On the structure of $(2+1)$–dimensional commutative and noncommutative integrable equations

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Abstract

We develop the symbolic representation method to derive the hierarchies of $(2+1)$-dimensional integrable equations from the scalar Lax operators and to study their properties globally. The method applies to both commutative and noncommutative cases in the sense that the dependent variable takes its values in $\mathbb{C}$ or a noncommutative associative algebra. We prove that these hierarchies are indeed quasi-local in the commutative case as conjectured by Mikhailov and Yamilov in 1998, [MY98]. We propose a ring extension based on the symbolic representation. As examples, we give noncommutative versions of KP, mKP and Boussinesq equations.

1   Introduction

Integrable $(1+1)$–dimensional nonlinear evolution equations, i.e. equations of the form

$$u_t = F(u, u_x, u_{xx}, ..., u_{xx...x})$$

possess rich algebraic and geometrical structures such as Lax representations, a bi-Hamiltonian formulation, infinitely many symmetries and conservation laws, etc. A lot of work has been devoted to the study of such equations and comprehensive classification results have been obtained.

The extension of these remarkable structures to the $(2+1)$–dimensional case is not straightforward. It was proved in [ZK84] that there is no bi-Hamiltonian formulation for $(2+1)$–dimensional integrable equations of the same type as those for the $(1+1)$–dimensional case like the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + 6uu_x.$$

In 1988, Fokas and Santini, cf. [SF88, FS88], constructed a bi-Hamiltonian structure for the Kadomtsev-Petviashvili (KP) equation

$$u_t = u_{xxx} + 6uu_x + 3D_x^{-1}u_{yy}$$

by considering it as a reduction of $(3+1)$-dimensional system. Meanwhile, Magri and his coauthors, [MMT88], explained this structure from the geometric point of the view by developing the concept of Nijenhuis G-manifolds, which amazingly works for both one and two space dimensions. Dorfman and her coauthors introduced the noncommutative ring of formal pseudo-differential operators, cf. [DF92, AD93]. They proved that the Fokas-Santini bi-Hamiltonian structure of the KP can be obtained from the Adler-Gel'fand-Dikii (AGD) scheme by considering the second order Lax operator

$$L = D_x^2 + u - \partial_y.$$

The bi-Hamiltonian structure naturally leads to a recursion operator for the equation. However, the operator (so-called operand) takes its value in the ring of formal differential operators and is of a different type from the one for KdV (and therefore does not contradict the result of [ZK84]). A hierarchy of infinitely many symmetries can only be produced by making suitable combinations of the operator acting on distinct seeds [MMT88].

One of the main obstacles to extend the spectacular results of $(1+1)$-dimensional integrable equations to the $(2+1)$-dimensional case is that the equations themselves, their higher symmetries and conservation laws are non-local, i.e. the appearance of the formal integral D_x^{-1} or D_y^{-1}. In 1998, Mikhailov and Yamilov, [MY98], introduced the concept of quasi-local functions based on the observation that the operators D_x^{-1} and D_y^{-1}

never appear alone but always in pairs like $D_x^{-1}D_y$ and $D_y^{-1}D_x$ for all known integrable equations and their hierarchies of symmetries and conservation laws, which enables them to extend the symmetry approach of testing integrability [MSS91] to the $(2+1)$-dimensional case.

In this paper, we develop the symbolic representation method to produce hierarchies of $(2+1)$-dimensional integrable equations from scalar Lax operators and to study their properties globally. The method applies both to the commutative and noncommutative case in the sense that the dependent variable takes its values in $\mathbb{C}$ or in a noncommutative associative algebra. We prove that these hierarchies are indeed quasi-local in the commutative case as conjectured by Mikhailov and Yamilov in 1998, [MY98]. This concept of quasi-locality has to be extended in the noncommutative case. Here we propose a ring extension based on the symbolic representation. As examples, we give noncommutative versions of the KP, mKP and Boussinesq equations.

2 Quasi-local polynomials and symbolic representation

The symbolic method was introduced by Gel'fand and Dikii in 1975 [GD75]. However, its basic idea can be dated back to the middle of 19th century. Recently, we successfully applied this method to the classification of both commutative and noncommutative $(1+1)$-dimensional homogeneous evolution equations in a series of papers, cf. [SW98, BSW98, OW00]. With the help of number theory, it enables us to give a global description of their integrable hierarchies [Wan98]. The symbolic method is also powerful in dealing with differential (cf. [SW02]) and pseudo-differential (cf. [MN02]) operators. The authors of [MN02] generalised the standard symmetry approach [MSS91] and made it suitable for the study of nonlocal and non–evolutionary equations, see also [MNW06].

In this section, we'll extend the symbolic method to the case of two spatial variables x and y. For simplicity, we restrict our attention to a single dependent variable u. Extensions to several independent variables and dependent variables are straightforward.

2.1 Quasi-local polynomials

We begin with basic definitions and notations of the ring of commutative and non-commutative differential polynomials. The derivatives of dependent variable u with respect to its independent variables x and y are denoted by $u_{ij} = \partial_x^i \partial_y^j u$. For smaller i and j, we sometimes write the indices out explicitly, that is u_{xxy} and u instead of u_{21} and u_{00}. A differential monomial takes the form

$$u_{i_1 j_1} u_{i_2 j_2} \cdots u_{i_k j_k} \ .$$

We call k the degree of the monomial. We let $\mathcal{U}^k$ denote the set of differential polynomials of degree k. The ring of differential polynomials is denoted by $\mathcal{U} = \oplus_{k \geq 1} \mathcal{U}^k$. Notice that $1 \notin \mathcal{U}$ and it is a differential ring with total x-derivation and y-derivation

$$D_x = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} u_{i+1,j} \frac{\partial}{\partial u_{ij}} \quad \text{and} \quad D_y = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} u_{i,j+1} \frac{\partial}{\partial u_{ij}}.$$

The ring $\mathcal{U}$ is commutative if the dependent variable u takes its values in a commutative algebra, for example the ring of smooth functions in x and y. Let us denote

$$\Theta = D_x^{-1}D_y, \qquad \Theta^{-1} = D_y^{-1}D_x. \tag{1}$$

The concept of **quasi-local (commutative) polynomials** $\mathcal{U}(\Theta)$ was introduced in [MY98] to test integrability of a given equation. To define it, we consider a sequence of extensions of $\mathcal{U}$. Let $\Theta\mathcal{U} = \{\Theta f : f \in \mathcal{U}\}$ and $\Theta^{-1}\mathcal{U} = \{\Theta^{-1}f : f \in \mathcal{U}\}$. We define $\mathcal{U}_0(\Theta) = \mathcal{U}$ and $\mathcal{U}_k(\Theta)$ is the ring closure of the union

$$\mathcal{U}_{k-1}(\Theta) \cup \Theta\mathcal{U}_{k-1}(\Theta) \cup \Theta^{-1}\mathcal{U}_{k-1}(\Theta).$$

Here the index k indicates the maximal depth of nesting the operator $\Theta^{\pm 1}$ in the expression. Clearly, we have $\mathcal{U}_{k-1}(\Theta) \subset \mathcal{U}_k(\Theta)$. We now define $\mathcal{U}(\Theta) = \lim_{k \to \infty} \mathcal{U}_k(\Theta)$. However, for a given $f \in \mathcal{U}(\Theta)$, there exists k such that $f \in \mathcal{U}_k(\Theta)$. The extension $\mathcal{U}(\Theta)$ has a natural gradation according to the number of u and its derivatives

$$\mathcal{U}(\Theta) = \oplus_{l \geq 1} \mathcal{U}^l(\Theta), \quad \mathcal{U}^p(\Theta) \cdot \mathcal{U}^q(\Theta) \in \mathcal{U}^{p+q}(\Theta).$$

Note that $\mathcal{U}_k(\Theta)$ is not invariant under transformation of variables. For example, a simple transformation

$$x' = x + y, \qquad y' = y \tag{2}$$

transforms $\Theta = D_{x'}^{-1} D_{y'} \mapsto 1 - \Theta = 1 - D_x^{-1} D_y$. Hence $\Theta^{-1} \mapsto (1 - \Theta)^{-1}$, which is not in $\mathcal{U}_k(\Theta)$.

If u takes its value in a noncommutative associative algebra, the ring $\mathcal{U}$ is noncommutative. Typical examples are the algebras of $n \times n$ matrices and Clifford algebras (see [OS98] for more examples and noncommutative $(1+1)$-dimensional integrable evolution equations). For any element f, g and h in a noncommutative associative algebra, we use the notation

$$L_f(g) = fg, \qquad R_f(g) = gf \tag{3}$$

for the operators of left and right multiplication and

$$C_f(g) = L_f(g) - R_f(g) = fg - gf \tag{4}$$

for the commutator (notation ad_f is also commonly used). It is a derivation since it satisfies the Leibniz rule: $C_f(gh) = C_f(g)h + gC_f(h)$.

So far there is no proper extension from noncommutative differential polynomials to quasi-local noncommutative polynomials, mainly because the study of noncommutative $(2+1)$-dimensional integrable equations is a new and challenging topic. We can take the corresponding extension of $\mathcal{U}(\Theta)$ as the starting point. However, this turns out to be too restrictive. This will be discussed further in Section 4.2.

2.2   Symbolic representation

The basic idea of the symbolic representation is to replace u_{ij} by $u\xi^i \eta^j$, where ξ and η are symbols. Now the total differentiation with respect to x, that is, mapping u_{ij} to $u_{i+1,j}$, is replaced by multiplication with ξ, as in the Fourier transform. Similarly, the total differentiation with respect to y mapping u_{ij} to $u_{i,j+1}$, is replaced by multiplication with η. For higher degree terms with multiple u's, one uses different symbols to denote each of them. Its symbolic representation depends on where u takes its value, i.e. whether it is commutative or not. For example, the noncommutative binomial $u_{ij}u_{kl}$ has symbolic representation $u^2 \xi_1^i \eta_1^j \xi_2^k \eta_2^l$. In the commutative case, one has $u_{ij}u_{kl} = u_{kl}u_{ij}$. We therefore need to average its symbolic representation over the permutation group Σ_2 so that $u_{ij}u_{kl}$ and $u_{kl}u_{ij}$ have the same symbolic form. So the symbolic representation of $u_{ij}u_{kl}$ is $\frac{u^2}{2}(\xi_1^i \eta_1^j \xi_2^k \eta_2^l + \xi_2^i \eta_2^j \xi_1^k \eta_1^l)$. Symmetrisation makes the symbol representation of monomials unique.

Let $\mathcal{A}^k$ be the space of algebraic polynomials f in $2k$ variables, ξ_i and η_i, where $i = 1, \cdots, k$.

The symbolic representation defines a linear isomorphism between the space $\mathcal{U}^k$ of (non)-commutative differential polynomials of degree k and the space $\mathcal{A}^k$. It is uniquely defined by its action on monomials.

Definition 1. *The symbolic representation of a differential monomial is defined as*

$$u_{i_1,j_1} u_{i_2,j_2} \cdots u_{i_k,j_k} \longmapsto \begin{cases} u^k \xi_1^{i_1} \eta_1^{j_1} \xi_2^{i_2} \eta_2^{j_2} \cdots \xi_k^{i_k} \eta_k^{j_k}, & (noncommutative); \\ \frac{u^k}{k!} \sum_{\sigma \in \Sigma_k} \xi_{\sigma(1)}^{i_1} \eta_{\sigma(1)}^{j_1} \cdots \xi_{\sigma(k)}^{i_k} \eta_{\sigma(k)}^{j_k}, & (commutative). \end{cases}$$

There are two parts in the symbolic representation of a monomial: the first part, u^k, indicating its degree; the second part being in $\mathcal{A}^k$. The symbols of Gel'fand and Dikii only contain the second part. We emphasise that the first part is very important in dealing with the monomials u^l the case of several noncommutative dependent variables.

In general, we denote the symbolic representation of $P \in \mathcal{U}^k$, whether it is commutative or not, by $\hat{P}$ and $Q \in \mathcal{U}^l$ by $\hat{Q}$. The multiplication of P and Q corresponds to the symbol

$$\widehat{PQ}(\xi_1, \eta_1, \cdots, \xi_k, \eta_k, \xi_{k+1}, \eta_{k+1}, \cdots, \xi_{k+l}, \eta_{k+l}) = \hat{P}(\xi_1, \eta_1, \cdots, \xi_k, \eta_k)\hat{Q}(\xi_{k+1}, \eta_{k+1}, \cdots, \xi_{k+l}, \eta_{k+l}); \tag{5}$$

when P and Q are commutative differential polynomials, the right-hand side needs to be symmetrised

$$\begin{aligned} &\widehat{PQ}(\xi_1, \eta_1, \cdots, \xi_k, \eta_k, \xi_{k+1}, \eta_{k+1}, \cdots, \xi_{k+l}, \eta_{k+l}) \\ = \ &\frac{1}{(k+l)!} \sum_{\sigma \in \Sigma_{k+l}} \hat{P}(\xi_{\sigma(1)}, \eta_{\sigma(1)}, \cdots, \xi_{\sigma(k)}, \eta_{\sigma(k)})\hat{Q}(\xi_{\sigma(k+1)}, \eta_{\sigma(k+1)}, \cdots, \xi_{\sigma(k+l)}, \eta_{\sigma(k+l)}). \end{aligned} \tag{6}$$

The total derivatives D_x and D_y have the following representations:

$$\widehat{D_x P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k) = (\xi_1 + \cdots + \xi_k)\hat{P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k);$$
$$\widehat{D_y P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k) = (\eta_1 + \cdots + \eta_k)\hat{P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k).$$

Naturally, the actions of $\Theta = D_x^{-1}D_y$ and Θ^{-1} on $P \in \mathcal{U}^k$ can be represented as

$$\widehat{\Theta P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k) = \frac{\xi_1 + \cdots + \xi_k}{\eta_1 + \cdots + \eta_k} \hat{P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k);$$

$$\widehat{\Theta^{-1} P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k) = \frac{\eta_1 + \cdots + \eta_k}{\xi_1 + \cdots + \xi_k} \hat{P}(\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k).$$

Similar to the multiplication of P and Q, we have

Definition 2. *Let $P \in \mathcal{U}^k$ and $Q \in \mathcal{U}^l$. Then*

$$\widehat{P\Theta Q} = \hat{P}(\xi_1, \eta_1, \cdots, \xi_k, \eta_k) \frac{\xi_{k+1} + \cdots + \xi_{k+l}}{\eta_{k+1} + \cdots + \eta_{k+l}} \hat{Q}(\xi_{k+1}, \eta_{k+1}, \cdots, \xi_{k+l}, \eta_{k+l}).$$

When P and Q are commutative differential polynomials, the right-hand side needs to be symmetrised.

In the same way, we can uniquely define the symbol of $P\Theta^{-1}Q$. Together with the multiplication rule (5) or (6), we have now completely determined the symbolic representations of the elements in $\mathcal{U}_1(\Theta)$. By induction, we can define the symbolic representation of any element in $\mathcal{U}(\Theta)$, which is a rational function with its denominator being the products of the linear factors. The expression of the denominator uniquely determines how the operator $\Theta^{\pm 1}$ is nested.

Example 1. *The expressions $u^3 \xi_1 \frac{\eta_2 + \eta_3}{\xi_2 + \xi_3} \frac{\xi_3}{\eta_3}$ and $u^3 \xi_1 \frac{\eta_2}{\xi_2} \frac{\xi_3}{\eta_3}$ are the symbolic representations of $u_x \Theta(u\Theta^{-1}u) \in \mathcal{U}_2(\Theta)$ and $u_x(\Theta u)\Theta^{-1}u \in \mathcal{U}_1(\Theta)$ respectively.*

Let us define the symbolic representations of pseudo-differential operators motivated by their Fourier transforms. First we assign a special symbol X to the operator D_x. Its formal inverse D_x^{-1} has the symbol $\frac{1}{X}$. Then we have the following rules for $f \in \mathcal{U}^k(\Theta)$ with symbol $\hat{f}$ and $g \in \mathcal{U}^l(\Theta)$ with symbol $\hat{g}$:

$$X \circ \hat{f}(\xi_1, \eta_1, \ldots, \xi_k, \eta_k) = (\xi_1 + \cdots + \xi_k + X)\hat{f}(\xi_1, \eta_1, \ldots, \xi_k, \eta_k),$$

which represents the Leibniz rule $D_x \circ f = D_x(f) + f \circ D_x$; and the composition rule

$$\hat{f}X^p \circ \hat{g}X^q = \hat{f}(\xi_1, \eta_1, \ldots, \xi_k, \eta_k)(\xi_{k+1} + \cdots + \xi_{k+l} + X)^p \hat{g}(\xi_{k+1}, \eta_{k+1}, \ldots, \xi_{k+l}, \eta_{k+l})X^q.$$

This composition rule is valid for both positive and negative powers p and q. For positive powers, it is a polynomial of X. For negative powers, one can expand it at $X = \infty$ to identify it.

Example 2. *The symbol of $D_x^{-1}u$ is $\frac{u}{X+\xi}$ if we assign ξ as the symbol for u. Then in the noncommutative case we have*

$$D_x^{-1}uD_x^{-1}u = u^2 D_x^{-2} - (2uu_x + u_xu)D_x^{-3} + (3uu_{xx} + 3u_x^2 + u_{xx}u)D_x^{-4} + \cdots \cdots$$

Symbolically, we can compute it by

$$\frac{u}{X+\xi} \circ \frac{u}{X+\xi} = \frac{u^2}{(X+\xi_1+\xi_2)(X+\xi_2)} = u^2\left(\frac{1}{X^2} - \frac{\xi_1 + 2\xi_2}{X^3} + \frac{\xi_1^2 + 3\xi_1\xi_2 + 3\xi_2^2}{X^4} + \cdots\right).$$

To extend the symbolic representation from one dependent variable to several noncommutative dependent variables is straightforward. We need to assign new symbols for each of them such as assigning $\xi^{(1)}, \eta^{(1)}$ for u and $\xi^{(2)}, \eta^{(2)}$ for v and so on. As one can imagine, the abstract formulas for arbitrary n dependent variables get very long and it is difficult to present them in a compact way. We have no problems in handling small n or given expressions.

Example 3. *The symbolic representation of $u_x v_y$ is $uv\xi^{(1)}\eta^{(2)}$ and that of $v_y u_x$ is $vu\xi^{(1)}\eta^{(2)}$.*

Notice that the only difference in the symbolic representations for $u_x v_y$ and $v_y u_x$ is uv instead of vu, that is, the non–commutativity of dependent variables lies in the first part of the symbolic representation indicating the degree of the expression.

3 Lax formulation of $(2+1)$–dimensional integrable equations

In this section, we give a short description of construction of (2+1)-dimensional integrable equations from a given scalar Lax operator based on the well-known Sato approach. For details, see the recent books [BS01, Kup00] and related references in them.

Consider an m-th order formal pseudo-differential operator

$$A = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} + \cdots, \quad m \geq 0,$$

where the coefficients a_k are functions of x and D_x^{-1} is the formal inverse of the total x-derivative D_x. Here the functions a_k take their values in commutative (for instance complex numbers $\mathbb{C}$) or noncommutative (for instance the algebra of $n \times n$ matrices) associative algebras.

To define the product of two pseudo-differential operators requires the action of differential operator D_x^n on a multiplication operator f (given by a function of x):

$$D_x^n \circ f = \sum_{i \geq 0} \frac{n(n-1) \cdots (n-i+1)}{i!} (D_x^i f) D_x^{n-i}$$

Let the commutator be the bracket on the set of pseudo-differential operators. Thus, the set of pseudo-differential operators forms a Lie algebra. For an integer $k < m$, we introduce notations:

$$A_{\geq k} = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_k D_x^k$$
$$A_{<k} = A - A_{\geq k} = a_{k-1} D_x^{k-1} + \cdots$$

When $k \in \{0, 1\}$, the Lie algebra decomposes as a direct sum of two subalgebras in both commutative and noncommutative cases. We denote the projections onto these subalgebras by $\mathcal{P}_{\pm}$. Such decompositions are naturally related with integrability and lead to admissible scalar Lax operators for $(1+1)$-dimensional Lax dynamics (cf. [Bla98] pp. 270 for commutative case), namely

$$k = 0: \quad \tilde{L} = D_x^n + u^{(n-2)} D_x^{n-2} + u^{(n-3)} D_x^{n-3} + \cdots + u^{(0)}, \quad n \geq 2;$$
$$k = 1: \quad \tilde{L} = D_x^n + u^{(n-1)} D_x^{n-1} + u^{(n-2)} D_x^{n-2} + \cdots + u^{(0)} + D_x^{-1} u^{(-1)}, \quad n \geq 2;$$

where $u^{(i)}$ are functions of a spatial variable x.

We know every pseudo-differential operator of order $n > 0$ has an n-th root. This allows us to define the fractional powers $\tilde{L}^{\frac{i}{n}}, i \in \mathbb{N}$. Let $\tilde{B}_i = \mathcal{P}_+(\tilde{L}^{\frac{i}{n}})$. For each choice of i, we introduce a different time variable t_i. Thus, the flows defined by

$$\frac{\partial \tilde{L}}{\partial t_i} = [B_i, \tilde{L}], \quad i \in \mathbb{N}$$

commute, [GD76].

This setting up can be generalised into the $(2+1)$-dimensional case as follows: An m-th order pseudo-differential operator of two spatial variables is of the form

$$H = -D_y + a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} + \cdots, \quad m \geq 0,$$

where coefficients a_k are functions of x, y and for an integer $k < m$, we split into

$$H_{\geq k} = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_k D_x^k$$
$$H_{<k} = H - H_{\geq k} = -D_y + a_{k-1} D_x^{k-1} + \cdots$$

Similar to $(1+1)$-dimensional case, this operator algebra decomposes as a direct sum of two subalgebras in both commutative and noncommutative cases when $k \in \{0, 1\}$. The admissible types of scalar Lax operators for the case of $(2+1)$ dimensions are $\tilde{L} - D_y$, i.e.

a. $k = 0:$ $L = D_x^n + u^{(n-2)} D_x^{n-2} + u^{(n-3)} D_x^{n-3} + \cdots + u^{(0)} - D_y, \quad n \geq 2;$

b. $k = 1:$ $L = D_x^n + u^{(n-1)} D_x^{n-1} + u^{(n-2)} D_x^{n-2} + \cdots + u^{(0)} + D_x^{-1} u^{(-1)} - D_y, \quad n \geq 2;$

c. $k = 1:$ $L = u^{(0)} + D_x^{-1} u^{(-1)} - D_y;$

where $u^{(i)}$ are functions of two spatial variables x, y. We often use $u, v, w, \cdots$ in the examples. When $n = 1$ for case **b**, we have $L = D_x + u^{(0)} + D_x^{-1} u^{(-1)} - D_y$, which can be transformed into case **c** by transformation (2). Therefore, we exclude it from our consideration.

Let $S = D_x + a_0 + a_{-1} D_x^{-1} + \cdots$. For any operator L as listed in cases **a**, **b** and **c**, the relation

$$[S, \ L] := SL - LS = 0, \tag{7}$$

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uniquely determines the operator S by taking the integration constants as zero. Furthermore, we have $[S^n, L] = 0$ for any $n \in \mathbb{N}$. For each choice of i, we introduce a different time variable t_i and define the Lax equation by

$$\frac{\partial L}{\partial t_i} = [S^i_{\geq k}, L], \tag{8}$$

where k is determined by the operator L as listed in cases **a**, **b** and **c**.

Theorem 1. *For the operator S uniquely determined as above by (7), the flows defined by Lax equations (8) commute, that is, $\partial_{t_j}\partial_{t_i} L = \partial_{t_i}\partial_{t_j} L$.*

Proof. Using Lax equations, we have

$$\partial_{t_j}\partial_{t_i} L = [\partial_{t_j} S^i_{\geq k}, L] + [S^i_{\geq k}, \partial_{t_j} L] = [\partial_{t_j} S^i_{\geq k}, L] + [S^i_{\geq k}, [S^j_{\geq k}, L]]$$

Hence

$$\partial_{t_j}\partial_{t_i} L - \partial_{t_i}\partial_{t_j} L = [\partial_{t_j} S^i_{\geq k} - \partial_{t_i} S^j_{\geq k} - [S^j_{\geq k}, S^i_{\geq k}], L].$$

Meanwhile, from formula (7), we get

$$[\partial_{t_i} S, \ L] = -[S, \partial_{t_i} L] = -[S, [S^i_{\geq k}, L]] = [[S^i_{\geq k}, S], L]$$

implying

$$\partial_{t_i} S = [S^i_{\geq k}, S]. \tag{9}$$

Thus

$$\partial_{t_j} S^i_{\geq k} - \partial_{t_i} S^j_{\geq k} - [S^j_{\geq k}, S^i_{\geq k}]$$
$$= (\partial_{t_j} S^i)_{\geq k} - (\partial_{t_i} S^j)_{\geq k} - [S^j_{\geq k}, S^i_{\geq k}] = [S^j_{\geq k}, S^i]_{\geq k} - [S^i_{\geq k}, S^j]_{\geq k} - [S^j_{\geq k}, S^i_{\geq k}]$$
$$= -[S^j_{<k}, S^i_{\geq k} + S^i_{<k}]_{\geq k} - [S^i_{\geq k}, S^j_{\geq k} + S^j_{<k}]_{\geq k} - [S^j_{\geq k}, S^i_{\geq k}] = -[S^j_{<k}, S^i_{<k}]_{\geq k}$$
$$= 0.$$

This leads to $\partial_{t_j}\partial_{t_i} L = \partial_{t_i}\partial_{t_j} L$. ◇

Remark 1. *In the commutative case, there is one more admissible type of scalar Lax operator, that is,*

$$k = 2: \quad L = u^{(n)^n} D_x^n + u^{(n-1)} D_x^{n-1} + u^{(n-2)} D_x^{n-2} + \cdots + u^{(0)} + D_x^{-1} u^{(-1)} + D_x^{-2} u^{(-2)} - D_y.$$

Theorem 1 is also valid for this type by taking $S = u^{(n)} D_x + a_0 + a_{-1} D_x^{-1} + \cdots$ instead.

Example 4. *Consider the Lax operator $L = u^2 D_x^2 + v D_x + w + D_x^{-1} p + D_x^{-2} q - D_y$, where all dependent variables take their values in $\mathbb{C}$. Using formula (7), we have $S = u D_x + a_0 + a_{-1} D_x^{-1} + \cdots$ with*

$$a_0 = -\frac{1}{2}\left(u_x - \frac{v}{u} + \Theta\frac{1}{u}\right)$$

$$a_{-1} = \frac{1}{8}\left(2u_{xx} - \frac{u_x^2}{u} - \frac{2v_x}{u} + \frac{4u_x v}{u^2} - \frac{v^2}{u^3} - \frac{4u_y}{u^2} + \frac{2}{u}D_x^{-1}D_y\frac{v}{u^2} + \frac{4w}{u} - \frac{2}{u}\Theta\left(\frac{1}{u}\Theta\frac{1}{u}\right) + \frac{1}{u}\left(\Theta\frac{1}{u}\right)^2\right).$$

and $S^1_{\geq 2} = 0$, $S^2_{\geq 2} = u^2 D_x^2$, $S^3_{\geq 2} = u^3 D_x^3 + 3u^2(u_x + a_0)D_x^2$. By the Lax equation, we obtain

$$\begin{cases} u_{t_3} = \frac{1}{4}(u^3 u_{xxx} - 6u_y\Theta\frac{1}{u} - 3u(\Theta\frac{1}{u})_y) + 3u^2 v_{xx} + 3u^2 w_x + 3v_y + 3vv_x - 3\frac{u_x}{u}v^2 - 6(uv)_x\Theta\frac{1}{u}) \\ v_{t_3} = u^3 v_{xxx} + 3u^2(w_{xx} + up_x + u_x p + (v_{xx} + 2w_x)(u_x + a_0)) \\ w_{t_3} = u^3 w_{xxx} + 3u^2(w_{xx} + p_x)(u_x + a_0) + 3p(u^2 a_0)_x + 3u^2(uq_x - 2u_x p_x + 2u_x q) \\ p_{t_3} = (u^3 p)_{xxx} - 3\left([u^2(u_x p + a_0 p + q)]_{xx} + 2[u^2 q(u_x + a_0)]_x\right) \\ q_{t_3} = (u^3 q)_{xxx} - 3[u^2 q(u_x + a_0)]_{xx} \end{cases}$$

Notice that the reduction of $v = w = p = q = 0$ leads to well-known $(2+1)$-Harry-Dym equation

$$u_{t_3} = \frac{1}{4}\left(u^3 u_{xxx} - 6u_y\Theta\frac{1}{u} - 3u(\Theta\frac{1}{u})_y\right).$$

6

4 Lax formulation in symbolic representation

In this section, we carry out the formalism of section 3 in symbolic form. We first consider noncommutative dependent variables to simplify our formulae. The strategy for the commutative case is to do the calculation as much as possible by treating it as noncommutative and only do symmetrisation at the last stage to get the uniqueness of the symbolic representation since the symmetrisation complicates the calculation dramatically.

Let us assign the symbols $\xi^{(i)}, \eta^{(i)}$ for dependent variable $u^{(i)}$ and the symbol Y for the operator D_y. The symbolic representations of the admissible scalar Lax operators are

    a. $k = 0:$ $\hat{L} = X^n - Y + u^{(n-2)}X^{n-2} + u^{(n-3)}X^{n-3} + \cdots + u^{(0)}, \quad n \geq 2;$

    b. $k = 1:$ $\hat{L} = X^n - Y + u^{(n-1)}X^{n-1} + u^{(n-2)}X^{n-2} + \cdots + u^{(0)} + u^{(-1)}\frac{1}{X+\xi^{(-1)}}, \quad n \geq 2;$

    c. $k = 1:$ $\hat{L} = -Y + u^{(0)} + u^{(-1)}\frac{1}{X+\xi^{(-1)}};$

We first treat case **a**. It is convenient to consider formal series in the form

$$S = X + \sum_{i=0}^{n-2} u^{(i)} a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) + \sum_{i_1=0}^{n-2}\sum_{i_2=0}^{n-2} u^{(i_1)}u^{(i_2)} a_2^{(i_1 i_2)}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X) + \cdots, n \geq 2 \quad (10)$$

where a_i are functions of their specific arguments and j_k is defined by the number of i_k in the list of $[i_1, i_2, \cdots, i_l]$, which implies that $j_1 = 1$ and $j_k \geq 1$, $k = 1, 2, \cdots, l$. For example, when $i_1 = i_2$, the arguments of function $a_2^{i_1 i_1}$ are $\xi_1^{(i_1)}, \eta_1^{(i_1)}, \xi_2^{(i_1)}, \eta_2^{(i_1)}$ and X.

It is easy to check that

$$[X^n - Y, \ \phi(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)] = N_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)\phi, \quad (11)$$

where the polynomial N_l is defined by

$$N_l(x_1, y_1, x_2, y_2, \ldots, x_l, y_l; X) = (\sum_{i=1}^{l} x_i + X)^n - X^n - \sum_{i=1}^{l} y_i \quad (12)$$

Proposition 1. *For any operator L in case* **a**, *if the formal series (10) satisfies the relation $[S, \ L] = 0$ (cf. (7)), we have for $l \geq 1$,*

$$a_l^{(i_1 i_2 \cdots i_l)} = a_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) = \prod_{r=1}^{l}(X + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} b_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X), \quad (13)$$

where the superindex $i_s \in \{0, 1, 2, \cdots, n-2\}$ and the subindex j_k is defined by the number of i_k in the list of $[i_1, i_2, \cdots, i_k]$. The function b_l, $l \geq 1$, is defined by

$$b_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) = \frac{c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}{N_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}. \quad (14)$$

with

$$c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) =$$
$$= b_{l-1}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)} \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) - b_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X), \quad l > 1 \quad (15)$$

and the initial function $c_1(\xi_1^{(i)}, \eta_1^{(i)}, X) = \xi_1^{(i)}$.

Proof. We compute $[L, \ S]$ symbolically and collect the coefficients of each algebraically independent monomial of the dependent variables. For the linear terms, we have for any $i \in \{0, \cdots, n-2\}$,

$$a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) = \frac{\xi_1^{(i)} X^i}{N_1(\xi_1^{(i)}, \eta_1^{(i)}, X)} = b_1(\xi_1^{(i)}, \eta_1^{(i)}, X)X^i. \quad (16)$$

For the quadratic terms, when $i \neq j$, we need to compute

$$
\begin{aligned}
A^{ij} : \;&=\; -X^i \circ a_1^{(j)}(\xi_1^{(j)}, \eta_1^{(j)}, X) + a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) \circ X^j \\
&=\; -\frac{(X + \xi_1^{(j)})^i \xi_1^{(j)} X^j}{N_1(\xi_1^{(j)}, \eta_1^{(j)}, X)} + \frac{\xi_1^{(i)}(X + \xi_1^{(j)})^i X^j}{N_1(\xi_1^{(i)}, \eta_1^{(i)}, X + \xi_1^{(j)})} \\
&=\; (X + \xi_1^{(j)})^i X^j \Big(\frac{\xi_1^{(i)}}{N_1(\xi_1^{(i)}, \eta_1^{(i)}, X + \xi_1^{(j)})} - \frac{\xi_1^{(j)}}{N_1(\xi_1^{(j)}, \eta_1^{(j)}, X)} \Big) \\
&=\; (X + \xi_1^{(j)})^i X^j (b_1(\xi_1^{(i)}, \eta_1^{(i)}, X + \xi_1^{(j)}) - b_1(\xi_1^{(j)}, \eta_1^{(j)}, X)) \\
&=\; (X + \xi_1^{(j)})^i X^j c_2(\xi_1^{(i)}, \eta_1^{(i)}, \xi_1^{(j)}, \eta_1^{(j)}, X)
\end{aligned}
$$

and when $i = j$, we have

$$
\begin{aligned}
A^{ii} : \;&=\; -[X^i, \, a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X)] = -X^i \circ a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) + a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) \circ X^i \\
&=\; -\frac{(X + \xi_2^{(i)})^i \xi_2^{(i)} X^i}{N_1(\xi_2^{(i)}, \eta_2^{(i)}, X)} + \frac{\xi_1^{(i)}(X + \xi_2^{(i)})^i X^i}{N_1(\xi_1^{(i)}, \eta_1^{(i)}, X + \xi_2^{(i)})}.
\end{aligned}
$$

This can be obtained by substituting $\xi_1^{(j)} = \xi_2^{(i)}$ and $\eta_1^{(j)} = \eta_2^{(i)}$ in A^{ij}. It follows that

$$
\begin{aligned}
a_2^{(i_1 i_2)}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X) &= \frac{A^{i_1 i_2}}{N_2(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X)} \\
&= (X + \xi_{j_2}^{(i_2)})^{i_1} X^{i_2} b_2(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X). \tag{17}
\end{aligned}
$$

Assume that formula (13) is valid for degree $l - 1$. Let us compute $a_l^{(i_1 i_2 \cdots i_l)}$ in the same way as computing the quadratic terms. Then we have

$$
\begin{aligned}
& N_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) a_l^{(i_1 i_2 \cdots i_l)} = -X^{i_1} \circ a_{l-1}^{(i_2 \cdots i_l)} + a_{l-1}^{(i_1 i_2 \cdots i_{l-1})} \circ X^{i_l} \\
&= -(X + \sum_{p=2}^{l} \xi_{j_p}^{(i_p)})^{i_1} \prod_{r=2}^{l}(X + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} b_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) \\
&\quad + \prod_{r=1}^{l-1}(X + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} X^{i_l} b_{l-1}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) \\
&= \prod_{r=1}^{l}(X + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} (b_{l-1}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) - b_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)). \\
&= \prod_{r=1}^{l}(X + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X).
\end{aligned}
$$

According to definition (14), we proved that formula (13) is valid for degree l. $\diamond$

Similarly, we prove the following result for any operator L in cases **b** and **c**.

Proposition 2. *For any operator L in case* **b**, *consider a formal series of the form*

$$
S = X + \sum_{i=-1}^{n-1} u^{(i)} a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) + \sum_{i_1, i_2 = -1}^{n-1} u^{(i_1)} u^{(i_2)} a_2^{(i_1 i_2)}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X) + \cdots, n \geq 2 \tag{18}
$$

Here a_i are functions of their specific arguments, the superindex $i_s \in \{-1, 0, 1, \cdots, n-1\}$ and the subindex j_k is defined by the number of i_k in the list of $[i_1, i_2, \cdots, i_k]$. If it satisfies the relation $[S, L] = 0$ (cf. (7)), we have for $l \geq 1$,

$$
a_l^{(i_1 i_2 \cdots i_l)} = a_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) = \prod_{r=1}^{l}(h(i_r) + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)})^{i_r} b_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X), \tag{19}
$$

where $h(i_r) = X + \xi_{j_r}^{(-1)}$ if $i_r = -1$, otherwise $h(i_r) = X$. The functions b_l are defined in Proposition 1, cf. formula (14) and (15).

Notice that for operator L listed in case **c**, we have $N_l(x_1, y_1, x_2, y_2, \ldots, x_l, y_l; X) = -\sum_{i=1}^{l} y_i$, cf. (11), which corresponds to the action of total y-derivative D_y. Thus

Proposition 3. *For operator L listed in case **c**, if a formal series of the form*

$$S = X + \sum_{i=-1}^{0} u^{(i)} a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X) + \sum_{i_1, i_2=-1}^{0} u^{(i_1)} u^{(i_2)} a_2^{(i_1 i_2)}(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X) + \cdots, \tag{20}$$

where the superindex $i_s \in \{-1, 0\}$ and the subindex j_k is defined by the number of i_k in the list of $[i_1, i_2, \cdots, i_k]$, satisfies the relation $[S, L] = 0$ (cf. (7)), we have for $l \geq 1$, the functions a_l are defined in Proposition 2, cf. formula (19) with $N_l(x_1, y_1, x_2, y_2, \ldots, x_l, y_l; X) = -\sum_{i=1}^{l} y_i$.

To construct the hierarchy of the Lax equations we need to expand the coefficients of operator (10) or (18) or (20) at $X = \infty$ and truncate at the required degree.

Definition 3. *We say that a function $h(x_1, y_1, x_2, y_2, \cdots, x_l, y_l, X)$ is kth (quasi-local) polynomial if the first k coefficients of its expansion at $X = \infty$ are symbols of (quasi-local) polynomials. If a function is kth (quasi-local) polynomial for any k, we say it is (quasi-local) polynomial.*

When $n \geq 2$, that is, operator L in cases **a** and **b**, the expansion of $N_l(x_1, y_1, x_2, y_2, \ldots, x_l, y_l; X)$ at $X = \infty$ is of the form

$$N_l(x_1, y_1, x_2, y_2, \ldots, x_l, y_l; X)^{-1}$$
$$= \frac{1}{nX^{n-1}(\sum_{i=1}^{l} x_i)} \sum_{j \geq 0} \left[\frac{\sum_{i=1}^{l} y_i}{nX^{n-1}(\sum_{i=1}^{l} x_i)} - \frac{1}{n} \sum_{k=0}^{n-2} \binom{n}{k} X^{k+1-n} (\sum_{i=0}^{l} x_i)^{n-k-1} \right]^j, \quad n \geq 2.$$

For operator L listed in case **c**, we know $N_l(x_1, y_1, x_2, y_2, \ldots, x_l, y_l; X) = -\sum_{i=1}^{l} y_i$. Therefore, if we want to prove that the coefficients of operators (10), (18) and (20), i.e. the functions a_l, are quasi-local, we need to show that the functions c_l can be split into the sum of the image of D_x and the image of D_y. It is clear from formula (16) that $a_1^{(i)}(\xi_1^{(i)}, \eta_1^{(i)}, X)$ are quasi-local since we have $c_1(\xi_1^{(i)}, \eta_1^{(i)}, X) = \xi_1^{(i)}$. Now we concentrate on the cases when $l > 1$.

Proposition 4. *The functions $c_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)$ for $l > 1$ vanish after substitution*

$$\xi_{j_1}^{(i_1)} = -\xi_{j_2}^{(i_2)} - \cdots - \xi_{j_{l-1}}^{(i_{l-1})} \quad and \quad \eta_{j_1}^{(i_1)} = -\eta_{j_2}^{(i_2)} - \cdots - \eta_{j_{l-1}}^{(i_{l-1})}.$$

We give its proof in the Appendix. In fact, this proposition does not lead to our intended conclusion that b_l and thus a_l are quasi-local since the objects are rational, not polynomial. For example, the expression $u^2(\frac{\eta_2}{\xi_2} - \frac{\eta_1}{\xi_1})$ representing $u\Theta u - (\Theta u)u$ satisfies the above proposition. However, we can't write $u\Theta u - (\Theta u)u = D_x f_1 + D_y f_2$, where both f_1 and f_2 are in $\mathcal{U}(\Theta)$.

We first have a close look at the quadratic terms. For any operator L in cases **a** and **b**, let $\eta_{j_r}^{i_r} = \theta_{j_r}^{i_r} \xi_{j_r}^{i_r}$. Then the function $c_2(\xi_{j_1}^{i_1}, \eta_{j_1}^{i_1}, \xi_{j_2}^{i_2}, \eta_{j_2}^{i_2}, X)$, where $\xi_{j_1}^{i_1} \neq 0$ and $\xi_{j_2}^{i_2} \neq 0$, is a polynomial of $\theta_{j_r}^{i_r}, \xi_{j_r}^{i_r}, r = 1, 2$. Notice that

$$(\eta_{j_1}^{i_1} + \eta_{j_2}^{i_2}) = \theta_{j_1}^{i_1} \xi_{j_1}^{i_1} + \theta_{j_2}^{i_2} \xi_{j_2}^{i_2} = (\theta_{j_1}^{i_1} - \theta_{j_2}^{i_2}) \xi_{j_1}^{i_1} + \theta_{j_2}^{i_2}(\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}) = \theta_{j_1}^{i_1}(\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}) - (\theta_{j_1}^{i_1} - \theta_{j_2}^{i_2}) \xi_{j_2}^{i_2}. \tag{21}$$

The affine variety (cf. [CLO97]) defined by $\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}$ and $\eta_{j_1}^{i_1} + \eta_{j_2}^{i_2}$ is

$$V(\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}, \eta_{j_1}^{i_1} + \eta_{j_2}^{i_2}) = V(\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}, (\theta_{j_1}^{i_1} - \theta_{j_2}^{i_2}) \xi_{j_1}^{i_1}, (\theta_{j_1}^{i_1} - \theta_{j_2}^{i_2}) \xi_{j_2}^{i_2})$$
$$= V(\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}, \theta_{j_1}^{i_1} - \theta_{j_2}^{i_2}) \cup V(\xi_{j_1}^{i_1}, \xi_{j_2}^{i_2})$$

From Proposition 4, it can be written as

$$c_2(\xi_{j_1}^{i_1}, \theta_{j_1}^{i_1} \xi_{j_1}^{i_1}, \xi_{j_2}^{i_2}, \theta_{j_2}^{i_2} \xi_{j_2}^{i_2}, X) = (\xi_{j_1}^{i_1} + \xi_{j_2}^{i_2}) f_1 + (\theta_{j_1}^{i_1} - \theta_{j_2}^{i_2}) f_2,$$

where both f_1 and f_2 are polynomials of $\theta_{j_r}^{i_r}, \xi_{j_r}^{i_r}, r = 1, 2$. From formula (21), the part that is not in the image of D_x or D_y only depends on $\theta_{j_r}^{i_r}, r = 1, 2$.

For operator L listed in case **c**, since the function $N_l(x_1, y_1, \cdots, x_l, y_l, X) = -\sum_{i=1}^{l} y_i$, cf. Proposition 3, we let $\xi_{j_r}^{i_r} = \theta_{j_r}^{i_r} \eta_{j_r}^{i_r}$ instead. Then the function $c_2(\xi_{j_1}^{i_1}, \eta_{j_1}^{i_1}, \xi_{j_2}^{i_2}, \eta_{j_2}^{i_2}, X)$, where $\eta_{j_1}^{i_1} \neq 0$ and $\eta_{j_2}^{i_2} \neq 0$, is a polynomial of $\theta_{j_r}^{i_r}, \xi_{j_r}^{i_r}, r = 1, 2$. The above discussion is valid.

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Setting $\xi_{j_1}^{i_1} = \xi_{j_2}^{i_2} = 0$, we obtain the function $c_2(\xi_{j_1}^{i_1}, \eta_{j_1}^{i_1}, \xi_{j_2}^{i_2}, \eta_{j_2}^{i_2}, X)$ equals

$$\frac{1}{nX^{n-1} - \theta_{j_1}^{i_1}} - \frac{1}{nX^{n-1} - \theta_{j_2}^{i_2}} \quad \text{for cases } \mathbf{a} \text{ and } \mathbf{b}; \quad \theta_{j_2}^{i_2} - \theta_{j_1}^{i_1} \quad \text{for case } \mathbf{c}, \tag{22}$$

which is zero after we symmetrise it with the permutation group Σ_2. This implies that the functions b_2 and thus a_2 are quasi-local in the commutative case.

4.1 The commutative case

We first give the formulae to compute the high degree terms of operator S for the commutative case directly from the formulae we obtained in Proposition 1, 2 and 3.

We denote the list $(i_1, i_2, \cdots, i_l)$ by I and $I_1 = (i_{s_1}, i_{s_2}, \cdots, i_{s_r})$, where $1 \le s_1 < s_2 < \cdots < s_r \le l$ and $1 \le r \le l - 1$. We use $I_2 = (i_{p_1}, i_{p_2}, \cdots, i_{p_{l-r}})$ to denote the list by removing the elements of list I_1 from list I. For any element i_r in the set $\{i_1, i_2, \cdots, i_l\}$, we denote the number of i_r in the list I by $\#i_r$. Then the coefficient of $\prod_{i_r \in \{i_1 i_2 \cdots i_l\}} (u^{(i_r)})^{\#i_r}$ in operator s, cf. formula (10), (18) and (20), can be computed by

$$\frac{1}{\prod_{i_r \in \{i_1 i_2 \cdots i_l\}} (\#i_r)!} \sum_{\sigma \in \Sigma_l} a_l(\xi_{j_{\sigma(1)}}^{(i_{\sigma(1)})}, \eta_{j_{\sigma(1)}}^{(i_{\sigma(1)})}, \cdots, \xi_{j_{\sigma(l)}}^{(i_{\sigma(l)})}, \eta_{j_{\sigma(l)}}^{(i_{\sigma(l)})}, X). \tag{23}$$

We denote it by

$$\frac{d_l^{i_1 i_2 \cdots i_l}}{N_l(\xi_{j_1}^{(i_1)}, \eta_{j_1}^{(i_1)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}.$$

In particular, for any operator L in case $\mathbf{a}$, we have

$$d_l^{i_1 i_2 \cdots i_l} = \frac{1}{\prod_{i_r \in \{i_1 i_2 \cdots i_l\}} (\#i_r)!} \sum_{\sigma \in \Sigma_l} \prod_{r=1}^{l} (X + \sum_{s=r+1}^{l} \xi_{j_{\sigma(s)}}^{(i_{\sigma(s)})})^{i_{\sigma(r)}} c_l(\xi_{j_{\sigma(1)}}^{(i_{\sigma(1)})}, \eta_{j_{\sigma(1)}}^{(i_{\sigma(1)})}, \cdots, \xi_{j_{\sigma(l)}}^{(i_{\sigma(l)})}, \eta_{j_{\sigma(l)}}^{(i_{\sigma(l)})}, X). \tag{24}$$

For cases $\mathbf{b}$ and $\mathbf{c}$, the expression $(X + p)^{-1}$, where $p = \xi_{j_r}^{(-1)} + \sum_{s=r+1}^{l} \xi_{j_s}^{(i_s)}$ appears in formula (19). Its expansion at $X = \infty$ is of the form

$$(X + p)^{-1} = \frac{1}{X} \sum_{j=0}^{+\infty} (-\frac{p}{X})^s, \tag{25}$$

whose coefficients are polynomials in p. This does not affect the proof and result. For case $\mathbf{c}$, we need to exchange the symbols $\xi_{j_r}^{i_r}$ and $\eta_{j_r}^{i_r}$ in the proof as the discussion we did for quadratic terms. Therefore, we'll only give the proof for this case.

The immediate consequence of the previous discussion on the quadratic terms is the following statement.

Proposition 5. *All quadratic terms in operator S are quasi-local.*

In fact, we can prove this statement is valid for all $l > 0$.

Theorem 2. *Every term in the operator S, cf. formula (10), (18) and (20), obtained via (23) is quasi-local.*

Proof. We proved the linear and quadratic terms in S are quasi-local, that is, the statement is true for $l = 1, 2$. Assume it is also true for $l - 1$. By induction, the function $d_l^{i_1 i_2 \cdots i_l}$, cf. (24), is polynomial of $\xi_{j_r}^{(i_r)}$ and $\theta_{I_1} = \frac{\eta_{j_{s_1}}^{(i_{s_1})} + \cdots \eta_{j_{s_r}}^{(i_{s_r})}}{\xi_{j_{s_1}}^{(i_{s_1})} + \cdots \xi_{j_{s_r}}^{(i_{s_r})}}$, $r = 1, 2, \cdots, l - 1$, which obviously satisfies Proposition 4. This implies that the function $d_l^{i_1 i_2 \cdots i_l}$ is the sum of the parts with factors $\xi_{j_1}^{(i_1)} + \xi_{j_2}^{(i_2)} + \cdots + \xi_{j_l}^{(i_l)}$ or $\theta_{I_1} - \theta_{I_2}$ using the argument of the affine variety. From the recurrent relation in Proposition 1, we can compute the factor for any $\theta_{I_1} - \theta_{I_2}$ (without symmetrisation), which equals

$$\prod_{q=1}^{r} (X + \xi_{j_{s_{q+1}}}^{(i_{s_{q+1}})} + \cdots + \xi_{j_{s_r}}^{(i_{s_r})} + \xi_{j_{p_1}}^{(i_{p_1})} + \cdots + \xi_{j_{p_{l-r}}}^{(i_{p_{l-r}})})^{i_{s_q}} \prod_{q=1}^{l-r} (X + \xi_{j_{p_{q+1}}}^{(i_{p_{q+1}})} + \cdots + \xi_{j_{p_{l-r}}}^{(i_{p_{l-r}})})^{i_{p_q}}$$

$$- \prod_{q=1}^{l-r} (X + \xi_{j_{p_{q+1}}}^{(i_{p_{q+1}})} + \cdots + \xi_{j_{p_{l-r}}}^{(i_{p_{l-r}})} + \xi_{j_{s_1}}^{(i_{s_1})} + \cdots + \xi_{j_{s_r}}^{(i_{s_r})})^{i_{p_q}} \prod_{q=1}^{r} (X + \xi_{j_{s_{q+1}}}^{(i_{s_{q+1}})} + \cdots + \xi_{j_{s_r}}^{(i_{s_r})})^{i_{s_q}}.$$

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This expression is zero under the substitution $\Xi_{I_2} = \xi_{j_{p_1}}^{(i_{p_1})} + \cdots + \xi_{j_{p_{l-r}}}^{(i_{p_{l-r}})} = 0$ and $\Xi_{I_1} = \xi_{j_{s_1}}^{(i_{s_1})} + \cdots + \xi_{j_{s_r}}^{(i_{s_r})} = 0$ implying we have either $\Xi_{I_2}(\theta_{I_1} - \theta_{I_2})$ or $\Xi_{I_1}(\theta_{I_1} - \theta_{I_2})$ as a factor. We know

$$\Xi_{I_2}(\theta_{I_1} - \theta_{I_2}) = \theta_{I_1}(\xi_{j_1}^{(i_1)} + \cdots + \xi_{j_l}^{(i_l)}) - (\eta_{j_1}^{(i_1)} + \cdots + \eta_{j_l}^{(i_l)});$$
$$\Xi_{I_1}(\theta_{I_1} - \theta_{I_2}) = (\eta_{j_1}^{(i_1)} + \cdots + \eta_{j_l}^{(i_l)}) - \theta_{I_2}(\xi_{j_1}^{(i_1)} + \cdots + \xi_{j_l}^{(i_l)}).$$

Thus the l-th degree term of S is quasi-local. $\diamond$

This theorem implies that every terms in S^n is quasi-local. From Theorem 1, we have

Theorem 3. *The hierarchies of $(2+1)$-integrable equation with scalar Lax operators are quasi-local.*

As mentioned in Remark 1, there is one more type of scalar Lax operator in the commutative case. The same result can be proved by extending the symbolic approach used in [SW00] for the $(1+1)$-dimensional case.

4.2 The noncommutative case

The extension of the concept of quasi-locality to the noncommutative case is rather complicated. D_x and D_y are the only derivations for the commutative differential ring. The extension simply enables us to apply D_x^{-1} and D_y^{-1} on the derivations. We know that the C_f, cf. (4), are also derivations for a noncommutative associative algebra and we need to take them into consideration. Based on Proposition 4, we propose the following extension using the symbolic representation. We start with $\mathcal{A}^k$, the space of polynomials in $2k$ variables, ξ_i and η_i, $i = 1, \cdots, k$. Let $\xi_i = \lambda z_i$ and $\eta_i = \mu z_i$, where λ, μ are constants. We define $I(\mathcal{A}^k)$ as a set of all $f \in \mathcal{A}^k$ satisfying $f\big|_{z_1+\cdots z_k=0} = 0$. For the elements in $g \in I(\mathcal{A}^k)$, we compute $\frac{g}{\xi_1+\cdots+\xi_k}$ and $\frac{g}{\eta_1+\cdots\eta_k}$ and collect them in the set $\mathcal{E}xt(\mathcal{A}^k)$. Now we define a sequence of extensions of $\mathcal{A} = \cup_k \mathcal{A}^k$. Let $\mathcal{A}_0^{ext} = \mathcal{A}$ and $\mathcal{A}_1^{ext}$ be the ring closure of the union $\mathcal{A}_0^{ext} \cup (\cup_k \mathcal{E}xt(\mathcal{A}^k))$. The number of variables is a natural grade on $\mathcal{A}_1^{ext}$ and again denote $\mathcal{A}_1^{ext\,k}$ the space of elements in $\mathcal{A}_1^{ext}$ with $2k$ variables, ξ_i and η_i, $i = 1, \cdots, k$. In general, we can define $\mathcal{A}_l^{ext}$ is the ring closure of the union $\mathcal{A}_{l-1}^{ext} \cup (\cup_k \mathcal{E}xt(\mathcal{A}_{l-1}^{ext\,k}))$. Clearly, we have $\mathcal{A}_{l-1}^{ext} \subset \mathcal{A}_l^{ext}$. We now define $\mathcal{A}^{ext} = \lim_{l \to \infty} \mathcal{A}_l^{ext}$.

The above ring extension is bigger than we require. For example, the expression $\frac{\eta_2}{\xi_2} - \frac{\eta_1}{\xi_1}$ is not equal to zero under the substitution $\xi_1 + \xi_2 + \xi_3 = 0$ and $\eta_1 + \eta_2 + \eta_3 = 0$, that is, not satisfying Proposition 4. But it is in $I(\mathcal{A}_1^{ext^3})$. If we start from the symmetric polynomial, we hope to end up the extension of commutative differential polynomial, i.e. quasi-local polynomials we defined in section 2. It is not difficult to notice that this extension is bigger than the concept of quasi-local commutative polynomials. However, neither can we formulate the such extension using symbolic representation nor starting from noncommutative differential polynomials as in commutative case. It is still an open problem.

Here we compute explicitly some Lax flows for lower order scalar Lax operators of noncommutative dependent variable, which will help the reader to see the structures.

4.2.1 Noncommutative Boussinesq equation

Let us compute the hierarchy of Lax operator $L = D_x^3 + uD_x + v - D_y$. From Proposition 1, we have

$$S = X + u\frac{\xi_1^{(1)}X}{(X+\xi_1^{(1)})^3 - X^3 - \eta_1^{(1)}} + v\frac{\xi_1^{(0)}}{(X+\xi_1^{(0)})^3 - X^3 - \eta_1^{(0)}}$$

$$+ \frac{u^2(X+\xi_2^{(1)})X}{(X+\xi_1^{(1)}+\xi_2^{(1)})^3 - X^3 - \eta_1^{(1)} - \eta_2^{(1)}}\left(\frac{\xi_1^{(1)}}{(X+\xi_1^{(1)}+\xi_2^{(1)})^3 - (X+\xi_2^{(1)})^3 - \eta_1^{(1)}} - \frac{\xi_2^{(1)}}{(X+\xi_2^{(1)})^3 - X^3 - \eta_2^{(1)}}\right)$$

$$+ \frac{uv(X+\xi_1^{(0)})}{(X+\xi_1^{(1)}+\xi_1^{(0)})^3 - X^3 - \eta_1^{(1)} - \eta_1^{(0)}}\left(\frac{\xi_1^{(1)}}{(X+\xi_1^{(1)}+\xi_1^{(0)})^3 - (X+\xi_1^{(0)})^3 - \eta_1^{(1)}} - \frac{\xi_1^{(0)}}{(X+\xi_1^{(0)})^3 - X^3 - \eta_1^{(0)}}\right)$$

$$+ \frac{vuX}{(X+\xi_1^{(1)}+\xi_1^{(0)})^3 - X^3 - \eta_1^{(1)} - \eta_1^{(0)}}\left(\frac{\xi_1^{(0)}}{(X+\xi_1^{(1)}+\xi_1^{(0)})^3 - (X+\xi_1^{(1)})^3 - \eta_1^{(0)}} - \frac{\xi_1^{(1)}}{(X+\xi_1^{(1)})^3 - X^3 - \eta_1^{(1)}}\right)$$

$$+ \cdots\cdots$$

$$= X + \frac{u}{3X} + \frac{v - u\xi_1^{(1)}}{3X^2} + \left(\frac{2u\xi_1^{(1)^2}}{9} + \frac{u\eta_1^{(1)}}{9\xi_1^{(1)}} - \frac{v\xi_1^{(0)}}{3} - \frac{u^2}{9}\right)\frac{1}{X^3}$$

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$$+(\frac{-u\xi_1^{(1)3}}{9} - \frac{2u\eta_1^{(1)}}{9} + \frac{2v\xi_1^{(0)2}}{9} + \frac{v\eta_1^{(0)}}{9\xi_1^{(0)}} - \frac{uv}{9} - \frac{vu}{9} + u^2(\frac{7\xi_2^{(1)}}{27} + \frac{5\xi_1^{(1)}}{27} + \frac{\frac{\eta_1^{(1)}}{27\xi_1^{(1)}} - \frac{\eta_2^{(1)}}{27\xi_2^{(1)}}}{\xi_1^{(1)} + \xi_2^{(1)}}))\frac{1}{X^4} + \cdots,$$

which corresponds to

$$S = D_x + \frac{u}{3}D_x^{-1} + (\frac{v}{3} - \frac{u_x}{3})D_x^{-2} + (\frac{2u_{xx}}{9} + \frac{\Theta u}{9} - \frac{v_x}{3} - \frac{u^2}{9})D_x^{-3}$$

$$+(-\frac{u_{xxx}}{9} - \frac{2u_y}{9} + \frac{2v_{xx}}{9} + \frac{\Theta v}{9} - \frac{uv}{9} - \frac{vu}{9} + \frac{7uu_x}{27} + \frac{5u_x u}{27} + \frac{D_x^{-1}((\Theta u)u)}{27} - \frac{D_x^{-1}(u\Theta u)}{27})D_x^{-4} + \cdots$$

This leads to

$$S_{\geq 0} = D_x; \qquad S_{\geq 0}^2 = D_x^2 + \frac{2}{3}u; \qquad S_{\geq 0}^3 = D_x^3 + uD_x + v;$$

$$S_{\geq 0}^4 = D_x^4 + \frac{4}{3}uD_x^2 + \frac{2}{3}(u_x + 2v)D_x + \frac{4}{9}\Theta u + \frac{2}{9}u^2 + \frac{2}{9}u_{xx} + \frac{2}{3}v_x;$$

$$S_{\geq 0}^5 = D_x^5 + \frac{5}{3}uD_x^3 + \frac{5}{3}(u_x + v)D_x^2 + (\frac{5}{9}\Theta u + \frac{5}{9}u^2 + \frac{10}{9}u_{xx} + \frac{5}{3}v_x)D_x$$

$$+ \frac{10}{9}v_{xx} + \frac{5}{9}(\Theta v + uv + vu) + \frac{5}{27}(uu_x - u_x u + D_x^{-1}((\Theta u)u - u\Theta u)).$$

And we have

$$\begin{cases} u_{t_2} = -u_{xx} + 2v_x \\ v_{t_2} = v_{xx} + \frac{2}{3}(u_y - u_{3x} - uu_x + uv - vu) \end{cases}$$

Eliminating the dependent variable v from this equation and writing $t_2 = t$, we can derive the (2+1)-dimensional Boussinesq equation

$$u_{tt} = -\frac{1}{3}u_{4x} + \frac{4}{3}u_{xy} - \frac{2}{3}(u^2)_{xx} + \frac{2}{3}D_x[u, \ D_x^{-1}u_t].$$

Under the scaling transformation $x \mapsto \sqrt{6}x, \ y \mapsto \sqrt{\frac{3}{2}}y, \ u \mapsto 3u, \ t \mapsto 2t$, it can be rewritten as

$$u_{tt} = -u_{4x} + u_{xy} - 3(u^2)_{xx} + D_x[u, \ D_x^{-1}u_t].$$

Let $w = D_x^{-1}u$. We get $(2+1)$-dimensional noncommutative potential Boussinesq equation

$$w_{tt} = -w_{4x} + w_{xy} - 3(w_x^2)_x + [w_x, \ w_t]$$

4.2.2 Noncommutative KP equation

Let us compute the hierarchy of the Lax operator $L = D_x^2 + u - D_y$. Since the order of L is low, it is easier to compute the operator S directly order by order as in Exercise 4 than to apply Proposition 2. Let

$$S = D_x + a_{-1}D_x^{-1} + a_{-2}D_x^{-2} + a_{-3}D_x^{-3} \cdots.$$

Using formula (10), we obtain

$$a_{-1} = \frac{u}{2}$$

$$a_{-2} = -\frac{u_x}{4} + \frac{\Theta u}{4}$$

$$a_{-3} = \frac{\Theta^2 u}{8} + \frac{u_{xx}}{8} - \frac{u_y}{4} - \frac{u^2}{8} - \frac{1}{8}D_x^{-1}C_u\Theta u$$

This leads to

$$S_{\geq 0}^3 = D_x^3 + \frac{3u}{2}D_x + \frac{3}{4}(u_x + \Theta u)$$

$$S_{\geq 0}^4 = D_x^4 + 2uD_x^2 + (2u_x + \Theta u)D_x + \frac{u_y}{2} + u^2 + u_{xx} + \frac{\Theta^2 u}{2} - \frac{1}{2}D_x^{-1}C_u\Theta u$$

$$= 2D_x^2 D_y - D_y^2 + 2uD_y + \Theta uD_x + \frac{3u_y}{2} + \frac{\Theta^2 u}{2} - \frac{1}{2}D_x^{-1}C_u\Theta u + L^2.$$

And we have

$$u_{t_3} = \frac{1}{4}(u_{xxx} + 3uu_x + 3u_xu + 3\Theta u_y - 3C_u\Theta u),$$

which is the noncommutative KP equation and can be obtained from the recursion operator of KP, cf. [DF92], and

$$u_{t_4} = \frac{1}{2}(u_{xxy} + \Theta^2 u_y + 2uu_y + 2u_yu + u_x\Theta u + (\Theta u)u_x + C_u D_x^{-1}C_u\Theta u - \Theta C_u\Theta u - C_u\Theta^2 u),$$

which appeared in [HT03], cf. formula (2.14), with a mistake: the last two terms are missing.

4.2.3   Noncommutative mKP equation

Let us compute the hierarchy of the Lax operator $L = D_x^2 + uD_x + v + D_x^{-1}w - D_y$. Let

$$S = D_x + a_0 + a_{-1}D_x^{-1} + \cdots.$$

Using formula (10), we obtain

$$a_0 = \frac{u}{2}$$

$$a_{-1} = -\frac{u_x}{4} - \frac{u^2}{8} + \frac{v}{2} + (D_x + \frac{C_u}{2})^{-1}\frac{u_y}{4}$$

This leads to

$$S_{\geq 1} = D_x; \qquad S_{\geq 1}^2 = D_x^2 + uD_x; \qquad S_{\geq 1}^3 = D_x^3 + \frac{3u}{2}D_x^2 + \frac{3}{8}(2u_x + u^2 + 4v + 2(D_x + \frac{C_u}{2})^{-1}u_y)D_x.$$

And we have

$$\begin{cases} u_{t_2} = u_y + 2v_x + uv - vu \\ v_{t_2} = v_x + 2w_x + uv_x + uw - wu \\ w_{t_2} = -w_{xx} + (wu)_x \end{cases}$$

$$\begin{cases} u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{8}C_u u_{xx} - \frac{3}{8}uu_xu + \frac{3}{4}u_{xy} + \frac{3}{8}(u^2)_y + \frac{3}{2}v_{xx} + \frac{3}{2}v_y + \frac{3}{2}uv_x + \frac{3}{2}vu_x - \frac{3}{8}C_v u^2 \\ \qquad -\frac{3}{4}C_v u_x + 3w_x + \frac{3}{2}C_u w + \frac{3}{4}(D_y - D_x^2 - uD_x + R_{u_x} - C_v)(D_x + \frac{C_u}{2})^{-1}u_y \\ v_{t_3} = v_{xxx} + \frac{3}{2}uv_{xx} + \frac{3}{4}u_xv_x + \frac{3}{8}u^2v_1 + \frac{3}{2}vv_1 + \frac{3}{2}uw_x + \frac{3}{2}w_xu + \frac{3}{4}wu_x + \frac{3}{4}u_xw - \frac{3}{8}C_w u^2 \\ \qquad -\frac{3}{2}C_w v + \frac{3}{4}(R_{v_x} - C_w)(D_x + \frac{C_u}{2})^{-1}u_y \\ w_{t_3} = w_{xxx} - (wa_0)_{xx} + (wa_{-1})_x \end{cases}$$

The reduction $v = w = 0$ leads to the noncommutative mKP equation

$$u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{8}C_u u_{xx} - \frac{3}{8}uu_xu + \frac{3}{4}u_{xy} + \frac{3}{8}(u^2)_y + \frac{3}{4}(D_y - D_x^2 - L_uD_x + R_{u_x})(D_x + \frac{C_u}{2})^{-1}u_y. \qquad (26)$$

If we introduce new variable p satisfying $u_y = p_x + \frac{1}{2}C_u p$, The system for u and p is equivalent to the matrix mKP in [Adl].

If u does not depend on y, i.e. $u_y = 0$, this gives us the noncommutative mKdV equation, [OS98, OW00]

$$u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{8}C_u u_{xx} - \frac{3}{8}uu_xu.$$

If u takes its value over $\mathbb{C}$ (commutative), this gives us the well known mKP equation

$$u_{t_3} = \frac{1}{4}u_{xxx} - \frac{3}{8}u^2 u_x + \frac{3}{4}\Theta u_y + \frac{3}{4}u_x\Theta u. \qquad (27)$$

We know Miura transformation $V = u_x - \frac{u^2}{2} + \Theta u$ transforms the mKP equation (27) into the KP equation $V_{t_3} = \frac{1}{4}V_{xxx} + \frac{3}{4}VV_x + \frac{3}{4}\Theta V_y$. However, we do not know a Miura transformation for noncommutative mKP (26).

4.2.4 Nonsymmetric Novikov-Veselov equation

Consider the Lax operator $L = u + D_x^{-1}v - D_y$ (case **c**) [1]. Let $S = D_x + a_0 + a_{-1}D_x^{-1} + \cdots$. Using formula (7), we have

$$a_0 = (C_u - D_y)^{-1}u_x;$$
$$a_{-1} = (C_u - D_y)^{-1}(v_x + C_v a_0),$$

leading to $S^1_{\geq 1} = D_x$, $\quad S^2_{\geq 1} = D_x^2 + 2a_0 D_x$, $\quad S^3_{\geq 1} = D_x^3 + 3a_0 D_x^2 + 3(a_{0x} + a_{-1} + a_0^2)D_x$. From the Lax equation, we obtain

$$
\begin{cases}
u_{t_2} = u_{xx} + 2v_x + 2a_0 u_x + 2a_0 v - 2v a_0 \\
v_{t_2} = -v_{xx} + 2(v a_0)_x
\end{cases}
$$
$$
\begin{cases}
u_{t_3} = u_{xxx} + 3(a_0 u_x)_x + 3a_{-1}u_x + 3a_0^2 u_x + 3a_0 v_x + 3(v a_0)_x - 3C_v(a_{0x} + a_{-1} + a_0^2) \\
v_{t_3} = v_{xxx} - 3(v_x a_0)_x + 3(v a_{-1})_x + 3(v a_0^2)_x
\end{cases}
$$

The reduction $v = 0$ leads to

$$u_{t_2} = u_{xx} - 2\{(D_y - C_u)^{-1}u_x\}u_x.$$

Let $w = (D_y - C_u)^{-1}u_x$. This equation transforms into the noncommutative Burgers equation $w_{t_2} = w_{xx} - 2ww_x$, which is linearisable, that is, we obtain $p_t = p_{xx}$ by $pw = -p_x$.

There is no reduction $u = 0$ in noncommutative case. In the commutative case, the reduction $u = 0$ leads to the nonsymmetric Novikov–Veselov equation [NV86]

$$v_{t_3} = v_{xxx} - 3(v\Theta^{-1}v)_x.$$

5 Conclusion

We have demonstrated the power of the symbolic representation in the study of $(2+1)$-dimensional integrable equations. It enables us to produce the hierarchies of $(2+1)$-dimensional integrable equations from the scalar Lax operators and to study their properties globally. We proved the conjecture of Mikhailov and Yamilov on the ring extension for $(2+1)$-dimensional commutative integrable equations derived from scalar Lax operators. The method can easily adapt to the case of $(2+1)$-dimensional differential-difference integrable equation with scalar Lax operators [BS01].

Noncommutative integrable equations can be considered as quantised analogs of classical integrable equations, which is very important in string theory. Right after finishing the paper, we noticed a few papers on constructing noncommutative integrable equations in the framework of the Sato theory motivated by noncommutative gauge theories, cf. [HT03, Ham05] and references therein. However, the authors only used an ansatz to derive the equations from some scalar Lax operators in case **a** instead of this systematical manner. There are few papers on noncommutative $(2+1)$-dimensional integrable equations derived from the scalar Lax operators in cases **b** and **c** such as noncommutative mKP, where the new type of nonlocal terms such as $(D_x + \frac{C_u}{2})^{-1}u_y$ and $(D_y - C_u)^{-1}u_x$ appear. To study these equations further such as constructing their Hamiltonian operators and soliton solutions may provide a deeper understanding of noncommutative geometry.

Acknowledgments

Appendix: Proof of Proposition 4

Proof. We need to show (cf. (14) for all $l \geq 2$ that

$$b_{l-1}\left(-\sum_{s=2}^{l}\xi_{j_s}^{(i_s)}, -\sum_{s=2}^{l}\eta_{j_s}^{(i_s)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}\right) = b_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X). \quad (28)$$

---

[1] In the commutative case, this worked out in paper [BS01] by both central extension approach and the operand approach.

We prove it by induction. It is true for $l = 2$ since

$$b_1(-\xi_{j_2}^{(i_2)}, -\eta_{j_2}^{(i_2)}, X + \xi_{j_2}^{(i_2)}) = \frac{-\xi_{j_2}^{(i_2)}}{N_1(-\xi_{j_2}^{(i_2)}, -\eta_{j_2}^{(i_2)}, X + \xi_{j_2}^{(i_2)})} = \frac{\xi_{j_2}^{(i_2)}}{N_1(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X)} = b_1(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, X).$$

Assume that formula (28) is valid for $l - 1$. Then its the left-hand side equals

$$
\begin{aligned}
&\frac{b_{l-2}(-\sum_{s=2}^l \xi_{j_s}^{(i_s)}, -\sum_{s=2}^l \eta_{j_s}^{(i_s)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)} \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, X + \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)})}{N_{l-1}(-\sum_{s=2}^l \xi_{j_s}^{(i_s)}, -\sum_{s=2}^l \eta_{j_s}^{(i_s)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})} \\
&\quad - \frac{b_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})}{N_{l-1}(-\sum_{s=2}^l \xi_{j_s}^{(i_s)}, -\sum_{s=2}^l \eta_{j_s}^{(i_s)}, \xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})} \\
=\ &- \frac{b_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)} \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)}, \eta_{j_{l-1}}^{(i_{l-1})} + \eta_{j_l}^{(i_l)}, X)}{N_1(\xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
&\quad + \frac{b_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})}{N_1(\xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
=\ &\frac{b_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)} \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, X + \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)}) - b_{l-3}(\xi_{j_3}^{(i_3)}, \eta_{j_3}^{(i_3)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})}{N_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) N_1(\xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
&\quad - \frac{b_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)} \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, X + \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)})}{N_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)}, \eta_{j_{l-1}}^{(i_{l-1})} + \eta_{j_l}^{(i_l)}, X) N_1(\xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
&\quad + \frac{b_{l-3}(\xi_{j_3}^{(i_3)}, \eta_{j_3}^{(i_3)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)}, \eta_{j_{l-1}}^{(i_{l-1})} + \eta_{j_l}^{(i_l)}, X)}{N_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)}, \eta_{j_{l-1}}^{(i_{l-1})} + \eta_{j_l}^{(i_l)}, X) N_1(\xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)};
\end{aligned}
$$

Meanwhile, the right side gives us

$$
\begin{aligned}
&\frac{b_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)} \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) - b_{l-2}(\xi_{j_3}^{(i_3)}, \eta_{j_3}^{(i_3)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}{N_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
=\ &\frac{b_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, X + \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)})}{N_1(\xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) N_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
&\quad - \frac{b_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-2}}^{(i_{l-2})} + \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-2}}^{(i_{l-2})} + \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)})}{N_1(\xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) N_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)} \\
&\quad - \frac{b_{l-3}(\xi_{j_3}^{(i_3)}, \eta_{j_3}^{(i_3)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X + \xi_{j_l}^{(i_l)}) - b_{l-3}(\xi_{j_3}^{(i_3)}, \eta_{j_3}^{(i_3)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})} + \xi_{j_l}^{(i_l)}, \eta_{j_{l-1}}^{(i_{l-1})} + \eta_{j_l}^{(i_l)}, X)}{N_1(\xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X) N_{l-1}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_l}^{(i_l)}, \eta_{j_l}^{(i_l)}, X)}.
\end{aligned}
$$

The difference between the above two expression vanishes due to the induction assumption, which implies that

$$
\begin{aligned}
&\frac{N_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, X + \xi_{j_{l-1}}^{(i_{l-1})}) b_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)} \cdots, \xi_{j_{l-2}}^{(i_{l-2})}, \eta_{j_{l-2}}^{(i_{l-2})}, X + \xi_{j_{l-1}}^{(i_{l-1})})}{N_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X) N_1(\xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X)} \\
=\ &\frac{b_{l-3}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-2}}^{(i_{l-2})} + \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-2}}^{(i_{l-2})} + \eta_{j_{l-1}}^{(i_{l-1})}, X)}{N_1(\xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X)} - \frac{b_{l-3}(\xi_{j_3}^{(i_3)}, \eta_{j_3}^{(i_3)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X)}{N_{l-2}(\xi_{j_2}^{(i_2)}, \eta_{j_2}^{(i_2)}, \cdots, \xi_{j_{l-1}}^{(i_{l-1})}, \eta_{j_{l-1}}^{(i_{l-1})}, X)}.
\end{aligned}
$$

By now we proved our proposition. $\diamond$

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