K_{ni} - \sum_{i=1}^{n} \rho(Y_i;\mu_{ni}^\top(\beta+\beta_p(\underline{x})))K_{ni} \right.$$

$$\left. + nh^d(W_p^{-1}\beta_n^*(\underline{x}) - H_n\beta)^\top S_{np}(\underline{x}) H_n\alpha \right\} > 0 \quad \text{almost surely}. \tag{A.38}$$

Note that by (A.40), Lemma 10, and Proposition 1, we have $|\beta_n^*(\underline{x})| \le C_3(nh^d/\log n)^{-\lambda_2}$ uniformly in $\underline{x} \in \mathcal{D}$ almost surely. Namely, $\beta_n^*(\underline{x}) \in B_n^{(2)}$ for all $\underline{x} \in \mathcal{D}$, if $M > C_3^4$. This implies that if $M > \max(C_3^4, C_4)$, (A.38) still holds with β replaced with $H_n^{-1} W_p^{-1} \beta_n^*(\underline{x})$. Therefore,

$$\inf_{\underline{x}\in\mathcal{D}} \inf_{\alpha \in B_n^N} \Big\{ \sum_{i=1}^n K_{ni}\rho(Y_i; \mu_{ni}^\top(\alpha + H_n^{-1}W_p^{-1}\beta_n^*(\underline{x}) + \beta_p(\underline{x}))) \\ - \sum_{i=1}^n K_{ni}\rho(Y_i; \mu_{ni}^\top(H_n^{-1}W_p^{-1}\beta_n^*(\underline{x}) + \beta_p(\underline{x})))\Big\} > 0,$$

which is equivalent to Theorem 1. ∎

Proof of (13). Let $\tilde{d}_n = (nh^d)^{1-2\lambda_1}(\log n)^{2\lambda_1}$. Following the proof lines of Theorem 1, we can see that (13) will follow if

$$\sup_{\underline{x}\in\mathcal{D}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} |\sum_{i=1}^n R_{ni}(\underline{x};\alpha,\beta)| \le M^{3/2}\tilde{d}_n \quad \text{almost surely,}$$

with $\lambda_1 = 1,\ \lambda_2 = 1/2$, and $B_n^{(i)},\ i = 1,2$ defined as in Lemma 1.

To prove this, cover $\mathcal{D}$ by a finite number $\tilde{\mathrm{T}}_n = \{(nh^d/\log n)^{1/2}/h\}^d$ of cubes $\mathcal{D}_k = \mathcal{D}_{nk}$ with side length $\tilde{l}_n = O\{h(nh^d/\log n)^{-1/2}\}$ and centers $\underline{x}_k = \underline{x}_{n,k}$. Write

$$\begin{aligned}
&\sup_{\underline{x}\in\mathcal{D}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} |\sum_{i=1}^n R_{ni}(\underline{x};\alpha,\beta)| \\
&\quad \le \max_{1\le k\le \tilde{\mathrm{T}}_n} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^n \Phi_{ni}(\underline{x}_k;\alpha,\beta) - \mathrm{E}\Phi_{ni}(\underline{x}_k;\alpha,\beta)\right| \\
&\qquad + \max_{1\le k\le \tilde{\mathrm{T}}_n} \sup_{\underline{x}\in\mathcal{D}_k} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^n \Big\{\Phi_{ni}(\underline{x}_k;\alpha,\beta) - \Phi_{ni}(\underline{x};\alpha,\beta)\Big\}\right| \\
&\qquad + \max_{1\le k\le \tilde{\mathrm{T}}_n} \sup_{\underline{x}\in\mathcal{D}_k} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left|\sum_{i=1}^n \Big\{\mathrm{E}\Phi_{ni}(\underline{x}_k;\alpha,\beta) - \mathrm{E}\Phi_{ni}(\underline{x};\alpha,\beta)\Big\}\right| \\
&\quad \equiv Q_1 + Q_2 + Q_3.
\end{aligned}$$

We will show that with probability 1, $Q_k \le M^{3/2}\tilde{d}_n/3,\ k = 1,2,3$.

Define ξ_{ij} as in Lemma 1. As $P(Q_2 > M^{3/2}\tilde{d}_n/2) \le \tilde{\mathrm{T}}_n(P_{n1} + P_{n2} + P_{n3})$, where

$$P_{nj} \equiv \max_{1\le k\le \tilde{\mathrm{T}}_n} P\left(\sup_{\underline{x}\in\mathcal{D}_k} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} |\sum_{i=1}^n \xi_{ij}| \ge M^{3/2}\tilde{d}_n/9\right), \quad j = 1,2,3.$$

Then, by the Borel-Cantelli lemma, $Q_2 \le M^{3/2}\tilde{d}_n/2$ almost surely, if $\sum_n \tilde{\mathrm{T}}_n P_{nj} < \infty$, for $j = 1, 2, 3$. We only prove that for P_{n1} to illustrate. Recall that

$$\xi_{i1} = \Big(K_{ik}\mu_{ik} - K_{ix}\mu_{ix}\Big)^\top \alpha \int_0^1 \Big\{\varphi_{ni}(\underline{x}_k; \mu_{ik}^\top(\beta + \alpha t)) - \varphi_{ni}(\underline{x}_k; 0)\Big\}\, dt.$$

Because $|(K_{ik}\mu_{ik} - K_{ix}\mu_{ix})^\top \alpha| \le C_2 M_n^{(1)} \tilde{l}_n/h$, $|\mu_{ik}^\top(\beta + \alpha t)| \le C M_n^{(2)}$ and $\varphi(.)$ is Lipschitz continuous, we have $|\xi_{i1}| \le C M_n^{(2)} M_n^{(1)} \tilde{l}_n/h$. Define $U_{ik} = I\{|\underline{X}_{ik}| \le 2h\}$. As $\tilde{l}_n = o(h)$, we can see that $\xi_{i1} = \xi_{i1} U_{ik}$ and, similar to (A.18), we have

$$\begin{aligned} P\Bigg(\sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \sup_{\underline{x}\in\mathcal{D}_k} \bigg|\sum_{i=1}^n \xi_{i1}\bigg| > \frac{M^{3/2}\tilde{d}_n}{9}\Bigg) &\le P\Bigg(\sum_{i=1}^n U_{ik} > \frac{M^{1/4} n h^d}{9C}\Bigg) \\ &\le P\Bigg(\bigg|\sum_{i=1}^n U_{ik} - \mathrm{E}U_{ik}\bigg| > \frac{M^{1/4} n h^d}{18C}\Bigg), \end{aligned}$$

and $\sum_n \tilde{\mathrm{T}}_n P_{nj} < \infty$ thus follows from similar arguments as those lying between (A.18) and (A.19).

The proof of $Q_1 \le M^{3/2}\tilde{d}_n/2$ almost surely is much easier than in Lemma 1, if $\varphi(.)$ is Lipschitz continuous. Instead of the iterative partition approach adopted there, we once and for all partition $B_n^{(i)}$, $i = 1, 2$, into a sequence of disjoint subrectangles $D_1^{(i)}, \dots, D_{J_1}^{(i)}$ such that

$$|D_{j_1}^{(i)}| = \sup\Big\{|H_n(\alpha - \beta)| : \alpha, \beta \in D_{j_1}^{(i)}\Big\} \le M_n^{(i)} (\log n/n)^{1/2}, \qquad 1 \le j_1 \le J_1.$$

Obviously, $J_1 \le (n/\log n)^{N/2}$. Choose a point $\alpha_{j_1} \in D_{j_1}^{(1)}$ and $\beta_{k_1} \in D_{k_1}^{(2)}$. Then

$$\begin{aligned} Q_1 \le{}& \max_{\substack{1\le k\le \tilde{\mathrm{T}}_n\\ 1\le j_1,k_1\le J_1}} \sup_{\substack{\alpha\in D_{j_1}^{(1)},\\ \beta\in D_{k_1}^{(2)}}} \bigg|\sum_{i=1}^n \{R_{ni}(\underline{x}_k; \alpha_{j_1}, \beta_{k_1}) - R_{ni}(\underline{x}_k; \alpha, \beta)\}\bigg| \\ &+ \max_{\substack{1\le k\le \mathrm{T}_n\\ 1\le j_1,k_1\le J_1}} \bigg|\sum_{i=1}^n R_{ni}(\underline{x}_k; \alpha_{j_1}, \beta_{k_1})\bigg| = H_{n1} + H_{n2}. \end{aligned} \tag{A.39}$$

By Lipschitz continuity of $\varphi(.)$, we have for any $\alpha \in D_{j_1}^{(1)}$ and $\beta \in D_{k_1}^{(2)}$,

$$|\Phi_{ni}(\underline{x}_k; \alpha_{j_1}, \beta_{k_1}) - \Phi_{ni}(\underline{x}_k; \alpha, \beta)|^2 = O(\{M_n^{(2)}\}^3 \log n/n) < M^{3/2}\tilde{d}_n/(4n).$$

Therefore, it remains to show that $P(H_{n2} > M^{3/2}\tilde{d}_n/4)$ is summable over n.

First, note that by Cauchy inequality $|R_{ni}(\underline{x}; \alpha, \beta)|^2 = O(\{M_n^{(1)} M_n^{(2)}\}^2)$ and $\mathrm{E}|R_{ni}(\underline{x}; \alpha, \beta)|^2 = O(h^d\{M_n^{(1)} M_n^{(2)}\}^2)$ uniformly in $\underline{X}_i$, $\underline{x}$, $\alpha \in M_n^{(1)}$, and $\beta \in M_n^{(2)}$. Next, for any $\eta > 0$,

$$P(H_{n2} > \eta) \le \tilde{\mathrm{T}}_n J_1^2 P\Bigg(\bigg|\sum_{i=1}^n R_{ni}(\underline{x}; \alpha_{j_1}, \beta_{k_1})\bigg| > \eta\Bigg).$$

We apply Lemma 4 with $r_n = (nh^d/\log n)^{1/2}$, $B_1 = 2C_1 M_n^{(1)} M_n^{(2)}$, $B_2 = C_2 nh^d (M_n^{(1)} M_n^{(2)})^2$, $\lambda_n = (4C_1 r_n\{M_n^{(2)}\}^2)^{-1}$, and $\eta = M^{3/2}\tilde{d}_n/4$. It is easy to see that $nB_1/\eta \to \infty$ and

$$\lambda_n \eta/4 = M\log n/(16C_1), \qquad \lambda_n^2 B_2 = o(\lambda_n \eta),$$
$$\Psi(n) \equiv q_n\{nB_1/\eta\}^{1/2}\gamma[r_n] = n^{3/2}(\log n)^{-1/2}\gamma[r(n)]/r(n).$$

As $\tilde{\mathrm{T}}_n J_1^2 \Psi(n)$ is summable over n by condition (A.4), so is $P(H_{n2} > M^{3/2}\tilde{d}_n/4)$. ∎

Proof of Corollary 1. As $1+\lambda_2 \geq 2\lambda_1$, it is sufficient to prove that, with probability 1,

$$\beta_n^*(\underline{x}) - \mathrm{E}\beta_n^*(\underline{x}) - \frac{1}{nh^d} W_p S_{np}^{-1}(\underline{x}) H_n^{-1} \sum_{i=1}^n K_h(\underline{X}_i - \underline{x})\varphi(\varepsilon_i)\mu(\underline{X}_i - \underline{x})$$
$$= O\Big\{\Big(\frac{\log n}{nh^d}\Big)^{(1+\lambda_2)/2}\Big\}, \qquad \textbf{(A.40)}$$

uniformly in $\underline{x} \in \mathcal{D}$. As $\varphi(\varepsilon_i) \equiv \varphi(Y_i, m(X_i))$ and $\mathrm{E}\varphi(\varepsilon_i) = 0$, the term on the left-hand side of (A.40) stands for

$$W_p S_{n,p}^{-1}(\underline{x}) \frac{1}{nh^d} \sum_{i=1}^n \{Z_{ni}(\underline{x}) - \mathrm{E}Z_{ni}(\underline{x})\},$$

where

$$Z_{ni}(\underline{x}) = H_n^{-1} K_h(\underline{X}_i - \underline{x})\mu(\underline{X}_i - \underline{x})\Big\{\varphi(Y_i, \mu(\underline{X}_i - \underline{x})^\top \beta_p(\underline{x})) - \varphi(\varepsilon_i)\Big\}.$$

Next, similar to what we did in Lemma 1, we cover $\mathcal{D}$ with number T_n cubes $\mathcal{D}_k = \mathcal{D}_{n,k}$ with side length $l_n = O(\mathrm{T}_n^{-1/d})$ and centers $\underline{x}_k = \underline{x}_{n,k}$. Write

$$\begin{aligned}
\sup_{\underline{x}\in\mathcal{D}} \Big|\sum_{i=1}^n Z_{ni}(\underline{x}) - \mathrm{E}Z_{ni}(\underline{x})\Big| \leq{}& \max_{1\leq k\leq \mathrm{T_n}} \Big|\sum_{i=1}^n Z_{ni}(\underline{x}_k) - \mathrm{E}Z_{ni}(\underline{x}_k)\Big| \\
&+ \max_{1\leq k\leq \mathrm{T}_n} \sup_{\underline{x}\in\mathcal{D}_k} \Big|\sum_{i=1}^n Z_{ni}(\underline{x}) - Z_{ni}(\underline{x}_k)\Big| \\
&+ \max_{1\leq k\leq \mathrm{T}_n} \sup_{\underline{x}\in\mathcal{D}_k} \Big|\sum_{i=1}^n \mathrm{E}Z_{ni}(\underline{x}) - \mathrm{E}Z_{ni}(\underline{x}_k)\Big| \\
\equiv{}& Q_1 + Q_2 + Q_3.
\end{aligned}$$

As $Z_{ni}(\underline{x}) - Z_{ni}(\underline{x}_k) = H_n^{-1}K_h(\underline{X}_i - \underline{x})\mu(\underline{X}_i - \underline{x})\{\varphi_{ni}(\underline{x};0) - \varphi_{ni}(\underline{x}_k;0)\}$, through approaches similar to that for ξ_{i3} in the proof of Lemma 2, we can show that

$$Q_2 = O\Big\{\Big(\frac{nh^d}{\log n}\Big)^{(1-\lambda_2)/2}\log n\Big\} \quad \text{almost surely,}$$

and the same result for Q_3 also holds. To bound Q_1, first note that $\mathrm{E}Z_{ni}^2(\underline{x}_k) = O(h^{p+1+d})$ uniformly in i and k. As $|Z_{ni}(\underline{x})| \leq C$ for some constant C by Assumption A2, we can see that from Lemma 5,

$$\sum_{i=1}^n \mathrm{E}Z_{ni}^2(\underline{x}_k) + \sum_{i<j} |\mathrm{Cov}(Z_{ni}(\underline{x}_k), Z_{nj}(\underline{x}_k))| \leq C_2 nh^{p+1+d}.$$

Finally, by Lemma 4 with $B_1 = C_1$, $B_2 \equiv Cnh^{p+1+d}$, $\eta = A_3(nh^d/\log n)^{(1-\lambda_2)/2}\log n$, and $r_n = r(n)$, we have, as $nB_1/\eta \to \infty$, that

$$\lambda_n\eta = A_3/(2C_1)\log n, \qquad \lambda_n^2 B_2 = C_2/(4C_1^2)\log n.$$

Therefore,

$$P\Big(\max_{1\le k\le \mathrm{T_n}}\Big|\sum_{i=1}^n Z_{ni}(\underline{x}_k)-\mathrm{E}Z_{ni}(\underline{x}_k)\Big| \ge A_3(nh^d/\log n)^{(1-\lambda_2)/2}\log n\Big) \le \mathrm{T}_n/n^a + C\mathrm{T}_n\Psi_n,$$

where $a = A_3/(8C_1) - C_2/(4C_1^2)$. By selecting A_3 large enough, we can ensure that T_n/n^a is summable over n. As $\mathrm{T}_n\Psi_n$ is summable over n from (A.4), we can conclude that

$$Q_1 = O\Big\{\Big(\frac{nh^d}{\log n}\Big)^{(1-\lambda_2)/2}\log n\Big\} \quad \text{almost surely.}$$

This together with Lemma 8 completes the proof. ■

Proof of Corollary 2. Through the proof lines for Theorem 1 and Corollary 1, it is not difficult to see that Corollary 2 still holds under the conditions imposed here. Under the additive structure (4), we thus have

$$\begin{aligned}\phi_{n1}(x_1) &= \phi_1(x_1) + \frac{1}{n}\sum_{i=1}^n m_2(\underline{X}_{2i}) - h^{p+1}e_1 W_p S_p^{-1} B_1 \frac{1}{n}\sum_{i=1}^n \mathbf{m}_{p+1}(x_1,\underline{X}_{2i})\\ &\quad + \frac{1}{n^2 h_1 h^{d-1}} e_1 \sum_{j=1}^n \varphi(\varepsilon_j)\\ &\quad \times \sum_{i=1}^n S_{np}^{-1}(x_1,\underline{X}_{2i}) K(X_{1,xj}/h_1, \underline{X}_{2,ij}/h)\mu(X_{1,xj}/h_1,\underline{X}_{2,ij}/h)\\ &\quad + o_p(\{\max(h_1,h)\}^{p+1}) + O_p\{(nh_1h^{d-1}/\log n)^{-3/4}\}, \end{aligned} \tag{A.41}$$

where $X_{1,xj} = X_{1j} - x$, $\underline{X}_{2,ij} = \underline{X}_{2i} - \underline{X}_{2j}$, and e_1 is as in Proposition 1. Note that by (17), $(nh_1)^{1/2}(nh_1h^{d-1}/\log n)^{-3/4} \to 0$, the $O_p(.)$ term can thus be safely ignored.

By the central limit theorem for strongly mixing processes (Bosq, 1998, Thm. 1.7), we have

$$\frac{1}{n}\sum_{i=1}^n m_2(\underline{X}_{2i}) = O_p(n^{-1/2}), \qquad \frac{1}{n}\sum_{i=1}^n \mathbf{m}_{p+1}(x_1,\underline{X}_{2i}) = \mathrm{E}\mathbf{m}_{p+1}(x_1,\underline{X}_2) + O_p(n^{-1/2}).$$

As the expectations of all other terms in (A.41) are 0, the leading term in the asymptotic bias of $\tilde{\phi}_1(x_1) - \phi_1(x_1)$ is thus given by

$$-\{\max(h_1,h)\}^{p+1} e_1 W_p S_p^{-1} B_1 \mathrm{E}\mathbf{m}_{p+1}(x_1,\underline{X}_2).$$

Again through standard arguments in Masry (1996), we can see that

$$\begin{aligned}&\frac{1}{nh^{d-1}}\sum_{i=1}^n S_{np}^{-1}(x_1,\underline{X}_{2i})K_h(X_{1,xj},\underline{X}_{2,ij})\mu(X_{1,xj}/h_1,\underline{X}_{2,ij}/h)\\ &\quad = S_{np}^{-1}(x_1,\underline{X}_{2j}) f_2(\underline{X}_{2j})\int_{[0,1]^{\otimes d-1}}\{K\mu\}(X_{1,xj}/h_1,\underline{v})\,d\underline{v}\Big\{1+O\Big(\Big\{\frac{\log n}{nh^{d-1}}\Big\}^{1/2}\Big)\Big\}\end{aligned}$$

uniformly in $1 \le i \le n$. Therefore, the leading term in the asymptotic variance of $\phi_{n1}(x_1) - \phi_1(x_1)$ is the variance of the term

$$(nh_1)^{-1} e_1 \sum_{j=1}^{n} \varphi(\varepsilon_j) S_{np}^{-1}(x_1, \underline{X}_{2j}) f_2(\underline{X}_{2j}) \int_{[0,1]^{\otimes d-1}} \{K\mu\}(X_{1,xj}/h_1, \underline{v})\, d\underline{v},$$

which is asymptotically

$$(nh_1)^{-1} \left\{ \int_{[0,1]^{\otimes d-1}} \{fg^2\}^{-1}(x_1, \underline{X}_2) f_2^2(\underline{X}_2) \sigma^2(x_1, \underline{X}_2)\, d\underline{X}_2 \right\} e_1 S_p^{-1} K_2 K_2^\top S_p^{-1} e_1^\top. \tag{A.42}$$

If $\rho(y;\theta) = (2q-1)(y-\theta) + |y-\theta|$ and $\varphi(\theta) = 2qI\{\theta > 0\} + (2q-2)I\{\theta < 0\}$, we have $g(\underline{x}) = 2f_\varepsilon(0|\underline{x})$ and

$$\sigma^2(\underline{x}) = \mathrm{E}[\varphi^2(\varepsilon)|\underline{X} = \underline{x}] = 4q^2(1 - F_\varepsilon(0)) + 4(1-q)^2 F_\varepsilon(0) = 4q(1-q),$$

which when substituted into (A.42), yields the asymptotic variance of the quantile regression estimator,

$$\tilde{\sigma}^2(x_1) = q(1-q) \left\{ \int_{[0,1]^{\otimes d-1}} f^{-1}(x_1, \underline{X}_2) f_\varepsilon^{-2}(0|x_1, \underline{X}_2) f_2^2(\underline{X}_2)\, d\underline{X}_2 \right\} e_1 \times S_p^{-1} K_2 K_2^\top S_p^{-1} e_1^\top.$$

The next lemma is due to Davydov (Hall and Heyde, 1980, Cor. A.2). ■

LEMMA 3. *Suppose X and Y are random variables that are respectively $\mathcal{G}-$ and $\mathcal{H}-$ measurable, where $\mathcal{G}-$ and $\mathcal{H}-$ are two σ-algebras.* $\mathrm{E}|X|^p < \infty$, $\mathrm{E}|Y|^q < \infty$, *with $p > 1$, $q > 1$, and $p^{-1} + q^{-1} < 1$. Then*

$$|\mathrm{E}XY - \mathrm{E}X\mathrm{E}Y| \le 8\|X\|_p \|Y\|_q \left\{ \sup_{A \in \mathcal{G}, B \in \mathcal{H}} |P(AB) - P(A)P(B)| \right\}^{1-p^{-1}-q^{-1}}.$$

The next lemma is a generalization of some results in the proof of Theorem 2 in Masry (1996).

LEMMA 4. *Suppose $\{Z_i\}_{i=1}^{\infty}$ is a zero-mean strictly stationary process with strong mixing coefficient $\gamma[k]$, and that $|Z_i| \le B_1$, $\sum_{i=1}^n \mathrm{E}Z_i^2 + \sum_{i<j} |\mathrm{Cov}(Z_i, Z_j)| \le B_2$. Then for any $\eta > 0$ and integer series $r_n \to \infty$, if $nB_1/\eta \to \infty$ and $q_n \equiv [n/r_n] \to \infty$, we have*

$$P\left(\left| \sum_{i=1}^{n} Z_i \right| \ge \eta \right) \le 4\exp\left\{ -\frac{\lambda_n \eta}{4} + \lambda_n^2 B_2 \right\} + C\Psi(n),$$

where $\Psi(n) = q_n\{nB_1/\eta\}^{1/2}\gamma[r_n]$, $\lambda_n = 1/\{2r_n B_1\}$.

Proof. We partition the set $\{1, \ldots, n\}$ into $2q \equiv 2q_n$ consecutive blocks of size $r \equiv r_n$ with $n = 2qr + v$ and $0 \le v < r$. Write

$$V_n(j) = \sum_{i=(j-1)r+1}^{jr} Z_i, \qquad j = 1, \ldots, 2q$$

and

$$W_n' = \sum_{j=1}^{q} V_n(2j-1), \qquad W_n'' = \sum_{j=1}^{q} V_n(2j), \qquad W_n''' = \sum_{i=2qr+1}^{n} Z_i.$$

Then $W_n \equiv \sum_{i=1}^n Z_i = W_n' + W_n'' + W_n'''$. The contribution of W_n''' is negligible, as it consists of at most r terms compared with qr terms in W_n' or W_n''. Then by the stationarity of the processes, for any $\eta > 0$,

$$P(W_n > \eta) \le P(W_n' > \eta/2) + P(W_n'' > \eta/2) = 2P(W_n' > \eta/2). \tag{A.43}$$

To bound $P(W_n' > \eta/2)$, using recursively Bradley's Lemma, we can approximate the random variables $V_n(1), V_n(3), \dots, V_n(2q-1)$ by independent random variables $V_n^*(1)$, $V_n^*(3), \dots, V_n^*(2q-1)$, which satisfy that for $1 \le j \le q$, $V_n^*(2j-1)$ has the same distribution as $V_n(2j-1)$ and

$$\begin{aligned} &P\Big(|V_n^*(2j-1) - V_n(2j-1)| > u\Big) \\ &\qquad \le 18(\|V_n(2j-1)\|_\infty/u)^{1/2} \sup|P(AB) - P(A)P(B)|, \end{aligned} \tag{A.44}$$

where u is any positive value such that $0 < u \le \|V_n(2j-1)\|_\infty < \infty$ and the supremum is taken over all sets of A and B in the σ-algebras of events generated by $\{V_n(1), V_n(3), \dots, V_n(2j-3)\}$ and $V_n(2j-1)$, respectively. By the definition of $V_n(j)$, we can see that $\sup|P(AB) - P(A)P(B)| = \gamma[r_n]$. Write

$$\begin{aligned} P\Big(W_n' > \frac{\eta}{2}\Big) &\le P\Big(\Big|\sum_{j=1}^{q} V_n^*(2j-1)\Big| > \frac{\eta}{4}\Big) + P\Big(\Big|\sum_{j=1}^{q} V_n(2j-1) - V_n^*(2j-1)\Big| > \frac{\eta}{4}\Big) \\ &\equiv I_1 + I_2. \end{aligned} \tag{A.45}$$

We bound I_1 as follows: Let $\lambda = 1/\{2B_1 r\}$. Since $|Z_i| \le B_1$, $\lambda|V_n(j)| \le 1/2$, then using the fact that $e^x \le 1 + x + x^2/2$ holds for $|x| \le 1/2$, we have

$$\mathrm{E}\Big\{e^{\pm\lambda V_n^*(2j-1)}\Big\} \le 1 + \lambda^2 \mathrm{E}\{V_n(j)\}^2 \le e^{\lambda^2 \mathrm{E}\{V_n^*(2j-1)\}^2}. \tag{A.46}$$

By Markov inequality, (A.46), and the independence of the $\{V_n^*(2j-1)\}_{j=1}^q$, we have

$$\begin{aligned} I_1 &\le e^{-\lambda\eta/4}\Big[\mathrm{E}\exp\Big(\lambda \sum_{j=1}^{q} V_n^*(2j-1)\Big) + \mathrm{E}\exp\Big(-\lambda \sum_{j=1}^{q} V_n^*(2j-1)\Big)\Big] \\ &\le 2\exp\Big(-\lambda\eta/4 + \lambda^2 \sum_{j=1}^{q} \mathrm{E}\{V_n^*(2j-1)\}^2\Big) \\ &\le 2\exp\Big\{-\lambda\eta/4 + C_2\lambda^2 B_2\Big\}. \end{aligned} \tag{A.47}$$

We now bound the term I_2 in (A.45). Notice that

$$I_2 \le \sum_{j=1}^{q} P\Big(\Big|V_n(2j-1) - V_n^*(2j-1)\Big| > \frac{\eta}{4q}\Big).$$

If $\|V_n(2j-1)\|_\infty \geq \eta/(4q)$, substitute $\eta/(4q)$ for u in (A.44),

$$I_2 \leq 18q\{\|V_n(2j-1)\|/\eta/(4q)\}^{1/2}\gamma[r_n] \leq Cq^{3/2}/\eta^{1/2}\gamma[r_n](r_n B_1)^{1/2}, \tag{A.48}$$

If $\|V_n(2j-1)\|_\infty < \eta/(4q)$, let $u \equiv \|V_n(2j-1)\|_\infty$ in (A.44) and we have

$$I_2 \leq Cq\gamma[r_n],$$

which is of smaller order than (A.48), if $nB_1/\eta \to \infty$. Thus, by (A.43), (A.45), (A.47), and (A.48),

$$P(W_n > \eta) \leq 4\exp\{-\lambda_n\eta/4 + C_2 B_2 \lambda_n^2\} + C\Psi_n,$$

where the constant C is independent of n. ■

LEMMA 5. *For any $\underline{x} \in R^d$, let $\psi_{\underline{x}}(\underline{X}_i, Y_i) = I(|\underline{X}_{ix}| \leq h)\psi_x(\underline{X}_{ix}, Y_i)$, a measurable function of $(\underline{X}_i, Y_i)$ with $|\psi_{\underline{x}}(\underline{X}_i, Y_i)| \leq B$ and $V = \mathrm{E}\psi^2_{\underline{x}}(\underline{X}_i, Y_i)$. Suppose the mixing coefficient $\gamma[k]$ satisfies (A.3). Then*

$$\mathrm{Cov}\left(\sum_{i=1}^n |\psi_{\underline{x}}(\underline{X}_i, Y_i)|\right) = nV\left[1 + o\left\{\left(B^2 h^{p+d+1}/V\right)^{1-2/\nu_2}\right\}\right].$$

Proof. Denote $\psi_{\underline{x}}(\underline{X}_i, Y_i)$ by ψ_{ix}. First note that

$$V = \mathrm{E}\psi_{ix}^2 = h^d \int_{|\underline{u}|\leq 1} \mathrm{E}(\psi_{ix}^2 | \underline{X}_i = \underline{x} + h\underline{u}) f(\underline{x} + h\underline{u})\, d\underline{u},$$

$$\sum_{i<j} |\mathrm{Cov}(\psi_{ix}, \psi_{jx})| = \sum_{l=1}^{n-d} (n-l-d+1)|\mathrm{Cov}(\psi_{0x}, \psi_{lx})| \leq n \sum_{l=1}^{n-d} |\mathrm{Cov}(\psi_{0x}, \psi_{lx})|$$
$$= n\sum_{l=1}^{d-1} + n\sum_{l=d}^{\pi_n} + n\sum_{l=\pi_n+1}^{n-d} \equiv nJ_{21} + nJ_{22} + nJ_{23},$$

where $\pi_n = h^{(p+d+1)(2/\nu_2-1)/a}$. For J_{21}, there might be an overlap between the components of $\underline{X}_0$ and $\underline{X}_l$, for example, when $\underline{X}_i = (X_{i-d}, \ldots, X_{i-1})$, where $\{X_i\}$ is a univariate time series. Without loss of generality, let $\underline{u}', \underline{u}''$, and $\underline{u}'''$ of dimensions $l, d-l$, and l, respectively, be the $d+l$ distinct random variables in $(\underline{X}_{0x}/h, \underline{X}_{lx}/h)$. Write $\underline{u}_1 = (\underline{u}'^\top, \underline{u}''^\top)^\top$ and $\underline{u}_2 = (\underline{u}''^\top, \underline{u}'''^\top)^\top$. Then by Cauchy inequality, we have

$$\left|\mathrm{E}\left(\psi_{0x}, \psi_{lx}\middle| \begin{smallmatrix}\underline{X}_0 = \underline{x} + h\underline{u}_1 \\ \underline{X}_l = \underline{x} + h\underline{u}_2\end{smallmatrix}\right)\right| \leq \left\{\mathrm{E}(\psi_{0x}^2 | \underline{X}_0 = \underline{x} + h\underline{u}_1)\mathrm{E}(\psi_{jx}^2 | \underline{X}_j = \underline{x} + h\underline{u}_2)\right\}^{1/2}$$
$$= V/h^d, \tag{A.49}$$

and through a transformation of variables, we have

$$|\mathrm{Cov}(\psi_{0x}, \psi_{lx})| \leq h^l V \int_{\substack{|\underline{u}_1| \leq 1 \\ |\underline{u}_2| \leq 1}} |f(\underline{x} + h\underline{u}_1, \underline{x} + h\underline{u}_2; l)$$
$$- f(\underline{x} + h\underline{u}_1) f(\underline{x} + h\underline{u}_2; l+d-1)|\, d\underline{u}'\, d\underline{u}''\, d\underline{u}''',$$

where, by Assumptions A4 and A5, the integral is bounded. Therefore,

$$nJ_{21} \le CnV \sum_{l=1}^{d-1} h^l = o(nV).$$

For J_{22}, there is no overlap between the components of $\underline{X}_0$ and $\underline{X}_l$. Let $\underline{X}_{0x} = h\underline{u}$ and $\underline{X}_{lx} = h\underline{v}$, and we have

$$\begin{aligned}|\mathrm{Cov}(\psi_{0x}, \psi_{lx})| &\le h^{2d} \int_{\substack{|\underline{u}| \le 1 \\ |\underline{v}| \le 1}} \mathrm{E}\left(\psi_{0x}, \psi_{lx} \middle| \begin{matrix}\underline{X}_0 = \underline{x} + h\underline{u} \\ \underline{X}_l = \underline{x} + h\underline{v}\end{matrix}\right) d\underline{u}\, d\underline{v} \\ &\qquad \times [f(\underline{x} + h\underline{u}, \underline{x} + h\underline{v}; l + d - 1) - f(\underline{x} + h\underline{u}) f(\underline{x} + h\underline{v})] \\ &= Ch^d V,\end{aligned}$$

where the last equality follows from Assumptions A4 and A5 and (A.49). Therefore, as $\pi_n h^d \to 0$,

$$nJ_{22} = O\{n\pi_n h^d V\} = o(nV).$$

For J_{23}, using Davydov's lemma (Lemma 3), we have

$$|\mathrm{Cov}(\psi_{0x}, \psi_{lx})| \le 8\{\gamma[l - d + 1]\}^{1-2/\nu_2} \{\mathrm{E}|\psi_{ix}|^{\nu_2}\}^{2/\nu_2}, \quad \text{as } \nu_2 > 2. \tag{A.50}$$

As $|\psi_{ix}| \le B$, $\mathrm{E}|\Phi_{ni}|^{\nu_2} \le B^{\nu_2 - 2} V$,

$$J_{23} \le CB^{(\nu-2)2/\nu_2} V^{2/\nu_2} / \pi_n^a \sum_{l=\pi_n+1}^{\infty} l^a \{\gamma[l - d + 1]\}^{1-2/\nu_2},$$

where the summation term is $o(1)$, as $\pi_n \to \infty$. Thus $J_{23} = o\left\{V\left(B^2 h^{p+d+1}/V\right)^{1-2/\nu_2}\right\}$, which completes the proof. ∎

LEMMA 6. *Suppose Assumptions A2–A6 hold. Then for $U_{ni}^l, l = 1, \ldots, m$ defined in (A.32) and $Z_{ni}, l = 1, \ldots, \mathrm{L}_n$ defined in (A.13), we have*

$$\sum_{i=1}^{n} \mathrm{E}(U_{ni}^l)^2 + \sum_{i<j} |\mathrm{Cov}(U_{ni}^l, U_{nj}^l)| \le Cnh^d M_n^{(1)} \{M_n^{(2)}/M_n^{(1)}\}^{1-2/\nu_2}, \tag{A.51}$$

$$\sum_{i=1}^{n} \mathrm{E}Z_{ni}^2 + \sum_{i<j} |\mathrm{Cov}(Z_{ni}, Z_{nj})| = nh^d (M_n^{(1)})^2 M_n^{(2)} \{M^l \log n\}^{-2/\nu_2}, \tag{A.52}$$

uniformly in $\underline{x}_k$, $1 \le k \le \mathrm{T}_n$.

Proof. We only prove (A.52), which is more involved than (A.51). To simplify the notations, denote $\alpha_{j_l}, \beta_{k_l}, \alpha_{j_l}$, and β_{j_l} by $\alpha_1, \beta_1, \alpha_2$, and β_2, respectively. Clearly,

$$\begin{aligned}&\int_{\underline{u}^\top H_n \beta_2}^{\underline{u}^\top H_n(\alpha_2+\beta_2)} \{\varphi_{ni}(\underline{x}_k; t) - \varphi_{ni}(\underline{x}_k; 0)\} dt \\ &\qquad = \int_{\underline{u}^\top H_n \beta_1}^{\underline{u}^\top H_n(\alpha_2+\beta_1)} \{\varphi_{ni}(\underline{x}_k; t + \underline{u}^\top H_n(\beta_2 - \beta_1)) - \varphi_{ni}(\underline{x}_k; 0)\} dt,\end{aligned}$$

and

$$
\begin{aligned}
Z_{ni} &= \int_{\underline{u}^\top H_n\beta_1}^{\underline{u}^\top H_n(\alpha_1+\beta_1)} \{\varphi_{ni}(\underline{x}_k;t)-\varphi_{ni}(\underline{x}_k;0)\}dt \\
&\quad - \int_{\underline{u}^\top H_n\beta_2}^{u^\top H_n(\alpha_2+\beta_2)} \{\varphi_{ni}(\underline{x}_k;t)-\varphi_{ni}(\underline{x}_k;0)\}dt \\
&= \int_{\underline{u}^\top H_n\beta_1}^{\underline{u}^\top H_n(\alpha_1+\beta_1)} \{\varphi_{ni}(\underline{x}_k;t)-\varphi_{ni}(\underline{x}_k;t+\underline{u}^\top H_n(\beta_2-\beta_1))\}dt \\
&\quad - \int_{\underline{u}^\top H_n(\alpha_1+\beta_1)}^{\underline{u}^\top H_n(\alpha_2+\beta_1)} \{\varphi_{ni}(\underline{x}_k;t+\underline{u}^\top H_n(\beta_2-\beta_1))-\varphi_{ni}(\underline{x}_k;0)\}dt \equiv \Delta_1+\Delta_2.
\end{aligned}
$$

Therefore, $\mathrm{E}\{Z_{ni}\}^2 = h^d\int K^2(\underline{u})f(\underline{x}_k+h\underline{u})\mathrm{E}\{(\Delta_1+\Delta_2)^2|X_i=\underline{x}_k+h\underline{u}\}d\underline{u}$. The conclusion is thus obvious, observing that by Cauchy inequality and (A.5),

$$
\begin{aligned}
\mathrm{E}(\Delta_1^2|X_i=\underline{x}_k+h\underline{u}) &\le |\underline{u}^\top H_n\alpha_1\underline{u}^\top H_n(\beta_2-\beta_1)\underline{u}^\top H_n\alpha_1| \le 2(M_n^{(1)})^2M_n^{(2)}/(M^l\log n), \\
E(\Delta_2^2|X_i=\underline{x}_k+h\underline{u}) &\le \{\underline{u}^\top H_n(\alpha_2-\alpha_1)\}^2(|\underline{u}^\top H_n\alpha_2|+|\underline{u}^\top H_n\alpha_1|+2|\underline{u}^\top H_n\beta_2|) \\
&\le 4(M_n^{(1)})^2M_n^{(2)}/(M^l\log n)^2,
\end{aligned}
$$

where we used the facts that $|\alpha_1-\alpha_2|\le 2M_n^{(1)}/(M^l\log n)$ and $|\beta_1-\beta_2|\le 2M_n^{(2)}/(M^l\log n)$. Therefore, $\mathrm{E}\{Z_{ni}\}^2 = Ch^d(M_n^{(1)})^2M_n^{(2)}/(M^l\log n)$. As $|Z_{ni}|\le CM_n^{(1)}$ and $h^{p+1}/M_n^{(2)} < \infty$, the rest of the proof can be completed following the proof of Lemma 5. ∎

LEMMA 7. *Suppose Assumptions A2–A6 hold.*

$$
\sum_{i=1}^n \mathrm{E}\Phi_{ni}^2 + \sum_{i<j}|\mathrm{Cov}(\Phi_{ni},\Phi_{nj})| \le Cnh^d(M_n^{(1)})^2M_n^{(2)}, \tag{A.53}
$$

uniformly in $\underline{x}\in\mathcal{D}, \alpha\in B_n^{(1)}$ and $\beta\in B_n^{(2)}$.

Proof. By Cauchy inequality and (A.5), we have

$$
\begin{aligned}
&\mathrm{E}\Phi_{ni}^2 \\
&\quad = h^d\int K^2(\underline{u})\mathrm{E}\Big[\Big\{\int_{\mu(\underline{u})^\top H_n\beta}^{\mu(\underline{u})^\top H_n(\alpha+\beta)}\Big(\varphi_{ni}(\underline{x};t)-\varphi_{ni}(\underline{x};0)\Big)dt\Big\}^2\Big|\underline{X}_i=\underline{x}+h\underline{u}\Big] \\
&\qquad \times f(\underline{x}+h\underline{u})\,d\underline{u} \\
&\quad \le h^d\int f(\underline{x}+h\underline{u})K^2(\underline{u})\mu(\underline{u})^\top H_n\alpha\int_{\underline{u}^\top H_n\beta}^{\mu(\underline{u})^\top H_n(\alpha+\beta)} \\
&\qquad \mathrm{E}\Big[\Big(\varphi_{ni}(\underline{x};t)-\varphi_{ni}(\underline{x};0)\Big)^2|\underline{X}_i=\underline{x}+h\underline{u}\Big]\,dt\,d\underline{u} \\
&\quad \le h^d\int K^2(\underline{u})\mu(\underline{u})^\top H_n\alpha\int_{\mu(\underline{u})^\top H_n\beta}^{\mu(\underline{u})^\top H_n(\alpha+\beta)} C|t|dtf(\underline{x}+h\underline{u})\,d\underline{u} = O\Big\{h^d(M_n^{(1)})^2M_n^{(2)}\Big\},
\end{aligned}
\tag{A.54}
$$

uniformly in $\underline{x}\in\mathcal{D}$, $\alpha\in B_n^{(1)}$ and $\beta\in B_n^{(2)}$. Then (A.53) follows from (A.54) and Lemma 5. ∎

LEMMA 8. *Let Assumptions A3–A6 hold. Then*

$$\sup_{\underline{x}\in\mathcal{D}} |S_{np}(\underline{x}) - g(\underline{x})f(\underline{x})S_p| = O(h + (nh^d/\log n)^{-1/2}) \quad \textit{almost surely.}$$

Proof. The result is almost the same as Theorem 2 in Masry (1996). Especially if (A.4) holds, then the condition (3.8a) there on the mixing coefficient $\gamma[k]$ is true. ■

LEMMA 9. *Denote $d_{n1} = (nh^d)^{1-\lambda_1-2\lambda_2}(\log n)^{\lambda_1+2\lambda_2}$ and let λ_1 and $B_n^{(i)}$, $i = 1,2$, be as in Lemma 1. Suppose that Assumptions A1–A5 and (A.2) hold. Then there is a constant $C > 0$ such that, for each $M > 0$ and all large n,*

$$\sup_{\underline{x}\in\mathcal{D}} \sup_{\substack{\alpha\in B_n^{(1)},\\ \beta\in B_n^{(2)}}} \left| \sum_{i=1}^{n} \mathrm{E}\Phi_{ni}(\underline{x};\alpha,\beta) - \frac{nh^d}{2}(H_n\alpha)^\top S_{np}(\underline{x})H_n(\alpha+2\beta)\right| \le CM^{3/2}d_{n1}.$$

Proof. Recall that $G(t,\underline{u}) = \mathrm{E}(\varphi(Y;t)|\underline{X} = \underline{u})$,

$$\mathrm{E}\Phi_{ni}(\underline{x};\alpha,\beta) = h^d \int K(\underline{u}) f(\underline{x}+h\underline{u})\,d\underline{u} \times \int_{\mu(\underline{u})^\top H_n\beta}^{\mu(\underline{u})^\top H_n(\alpha+\beta)} \tag{A.55}$$

$$\left\{G(t+\mu(\underline{u})^\top H_n\beta_p(\underline{x}), \underline{x}+h\underline{u}) - G(\mu(\underline{u})^\top H_n\beta_p(\underline{x}), \underline{x}+h\underline{u})\right\}dt.$$

By Assumptions A3 and A5, we have

$$G(t+\mu(\underline{u})^\top H_n\beta_p(\underline{x}), \underline{x}+h\underline{u}) - G(\mu(\underline{u})^\top H_n\beta_p(\underline{x}), \underline{x}+h\underline{u})$$

$$= tG_1(\mu(\underline{u})^\top H_n\beta_p(\underline{x}), \underline{x}+h\underline{u}) + \frac{t^2}{2}G_2(\xi_n(t,\underline{u};\underline{x}), \underline{x}+h\underline{u}),$$

$$G_1(\mu(\underline{u})^\top H_n\beta_p(\underline{x}), \underline{x}+h\underline{u}) = g(\underline{x}+h\underline{u}) + O(h^{p+1}),$$

where $\xi_n(t,\underline{u};\underline{x})$ falls between $\mu(\underline{u})^\top H_n\beta_p(\underline{x})$ and $t + \mu(\underline{u})^\top H_n\beta_p(\underline{x})$, and the term $O(h^{p+1})$ is uniform in $\underline{x}\in\mathcal{D}$. Therefore, the inner integral in (A.55) is given by

$$\frac{1}{2}g(\underline{x}+h\underline{u})(H_n\alpha)^\top \mu(\underline{u})\mu(\underline{u})^\top H_n(\alpha+2\beta) + O\left\{M^{3/2}\left(\frac{\log n}{nh^d}\right)^{\lambda_1+2\lambda_2}\right\}$$

uniformly in $\underline{x}\in\mathcal{D}$, where we have used the fact that $nh^{d+(p+1)/\lambda_2}/\log n < \infty$. By the definition of $S_{np}(\underline{x})$, the proof is thus completed. ■

LEMMA 10. *Under conditions in Theorem 1, we have*

$$\sup_{\underline{x}\in\mathcal{D}} \left|\frac{1}{nh^d} W_p S_{np}^{-1}(\underline{x}) H_n^{-1} \sum_{i=1}^{n} K_h(\underline{X}_i - \underline{x})\varphi(\varepsilon_i)\mu(\underline{X}_i - \underline{x})\right|$$

$$= O\left\{\left(\frac{\log n}{nh^d}\right)^{1/2}\right\} \quad \textit{almost surely.}$$

Proof. Note that under conditions of Theorem 1, the assumptions imposed by Masry (1996) in Theorem 5 hold. Specifically, (4.5) there follows from (A.2), and (4.7b) there from (A.4). Therefore, mimicking the proof lines there, we can show that

$$\sup_{\underline{x}\in\mathcal{D}} \left| \frac{1}{nh^d} H_n^{-1} \sum_{i=1}^{n} K_h(\underline{X}_i - \underline{x})\varphi(\varepsilon_i)\mu(\underline{X}_i - \underline{x}) \right| = O\left\{ \left(\frac{\log n}{nh^d} \right)^{1/2} \right\},$$

which together with Lemma 8 yields the desired results. ■